CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **DIS 2**

1 Induction

Prove the following using induction:

(a) For all natural numbers $n \geq 1$, if $A = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1-3n & -9n \\ n & 1+3n \end{bmatrix}$.

(b) For real numbers a_i where $-1 < a_i \leq 0$, $i \in \mathbb{N}$, $\prod_{i=0}^{i=n}(1+a_i) \geq 1 + \sum_{i=0}^{i=n} a_i$.

Solution:

(a)
- Base case (n=1): $P(0)$ asserts that $A^1 = \begin{bmatrix} 1-3(1) & -9(1) \\ 1 & 1+3(1) \end{bmatrix} = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix}$. Thus, the base case is correct.
- Inductive Hypothesis: For arbitrary $n = k \geq 1$, assume that $P(k)$ is true: $A^k = \begin{bmatrix} 1-3k & -9k \\ k & 1+3k \end{bmatrix}$.
- Inductive Step: Prove the statement for $n = k+1$: $A^{k+1} = \begin{bmatrix} 1-3(k+1) & -9(k+1) \\ k+1 & 1+3(k+1) \end{bmatrix}$.

$$\begin{aligned}
A^{k+1} &= A^k A \\
&= \begin{bmatrix} 1-3k & -9k \\ k & 1+3k \end{bmatrix} A \qquad \text{(Inductive Hypothesis)} \\
&= \begin{bmatrix} 1-3k & -9k \\ k & 1+3k \end{bmatrix} \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} -2-3k & -9-9k \\ k+1 & 4+3k \end{bmatrix} \\
&= \begin{bmatrix} 1-3(k+1) & -9(k+1) \\ k+1 & 1+3(k+1) \end{bmatrix}.
\end{aligned}$$

Thus, $A^{k+1} = \begin{bmatrix} 1-3(k+1) & -9(k+1) \\ k+1 & 1+3(k+1) \end{bmatrix}$.

Hence, $A^n = \begin{bmatrix} 1-3n & -9n \\ n & 1+3n \end{bmatrix}$ holds for all $n \geq 1$ by induction.

(b)
- Base case (n=0): $P(0)$ asserts that $\prod_{i=0}^{i=0}(1+a_i) = 1 + a_0 = 1 + \sum_{i=0}^{i=0} a_i$. This verifies the base case.

- Inductive Hypothesis: For arbitrary $n = k \geq 0$, assume that $P(k)$ is true: $\prod_{i=0}^{i=k}(1+a_i) \geq 1+\sum_{i=0}^{i=k} a_i$.
- Inductive Step: Prove the statement for $n = k+1$: $\prod_{i=0}^{i=k+1}(1+a_i) \geq 1+\sum_{i=0}^{i=k+1} a_i$.

$$\begin{aligned}\prod_{i=0}^{i=k+1}(1+a_i) &= (1+a_{k+1})\prod_{i=0}^{i=k}(1+a_i) \\ &\geq (1+a_{k+1})\left[1+\sum_{i=0}^{i=k} a_i\right]. \qquad \text{(Inductive Hypothesis)}\end{aligned}$$

$$\begin{aligned}\Rightarrow \prod_{i=0}^{i=k+1}(1+a_i) &\geq 1+\sum_{i=0}^{i=k+1} a_i + a_{k+1}\sum_{i=0}^{i=k} a_i \\ &\geq 1+\sum_{i=0}^{i=k+1} a_i. \qquad \text{since} \qquad a_i \leq 0, a_{k+1} \leq 0 \Rightarrow a_i a_{k+1} \geq 0.\end{aligned}$$

Thus, $\prod_{i=0}^{i=k+1}(1+a_i) \geq 1+\sum_{i=0}^{i=k+1} a_i$.
Hence, $\prod_{i=0}^{i=n}(1+a_i) \geq 1+\sum_{i=0}^{i=n} a_i$ holds for all $n \geq 0$ by induction.

2 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0,1\}$.

Solution:

Prove by strong induction on n. (Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.)

The key insight here is that if n is divisible by 2, then it is easy to get a bit string representation of $(n+1)$ from that of n. However, if n is not divisible by 2, then $(n+1)$ will be, and its binary representation will be more easily derived from that of $(n+1)/2$. More formally:

- Base Case: $n = 1$ can be written as 1×2^0.
- Inductive Step: Assume that the statement is true for all $1 \leq m \leq n$, where n is arbitrary. Now, we need to consider $n+1$. If $n+1$ is divisible by 2, then we can apply our inductive hypothesis to $(n+1)/2$ and use its representation to express $n+1$ in the desired form.

$$\begin{aligned}(n+1)/2 &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0 \\ n+1 = 2 \cdot (n+1)/2 &= c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0.\end{aligned}$$

Otherwise, n must be divisible by 2 and thus have $c_0 = 0$. We can obtain the representation of $n+1$ from n as follows:

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 0 \cdot 2^0$$
$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 1 \cdot 2^0$$

Therefore, the statement is true.

