APPROXIMATION ALGORITHM

Vertex Cover

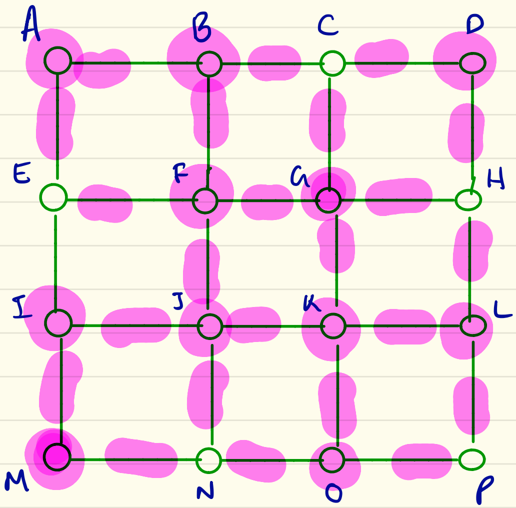
Def: An α-approximation algorithm (ALG) for a minimization problem

maximisation

$ALG(I) \geq \alpha \cdot OPT(I)$

$\forall$ instance I

$$ALG(I) \leq \alpha \cdot OPT(I)$$



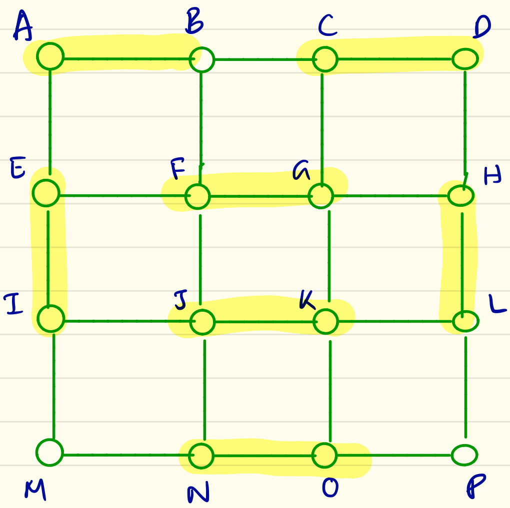
VERTEX COVER

INPUT: Graph $G = (V, E)$

Sol: A subset $S \subseteq V$ of minimum size that covers all edges.

$\forall$ edge (u,v), $u \in S$ or $v \in S$ or both

$S = \{$All but E$\}$



S = {all but M, P}

1) Pick a maximal matching M

"cannot add any more edges"

{add edges to M until you can't}

2) Output

S = {all endpoints of edges in M}

Thm: S is a vertex cover.

Proof: Suppose an edge (u,v) is NOT covered by S.

⇒ add edge (u,v) to matching M.

⇒ but M is maximal, a contradiction.

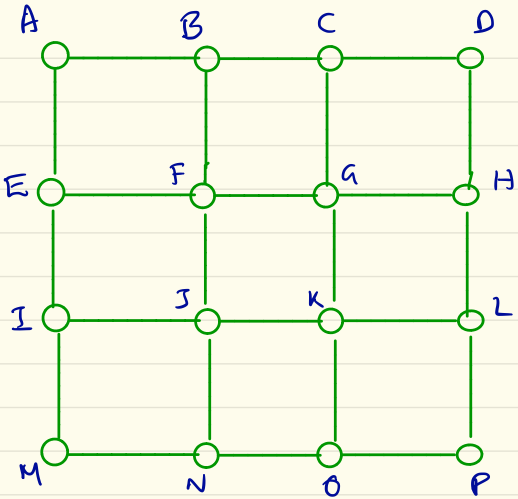
$$|S| = 2 \cdot \left| \text{Size of the maximal matching } M \right|$$

Obs: Optimal vertex Cover OPT $\geq$ Size of every maximal matching

Proof: $\forall$ each edge $(u,v) \in M$

To cover (u,v), pick u or v.

