Lecture in a Minute

Games

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  - Nash Equilibrium

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- Zero Sum Two Person Games
- Mixed Strategies.
- Checking Equilibrium.
- Best Response.
- Statement of Duality Theorem.

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- Generality of Linear Program.

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- Nash Equilibrium
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- Checking Equilibrium.
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- Statement of Duality Theorem.

Generality of Linear Program.
- Any circuit can be implemented by linear program!
- Any polynomial time algorithm
  - $\implies$ a poly sized linear program.

Strategic Games.

N players.

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Player 1: { **D**efect, **C**ooperate }.
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Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

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Nash Equilibrium:

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Nash Equilibrium:
neither player has incentive to change strategy.

Two person zero sum games.

$m \times n$ payoff matrix A.

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Payoff for strategy pair (x, y):

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$$p(x,y) = x^t Ay$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j}$$

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(No better column strategy,

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Equilibrium pair: (x^*, y^*)?

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(No better column strategy, no better row strategy.)

Zero Sum Games. $R = \min_{y} \max_{x} (x^t A y)$.

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Weak Duality: $R \geq C$.
Proof: Better to go second. $\square$

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row payoffs (Ay^*) all $\leq v$

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Equilibrium $\implies R = C$!

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Strong Duality: There is an equilibrium point!

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Doesn't matter who plays first!

Roshambo Example.

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

How do you play?

Roshambo Example.

		R	P	S
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How do you play?

Player 1: play each strategy with equal probability.

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Definitions.

Mixed strategies: Each player plays distribution over strategies.

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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Playing the boss...

Row has extra strategy:Cheat.

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Note: column knows row cheats.

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Equilibrium pair: (x^*, y^*)?

---

[1] $A^{(i)}$ is ith row.

Equilibrium.

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$$p(x,y) = (x^*)^T A y^* = \min_y (x^*)^T A y = \max_x \; x^T A y^*.$$

(No better column strategy, no better row strategy.)

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Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^T Ay^* = \min_y (x^*)^T Ay = \max_x \; x^T Ay^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\max_i A^{(i)} \cdot y^* = (x^*)^T Ay^*$.[1]

[1] $A^{(i)}$ is ith row.

Equilibrium.

Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^T A y^* = \min_y (x^*)^T A y = \max_x \; x^T A y^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\max_i A^{(i)} \cdot y^* = (x^*)^T A y^*$.[1]

No column is better:
$\min_j (A^T)^{(j)} \cdot x^* = (x^*)^T A y^*$.

[1] $A^{(i)}$ is ith row.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium:

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$.

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Payoff?

Equilibrium: play the boss...

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Payoff? Remember: weighted average of pure strategies.

Equilibrium: play the boss...

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Payoff? Remember: weighted average of pure strategies.

Row Player.

Equilibrium: play the boss...

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Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}$

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}$
Strategy 2: $\frac{1}{3} \times 1 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1$

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Payoff is $0 \times -\frac{1}{3} + \frac{1}{3} \times (\frac{1}{6}) + \frac{1}{6} \times (\frac{1}{6}) + \frac{1}{2} \times (\frac{1}{6})$

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Column player: every column payoff is $\frac{1}{6}$.

Equilibrium: play the boss...

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Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

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Column player: every column payoff is $\frac{1}{6}$.

Both only play optimal strategies!

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

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Column player: every column payoff is $\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

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Payoff is $0 \times -\frac{1}{3} + \frac{1}{3} \times (\frac{1}{6}) + \frac{1}{6} \times (\frac{1}{6}) + \frac{1}{2} \times (\frac{1}{6}) = \frac{1}{6}$

Column player: every column payoff is $\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.
Why play more than one?

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times -1 + \frac{1}{6} \times 1 = -\frac{1}{3}$
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Payoff is $0 \times -\frac{1}{3} + \frac{1}{3} \times (\frac{1}{6}) + \frac{1}{6} \times (\frac{1}{6}) + \frac{1}{2} \times (\frac{1}{6}) = \frac{1}{6}$

Column player: every column payoff is $\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Equilibrium: always?

Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^T A y^* = \min_y (x^*)^T A y = \max_x \; x^T A y^*.$$

Equilibrium: always?

Equilibrium pair: (x^*, y^*)?

$$p(x,y) = (x^*)^T A y^* = \min_y (x^*)^T A y = \max_x \; x^T A y^*.$$

Does an equilibrium pair:, (x^*, y^*), exist?

Equilibrium value is unique.

Zero sum game:

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x
... probability distribution over rows.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

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column minimizes.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x
... probability distribution over rows.

column minimizes. strategy: vector n-dimensional vector y
... probability distribution over columns.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x
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Payoff (x, y): $x^T Ay$.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

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Payoff (x, y): $x^T Ay$.

Nash equilibrium (x^*, y^*):

Equilibrium value is unique.

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Payoff (x, y): $x^T Ay$.

Nash equilibrium (x^*, y^*):
neither player has better response against others.

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If there is an equilibrium: no disadvantage in announcing strategy!

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All equilibrium points all have same payoff.

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Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.

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Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.
$\implies \max_i (Ay_1)_i > \max_i (Ay_2)_i$

Equilibrium value is unique.

