Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Taking a Dual

Consider the following linear program:

$$\begin{aligned} \max\ & 4x_1 + 7x_2 \\ x_1 + 2x_2 &\leq 10 \\ 3x_1 + x_2 &\leq 14 \\ 2x_1 + 3x_2 &\leq 11 \\ x_1, x_2 &\geq 0 \end{aligned}$$

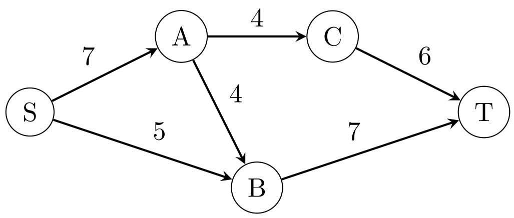
Construct the dual of the above linear program.

Solution: If we scale the first constraint by $y_1 \geq 0$, the second by $y_2 \geq 0$, the third by $y_3 \geq 0$, and we add them up, we get an upperbound of $(y_1 + 3y_2 + 2y_3)x_1 + (2y_1 + y_2 + 3y_3)x_2 \leq (10y_1 + 14y_2 + 11y_3)$. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned} \min\ & 10y_1 + 14y_2 + 11y_3 \\ y_1 + 3y_2 + 2y_3 &\geq 4 \\ 2y_1 + y_2 + 3y_3 &\geq 7 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

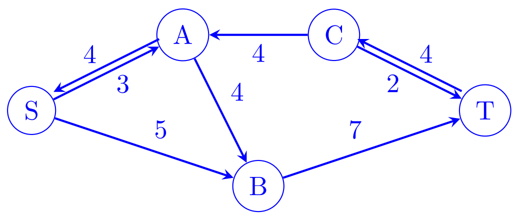
2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

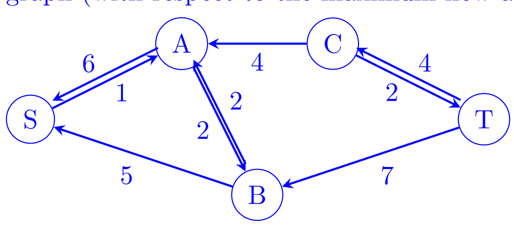


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

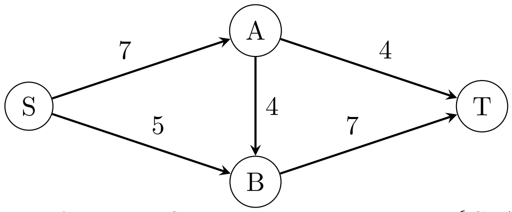
(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:



A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

3 Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.
Consider this instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$\begin{aligned}
\max \quad & f_1 + f_2 + f_3 & \\
& f_1 + f_2 \leq 7 & \text{(Constraint for } (S, A)) \\
& f_3 \leq 5 & \text{(Constraint for } (S, B)) \\
& f_1 \leq 4 & \text{(Constraint for } (A, T)) \\
& f_2 \leq 4 & \text{(Constraint for } (A, B)) \\
& f_2 + f_3 \leq 7 & \text{(Constraint for } (B, T)) \\
& f_1, f_2, f_3 \geq 0 &
\end{aligned}$$

The objective is to maximize the flow being pushed, with the constraint that for every edge, we can't push more flow through that edge than its capacity allows.

(a) Find the dual of this linear program, where the variables in the dual are x_e for every edge e in the graph.

(b) Consider any cut in the graph. Show that setting $x_e = 1$ for every edge crossing this cut and $x_e = 0$ for every edge not crossing this cut gives a feasible solution to the dual program.

(c) Based on your answer to the previous part, what problem is being modelled by the dual program? By LP duality, what can you argue about this problem and the max flow problem?

Solution:

(a) The dual is:

$$\begin{aligned}
\min \quad & 7x_{SA} + 5x_{SB} + 4x_{AT} + 4x_{AB} + 7x_{BT} \\
& x_{SA} + x_{AT} \geq 1 \text{ - Constraint for } f_1 \\
& x_{SA} + x_{AB} + x_{BT} \geq 1 \text{ - Constraint for } f_2 \\
& x_{SB} + x_{BT} \geq 1 \text{ - Constraint for } f_3 \\
& x_e \geq 0 \quad \forall e \in E
\end{aligned}$$

(b) Notice that each constraint contains all variables x_e for every edge e in the corresponding path. For any $s-t$ cut, every $s-t$ path contains an edge crossing this cut. So for any cut, the suggested solution will set at least one x_e to 1 on each path, giving that each constraint is satisfied.

(c) The dual LP is an LP for the min-cut problem. By the previous answer, we know the constraints describe solutions corresponding to cuts. The objective then just says to find the cut of the smallest size. By LP duality, the dual and primal optima are equal, i.e. the max flow and min cut values are equal.

4 Standard Form LP

Recall that any Linear Program can be reduced to a more constrained *standard form* where all variables are nonnegative, the constraints are given by equations and the objective is that of minimizing a cost function. For each of the subparts, what system of variables, constraints, and objectives would be equivalent to the following:

(a) Max Objective: $\max c^\top x$

(b) Min Max Objective: $\min \max(y_1, y_2)$

(c) Upper Bound on Variable: $x_1 \leq b_1$

(d) Lower Bound on Variable: $x_2 \geq b_2$

(e) Bounded Variable: $b_2 \leq x_3 \leq b_1$

(f) Inequality Constraint: $x_1 + x_2 + x_3 \leq b_3$

(g) Unbounded Variable: $x_4 \in R$

Solution:

(a) $\min -c^\top x$

(b) $\min t\,, \quad x \leq t\,, \quad y \leq t$

(c) $x_1 + s_1 = b_1\,, \quad s_1 \geq 0$

(d) $-x_2 + s_2 = -b_2 \quad s_2 \geq 0$

(e) Break it into two inequalities $x_3 \leq b_1$ and $x_3 \geq b_2$ and use the parts above

(f) $x_1 + x_2 + x_3 + s_1 = b_3\,, \quad s_1 \geq 0$

(g) Replace x_4 by $x^+ - x^-$ along with $x^+ \geq 0\,, \quad x^- \geq 0$