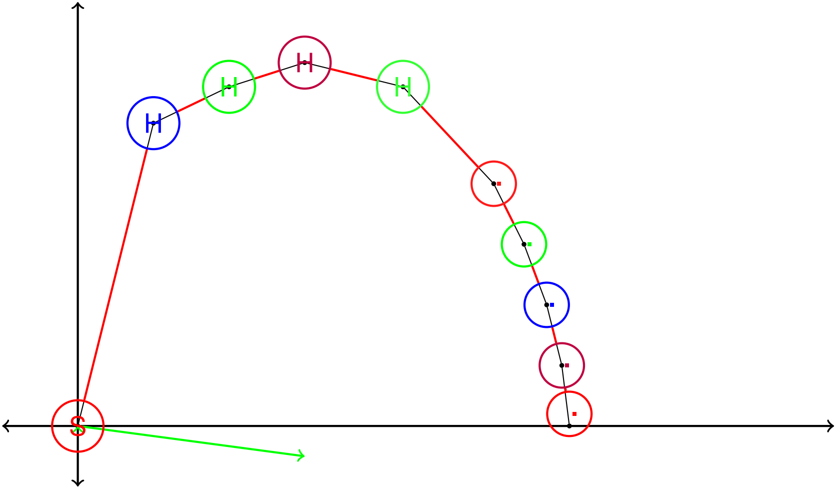
CS 170: Algorithms



Lecture in a Minute

Simplex Implementation:
Start at a (feasible) vertex.

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(defined by linear system $A'x = [b', 0, \cdots, 0]$).

Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Vertex at origin in coordinate system.

$O(mn)$ time to update linear system.

Until no better neighboring vertex.

Objective function in coordinate system is non-positive.

Dual Variables: new objective function!

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      Coordinate system: distance from tight constraints.
      Vertex at origin in coordinate system.
      $O(mn)$ time to update linear system.
  Until no better neighboring vertex.
  Objective function in coordinate system is non-positive.
  Dual Variables: new objective function!

Maximum flow.
  "Greedy" augment path...
    Except reverse old decisions ..
    Reverse residual capacities.
  (Friday): Optimality?
    No augmenting path $\Longrightarrow$
      $s-t$ cut size = flow value.
    Find flow and $s-t$ cut with equal value!

Simplex algorithm.

Two tasks:

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

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Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Simplex algorithm.

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$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Simplex algorithm.

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Optimal?

If all $c_i \leq 0$

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Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value

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Optimal?

If all $c_i \le 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$

Simplex algorithm.

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If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

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if there is $c_i > 0$ increasing x_i increases value $\implies$ not optimal.

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$$x \geq 0$$

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If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$ increasing x_i increases value $\implies$ not optimal.

Done with task 1.

Going to a better place..

Two tasks:

Going to a better place..

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1. Check optimality of vertex?
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Two tasks:

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At origin, there is positive c_i, so increase x_i.

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At origin, there is positive c_i, so increase x_i.

...until you hit another constraint.

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$x_i \geq 0$ is no longer tight, but new constraint is.

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...until you hit another constraint.

$x_i \geq 0$ is no longer tight, but new constraint is.
$\implies$ n constraints!

At vertex!

Example.

$$\max 2x_1 + 5x_2$$
$$2x_1 - x_2 \leq 4 \qquad (1)$$
$$x_1 + 2x_2 \leq 9 \qquad (2)$$
$$-x_1 + x_2 \leq 3 \qquad (3)$$
$$x_1 \geq 0 \qquad (4)$$
$$x_2 \geq 0 \qquad (5)$$

Example.

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$$2x_1 - x_2 \leq 4 \qquad ①$$

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Origin: feasible, value 0.

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Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

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Relax constraint $x_2 = 0$.

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Relax constraint $x_2 = 0$.
Increase x_2 until

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...Inequality (3) becomes tight constraint.

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...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).

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Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).
...new vertex: (0,3) with value 15.

Example.

$$\begin{aligned} \max 2x_1 + 5x_2 & \\ 2x_1 - x_2 \leq 4 & \qquad 1 \\ x_1 + 2x_2 \leq 9 & \qquad 2 \\ -x_1 + x_2 \leq 3 & \qquad 3 \\ x_1 \geq 0 & \qquad 4 \\ x_2 \geq 0 & \qquad 5 \end{aligned}$$

Origin: feasible, value 0.
Inequalities 4 and 5 are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality 3 becomes tight constraint.
...Tight constraints: 3 and 4.
...new vertex: (0,3) with value 15.

Easy process from origin:

Example.

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Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
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Easy process from origin: just increase one variable with positive c.

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Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

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Increase x_2 until
...Inequality (3) becomes tight constraint.
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Now what?

A new coordinate system.

New coordinates: Distance from new tight constraints.

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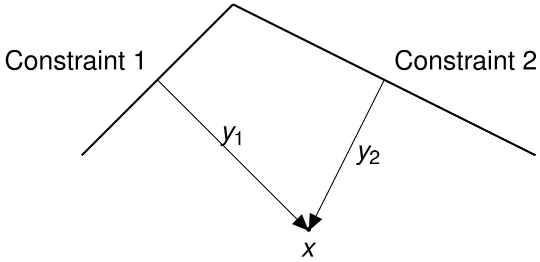
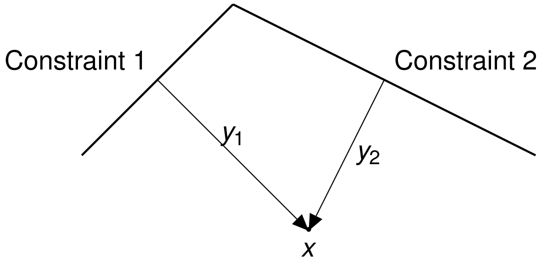


A new coordinate system.

New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

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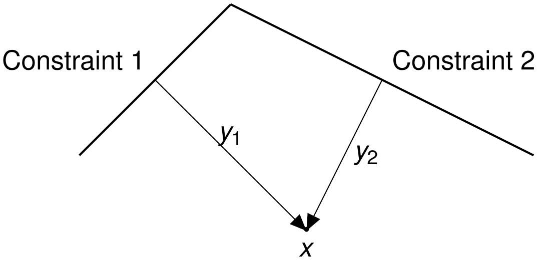
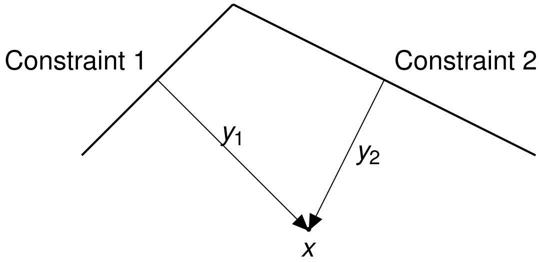


y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

A new coordinate system.

New coordinates: Distance from new tight constraints.



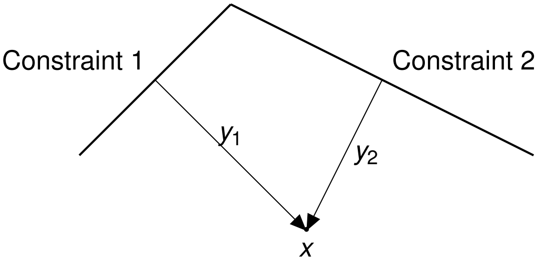
y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

A new coordinate system.

