CS 170: Algorithms

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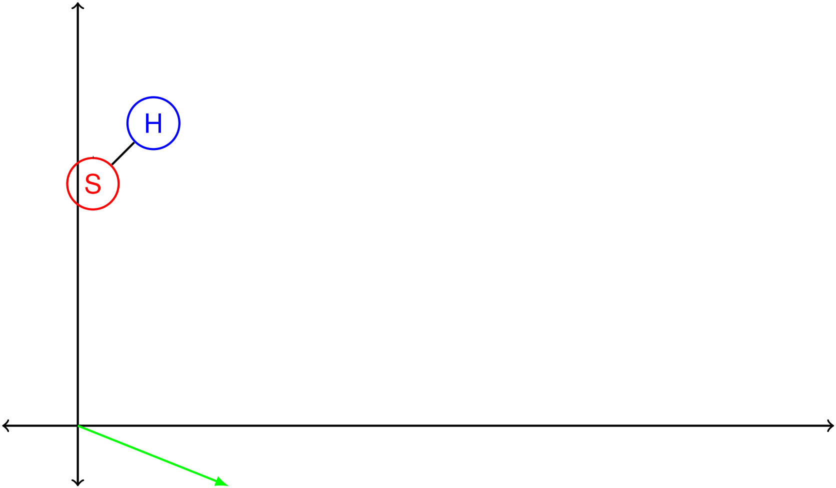
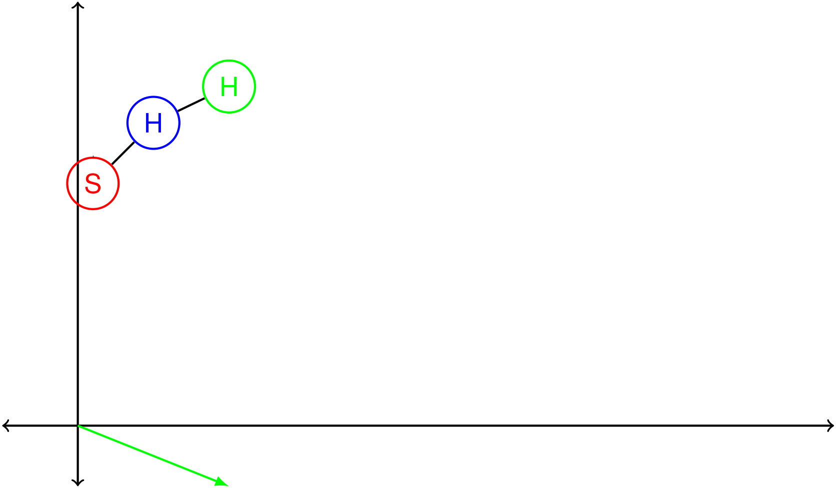
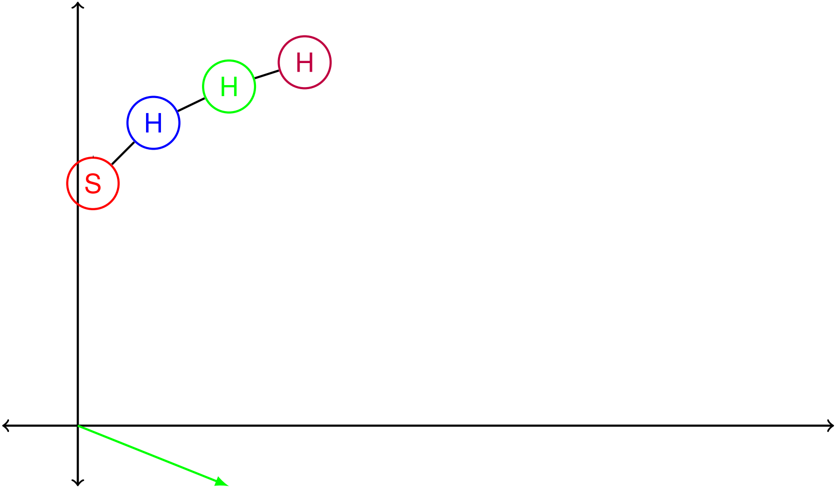


