Double Integration (AKA "Goodbye Washer (Disc) Method")

Let $A(x)$ be the area at x.

$$V = \int_a^b A(x)\,dx$$

$A(x)$ can be an integral

$$A(x_0) = \int_c^d f(x_0, y)\,dy$$

Then

$$V = \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$$

① $$\int_0^2 \int_1^3 (x^2+y)\,dy\,dx$$

Solution

$$\int_0^2\int_1^3 (x^2+y)\,dy\,dx = \int_0^2 \left[x^2y + \tfrac{1}{2}y^2\right]\Big|_{y=1}^{y=3} dx$$

$$= \int_0^2 \left(3x^2 + \tfrac{9}{2}\right) - \left(x^2 + \tfrac{1}{2}\right) dx = \int_0^2 2x^2 + 4\,dx$$

I

$$= \left[\frac{2}{3}x^3 + 4x\right]\Bigg|_{x=0}^{x=2} = \frac{40}{3}$$

(2) $$\int_{y=0}^{y=\pi/2} \int_{x=0}^{x=1} (e^x \cos y)\,dx\,dy$$

FACT:

$$\frac{d}{dx} e^x = e^x$$

$$\int e^x\,dx = e^x + C$$

$$= \int_0^{\pi/2} \underbrace{\left[e^x \cos y\right]\Bigg|_{x=0}^{x=1}} dy$$ ← evaluate

$$= \int_0^{\pi/2} e\cos y - e^0 \cos y\,dy$$

$$= \int_0^{\pi/2} \cos y\,(e-1)\,dy$$

FACT II

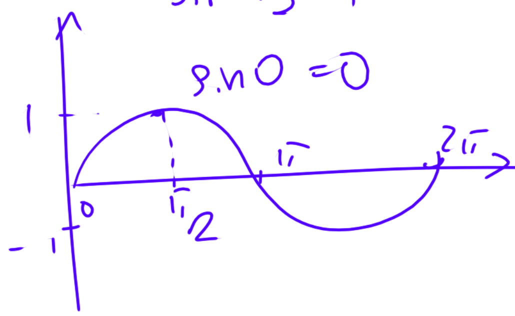
$$\frac{d}{dx}\cos x = -\sin x$$

$$\int \cos x\,dx = \sin x + C$$

$$= \Big[\sin y\,(e-1)\Big]_{y=0}^{y=\pi/2}$$

$\sin \pi/2 = 1$

$\sin 0 = 0$



$$= e - 1$$

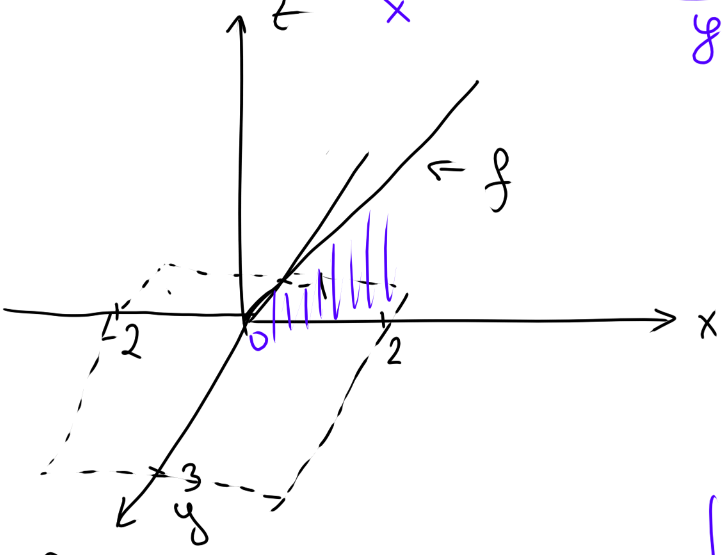
Multiplying something by zero: vanishing (Bad Influence Theorem)

II

Double integral can compute volume of $f(x,y)$ over some area of xy.

Quick: $dA = dx\,dy$

Let $z = f(x,y) = x$. Find the volume enclosed by $\underbrace{[-2,2]}_{x} \times \underbrace{[-1,3]}_{y} = R$



$V = \iint_R f\, dA$ ← general notes

So,

$$V = \int_{-1}^{3} \int_{-2}^{2} x \, dA = 0$$

III

Double Integral Properties:

1. $f+g$ is also int. on R

$$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$$

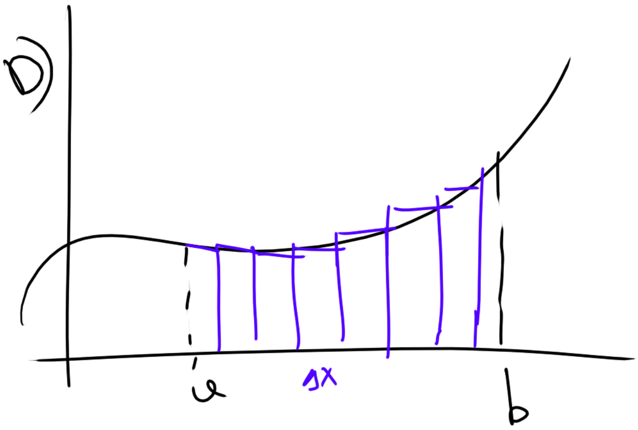
2. constant c,

$$\iint_R cf\,dA = c\iint_R f\,dA$$

3. If $f \le g$ on R

$$\iint_R f\,dA \le \iint_R g\,dA$$

Riemann (2D)



