Series and Sequences

① ~~Series~~ Sequences: $(a_n)_{n=1}^{\infty}$

$a_0 = \text{some } \ell$

$a_1 = \cdots$

$\vdots$

Series: $S_{\cancel{n}K} = \sum_{n=0}^{K} a_n$

Converging: $\lim_{n \to \infty} a_n$ ~~is~~ is defined and not $\pm\infty$

Diverging: otherwise

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① Sequence where $\lim a_n \neq \pm\infty$ but it's diverging

$m_n = \sin(n)$ ← diverges, oscillates!

$V_n = (-1)^n$

$1, -1, 1, -1$

② $\lim_{n\to\infty} \frac{n^n}{n!} = ?$

Solution

(One): We know $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$$\Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{1}{\frac{n!}{n^n}} = \infty$$

(Full Solution)

$$\frac{n^n}{n!} = \frac{\overbrace{n\times n\times n\times \dots \times n}^{n \text{ times}}}{\underbrace{1\times 2\times 3\times \dots \times n}_{n \text{ times}}} = \underbrace{\frac{n}{1}}_{=n} \times \underbrace{\frac{n}{2}\times\frac{n}{3}\times \dots}_{>1} \times \underbrace{\frac{n}{n}}_{=1}$$

= (whole thing) $> n$

We know that $\lim_{n\to\infty} n = \infty$

$$\lim_{n\to\infty} \frac{n^n}{n!} > \lim_{n\to\infty} n = \infty \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

$\boxed{\text{for all } n \geq 1, \ \frac{n^n}{n!} > n}$ Squeeze Theorem

③ Let R be a positive number (whole)

$$\lim_{n\to\infty}\frac{R^n}{n!} = \underset{\downarrow}{\frac{R}{n}} \times \underbrace{\frac{R}{n-1}\times\cdots\times\frac{R}{R+1}}_{\leq 1} \times \boxed{\underbrace{\frac{R}{R}}_{1}\times\underbrace{\frac{R}{R-1}\times\cdots\times\frac{R}{2}\times\frac{R}{1}}_{>1}}$$

$$\leq \frac{R}{n}\times\underbrace{\frac{R}{R-1}\times\cdots\times\frac{R}{2}\times\frac{R}{1}}_{R-1\text{ times}} = \frac{R^R}{(R-1)!\,n}$$

<u>Recognize</u>: $0 \leq \frac{R^n}{n!} \leq \frac{R^R}{(R-1)!\,n}$

constant!!!

$$0=\lim_{n\to\infty}0 \leq \lim_{n\to\infty}\frac{R^n}{n!} \leq \lim_{n\to\infty}\frac{R^R}{(R-1)!\,n} = 0$$

$\therefore$ By Squeeze theorem: $\lim_{n\to\infty}\frac{R^n}{n!}=0$

(This Problem made by Dr. Brennan)

4. $\lim_{n\to\infty}\left(\frac{50}{\pi}\right)^n = \infty$, as $\frac{50}{\pi} > 1$

5. $\lim_{n\to\infty}\left(\frac{e}{\pi}\right)^n = \cancel{\infty}\ 0$, as $\frac{e}{\pi} \approx \frac{2.7}{3.1} < 1$

No!

Fact! $\lim_{n\to\infty}(<1)^n = 0$

6. $\lim_{n\to\infty}\left(\frac{n^2+n}{n+n^2}\right)^n = 1$

Fact!!! $(1)^k = 1$ for **any** k