CS 170: Algorithms

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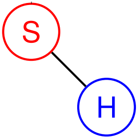
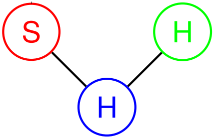
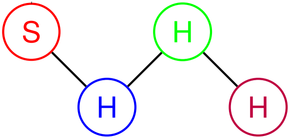
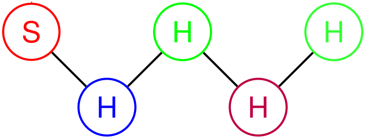


CS 170: Algorithms



CS 170: Algorithms

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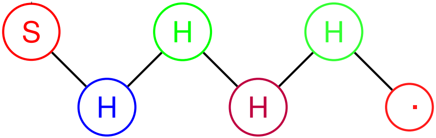
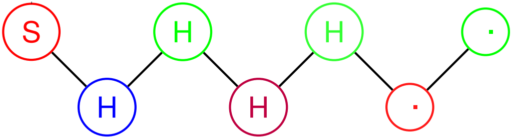
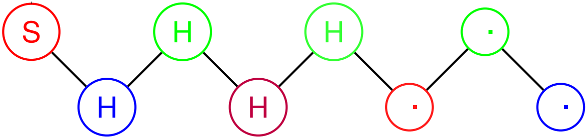
CS 170: Algorithms



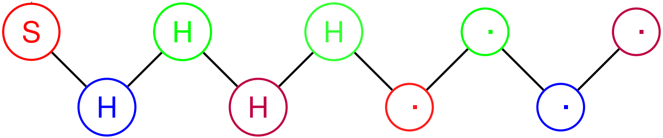
CS 170: Algorithms



CS 170: Algorithms



CS 170: Algorithms



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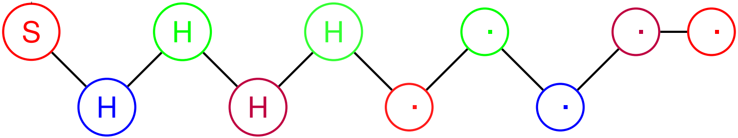

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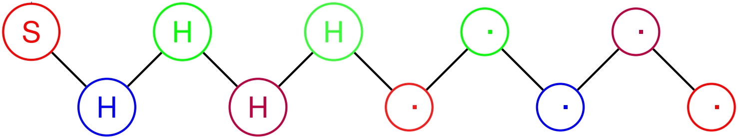

Lecture in a minute

Maximum flow.
  "Greedy" augment path...
    Except reverse old decisions ..
    Reverse residual capacities.
  : Optimality?
    No augmenting path $\implies$
     $s-t$ cut size = flow value.
   Find flow and $s-t$ cut with equal value!

Lecture in a minute

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  - Reverse residual capacities.
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    - $s-t$ cut size = flow value.
  - Find flow and $s-t$ cut with equal value!

Maximum Matching.
- $G=(V,E)$, find subset of one-to-one matches.
- Reduction to max flow.
- Augmenting Alternating Path
- Algorithm $\equiv$ Simplex.

Maximum flow

Flow network $G = (V, E)$, source s, sink $t \in V$, capacities $c_e > 0$.

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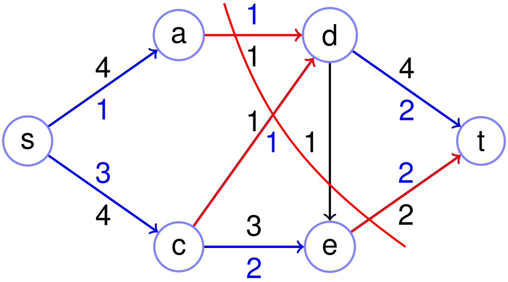

Find Flow: f_e

1. $0 \le f_e \le c_e$. "Capacity constraints." $3 = f_{s,c} \le c_{s,c} = 4$.
2. If u is not s or t
   $\sum_{(w,u) \in E} f_{wu} = \sum_{(u,w) \in E} f_{uw}$. $3 = f_{s,c} = f_{c,d} + f_{c,e} = 2 + 1$.

Maximize: $\text{size}(f) = \sum_{(s,u) \in E} f_{su}$. $f_{sa} + f_{sc} = 1 + 3 = 4$

Optimal? $c_{ad} + c_{cd} + c_{et} = 1 + 1 + 2 = 4$.

S-T cut.

An $s - t$ cut is a partition of V into S and T where $s \in S$ and $t \in T$. Its capacity is the total capacity of edges from S to T.

Do you know the definition?

Find Flow: f_e

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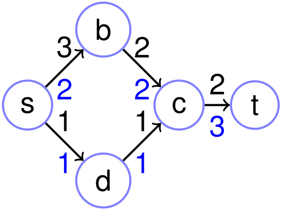

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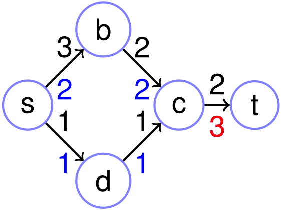

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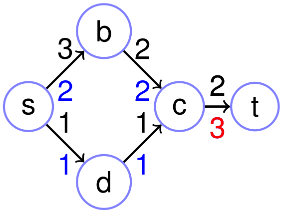

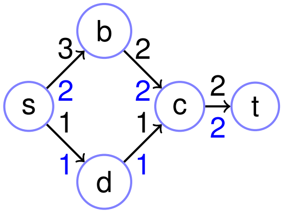

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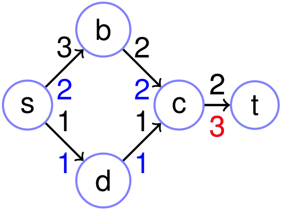

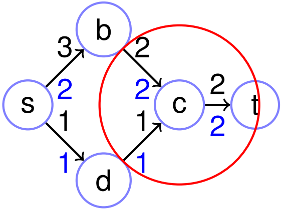

$2+1 \neq 2$

Algorithms.

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Linear program!

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Linear program!

Variables f_e, linear constraints, linear optimization function.

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Cool!

Note...

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Yes. There is an integer vertex solution!

Constraint matrix has every subdeterminant being 1, 0, -1.

Vertex solution to linear program must be integral!

Ford-Fulkerson.

"Simplex" method.

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Find s to t path with remaining capacity.

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Add to flow variables along path.

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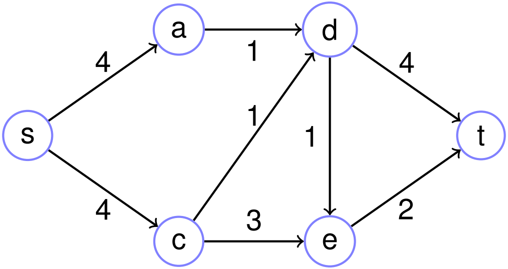

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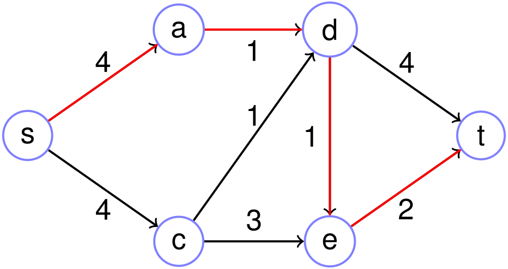

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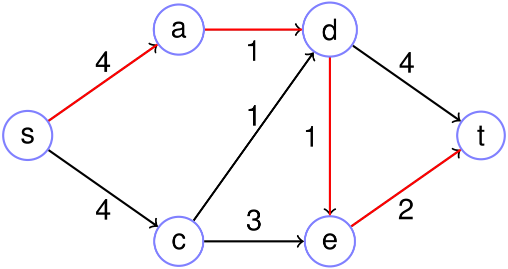

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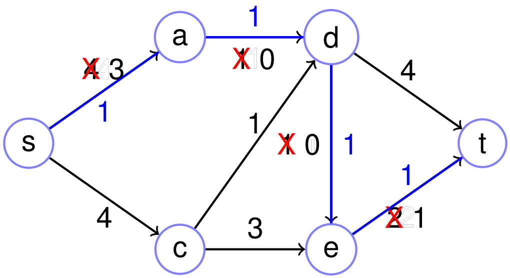
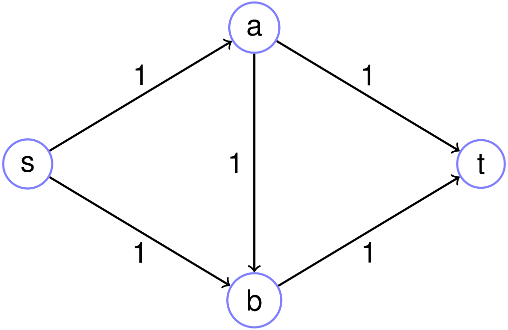


Residual Capacity.

Find s to t path with remaining capacity.
Add to flow along path.
Update remaining capacity.

Repeat.



