CS 170: Algorithms

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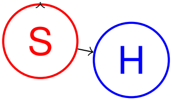
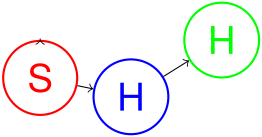
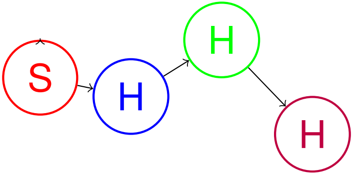


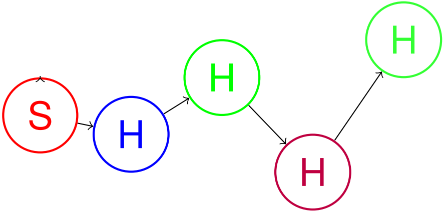
CS 170: Algorithms



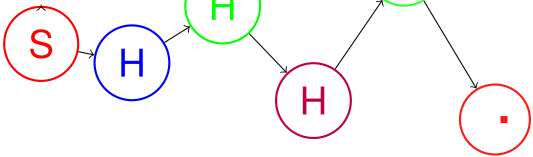
CS 170: Algorithms



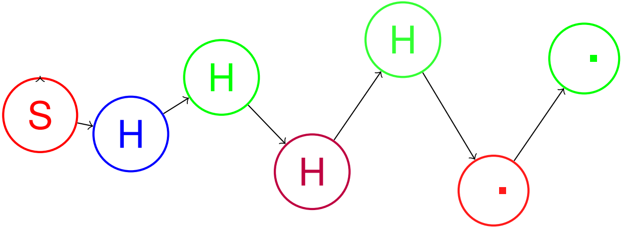
CS 170: Algorithms



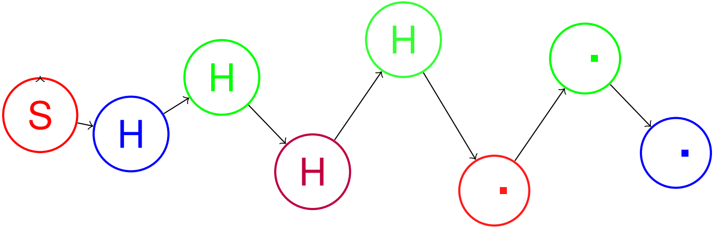
CS 170: Algorithms



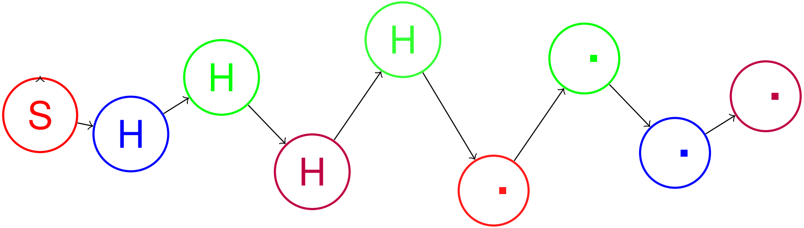
CS 170: Algorithms



CS 170: Algorithms



CS 170: Algorithms



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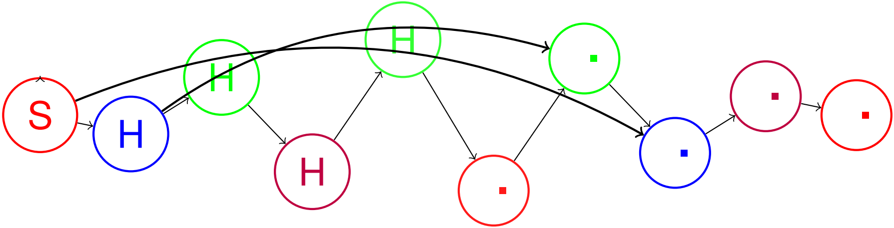
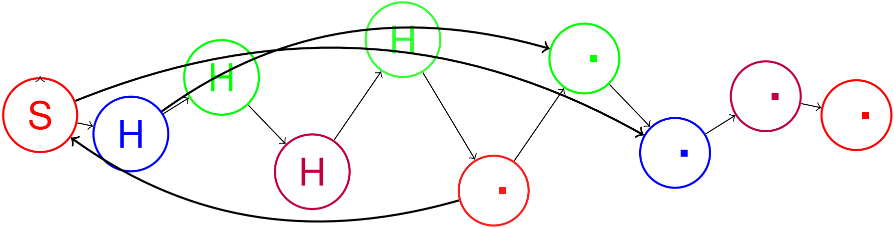


CS 170: Algorithms



Directed graphs...with cycles.

Lecture in a minute!

Quick Review:

DFS so far $\equiv$

Lecture in a minute!

Quick Review:
- DFS so far $\equiv$ how I learned to love the stack.
  - pre/post = time on stack.
- Topological Ordering:
  - Inverse post ordering $\equiv$ topological ordering.
  - Remove source, repeat.

Lecture in a minute!

Quick Review:
  DFS so far $\equiv$ how I learned to love the stack.
   pre/post = time on stack.
  Topological Ordering:
    Inverse post ordering $\equiv$ topological ordering.
    Remove source, repeat.

Strongly Connected Components: directed graphs.
 Strong Connectivity for u and v.
   On a cycle together.
 Easy: $O(|V||E|)$ algorithm.
 Linear time algorithm!

Lecture in a minute!

Quick Review:
DFS so far $\equiv$ how I learned to love the stack.
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Remove source, repeat.

Strongly Connected Components: directed graphs.
Strong Connectivity for u and v.
On a cycle together.
Easy: $O(|V||E|)$ algorithm.
Linear time algorithm!
Observation: Highest post in "source component".
Find vertex in sink component.
Explore.
Repeat.

DFS: so far.

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How I learned to stop worrying

DFS: so far.

How I learned to stop worrying
...and love the stack.

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$[pre(u), post(u)]$ is “clock interval on stack.”

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$\vdots$

u

$\vdots$

v

$\vdots$

Who is an ancestor of whom?

DFS: so far.

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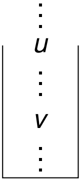

Who is an ancestor of whom?

v is an ancestor of *u*

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$[pre(u), post(u)]$ is “clock interval on stack.”

$\vdots$
u
$\vdots$
v
$\vdots$

Who is an ancestor of whom?

v is an ancestor of u

No cycles: no back edge

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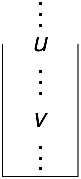

Who is an ancestor of whom?

v is an ancestor of u

No cycles: no back edge
Back edge – edge to ancestor in "tree" of explore calls."

DFS: so far.

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$[pre(u), post(u)]$ is "clock interval on stack."

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u
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$$\begin{array}{|c|} \vdots \\ u \\ \vdots \\ v \\ \vdots \\ \hline \end{array}$$

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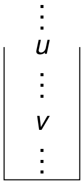

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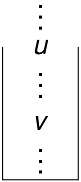

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Bonus Topological Order Algorithm:

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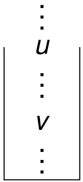

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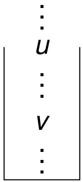

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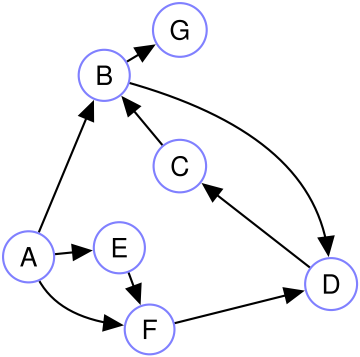
remove source, repeat.

Depth first search: directed.