$n = \#\text{rect}$

if $n \to \infty$, you have an integral

Riemann (3D)

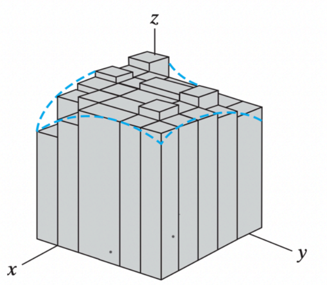
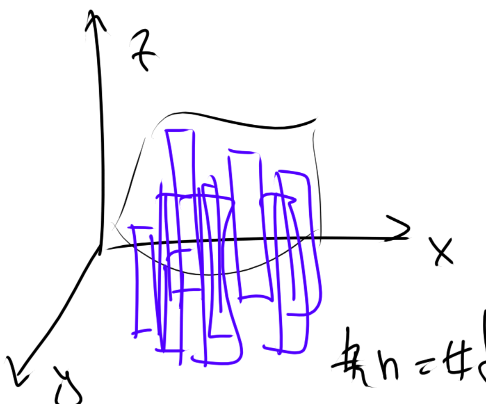


Figure 5.11 The volume under the graph of f is approximated by the Riemann sum.

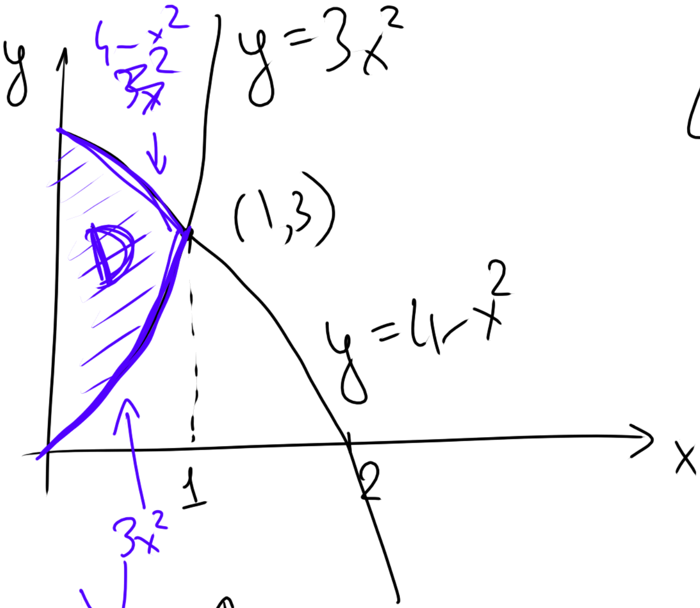


$n \to \infty$, double integral

$n = \#\text{forms}$

IV

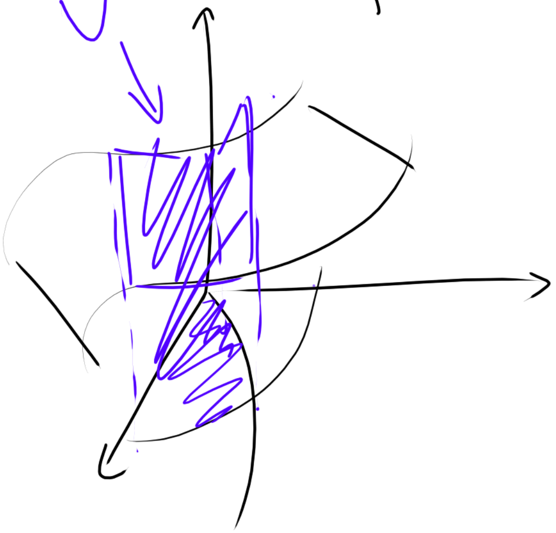
3



Let D be the region bounded by parabolas. Find me the Volume of

$$f(x,y) = x^2 y$$

on D



Bounds of x : (on D) $= [0,1]$

Bounds of y (on D) $= [3x^2, 4-x^2]$

$$V = \int_{x=0}^{x=1} \int_{y=3x^2}^{y=4-x^2} x^2 y \, dy \, dx$$

$$\boxed{V} = \int_{x=0}^{x=1} \left[\frac{x^2 y^2}{2} \right] \Bigg|_{y=3x^2}^{y=4-x^2} dx$$

$$= \int_0^1 \frac{x^2}{2}(4-x^2)^2 - \frac{x^2}{2}(3x^2)^2 \, dx$$

$$= \int_0^1 \frac{x^2}{2}(16 - 8x^2 + x^4) - \frac{x^2}{2} \cdot 9x^4 \, dx$$

$$= \int_0^1 8x^2 - 4x^4 + \frac{x^6}{2} - \frac{9}{2}x^6 \, dx$$

$$= \int_0^1 -4x^6 - 4x^4 + 8x^2 \, dx$$

$$= \left[-\frac{4}{7}x^7 - \frac{4}{5}x^5 + \frac{8}{3}x^3 \right]_0^1$$

$$= -\frac{4}{7} - \frac{4}{5} + \frac{8}{3} = \frac{136}{105}$$

$\frac{136}{105}$ is the volume under $f(x,y) = x^2 y$ over an area D constrained by $y = 3x^2$ and $y = 4 - x^2$.

$\boxed{\text{VI}}$