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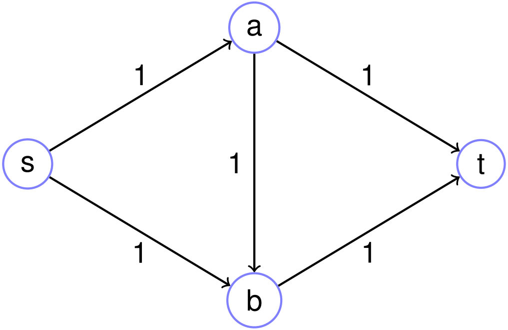

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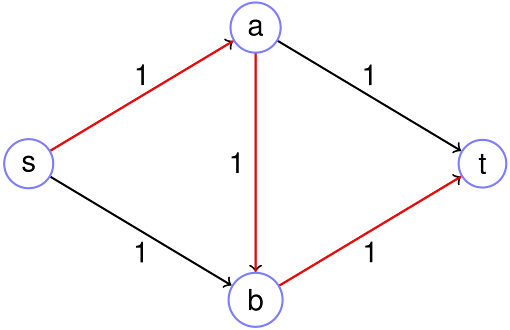

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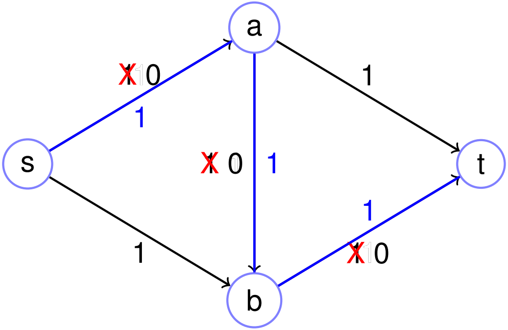

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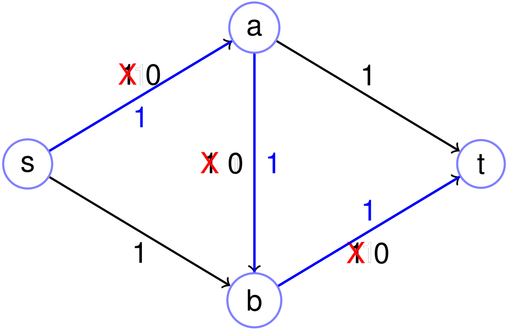

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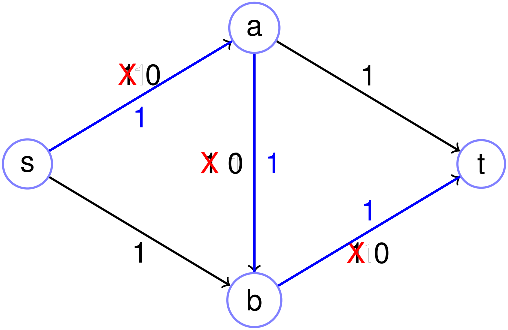

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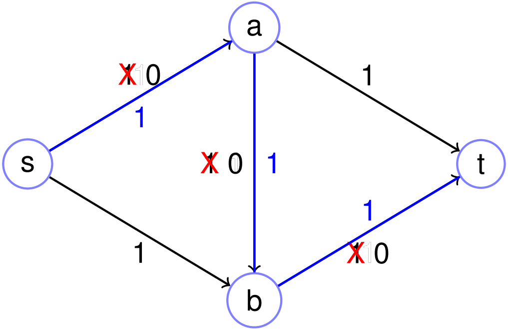

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Find s to t path with remaining capacity.
Add to flow along path. Or reduce flow on reverse edge.
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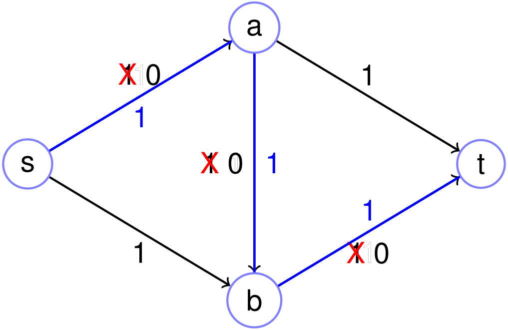

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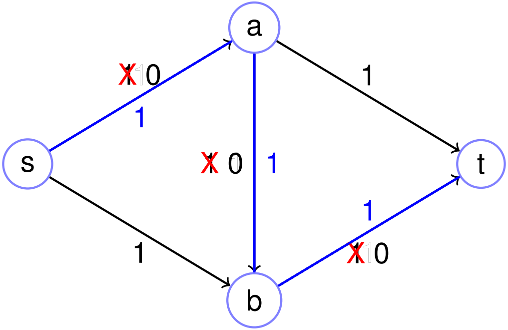

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Find s to t path with remaining capacity.
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  and add reverse $r_{uv} = f_{vu}$

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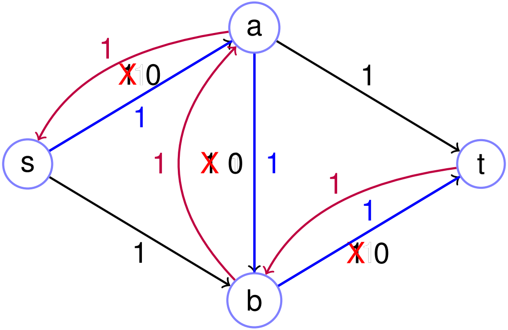

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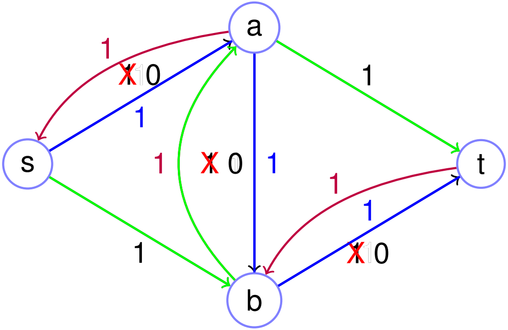

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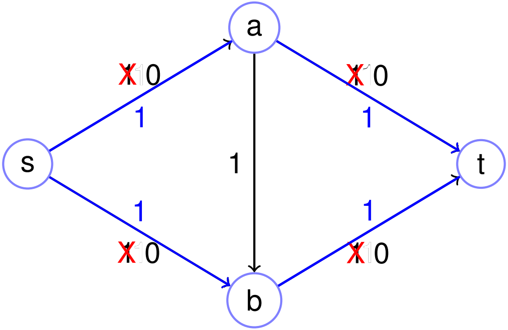

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Bigger Example.

Find s to t path with remaining capacity.

Add to flow along path.

Update residual capacities: : $r_e = c_e - f_e; r_{uv} = f_{vu}$.

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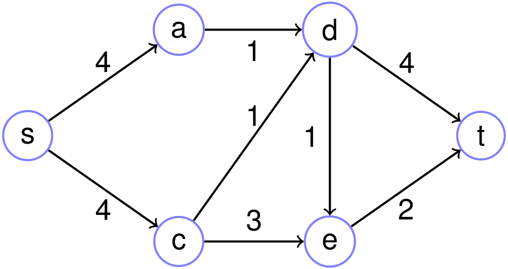

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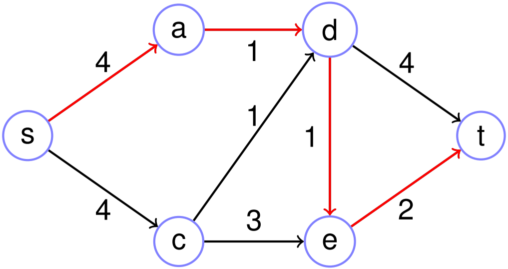

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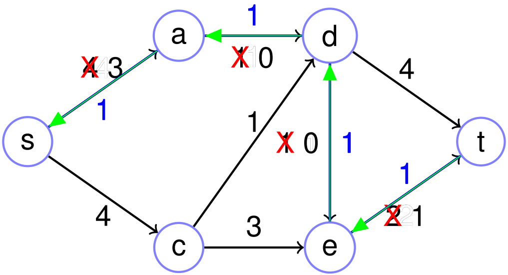

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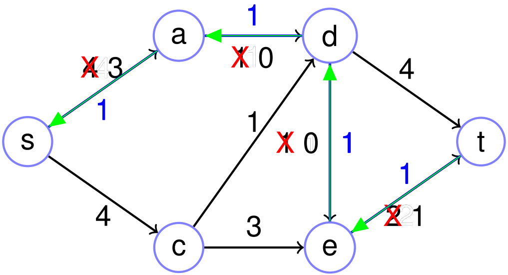

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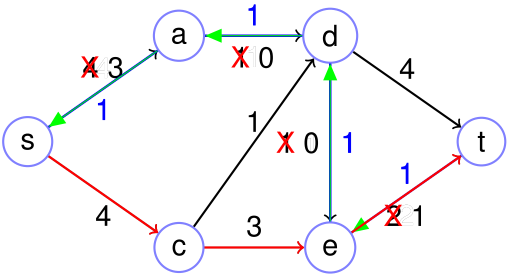