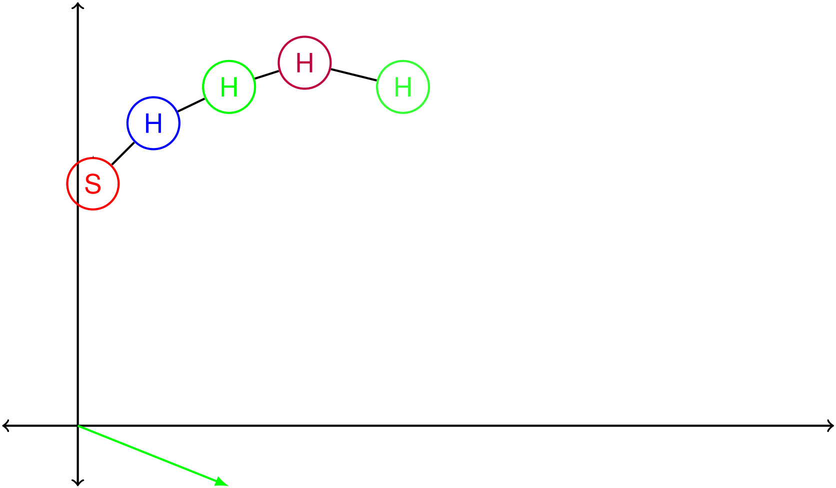
CS 170: Algorithms



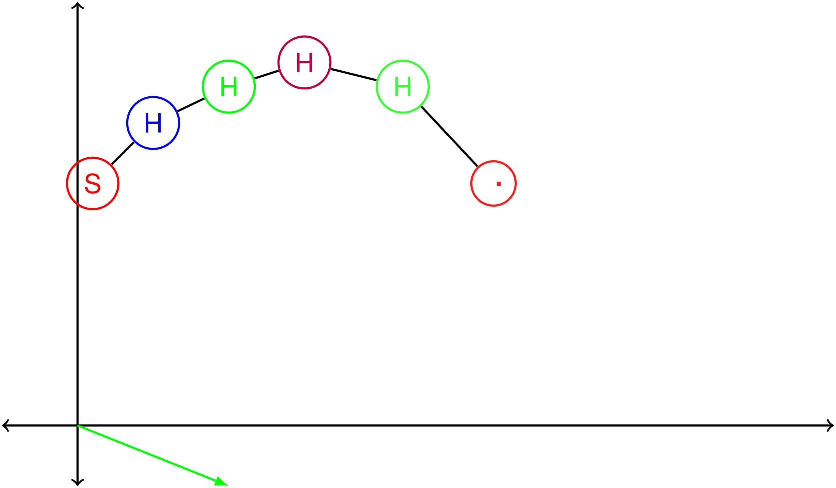
CS 170: Algorithms



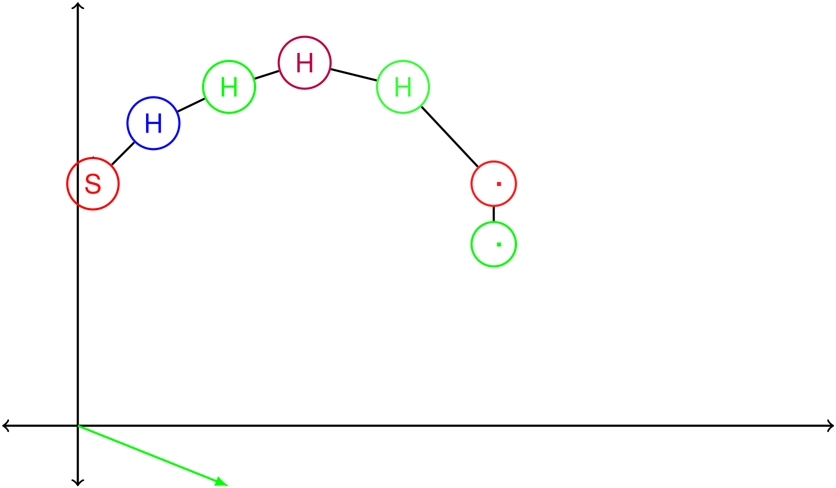
CS 170: Algorithms



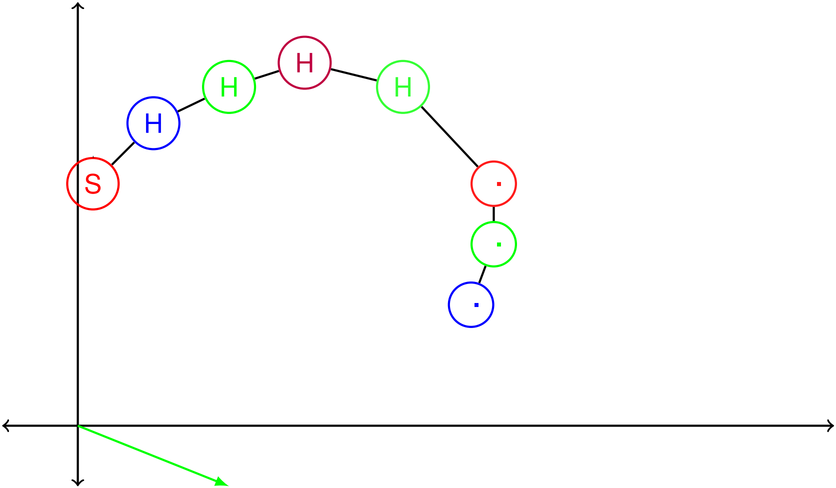
CS 170: Algorithms



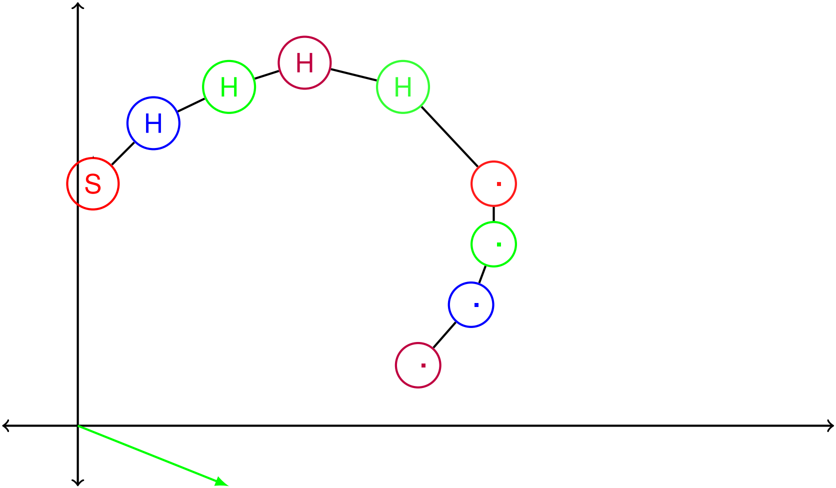
CS 170: Algorithms



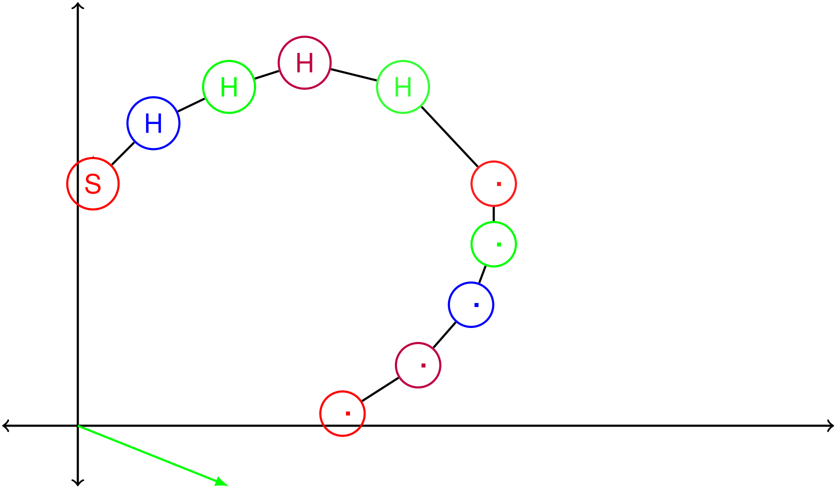
CS 170: Algorithms



CS 170: Algorithms



CS 170: Algorithms



Standard Form: $Ax \leq b, \max cx, x \geq 0$.

Lecture in a minute.

Duality:

Lecture in a minute.

Duality:

Primal: $Ax \leq b, \max cx, x \geq 0$

Dual: $A^T y \geq b, \min by, y \geq 0$

Linear combiniation of equations that dominates objective function.

Duality: (typically) have the same value.

Weak Duality: Primal $\leq$ Dual.

Feasible $x, y \implies cx \leq y^T Ax \geq y^T b$.

Lecture in a minute.

Duality:

Primal: $Ax \leq b, \max cx, x \geq 0$

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Feasible $x, y \implies cx \le y^T Ax \ge y^T b$.

Simplex Implementation:

Start at a (feasible) vertex.

Lecture in a minute.

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Start at a (feasible) vertex.

(defined by linear system $A'x = [b, 0, \cdots, 0]$).

Lecture in a minute.

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Start at a (feasible) vertex.

(defined by linear system $A'x = [b, 0, \cdots, 0]$).

Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Lecture in a minute.

Duality:

Primal: $Ax \leq b, \max cx, x \geq 0$

Dual: $A^T y \geq b, \min by, y \geq 0$

Linear combiniation of equations that dominates objective function.

Duality: (typically) have the same value.

Weak Duality: Primal $\leq$ Dual.

Feasible $x, y \implies cx \leq y^T Ax \geq y^T b$.

Simplex Implementation:

Start at a (feasible) vertex.

(defined by linear system $A'x = [b, 0, \cdots, 0]$).

Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Vertex at origin in coordinate system.

$O(mn)$ time to update linear system.

Until no better neighboring vertex.

Objective function in coordinate system is non-positive.

Dual Variables: new objective function!

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\le 4 \\ x_2 &\le 3 \\ x_1 + 2x_2 &\le 7 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
yields bound on objective..

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

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Added equation 1 and 8 times equation 2
yields bound on objective..
Better solution?
Better upper bound?

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

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For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
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Better solution?
Better upper bound?

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\le 4 \\ x_2 &\le 3 \\ x_1 + 2x_2 &\le 7 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \le 4 + 24 = 28$.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure:

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Thus, the value is at most 25.

Duality.

$$\max x_1 + 8x_2$$

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Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

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Sure: 6 times equation 2 and 1 times equation 3.
$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Thus, the value is at most 25.

The upper bound is same as solution!

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28.$

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$x_1 + 8x_2 \leq 6(3) + 7 = 25.$

Thus, the value is at most 25.

The upper bound is same as solution!

Proof of optimality!

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
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Adding equations thusly...

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The left hand side should “dominate” optimization function:

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1+y_3)x_1+(y_2+2y_3)x_2 \leq 4y_1+3y_2+7y_3.$$

The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

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The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \qquad \leq 4$
y_2	$\qquad x_2 \leq 3$
y_3	$x_1 + \; 2x_2 \leq 7$

Adding equations thusly...

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The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
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$$x_1+8x_2 \leq 4y_1+3y_2+7y_3$$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
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The left hand side should "dominate" optimization function:

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and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$

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and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

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A linear program.

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
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$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\begin{aligned} \min\ 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

The dual.

In general.

Primal LP

$$\max c \cdot x$$

$$Ax \leq b \; x \geq 0$$

Dual LP

$$\min y^T b$$

$$y^T A \geq c$$

$$y \geq 0$$

The dual.

In general.

Primal LP

$$\max c \cdot x$$

$$Ax \leq b \; x \geq 0$$

Dual LP

$$\min y^T b$$

$$y^T A \geq c$$

$$y \geq 0$$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

The dual.

In general.

Primal LP

$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

Dual LP

$$\min y^T b$$
$$y^T A \geq c$$
$$y \geq 0$$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal (P) $\leq$ dual (D)

The dual.

In general.

Primal LP

$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

Dual LP

$$\min y^T b$$
$$y^T A \geq c$$
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Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)

The dual.

In general.

Primal LP

$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

Dual LP

$$\min y^T b$$
$$y^T A \geq c$$
$$y \geq 0$$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)

$P(x)$

The dual.

In general.

Primal LP
$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

Dual LP
$$\min y^T b$$
$$y^T A \geq c$$
$$y \geq 0$$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)
$\quad P(x) = c \cdot x$

The dual.

In general.

Primal LP
$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

Dual LP
$$\min y^T b$$
$$y^T A \geq c$$
$$y \geq 0$$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)
$$P(x) = c \cdot x \leq y^T Ax$$

The dual.

In general.

Primal LP
$$\max c \cdot x$$
$$Ax \leq b \; x \geq 0$$

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Strong Duality: later.

Complementary Slackness

Primal LP	Dual LP
max $c \cdot x$	min $y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Complementary Slackness

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Solutions x and y are both optimal if and only if
$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

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Complementary Slackness

<u>Primal LP</u>	<u>Dual LP</u>
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$cx = by$.