New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

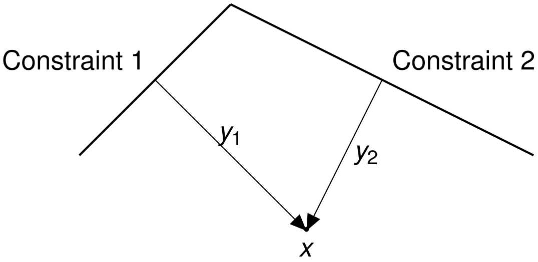
x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

Recall that for origin: x_i was distance from constraint $x_i \geq 0$.

A new coordinate system.

New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

Recall that for origin: x_i was distance from constraint $x_i \geq 0$.

At origin in new coordinate system!

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$
$$2x_1 - x_2 \leq 4 \qquad (1)$$
$$x_1 + 2x_2 \leq 9 \qquad (2)$$
$$-x_1 + x_2 \leq 3 \qquad (3)$$
$$x_1 \geq 0 \qquad (4)$$
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New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

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Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Rewrite linear program.

Rewrite linear program with new coordinates.

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\begin{aligned}
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2x_1 - x_2 \leq 4 & \quad (1) \\
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Plug in for x_1 and x_2:

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Plug in for x_1 and x_2: objective funcition

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$$\max \quad 2x_1 + 5x_2$$

Rewrite linear program.

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Plug in for x_1 and x_2: objective funcition

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \max \quad & 2(y_1) + 5(3 - y_2 + y1) \end{aligned}$$

Rewrite linear program.

Rewrite linear program with new coordinates.

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Plug in for x_1 and x_2: objective funcition

$$\max \quad 2x_1 + 5x_2$$
$$\max \quad 2(y_1) + 5(3 - y_2 + y1)$$
$$\max \quad 15 + 7y_1 - 5y_2$$

Rewrite linear program.

Rewrite linear program with new coordinates.

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$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal?

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$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes!

Rewrite linear program.

Rewrite linear program with new coordinates.

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Plug in for x_1 and x_2: objective funcition

max $2x_1 + 5x_2$
max $2(y_1) + 5(3 - y_2 + y1)$
max $15 + 7y_1 - 5y_2$ Are we optimal? Yes! Maybe not!

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$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes! Maybe not! No.

Positive coefficient for increasing y_1.

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & (1) \\
& 3y_1 - 2y_2 \leq 3 & (2) \\
& y_2 \geq 0 & (3) \\
& y_1 \geq 0 & (4) \\
& -y_1 + y_2 \leq 3 & (5)
\end{aligned}
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Rewriting example..

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\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 &\leq 7 & \quad (1) \\
3y_1 - 2y_2 &\leq 3 & \quad (2) \\
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y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

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& -y_1 + y_2 \leq 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & \text{①} \\
& 3y_1 - 2y_2 \leq 3 & \text{②} \\
& y_2 \geq 0 & \text{③} \\
& y_1 \geq 0 & \text{④} \\
& -y_1 + y_2 \leq 3 & \text{⑤}
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \le 7 & (1) \\
& 3y_1 - 2y_2 \le 3 & (2) \\
& y_2 \ge 0 & (3) \\
& y_1 \ge 0 & (4) \\
& -y_1 + y_2 \le 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)?

Rewriting example..

$$\begin{aligned} \max \quad 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \le 7 & \quad (1) \\ 3y_1 - 2y_2 \le 3 & \quad (2) \\ y_2 \ge 0 & \quad (3) \\ y_1 \ge 0 & \quad (4) \\ -y_1 + y_2 \le 3 & \quad (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & (1) \\
& 3y_1 - 2y_2 \leq 3 & (2) \\
& y_2 \geq 0 & (3) \\
& y_1 \geq 0 & (4) \\
& -y_1 + y_2 \leq 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)?

Rewriting example..

$$
\begin{aligned}
\max \quad 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \le 7 & \qquad (1) \\
3y_1 - 2y_2 \le 3 & \qquad (2) \\
y_2 \ge 0 & \qquad (3) \\
y_1 \ge 0 & \qquad (4) \\
-y_1 + y_2 \le 3 & \qquad (5)
\end{aligned}
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y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 \\
& y_1 + y_2 \le 7 && (1) \\
& 3y_1 - 2y_2 \le 3 && (2) \\
& y_2 \ge 0 && (3) \\
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$$\begin{aligned} \max \quad & 15 + 7y_1 - 5y_2 & \\ & y_1 + y_2 \leq 7 & (1) \\ & 3y_1 - 2y_2 \leq 3 & (2) \\ & y_2 \geq 0 & (3) \\ & y_1 \geq 0 & (4) \\ & -y_1 + y_2 \leq 3 & (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Rewriting example..

$$\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \le 7 & (1) \\
& 3y_1 - 2y_2 \le 3 & (2) \\
& y_2 \ge 0 & (3) \\
& y_1 \ge 0 & (4) \\
& -y_1 + y_2 \le 3 & (5)
\end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Rewriting example..

$$\max \quad 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad ①$$

$$3y_1 - 2y_2 \leq 3 \qquad ②$$

$$y_2 \geq 0 \qquad ③$$

$$y_1 \geq 0 \qquad ④$$

$$-y_1 + y_2 \leq 3 \qquad ⑤$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? ①? ②? ③? ④? ⑤?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 &\leq 7 & (1) \\
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y_2 &\geq 0 & (3) \\
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\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

Rewriting example..

$$\begin{aligned} \max \quad & 15+7y_1-5y_2 & \\ & y_1+y_2 \leq 7 & (1) \\ & 3y_1-2y_2 \leq 3 & (2) \\ & y_2 \geq 0 & (3) \\ & y_1 \geq 0 & (4) \\ & -y_1+y_2 \leq 3 & (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

Rewriting example..

$$\begin{aligned} \max \quad 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \le 7 & \quad (1) \\ 3y_1 - 2y_2 \le 3 & \quad (2) \\ y_2 \ge 0 & \quad (3) \\ y_1 \ge 0 & \quad (4) \\ -y_1 + y_2 \le 3 & \quad (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$.

Rewriting example..

$$\begin{aligned} \max \quad & 15 + 7y_1 - 5y_2 & \\ & y_1 + y_2 \leq 7 & (1) \\ & 3y_1 - 2y_2 \leq 3 & (2) \\ & y_2 \geq 0 & (3) \\ & y_1 \geq 0 & (4) \\ & -y_1 + y_2 \leq 3 & (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value:

Rewriting example..

$$\begin{aligned} \max \quad & 15+7y_1-5y_2 & \\ & y_1+y_2 \le 7 & (1) \\ & 3y_1-2y_2 \le 3 & (2) \\ & y_2 \ge 0 & (3) \\ & y_1 \ge 0 & (4) \\ & -y_1+y_2 \le 3 & (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value: $12+7(1)-5(0)$

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 &\le 7 & \text{①} \\
3y_1 - 2y_2 &\le 3 & \text{②} \\
y_2 &\ge 0 & \text{③} \\
y_1 &\ge 0 & \text{④} \\
-y_1 + y_2 &\le 3 & \text{⑤}
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? ①? ②? ③? ④? ⑤?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality ②!