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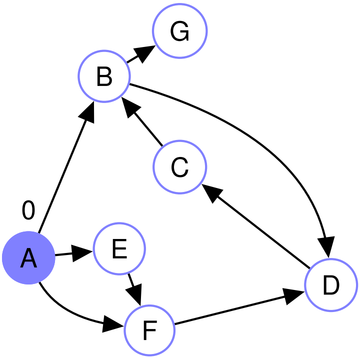

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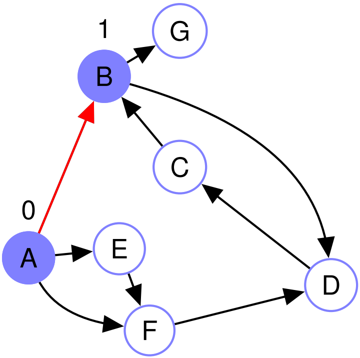

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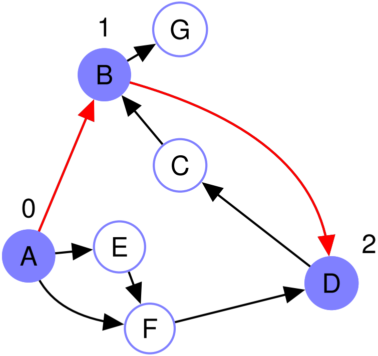

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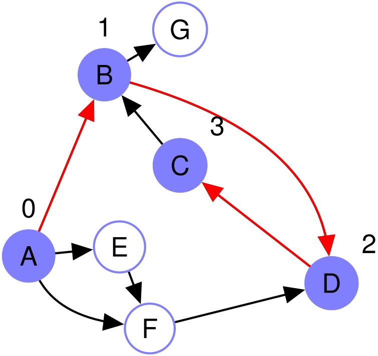

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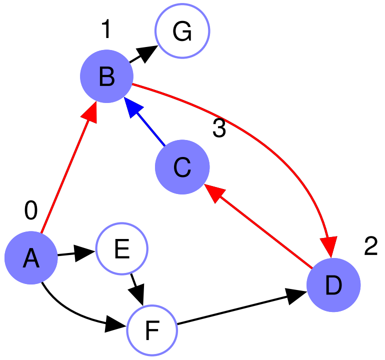

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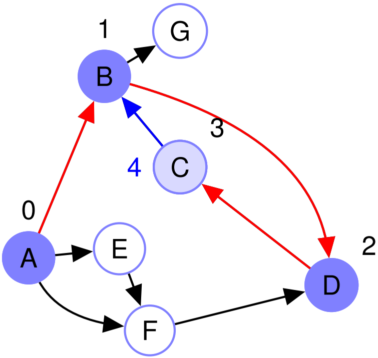

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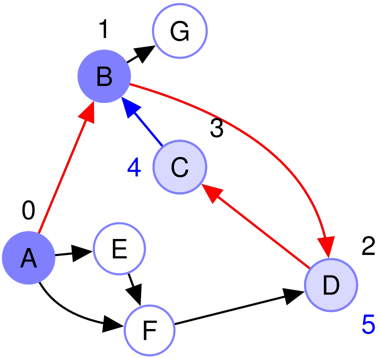

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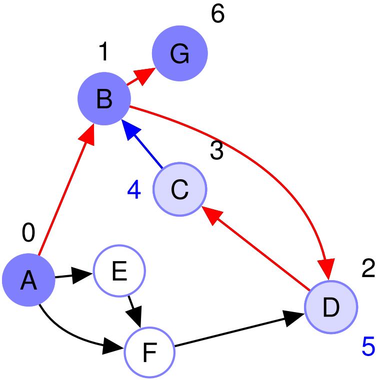

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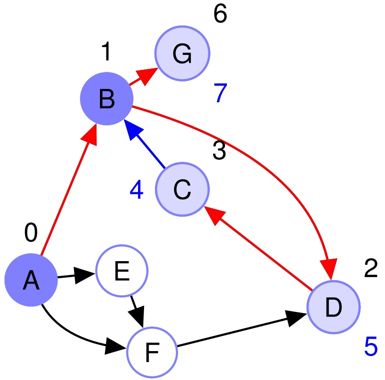

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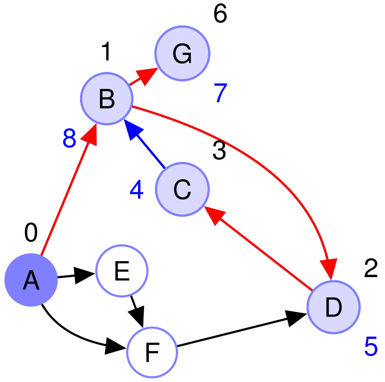

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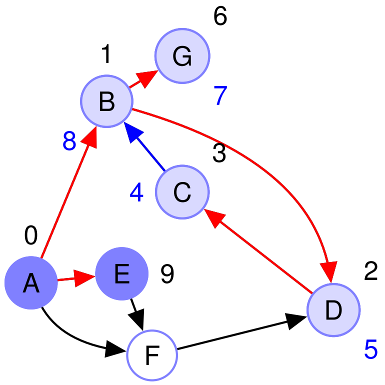

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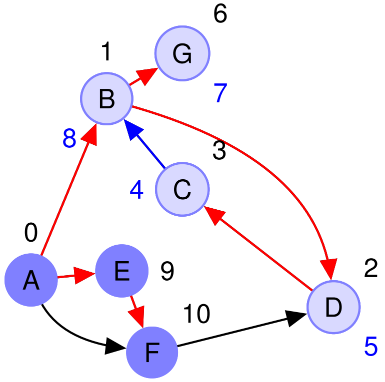

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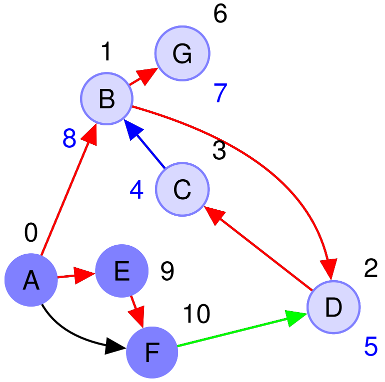

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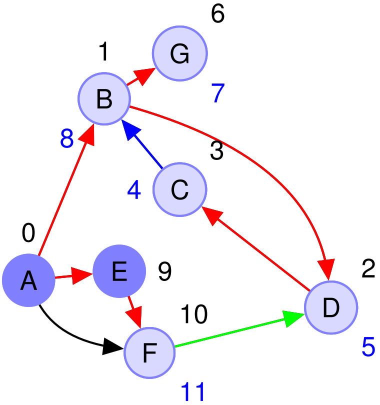

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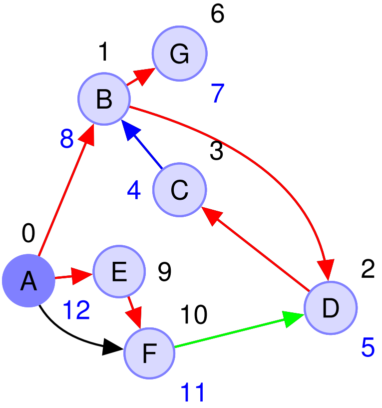

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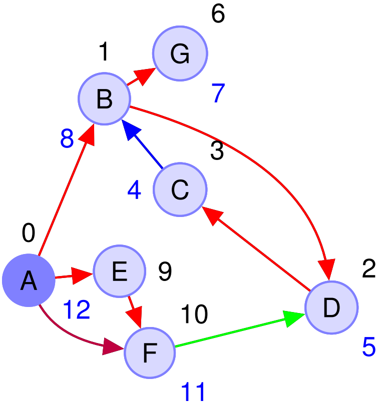

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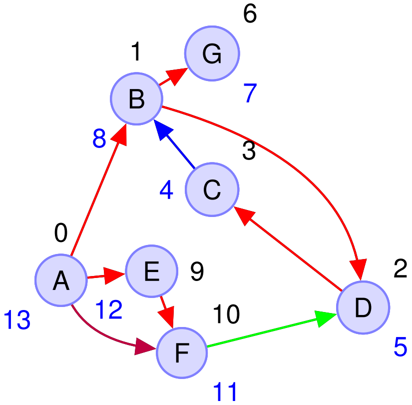

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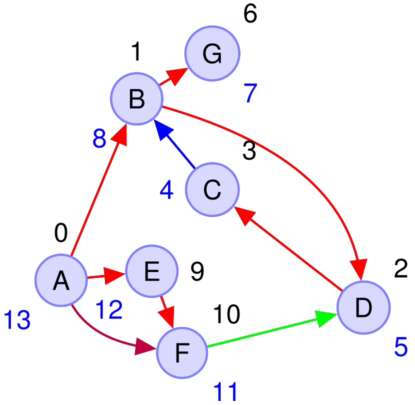

Tree/forward edge (u, v): $\text{int}(v)$ = [pre(v),post(v)] in $\text{int}(u)$ = [pre(u),post(u)].