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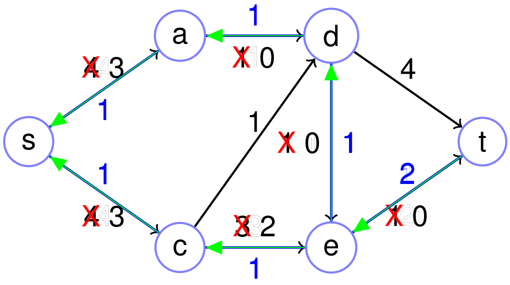

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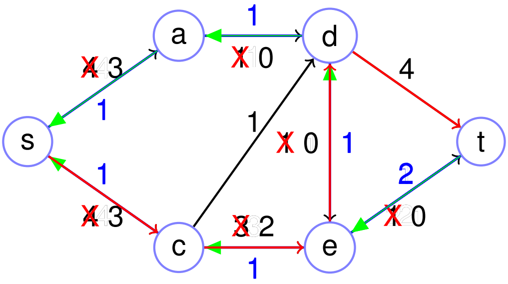

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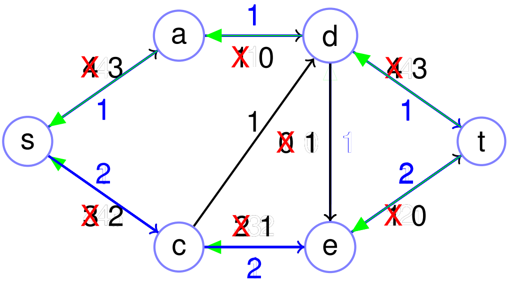

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Find s to t path with remaining capacity.

Add to flow along path.

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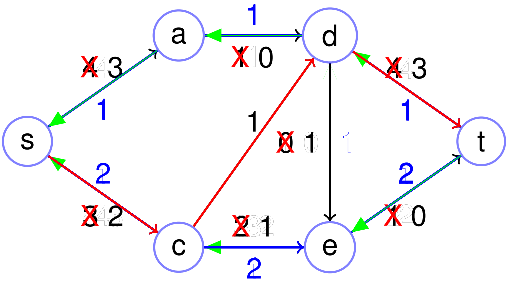

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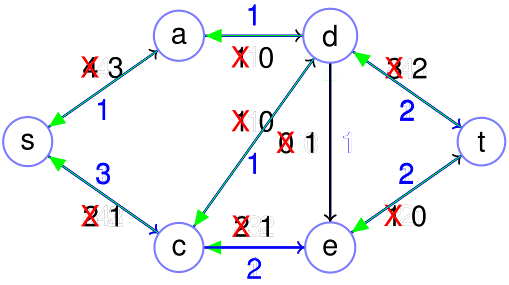

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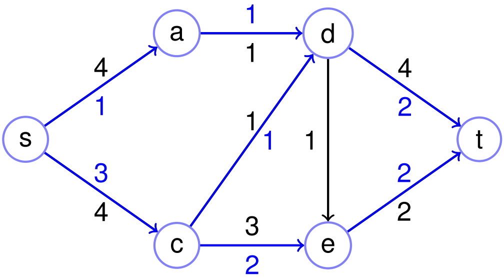

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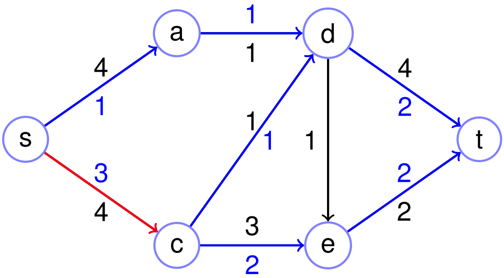

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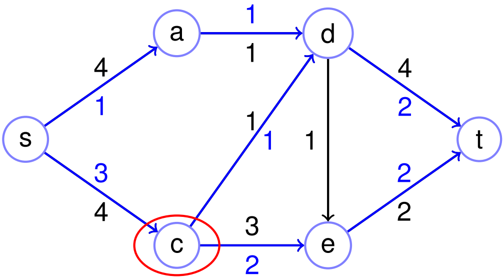

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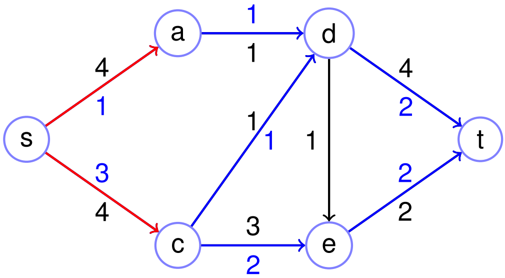

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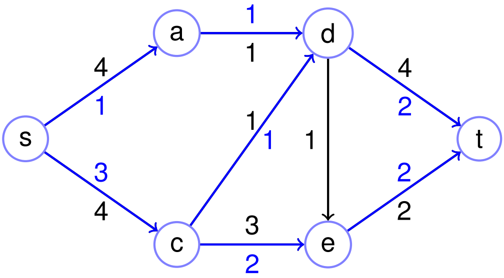

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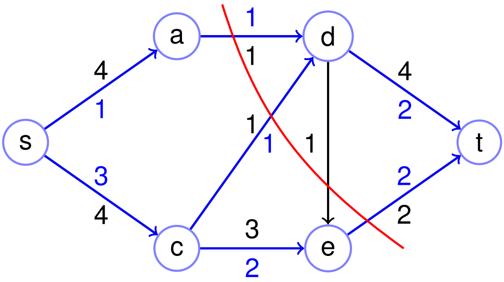

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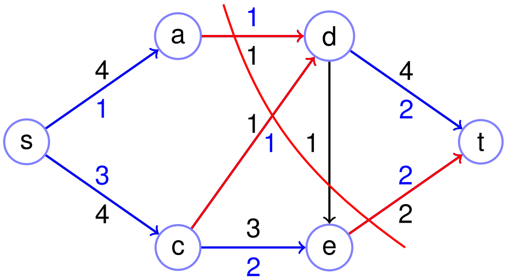

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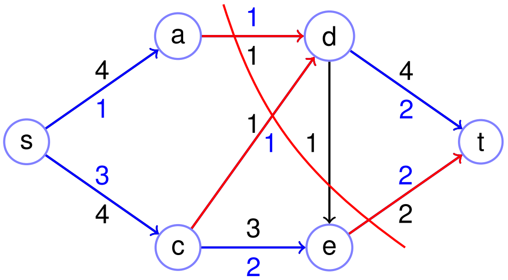

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Any $s - t$ cut gives an upper bound.

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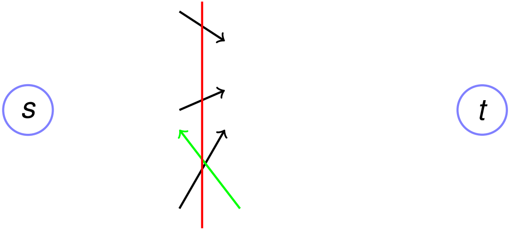
Algorithm adds flow, say f, to path from s to t.
Each internal node has f in, and f out.

Optimality: upper bound.

s-t Cut: $V = S \cup T$ and $s \in S$ and $t \in T$.



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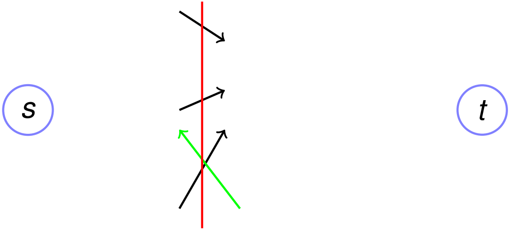

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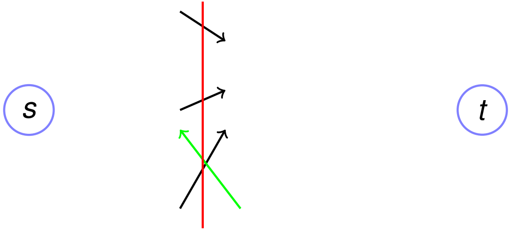

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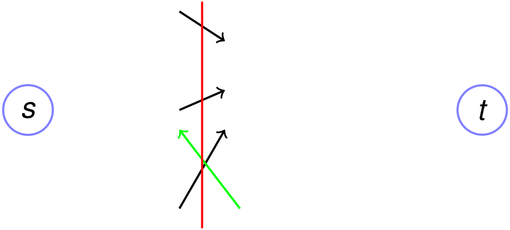

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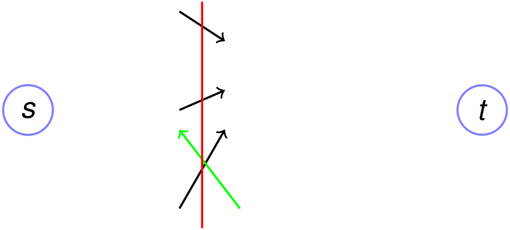