New vertex: tight constraints ③ and ②.

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value: $12 + 7(1) - 5(0) = 22$.

Rewriting example..

$$\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1)\\
3y_1 - 2y_2 \leq 3 & \qquad (2)\\
y_2 \geq 0 & \qquad (3)\\
y_1 \geq 0 & \qquad (4)\\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}$$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2$

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$

$z_1 = 3 - 3y_1 + 2y_2 \rightarrow$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \to y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
max $\quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

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Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

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$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal?

Rewriting example..

$$\begin{aligned} \max 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \le 7 & \qquad (1) \\ 3y_1 - 2y_2 \le 3 & \qquad (2) \\ y_2 \ge 0 & \qquad (3) \\ y_1 \ge 0 & \qquad (4) \\ -y_1 + y_2 \le 3 & \qquad (5) \end{aligned}$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \to y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes!

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \quad ①$$

$$3y_1 - 2y_2 \leq 3 \quad ②$$

$$y_2 \geq 0 \quad ③$$

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Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \to y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not!

Rewriting example..

$$\begin{aligned} \max 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \le 7 & \qquad (1) \\ 3y_1 - 2y_2 \le 3 & \qquad (2) \\ y_2 \ge 0 & \qquad (3) \\ y_1 \ge 0 & \qquad (4) \\ -y_1 + y_2 \le 3 & \qquad (5) \end{aligned}$$

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Objective function.
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$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not! Optimal point!

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad \text{(1)} \\
3y_1 - 2y_2 \leq 3 & \qquad \text{(2)} \\
y_2 \geq 0 & \qquad \text{(3)} \\
y_1 \geq 0 & \qquad \text{(4)} \\
-y_1 + y_2 \leq 3 & \qquad \text{(5)}
\end{aligned}
$$

Rewrite: $z_2 = y_2$
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Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not! Optimal point!
Increasing z_1, z_2 makes things worse.

Review.

In each step:

Review.

In each step:

LP in coordinate system from tight constraints.

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2.$

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2$.

Go to tight constraint along improving coordinate.

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2.$

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3.$

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2.$

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3.$

Express LP in coordinate system for new tight constraints.

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2$.

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3$.

Express LP in coordinate system for new tight constraints.

See previous slides!

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2$.

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3$.

Express LP in coordinate system for new tight constraints.

See previous slides!

Repeat.

Details: getting started.

What if origin is not feasible?

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Make a new linear program.

Details: getting started.

What if origin is not feasible?

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An x where $Ax \leq b$ and at vertex.

Make a new linear program.

Introduce positive variables z_i for inequality i.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Make a new linear program.

Introduce positive variables z_i for inequality i.

Constraints: $a_i x - z_i \leq b_i$.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Make a new linear program.

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Constraints: $a_i x - z_i \leq b_i$.

$$\max \sum -z_i.$$

Details: getting started.

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Make a new linear program.

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$$\max \sum -z_i.$$

Vertex solution (x, z) of value zero

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

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Make a new linear program.

Introduce positive variables z_i for inequality i.

Constraints: $a_i x - z_i \leq b_i$.

$$\max \sum -z_i.$$

Vertex solution (x, z) of value zero
$\implies$ all z's are zero

Details: getting started.

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Make a new linear program.

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Constraints: $a_i x - z_i \leq b_i$.

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Vertex solution (x,z) of value zero
$\implies$ all z's are zero
$\implies$ all inequalities are satisfied

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Make a new linear program.

Introduce positive variables z_i for inequality i.

Constraints: $a_i x - z_i \leq b_i$.

$$\max \sum -z_i.$$

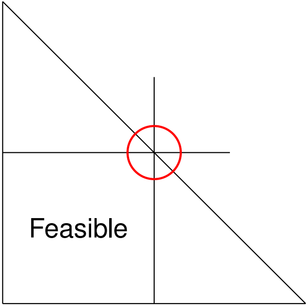
Vertex solution (x,z) of value zero
$\implies$ all z's are zero
$\implies$ all inequalities are satisfied
$\implies$ x is a feasible vertex of $Ax \leq b$.

Degeneracy.

Degenerate vertices.

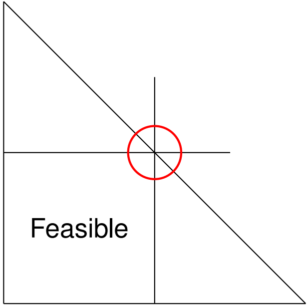
Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.



Degeneracy.

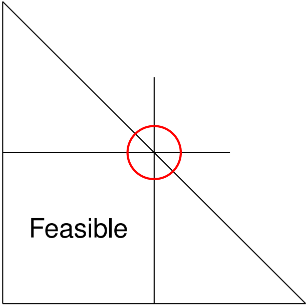
Degenerate vertices.
Intersection of more than n constraints.



Problem: all neighboring vertices are no better.

Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.

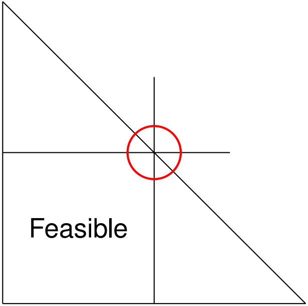


Problem: all neighboring vertices are no better.

Infinite looping: Bland's anticycling rule.

Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.



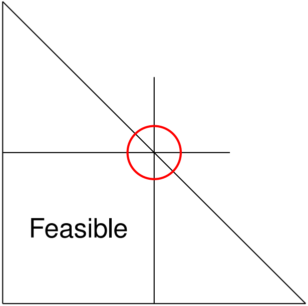
Problem: all neighboring vertices are no better.

Infinite looping: Bland's anticycling rule.

Or Perturb problem a bit.

Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.

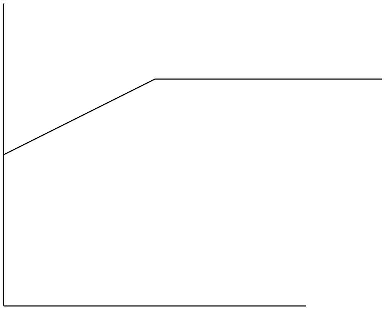


Problem: all neighboring vertices are no better.

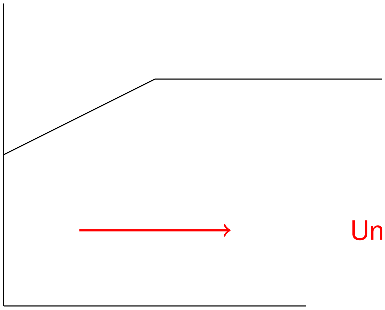
Infinite looping: Bland's anticycling rule.

Or Perturb problem a bit. Unlikely to intersect!

Unboundedness.

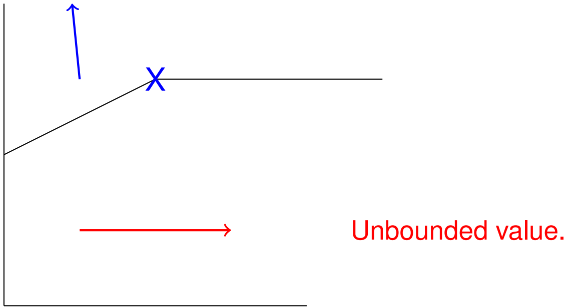


Unboundedness.



