limit improper

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \int_{-b}^{b} \frac{x}{1+x^2}\,dx$$

break down

$1+\tan^2 x = \sec^2 x$

$$= \lim_{a\to-\infty} \int_a^0 \frac{x}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

u-sub

$$= \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

solve indef. int.

plug u back

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2|$$

last page

start everything

$$\lim_{a\to-\infty}\frac{1}{2}\ln|1+x^2|\Big|_a^0+\lim_{b\to\infty}\frac{1}{2}\ln|1+x^2|\Big|_0^b$$

plug in ranges

$$=\frac{1}{2}\lim_{a\to-\infty}\left(\ln|1+0^2|-\ln|1+a^2|\right)+\frac{1}{2}\lim_{b\to\infty}\left(\ln|1+b^2|-\ln|1|\right)$$

$\ln\left(\frac{a}{b}\right)=\ln(a)-\ln(b)$

$$=\frac{1}{2}\lim_{a\to-\infty}\left(\ln\left|\frac{1}{1+a^2}\right|\right)+\frac{1}{2}\lim_{b\to\infty}\ln\left|\frac{1+b^2}{1}\right|$$

$\ln(ab)=\ln(a)+\ln(b)$

solve

$$=\frac{1}{2}\lim_{a\to\infty}\lim_{b\to\infty}\ln\left(\frac{1+b^2}{\cancel{1}+a^2}\right)=\frac{1}{2}\ln|1|=\underline{\underline{0}}$$