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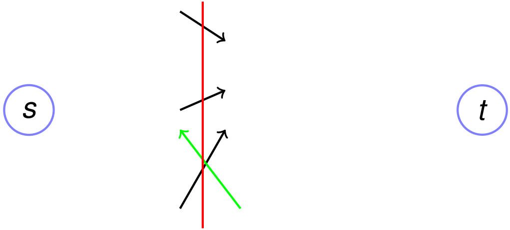

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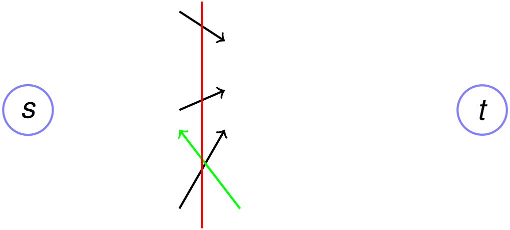

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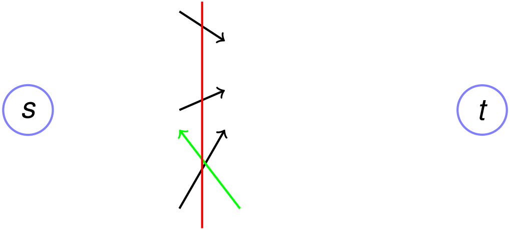

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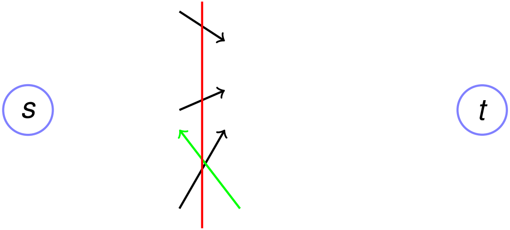

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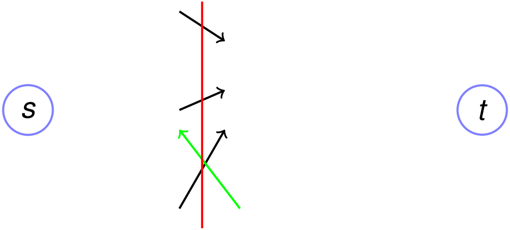

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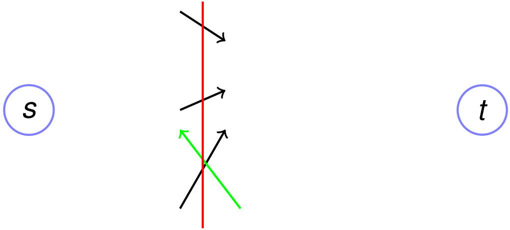

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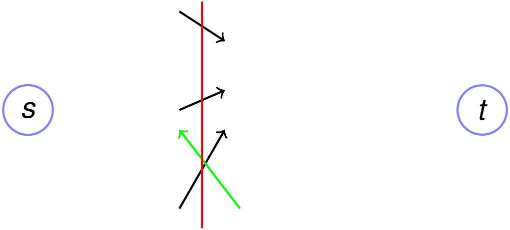

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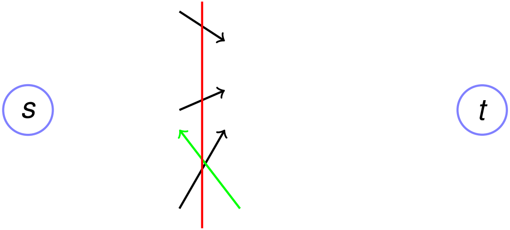

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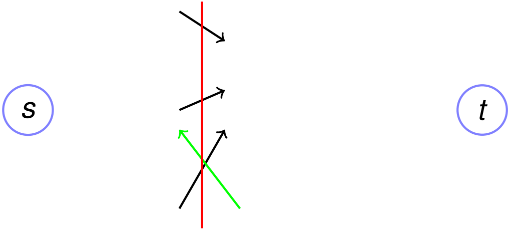

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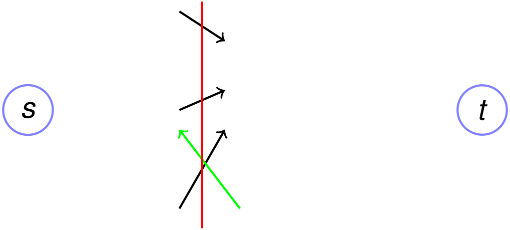

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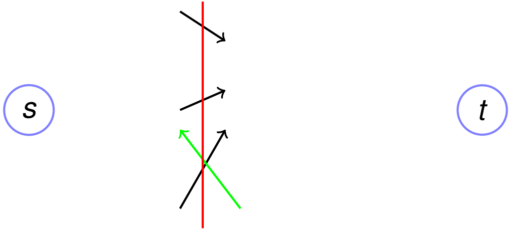

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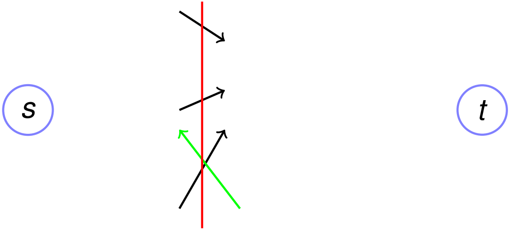

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$\sum_{v \in S} \sum_{(v,u) \in E} f(v,u) - \sum_{(u,v) \in E} f(u,v) = \sum_{e \in (S,T)} f(e)$
Flow into (T) = Flow into t.

For any valid flow, $f : E \to Z+$, the flow out of S (into T)
$\sum_{e \in S \times T} f_e - \sum_{e \in T \times S} f_e \le \sum_{e \in S \times T} c_e - \sum_{e \in T \times S} 0 = C(S,T)$.

$\to$ The value of any valid flow is at most $C(S,T)$! $\square$

Optimality: max flow = min cut.

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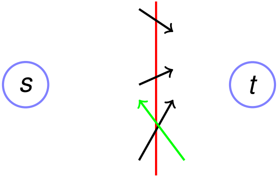

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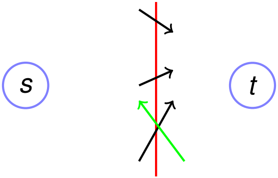

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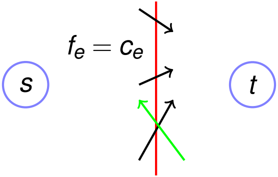

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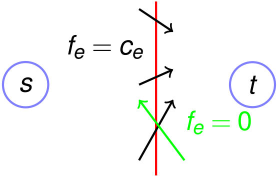

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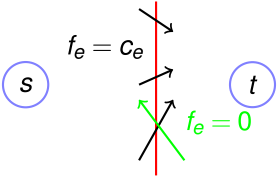

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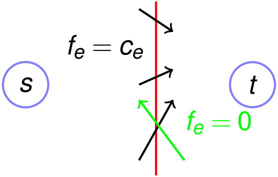

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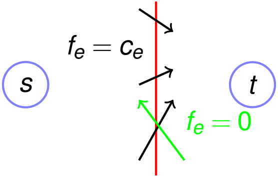

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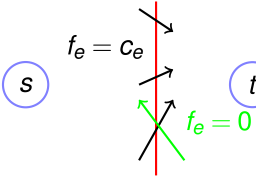

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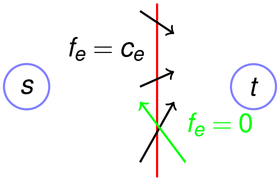

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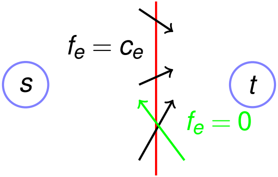

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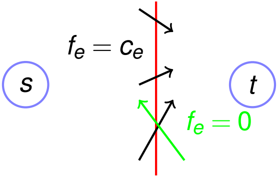

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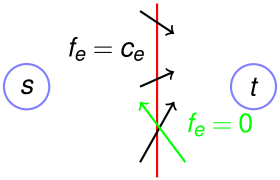

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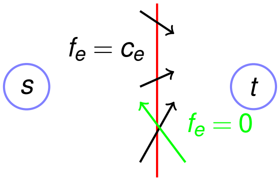

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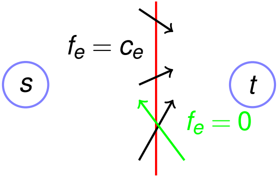

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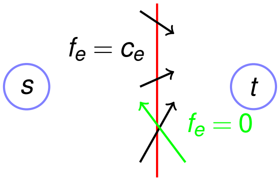

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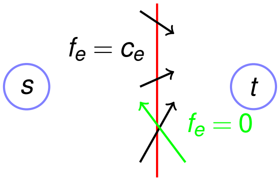

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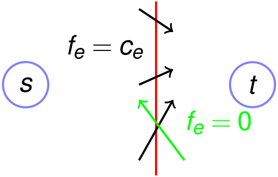

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Cut is minimum $s - t$ cut too!
"any flow" $\leq$ "any cut" and this flow $=$ this cut.
$\rightarrow$ Maximum flow and minimum $s - t$ cut!

