Let's talk about Jacobian

2D|

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

Would not work

Because small details during the conversion are lost.

$$\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| dA_{uv}$$

where $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

F)

Cartesian $(x, y) \longrightarrow$ Polar (r, θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \left\| \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right\|$$

$\frac{\partial x}{\partial r}$ $\frac{\partial x}{\partial \theta}$ $\frac{\partial y}{\partial r}$

$= r\cos^2\theta + r\sin^2\theta$
$= r$

Detour Matrices:

As multi-dimensional vector or a spreadsheet

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} a & b & e & g \\ c & d & f & h \end{pmatrix}$$

Can be any size $n \times m$, $n, m \in \mathbb{N}$

rows

columns

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = ad - bc$$

II

① Find the jacobian of

$$\begin{cases} x = u^2 v \\ y = 5u + \sin v \end{cases}$$

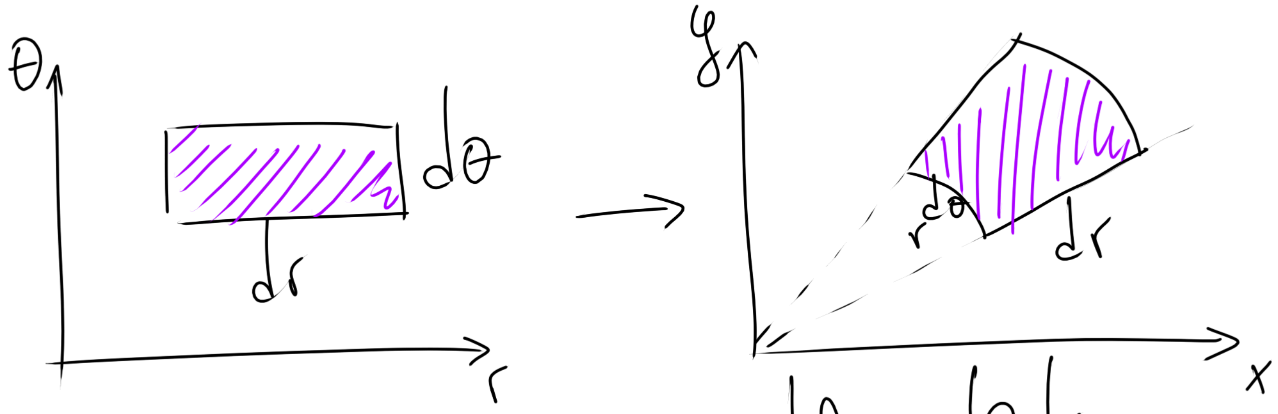
Find $\dfrac{\partial(x,y)}{\partial(u,v)}$

Solution

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 2uv & u^2 \\ 5 & \cos v \end{pmatrix}$$

$$\left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| = 2uv\cos v - 5u^2$$

Visualizing Jacobian



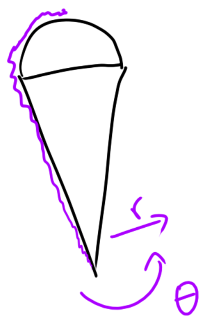
$$dA = r\,d\theta\,dr$$

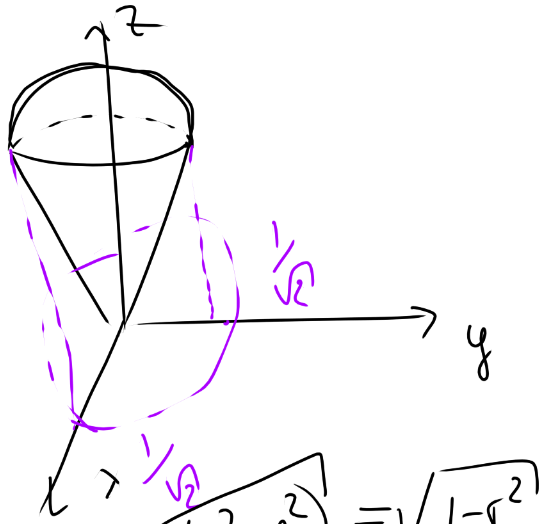
Recall Ice Cream

Bound Between

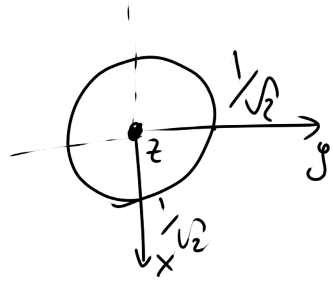
$$x^2+y^2+z^2=1 \quad \text{and} \quad x^2+y^2=z^2$$

left side view





TOP VIEW



$$V_{cart} = \int_{x=-1/\sqrt{2}}^{x=1/\sqrt{2}} \int_{y=-\sqrt{\frac{1}{2}-x^2}}^{y=\sqrt{\frac{1}{2}-x^2}} \int_{z=\sqrt{x^2+y^2}=r}^{z=\sqrt{1-(x^2+y^2)}=\sqrt{1-r^2}} 1 \; dz\,dy\,dx$$

sphere

cone

② Convert this to cylindrical coords

Claim:

$0 \le \theta \le 2\pi$

$0 \le r \le 1/\sqrt{2}$

$r \le z \le \sqrt{1-r^2}$

$$V_{cyl} = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^{\sqrt{1-r^2}} r \; dz\,dr\,d\theta$$

