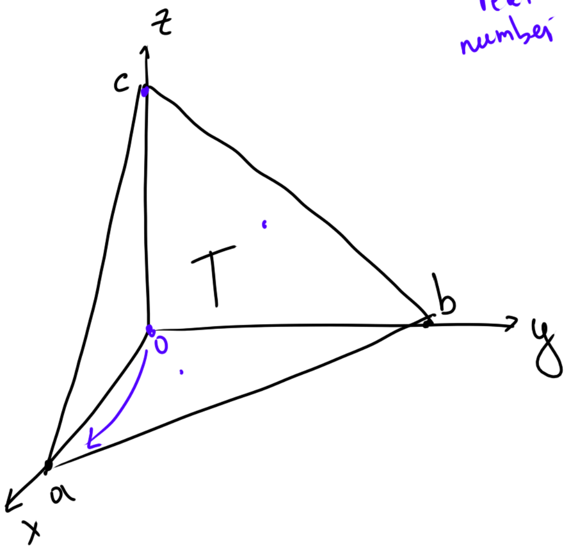
① Find the volume enclosed by planes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad x=0, \; y=0, \; z=0$$

where $a, b, c \in \mathbb{R} - \{0\}$

real number ↗   ↖ except zero



$$V_T = \iiint_T dV$$

Hints:

- Find boundaries of x, y, z
- $V = \iiint dz\,dy\,dx$

can depend on y, x

can only depend on x

can't depend on anything.

Solution

Elise Conjecture

Find z in terms of y and x, then

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \leftarrow \text{upper bound}$$

What are the bounds?

$$0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

I

Bryon's Lemma

Find y in terms of x

let $z=0 \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow y = b\left(1 - \frac{x}{a}\right)$$ ← upper bound

Then

$$0 \le y \le b\left(1 - \frac{x}{a}\right)$$

Mylee's Theorem

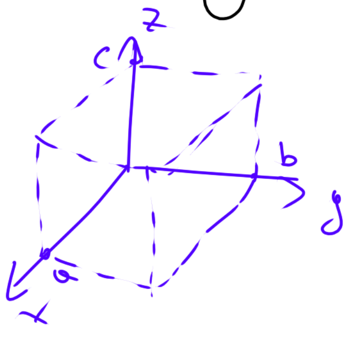
find x, so

$$0 \le x \le a$$

Claire's Proof

$$V_T = \iiint_T dV = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

Note: Why can't we say $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

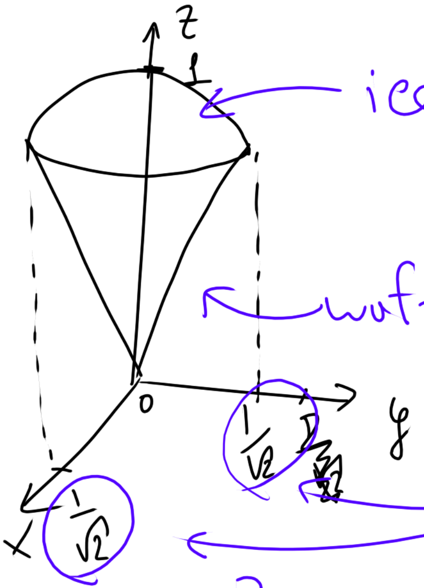


II

② [Ice-Cream] Problem

Express volume of a solid bounded by

$$x^2+y^2+z^2=1, \quad x^2+y^2=z^2$$



waffle: $x^2+y^2=z^2$

ice cream: $x^2+y^2+z^2=1$

$$V_I = \iiint_I 1\,dV$$

☆ $z^2=x^2+y^2 \Rightarrow x^2+y^2+(x^2+y^2)=1$

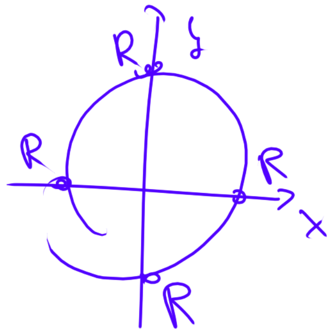
$$\Rightarrow x^2+y^2=1/2=\sqrt{\tfrac{1}{2}}^{\,2}=\left(\tfrac{1}{\sqrt{2}}\right)^2$$

$x^2+y^2=R^2$,

R is the radius

III

Recap on circles:



$x^2+y^2=R^2$

- $x^2+y^2=z^2$ is the lower part of the ice cream, therefore $z=\sqrt{x^2+y^2}$ is the lower bound.
- $x^2+y^2+z^2=1$ is the upper part of the ice cream. Then

$$z=\sqrt{1-y^2-x^2}$$

is the upper bound.