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$\implies \max_i (Ay_1)_i > \max_i (Ay_2)_i \quad x_i$ zero on non-best row of (Ay_1)
Best row is worse under y_2.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x
... probability distribution over rows.

column minimizes. strategy: vector n-dimensional vector y
... probability distribution over columns.

Payoff (x, y): $x^T Ay$.

Nash equilibrium (x^*, y^*):
neither player has better response against others.

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All equilibrium points all have same payoff.

Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.
$\implies$ $\max_i (Ay_1)_i > \max_i (Ay_2)_i$ $\quad x_i$ zero on non-best row of (Ay_1)
Best row is worse under y_2.
$\implies$ Column player strategy y_2 is better than y_1

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Zero sum game: $m \times n$ matrix A

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Payoff (x, y): $x^T A y$.

Nash equilibrium (x^*, y^*):
neither player has better response against others.

If there is an equilibrium: no disadvantage in announcing strategy!

All equilibrium points all have same payoff.

Why? Assume equilibriums: $x_1^T A y_1 > x_2^T A y_2$.
$\implies \max_i (Ay_1)_i > \max_i (Ay_2)_i$ $\quad x_i$ zero on non-best row of (Ay_1)
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If there is an equilibrium: no disadvantage in announcing strategy!

All equilibrium points all have same payoff.

Why? Assume equilibriums: $x_1^T Ay_1 > x_2^T Ay_2$.
$\implies \max_i (Ay_1)_i > \max_i (Ay_2)_i$ $\quad x_i$ zero on non-best row of (Ay_1)
Best row is worse under y_2.
$\implies$ Column player strategy y_2 is better than y_1
x_1, y_1 is not equilibrium.

Equilibrium value is unique.

Zero sum game: $m \times n$ matrix A

row maximizes strategy: m-dimensional vector x
... probability distribution over rows.

column minimizes. strategy: vector n-dimensional vector y
... probability distribution over columns.

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Zero Sum games and Linear Programs.

Matrix A, rows a_i, columns, $a^{(i)}$.

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(Adler, recently.)

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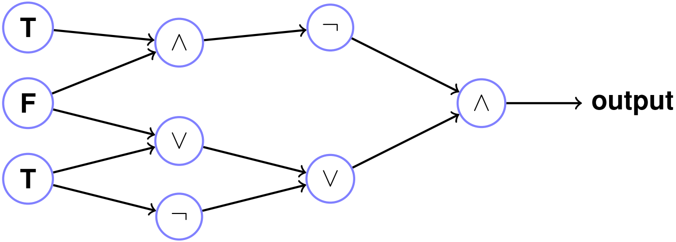

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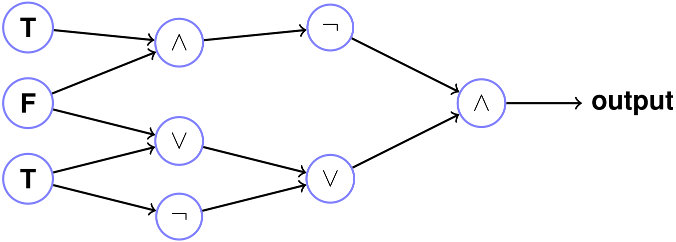

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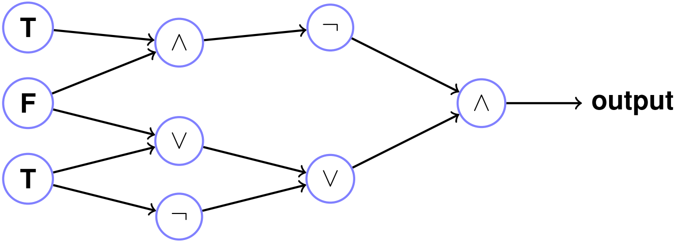

No really! What is the value of the output?

Translation to linear program.

Variable for gate g: x_g.

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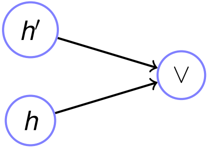

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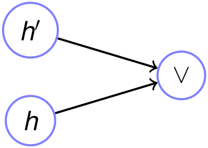

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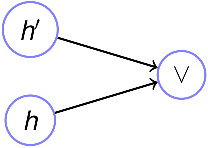

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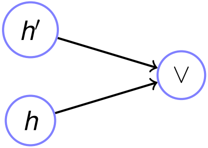

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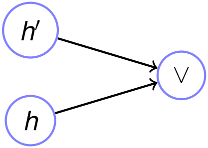

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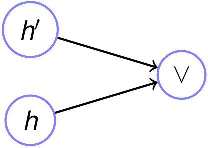

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$x_g \le x_h, \quad x_g \le x_h'$

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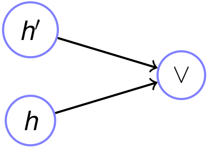

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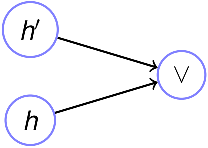

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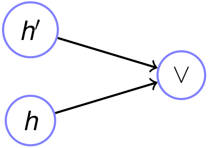

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x_o is 1 if and only if the circuit evaluates to true.

61C

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Next: NP completeness..more reductions.

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Generality of Linear Program.
- Any circuit can be implemented by linear program!
- Any polynomial time algorithm
  - $\implies$ a poly sized linear program.