Unbounded value.

Unboundedness.



Simplex can tell difference.

From X: either unbounded improvement or optimal.

Running Time

Check optimality? $O(n)$.

Running Time

Check optimality? $O(n)$.

Find tight constraint:

Running Time

Check optimality? $O(n)$.

Find tight constraint:
$O(m)$ constraints.

Running Time

Check optimality? $O(n)$.

Find tight constraint:
$O(m)$ constraints. $O(1)$ time per constraint.

Running Time

Check optimality? $O(n)$.

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Fast, in practice!

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The negations of coefficients of new function!

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Feasible!

Next Up: Maximum Flow.

Maximum Flow.

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Flow network $G = (V, E)$, source s, sink $t \in V$, capacities $c_e > 0$.

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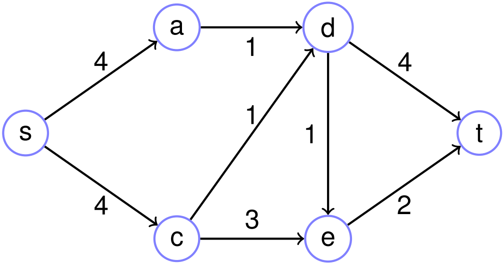

Find Flow: f_e

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2. If u is not s or t
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Maximize: $\text{size}(f) = \sum_{(s,u)\in E} f_{su}$.

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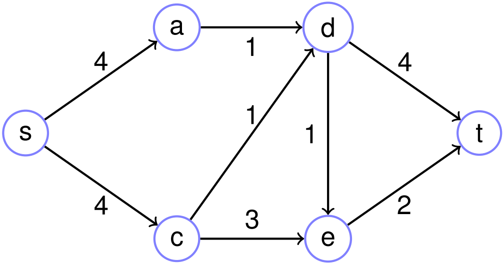

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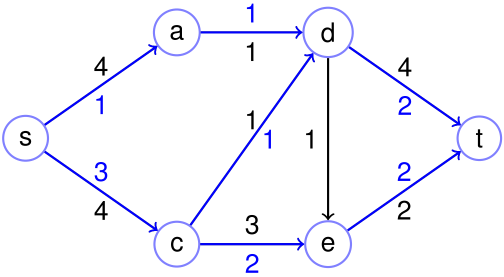

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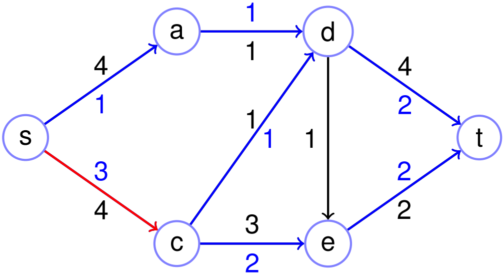

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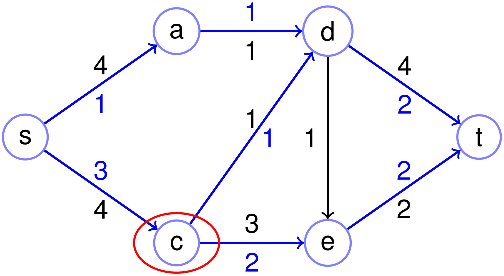

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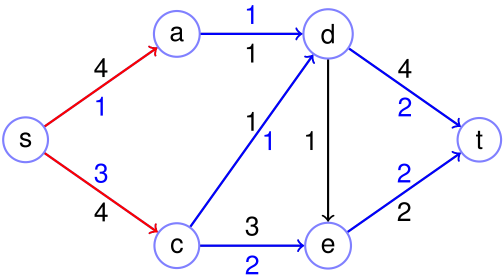

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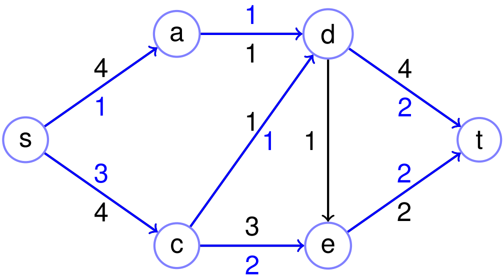

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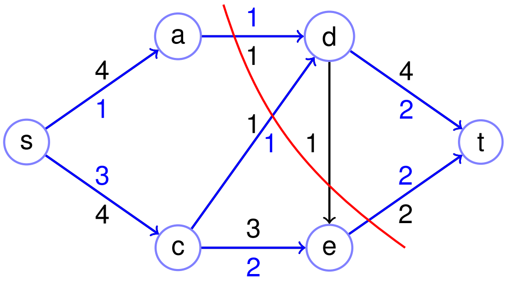

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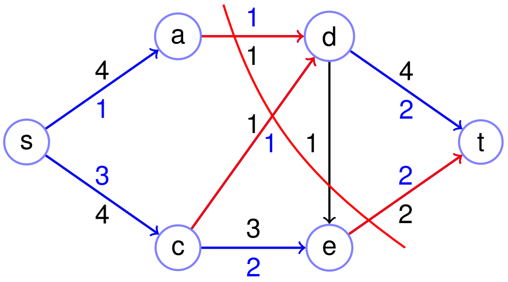

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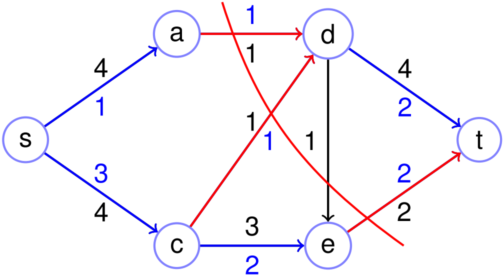

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Optimal? $c_{ad} + c_{cd} + c_{et} = 1 + 1 + 2 = 4$.
Any $s - t$ cut gives an upper bound.

S-T cut.

An $s-t$ cut is a partition of V into S and T where $s \in S$ and $t \in T$. Its capacity is the total capacity of edges from S to T.

Do you know the definition?

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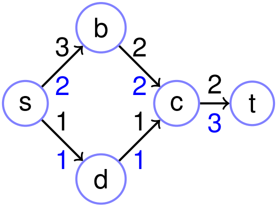

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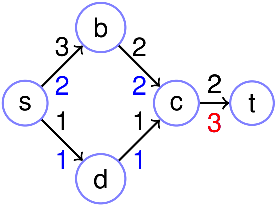

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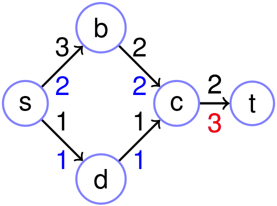

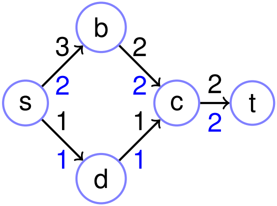

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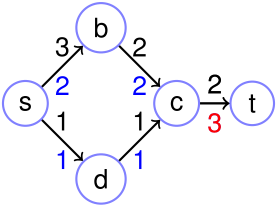

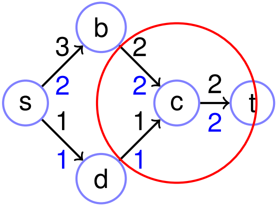

$2 + 1 \neq 2$

Algorithms.