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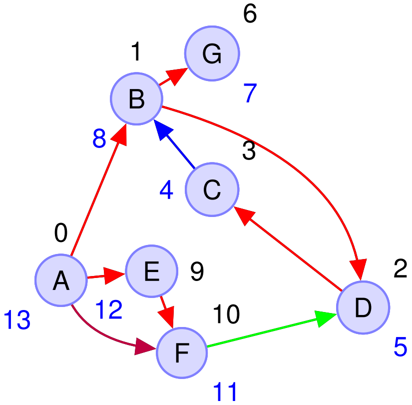

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Forward (A, F): [10,11] in [0,13] or [0,[10,11],13]

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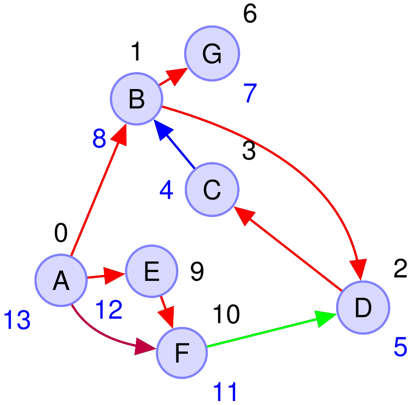

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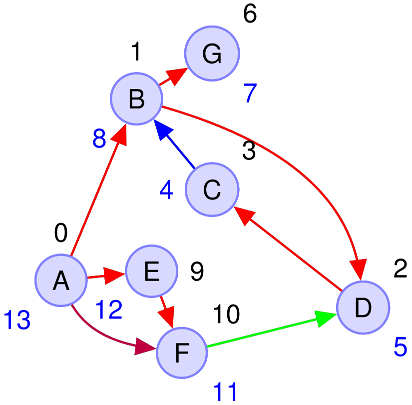

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(C, B): [3,4] in [1,8] or [1, [3, 4], 8]

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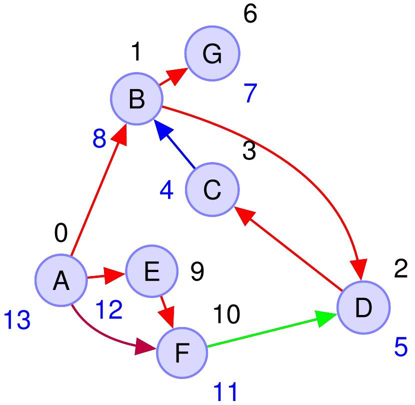

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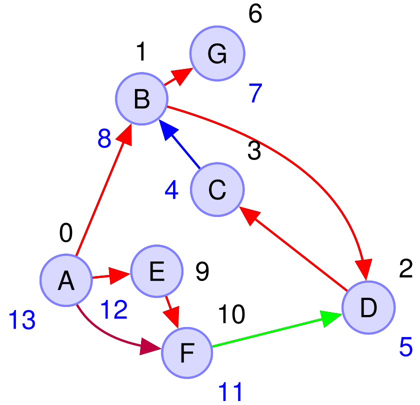
Cross edge (u, v): int(v) before int(u).

Depth first search: directed.



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Cross edge (u, v): int(v) before int(u).

(F, D): [2,5] before [10,11]

Testing for cycle/Topological sort.

Back Edge: (u, v) where $int(v)$ contains $int(u)$.

Testing for cycle/Topological sort.

Back Edge: (u, v) where $int(v)$ contains $int(u)$.

Thm: A graph has a cycle if and only if there is back edge.

Testing for cycle/Topological sort.

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Proof Idea:

Testing for cycle/Topological sort.

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Proof Idea: Back edge $\implies$ cycle!

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Edge to ancestor, plus path from ancestor is cycle.

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Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, v_0, contains all others.
Edge from "last" vertex, v_k, to v_0 is back edge.

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Topological Ordering: π, where for all edges (u, v), $\pi(u) < \pi(v)$.

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Proof: No back edges!
Tree/Forward edge: (u,v) : $[pre(u), post(u)] \in [pre(v), post(v)]$
$\implies post(u) > post(v)$.
Cross edge: (u,v) : $[pre(u), post(u)] > [pre(v), post(v)]$
$\implies post(u) > post(v)$.

Testing for cycle/Topological sort.

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Tree/Forward edge: (u,v) : $[pre(u), post(u)] \in [pre(v), post(v)]$
$\implies post(u) > post(v)$.
Cross edge: (u,v) : $[pre(u), post(u)] > [pre(v), post(v)]$
$\implies post(u) > post(v)$.

$\implies$ for every edge, (u,v), $post(u) > post(v)$

Testing for cycle/Topological sort.

Back Edge: (u,v) where $int(v)$ contains $int(u)$.

Thm: A graph has a cycle if and only if there is back edge.

Proof Idea: Back edge $\implies$ cycle!
Edge to ancestor, plus path from ancestor is cycle.

Interval of first explored vertex, v_0, contains all others.
Edge from "last" vertex, v_k, to v_0 is back edge.

Topological Ordering: π, where for all edges (u,v), $\pi(u) < \pi(v)$.

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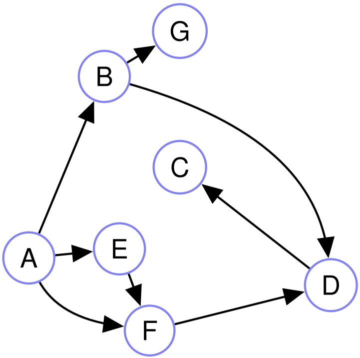
Topological Sort Example.



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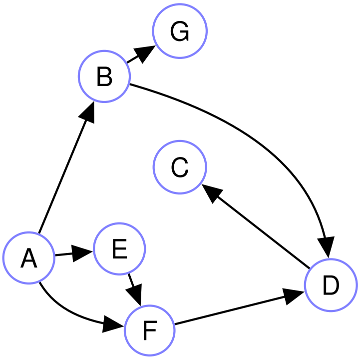
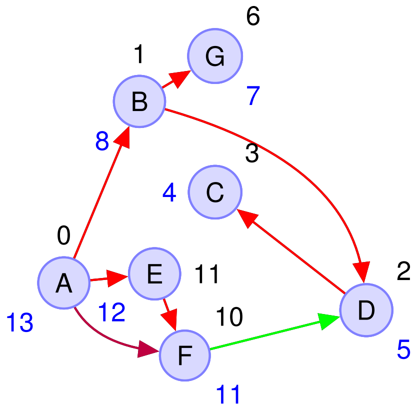


A linear order:

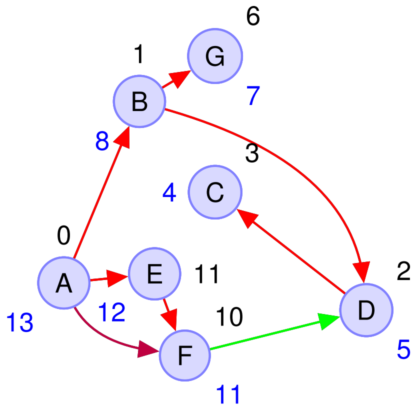
Topological Sort Example.



A linear order:

A, E, F, B, G, D, C

Topological Sort Example.



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In DFS: When is A popped off stack?