Celebrated max flow -minimum cut theorem.

Theorem: In any flow network, the maximum s-t flow is equal to the minimum cut.

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Celebrate!

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One more unit every step!

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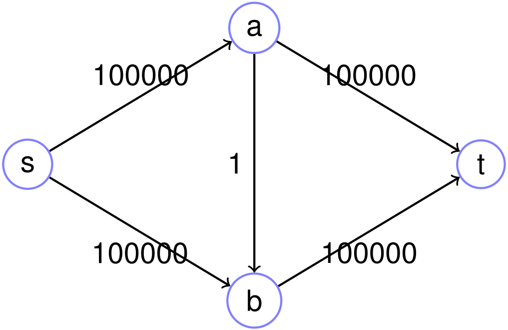

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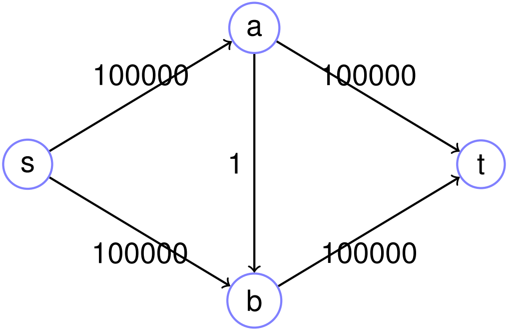

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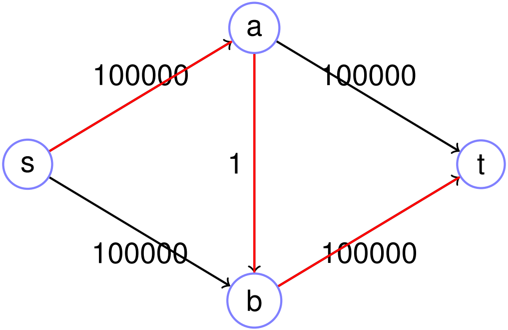

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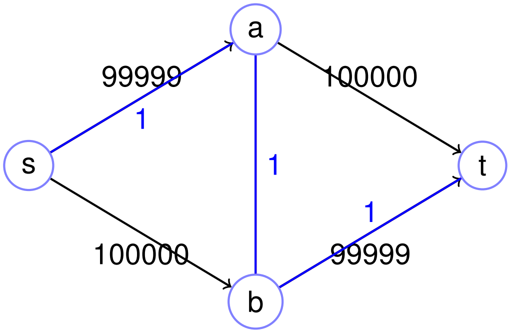

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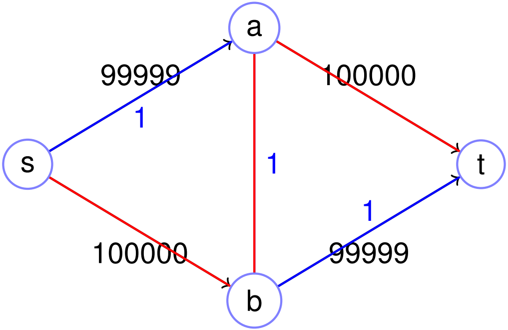

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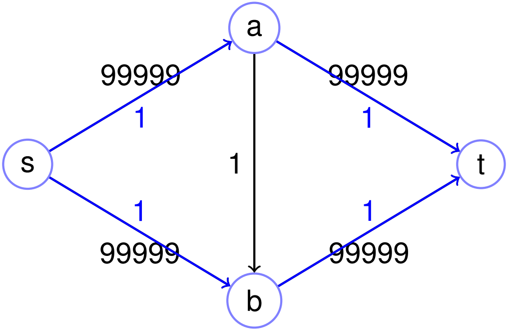

Edmonds-Karp

Augment along shortest path.

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Only route flow on (u,v) if $d(u) \geq d(v)$.

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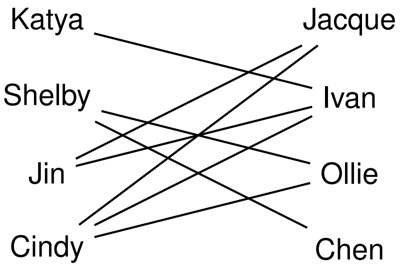

Find largest subset of edges ("matches") which are one to one.

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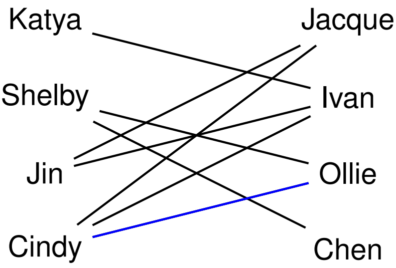

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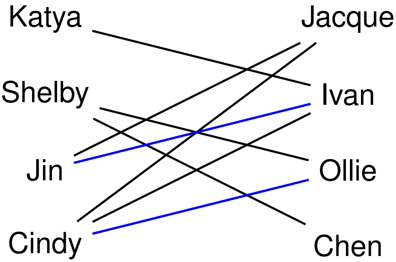

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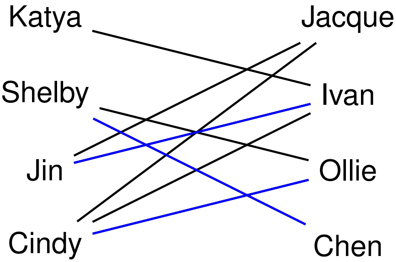

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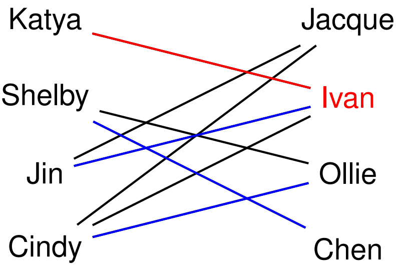

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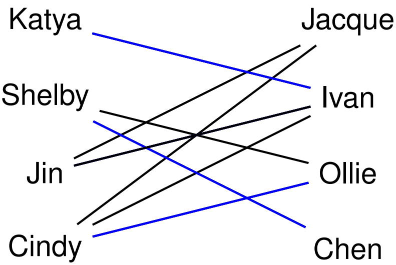

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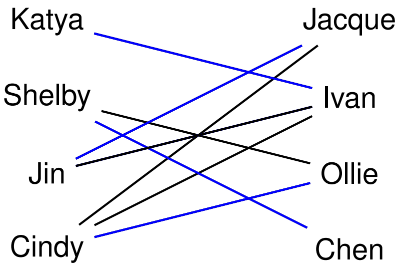

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From matching problem produce flow problem.

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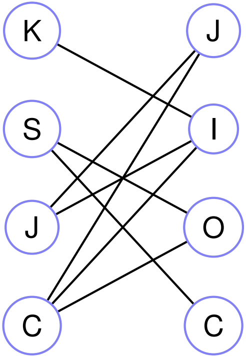

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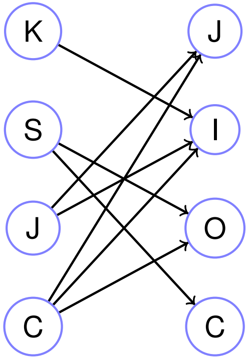

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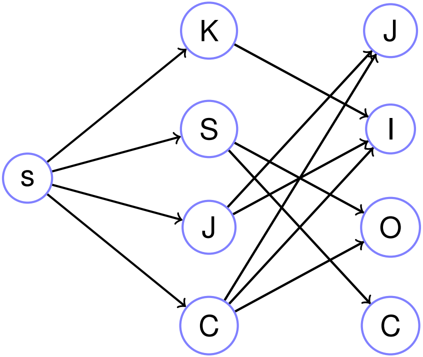

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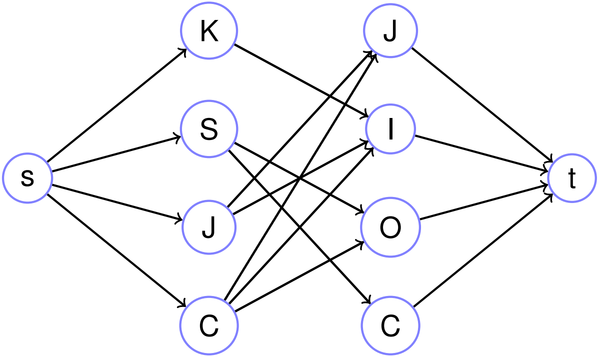

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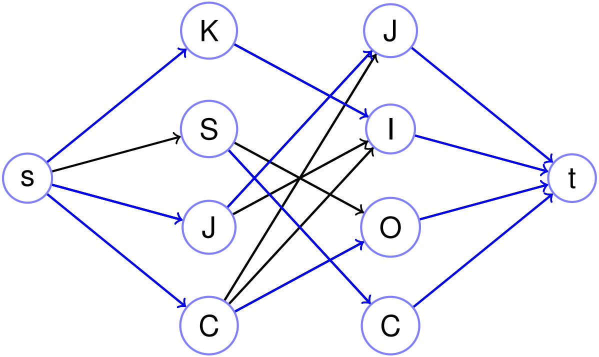

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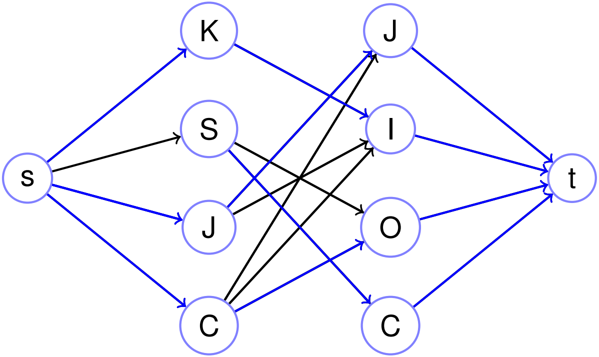

Max flow = Max Matching Size.

