Midterm II Post-Rom

(6) Pos. of a particle is $\left.\begin{aligned} x &= \cos(\pi t) \\ y &= \sin^2(\pi t) \end{aligned}\right\} t \geq 0$

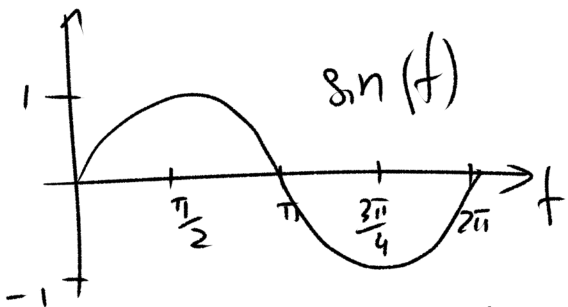
a) __Not__ E. Because it's a circle, $\langle \cos, \sin \rangle$ or $\langle \sin, \cos \rangle$

__Not__ A, D, F. Because $y<0$ but that's impossible.

C would work iff the x component were linear, but it's it's not. $\Rightarrow$ __Not__ C.

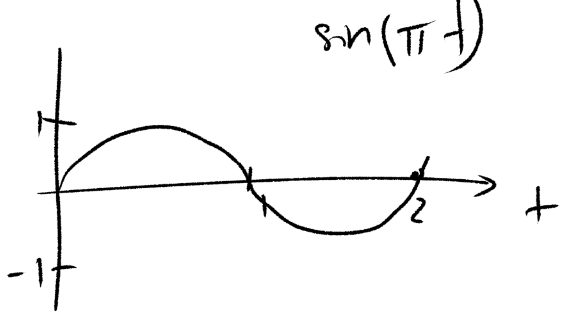
I need all $y \geq 0$ because $\sin^2(\pi t) \geq 0$

$\sin(t) \in [-1, 1]$



Requirements.

- Value 0 if $t=1$
- Decreasing $0 \leq t \leq 1$
- $y \geq 0$ for all t



I

① b) Evaluate $\displaystyle\int \frac{y+1}{(y-1)(y^2+1)}\,dy$

Let's do the Fraction Decomposition:

$$\frac{y+1}{(y-1)(y^2+1)} = \frac{A}{(y-1)} + \frac{By+C}{(y^2+1)}$$

$$= \frac{A(y^2+1)+(By+C)(y-1)}{(y-1)(y^2+1)}$$

$$= \frac{\underline{Ay^2}+\underline{A}+\underline{By^2}-\underline{By}+\underline{Cy}-\underline{C}}{(y-1)(y^2+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-B=1 \\ A-C=1 \end{cases} \Rightarrow \begin{cases} B=-A \\ C=1+B=1-A \\ A-(1-A)=1 \end{cases}$$

sub eq. 1 and 2 into 3.

$$\Rightarrow \begin{cases} \dots \\ \dots \\ A-1+A=1 \end{cases} \Rightarrow \begin{cases} \dots \\ \dots \\ 2A=2 \end{cases} \Rightarrow \begin{cases} B=-1 \\ C=0 \\ A=1 \end{cases}$$

$$\Rightarrow \frac{y+1}{(y-1)(y^2+1)} = \frac{1}{y-1} - \frac{y}{y^2+1}$$

II ⁊

$$\int \frac{y+1}{(y-1)(y^2+1)}\,dy = \int \frac{1}{y-1}\,dy - \int \frac{y}{y^2+1}\,dy$$

1) $\int \frac{1}{y-1}\,dy = \ln|y-1| + C$

2) $\int \frac{y}{y^2+1}\,dy$

let $u = y^2+1 \Rightarrow \frac{du}{dy} = 2y$

$\Rightarrow dy = \frac{d\cancel{y}u}{2y}$

$$= \int \frac{\cancel{y}}{u} \cdot \frac{d\cancel{y}u}{2\cancel{y}} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|y^2+1| + C$$

$$\Rightarrow \int \frac{y+1}{(y-1)(y^2+1)}\,dy = \ln|y-1| + \frac{1}{2}\ln|y^2+1| + C$$

