March 17th | Optimization

Given function f, find min/max if it's constrained by function g

Lagrange Multipliers

given f and g

$$\nabla f = \lambda \nabla g$$

we find extreme of $z = f(x,y)$, constrainted by $g(x,y) = k$

(1) Find all a, b, λ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

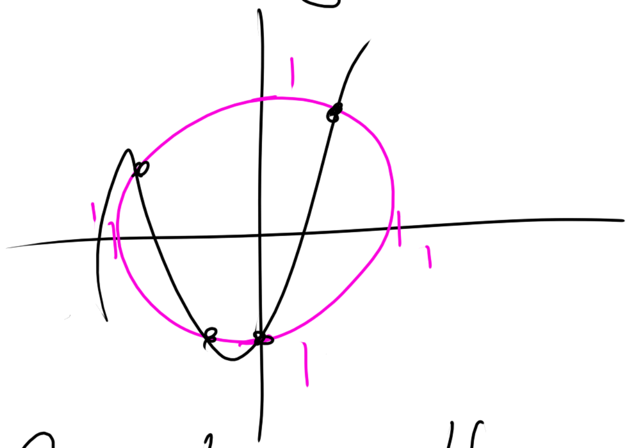
(2) Find all $f(a,b)$

(3) Largest $\Rightarrow$ ~~local~~ absolute maximum

Smallest $\Rightarrow$ absolute minimum

$f(x,y) = x^2 - 4xy + y^2$

$g(x,y) = x^2 + y^2 = 1$ ⟵ unit circle



① $\nabla f = \langle 2x - 4y, \; -4x + 2y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

② Solve $\nabla f = \lambda \nabla g$ and $g = 1$

$$\begin{cases} 2x - 4y = \lambda 2x \\ -4x + 2y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x - 2y = \lambda x \\ -2x + y = \lambda y \\ x^2 + y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda = \dfrac{x - 2y}{x} \\ \lambda = \dfrac{-2x + y}{y} \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} \dfrac{x - 2y}{x} = \dfrac{-2x + y}{y} \\ x^2 + y^2 = 1 \end{cases}$$

II

$$\Rightarrow \begin{cases} y(x-2y) = x(-2x+y) \\ x^2+y^2=1 \end{cases}$$

$$\Rightarrow \begin{cases} \cancel{xy} - \cancel{2}y^2 = -\cancel{2}x^2 \cancel{+xy} \\ x^2+y^2=1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2=y^2 \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} x^2 = 1/2 \\ y^2 = 1/2 \end{cases}$$

$$\Rightarrow \begin{cases} x = \pm \frac{1}{\sqrt{2}} \\ y = \pm \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \begin{cases} x = \pm \frac{\sqrt{2}}{2} \\ y = \pm \frac{\sqrt{2}}{2} \end{cases}$$

3 $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -1$ ← minimum

$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 3$ ← maximum

$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 3$ ← maximum

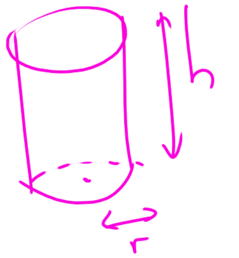
$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -1$ ← minimum

III

② Find the maximum volume of a cylindrical can to be made from $100\pi\ cm^2$ of metal.

Solution

Hint: We maximize volume of a cylinder against the area of a cylinder.



$$f(r,h) = \pi r^2 h$$

$$g(r,h) = 2\pi r^2 + 2\pi r h = 100\pi$$

$$\nabla f = \langle 2\pi r h,\ \pi r^2 \rangle$$

$$\nabla g = \langle 4\pi r + 2\pi h,\ 2\pi r \rangle$$

Solve $\nabla f = \lambda \nabla g$, $g = 100\pi$

$$\begin{cases} 2\cancel{\pi} r h = \lambda(4\cancel{\pi} r + 2\cancel{\pi} h) \\ \cancel{\pi} r^2 = \lambda 2 \cancel{\pi} r \\ 2\cancel{\pi} r^2 + 2\cancel{\pi} r h = 100\cancel{\pi} \end{cases}$$

$$\Rightarrow \begin{cases} rh = \lambda(2r + h) \\ r = 2\lambda \\ r^2 + rh = 100 \end{cases}$$

$\boxed{IV}$

$$\begin{cases} rh = \lambda(2r + \cancel{2}h) \\ r^{\cancel{2}} = 2\lambda \\ r^2 + rh = 100 \end{cases}$$

Elise Theorem

$$\begin{cases} rh = \frac{r^2}{2}(2r + \cancel{2}h) = r^2(r+h) \\ \lambda = \frac{r^2}{2} \\ r^2 + r^2(r+h) = 100 \end{cases}$$

Mylee's Conjecture

$$\begin{cases} rh = \lambda(2r + h) \\ r^{\cancel{2}} = 2\lambda \\ r^2 + rh = 100 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{rh}{2r+h} \\ \lambda = \frac{r}{2} \\ r^2 + rh = 100 \end{cases}$$

Solve for r, h

3cnf

$$\Rightarrow \begin{cases} \frac{rh}{2r+h} = \frac{r}{2} \\ r^2 + rh = 100 \end{cases}$$

left for the reader

$$\lambda = r/2$$

$$2rh = 2r^2 + rh \Rightarrow h = 2r$$

$$\therefore \quad r = \frac{5\sqrt{6}}{3}\text{ cm}, \quad h = \frac{10\sqrt{6}}{3}\text{ cm}$$

$\boxed{V}$