Extended Essay. Mathematics.

RSA: Encrypting, Decrypting, Hacking.

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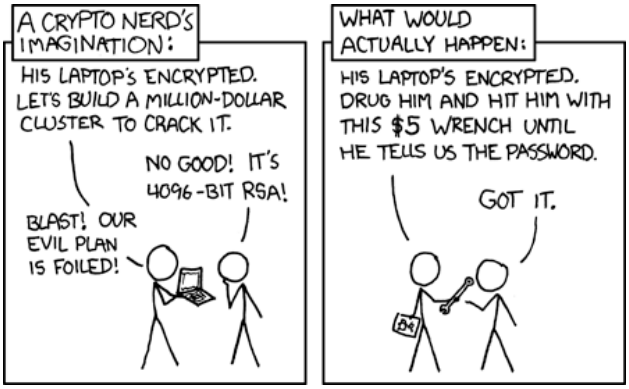


Figure 1: "Security" webcomic by Randall Munroe. Used with permission. [1]

Abstract

By starting this Extended Essay with a lovely webcomic by Randall Munroe, one can guess that the "4096-BIT RSA" sounds like it is tough to crack. However, what is "RSA" and how does it work that even a "MILLION-DOLLAR CLUSTER" is not enough to crack it? The research question of this Extended Essay is - "Does RSA provide secure data transmission and confidential certificate signatures?"

This Extended Essay will explore the history of RSA algorithm, the structure of the cryptographic technique, usage of it and ways to crack the algorithm. For the sake of demonstration and proof, I have written several programs in C Language [3] that can help us to understand the process "behind the curtains". After all these parts, at the end, we can conclude how RSA technique is secure.

Word Count : 133

1 Introduction

First time when I heard about RSA technique, was during the ITGS class in IB course. In ITGS Course Companion [4] there is a chapter about Security and Secret Key Encryptions. As an example, the author included two cases of encryption: Symmetric Key Encryption or Single Key Encryption and Asymmetric key encryption, more known as Public Key Encryption.

During the lesson, the fact that a message can be encrypted by one key and only a second key can decrypt the message intrigued me. After that, I did some research on asymmetric encryptions, particularly about RSA and other cryptographic technique based on it. I found that RSA is not only used for messages encryption, but also for signing online certificates to prove one's identity. I chose this topic for my Extended Essay because I believe the Mathematics in the algorithm is simple and elegant; also, it is safe to say that RSA is used everywhere and the Internet trust relies on this algorithm.

RSA encryption algorithm is essential if not the most critical security algorithm today, as nearly every system in the world uses this asymmetric encryption technique to encrypt and decrypt data. In this EE, I will be researching the methods of RSA: the way the algorithm works, why it is considered secure, where and how it is used, mathematical proof with examples and ways to break or brute-force the algorithm.

2 Historical background

Ever since people began to write down events or private information, there has been a need for cryptography. Cryptography is a study of techniques of encrypting a text in such a manner that outsiders to the code cannot understand the contents, but the desired reader can decrypt and read the message in its original form. Just like for an early man and modern man, there always has been a need for secrecy.

2.1 Caesar Cypher

First records of known cryptographic techniques go back to the times that were Before the Common Era, in the times Julius Caesar. Caesar cypher or Caesar's code is one the first methods of encrypting texts. It works in a way that there are a text and a key, where the key is an integer value.

The key shows a shift of letters in the document according to the Latin alphabet. For example, if the key is two, then the first letter of the alphabet will become third, the second will become forth and so on. During his lifetime, Julius Caesar mostly used three as a shift to protect messages of military significance. [5]

2.2 Enigma Machine

We can say for sure that the moment, when encryption was the tool of secretly transmitting data so important and valuable that lives of thousands of people depended on how well that information is transferred from one end to another, is World War II.

Enigma machine was the German perfection of cryptography and engineering, the device so small and so complex, that it produced code, which was considered as impossible to crack. However, during the war, a brilliant mathematician and cryptographer Alan Turing found a flaw in Enigma Machine and was able to build the first computer to exploit the system's vulnerability [6].

2.3 Flaw in Symmetric Cryptography

All existing algorithms had one most significant vulnerability in all of them-both ends should have known the key or secret passphrase, which is used to encrypt and decrypt messages. In the example with Enigma machine, German radio operators had a book with Enigma codes for the next month, and it was a massive risk if hostile troops possess the codes.

2.4 Whitfield Diffie and Martin Hellman

The first attempt at solving the problem of symmetric algorithms goes back to a pair of cryptologists: Whitfield Diffie and Martin Hellman. In the book by Simon Singh [5], there is a story about how these cryptographers came up with the idea of a public-key encryption: "I walked downstairs to get a Coke, and almost forgot about the idea. I remembered that I'd been thinking about something interesting, but couldn't quite recall what it was. Then it came back in a real adrenaline rush of excitement. I was aware for the first time in my work on cryptography of having discovered something really valuable."

They have discovered a revolutionary type of cyphering – asymmetric cryptography. This means that it solved the problem of single key encryption, where encoder and decoder needed to know the passphrase. Unfortunately, they did not find a one-way function for solving the problem, they have published the concept in 1976 and left it open.

