Integration by Parts is the Product Rule for integrals.

Let's say, you have functions f, g, Assume f', g'

$$(fg)' = f'g + fg'$$

$$\int fg\,dt = \int f\,dt \cdot g - \int \left[\int f\,dt \cdot \frac{dg}{dt} \right] dt$$

(1) $\int t^3 \sin(t)\,dt$ $\quad \begin{cases} u = t^3 \\ dv = \sin(t) \end{cases} \Rightarrow \begin{cases} \frac{du}{dt} = 3t^2 \\ v = -\cos(t) \end{cases}$

$= uv - \int v\,du$ ← Integration by Parts

$= (t^3)\cdot(-\cos(t)) - \int (-\cos(t))\cdot(3t^2)\,dt$

$= -t^3\cos(t) + 3\int t^2 \cos(t)\,dt$ $\quad \begin{cases} u = t^2 \\ dv = \cos(t) \end{cases} \Rightarrow \begin{cases} du = 2t \\ v = \sin(t) \end{cases}$

$= -t^3\cos(t) + 3\left[(t^2)\cdot(\sin(t)) - \int \sin(t)\cdot(2t)\,dt\right]$

$= -t^3\cos(t) + 3\left[t^2\sin(t) - 2\int t\sin(t)\,dt\right]$ $\quad \begin{cases} u = t \\ dv = \sin(t) \end{cases} \Rightarrow \begin{cases} du = 1 \\ v = -\cos(t) \end{cases}$

$\boxed{I}$

$$= -t^3\cos(t) + 3\left[t^2\sin(t) - 2\left(-t\cos(t) - \int -\cos(t)\cdot 1\, dt\right)\right]$$

$$= -t^3\cos(t) + 3\left[t^2\sin(t) - 2\left(-t\cos(t) + \sin(t)\right)\right]$$

$$= -t^3\cos(t) + 3\left[t^2\sin(t) + 2t\cos(t) - 2\sin(t)\right]$$

$$= -t^3\cos(t) + 3t^2\sin(t) + 6t\cos(t) - 6\sin(t) + C$$

Answer

② $\int e^x \sin(x)\, dx$

$$\begin{cases} u = e^x \\ dv = \sin(x) \end{cases} \Rightarrow \begin{cases} du = e^x \\ v = -\cos(x) \end{cases}$$

$$= uv - \int v\, du$$

$$= e^x(-\cos(x)) - \int (-\cos(x)) \cdot e^x\, dx$$

$$= -e^x\cos(x) + \int e^x\cos(x)\, dx \qquad \begin{cases} u = e^x \\ dv = \cos(x) \end{cases} \Rightarrow \begin{cases} du = e^x \\ v = \sin(x) \end{cases}$$

$$= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\, dx$$

Notice that:

$$\int e^x\sin(x)\, dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\, dx$$

$$2\int e^x\sin(x)\, dx = -e^x\cos(x) + e^x\sin(x)$$

$$\therefore \int e^x\sin(x)\, dx = \frac{e^x}{2}\left(\sin(x) - \cos(x)\right)$$

③ $\int \tan^7(x)\,dx$

$= \int \tan^6(x)\tan(x)\,dx$

$= \int (\tan^2(x))^3 \tan(x)\,dx$

$= \int (\sec^2(x)-1)^3 \underbrace{\tan(x)\,dx}$

u-sub

let $u = \sec(x)$

$\Rightarrow du = \sec(x)\tan(x)\,dx$

$\Rightarrow \frac{du}{u} = \underbrace{\tan(x)\,dx}$

$= \int (u^2-1)^3 \frac{du}{u}$

$= \int (u^6 - 3u^4 + 3u^2 - 1)\frac{du}{u}$

$= \int u^5 - 3u^3 + 3u - u^{-1}\,du$

$= \int u^5\,du - 3\int u^3\,du + 3\int u\,du - \int u^{-1}\,du$

$= \frac{1}{6}u^6 - \frac{3}{4}u^4 + \frac{3}{2}u^2 - \ln|u| + C$

Sub sec(x) back into u

$\therefore \frac{1}{6}\sec^6(x) - \frac{3}{4}\sec^4(x) + \frac{3}{2}\sec^2(x) -$

$- \ln|\sec(x)| + C$

III

④ $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

Hint: simple $\sin(x)$ or $\cos(x)$ Not gonna work.

Use $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4}$

$= \sqrt{4(\sec^2\theta - 1)}$

$= \sqrt{4\tan^2\theta} = 2\tan\theta$

$$= \int \frac{\cancel{2\sec\theta}\,\cancel{\tan\theta}\,d\theta}{4\sec^{\cancel{2}}\theta \cdot \cancel{2}\cancel{\tan\theta}}$$

$$= \int \frac{d\theta}{4\sec\theta}$$

$$= \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C$$

$$= \frac{1}{4}\cdot\left(\frac{\sqrt{x^2-4}}{x}\right) + C$$

Answer

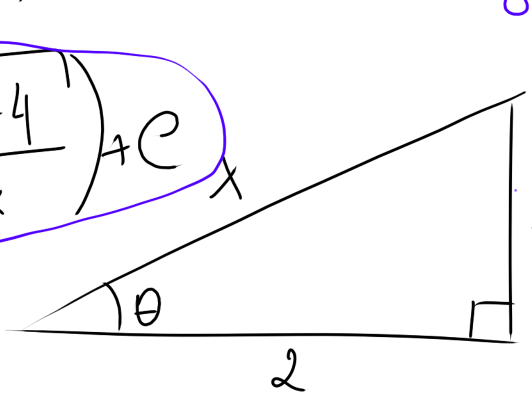
FACT:

- $\sin(x)$, $\cos(x)$
- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
- $\csc(x) = \dfrac{1}{\sin(x)}$
- $\sec(x) = \dfrac{1}{\cos(x)}$
- $\text{ctg}(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$

$x = 2\sec\theta = \dfrac{2}{\cos\theta}$

$x\cos\theta = 2$

$\cos\theta = \dfrac{2}{x}$ ← adjacent, ↖ hypothenuse



$\sqrt{x^2-2^2} = \sqrt{x^2-4}$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypothenuse}} = \frac{\sqrt{x^2-4}}{x}$$

$\boxed{\text{IV}}$