Pre-Announcements

Blockchain event is today 7 - 10 PM at International House at Chevron House.

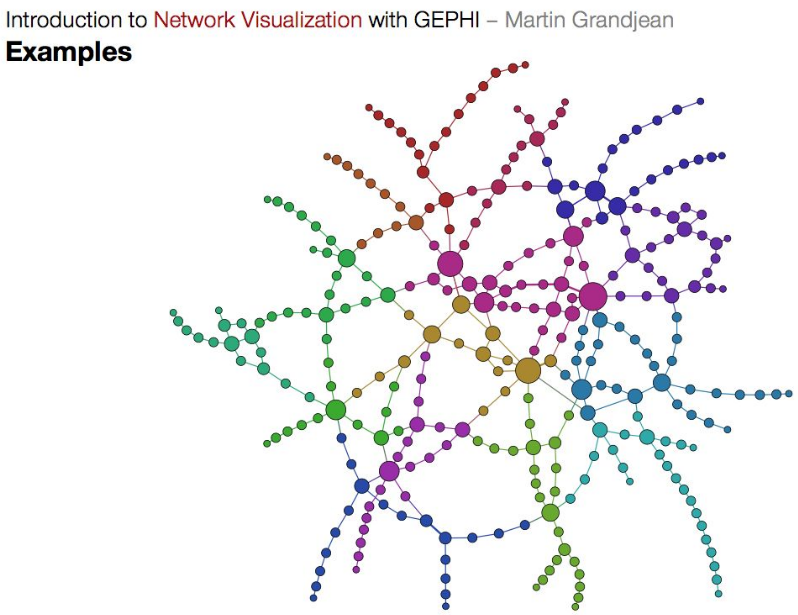
- Lots of fancy people from fancy places will be there.
- Blockchain is a topical concept.
- Pizza is a topical concept.

Flyers available.

Announcements

Exam solution exists, not sure why it’s not posted, but will be posted soon.

- Exam was really hard, but that’s how exams go.
- More later.



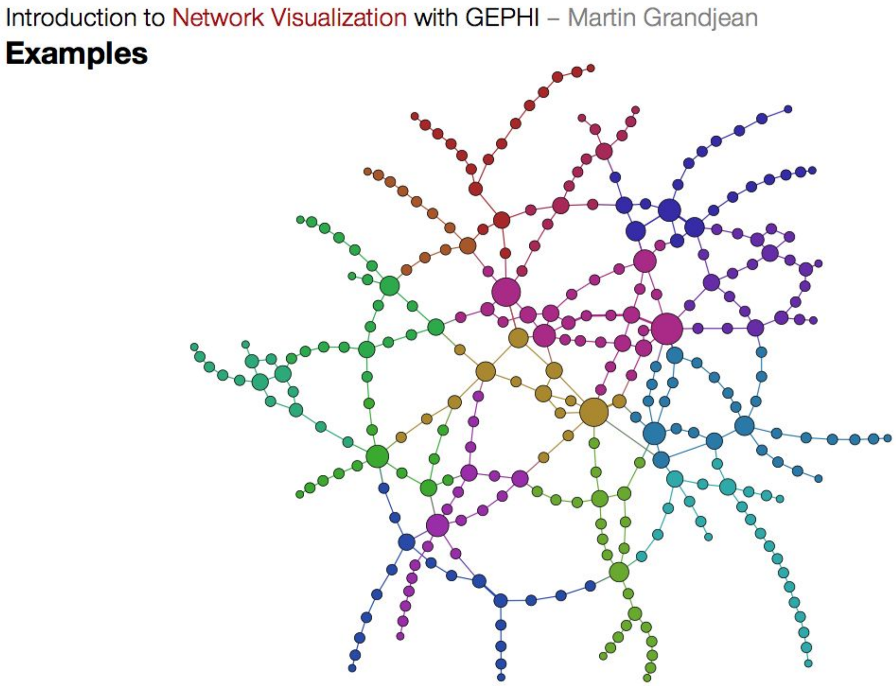
CS61B

Lecture 26: Graphs

- Intro
- Graph Implementations
- Depth First Traversal

Graph

Graph: A set of nodes (a.k.a. vertices) connected pairwise by edges.