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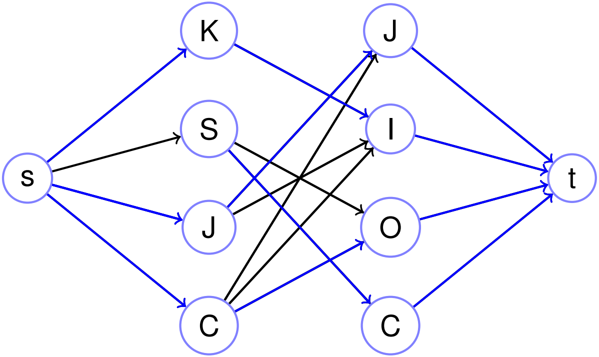

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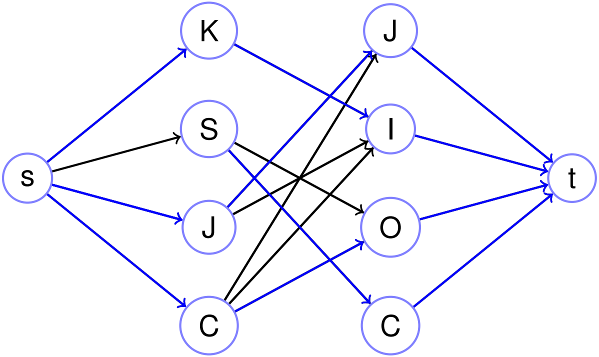

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Flow is not integer necessarily....

Augmenting path algorithm gives integer flow.

Maximum Matching Problem.

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Given a bipartite graph, $G = (U, V, E)$, find a maximum sized matching.

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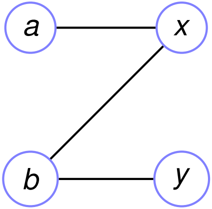

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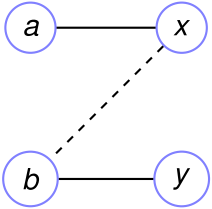

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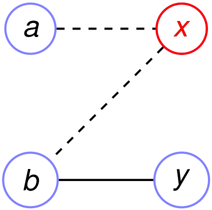

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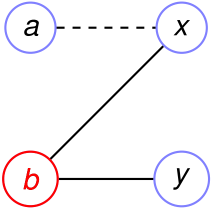

Start at unmatched node(s),
follow unmatched edge(s),
follow matched.
Repeat until an unmatched node.

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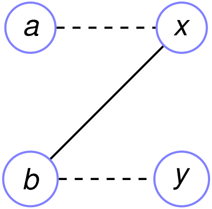

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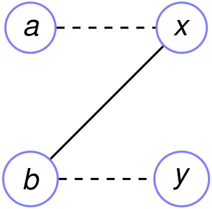

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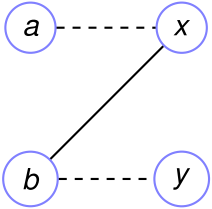

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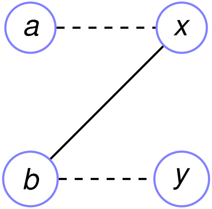

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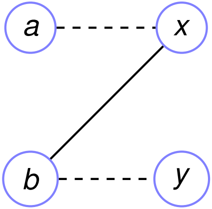

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No perfect matching

Algorithm:
Given matching.
Direct unmatched edges U to V, matched V to U.
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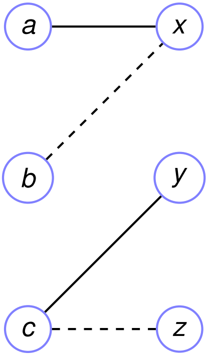
Until everything matched ... or output a cut.

No perfect matching



Can't increase matching size.
No alternating path from (a) to (y).

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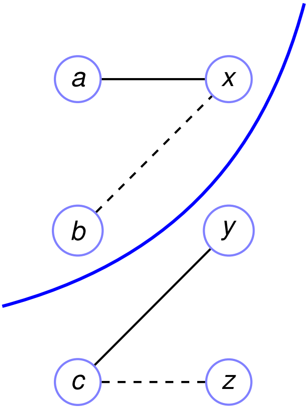

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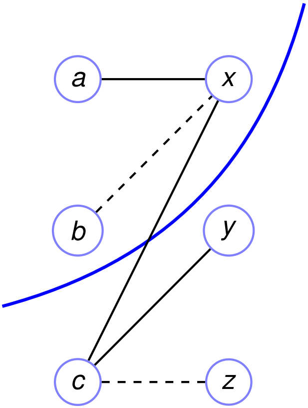

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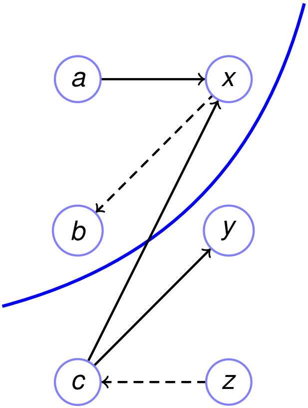

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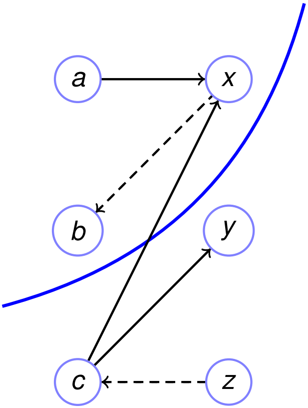

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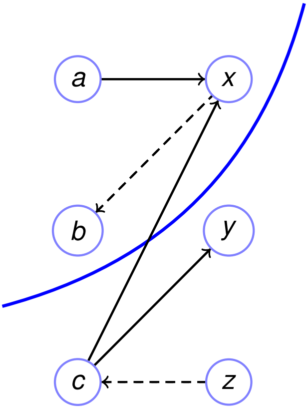

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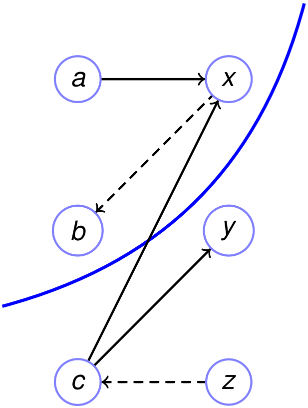

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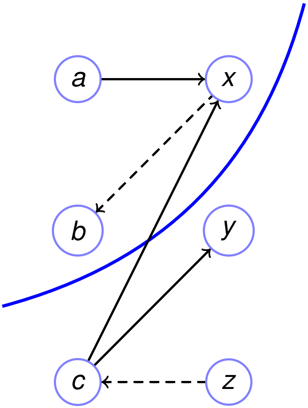

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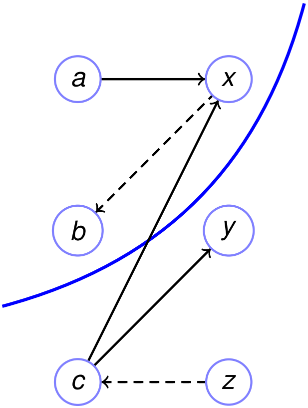

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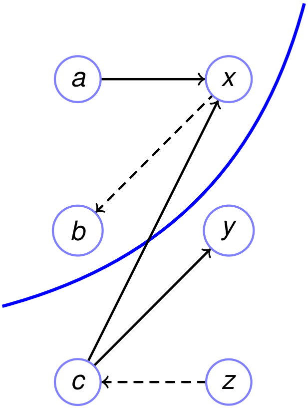

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$x + y + z \leq 1$

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$(a+1,b+1,c+1)-(a,b,c)=(1,1,1)$.