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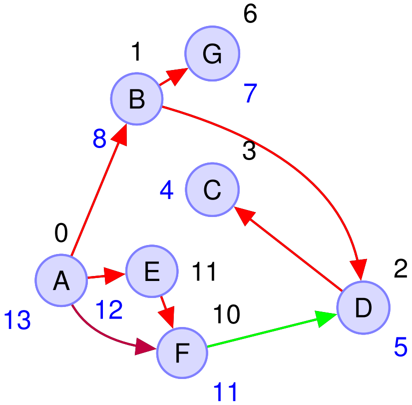

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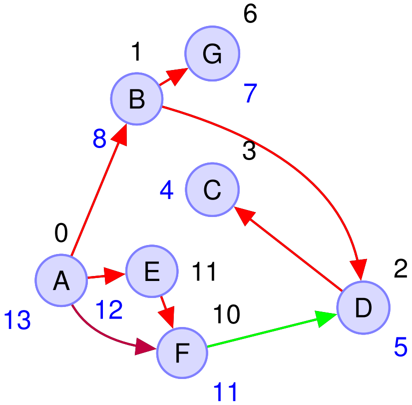

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In DFS: When is A popped off stack?

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Topological Sort: DFS

Last post order should..

(A) be first in linearization!

(B) be last in linearization!

Topological Sort: DFS

Last post order should..

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(A). First!

Connectivity in Directed Graphs.

Two nodes are connected...

Connectivity in Directed Graphs.

Two nodes are connected...when?

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When there is a path from u to v?

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Nodes u and v are **strongly connected**

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Note: Nodes are strongly connected to themselves.

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Path with zero edges in both directions!

Properties..

Nodes *u* and *v* are **strongly connected**
if there is a path from *u* to *v*
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Relation $\implies$ a partition into equivalence classes.

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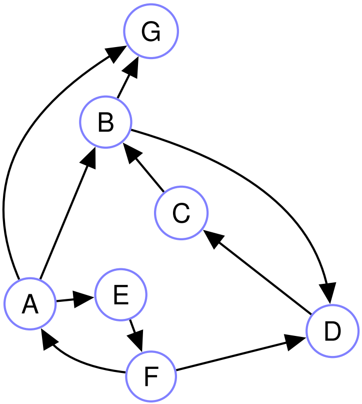
Relation $\implies$ a partition into equivalence classes.

Strongly connected components: sets of nodes which are strongly connected.

Example.



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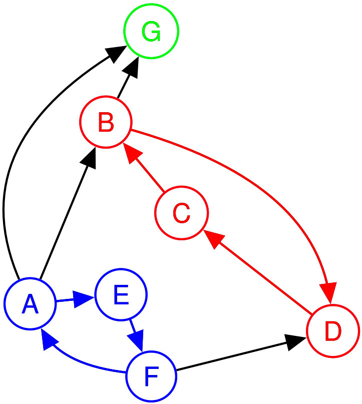

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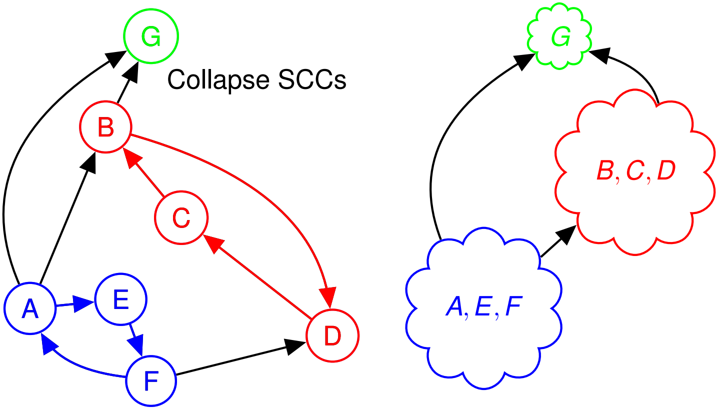

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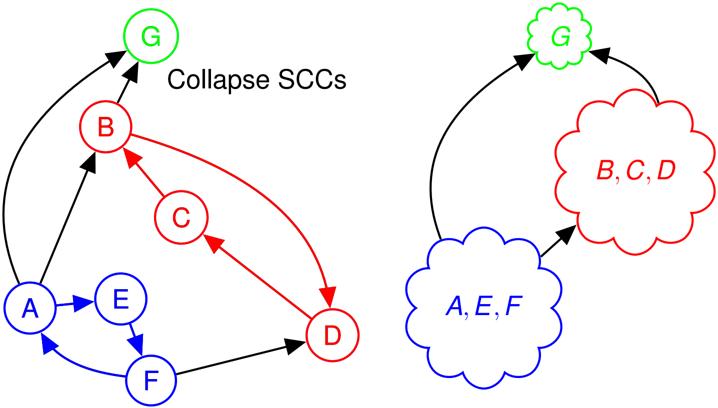
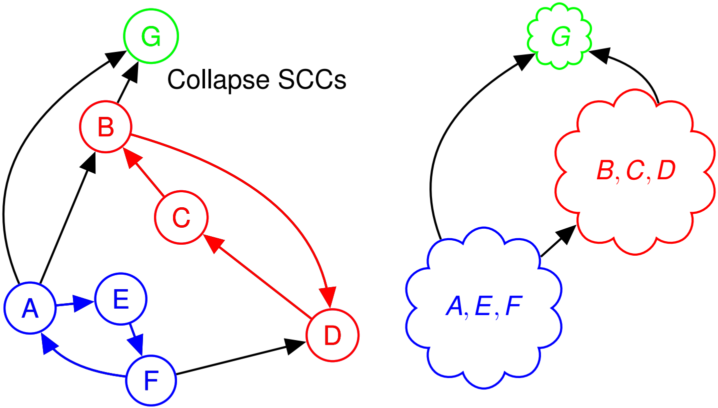


Collapsing strongly connected components (SCCs)..
..yields a DAG!

Example.



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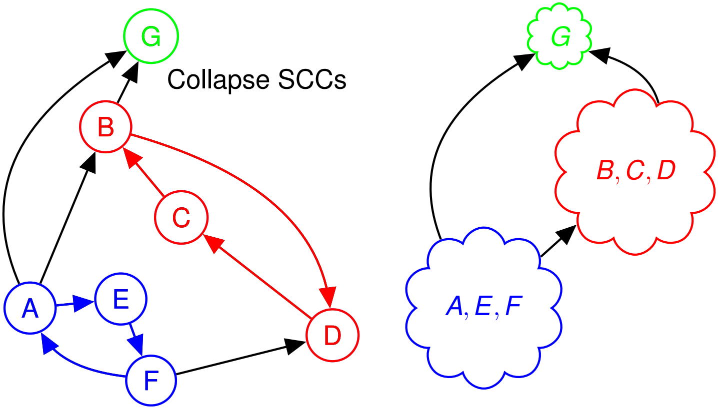
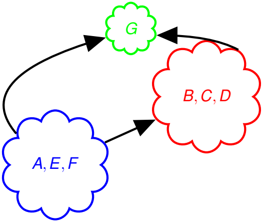


Collapsing strongly connected components (SCCs)..
..yields a DAG!

Why?
..any cycle collapses nodes into a single SCC.

Dag of SCCs

Property: Every directed graph is a DAG of strongly connected components.



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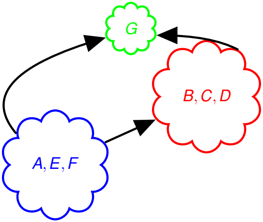

Finding the strongly connected components?

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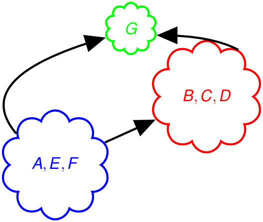

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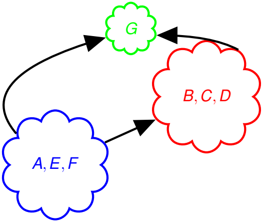

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Algorithm:

1. Run explore on node in sink component.

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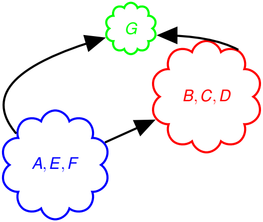

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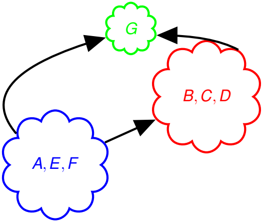

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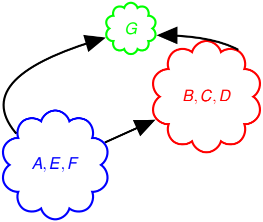

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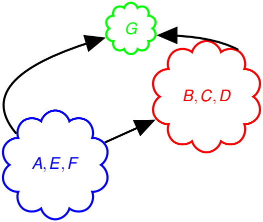

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How do we find a node in the sink component?