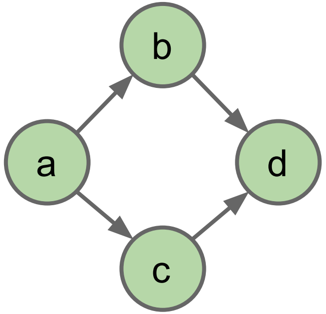


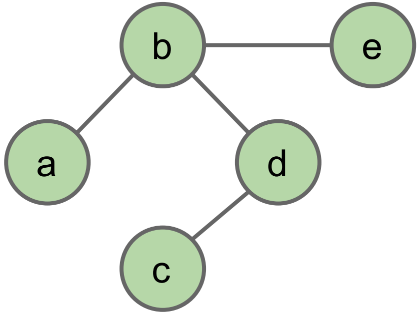
Graph Types

Directed

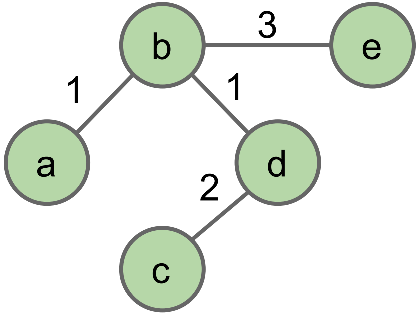
Undirected

Acyclic:

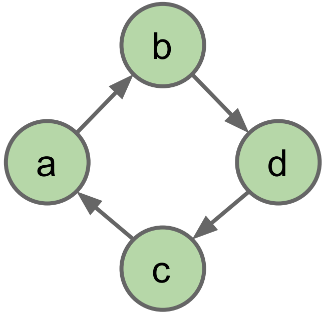


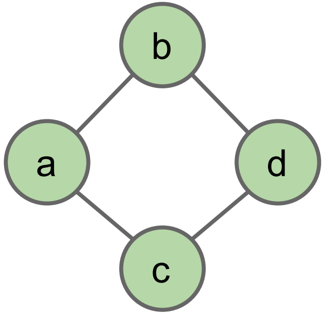


With Edge Labels



Cyclic:





Graph Terminology

- Graph:
  - Set of ***vertices***, a.k.a. ***nodes***.
  - Set of ***edges***: Pairs of vertices.
  - Vertices with an edge between are ***adjacent***.
  - Optional: Vertices or edges may have ***labels*** (or ***weights***).
- A ***path*** is a sequence of vertices connected by edges.
- A ***cycle*** is a path whose first and last vertices are the same.
  - A graph with a cycle is 'cyclic'.
- Two vertices are ***connected*** if there is a path between them. If all vertices are connected, we say the graph is connected.

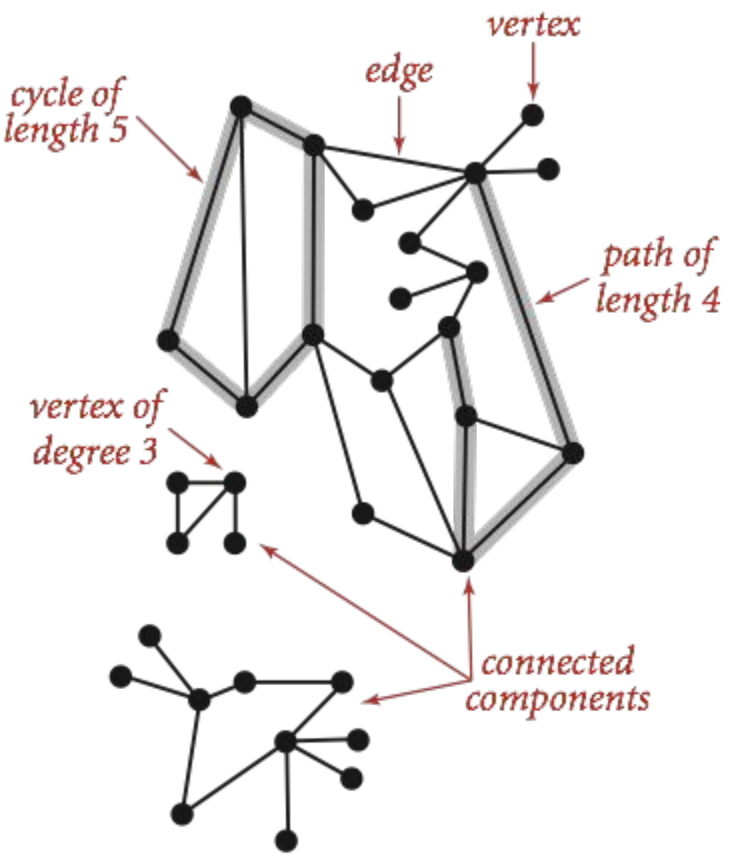


Figure from Algorithms 4th Edition

Some Graph-Processing Problems

s-t Path. Is there a path between vertices s and t?

