Polynomials:

$$P(x) = p_0 + p_1 x + p_2 x^2 + \dots p_d x^d$$

$$\deg(p) = d$$

Representation:

$\langle p_0 \dots p_d \rangle$

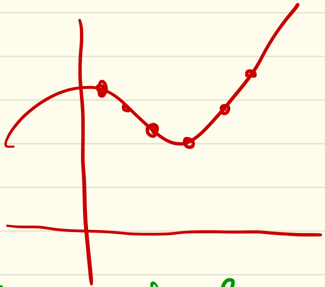
coefficients

COEFFICIENT REPRESENTATION

Specify points on the graph of p

$p(\alpha_1) \dots p(\alpha_n)$

VALUE REPRESENTATION



FACT: $d+1$ evaluations uniquely determine a deg d polynomial.

EVALUATION:

Given $P = \langle p_0, \ldots, p_d \rangle$ & $\alpha \in \mathbb{R}$

compute $p(\alpha) = \sum_{i=0}^{d} p_i \alpha^i$

$p(x) = 1 + 2x + 3x^2 + 4x^3$ $\quad \alpha = 5$

$p(5) = 1 + 2 \cdot 5 + 3 \cdot 5 \cdot 5 + 4 \cdot 5 \cdot 5 \cdot 5$ $\leftarrow$ Naive

Horner's Method

$p(\alpha) = 1 + \alpha(2 + \alpha \cdot (3 + \alpha \cdot 4))$

$p(x) = p_0 + \alpha \cdot (p_1 + \alpha \cdot (p_2 + \alpha(\ldots)))$

d-additions & d-multiplications

Assumption: Each arithmetic operation in $O(1)$ time.

ADDITION:

Given $p = \langle p_0 \ldots p_d \rangle$ $q = \langle q_0 \ldots q_d \rangle$

$$p+q(x) = \langle p_0+q_0, p_1+q_1, \ldots, p_d+q_d \rangle$$

$\uparrow$

$\Theta(d)$ time.

COEFFICIENT REPRESENTATION

Given $p = \langle p(\alpha_1), p(\alpha_2) \ldots p(\alpha_n) \rangle$

$q = \langle q(\alpha_1) \ldots q(\alpha_n) \rangle$

VALUE REPRESENTATION

$$p+q = \langle p(\alpha_1)+q(\alpha_1), p(\alpha_2)+q(\alpha_2) \ldots p(\alpha_n)+q(\alpha_n) \rangle$$

$\Theta(d)$ time

MULTIPLICATION

Given $p(x) = \langle p_0 \cdots p_d \rangle$ $\quad q(x) = \langle q_0, \ldots, q_d \rangle$

COEFFICIENT REPRESENTATION

$p(x) \cdot q(x)$

$$(\underbrace{1+2x+3x^2+4x^3}_{d \text{ terms}}) \cdot (\underbrace{5+6x+7x^2+8x^3}_{d \text{ terms}})$$

Naive Alg: Multiply all pairs and simplify

$\Theta(d^2)$ time

Given $p \leftarrow \langle p(\alpha_1) \ldots p(\alpha_n) \rangle$ $\quad n = 2d+2$

$q \leftarrow \langle q(\alpha_1) \ldots q(\alpha_n) \rangle$

VALUE REPRESENTATION

$$p \cdot q = \langle p(\alpha_1) \cdot q(\alpha_1), p(\alpha_2) \cdot q(\alpha_2) \ldots, p(\alpha_n) \cdot q(\alpha_n) \rangle$$

$\Theta(n)$ time

REPRESENTATION / OPERATION	COEFFICIENTS	EVALUATIONS
	$p = \langle p_0, \ldots p_d \rangle$	$\langle p(\alpha_1) \ldots p(\alpha_n) \rangle$
	$q = \langle q_0 \ldots, q_d \rangle$	$\langle q(\alpha_1) \ldots q(\alpha_n) \rangle$
EVALUATION	$O(d)$	
ADDITION	$O(d)$	$O(n)$
MULTIPLICATION	$O(d^2)$	$O(n)$

FAST POLY MULTIPLICATION

"FFT"

$\mathbb{C}$

? $\Theta(d\log d)$

$p(x)$ $\langle p_0, \ldots p_d \rangle$ $\longrightarrow$ $\langle p(\alpha_1) \ldots p(\alpha_n) \rangle$

$q(x)$ $\langle q_0, \ldots q_d \rangle$ $\qquad$ $\langle q(\alpha_1) \ldots q(\alpha_n) \rangle$

$\downarrow$ linear time

$\langle p(\alpha_1)\cdot q(\alpha_1), p(\alpha_2)\cdot q(\alpha_2) \ldots p(\alpha_n)\cdot q(\alpha_n) \rangle$

Coefficients of $p(x)\cdot q(x)$ $\longleftarrow$? $\Theta(d\log d)$

EVALUATE$(\langle p_0 \ldots p_d \rangle, \{\alpha_1 \ldots \alpha_n\})$:

GOAL: $(p(\alpha_1) \ldots p(\alpha_n))$

1) SPLIT

$$P_{odd}(z) = p_1 + p_3 z + p_5 z^2 + \ldots$$

$$P_{even}(z) = p_0 + p_2 z + p_4 z^2 \ldots$$

$$P(x) = P_{even}(x^2) + x \cdot P_{odd}(x^2)$$

2) Evaluate$(P_{odd}, \{\alpha_1^2 \ldots \alpha_n^2\})$

Evaluate$(P_{even}, \{\alpha_1^2 \ldots \alpha_n^2\})$

3) $\forall\ i = 1 \ldots n$

$$p(\alpha_i) = P_{even}(\alpha_i^2) + \alpha_i P_{odd}(\alpha_i^2)$$

$$p(x) = 0 + x + 2x^2 + 3x^3$$

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{5, 6, 7, 8\}$$

$P_{odd}(z) = (1, 3)$: $1 + 3z$

$P_{even}(z) = (0, 2)$: $0 + 2z$

$$p(x) = (0 + 2x^2) + x(1 + 3x^2)$$

$$p(\alpha_i) = P_{even}(\alpha_i^2) + \alpha_i P_{odd}(\alpha_i^2)$$

$P_{odd}(5^2)$	$P_{even}(5^2)$
$P_{odd}(6^2)$	$P_{even}(6^2)$
$P_{odd}(7^2)$	$P_{even}(7^2)$
$P_{odd}(8^2)$	$P_{even}(8^2)$

Base Case:

$$\left(r(z) = r_0 + r_1 z, \ \{\tilde{\alpha_1}, \tilde{\alpha_2} \ldots \tilde{\alpha_n}\} \right)$$

$\Theta(n)$ time

Analysis:

$$T[n, d] = 2T[n, d/2] + \Theta(n)$$

# of points

degree

+

$$T[n, 1) = \Theta(n)$$

Excercise: Show $T[n,d] = \Theta(nd)$