Complementary Slackness

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If both are feasible, $cx \leq by$, so must be optimal.

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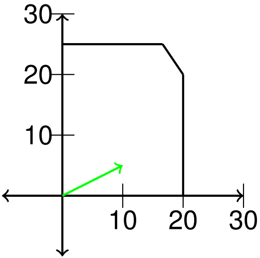
$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

In words: nonzero dual variables only for tight constraints!

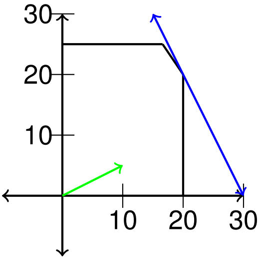
Again: simplex



Simplex: Start at vertex.

$$\max 4x_1 + 2x_2$$
$$3x_1 \le 60$$
$$3x_2 \le 75$$
$$3x_1 + 2x_2 \le 100$$
$$x_1, x_2 \ge 0$$

Again: simplex



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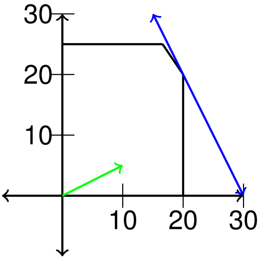
Simplex: Start at vertex. Move to better neighboring vertex.

Again: simplex



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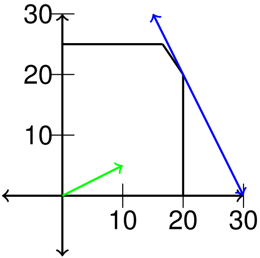
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor.

Again: simplex



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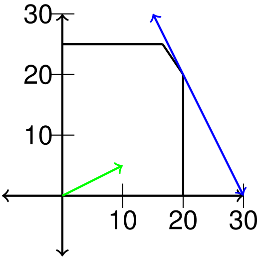
Until no better neighbor. Duality:

Again: simplex



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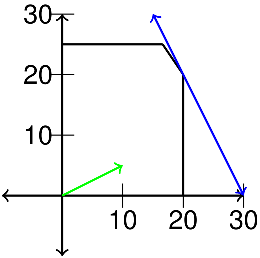
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?

Again: simplex



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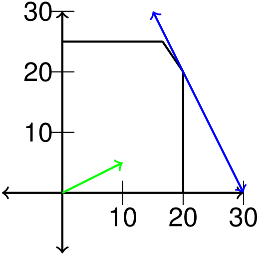
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.

Again: simplex



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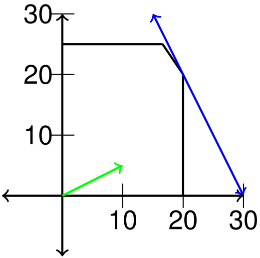
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$.

Again: simplex



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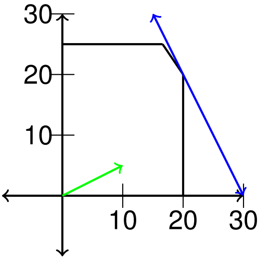
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!

Again: simplex



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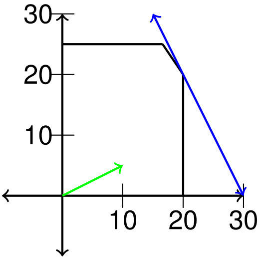

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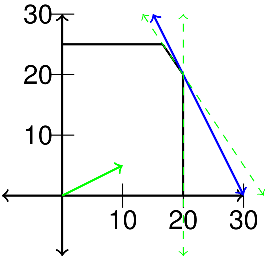
Geometrically and Complementary slackness:

Again: simplex



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$1/3$ times first plus second.
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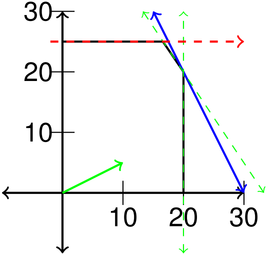
Geometrically and Complementary slackness:
Add tight constraints to “dominate objective function.”

Again: simplex



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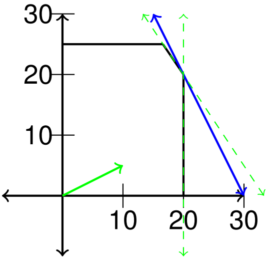
Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."
Don't add this equation!

Again: simplex



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Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."
Don't add this equation! Shifts.

Example: review.

$$\begin{gathered}\max x_1 + 8x_2 \\ x_1 \le 4 \\ x_2 \le 3 \\ x_1 + 2x_2 \le 7 \\ x_1, x_2 \ge 0\end{gathered}$$

$$\begin{gathered}\min\ 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + 2y_3 \ge 8 \\ y_1, y_2, y_3 \ge 0\end{gathered}$$

Example: review.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \leq 4 \\ x_2 \leq 3 \\ x_1 + 2x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{aligned}$$

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"Matrix form"

$$\max[1,8] \cdot [x_1, x_2]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$[x_1, x_2] \geq 0$$

$$\min[4,3,7] \cdot [y_1, y_2, y_3]$$

$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$[y_1, y_2, y_3] \geq 0$$

Matrix equations.

$$\max [1,8] \cdot [x_1, x_2]$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$
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$$\min [4,3,7] \cdot [y_1, y_2, y_3]$$
$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$
$$[y_1, y_2, y_3] \geq 0$$

We can rewrite the above in matrix form.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \qquad c = [1,8] \quad b = [4,3,7]$$

The primal is $Ax \leq b, \max c \cdot x, x \geq 0$.
The dual is $y^T A \geq c, \min b \cdot y, y \geq 0$.

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2]\geq 0.$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\geq\begin{bmatrix}1\\ 8\end{bmatrix}$$
$$[y_1,y_2,y_3]\geq 0$$

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
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$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\geq\begin{bmatrix}1\\ 8\end{bmatrix}$$
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Primal: $(x_1,x_2)=(1,3)$

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
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Primal: $(x_1,x_2)=(1,3)$
Feasible?

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2]\geq 0.$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\geq\begin{bmatrix}1\\ 8\end{bmatrix}$$
$$[y_1,y_2,y_3]\geq 0$$

Primal: $(x_1,x_2)=(1,3)$
Feasible? $1\times 1+0\times 3\leq 4$, $0\times 1+1\times 3\leq 3$, $1\times 1+2\times 3\leq 7$.

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2]\geq 0.$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\geq\begin{bmatrix}1\\ 8\end{bmatrix}$$
$$[y_1,y_2,y_3]\geq 0$$

Primal: $(x_1,x_2)=(1,3)$
Feasible? $1\times 1+0\times 3\leq 4$, $0\times 1+1\times 3\leq 3$, $1\times 1+2\times 3\leq 7$.
Value $=1\times 1+3\times 8=25$.

Solution(s)

$$\max [1,8] \cdot [x_1, x_2]$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$
$$[x_1, x_2] \geq 0.$$

$$\min [4,3,7] \cdot [y_1, y_2, y_3]$$
$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$
$$[y_1, y_2, y_3] \geq 0$$

Primal: $(x_1, x_2) = (1, 3)$
Feasible? $1 \times 1 + 0 \times 3 \leq 4$, $0 \times 1 + 1 \times 3 \leq 3$, $1 \times 1 + 2 \times 3 \leq 7$.
Value $= 1 \times 1 + 3 \times 8 = 25$.

Dual: $(y_1, y_2, y_3) = (0, 6, 1)$

Solution(s)

$$\max [1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \leq \begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2] \geq 0.$$

$$\min [4,3,7]\cdot[y_1,y_2,y_3]$$
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Value $= 1\times 1 + 3\times 8 = 25$.

Dual: $(y_1,y_2,y_3) = (0,6,1)$
Feasible?

Solution(s)

$$\max [1,8] \cdot [x_1, x_2]$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$
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Dual: $(y_1, y_2, y_3) = (0, 6, 1)$
Feasible? $1 \times 0 + 0 \times 6 + 1 \times 1 \geq 1$, $0 \times 0 + 1 \times 1 + 2 \times 3 \geq 8$.

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \leq \begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
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Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\leq\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
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Primal: $(x_1,x_2)=(1,3)$
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Dual: $(y_1,y_2,y_3)=(0,6,1)$
Feasible? $1\times 0+0\times 6+1\times 1\geq 1$, $0\times 0+1\times 1+2\times 3\geq 8$.
Value $=1\times 1+3\times 8=25$.

Complimentary Slackness: $(b_i-a_ix)(y_i)=0$.
Either slack for equation is 0 or dual variable is 0 or both.