2.5 Rivest, Shamir, and Adleman

The paper released by Whitfield Diffie and Martin Hellman showed that there was a solution for the problem of symmetric encryption in the form of a one-way function. Another trio of researchers made the actual discovery and proof: Ron Rivest, Adi Shamir, and Leonard Adleman. Rivest, Shamir, and Adleman were a perfect team, as Rivest was a computer scientist, who was tracking all the latest scientific papers and with an ability to implement

something new in new places. Shamir is also a computer scientist with a lightning intellect and ability to directly focus on the core of the problem. Adleman was a mathematician with broad knowledge and patience, so during the research, he was finding flaws in the works of Rivest and Shamir, and ensured that cryptographers did not follow a false lead. [5]

A year later Ron Rivest with the help of Adi Shamir and Leonard Adleman has made a breakthrough by finding the desired one-way function, thus solving the problem of symmetric encryptions. In 1977, the trio published a full paper about the new asymmetric encryption technique, which was named after its creators- R.S.A. [2]

3 The Usage of RSA

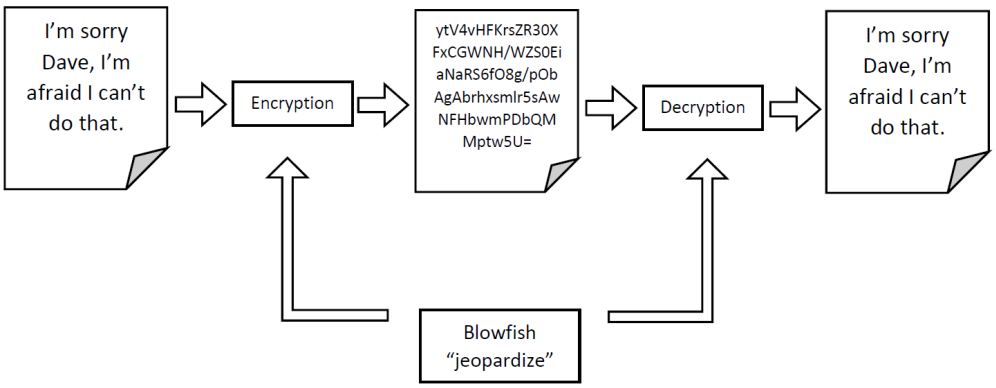
As an algorithm and cryptographic technique, RSA is brilliant without a doubt. It is safe to say that RSA and its versions are used everywhere in our digital world. It is crucial for modern cryptography techniques to be sophisticated for computers to brute-force but not too complicated and inefficient for computers to encrypt and decrypt data. RSA is located right in the middle, as the operations performed in part 4 do not need a lot of computational power but to attack it requires a lot of it. For more information please refer to part 6

To demonstrate the work and the functionalities of the algorithm, three

types of cryptography techniques will be displayed: single key encryption, assymetric encryption and certificate signatures.

3.1 Symmetric Encryption

This type of encrypting and decrypting algorithms is the oldest way to cypher data. It works with one key, one text and two ends. On the first end, there is an encoder, who cyphers the text with a known key and then sends the encrypted data to the other end, decoder, who later decrypts text with the same key. The example with Blowfish [7] algorithm with key "jeopardize" can be seen below in Figure 2.



1

Figure 2: Example of Symmetric Encryption

The downside is that both ends of the transmission should know the shared key. In the scenario where people have met and shared the encryption key in real life, there is no danger that someone can steal the shared key and decrypt transmissions. The problem arises when we have a scenario

where parties on both ends never talked with each other and are located on distinct physical locations. Both ends want to transmit data securely, so the eavesdropper in the middle will not understand the contents of the transmitted information.

In this scenario, it is impossible to transmit data securely, because to decrypt text messages, both ends should know the same key. Before the conversation, one end should send the key to the other end; however, the eavesdropper will be able to ”sniff” the key and will be able to understand all further encrypted transmissions, thus compromising all further interactions.

3.2 Asymmetric Encryption

Asymmetric encryption relies on two distinct keys, where one is used for encrypting data and the second one is for decrypting data that was decoded by its unique pair. Keys for encrypting and decrypting are called public and private, respectively. There are two ways of using RSA: to send an encrypted message to the decoder and signing certificates for proving one's identity.

3.2.1 Encrypting Messages

Every user has a pair of connected keys: a public key and private key. The public key is available to everyone while the private key should be kept private

to the owner of the unique pair of keys. It is a preferred way of encryption; because both ends do not need to exchange the secret key, they would already know the required part of the pair to transmit messages securely. Continuing from part 3.2, this is widely used for secure data transmissions.

Imagine a scenario with three users: Hal, Dave, and Eavesdropper. If Hal wants to send a message to Dave, Hal should use Dave's public key and encrypt the desired message, later Hal sends an encrypted message to Dave. As eavesdropper is sitting in the middle of Hal and Dave's transmission line, he can intercept the transmitted data, make a copy of it and send it forward to Dave, thus having a copy of the letter. Now, Dave and Eavesdropper have the encrypted message from Hal. Dave, using the private key that is available only to him, can decrypt the message from Hal and read the contents. Fortunately, the eavesdropper has access only to Dave's public key and not to his private key, so he is not able to read message's contents.

The transmission is shown in Figure 3.

3.2.2 Signing Certificates

RSA encryption and decryption works in both ways, so a message can be encrypted by a public key and decrypted by the private key and vice versa.