Shortest s-t Path. What is the shortest path between vertices s and t?

Cycle. Does the graph contain any cycles?

Euler Tour. Is there a cycle that uses every edge exactly once?

Hamilton Tour. Is there a cycle that uses every vertex exactly once?

Connectivity. Is the graph connected, i.e. is there a path between all vertex pairs?

Biconnectivity. Is there a vertex whose removal disconnects the graph?

Planarity. Can you draw the graph on a piece of paper with no crossing edges?

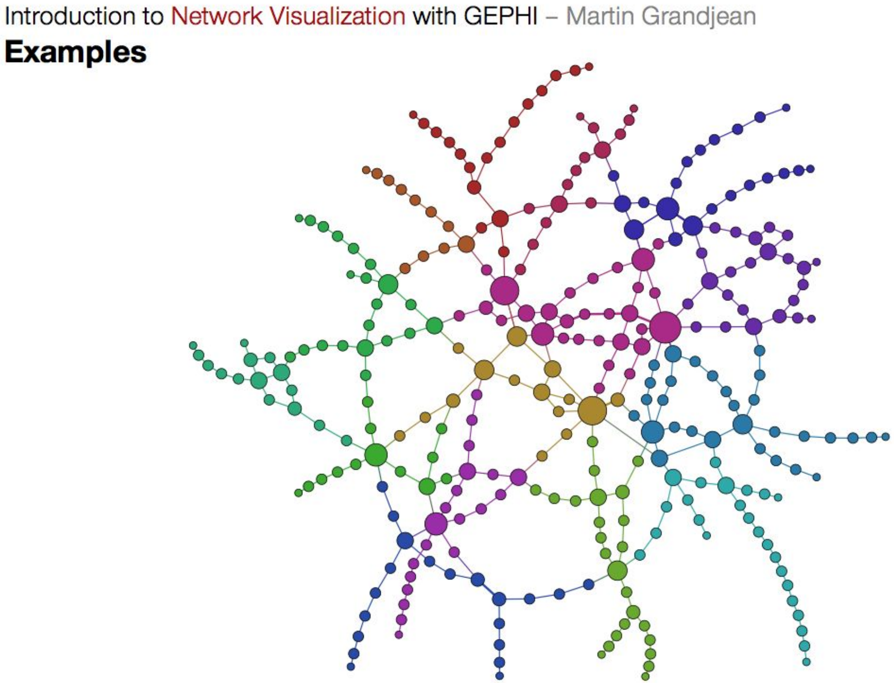
Isomorphism. Are two graphs isomorphic (the same graph in disguise)?

Graph problems: Unobvious which are easy, hard, or computationally intractable.

Graph Example: The Paris Metro

This subway map of Paris is:

- Undirected
- Connected
- Cyclic (not a tree!)
- Vertex-labeled



Graph Example: BART

Is the BART graph a tree?