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\le\begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
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Value $=1\times 1+3\times 8=25$.

Complimentary Slackness: $(b_i-a_ix)(y_i)=0$.
Either slack for equation is 0 or dual variable is 0 or both.

First equation for primal: $4-(1\times 1)+0\times 3=1$

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \leq \begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2] \geq 0.$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
$$[y_1,y_2,y_3]\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix} \geq \begin{bmatrix}1\\ 8\end{bmatrix}$$
$$[y_1,y_2,y_3] \geq 0$$

Primal: $(x_1,x_2) = (1,3)$
Feasible? $1\times 1 + 0\times 3 \leq 4$, $0\times 1 + 1\times 3 \leq 3$, $1\times 1 + 2\times 3 \leq 7$.
Value $= 1\times 1 + 3\times 8 = 25$.

Dual: $(y_1,y_2,y_3) = (0,6,1)$
Feasible? $1\times 0 + 0\times 6 + 1\times 1 \geq 1$, $0\times 0 + 1\times 1 + 2\times 3 \geq 8$.
Value $= 1\times 1 + 3\times 8 = 25$.

Complimentary Slackness: $(b_i - a_i x)(y_i) = 0$.
Either slack for equation is 0 or dual variable is 0 or both.

First equation for primal: $4 - (1\times 1) + 0\times 3 = 1$ and $y_1 = 0$.

Solution(s)

$$\max[1,8]\cdot[x_1,x_2]$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \leq \begin{bmatrix}4\\ 3\\ 7\end{bmatrix}$$
$$[x_1,x_2] \geq 0.$$

$$\min[4,3,7]\cdot[y_1,y_2,y_3]$$
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Feasible? $1\times 0 + 0\times 6 + 1\times 1 \geq 1$, $0\times 0 + 1\times 1 + 2\times 3 \geq 8$.
Value $= 1\times 1 + 3\times 8 = 25$.

Complimentary Slackness: $(b_i - a_i x)(y_i) = 0$.
Either slack for equation is 0 or dual variable is 0 or both.

First equation for primal: $4 - (1\times 1) + 0\times 3 = 1$ and $y_1 = 0$.

In dual, both equations are tight.
so both x_1 and x_2 can be non-zero in optimal.

Dual.

$$\begin{aligned}
\max(x_1 + x_2)\\
7x_1 + 5x_2 \leq 21\\
4x_1 + 5x_2 \leq 20\\
2x_1 + 10x_2 \leq 33\\
x_1 \geq 0, x_2 \geq 0
\end{aligned}$$

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
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Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \le 21$$
$$4x_1 + 5x_2 \le 20$$
$$2x_1 + 10x_2 \le 33$$
$$x_1 \ge 0, x_2 \ge 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
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Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides:

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
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Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2)$

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2)$

Dual.

$$\begin{aligned} \max(x_1 + x_2) \\ 7x_1 + 5x_2 \leq 21 \\ 4x_1 + 5x_2 \leq 20 \\ 2x_1 + 10x_2 \leq 33 \\ x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2) = x_1 + x_2$.

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
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$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2) = x_1 + x_2$.
Right Hand Sides:

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
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$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2) = x_1 + x_2$.
Right Hand Sides: $(\frac{1}{15})21 + (\frac{2}{15})20$

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
$$4x_1 + 5x_2 \leq 20$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Optimal value?

$$7x_1 + 5x_2 = 21$$
$$4x_1 + 5x_2 = 20$$

$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2) = x_1 + x_2$.
Right Hand Sides: $(\frac{1}{15})21 + (\frac{2}{15})20 = 4\frac{1}{15}$.

Dual.

$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 21$$
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Optimal value?

$$7x_1 + 5x_2 = 21$$
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$x_1 = 1/3, x_2 = 3\frac{11}{15}$
Value is $4\frac{1}{15}$.
Left hand sides: $\frac{1}{15}(7x_1 + 5x_2) + \frac{2}{15}(4x_1 + 5x_2) = x_1 + x_2$.
Right Hand Sides: $(\frac{1}{15})21 + (\frac{2}{15})20 = 4\frac{1}{15}$.
Value is no more than $4\frac{1}{15}$.

Geometry of Dual.

Solution: $x_1 = \frac{1}{3}, x_2 = 3\frac{11}{15}$

Geometry of Dual.

Solution: $x_1 = \frac{1}{3}, x_2 = 3\frac{11}{15}$
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Geometry of Dual.

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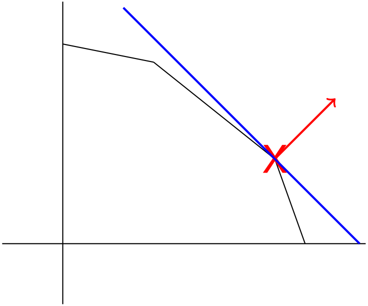
Dual: $y_1 = \frac{1}{15}, y_2 = \frac{2}{15}$



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Geometry of Dual.

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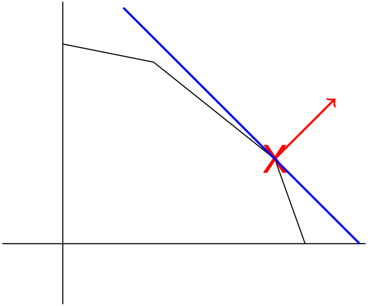

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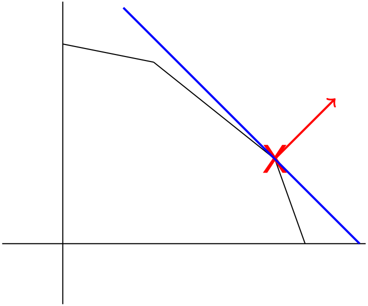
Dual adds tight constraints to get objective function.

Geometry of Dual.

Solution: $x_1 = \frac{1}{3}, x_2 = 3\frac{11}{15}$
Dual: $y_1 = \frac{1}{15}, y_2 = \frac{2}{15}$



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Dual adds tight constraints to get objective function.
Gemetrically:

Geometry of Dual.

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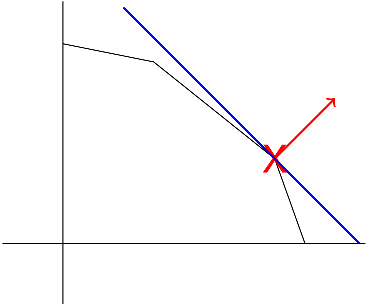

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Dual adds tight constraints to get objective function.

Gemetrically: can't get better!

Simplex Algorithm.

Simplex Algorithm.

Start at a vertex.

Simplex Algorithm.

Start at a vertex.

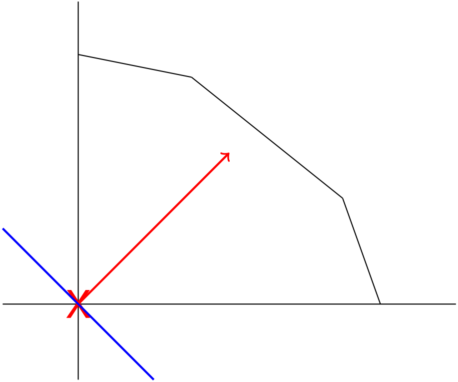
Move to better neighboring vertex.

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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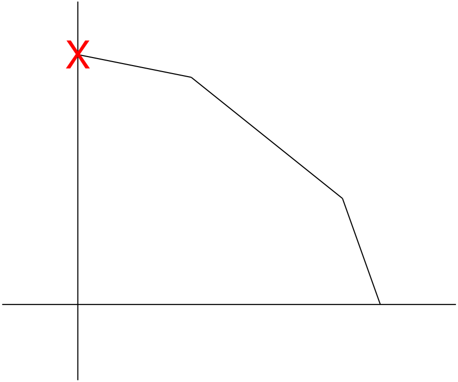

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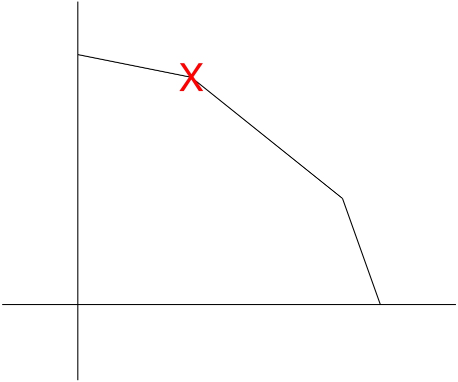

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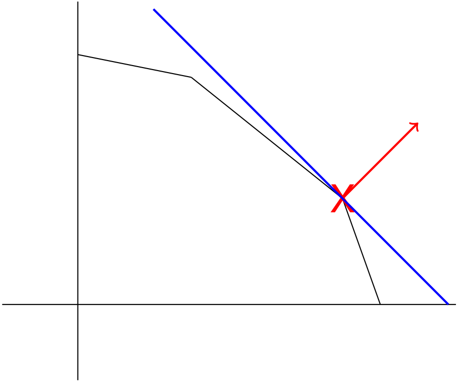

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Simplex Algorithm.

