1) $$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot n!}{5\cdot 8 \cdot \ldots \cdot (3n+2)}$$, Converges or Diverges?

(5 ↑ $n=1$, 8 ↑ $n=2$)

Solution

Let $$a_n = \frac{(-1)^n 2^n \cdot n!}{5\cdot 8 \cdot \ldots \cdot (3n+2)} \Rightarrow a_{n+1} = \frac{(-1)^{n+1} 2^{n+1} \cdot (n+1)!}{\underbrace{5\cdot 8 \cdot \ldots \cdot (3n+2)}\cdot (3(n+1)+2)}$$

$1\cdot 2 \cdot \ldots \cdot n$, what's the $n+1$?

$1\cdot 2 \cdot \ldots \cdot n \cdot (n+1)$

$$= \frac{(-1)\cdot(-1)^n \cdot 2 \cdot 2^n \cdot n! \cdot (n+1)}{5\cdot 8 \cdot \ldots \cdot (3n+2)\cdot(3n+5)}$$

Let's Apply the Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)\cdot\cancel{(-1)^n}\cdot 2\cdot\cancel{2^n}\cdot\cancel{n!}\cdot(n+1)\cdot\cancel{5\cdot 8 \cdot \ldots \cdot (3n+2)}}{\cancel{5\cdot 8 \cdot \ldots \cdot (3n+2)}\cdot(3n+5)\cdot\cancel{(-1)^n}\cdot\cancel{2^n}\cdot\cancel{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(n+1)\cdot 2}{3n+5}\right| = \lim_{n\to\infty}\left|\frac{2n+2}{3n+5}\right|$$

Recall!!

$n! = \underline{1\cdot 2 \cdot \ldots \cdot n}$

$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$

$(n+1)! = \underline{1\cdot 2 \cdot \ldots \cdot n}\cdot(n+1)$
$= n!\cdot(n+1)$

$$\overset{\text{L'Hopital's}}{=} \lim_{n\to\infty}\left|\frac{\frac{d}{dn}(2n+2)}{\frac{d}{dn}(3n+5)}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\right| = 2/3$$

Now, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2/3 < 1$, then the series Converges!!!

② $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{5^n \cdot \sqrt{n^2+3}}$ what values of x, does it converge? Provide radius of convergence.

Solution

$$L = \lim_{n\to\infty} \left| \frac{\cancel{(2x-3)^n} \cdot (2x-3)}{\cancel{5^n} \cdot 5 \cdot \sqrt{(n+1)^2+3}} \cdot \frac{\cancel{5^n} \sqrt{n^2+3}}{\cancel{(2x-3)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2x-3}{5} \cdot \frac{\sqrt{n^2+3}}{\sqrt{(n+1)^2+3}} \right| = \left| \frac{2x-3}{5} \right|$$

1

Set $L < 1 \Rightarrow -1 < x < 4$

Check Endpoints

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ ← Converges by Alternating Series Test

$x = 4$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ ← Diverges, because it looks like $1/n$, harmonic series diverge

Therefore:

- Interval: $[-1, 4)$
- Radius: $5/2$
- Center: $3/2$

(3) $\lim_{n\to\infty} \frac{(-5)^n}{n!}$, what's the limit?

Solution

Let's try to apply Squeeze "Sandwich" theorem. We have to find upper/lower bound.

$$-\frac{5^n}{n!} \leq \frac{(-5)^n}{n!} \leq \frac{5^n}{n!}$$

FACT!

$\lim_{n\to\infty} \frac{R^n}{n!} = 0,$

$R \in \mathbb{R}$

$$\underbrace{\lim_{n\to\infty} -\frac{5^n}{n!}}_{=0} \leq \lim_{n\to\infty} \frac{(-5)^n}{n!} \leq \underbrace{\lim_{n\to\infty} \frac{5^n}{n!}}_{=0}$$

$\therefore$ By Squeeze theorem, $\lim_{n\to\infty} \frac{(-5)^n}{n!} = 0$