Imagine a scenario, when Hal wants to send an email to Stanley. The problem is that Hal is not sure if Stanley is indeed Stanley and not a thief who stole Stanley's identity. To resolve the issue, RSA technique should be

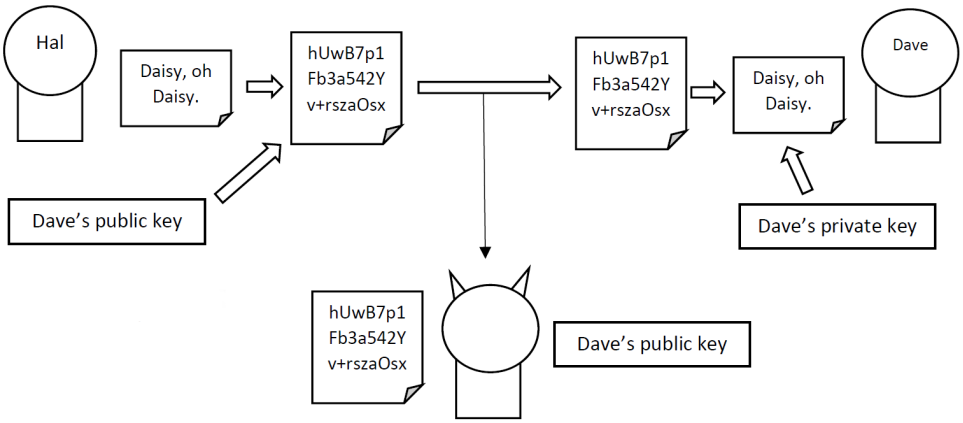


Figure 3: Example of Asymmetric Encryption

used.

Hal and Stanley have a common friend - Dave. Dave can prove Stanley's identity by signing Stanley's public key with his private key. Later, Hal can decrypt Stanley's public key with Dave's public key. With this, Hal would be confident about Stanley's identity.

This whole process is based on trust. If Person A will sign a key of an untrusted entity, Person B, all other parties would lose trust in Person A as he compromised secure network by signing unknown public key.

4 Mathematical Foundation of RSA

The mathematics involved in keys generation and RSA algorithm require knowledge on modular arithmetic, greatest common divisors, Euler's totient function, prime number theorem. In order to make the calculations understandable and not hard to follow, I will make an introduction to required theorems and formulas before the RSA steps.

4.1 Modular Arithmetic and Congruence

In mathematics, the modulus is the operation to get the remainder when diving one number by another. For example, if we want to find modulus of 31 to 9, we use the notation: 31 mod 9, this means that we divide 31 by 9 and take out only the remainder, in this case, it is going to be 4. Congruence means that if two numbers b and c have the property that their difference $b - c$ is integrally divisible by a number m, so that $(b - c)/m$ is an integer, then b and c are said to be "congruent modulo m", where m is called the modulus. The notation as follows:

$$b \equiv c \ (\mathrm{mod} m)$$

4.2 Euler's Totient Function

The Euler's totient function $\varphi(n)$ is defined as a function that returns a number of positive integers $< n$ that are relatively coprime to n, this means that they are not sharing any common factor. For example, the totatives of $n = 9$

are $1, 2, 4, 5, 7$ and 8. So that $\varphi(9) = 6$. For prime number p, the number of totatives will be $p-1$, as no integers share same multiples with p, except p.

4.3 Greatest Common Divisor

Greatest common divisor, using the notation $gcd(a, b) = c$, means that a and b share some common divisors and where c is the largest divisor. For example: $gcd(24, 9) = 3$. If we have an expression that $gcd(d, e) = 1$, it means that d and e are coprime because they are both integrally divisible only by one.

5 RSA

There are five main steps to generate public and private keys and two steps for encryption and decryption together. More detailed information about that in the next parts.

5.1 Key Generation

1. Generate two distinct primes p and q

    - Primes p and q are recommended to be from 100 digits to 200 digits long. Because of security reasons, primes p and q should be chosen at random, be similar in magnitude, but differ in digits number, so the factoring would be harder.

- The factoring problem will be discussed in chapter 5.

2. Find $n = p \times q$

   - It is not difficult for computers to multiply values of p and q, even if they are primes and big in size, in terms of amount of bits.
   - During the encryption and decryption steps, n would be used as a modulus for public and private keys.
   - Usually, the key length is referred to the length of n. The bigger the key size is, then keys are considered more secure.

3. Compute $\varphi(n) = \varphi(p) \times \varphi(q) = (p-1) \times (q-1)$.

   - Referring to part 4, we know how the Euler totient formula works and how it is applicable for prime numbers.
   - This value should be kept private.

4. Choose e, so that $1 < e < \varphi(n)$ and $gcd(e), \varphi(n)) = 1$.

   - e is our public key component, which is public and should be coprime to Euler totient function of n.

5. Determine d, as $d \times e \equiv 1 \bmod (\varphi(n))$.

   - Referring to part 4, this can be solved by knowing that

$$d \times e \equiv k\varphi(n) + 1$$

   All variables in the equation are positive integers.

5.2 Encryption and Decryption

After completing the necessary calculations, which are described in part 4.1, the large prime numbers p, q, and totient value of n- $\varphi(n)$ should be deleted or not revealed to anyone except the host, where keys were generated. We have two generated keys: a public key and private key. The host should share his/her public key, so it is available to anyone and keep the private in secret. As the RSA encryption method is described in part 3.2, we will continue now with the process of encryption and decryption, later on, accompanied by two examples.

Let us return to the scenario with Dave and Hal in part 3.2. Hal wants to send Dave a message, so he knows only Dave's public key. To encrypt the message and transmit it securely, Hal firstly converts his message from plaintext to numerical values, for example, binary, octal, decimal, and hexadecimal etc. Because of Dave's public key, Hal knows the public exponent e and modulus n. By using the encryption formula below

$$c = m^e \bmod n$$

Where m is converted text and c is the encrypted message, Hal now has a uniquely encrypted message and so can send it to Dave. On the other end of the communication line, Dave receives the encrypted message c and by knowing private exponent d and modulus n from his private key, Dave and only Dave can decrypt the following message using the formula below.

$$m = c^d \bmod n$$

It can be seen that the formula for encryption and decryption are inverse, the only exceptions are the public and private exponents, which are connected to each other, as only in right pairs, encryption and decryption would take place. Thus, this is considered a one-way function, so without knowing the public exponent and private exponent, at the time it is considered impossible to brute-force private exponent. The reason for this will be explained in part 6.