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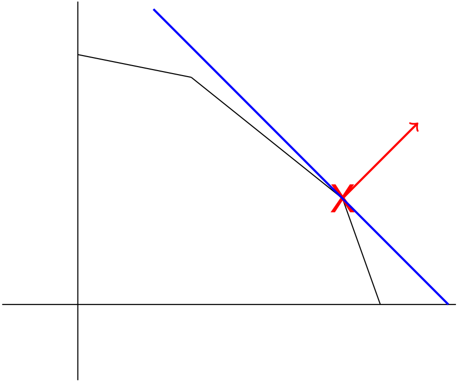

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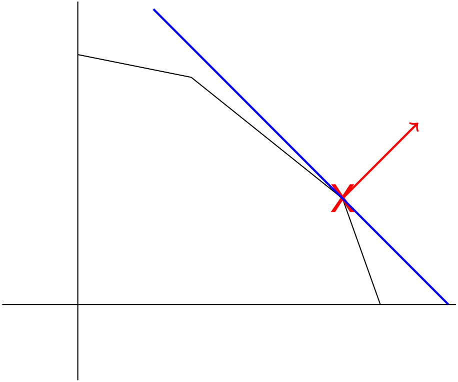
Why optimal?

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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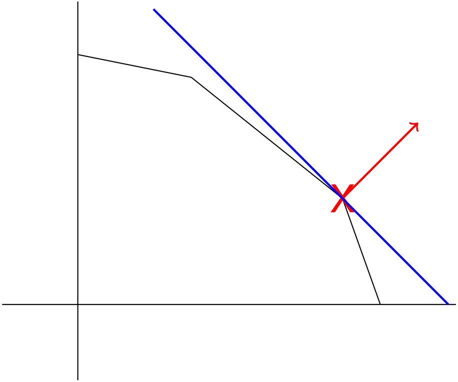
Why optimal? Draw line corresponding to $cx =$ current value.

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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$$x_1 \geq 0, x_2 \geq 0$$

Why optimal? Draw line corresponding to $cx =$ current value.
Entire feasible region on "wrong" side.

Three dimensions.

Vertex?

Three dimensions.

Vertex?

Two tight constraints?

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables $\implies$ n constraints define vertex.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables $\implies$ n constraints define vertex.

A constraint defines a *hyperplane*.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables $\implies$ n constraints define vertex.

A constraint defines a *hyperplane*.
Line in two dimensions.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables $\implies$ n constraints define vertex.

A constraint defines a *hyperplane*.
Line in two dimensions. Plane in three.

Three dimensions.

Vertex?

Two tight constraints?

$x_1 + x_2 + x_3 = 5$
$x_1 + 5x_2 + 2x_3 = 7$

Which point?

Three unknowns, two equations.

Defines a line, not a point.

Three tight constraints define a vertex!

n dimensions $\implies$ n variables $\implies$ n constraints define vertex.

A constraint defines a *hyperplane*.
Line in two dimensions. Plane in three.

In n dimensions, vertex is intersection of n hyperplanes.

Test

$m \times n$ matrix A.

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dude

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

Test

$m \times n$ matrix A. How many tight constraints at vertex?

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Dudette!

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

C.

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

C. dimension of space is n.

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

C. dimension of space is n. n constraints.

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

C. dimension of space is n. n constraints.

At least?

Test

$m \times n$ matrix A. How many tight constraints at vertex?

(A) At least m.

(B) At most n.

(C) At least n.

Dudette!

C. dimension of space is n. n constraints.

At least? May be redundant constraints!

Simplex Algorithm.

Simplex Algorithm.

Start at a vertex.

Simplex Algorithm.

Start at a vertex.

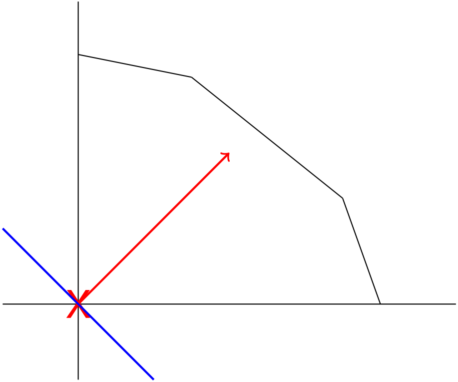
Move to better neighboring vertex.

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



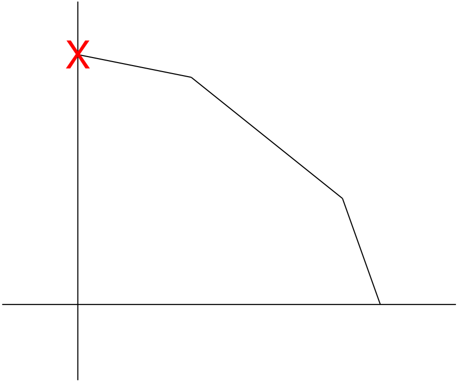
$$\max(x_1 + x_2)$$
$$7x_1 + 5x_2 \leq 20$$
$$4x_1 + 5x_2 \leq 21$$
$$2x_1 + 10x_2 \leq 33$$
$$x_1 \geq 0, x_2 \geq 0$$

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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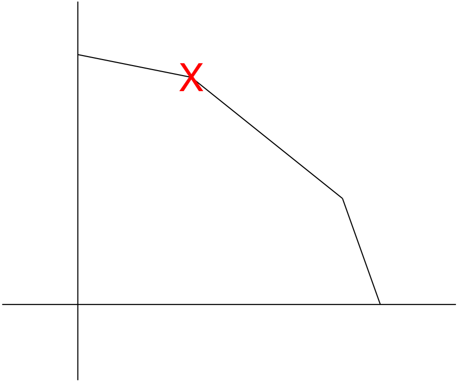

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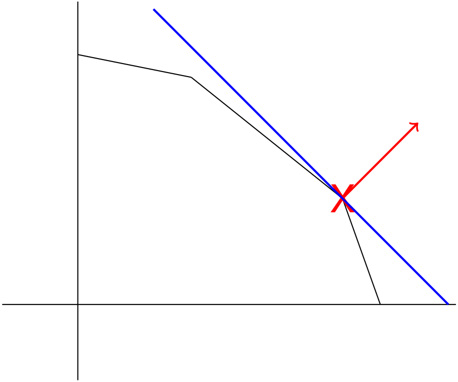

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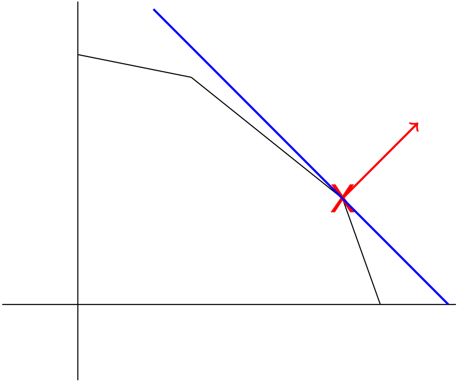

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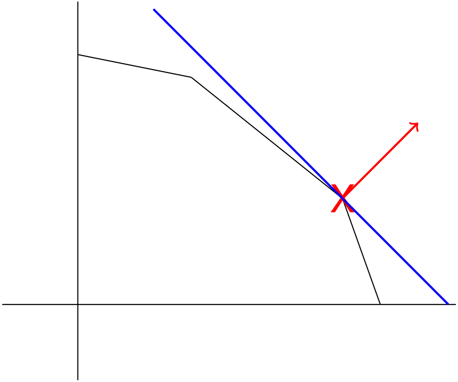
Why optimal?

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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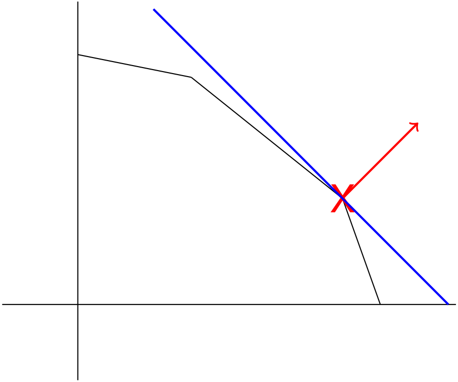
Why optimal? Draw line corresponding to $cx =$ current value.