Finding a source.

Property: The node with the highest post order number is in a source component.

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Property++: If C and C' are SCCs with an edge from C to C', highest post# of a node in C larger than post# of any node in C'.

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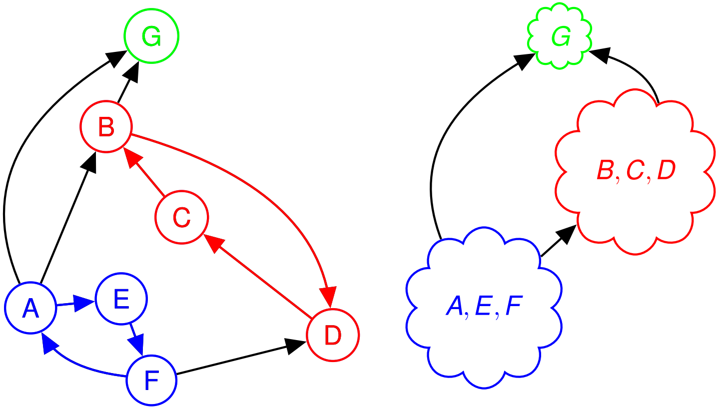

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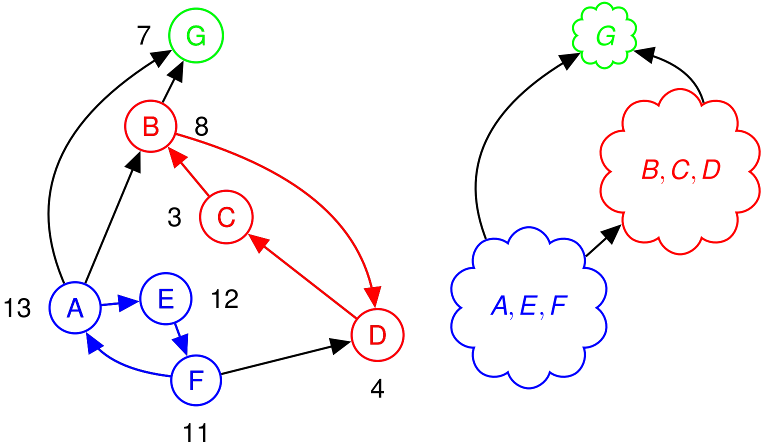

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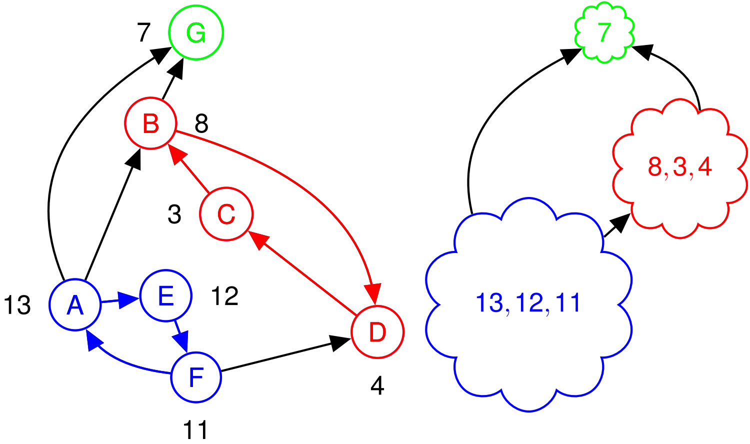

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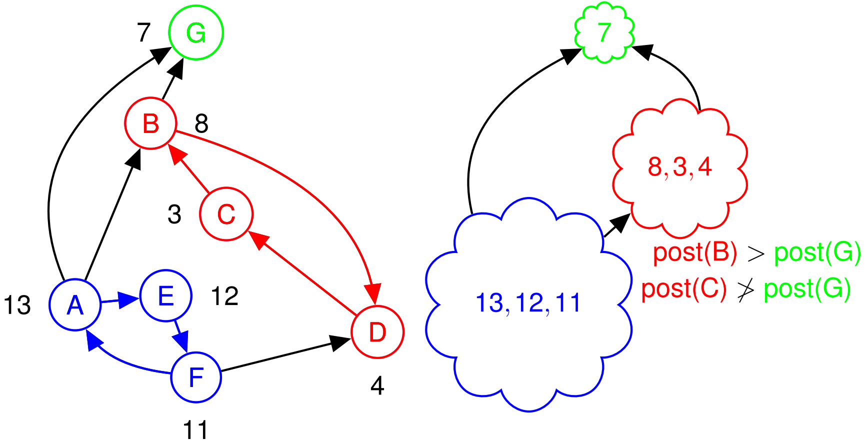

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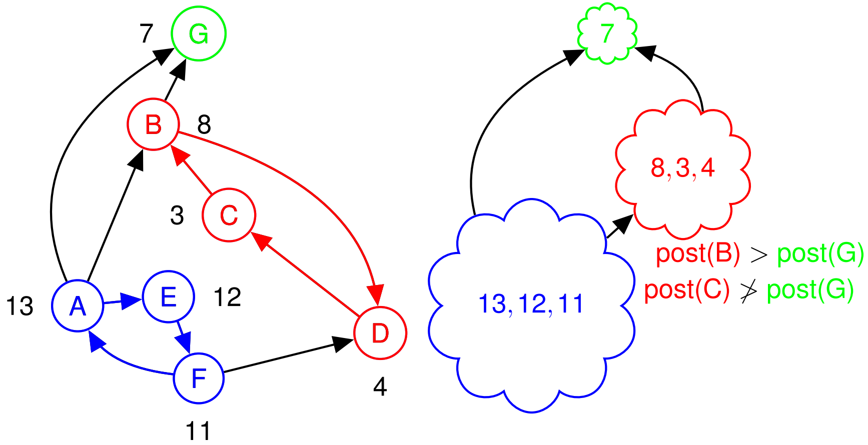
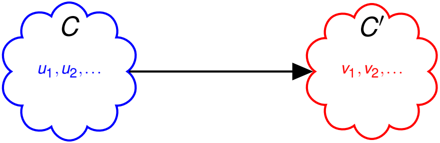


Highest post# in C bigger than any in C'

not true every post# in C greater than any in C'

Proof:



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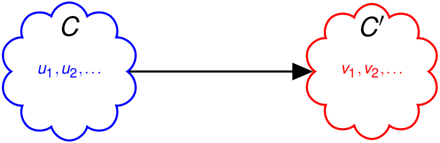

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Proof:

If a node v in C' is explored first.

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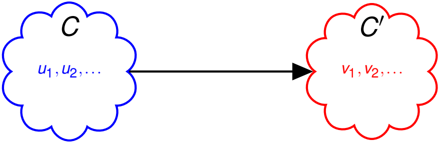

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If a node v in C' is explored first.
explore(v) gets to all of C'
and none of C!
So every node in C explored after every node in C'.