5.3 Examples of RSA

In this part I will include two RSA examples.

5.3.1 First Example with Small Primes

For the first example, we will use small primes in the range from 2 to 20. For this part, I will make some of the calculations by hand, after 4th step, I will use Wolfram Alpha to compute data.

1. We should choose our prime numbers p and q. In this example, the values are

$$p = 11 \text{ and } q = 13$$

2. In order to calculate modulus, I will multiply these two primes and the

result will be the modulus n

$$n = p \times q = 11 \times 13 = 143$$

3. Using Euler's totient function, we can find the number of integers that do not share any common multiples with modulus n.

   $\varphi(n) = \varphi(p) \times \varphi(q) = (p-1) \times (q-1) = (11-1) \times (13-1) = 10 \times 12 = 120$

4. Choosing the right public exponential e is very important because we want the encryption process to be fast, for that, we should pick a secure and reasonably small value. For this example, $e = 7$

5. Now we should find the private exponent

$$d \times e \equiv 1 \bmod (\varphi(n))$$
$$d \times e \bmod (\varphi(n)) \equiv 1 \bmod (\varphi(n)) = 1$$
$$d \times e = k \times \varphi(n) + 1$$

   Where k is some positive integer constant, so that $(k \times \varphi(n) + 1)/(e)$ is an integer

$$d = \frac{k \times \varphi(n) + 1}{e}$$

Substituting variables from our example, we will get that $d = 103$.

Now, we have calculated public and private keys:

$$\text{public key} = (e, n) = (7, 143)$$

$$\text{private key} = (d, n) = (103, 143)$$

Now, let us try to encrypt some message and then decrypt it. In order to prove the work of RSA, I will encrypt the message – "ATTACK AT DAWN".

Firstly, we need to convert the plaintext into numbers, so we can use the encryption and decryption formula. Table 1 below presents the way we will convert the plaintext.

Table 1: Table on How Individual Characters Will be Converted into Digits

A	B	C	D	E	F	G	H	I	J	K	L	M
01	02	03	04	05	06	07	08	09	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

After converting each letter, we will get the following: 012020010311012004012314. We consider that spaces between words are not convertible and we skip them. To encrypt the message efficiently, we will encrypt each pair using our public key, where our public exponent is 7 and modulus is 143. The following calculations have been completed by using Wolfram Alpha. I will calculate only unique letters' codes and then combine it to make an encrypted message. Notation $Encrypt(N)$ will be used for simplicity, where N is the number of

a letter or a letter from Latin alphabet. "???" will be used to represent a letter that cannot be converted from digits, according to the table above.

$$Encrypt(A) = \quad Encrypt(01) = 01^7 \bmod 143 = 1 \bmod 143 = 01 = A$$

$$Encrypt(C) = \quad Encrypt(03) = 03^7 \bmod 143 = 2187 \bmod 143 = 42 = ???$$

$$Encrypt(D) = \quad Encrypt(04) = 04^7 \bmod 143 = 16384 \bmod 143 = 82 = ???$$

$$Encrypt(K) = \quad Encrypt(11) = 11^7 \bmod 143 = 19487171 \bmod 143 = 132 = ???$$

$$Encrypt(N) = \quad Encrypt(14) = 14^7 \bmod 143 = 105413504 \bmod 143 = 53 = ???$$

$$Encrypt(T) = \quad Encrypt(20) = 20^7 \bmod 143 = 1280000000 \bmod 143 = 136 = ???$$

$$Encrypt(W) = \quad Encrypt(23) = 23^7 \bmod 143 = 3404825447 \bmod 143 = 23 = W$$

The original message's code is 012020010311012004012314, will be encrypted to 011361360142132011368201235 3, which obviously does not look like our initial input. This is the encrypted message; it is non-readable and non-understandable to all users. Only the receiver with the private key can decipher the message and find the initial content.

It is noticeable that encrypted version of A and W are the same. This can be viewed as a flaw in RSA, however, the probability of this correspondence is very low and individual letters would not compromise the message's initial contents.

As we have the private key with the private exponent, we can decipher the encrypted message and calculate the content of the initial message. To do that so, the decryption formula will be used. Rising numbers to power 103 will give us very big numbers. Calculations below include only the result, for the extended calculations please refer to A. Function called $Decrypt(N)$ will be used for simplicity to represent decryption process, where N is an encrypted number that should be deciphered with the appropriate private key.

$$
\begin{aligned}
Decrypt(01) = &\quad 01^{103} \bmod 143 = 01 = A \\
Decrypt(42) = &\quad 42^{103} \bmod 143 = 03 = C \\
Decrypt(82) = &\quad 82^{103} \bmod 143 = 04 = D \\
Decrypt(132) = &\quad 132^{103} \bmod 143 = 11 = K \\
Decrypt(53) = &\quad 53^{103} \bmod 143 = 14 = N \\
Decrypt(136) = &\quad 136^{103} \bmod 143 = 20 = T \\
Decrypt(23) = &\quad 23^{103} \bmod 143 = 23 = W
\end{aligned}
$$

After the decryption, we can rebuild the initial message- "ATTACK AT DAWN". RSA has proved its work with small values of p and q, we can test RSA for bigger values.