$$\Rightarrow \boxed{|S| \leq 2 \cdot |M| \leq 2 \cdot OPT}$$



Variables: $\{x_1, \dots, x_n\}$

$$x_i \begin{cases} 1 & \text{if } i \in \text{Vertex Cover} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Min} \sum_{i=1}^{n} x_i$$

$$0 \leq x_i \leq 1$$

Run this LP $\longrightarrow$ $x_i + x_j \geq 1 \quad \forall\ (i, j) \in E$

$\{x_i^*\}$ - OPTIMAL SOL to LP.

$$\text{LP-OPT} = \sum_i x_i^*$$

OBSERVATION:

$$\text{LP-OPT} \leq \text{OPT-VC}$$

Proof: if S is an optimal vertex cover

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$ is a feasible LP

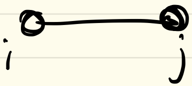
ROUND-LP

$\{x_i^*\}$ - optimal LP sol:

$$S = \{ i \in S \text{ if } x_i^* \geq 1/2 \}$$

$$\left\{ \begin{array}{l} \geq 1/2 \xrightarrow{\text{round}} 1 \\ < 1/2 \longrightarrow 0 \end{array} \right\}$$

1) S is a vertex Cover

Proof: 

$x_i^* + x_j^* \geq 1$

$\Rightarrow$ at least one of $x_i^*, x_j^* \geq 1/2$

$\Rightarrow$ at least one of $i, j \in S$.

$$\text{Cost of VC} = |S| \quad ; \{ i \mid x_i^* \geq 1/2 \}$$

$$\wedge\!|$$

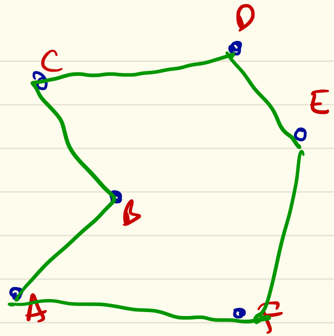
$$2 \cdot \text{LP Cost} = \sum_{i=1}^{n} x_i^*$$

$$\wedge\!\backslash$$

$$2 \text{ OPT} \cdot \text{VC}$$

$\forall i \in S$ Vertex Cover pays 1.

LP pays $x_i^* \geq 1/2$



METRIC TRAVELLING SALESMAN PROBLEM

(every pair)

INPUT: n cities with distances $\{d_{ij}\}$

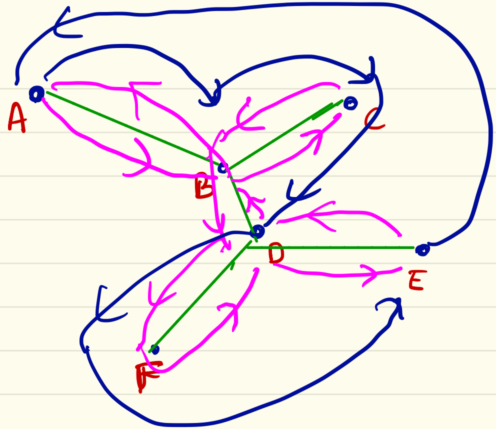
SOL: tour visiting all the cities and returning to starting point.

(Minimise the total distance covered)

ASSUMPTION: Distances satisfy $\triangle$'le inequality

$$d_{ij} + d_{jk} \geq d_{ik} \qquad \forall\, i, j, k.$$

1) Find an MST T on $\{d_{ij}\}$



$$\mathrm{cost}(T) \leq \mathrm{cost}\begin{pmatrix}\text{OPT TSP} \\ \text{tour}\end{pmatrix}$$

2| DFS Traversal of the tree T

$$A \to B \to C \to B \to D \to F \to D \to E \to D \to B \to A$$

$$\mathrm{cost}(\text{DFS Traversal}) = 2 \cdot \mathrm{Cost}(\text{Tree } T) \leq 2 \cdot \begin{pmatrix}\text{OPT} \\ \text{TSP} \\ \text{tour}\end{pmatrix}$$

3) Drop the repeated vertices from traversal.

$$A \to B \to C \to D \to F \to E \to A$$

$$\mathrm{Cost}(\text{Output Tour}) \leq \mathrm{Cost}(\text{DFS Traversal}) \leq 2 \cdot \text{OPT}$$