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Linear program!

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Cool!

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Vertex solution to linear program must be integral!

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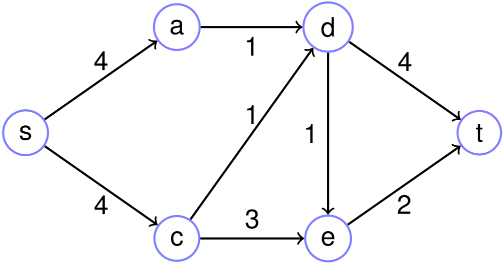

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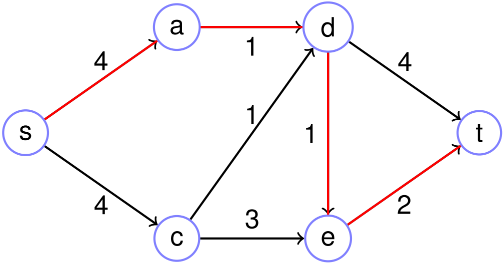

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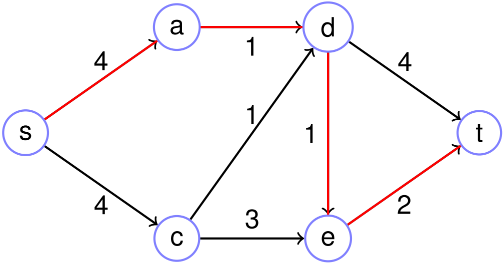

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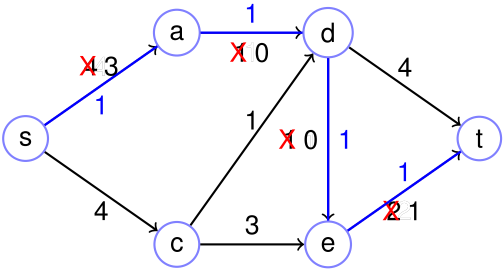

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Add to flow along path.
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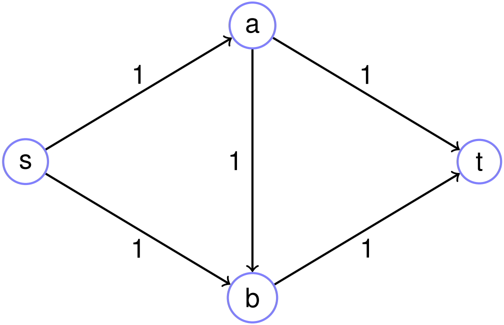

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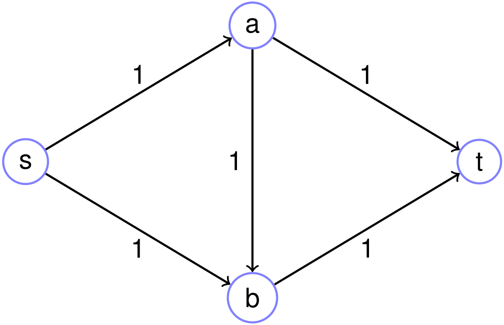

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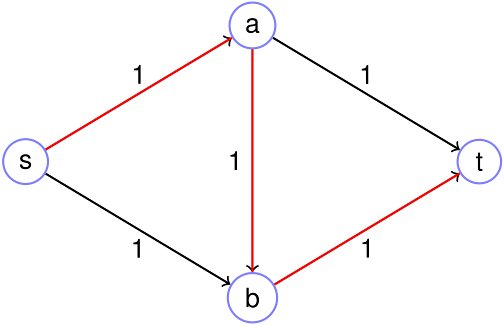

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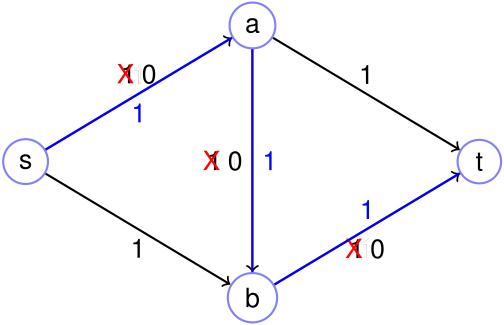

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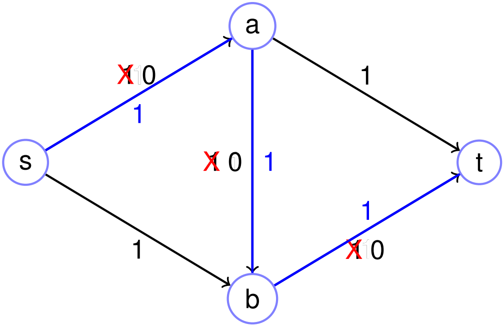

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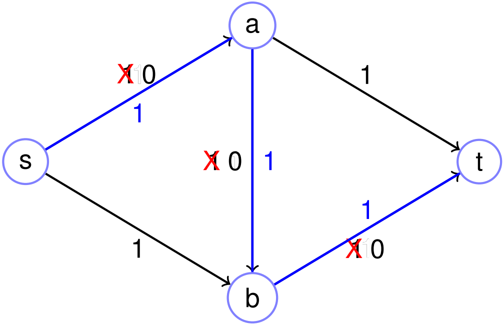

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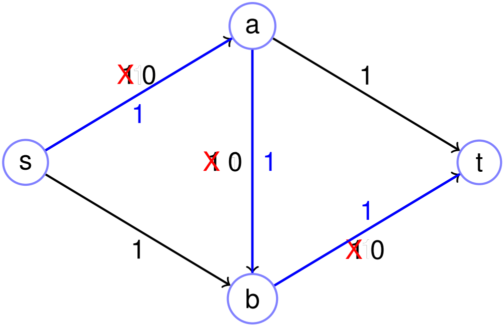

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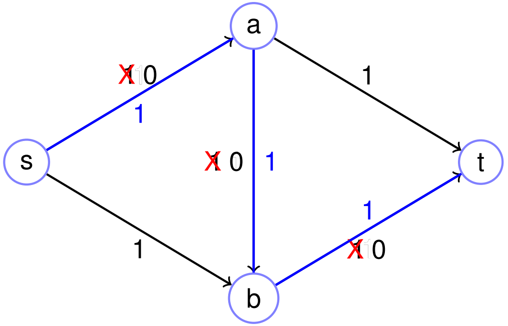

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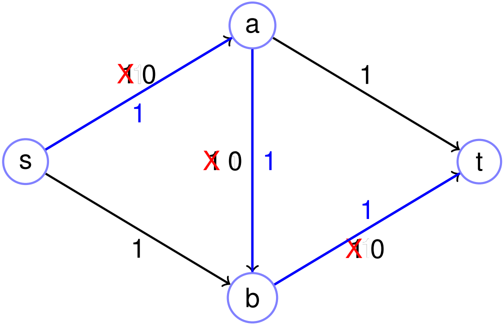