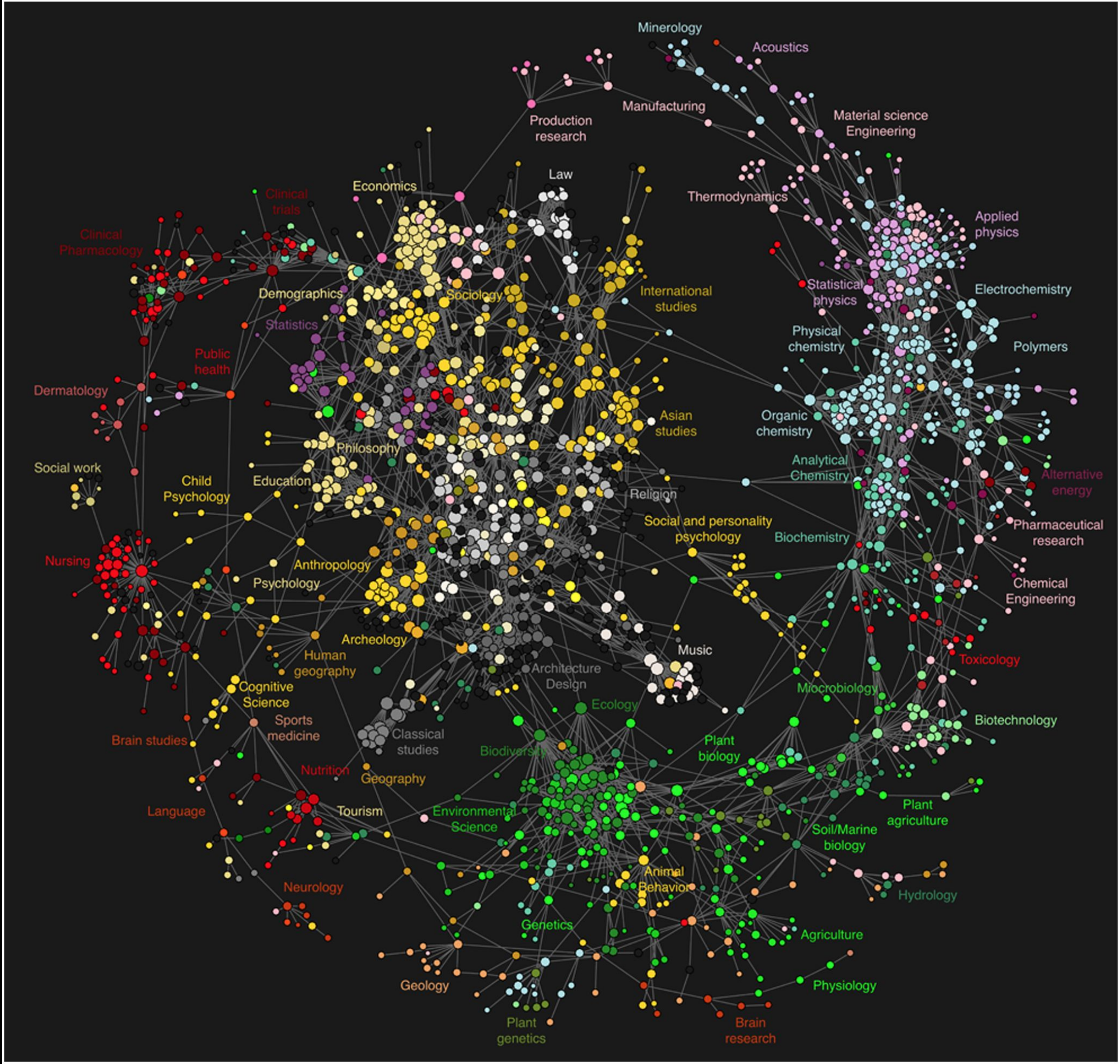
Nodes: Cities. Edge Weights: ~Number of friends between cities

