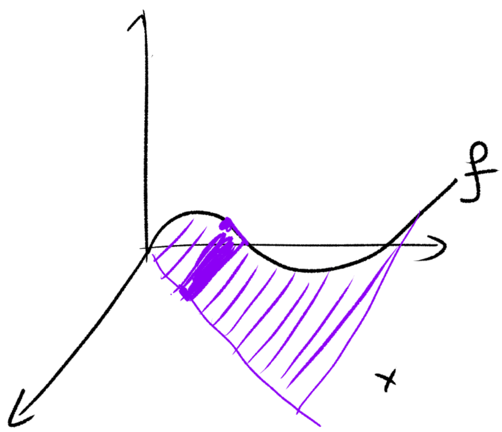
Line Integrals

$x:[a,b]\to\mathbb{R}^n$

The scalar line integral of f along x, denoted by $\int_x f\,ds$ is

$$\int_x f\,ds = \int_a^b \underbrace{f(x(t))}_{\text{f val. at some point}}\cdot\overbrace{\|x'(t)\|}^{\text{arclength}}\,dt$$

Simply, it's the sum of products of values of f with the arclength of pieces of part of x



What is arclength?

I

Vector Line Integrals

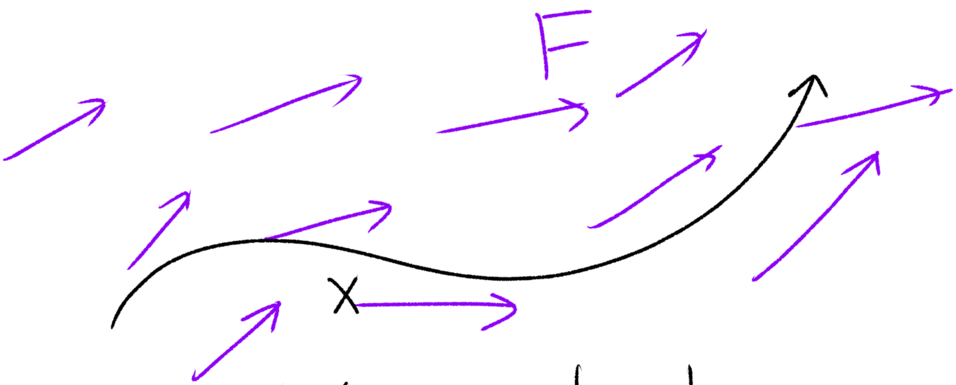
Let
- $x:[a,b] \to \mathbb{R}^n$ be a C path
- $F: X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, continuous along the image (path) of x

Vector line int. is

$$\int_x F \cdot ds = \int_a^b F(x(t)) \cdot x'(t)\,dt$$

dot product

Interpretation: Work being by F on a particle as it moves along the path x



Think as total work done.

① Evaluate $\int_C 2+x^2y\,ds$, where C is the unit ~~circle~~ circle.

Solution

1. Let C be def. as $\vec{r}(t)=\langle\cos(t),\sin(t)\rangle, t\in[0,2\pi]$
2. $\|\vec{r}'(t)\|=\sqrt{(-\sin(t))^2+(\cos(t))^2}=1$
3. Expand the arclength elem

$$ds=\|\vec{r}'(t)\|\,dt=dt$$

$$\Rightarrow \int_C 2+xy^2 ds=\int_0^{2\pi} 2+\cos(t)\,[\sin(t)]^2\,dt$$

Here's the Strategy:

1. Parametrize path C with $\vec{r}(t)$
2. Find $\|\vec{r}'(t)\|$
3. • $ds=\|\vec{r}'(t)\|dt$ and
   • $f(x,y)\Rightarrow f(\vec{r}(t))$.
4. Evaluate

② ~~Evaluate~~

Let $\vec{F}(x,y,z) = \langle x^2, y^2, yz \rangle$

Let $\underset{\text{path}}{C}$: vs $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle, t \in [0, \pi]$.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$. ← work done by F along the path of r

Solution

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{\pi} \langle \cos^2(t), \sin^2(t), t\sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle dt$$

$$= \int_0^{\pi} -\sin(t)\cos^2(t) + \sin^2(t)\cos(t) + t\sin(t)\, dt$$

$$= (\text{math}) = \pi - 2/3$$

Strat: • Put $F(\vec{r}(t))$
- Find $\vec{r}\,'(t)$
- Find $F(\vec{r}(t)) \cdot \vec{r}\,'(t)$
- Evaluate the int.