5.3.2 Second Example with Large Primes

In order to work with big primes, calculations by hand or Wolfram Alpha will become inefficient and quite difficult. To demonstrate the work of RSA, I wrote a program in C. The program takes an input of modulo - n, public exponent - e, private exponent – d and a message that needs to be encrypted. More detailed information about the code in AppendixC.
The input message will be encoded in hexadecimal by using ASCII table. Using the message's representation in hexadecimal, the program will encrypt it using the public key, later will make an attempt of reading the contents of the encrypted message and finally decrypting the message using the private key. I wrote this program for this Extended Essay to demonstrate the work of RSA encryption algorithm with big primes and proving the security of RSA later in section 6. The input and output of the program will be shown in boxes below. Calculations have been made on a computer with specifications that can be found in Appendix B.

I have entered values 516311845790656153499716760847001433441357, 65537 and 561784318784495317030846362223028337629868 5 for modulo n, public exponent e, and private exponent d, respectively. p and q values are unknown. We will encrypt message – "The cake is a lie". The executed program shows the hexadecimal version of our message and the encrypted value of our message. When we try to read it, we get "c?SX?|^T??tv??", which is not our initial message. The encrypted value will be deciphered, using the private key. We will get our initial message – "The cake is a lie." Further

details and examples can be found on my Github [8].

6 Security of RSA

As mentioned above, RSA encryption algorithm is considered secure. The key generation is not very expensive operation in terms of computer resources. For computers, multiplication is a very easy task and they can do it easily. On the other hand, if we know n, to find two prime numbers p and q is a tough challenge. There is a challenge, where people around the world are trying to factor values of n, thus meaning finding initial prime p and q. [9]

RSA Factoring Challenge is a challenge by RSA Laboratories. They have calculated multiple n on a computer with no network connection and all the hard are destroyed in order to keep p and q a secret.

The factoring is a very big mathematical challenge because in general we will need to brute-force all possible combination and find the correct match.

6.1 Full Brute-Force

Full brute-force algorithm will try all p and q number from 2 to n and will try to find a match. My implementation of this algorithm can be found in Appendix E. Generally, the algorithm runs two nested loops, each with n steps. The performance can be described as $O(n^2)$, where n^2 is the ex-

pected number of calculations required during the worst-case scenario. This algorithm is very slow, for example, some n with a value of 9237079 will require $9237079^2 = 85323628452241$ steps, which will take the system from Appendix B approximately

$$\frac{9237079^2 \text{ operations}}{310984726 \text{ operations per second}} = 274365 \text{ seconds}$$

And when converted to hours

$$274365 \text{ seconds} = \frac{274365 \text{ seconds}}{3600 \text{ seconds in one hour}} \approx 76.21 \text{ hours}$$

In a worst-case scenario, it would take the algorithm approximately 76 hours to factorize 9237079. The time spent on factorizing will be increasing exponentially and very rapidly, so full brute-force cannot be used to effectively factorize moduli.

For the sake of demonstration, I wrote a program in C that factorizes input number by using all integers. When the program finds the correct match, it exists, so it would not spend time on further calculations. [8]

```
$ ./factor_with_integers 9237079
Primes for 9237079 are : 2351 and 3929

Execution time in seconds : 60.857919

```

Listing 1: Demonstration of Prime Factorization with Integers

It took the computer only 60 seconds because the executed algorithm used best-case scenario and was optimized. [8]

6.2 Brute-Force Only With Prime Numbers

The algorithm in part 6.1 is too slow because it loops through all numbers less than n for all numbers that are less than n. We know that p and q are primes, so instead of looping through all numbers below n, we can parse only through prime numbers that are less than n.
Unfortunately, there is no formula to find the exact number of prime numbers before n - $\pi(x)$, we will need to use the formula from Prime Number Theorem that approximates the number of primes that less than n [10].

$$\pi(x) = \frac{x}{ln(x)} \tag{1}$$

I wrote another code in C [3] Language that factorizes n, by iterating only through prime numbers. My implementation of the code is in Appendix F and the system is from Appendix B. It is expected for the worst-case performance to be $O(\pi(n)^2)$ or $O((\frac{n}{ln(n)})^2)$. Because this algorithm loops only prime numbers, it would need to go through a fewer number of iterations, thus making the algorithm more efficient. In order to demonstrate the work of the algorithm, I have written another C program that will factorize input and find its primes.

```
$ ./factor_with_primes 9237079
Primes are : 2351 and 3929
```

```
3
4  The results and additional data :
5      Number of primes generated to factorize 9237079 : 617252
6      Prime index of p (2351) : 349
7      Prime index of q (3929) : 545
8
9  Execution time in seconds : 2.484568
10
```

Listing 2: Demonstration of Prime Factorization with Primes

Here is the difference and the reason why the second version runs faster than the full brute-force. For more details, please refer to my GitHub with the open code, comments and documentation.

6.3 Fast Factorizing Algorithm

There are fast factoring algorithms, for example as described in original RSA paper [2], they suggest factoring algorithm by Richard Schroeppel

$$(ln(n))^{\sqrt{\frac{ln(n)}{ln(ln(n))}}} \tag{2}$$

The average worst-case scenario is approximately $O(n^{1/4})$. Even for fastest algorithms, the difficulty of breaking the key will rise exponentially with each new digit. Table 2 shows the performance of one of the fastest factoring algorithms.

In this Extended Essay, other factorization methods will not be discussed.

Table 2: The Relationship between factoring time and a number of digits in moduli. [2]

Digits	*Number of operations*	*Time*
50	1.4×10^{10}	3.9 hours
75	9.0×10^{12}	104 days
100	2.3×10^{15}	74 years
200	1.2×10^{23}	3.8×10^{9} years
300	1.5×10^{29}	4.9×10^{15} years
500	1.3×10^{39}	4.2×10^{25} years

7 Conclusion

The main question of this Extended Essay is "Does RSA Provide Secure Data Transmission and Confidential Certificate Signatures?". Now, it is possible to the question as we have seen how the algorithm works in parts 3, 3.2, 4, 5 and how it can be hacked from part 6.