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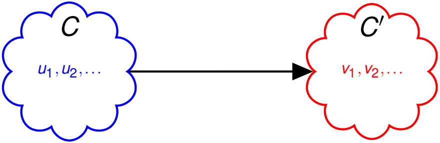

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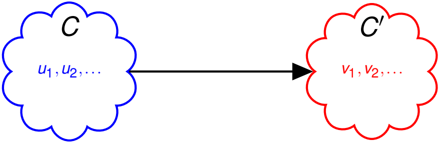

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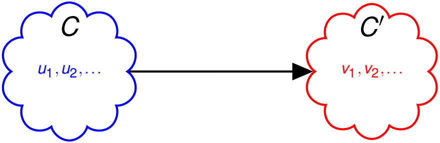
If a node u in C is explored first

Proof:



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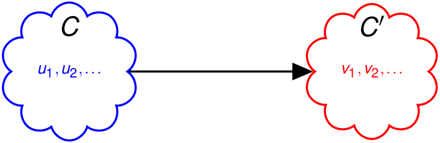
If a node u in C is explored first
All of C and C' will be explored before returning

Proof:



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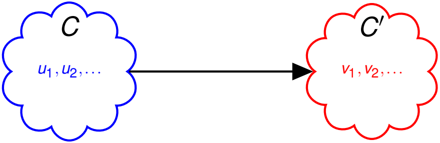
If a node u in C is explored first
    All of C and C' will be explored before returning
    from **explore**(u)

Proof:



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If a node u in C is explored first
All of C and C' will be explored before returning
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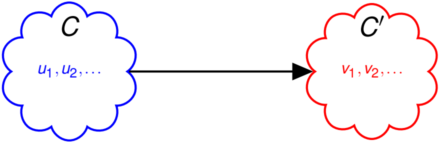
Proof:



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$\square$

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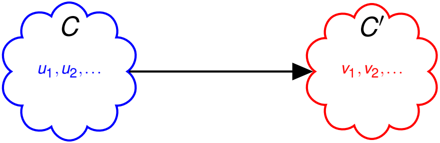

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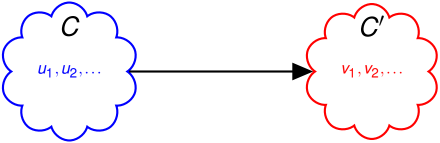
So u has higher post number than any node in C'.

$\square$

Implies highest post numbered node is in source component.

Proof:



Property++: If C and C' are SCCs with an edge from C to C', highest post# of a node in C larger than post# of any node in C'.
Proof:

If a node v in C' is explored first.
    explore(v) gets to all of C'
    and none of C!
So every node in C explored after every node in C'.
$\equiv$ every post # in C larger than every post # in C'.

If a node u in C is explored first
    All of C and C' will be explored before returning
    from **explore**(u)
So u has higher post number than any node in C'.

$\square$

Implies highest post numbered node is in source component.

Test your understanding..

Property++: If C and C' are SCCs with an edge from C to C', highest post# of a node in C larger than post# of any node in C'.

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Does every node in C have a higher post order number than every node in C'?

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Does every node in C have a higher post order number than every node in C'?

(A) Yes!

Test your understanding..

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(A) Yes! (B) Not Necessarily.

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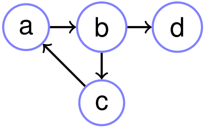

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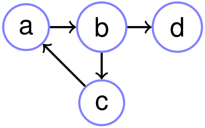

Explore(a)

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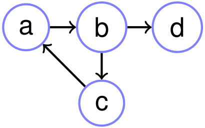

Explore(a) $\Longrightarrow$

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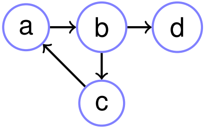

Explore(a) $\Longrightarrow$ Explore(b)

Test your understanding..

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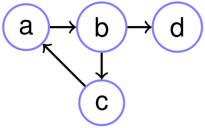

Explore(a) $\Longrightarrow$ Explore(b) $\Longrightarrow$ Explore (c)

Test your understanding..

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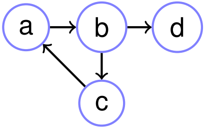

Explore(a) $\Longrightarrow$ Explore(b) $\Longrightarrow$ Explore (c)
$\Longrightarrow$ Return from (c)

Test your understanding..

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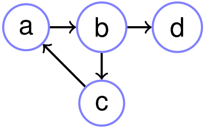

Explore(a) $\Longrightarrow$ Explore(b) $\Longrightarrow$ Explore (c)
$\Longrightarrow$ Return from (c) $\Longrightarrow$ Explore(d)

Test your understanding..

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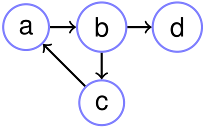

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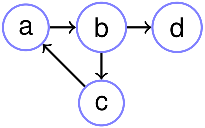

Explore(a) $\Longrightarrow$ Explore(b) $\Longrightarrow$ Explore (c)
$\Longrightarrow$ Return from (c) $\Longrightarrow$ Explore(d) $\Longrightarrow$ Return from (d) ...

(c) has lower post order number than (d).

Back to SCC Algorithm..

Property: The highest post numbered node is in source component.

Back to SCC Algorithm..

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Algorithm:

1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.

Back to SCC Algorithm..

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Uh...oh.

Back to SCC Algorithm..

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1. Run explore on node in sink component.
2. Output visited nodes.
3. Repeat.

Uh...oh.

How should we fix this?

SCC Algorithm.

Property: The highest post numbered node is in source component.

SCC Algorithm.

Property: The highest post numbered node is in source component.

Find node in sink component?

SCC Algorithm.

Property: The highest post numbered node is in source component.

Find node in sink component?

Reverse edges!

SCC Algorithm.

Property: The highest post numbered node is in source component.

Find node in sink component?

Reverse edges! G^R

SCC Algorithm.

Property: The highest post numbered node is in source component.

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Reverse edges! G^R

Source component in G^R is sink component in G.

SCC Algorithm.

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1. DFS on G^R to compute post($\cdot$)

SCC Algorithm.

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Source component in G^R is sink component in G.

Algorithm:

1. DFS on G^R to compute post$(\cdot)$
   Highest post # vertex, v, in G^R in sink comp. of G.

SCC Algorithm.

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Algorithm:

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   Highest post # vertex, v, in G^R in sink comp. of G.
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Find another node in sink of unvisited part of G!

SCC Algorithm.

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Recompute DFS in G^R...

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   Highest post # vertex, v, in G^R in sink comp. of G.
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Recompute DFS in G^R...or...

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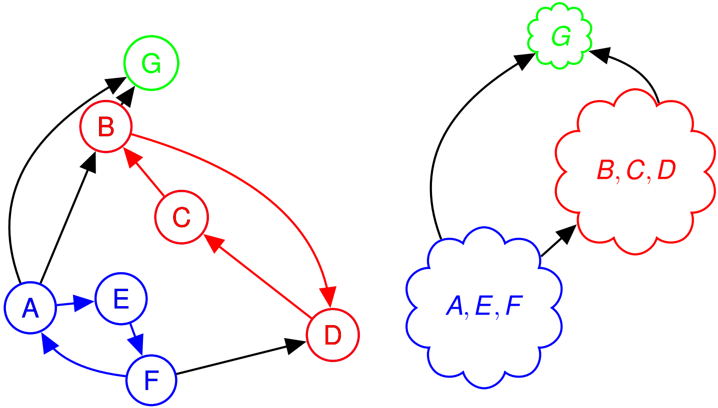
   Highest post # vertex, v, in G^R in sink comp. of G.
2. Output nodes visited in: explore(v)
   Then what?

Find another node in sink of unvisited part of G!

Recompute DFS in G^R...or...

.... use post($\cdot$) again!