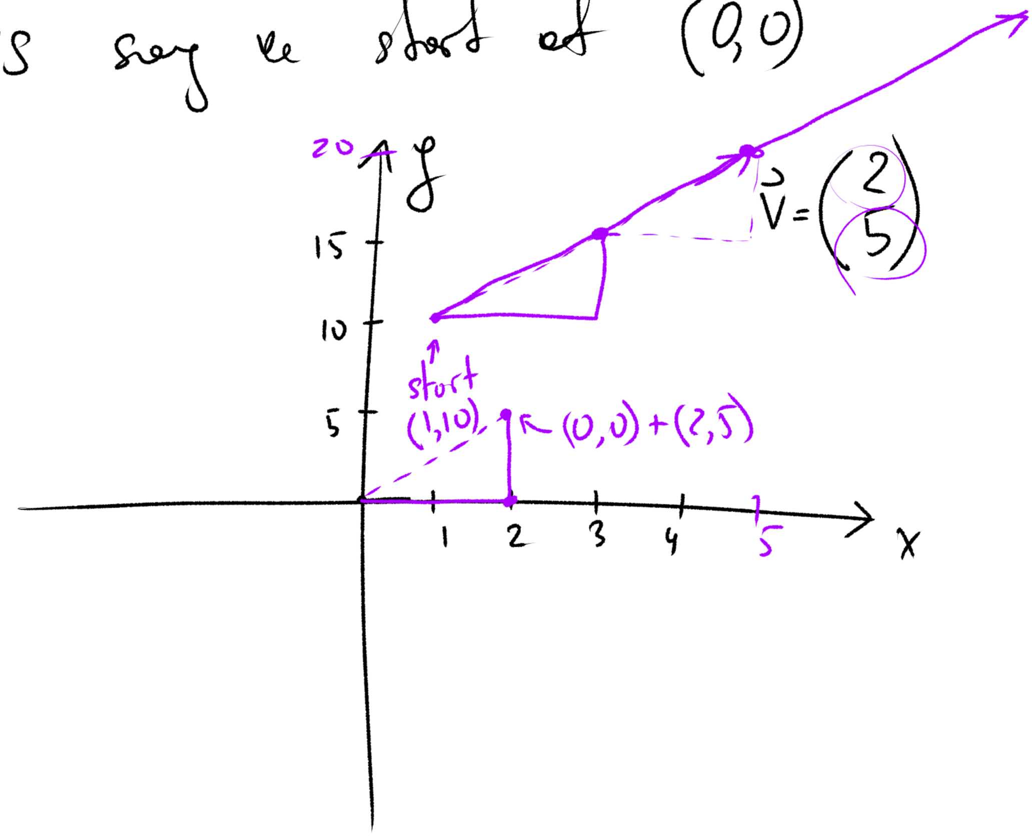
III

Vectors

Vectors define directions in space.

Vector $\begin{pmatrix}2\\5\end{pmatrix}$ or $(2,5)$ or $\langle 2;5 \rangle$ means that we move 2 values in x direction and at the same time we move 5 values in the y direction.

Let's say we start at $(0,0)$



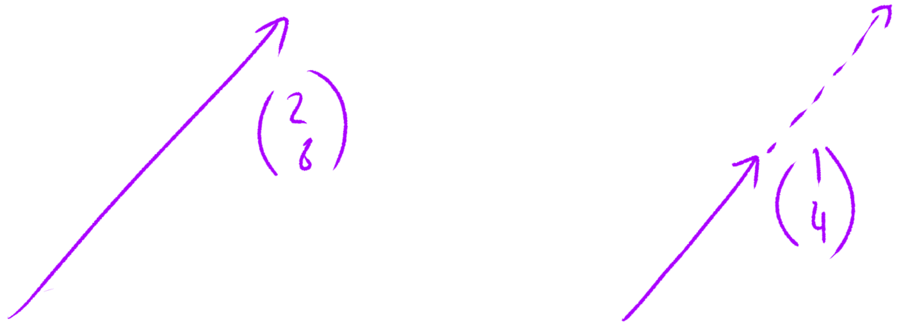
Vectors $\vec{u}$ and $\vec{v}$ are parallel iff

$$\vec{u} = \alpha \vec{v}$$

where α is a scalar

Vectors $\begin{pmatrix}2\\8\end{pmatrix}$ and $\begin{pmatrix}1\\4\end{pmatrix}$ are parallel

as $\begin{pmatrix}2\\8\end{pmatrix} = 2\cdot\begin{pmatrix}1\\4\end{pmatrix}$



Addition: $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}c\\d\end{pmatrix}$, then $\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix}$

$$=\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}$$

Scalar Mult. α scalar, $\begin{pmatrix}a\\b\end{pmatrix}$, then $\alpha\cdot\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\alpha a\\\alpha b\end{pmatrix}$

Scalar Addition: Illegal!

$5+\begin{pmatrix}1\\2\end{pmatrix}$ = impossible

Subtraction: same as addition

Addition is defined only for vectors of the same size!!! $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\4\\5\end{pmatrix}$ = ILLEGAL.

Dot Product

1) $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = a \cdot c + b \cdot d$ ← scalar

2) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = x \cdot a + y \cdot b + z \cdot c$ ← scalar

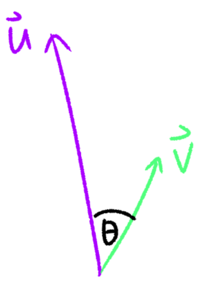
Both vectors have to have the same size.

3) Find $\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\text{vector}} \underbrace{+}_{\text{undefined illegal.}} \left[\underbrace{\begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} e \\ f \end{pmatrix}}_{\text{scalar}} \right]$

Prop. of dot Product

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$

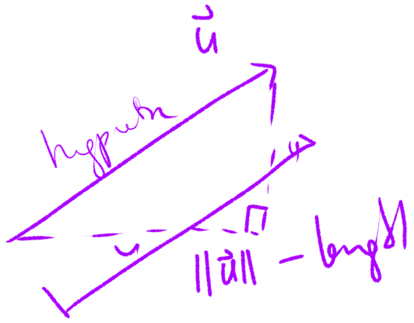
acute



$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

Magnitude of a vector

Given $\vec{u}$, $\|\vec{u}\|$ is the length of the vector.



- $\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| = \sqrt{a^2+b^2}$
- $\left\|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\| = \sqrt{x^2+y^2+z^2}$

Examples:

- $\left\|\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right\| = \sqrt{2^2+5^2} = \sqrt{4+25} = \sqrt{29}$
- $\left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = \sqrt{3^2+4^2} = \sqrt{25} = 5.$