Note: In proofs using simple induction, we only use $P(n)$ in order to prove $P(n+1)$. Simple induction gets stuck here because in order to prove $P(n+1)$ in the inductive step, we need to assume more than just $P(n)$. This is because it is not immediately clear how to get a representation for $P(n+1)$ using just $P(n)$, particularly in the case that $n+1$ is divisible by 2. As a result, we assume the statement to be true for all of $1,2,\ldots,n$ in order to prove it for $P(n+1)$.

3 Stable Marriage

The following questions refer to stable marriage instances with n men and n women, answer True/-False or provide an expression as requested.

(a) For $n = 2$, or any 2-man, 2-woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False.)

(b) In any stable marriage instance, in the pairing the Stable Marriage Algorithm produces there is some man who gets his favorite woman (the first woman on his preference list). (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third favorite, and so on, and every woman has the same preference list, how many days does it take for Stable Marriage Algorithm to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorite men, and different second-favorite men, and so on. How many days does the Stable Marriage Algorithm take to finish? (Form of Answer: An expression that may contain n.)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False.)

Solution:

(a) **False.** This says there is only one stable pairing. But if the preference list for man A is $(1,2)$ and for man B is $(2,1)$ and preference list for woman 1 is (B,A) and woman 2 is (A,B) the male and female optimal pairings are different.

(b) **False.** Let man A have preference list $(1,3,2)$, B have $(1,2,3)$, and C have $(2,1,3)$. We develop a "cyclic" chain of preferences, causing A to displace B to displace C who then displaces A.

  (a) If woman 1 prefers A over B, she puts A on a string and rejects B.

  (b) B does not get his favorite and proposes to 2, who prefers B over C and thus rejects C.

  (c) C does not get his favorite and proposes to 1, who prefers C over A and thus rejects A.

Thus, A also does not get his favorite, and no man gets his favorite.

(c) **1**.
On the first day every woman gets exactly one proposal since each man has a different woman in their first position. The algorithm terminates.

(d) **n**.
Every man proposes to their common favorite. One man is kept on a string. The rest propose to the second. And so on. After each day, a new woman gets a man on a string. After n days, we finish. Note that the men's preference lists (assuming they're the same for everyone) were irrelevant.

(e) **True.**
A and 1 are respectively all the women's and men's least favorite. Man A proposes to everyone in his list and gets rejected by all of them until he gets to his last option who is woman 1. On the other, hand no one proposes to woman 1 until the day that man A proposes to her.

4 Universal Preference

Suppose that preferences in a stable marriage instance are universal: all n men share the preferences $W_1 > W_2 > \cdots > W_n$ and all women share the preferences $M_1 > M_2 > \cdots > M_n$.

(a) What pairing do we get from running the algorithm with men proposing? Can you prove this happens for all n?

(b) What pairing do we get from running the algorithm with women proposing?

(c) What does this tell us about the number of stable pairings?

Solution:

(a) The pairing results in (W_i, M_i) for each $i \in \{1, 2, ..., n\}$.

This result can be proved by induction:

Our base case is when $n = 1$, so the only pairing is (W_1, M_1), and thus the base case is trivially true.
Now assume this is true for some $n \in \mathbb{N}$.
On the first day with $n+1$ men and $n+1$ women, all $n+1$ men will propose to W_1. W_1 prefers

M_1 the most, and the rest of the men will be rejected. This leaves a set of n unpaired men and n unpaired women who all have the same preferences (after the pairing of (W_1, M_1)). By the process of induction, this means that every i^{th} preferred woman will be paired with the i^{th} preferred man.

(b) The pairings will again result in (M_i, W_i) for each $i \in \{1, 2, ..., n\}$. This can be proved by induction in the same as above, but replacing "man" with "woman" and vice-versa.

(c) We know that male-proposing produces a female-pessimal stable pairing. We also know that female-proposing produces a female-optimal stable pairing. We found that female-optimal and female-pessimal pairings are the same. This means that there is only one stable pairing, since both the best and worst pairings (for females) are the same pairings.