$r = \sqrt{x^2+y^2}$

V

3 Spherical

$$0 \le \theta \le 2\pi$$

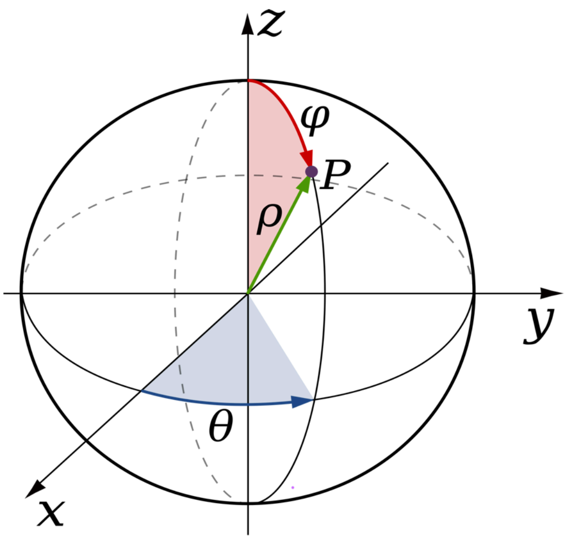
$$0 \le \rho \le 1$$

$$0 \le \varphi \le \pi/4$$

$$V_s = \int_0^{2\pi}\int_0^{1}\int_0^{\pi/4} \underbrace{\rho^2 \sin\varphi}_{\text{Jacobian}} \, d\varphi \, d\rho \, d\theta$$

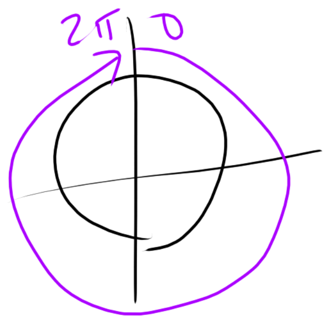
Spherical Coordinates

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$



TOP VIEW

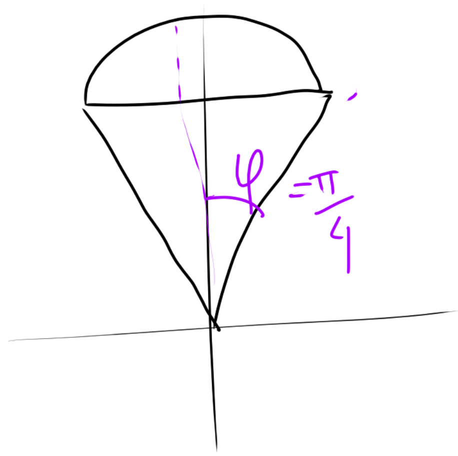
θ



SIDE VIEW

ρ

SIDE VIEW



VT

(4) Solid bounded by

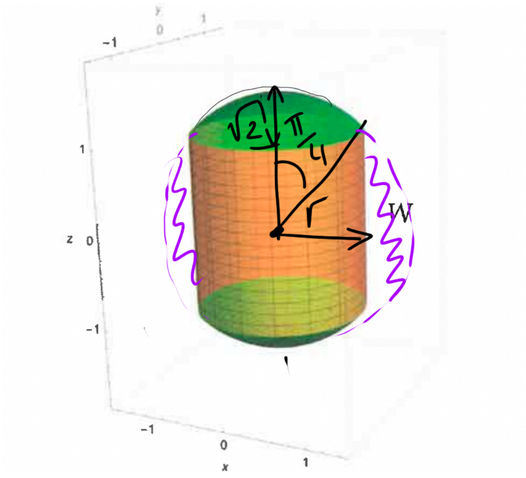
$$x^2+y^2+z^2=2 \quad \text{and} \quad x^2+y^2=1$$

sphere cylinder

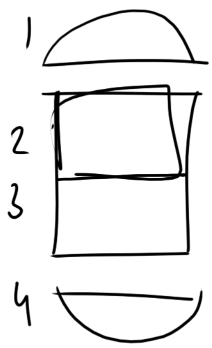
Abstract Volume of a sphere

Volume of of W

-Volume of



SIDE VIEW



Divide for Spherical

a) Set up an intg using cylindrical and spherical.

1) Cylindrical

$$\text{Volume} = \int_0^{2\pi}\int_0^{1}\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

Jacobian

2) Spherical

$$\text{Volume} = 2\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\frac{1}{\sin\varphi}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta + 2\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

SIDVIEW

$x^2+y^2=1$

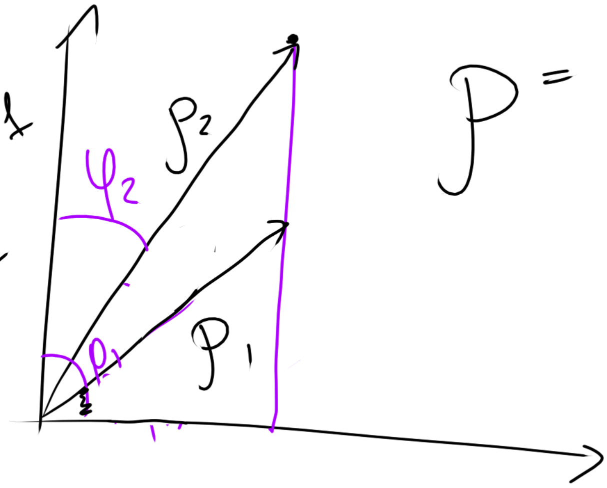
$\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=1$

$\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)=1$

$\rho^2\sin^2\varphi=1$

$\rho^2\sin^2\varphi$

$\rho=\frac{1}{\sin\varphi}$



$$\rho=\frac{1}{\sin\varphi}$$