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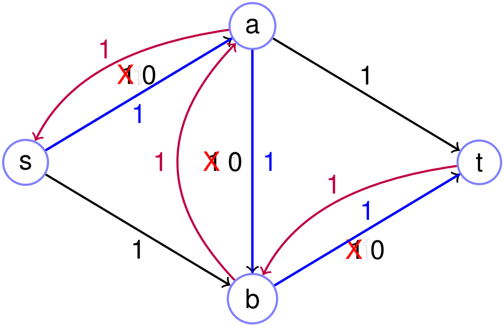

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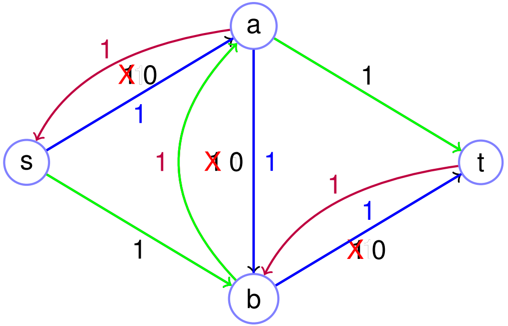

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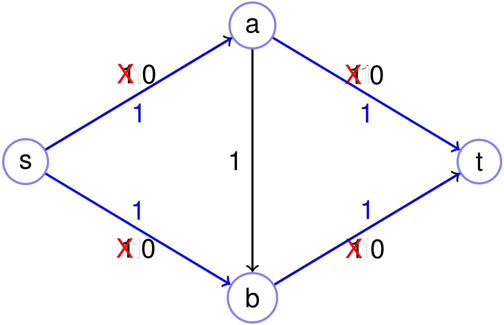

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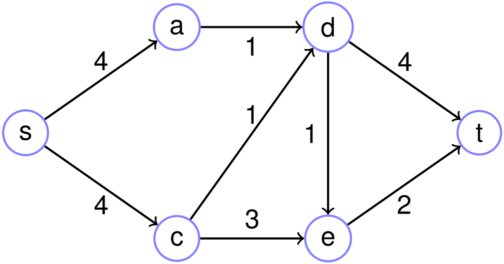

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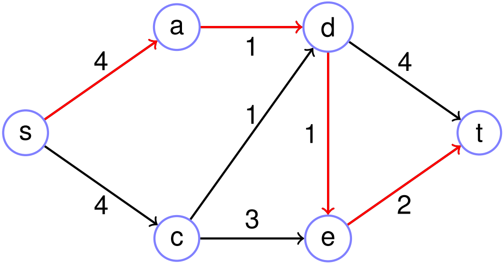

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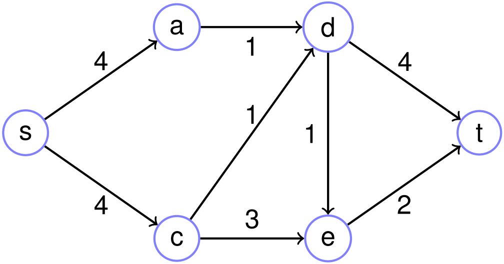

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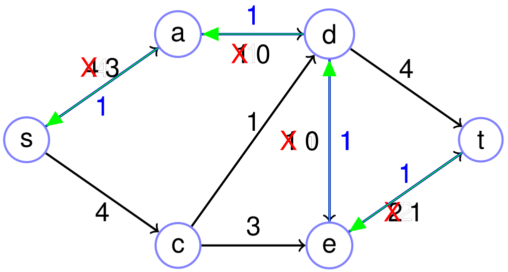

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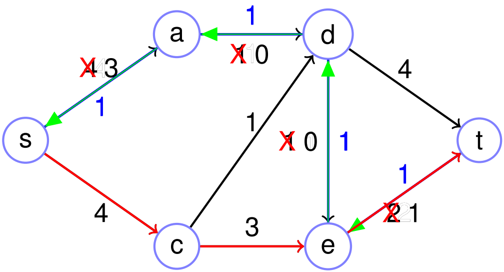

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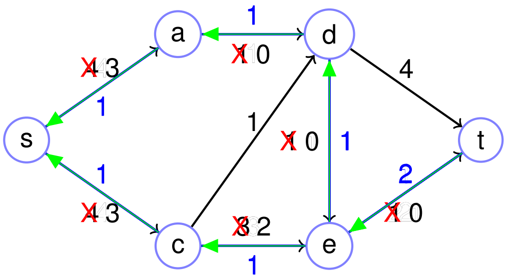

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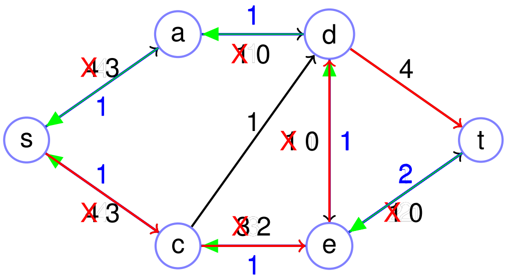

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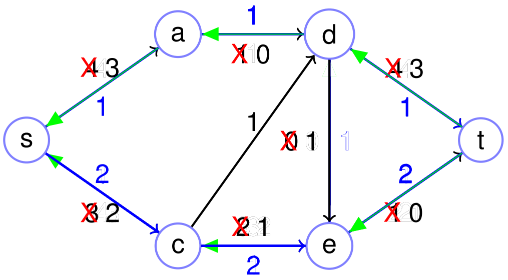

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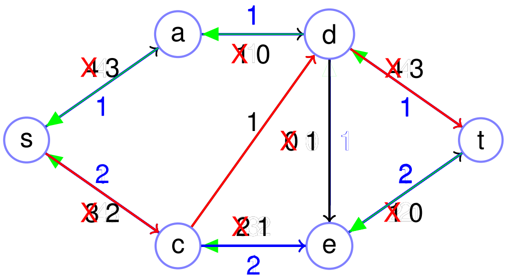

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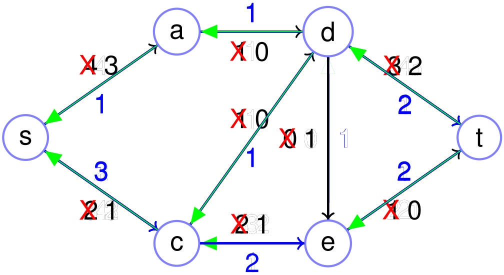

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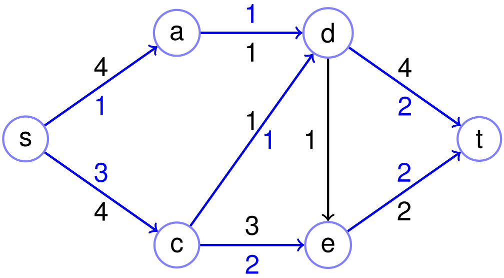

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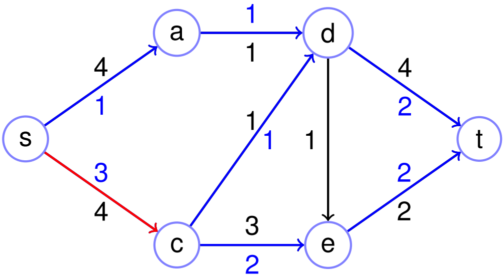