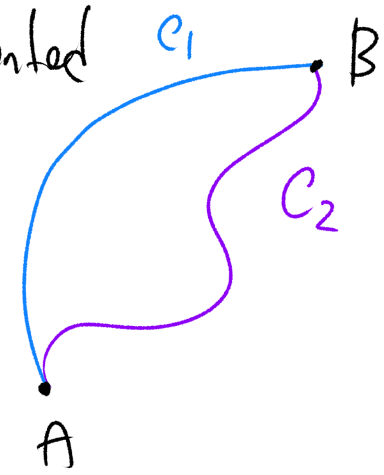
$\boxed{\text{IV}}$

Conservative Fields

Definition A vector field F has path-independence line integrals if

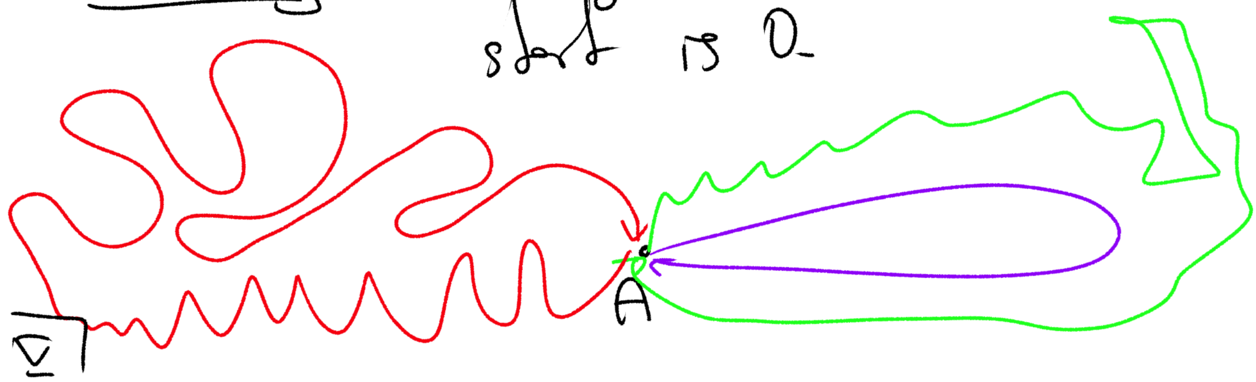
$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$$

for any two simple, oriented curves C_1 and C_2 with the same initial and terminal point



Intuition No matter what path you take from A to B, the work done will always be the same.

Corollary Moving from start to the same start is 0.



Theorem A cont. vector field $\vec{F}$ has path-independent line integrals iff

$$\oint_C \vec{F} \cdot d\vec{s} = \vec{0}$$

for all closed paths C.

As $\vec{F} = \nabla f$, let's find a new form for vector line integral

$$\int_x \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, dt \quad \text{by def.}$$

$$= \int_a^b \nabla f(\vec{x}(t)) \cdot \vec{x}(t)\, dt \quad \vec{F} = \nabla f$$

$$= \int_a^b \frac{d}{dt}(f(\vec{x}(t)))\, dt \quad \text{by the chain rule}$$

$$= f(\vec{x}(b)) - f(\vec{x}(a)) \quad \text{FTC}$$

VI

③ Suppose $\vec{F} = \nabla f$ where

$$f(x,y,z) = \frac{-1}{x^2+y^2+z^2}$$

Find the work done by $\vec{F}$ by moving along a smooth curve C from $(1,0,0)$ to $(0,0,2)$

Solution Apply above

$$\int_C \vec{F}\cdot ds = f(0,0,2) - f(1,0,0)$$
$$= 3/4$$

④ Compute the total mass of the wire given by

$$\vec{x}(t) = \begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = t \end{cases}, \quad \vec{x} : [0, 20\pi] \to \mathbb{R}^3$$

where density of mass is

$$f(x, y, z) = x^2 + y^2 + z, \quad f : \mathbb{R}^3 \to \mathbb{R}$$

Solution

Recall the Scalar Line Integral...

$$\int_{\vec{x}} f \, ds = \int_a^b f(\vec{x}(t)) \cdot \|\vec{x}'(t)\| dt$$

$$= \int_0^{20\pi} \left([x(t)]^2 + [y(t)]^2 + z(t) \right) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

$$= \int_0^{20\pi} (1+t)\sqrt{2} \, dt = \sqrt{2}\left(20\pi + \frac{(20\pi)^2}{2}\right).$$

"Windy Daisy Hill"

What's the work done to get to the top of Daisy Hill, if I follow path

$$\begin{cases} x = t\cos(t) \\ y = t\sin(t) \\ z = -t^2 \end{cases}$$

And the wind is blowing with force of

$$\vec{F}(x,y,z) = \left(-\frac{1}{x+2}, -\frac{1}{y+2}, -\frac{1}{z+2+4\pi^2}\right)$$

Home Exercise!

Function Notation

$$f : D \to R$$

function name

domain, the input values

range (image), the output of the func.

- $\mathbb{R}$ - a real number
- $\mathbb{R}^2$ - a 2D vector with real num
- $\mathbb{R}^n$ - an n-th dim. vector of reals.
- $[a,b]$ - range of values from a to b

$$g : [1,2] \to \mathbb{R}^4$$

notation tells us about func's dimension

$$g(x) = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} x \\ x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x^2 \\ 0^x \\ \ln x \\ \sin(x) \end{pmatrix}$$

$\overline{\underline{X}}$