Simplex Algorithm.

Start at a vertex.

Move to better neighboring vertex.

Until no better neighboring vertex.



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Why optimal? Draw line corresponding to $cx =$ current value.
Entire feasible region on “wrong” side.

Simplex algorithm.

Two tasks:

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \le b \\ x \ge 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\max c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0$

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\max c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$ increasing x_i increases value

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$ increasing x_i increases value $\implies$ not optimal.

Simplex algorithm.

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Canonical LP.

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Start at origin, supposing it is feasible.

Vertex since intersection of n constraints of form $x_i = 0$.

Optimal?

If all $c_i \leq 0 \implies$ increasing any x_i decreases value $\implies$ optimal!

if there is $c_i > 0$ increasing x_i increases value $\implies$ not optimal.

Done with task 1.

Going to a better place..

Two tasks:

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive c_i, so increase x_i.

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive c_i, so increase x_i.

...until you hit another constraint.

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive c_i, so increase x_i.

...until you hit another constraint.

$x_i \geq 0$ is no longer tight, but new constraint is.

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive c_i, so increase x_i.

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$x_i \geq 0$ is no longer tight, but new constraint is.
$\implies$ n constraints!

Going to a better place..

Two tasks:

1. Check optimality of vertex?
2. Where to go next?

At origin, there is positive c_i, so increase x_i.

...until you hit another constraint.

$x_i \geq 0$ is no longer tight, but new constraint is.
$\implies$ n constraints!

At vertex!

Example.

$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 \le 4 & \qquad \text{①} \\
x_1 + 2x_2 \le 9 & \qquad \text{②} \\
-x_1 + x_2 \le 3 & \qquad \text{③} \\
x_1 \ge 0 & \qquad \text{④} \\
x_2 \ge 0 & \qquad \text{⑤}
\end{aligned}
$$

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

Origin: feasible, value 0.

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

$$-x_1 + x_2 \leq 3 \quad (3)$$

$$x_1 \geq 0 \quad (4)$$

$$x_2 \geq 0 \quad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Example.

$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 \le 4 & \quad (1) \\
x_1 + 2x_2 \le 9 & \quad (2) \\
-x_1 + x_2 \le 3 & \quad (3) \\
x_1 \ge 0 & \quad (4) \\
x_2 \ge 0 & \quad (5)
\end{aligned}
$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

$$-x_1 + x_2 \leq 3 \quad (3)$$

$$x_1 \geq 0 \quad (4)$$

$$x_2 \geq 0 \quad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \le 4 \quad (1)$$

$$x_1 + 2x_2 \le 9 \quad (2)$$

$$-x_1 + x_2 \le 3 \quad (3)$$

$$x_1 \ge 0 \quad (4)$$

$$x_2 \ge 0 \quad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).

Example.

$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 \leq 4 & \quad (1) \\
x_1 + 2x_2 \leq 9 & \quad (2) \\
-x_1 + x_2 \leq 3 & \quad (3) \\
x_1 \geq 0 & \quad (4) \\
x_2 \geq 0 & \quad (5)
\end{aligned}
$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).
...new vertex: (0,3) with value 15.

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).
...new vertex: (0,3) with value 15.

Easy process from origin:

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

$$-x_1 + x_2 \leq 3 \quad (3)$$

$$x_1 \geq 0 \quad (4)$$

$$x_2 \geq 0 \quad (5)$$

Origin: feasible, value 0.
Inequalities 4 and 5 are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality 3 becomes tight constraint.
...Tight constraints: 3 and 4.
...new vertex: (0,3) with value 15.

Easy process from origin: just increase one variable.

Example.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

$$-x_1 + x_2 \leq 3 \quad (3)$$

$$x_1 \geq 0 \quad (4)$$

$$x_2 \geq 0 \quad (5)$$

Origin: feasible, value 0.
Inequalities (4) and (5) are tight.

Relax constraint $x_2 = 0$.
Increase x_2 until
...Inequality (3) becomes tight constraint.
...Tight constraints: (3) and (4).
...new vertex: (0,3) with value 15.

Easy process from origin: just increase one variable.

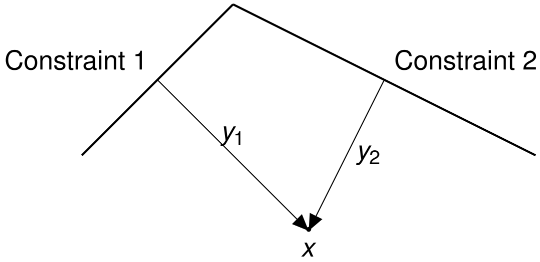
Now what?

A new coordinate system.

New coordinates: Distance from new tight constraints.

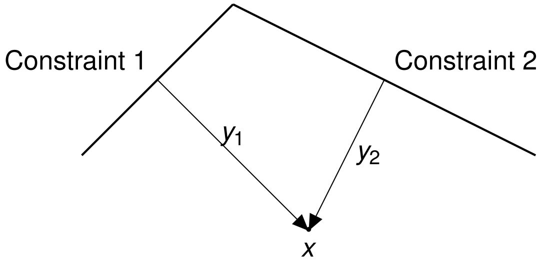
A new coordinate system.

New coordinates: Distance from new tight constraints.



A new coordinate system.

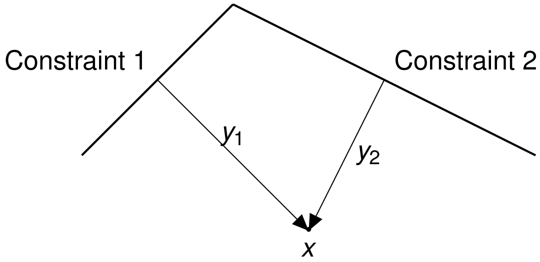
New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

A new coordinate system.

New coordinates: Distance from new tight constraints.

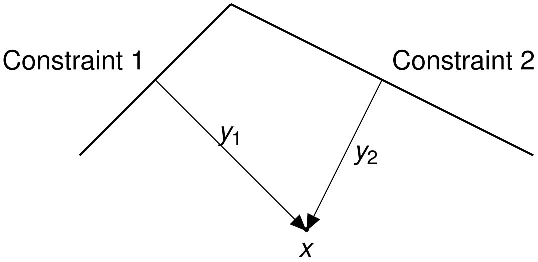


y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

A new coordinate system.

New coordinates: Distance from new tight constraints.



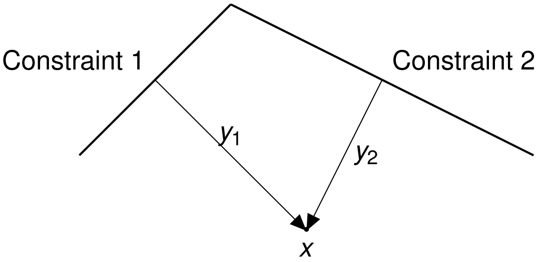
y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

A new coordinate system.

New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

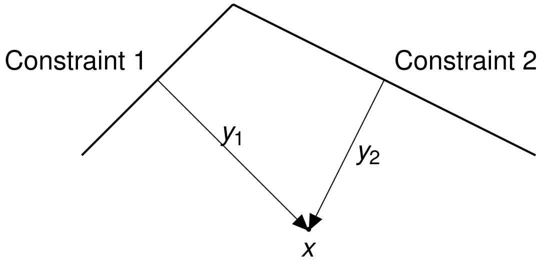
x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

Recall that for origin: x_i was distance from constraint $x_i \geq 0$.

A new coordinate system.

New coordinates: Distance from new tight constraints.



y_i is distance from constraint i

x is at (y_1, y_2) in new coordinate system.

For constraint i: $y_i = b_i - a_i x$

Recall that for origin: x_i was distance from constraint $x_i \geq 0$.

At origin in new coordinate system!

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

$$-x_1 + x_2 \leq 3 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$

New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Rewrite linear program.

Rewrite linear program with new coordinates.

$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 \leq 4 & \qquad (1) \\
x_1 + 2x_2 \leq 9 & \qquad (2) \\
-x_1 + x_2 \leq 3 & \qquad (3) \\
x_1 \geq 0 & \qquad (4) \\
x_2 \geq 0 & \qquad (5)
\end{aligned}
$$

New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Plug in for x_1 and x_2:

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \le 4 \qquad (1)$$

$$x_1 + 2x_2 \le 9 \qquad (2)$$

$$-x_1 + x_2 \le 3 \qquad (3)$$

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Plug in for x_1 and x_2: objective funcition

Rewrite linear program.