Post from Reverse graph



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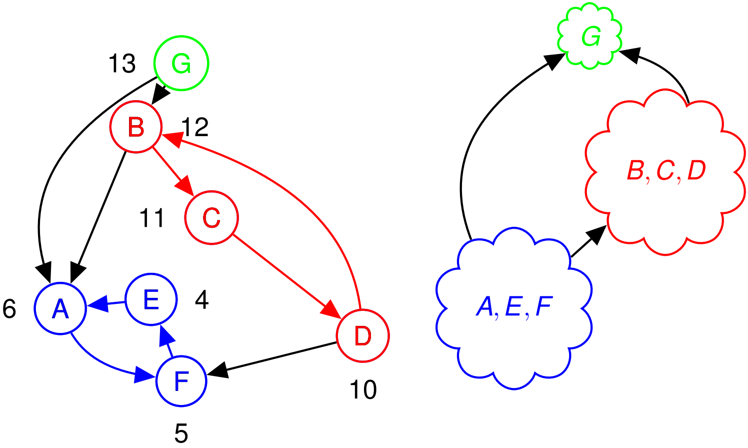

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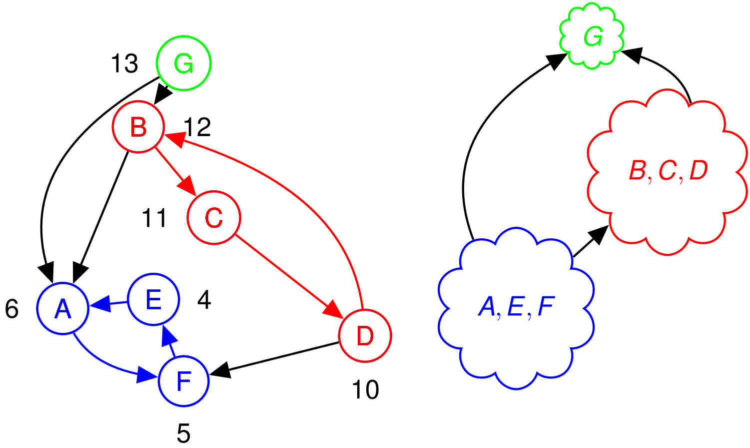

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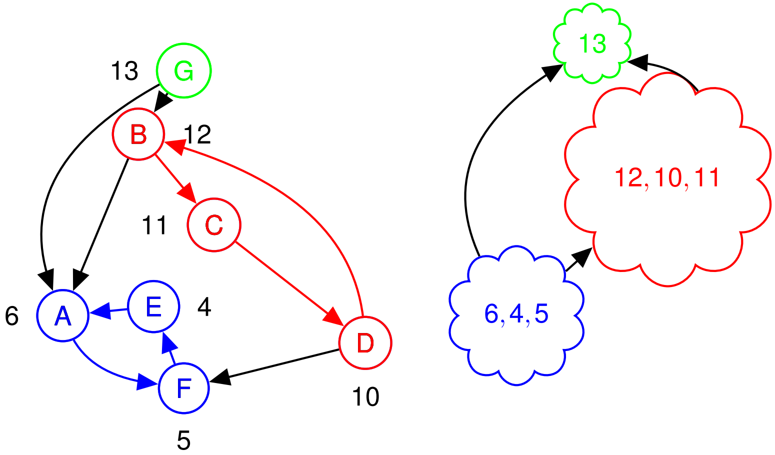

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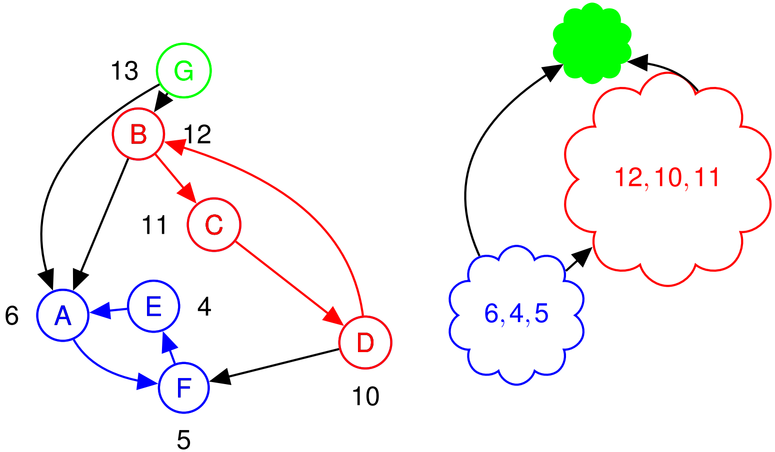

Property++: If C and C' are SCCs with an edge from C to C', highest post# of a node in C larger than post# of any node in C'.

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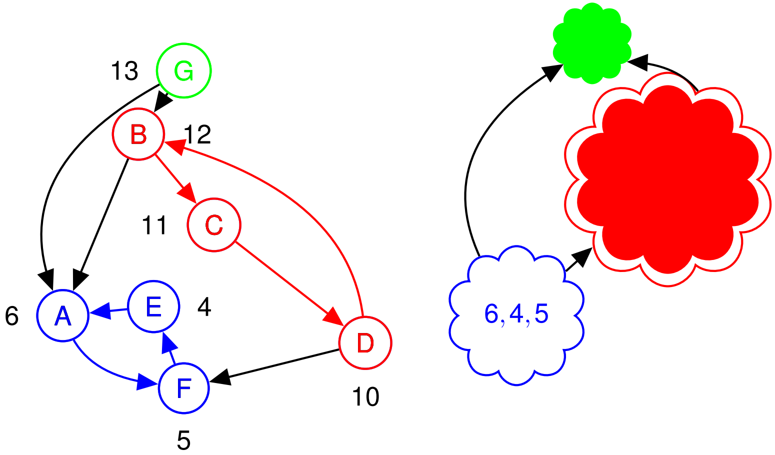

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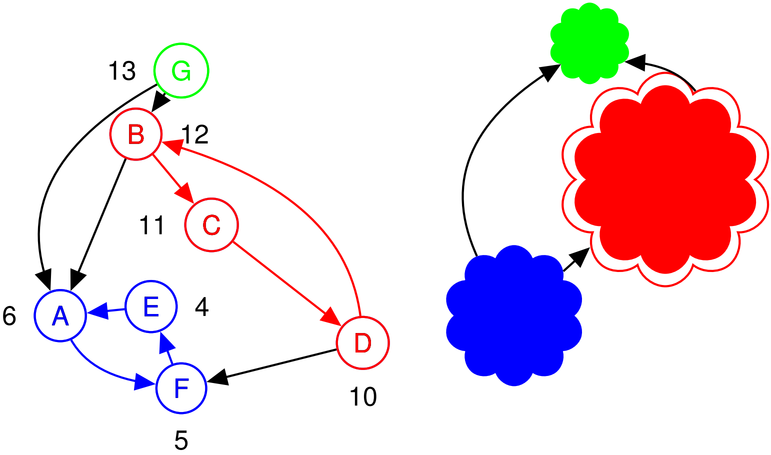

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All sinks from one dfs.

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First explore of G:

All sinks from one dfs.

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First explore of G:
Removes sink component of G.

All sinks from one dfs.

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First explore of G:
Removes sink component of G.
$\implies$ removes source component of G^R.

All sinks from one dfs.

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$\implies$ highest rem. post # vertex, v,

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$\implies$ v in sink component of G!

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SCC Algorithm:

All sinks from one dfs.

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SCC Algorithm:
1. DFS of G^R.

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SCC Algorithm:
1. DFS of G^R.
2. Run undirected components algorithm on G

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$O(|V| + |E|)$ time ...

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$O(|V| + |E|)$ time ...

Compute G^R in linear time?..

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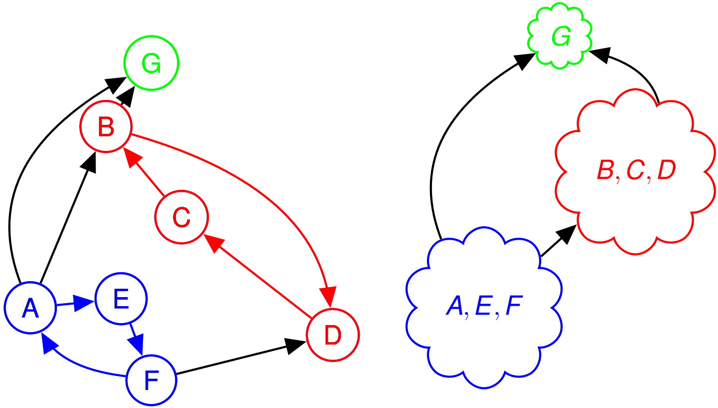
$O(|V| + |E|)$ time ...

