Math 126 · March 1st

① $\lim\limits_{n\to\infty} \frac{(-5)^n}{n!} = \lim\limits_{n\to\infty} a_n$

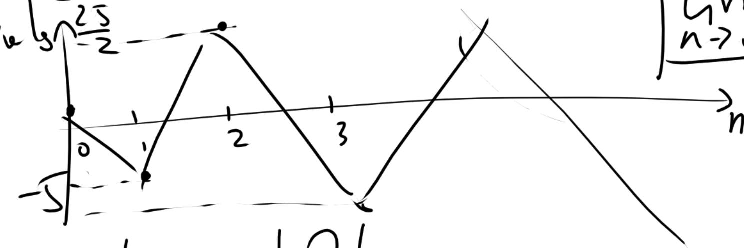
True fact: (☆)
$\lim\limits_{n\to\infty} \frac{R^n}{n!} = 0, R\in\mathbb{R}$

True fact:
$\lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty$

Solution

$(a_n)_{n=0}^{\infty} = \frac{(-5)^n}{n!} = \left\{1, -5, \frac{25}{2}, \frac{-125}{6}, \dots\right\}$

Visually:



How do solve it?!

$$\frac{-5^n}{n!} \le \frac{(-5)^n}{n!} \le \frac{5^n}{n!}$$

$$\Rightarrow \lim_{n\to\infty} \frac{-5^n}{n!} \le \lim_{n\to\infty} \frac{(-5)^n}{n!} \le \lim_{n\to\infty} \frac{5^n}{n!}$$

By (☆), $\lim\limits_{n\to\infty} \frac{R^n}{n!} = 0, R\in\mathbb{R}$, then

$$0 \le \lim_{n\to\infty} \frac{(-5)^n}{n!} \le 0$$

$\therefore \lim\limits_{n\to\infty} \frac{(-5)^n}{n!} = 0$ by Squeeze theorem

(2) $\sum_{n=1}^{\infty} \frac{(3-2x)^n}{4^n \cdot n}$, find x for which it converges

Solution let $a_n = \frac{(3-2x)^n}{4^n \cdot n}$, then $a_{n+1} = \frac{(3-2x)^{n+1}}{4^{n+1} \cdot (n+1)}$

Let's Apply the Ratio Test.

The sum will converge iff

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

So then let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3-2x)^{n+1}}{4^{n+1} \cdot (n+1)} \cdot \frac{4^n \cdot n}{(3-2x)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(3-2x)^n} \cdot (3-2x) \cdot \cancel{4^n} \cdot n}{\cancel{4^n} \cdot 4 \cdot (n+1) \cdot \cancel{(3-2x)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3-2x)n}{4(n+1)} \right| = \left| \frac{3-2x}{4} \right| \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$$

L'Hôpital's

$$\Rightarrow = \left| \frac{3-2x}{4} \right|$$

The sum converges $\Leftrightarrow$ $|L| < 1 \Rightarrow -1 < \frac{3-2x}{4} < 1$

(Skip trivialities) $\Longrightarrow$ $-\frac{1}{2} < x \leq \frac{7}{2}$

Check Endpoints:

$x = -\frac{1}{2}$, it's harmonic $\Rightarrow$ diverges

$x = \frac{7}{2}$, Converges by Alternating Series Test

$\therefore$ Center: $\frac{3}{2}$

Radius: 2

Interval: $\left(-\frac{1}{2}, \frac{7}{2}\right]$

③ $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n \cdot n!}{\underset{\uparrow\, n=1}{5} \cdot \underset{\uparrow\, n=2}{8} \cdot \ldots \cdot (3n+2)}$, Diverges or Converges?

Solution

Let $a_n = \frac{(-1)^n 2^n n!}{5 \cdot 8 \cdot \ldots \cdot (3n+2)}$, then $a_{n+1} = \frac{(-1)^{n+1} \cdot 2^{n+1} \cdot (n+1)!}{5 \cdot 8 \cdot \ldots \cdot (3n+2) \cdot (3(n+1)+2)}$

$$= \frac{(-1) \cdot (-1)^n \cdot 2^n \cdot 2 \cdot n! \cdot (n+1)}{5 \cdot 8 \cdot \ldots \cdot (3n+2) \cdot (3n+5)}$$

Let's apply the Ratio Test!

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1) \cdot \cancel{(-1)^n} \cdot \cancel{2^n} \cdot 2 \cdot \cancel{n!} \cdot (n+1) \cdot \cancel{5 \cdot 8 \cdot \ldots \cdot (3n+2)}}{\cancel{5 \cdot 8 \cdot \ldots \cdot (3n+2)} \cdot (3n+5) \cdot \cancel{(-1)^n} \cdot \cancel{2^n} \cdot \cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1) \cdot 2 \cdot (n+1)}{3n+5} \right| = \lim_{n\to\infty} \left| \frac{2n+2}{3n+5} \right| = 2/3$$

Notice that $L = 2/3 < 1$,

$\therefore$ The series converges by the Ratio Test Absolutely!!!

(4) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n+1}{n}\right)^{n}$, Converges or Diverges?

"1^{∞}"

Solution

Let's use Test for Divergence, it says that $\displaystyle\lim_{x\to\infty} a_x$ is not 0, then series diverges.

So

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x} = e^{\lim_{x\to\infty} \ln\left(\left(\frac{x+1}{x}\right)^{x}\right)}$$

cancel out

$$= e^{\lim_{x\to\infty} x \ln\left(\frac{x+1}{x}\right)} \quad \leftarrow \infty \cdot 0 \text{ problem}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x}} \quad \frac{0}{0} \text{ l'Hopital's}$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}} = e$$

Since $\displaystyle\lim_{n\to\infty} a_n = e \neq 0$, $\displaystyle\sum_{n=2}^{\infty} a_n$ diverges by Test for Divergence