Throughout the investigation, we see how the RSA algorithm is sophisticated and logical. Not too complicated for computers to calculate it and exponentially difficult to brute-force. My program from Appendix C and D demonstrated how quickly a relatively slow computer B is able to encrypt and decrypt digital data. On the other hand, programs from Appendix E and F demonstrated the complications that can arise even by the slightly increasing size of the moduli.

Over the last decades, RSA has proven its stability and efficiency, therefore it is actively used by almost all companies and organizations to ensure one's privacy and prove the identity of users, thus increasing the overall trust

on the web and making the world a safer place.

A Extended Calculations from Decryption Process

Extended decryption process from 5.3.1. Calculated by using Wolfram Alpha. [11]

$1^{103} \bmod 143 = 1 \bmod 143 = 1$

$42^{103} \bmod 143 =$
$156558593484049694248037903244716469416986497765907719578962641203202261783735879353364477197259663386113307908088797795612644883768455171861184892841357579841425113088 \bmod 143 = 3$

$82^{143} =$
$132686517441513244111225587492420539729722593313714217827377620906644373338237359944452727878988311651377571872288819722347633861743847454494908819785858229566567822905304408192731118486558624186368 \bmod 143 = 4$

$132^{103} \bmod 143 =$
$262491300277461517692742508369989088620996574052001660598047866636259043541351641556151649081914219419551303756509079237489233646749010382622071994394592374767244175086181744513186988002964207522501577241039475172179968 \bmod 143 = 11$

$53^{103} \bmod 143 =$
$398487377186516560079475923337097952710442774239626447456189754533149897926474993995764869013491113276517214848430890042669891112368896492613013653374784412947984191133953236287 7 \bmod 143 = 14$

$136^{103} \bmod 143 =$
$568208285374026424445035277191997294623042602130796035434225124563992615875626229261345040816483042730870361180374085935670061175669161047759186221555826806044013255300470788548878847931178036550242760793369026888309145 6 \bmod 143 = 20$

$23^{103} \bmod 143 =$
$181120292057919100304261868988672529740587284827801252809582215541609345699988044561638130873255478567108709697047335431227782547521001304167 \bmod 143 = 23$

B About the Testing System

The computer with the specifications and OS [12] [13] was used to develop, test and run all demonstration programs [8] in this Extended Essay.

```
  ~ neofetch
        `++              thecsw@hpstream
      -yMMs              ----------------
    `yMMMMN`             OS: BunsenLabs GNU/Linux 8.9 (Hydrogen)
   -NMMMMMMm.            Host: HP Stream Notebook PC 11 Type1
  :MMMMMMMMMN-           Kernel: 3.16.0-4-amd64
 .NMMMMMMMMMMM/          Uptime: 3 hours, 53 mins
 yMMMMMMMMMMMMM/         Packages: 1748
`MMMMMMNNMMMMMMN.        Shell: zsh 5.0.7
-MMMMN+  /mMMMMMMMy      Resolution: 1366x768
-MMMm`     `dMMMMMM      WM: i3
`MMN.        .NMMMMM.    Theme: Bunsen-Blackish [GTK2/3]
 hMy          yMMMMMM`   Icons: Faenza-Bunsen [GTK2/3]
 -Mo          +MMMMN     Terminal: x-term
  /o          +MMMMs     CPU: Intel Celeron N2840 (2) @ 2.582GHz
              +MMMN      GPU: Intel Integrated Graphics
              hMMM:      Memory: 1589MiB / 1901MiB
             `NMM/
             +MN:
             mh.
            -/

```

Listing 3: System Specifications

C Encryption and Decryption with RSA (Source Code)

The program is written in C [3] language to demonstrate RSA cryptographic technique with GMP [14], the library allows C to allocate memory for variables with values more than $2^{32} - 1$. [15]

