March 8th. Midterm Session

① $S = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$, Converges by AST

a) Show $|S - S_4| \leq \frac{1}{\ln 3125}$

Solution

$n = 4$ $\quad |S - S_n| < b_{n+1}$, where $b_n =$ non alternating part of

$$|S - S_4| \leq b_{4+1} = b_5 = \frac{1}{5 \ln 5} = \frac{1}{\ln 5^5} = \frac{1}{\ln 3125}$$

b) Find N (the least n), such that

$$|S - S_N| < 0.05$$

$$\Rightarrow \quad |S - S_N| < b_{N+1} \leq 0.05$$

$$\Rightarrow \frac{1}{(N+1)\ln(N+1)} \leq 0.05$$

$$\Rightarrow \frac{1}{(N+1)\ln(N+1)} \leq \frac{1}{20}$$

$$\Rightarrow (N+1)\ln(N+1) \geq 20$$

$$\Rightarrow \ln((N+1)^{N+1}) \geq 20$$

$$\Rightarrow (N+1)^{N+1} \geq e^{20}$$

FACT!

$4 > 2$ $\quad n > n-1$

$\frac{1}{4} < \frac{1}{2}$ $\quad \Leftrightarrow \quad \frac{1}{n} < \frac{1}{n-1}$

~~$\ln 1 = 0$~~

~~$\ln(N \cdot 1) = \ln(N) \cdot \ln(1) = \ln(N) \cdot 0$~~

Let $m = N+1$, then

$$m^m \geq e^{20}$$

On a crappy calculator, find that

$$e^{20} \approx 4.85 \cdot 10^8$$

Trial by fire

$5^5 \approx 3.13 \cdot 10^3$ NO

$8^8 \approx 1.68 \cdot 10^7$ NO

$9^9 \approx 3.87 \cdot 10^8$ NO

$10^{10} = 10^{10}$ YES

Then we find $m = 10 = N+1 \Rightarrow N = 9$

$\therefore N = 9$

② $f(x) = \sqrt[4]{x}$, centered around $\cancel{x} = 16$.

Approximate $\sqrt[4]{17}$, give error

$\boxed{\text{Solution}}$

$f(x) = x^{1/4} \Rightarrow f(16) = 16^{1/4} = 2$

$f'(x) = \frac{1}{4} x^{-3/4} \quad f'(16) = \frac{1}{4} \cdot 16^{-3/4} = \frac{1}{32}$

$f''(x) = -\frac{3}{16} x^{-7/4} \quad f''(16) = -\frac{3}{16} \cdot \underbrace{16^{-7/4}}_{= \frac{1}{128}} = -\frac{3}{2048}$

$2^{10} = 1024$

$2^9 = 512$

Then 2^{nd} Taylor is:

$$T_2(x) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2$$

$$= 2 + \tfrac{1}{32}(x-16) - \frac{3}{4096}(x-16)^2$$

$$\Rightarrow \sqrt[4]{17} = f(17) \approx T_2(17) = 2 + \tfrac{1}{32} - \frac{3}{4096} = \frac{8317}{4096} \approx 2.03$$

You can leave it like this

$$\text{error} = \left| f(17) - T_2(17) \right| \le K \frac{|17-16|^{2+1}}{(2+1)!},$$

where $\left| f^{(2+1)}(u) \right| \le K$ for $16 < u < 17$

a) $f'''(x) = \frac{21}{64} x^{-11/4}$, maximized when x is minimized $\Rightarrow K = \frac{21}{64 \cdot 16^{11/4}}$ leave it like this

$$= \frac{21}{131072}$$

b) Plug in K,

$$\text{error} = \left| f(17) - T_2(17) \right| \le \frac{21}{64 \cdot 16^{11/4}} \cdot \frac{1}{6}$$