Compute G^R in linear time?.. exercise.

Example Again: think runtime.



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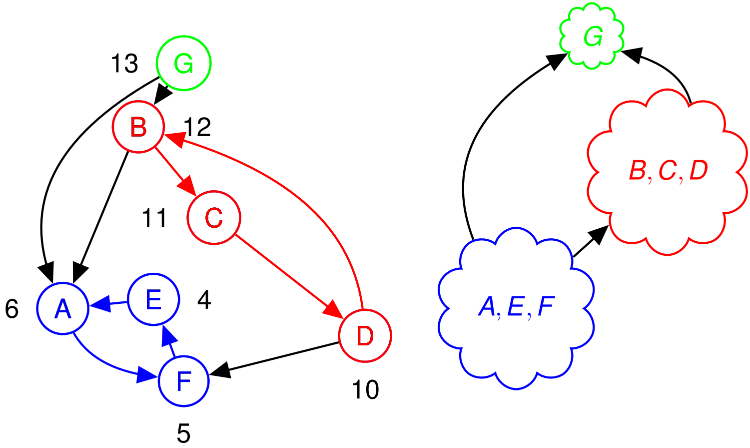

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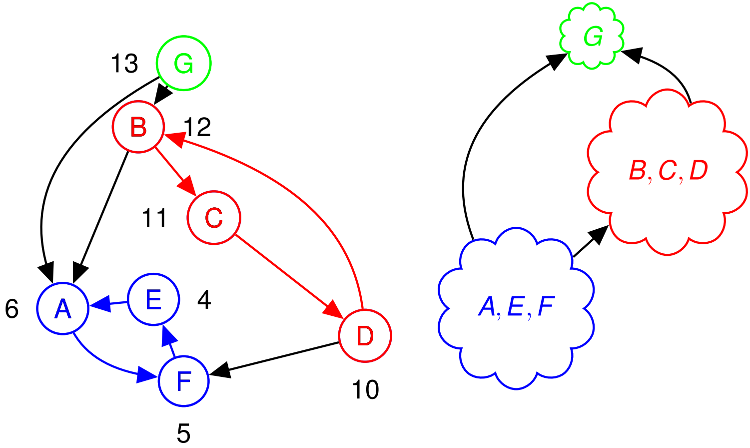

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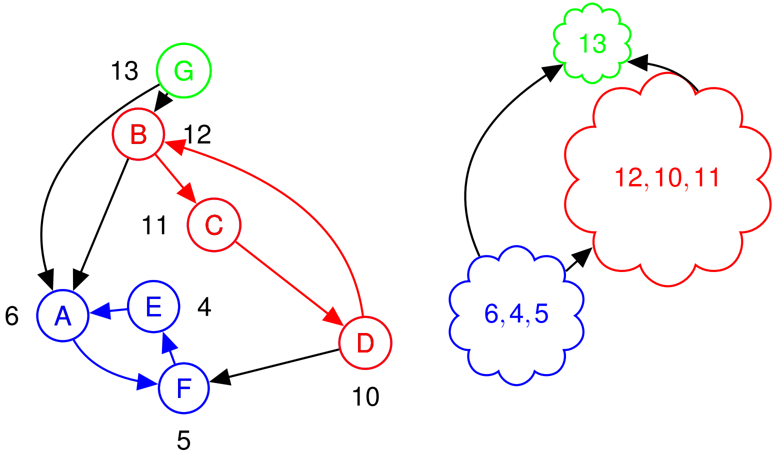

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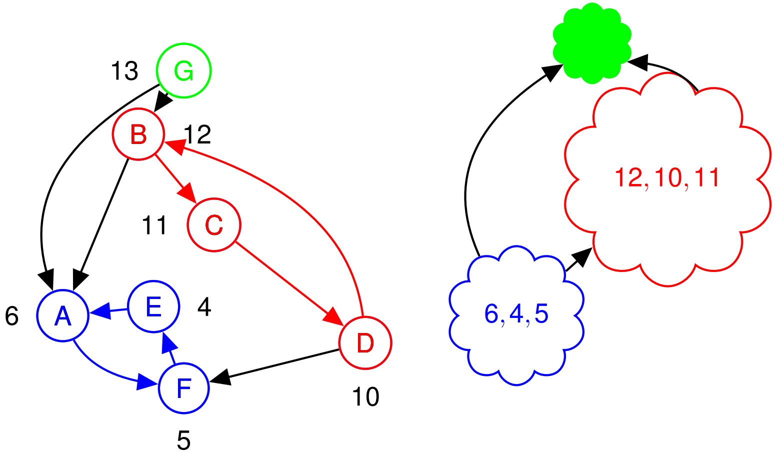

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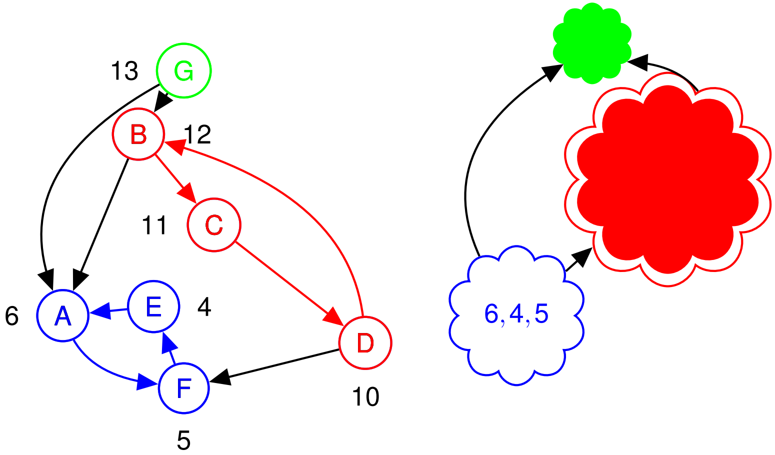

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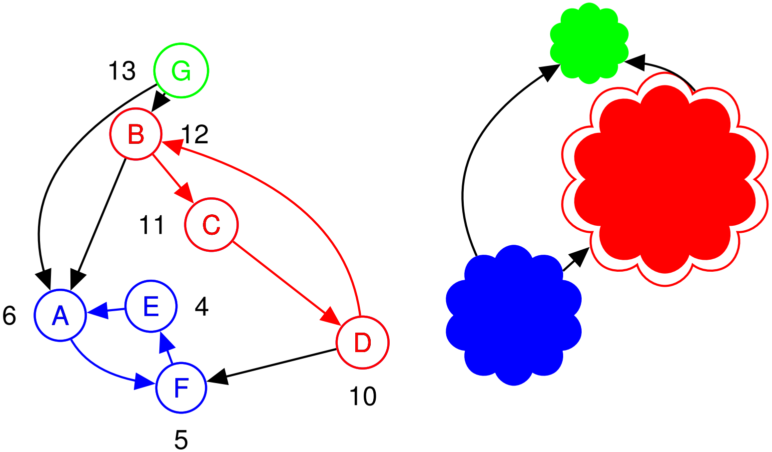

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Lecture in a minute!

Quick Review:
DFS so far $\equiv$

Lecture in a minute!

Quick Review:

DFS so far $\equiv$ how I learned to love the stack.

pre/post = time on stack.

Topological Ordering:

Inverse post ordering $\equiv$ topological ordering.

Remove source, repeat.

Lecture in a minute!

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DFS so far $\equiv$ how I learned to love the stack.
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Strongly Connected Components: directed graphs.
Strong Connectivity for u and v.
On a cycle together.
Easy: $O(|V||E|)$ algorithm.
Linear time algorithm!

Lecture in a minute!

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- DFS so far $\equiv$ how I learned to love the stack.
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Strongly Connected Components: directed graphs.
- Strong Connectivity for u and v.
  - On a cycle together.
- Easy: $O(|V||E|)$ algorithm.
- Linear time algorithm!
  - Observation: Highest post in "source component".
  - Find vertex in sink component.
    - Explore.
  - Repeat.