The extended version of the program with comments and documentation is available [8].

```c
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include "gmp.h"

int main(int argc, char** argv) {
    gmp_printf("\nProgram to demonstrate the work of RSA encryption
    ↪  algorithm.\n\n");
    gmp_printf("Checking for the terminal input... ");
    // We need exactly 5 variables. If not, abort.
    char n_char[512], e_char[512], d_char[512], m_char[512];
    if (argc != 5) {
        printf("Please enter values of n, e and d, also the
        ↪  message.\nExiting...\n");
        printf("Please enter the value of modulo in decimal : ");
        scanf("%s", n_char);
        printf("Please enter the value of public exponent in decimal : ");
        scanf("%s", e_char);
        printf("Please enter the value of private exponent in decimal :
        ↪  ");
        scanf("%s", d_char);
        printf("Please enter your message in ASCII : ");
        scanf("%s", m_char);
        printf("Thank you for the input.\n");
    } else {
        strcpy(n_char, argv[1]);
        strcpy(e_char, argv[2]);
        strcpy(d_char, argv[3]);
        strcpy(m_char, argv[4]);
    }
    gmp_printf("Success.\n");

    // Declaring varibles, ned is obvious, while:
    // m is the original converted message
    // c is the original converted message that is encrypted
    // mt is the decrypted original converted message that was encrypted
    gmp_printf("Creating local variables... ");
```

```
35      mpz_t n, e, d, converted_message, encrypted_message,
        ↪  decrypted_message;
36      gmp_printf("Success.\n");
37
38      // Initializing variables, thus allocating memery for them
39      gmp_printf("Initializing local variables... ");
40      mpz_init(converted_message);
41      mpz_init(encrypted_message);
42      mpz_init(decrypted_message);
43      gmp_printf("Success.\n");
44
45      // Initializing and setting variables with values from the terminal
        ↪  input
46      gmp_printf("Initializing and setting up local variables... ");
47      mpz_init_set_str(n, n_char, 10);
48      mpz_init_set_str(e, e_char, 10);
49      mpz_init_set_str(d, d_char, 10);
50      gmp_printf("Success.\n");
51
52      // Converting text into hexadecimal
53      gmp_printf("Converting text into hexadecimal... ");
54      mpz_import(converted_message, strlen(m_char), 1, 1, 0, 0, m_char);
55      gmp_printf("Success.\n");
56
57      // If the converted message's value is bigger than n, abort.
58      gmp_printf("Checking if text value is bigger than the modulo... ");
59      if (mpz_cmp(converted_message, n) > 0) {
60          printf("The text value is bigger than modulo
            ↪  parameter.\nExiting...\n");
61          return EXIT_FAILURE;
62      }
63      gmp_printf("Success.\n");
64
65      // Encrypting the message
66      gmp_printf("Encrypting the message... ");
67      mpz_powm(encrypted_message, converted_message, e, n); //
        ↪  encrypted_message = pow(m, e) % n
68      gmp_printf("Success.\n");
69
70      // Trying to make the cypher into a text
71      gmp_printf("Trying to directly translate the encrypted text... ");
72      char attempted_decryption[256];
73      mpz_export(attempted_decryption, NULL, 1, 1, 0, 0, encrypted_message);
74      gmp_printf("Success.\n");
75
76      // Decrypting the message
77      gmp_printf("Decrypting the message... ");
78      mpz_powm(decrypted_message, encrypted_message, d, n); //
        ↪  decrypted_message = pow(encrypted_message, d) % n
```

```
79      gmp_printf("Success.\n");
80
81      // Converting the message from digits to characters
82      gmp_printf("Translating the decypted text into string... ");
83      char decrypted_final_text[256];
84      mpz_export(decrypted_final_text, NULL, 1, 1, 0, 0, decrypted_message);
85      gmp_printf("Success.\n");
86
87      // Print out all the necessary data
88      gmp_printf("\nInitial settings:\n");
89      gmp_printf("\tThe n value: %Zd\n", n);
90      gmp_printf("\tThe e value: %Zd\n", e);
91      gmp_printf("\tThe d value: %Zd\n", d);
92
93      gmp_printf("\nAll following data will be shown as hexadecimal
        ↪  values.");
94      gmp_printf("\nIt is easier to present in this way than decimal.\n");
95
96      gmp_printf("\nRSA keys:\n");
97      gmp_printf("\tPublic key  :   (%#Zx, \n\t\t\t %#Zx)\n", e, n);
98      gmp_printf("\tPrivate key :   (%#Zx, \n\t\t\t %#Zx)\n", d, n);
99
100     gmp_printf("\nThe results:\n");
101     gmp_printf("\tInput text             : %s\n", m_char);
102     gmp_printf("\tConverted text         : %Zx\n", converted_message);
103     gmp_printf("\tAttempt of decrypting : %Zx\n", encrypted_message);
104     gmp_printf("\tEncrypted text         : %s\n", attempted_decryption);
105     gmp_printf("\tDecrypted text         : %Zx\n", decrypted_message);
106     gmp_printf("\tOutput text            : %s\n", decrypted_final_text);
107
108     // Clearing the memory and resetting it to NULL
109     mpz_clears(n, e, d, converted_message, encrypted_message,
        ↪  decrypted_message, NULL);
110
111     printf("\nThank you for using rsa-ee!\nExiting...\n");
112     return EXIT_SUCCESS;
113 }
```

D Result of Execution of rsa.c Demonstration Program

```
$ ./rsa 95163118457906561534997167608470014334413 57 65537
    56178431878449531703084636222302833762986 85 "The cake is a
    lie."

Program to demonstrate the work of RSA encryption algorithm.

Checking for the terminal input... Success.
Creating local variables... Success.
Initializing local variables... Success.
Initializing and setting up local variables... Success.
Converting text into hexadecimal... Success.
Checking if text value is bigger than the modulo... Success.
Encrypting the message... Success.
Trying to directly translate the encrypted text... Success.
Decrypting the message... Success.
Translating the decypted text into string... Success.

Initial settings:
 The n value: 95163118457906561534997167608470014334413 57
 The e value: 65537
 The d value: 56178431878449531703084636222302833762986 85

All following data will be shown as hexadecimal values.
It is easier to present in this way than decimal.

RSA keys:
 Public key  :  (0x10001,
    0x6d3ded5264bdacea5cc076e62ae5676c844d)
 Private key :  (0x407d5b79d59107e07e4086752d72897e8abd,
    0x6d3ded5264bdacea5cc076e62ae5676c844d)

The results:
 Input text           : The cake is a lie.
 Converted text       : 5468652063616b652069732061206c69652e
 Attempt of decrypting : 639e13dc8bb6d382af7c1480c37476f7e4e2
 Encrypted text       : c? SX}{?|^T??tv??
 Decrypted text       : 5468652063616b652069732061206c69652e
 Output text          : The cake is a lie.

Thank you for using rsa-ee!
Exiting...
```

Listing 4: Output from the Execution of rsa.c

E Factorization with Integers (Source Code)