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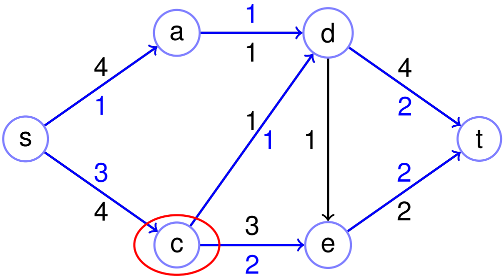

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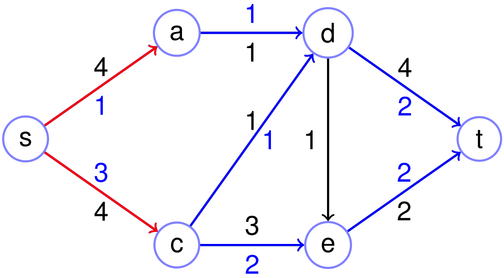

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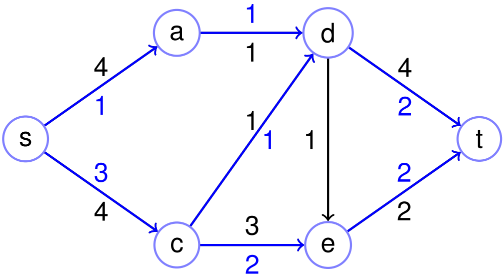

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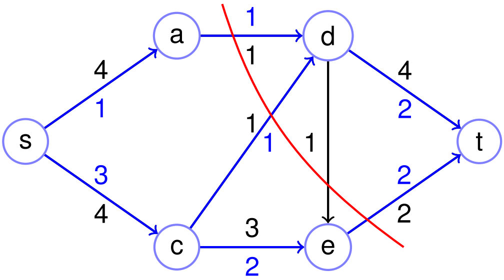

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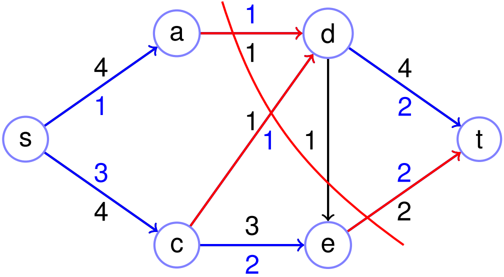

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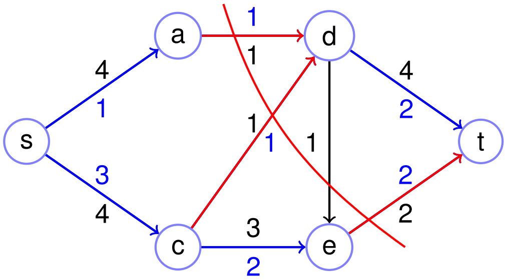

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Optimal? $c_{ad} + c_{cd} + c_{et} = 1 + 1 + 2 = 4.$
Any $s - t$ cut gives an upper bound.

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"flow into v" = "flow out of v" (if not s or t.)

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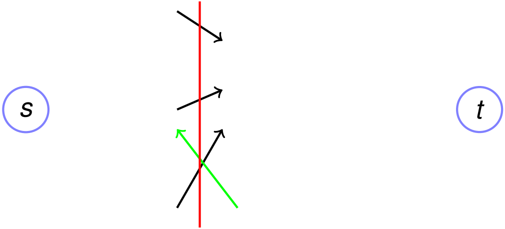
Algorithm adds flow, say f, to path from s to t.
Each internal node has f in, and f out.

Optimality: upper bound.

s-t Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.



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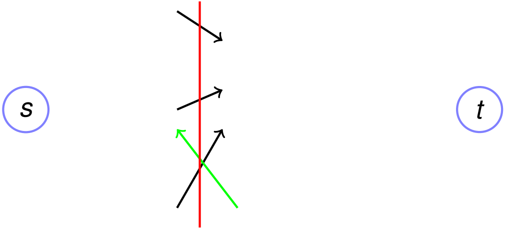
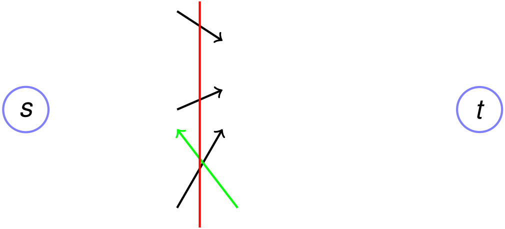


Lemma: Capacity of any $s - t$ cut is an upper bound on the flow.

Optimality: upper bound.

s-t Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.



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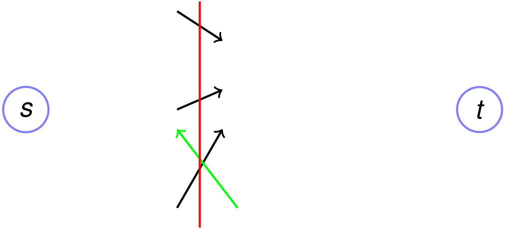

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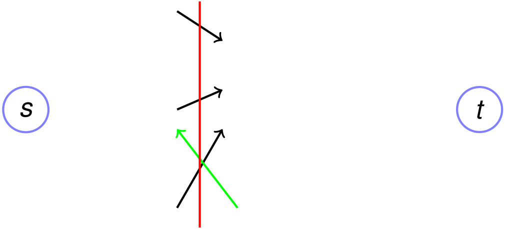

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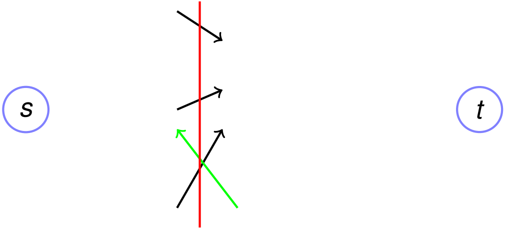

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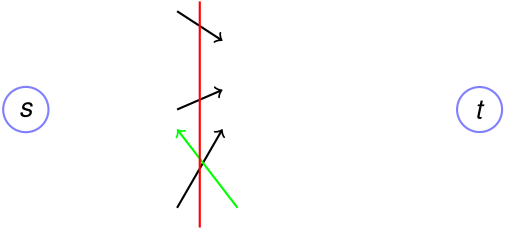

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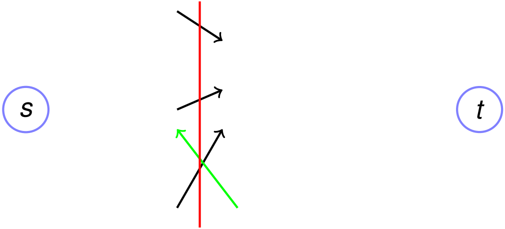

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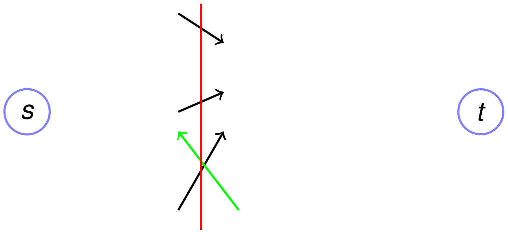