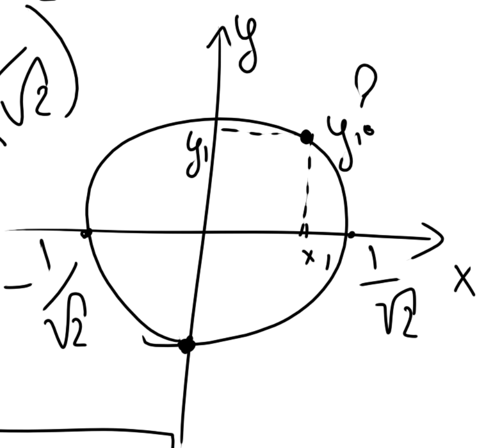
Our Volume looks as follow:

$$V_I=\iiint_I dV=\iint_{\substack{\text{shadow}\\ x^2+y^2\le \frac{1}{2}}}\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dA.$$

$\boxed{IV}$

Here's the shadow

$$x^2+y^2 = 1/2 = \left(1/\sqrt{2}\right)^2$$



$$-1/\sqrt{2} \le x \le 1/\sqrt{2}$$

$$-\sqrt{\tfrac{1}{2}-x^2} \le y \le \sqrt{\tfrac{1}{2}-x^2}$$

$$y_1 = \sqrt{1/2 - x_1^2}$$

Recap:

$$-1/\sqrt{2} \le x \le 1/\sqrt{2}$$

$$-\sqrt{1/2-x^2} \le y \le \sqrt{1/2-x^2}$$

Back to Volume

$$V_I = \iiint_I dV$$

$$= \iint_{x^2+y^2\le 1/2} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dA$$

Volume of the Ice Cream

$$\underline{V} = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dy\, dx$$

Density of the Ice Crown:

$$\delta(x,y,z) = z$$

Find the Mass of the Ice Crown:

$$M_I = \iiint_I \delta(x,y,z)\,dV$$

$$= \int_{x=-1/\sqrt{2}}^{x=1/\sqrt{2}} \int_{y=-\sqrt{1/2-x^2}}^{y=\sqrt{1/2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{1-x^2-y^2}} z \, dz\, dy\, dx$$

$$= \int_{x=-1/\sqrt{2}}^{x=1/\sqrt{2}} \int_{y=-\sqrt{1/2-x^2}}^{y=\sqrt{1/2-x^2}} \frac{1}{2}\left[(1-x^2-y^2)-(x^2+y^2)\right] dy\, dx$$

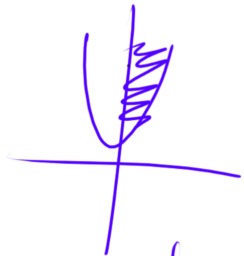
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \left[\frac{1}{2}-x^2\right]-y^2 \, dy\, dx$$

$$= \int_{x=-1/\sqrt{2}}^{x=1/\sqrt{2}} \left[\left(\frac{1}{2}-x^2\right)y - \frac{y^3}{3}\right]\Bigg|_{y=-\sqrt{\frac{1}{2}-x^2}}^{y=\sqrt{\frac{1}{2}-x^2}} dx$$

$$= 2\int_{x=-1/\sqrt{2}}^{x=1/\sqrt{2}} \left(\frac{1}{2}-x^2\right)^{3/2} - \frac{\left(\frac{1}{2}-x^2\right)^{3/2}}{3}\, dx$$

$$= \frac{4}{3}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \underbrace{\left(\frac{1}{2}-x^2\right)}\,^{3/2} dx$$

parabole



(we make an observation the function is symmetric)

symmetric

$$\int_{-b}^{b} f\,dx = 2\int_{0}^{b} f\,dx$$

$$= \frac{8}{3}\int_{0}^{1/\sqrt{2}} \left(\frac{1}{2}-x^2\right)^{3/2} dx$$

$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$45^\circ = \pi/4$$

Trig sub!!! let $x = \sin t \frac{1}{\sqrt{2}}$, $x^2 = \frac{1}{2} \cdot \sin^2(t)$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\left(1 - \sin^2 t\right) \right]^{3/2} \frac{1}{\sqrt{2}} \cos t \, dt$$

$$= \frac{8}{3} \cdot \frac{1}{4} \int_0^{\frac{\pi}{2}} \cos^4(t) \, dt$$

= (magic, integration by parts)

$$= \frac{\pi}{8}.$$

$\overline{\text{VIII}}$