Full code with documentation is available on my Github page. [8]

```c
#include <stdio.h>
#include <time.h>
#include <stdlib.h>
voidnot(){}
int find_val(int* arr, int size, int val) {
    int i = 0;
    for(i = 0; i < size; i++) if(arr[i] == val) return 1;
    return 0;
}

void brute_force(int n) {
    int p, q, i = 0;
    int arr[100];
    for(p = 2; p < n; p++)
    for(q = 2; q < n; q++)
    if ((p * q == n) && (find_val(arr, i, p) == 0)) {
                printf("Primes for %d are : %d and %d\n", n, p, q);
                arr[i++] = p;
                arr[i++] = q;
            }
}

int main(int argc, char** argv) {
    if(argc != 2) {
        printf("Please enter n.\n");
        return EXIT_FAILURE;
    }
    clock_t begin = clock();
    int n;
    n = atoi(argv[1]);
    brute_force(n);
    clock_t end = clock();
    double time_spent = (double)(end - begin) / CLOCKS_PER_SEC;
    printf("\nExecution time in seconds : %f\n", time_spent);
    return EXIT_SUCCESS;
}
```

F Factorization with Primes (Source Code)

Program to factor input and find two primes.

Full code with documentation is available on my Github page. [8]

```c
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#include <math.h>
#include <string.h>
void not(){}
unsigned int performance() {
    unsigned int i = 0;
    clock_t begin = clock();
    while (i < pow(10, 9)) i++;
    clock_t end = clock();
    double time = (double)(end - begin) / CLOCKS_PER_SEC;
    return pow(10, 9) / time;
}

// By using sieve of eratosthenes, all non-primes in array arr are turned to 0
void find_primes(unsigned int* arr, unsigned int size) {
    unsigned int i, j = 0;
    for (i = 0; i < size; i++) arr[i] = i;
    for (i = 2; i < size; i++) if (arr[i] != 0) {
            j = i;
            while (j + arr[i] < size) arr[j += arr[i]] = 0;
    }
}

// Counting number of non-zero values in array arr
int count(unsigned int* arr, unsigned int size) {
    unsigned int i, k = 0;
    for (i = 0; i < size; i++) if (arr[i] != 0) k++;
    return k;
}

// Copies all primes from array arr to new array arg
void primes(unsigned int* arr, unsigned int size, unsigned int k, unsigned int* arg) {
    unsigned int i, l = 0;
    unsigned int j = 0;
    for (i = 0; i < size; i++) if (arr[i] != 0) arg[j++] = arr[i];
}

```

```c
// Brute
void brute_force(unsigned int n, unsigned int speed) {
    unsigned int p, q, i, j = 0;
    unsigned int* arr = (unsigned int*)malloc(sizeof(unsigned int)*n);
    find_primes(arr, n);
    unsigned int s = count(arr, n);
    double secs = pow(s, 2) / speed;
    int days = ceil(secs / (3600 * 24));
    (secs < 3600 * 24) ? days = 0 : not();
    printf("\tIn reality : \n\t\tIn worst-case scenario, it will take %f
    ↪  seconds or %d day(s)\n", secs, days);
    unsigned int* arg = (unsigned int*)malloc(sizeof(unsigned int)*s);
    primes(arr, n, s, arg);
    free(arr);
    int a, b = 0;
    for (p = 0; p < s; p++) for (q = 0; q < s; q++) if ((arg[p] * arg[q]
    ↪  == n) && (n / arg[p] == arg[q])) {
                printf("\nPrimes are : %u and %u\n", arg[p], arg[q]);
                printf("\nThe results and additional data :\n");
                printf("\tNumber of primes generated to factorize %u :
                ↪  %u\n", n, s);
                printf("\tPrime index of p (%u) : %u\n", arg[p], p);
                printf("\tPrime index of q (%u) : %u\n", arg[q], q);
                free(arg);
                return;
            }
    printf("\nThe entered modulo cannot be factorized!(Bad modulo)\n");
}

int main(int argc, char** argv) {
    unsigned int n;
    if (argc != 2) {
        printf("Please enter n : ");
        scanf(" %u", &n);
    } else if (argc == 2) n = atoi(argv[1]);
    else {
        printf("Please recheck your input.\nExiting...\n");
        return EXIT_FAILURE;
    }
    char performance_test;
    printf("\nThis is a program to factorize input n.\nIf you receive
    ↪  Segmentation error when executing, it means that you don\'t have
    ↪  enough RAM capacity to hold prime numbers\n");
    printf("\nBefore starting the factorization, the system performance
    ↪  test shoukd be performed.\n");
    printf("It is necessary for accurate approximations of runtime.\n");
    printf("However if you don't want to run the performance test, it\'s
    ↪  up to you. Run it? (Y/n) ");
    scanf(" %c", &performance_test);
```

```
82      unsigned int SPEED;
83      char reliability[15] = "";
84      if (performance_test == 'Y') {
85          printf("Starting the performance test...\n");
86          SPEED = performance();
87          printf("Finished successfully.\n");
88          printf("Operations per second for your computer : %d.\n", SPEED);
89      } else {
90          SPEED = 400000000;
91          strcat(reliability, "(UNRELIABLE!)");
92      }
93      clock_t begin = clock();
94      double secs = (pow(n/log(n), 2)) / SPEED;
95      int days = ceil(secs / (3600 * 24));
96      (secs < 3600 * 24) ? days = 0 : not();
97      printf("\nExpectations: %s\n", reliability);
98      printf("\tApproximation : \n\t\tIn worst-case scenario, it will take
        ↪  %f seconds or %d day(s)\n", secs, days);
99      brute_force(n, SPEED);
100     clock_t end = clock();
101     double time_spent = (double)(end - begin) / CLOCKS_PER_SEC;
102     printf("\nExecution time in seconds : %f\n", time_spent);
103     return EXIT_SUCCESS;
104 }
```

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