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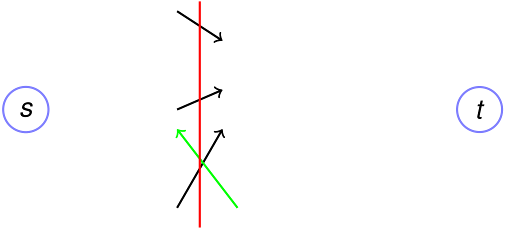

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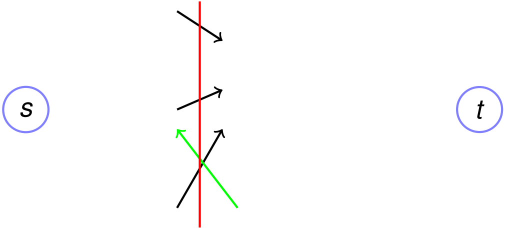

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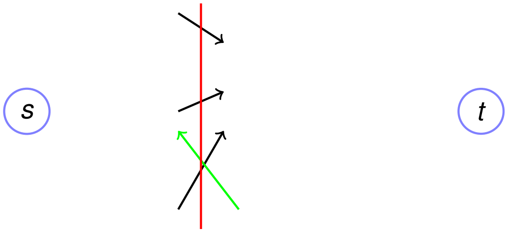

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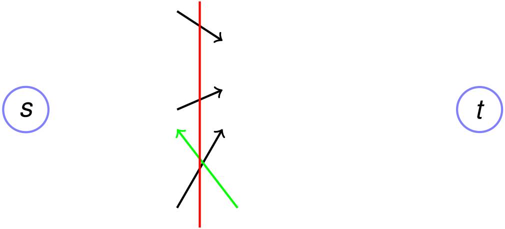

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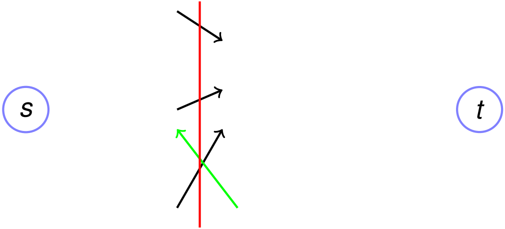

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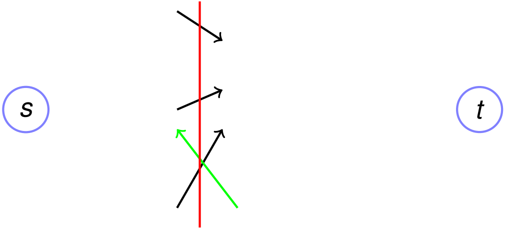

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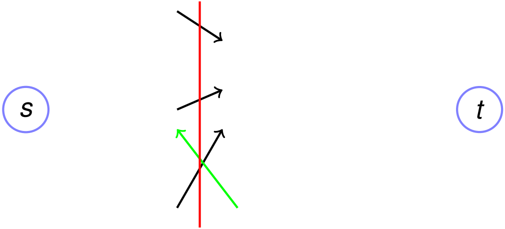

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$\rightarrow$ The value of any valid flow is at most $C(S,T)$! $\square$

Optimality: max flow = min cut.

At termination of augmenting path algorithm.

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No path with residual capacity!

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Depth first search only starting at s does not reach t.

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Depth first search only starting at *s* does not reach *t*.

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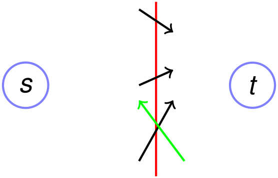

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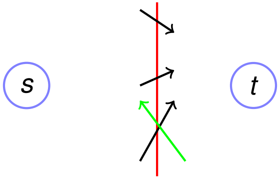

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No arc with positive residual capacity leaving *S*

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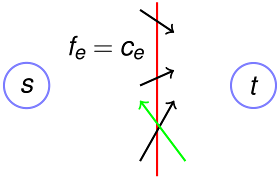

S be reachable nodes.

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$\implies$ All arcs leaving S are full.

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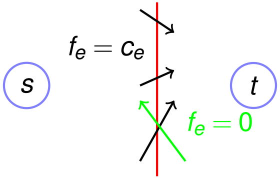

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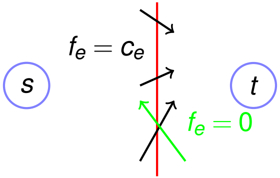

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$\Longrightarrow$ All arcs leaving S are full.

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Total flow leaving S is $C(S, T)$.

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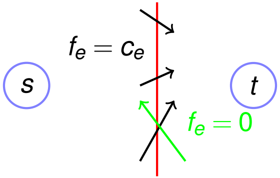

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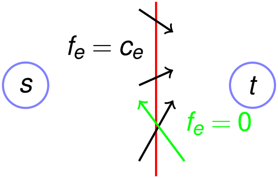

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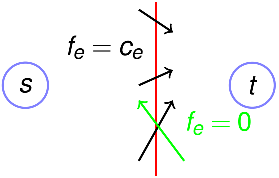

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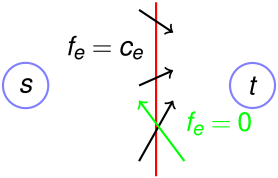

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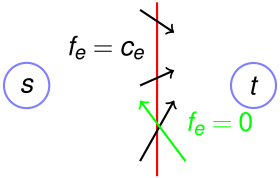

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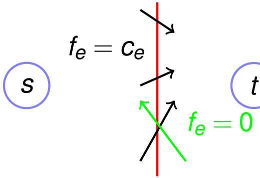

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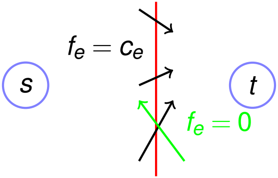

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Cut is minimum $s - t$ cut too!

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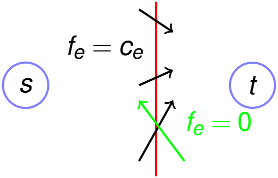

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"any flow" $\leq$ "any cut"

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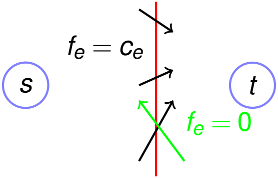

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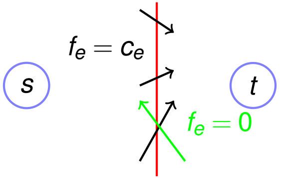

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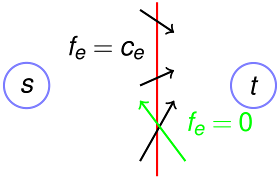

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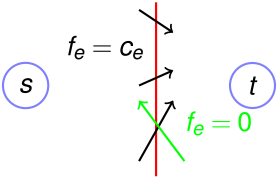

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Cut is minimum $s-t$ cut too!
"any flow" $\leq$ "any cut" and this flow $=$ this cut.
$\rightarrow$ Maximum flow and minimum $s-t$ cut!

Celebrated max flow -minimum cut theorem.

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Celebrate!

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Maximum flow.

"Greedy" augment path...

Except reverse old decisions ..

Reverse residual capacities.

(Friday): Optimality?

No augmenting path $\implies$

$s - t$ cut size = flow value.

Find flow and $s - t$ cut with equal value!