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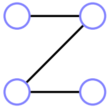
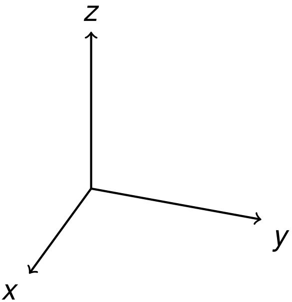
Points in hyperplane are related by nullspace of row.

Maximum matching and simplex.





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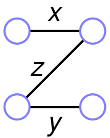

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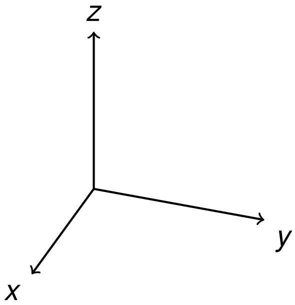

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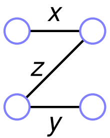

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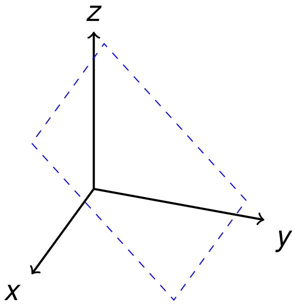
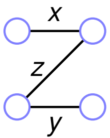


Blue constraints intersect.

Maximum matching and simplex.



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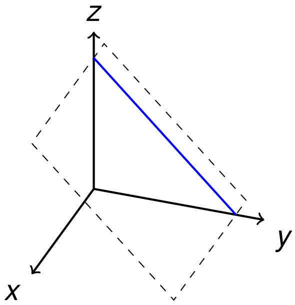

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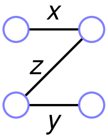

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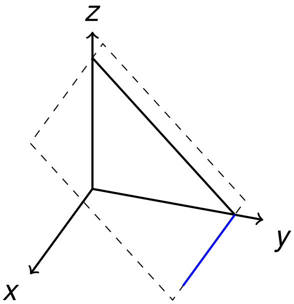

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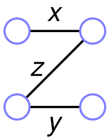

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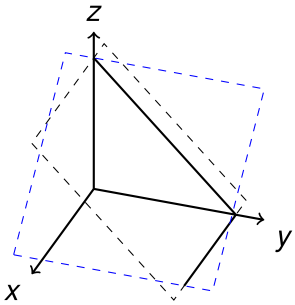
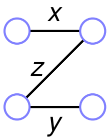
$$y \ge 0$$
$$z \ge 0$$

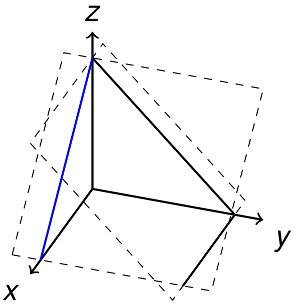


Blue constraints intersect.

Maximum matching and simplex.

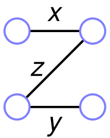


$$
\begin{aligned}
\max x+y+z \\
x \le 1 \\
{\color{blue} x+z \le 1} \\
z+y \le 1 \\
y \le 1 \\
x \ge 0 \\
{\color{blue} y \ge 0} \\
z \ge 0
\end{aligned}
$$

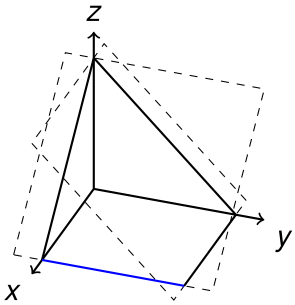


Blue constraints intersect.

Maximum matching and simplex.

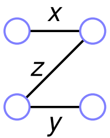


$$
\begin{aligned}
\max x+y+z \\
x \le 1 \\
x+z \le 1 \\
z+y \le 1 \\
y \le 1 \\
x \ge 0 \\
y \ge 0 \\
z \ge 0
\end{aligned}
$$

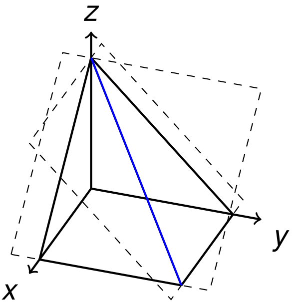


Blue constraints intersect.

Maximum matching and simplex.

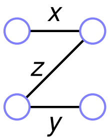


$$\max x + y + z$$
$$x \leq 1$$
$$x + z \leq 1$$
$$z + y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

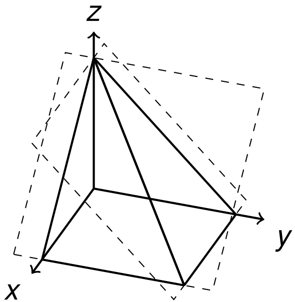


Blue constraints intersect.

Maximum matching and simplex.

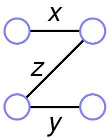


$$
\begin{aligned}
\max x+y+z & \\
x &\le 1\\
x+z &\le 1\\
z+y &\le 1\\
y &\le 1\\
x &\ge 0\\
y &\ge 0\\
z &\ge 0
\end{aligned}
$$

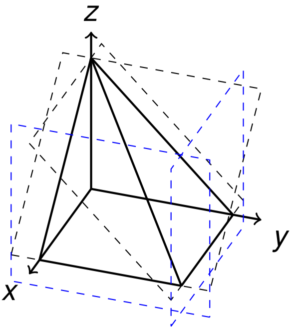


Blue constraints intersect.

Maximum matching and simplex.

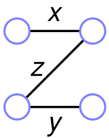


$$\max x + y + z$$
$$x \le 1$$
$$x + z \le 1$$
$$z + y \le 1$$
$$y \le 1$$
$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$



Blue constraints redundant.

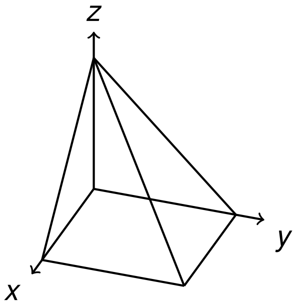
Maximum matching and simplex.



$$\max x + y + z$$

$$x + z \le 1$$
$$z + y \le 1$$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

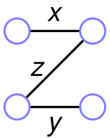


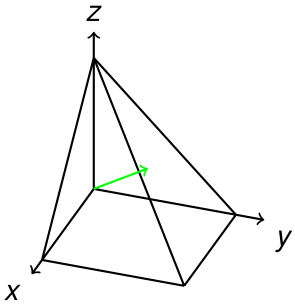
Maximum matching and simplex.

$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

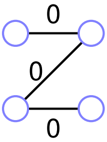
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$





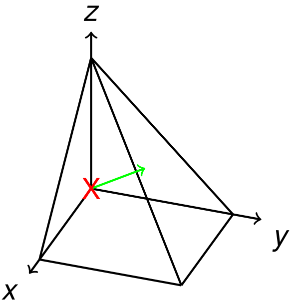
Maximum matching and simplex.

$$\max x + y + z$$



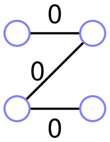
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

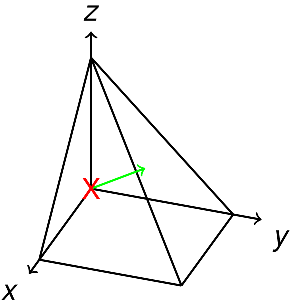
Maximum matching and simplex.



$$\max x + y + z$$

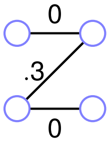
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

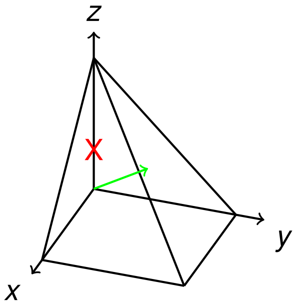
Maximum matching and simplex.



$$\max x + y + z$$

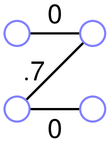
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

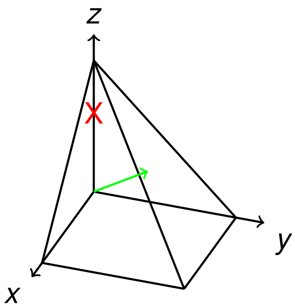
Maximum matching and simplex.



$$\max x + y + z$$

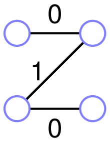
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

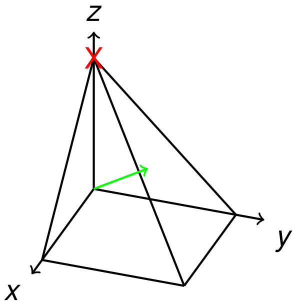
Maximum matching and simplex.



$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

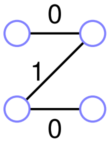


Blue constraints tight.

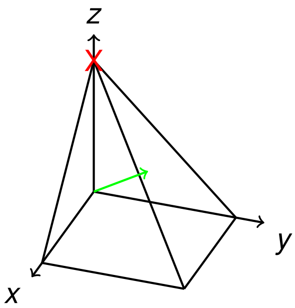
Maximum matching and simplex.