Rewrite linear program with new coordinates.

$$
\begin{aligned}
\max 2x_1 + 5x_2 & \\
2x_1 - x_2 \leq 4 & \qquad (1) \\
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Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Plug in for x_1 and x_2: objective funcition

$$\max \quad 2x_1 + 5x_2$$

$$\max \quad 2(y_1) + 5(3 - y_2 + y1)$$

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

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New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Plug in for x_1 and x_2: objective funcition

$$\max \quad 2x_1 + 5x_2$$
$$\max \quad 2(y_1) + 5(3 - y_2 + y1)$$
$$\max \quad 15 + 7y_1 - 5y_2$$

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad ①$$

$$x_1 + 2x_2 \leq 9 \quad ②$$

$$-x_1 + x_2 \leq 3 \quad ③$$

$$x_1 \geq 0 \quad ④$$

$$x_2 \geq 0 \quad ⑤$$

New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Plug in for x_1 and x_2: objective funcition

$\max \quad 2x_1 + 5x_2$
$\max \quad 2(y_1) + 5(3 - y_2 + y1)$
$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal?

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

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Plug in for x_1 and x_2: objective funcition

$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes!

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad (1)$$

$$x_1 + 2x_2 \leq 9 \quad (2)$$

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Plug in for x_1 and x_2: objective funcition

$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes! Maybe not!

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \qquad (1)$$

$$x_1 + 2x_2 \leq 9 \qquad (2)$$

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Plug in for x_1 and x_2: objective funcition

$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes! Maybe not! No.

Rewrite linear program.

Rewrite linear program with new coordinates.

$$\max 2x_1 + 5x_2$$

$$2x_1 - x_2 \leq 4 \quad ①$$

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New variables: $y_1 = x_1$, $y_2 = 3 + x_1 - x_2$.

Solve for x_i's: $x_1 = y_1$ and $x_2 = 3 - y_2 + y_1$.

Plug in for x_1 and x_2: objective funcition

$\max \quad 2x_1 + 5x_2$

$\max \quad 2(y_1) + 5(3 - y_2 + y1)$

$\max \quad 15 + 7y_1 - 5y_2$ Are we optimal? Yes! Maybe not! No.

Positive coefficient for increasing y_1.

Rewriting example..

$$\begin{aligned}
\max \quad 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}$$

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & \text{(1)} \\
& 3y_1 - 2y_2 \leq 3 & \text{(2)} \\
& y_2 \geq 0 & \text{(3)} \\
& y_1 \geq 0 & \text{(4)} \\
& -y_1 + y_2 \leq 3 & \text{(5)}
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & (1) \\
& 3y_1 - 2y_2 \leq 3 & (2) \\
& y_2 \geq 0 & (3) \\
& y_1 \geq 0 & (4) \\
& -y_1 + y_2 \leq 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Rewriting example..

$$\begin{aligned}
\max \quad 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \quad (1) \\
3y_1 - 2y_2 \leq 3 & \quad (2) \\
y_2 \geq 0 & \quad (3) \\
y_1 \geq 0 & \quad (4) \\
-y_1 + y_2 \leq 3 & \quad (5)
\end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \le 7 & (1) \\
& 3y_1 - 2y_2 \le 3 & (2) \\
& y_2 \ge 0 & (3) \\
& y_1 \ge 0 & (4) \\
& -y_1 + y_2 \le 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)?

Rewriting example..

$$
\begin{aligned}
\max \quad 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & \text{(1)} \\
& 3y_1 - 2y_2 \leq 3 & \text{(2)} \\
& y_2 \geq 0 & \text{(3)} \\
& y_1 \geq 0 & \text{(4)} \\
& -y_1 + y_2 \leq 3 & \text{(5)}
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)?

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & \quad (1) \\
& 3y_1 - 2y_2 \leq 3 & \quad (2) \\
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Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)?

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Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Rewriting example..

$$
\begin{array}{rr}
\max \quad 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & (1) \\
3y_1 - 2y_2 \leq 3 & (2) \\
y_2 \geq 0 & (3) \\
y_1 \geq 0 & (4) \\
-y_1 + y_2 \leq 3 & (5)
\end{array}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Rewriting example..

$$
\begin{aligned}
\max \quad & 15 + 7y_1 - 5y_2 & \\
& y_1 + y_2 \leq 7 & (1) \\
& 3y_1 - 2y_2 \leq 3 & (2) \\
& y_2 \geq 0 & (3) \\
& y_1 \geq 0 & (4) \\
& -y_1 + y_2 \leq 3 & (5)
\end{aligned}
$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

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$$\max \quad 15 + 7y_1 - 5y_2$$

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y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

Rewriting example..

$$\begin{aligned} \max \quad 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \le 7 & \qquad (1) \\ 3y_1 - 2y_2 \le 3 & \qquad (2) \\ y_2 \ge 0 & \qquad (3) \\ y_1 \ge 0 & \qquad (4) \\ -y_1 + y_2 \le 3 & \qquad (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

Rewriting example..

$$\begin{aligned} \max \quad & 15 + 7y_1 - 5y_2 & \\ & y_1 + y_2 \leq 7 & (1) \\ & 3y_1 - 2y_2 \leq 3 & (2) \\ & y_2 \geq 0 & (3) \\ & y_1 \geq 0 & (4) \\ & -y_1 + y_2 \leq 3 & (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$.

Rewriting example..

$$\max \quad 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \le 7 \qquad (1)$$

$$3y_1 - 2y_2 \le 3 \qquad (2)$$

$$y_2 \ge 0 \qquad (3)$$

$$y_1 \ge 0 \qquad (4)$$

$$-y_1 + y_2 \le 3 \qquad (5)$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value:

Rewriting example..

$$\begin{aligned}
\max \quad & 15+7y_1-5y_2 & \\
& y_1+y_2 \leq 7 & \text{(1)} \\
& 3y_1-2y_2 \leq 3 & \text{(2)} \\
& y_2 \geq 0 & \text{(3)} \\
& y_1 \geq 0 & \text{(4)} \\
& -y_1+y_2 \leq 3 & \text{(5)}
\end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value:
$12+7(1)-5(0)$

Rewriting example..

$$\begin{aligned} \max \quad 15 + 7y_1 - 5y_2 & \\ y_1 + y_2 \leq 7 & \qquad (1) \\ 3y_1 - 2y_2 \leq 3 & \qquad (2) \\ y_2 \geq 0 & \qquad (3) \\ y_1 \geq 0 & \qquad (4) \\ -y_1 + y_2 \leq 3 & \qquad (5) \end{aligned}$$

y_1, y_2 are non-negative just like x_i's. (Constraints are satisfied!)

Improve by increasing y_1.

Which is tight? (1)? (2)? (3)? (4)? (5)?

Note: $y_2 = 0$.

Smallest right hand side divided by (positive) coefficient of y_2!

Inequality (2)!

New vertex: tight constraints (3) and (2).

New solution: $y_1 = 1$, $y_2 = 0$. New Objective Value: $12 + 7(1) - 5(0) = 22$.

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2$

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow$

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$
$$y_1 + y_2 \leq 7 \qquad (1)$$
$$3y_1 - 2y_2 \leq 3 \qquad (2)$$
$$y_2 \geq 0 \qquad (3)$$
$$y_1 \geq 0 \qquad (4)$$
$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

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Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \leq 7 & \qquad (1) \\
3y_1 - 2y_2 \leq 3 & \qquad (2) \\
y_2 \geq 0 & \qquad (3) \\
y_1 \geq 0 & \qquad (4) \\
-y_1 + y_2 \leq 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal?

Rewriting example..

$$
\begin{aligned}
\max 15 + 7y_1 - 5y_2 & \\
y_1 + y_2 \le 7 & \qquad (1) \\
3y_1 - 2y_2 \le 3 & \qquad (2) \\
y_2 \ge 0 & \qquad (3) \\
y_1 \ge 0 & \qquad (4) \\
-y_1 + y_2 \le 3 & \qquad (5)
\end{aligned}
$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \to y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes!

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not!

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \to y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not! Optimal point!