$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

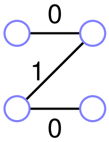


$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

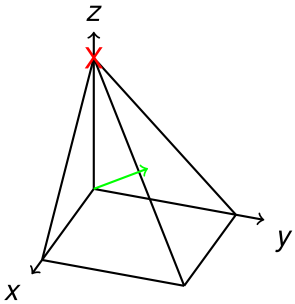
Maximum matching and simplex.



$$\max x + y + z$$

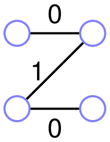
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

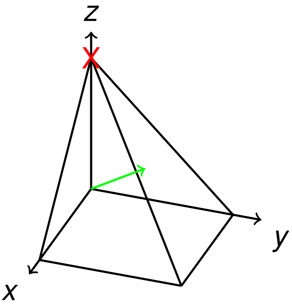
Maximum matching and simplex.



$$\max x + y + z$$

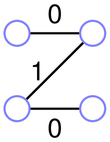
$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Blue constraints tight.

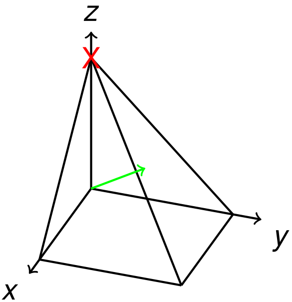
Maximum matching and simplex.



$$\max x + y + z$$

$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

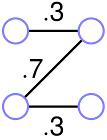


Blue constraints tight.

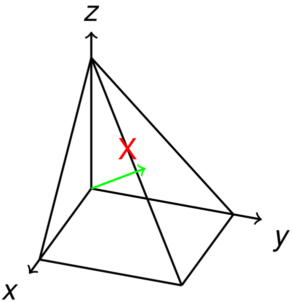
Maximum matching and simplex.

$$\max x + y + z$$

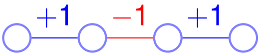
$$x + z \le 1$$
$$z + y \le 1$$



$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

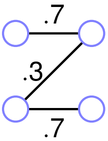


Blue constraints tight.



Augmenting Path.

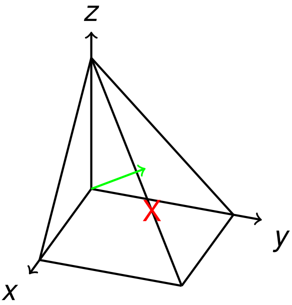
Maximum matching and simplex.



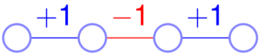
$$\max x + y + z$$

$$x + z \le 1$$
$$z + y \le 1$$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$



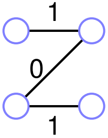
Blue constraints tight.



Augmenting Path. Via Gaussian Elimination!

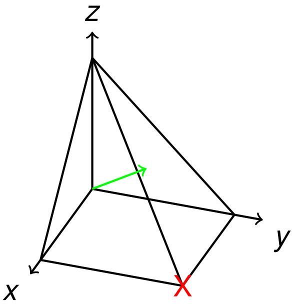
Maximum matching and simplex.

$$\max x + y + z$$

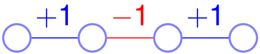


$$x + z \leq 1$$
$$z + y \leq 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



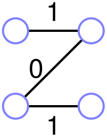
Blue constraints tight.



Augmenting Path. Via Gaussian Elimination!

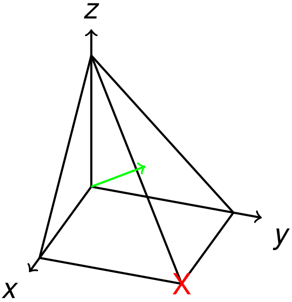
Maximum matching and simplex.

$$\max x + y + z$$

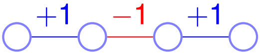


$$x + z \le 1$$
$$z + y \le 1$$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$



Blue constraints tight.



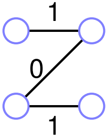
Augmenting Path. Via Gaussian Elimination!

Maximum matching and simplex.

$$\max x + y + z$$

$$x + z \leq 1$$

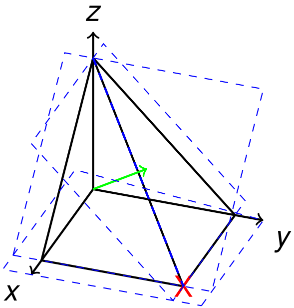
$$z + y \leq 1$$



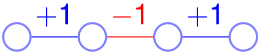
$$x \geq 0$$

$$y \geq 0$$

$$z \geq 0$$

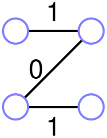


Blue constraints tight.



Augmenting Path. Via Gaussian Elimination!

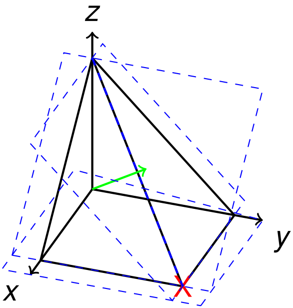
Maximum matching and simplex.



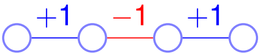
$$\max x + y + z$$

$$x + z \leq 1 \quad a$$
$$z + y \leq 1 \quad b$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0 \quad c$$

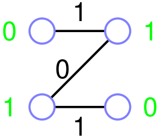


Blue constraints tight.



Augmenting Path. Via Gaussian Elimination!

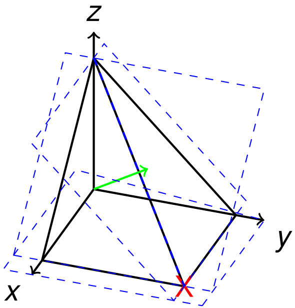
Maximum matching and simplex.



$$\max x + y + z$$

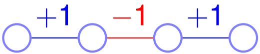
$$x + z \le 1 \quad a = 1$$
$$z + y \le 1 \quad b = 1$$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0 \quad c = 1$$



Blue constraints tight.

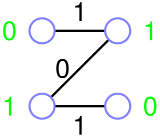
Sum: $x + z + y$.



Augmenting Path. Via Gaussian Elimination!

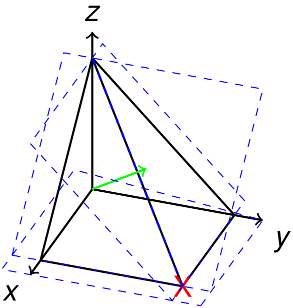
Maximum matching and simplex.

$$\max x + y + z$$



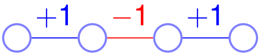
$$x + z \leq 1 \quad a = 1$$
$$z + y \leq 1 \quad b = 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0 \quad c = 1$$



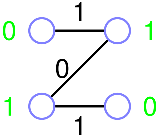
Blue constraints tight.

Sum: $x + z + y$.



Augmenting Path. Via Gaussian Elimination!

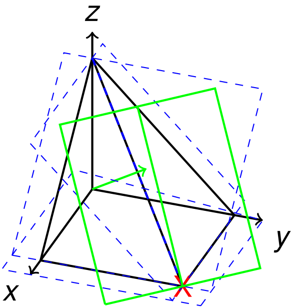
Maximum matching and simplex.



$$\max x + y + z$$

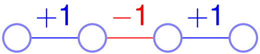
$$x + z \leq 1 \quad a = 1$$
$$z + y \leq 1 \quad b = 1$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0 \quad c = 1$$



Blue constraints tight.

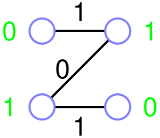
Sum: $x + z + y$.



Augmenting Path. Via Gaussian Elimination!

Maximum matching and simplex.

$$\max x + y + z$$



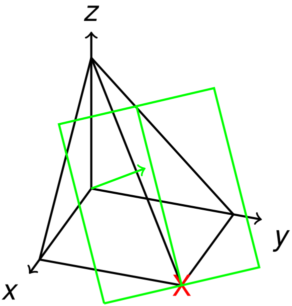
$$x + z \le 1 \quad a = 1$$

$$z + y \le 1 \quad b = 1$$

$$x \ge 0$$

$$y \ge 0$$

$$z \ge 0 \quad c = 1$$



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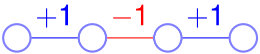

Augmenting Path. Via Gaussian Elimination!

Lecture in a minute

Maximum flow.

"Greedy" augment path...

Except reverse old decisions ..

Reverse residual capacities.

: Optimality?

No augmenting path $\implies$

$s-t$ cut size = flow value.

Find flow and $s-t$ cut with equal value!

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: Optimality?

No augmenting path $\implies$

$s-t$ cut size = flow value.

Find flow and $s-t$ cut with equal value!

Maximum Matching.

$G=(V,E)$, find subset of one-to-one matches.

Reduction to max flow.

Augmenting Alternating Path

Algorithm $\equiv$ Simplex.