Rewriting example..

$$\max 15 + 7y_1 - 5y_2$$

$$y_1 + y_2 \leq 7 \qquad (1)$$

$$3y_1 - 2y_2 \leq 3 \qquad (2)$$

$$y_2 \geq 0 \qquad (3)$$

$$y_1 \geq 0 \qquad (4)$$

$$-y_1 + y_2 \leq 3 \qquad (5)$$

Rewrite: $z_2 = y_2$
$z_1 = 3 - 3y_1 + 2y_2 \rightarrow y_1 = -\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1$
Objective function.
$\max \quad 15 + 7(-\frac{1}{3}z_1 + \frac{2}{3}z_2 + 1) - 5z_2$
$\max 22 - \frac{7}{3}z_1 - \frac{1}{3}z_2$ Optimal? Yes! Maybe not! Optimal point!
Increasing z_1, z_2 makes things worse.

Review.

In each step:

Review.

In each step:

LP in coordinate system from tight constraints.

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$$\max 15 + 7y_1 - 5y_2.$$

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$$\max 15 + 7y_1 - 5y_2.$$

Go to tight constraint along improving coordinate.

Review.

In each step:

LP in coordinate system from tight constraints.

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Does objective function have nonnegative multiplier?

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$3y_1 - 2y_2 \le 3$.

Review.

In each step:

LP in coordinate system from tight constraints.

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Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2$.

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3$.

Express LP in coordinate system for new tight constraints.

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2$.

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3$.

Express LP in coordinate system for new tight constraints.

See previous slides!

Review.

In each step:

LP in coordinate system from tight constraints.

Optimal?
Does objective function have nonnegative multiplier?

$\max 15 + 7y_1 - 5y_2.$

Go to tight constraint along improving coordinate.

$3y_1 - 2y_2 \leq 3.$

Express LP in coordinate system for new tight constraints.

See previous slides!

Repeat.

Details: getting started.

What if origin is not feasible?

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

An x where $Ax \leq b$ and at vertex.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

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Make a new linear program.

Details: getting started.

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Make a new linear program.

Introduce positive variables z_i for inequality i.

Details: getting started.

What if origin is not feasible?

How do you find a feasible vertex?

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Make a new linear program.

Introduce positive variables z_i for inequality i.

Constraints: $a_i x - z_i \leq b_i$.

Details: getting started.

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$$\max \sum -z_i.$$

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Vertex solution (x, z) of value zero

Details: getting started.

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Vertex solution (x, z) of value zero
$\implies$ all z's are zero

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$$\max \sum -z_i.$$

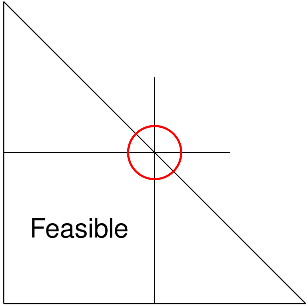
Vertex solution (x, z) of value zero
$\implies$ all z's are zero
$\implies$ all inequalities are satisfied
$\implies$ x is a feasible vertex of $Ax \leq b$.

Degeneracy.

Degenerate vertices.

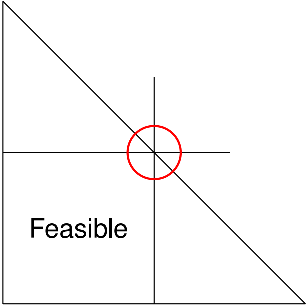
Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.



Degeneracy.

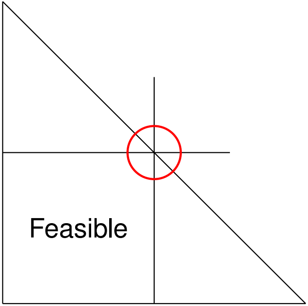
Degenerate vertices.
Intersection of more than n constraints.



Problem: all neighboring vertices are no better.

Degeneracy.

Degenerate vertices.
Intersection of more than *n* constraints.



Problem: all neighboring vertices are no better.

Infinite looping: Bland's anticycling rule.

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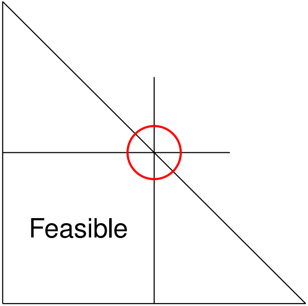
Degenerate vertices.
Intersection of more than n constraints.



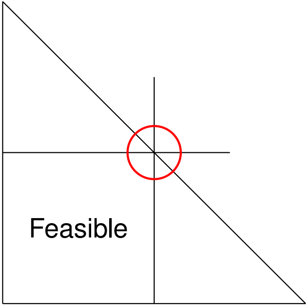
Problem: all neighboring vertices are no better.

Infinite looping: Bland's anticycling rule.

Or Perturb problem a bit.

Degeneracy.

Degenerate vertices.
Intersection of more than n constraints.

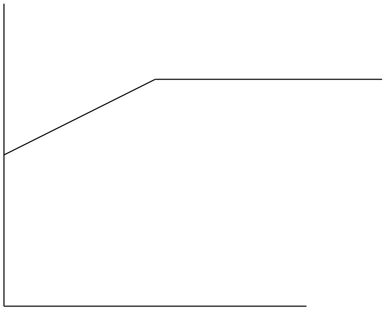


Problem: all neighboring vertices are no better.

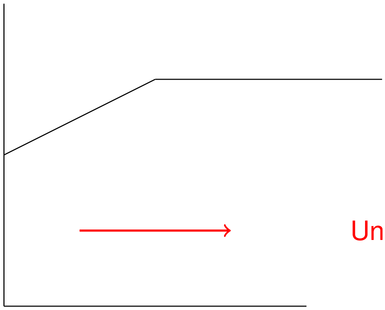
Infinite looping: Bland's anticycling rule.

Or Perturb problem a bit. Unlikely to intersect!

Unboundedness.

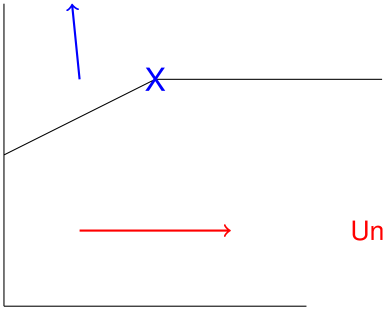


Unboundedness.



Unbounded value.

Unboundedness.



Unbounded value.

Simplex can tell difference.

From X: either unbounded improvement or optimal.

Running Time

Check optimality? $O(n)$.

Running Time

Check optimality? $O(n)$.

Find tight constraint:

Running Time

Check optimality? $O(n)$.

Find tight constraint:
$O(m)$ constraints.

Running Time

Check optimality? $O(n)$.

Find tight constraint:
$O(m)$ constraints. $O(1)$ time per constraint.

Running Time

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Running Time

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Find new coordinate system, rewrite LP.

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Recall $y_i = b_i - a_i x$

Running Time

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$O(m)$ total.

Find new coordinate system, rewrite LP.

Recall $y_i = b_i - a_i x$

Rewrite in terms of y_i.

Running Time

Check optimality? $O(n)$.

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Recall $y_i = b_i - a_i x$

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  Solve for x_i in terms of y_i.

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Naively: $O(n^3)$ time.

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$O(m)$ total.

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Rewrite in terms of y_i.
- Solve for x_i in terms of y_i.
- Plug in.

Naively: $O(n^3)$ time.

Only one new constraint.

Running Time

Check optimality? $O(n)$.

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Only one new constraint. One new y_i.

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Naively: $O(n^3)$ time.

Only one new constraint. One new y_i.
Only one unknown.

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Backsolve.

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Backsolve.
$O(nm)$ time to update LP.

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How many steps?

Running Time

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How many steps?

Could be large.

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How many steps?

Could be large. Exponential in worst case!

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How many steps?

Could be large. Exponential in worst case!

Fast, in practice!

Lecture in a minute.

Duality:

Lecture in a minute.

Duality:

Primal: $Ax \leq b, \max cx, x \geq 0$

Dual: $A^T y \geq b, \min by, y \geq 0$

Linear combiniation of equations that dominates objective function.

Duality: (typically) have the same value.

Weak Duality: Primal $\leq$ Dual.

Feasible $x, y \implies cx \leq y^T Ax \geq y^T b$.

Lecture in a minute.

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Simplex Implementation:

Start at a (feasible) vertex.

Lecture in a minute.

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(defined by linear system $A'x = [b, 0, \cdots, 0]$).

Lecture in a minute.

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Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Lecture in a minute.

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Linear combiniation of equations that dominates objective function.

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Simplex Implementation:

Start at a (feasible) vertex.

(defined by linear system $A'x = [b, 0, \cdots, 0]$).

Begin at origin. Move to better neighboring vertex.

Coordinate system: distance from tight constraints.

Vertex at origin in coordinate system.

$O(mn)$ time to update linear system.

Until no better neighboring vertex.

Objective function in coordinate system is non-positive.

Dual Variables: new objective function!