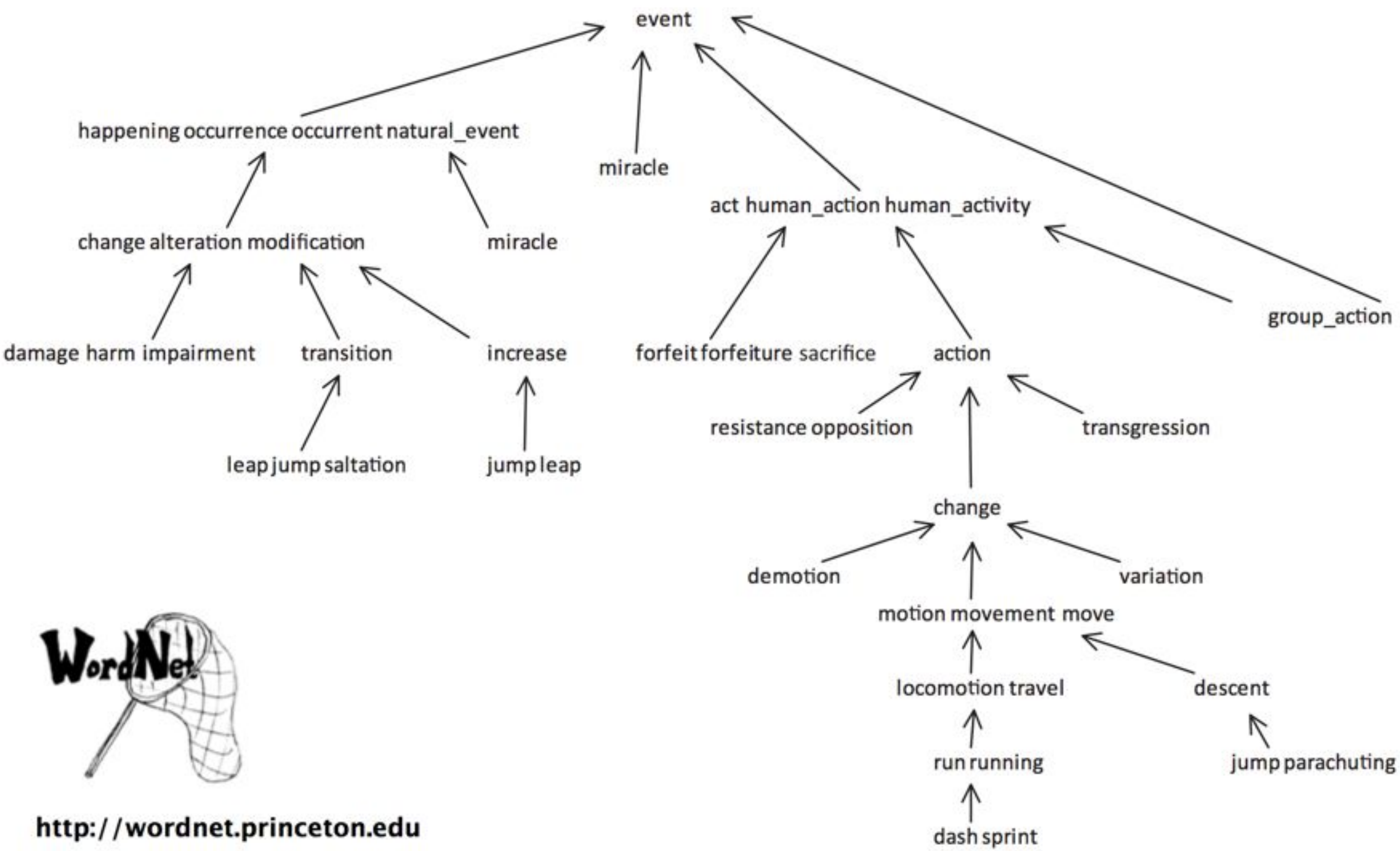
Nodes: Scientific Journals.

- Label: AAT classification (the topic that it covers)

Edges:

- Based on clickthrough data.
- Clickthrough from v to w means that someone reading an article in journal v clicked on a link to an article in journal w.
- Edge assigned from v to w if clickthrough rate from v to w is above some arbitrary threshold.



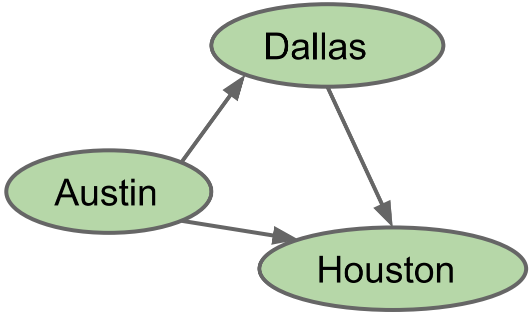


Edge captures ‘is-a-type-of’ relationship. Example: descent is-a-type-of movement.

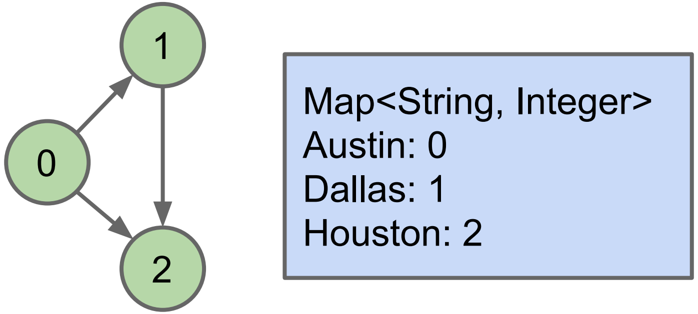
Graph Representations

Common Simplification: Integer Vertices

Common convention: Number nodes irrespective of label, and use number throughout the graph implementation. To lookup a vertex by label, use a Map<Label, Integer>.



Intended graph.



What you get.

Graph API

Using a graph in Java:

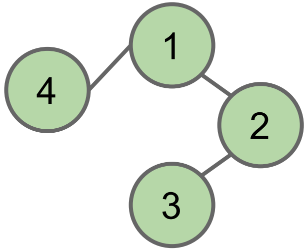
```
public class Graph {
  public Graph(int V):                    Create empty graph with v vertices
  public void addEdge(int v, int w): add an edge v-w
  Iterable<Integer> adj(int v):          vertices adjacent to v
  int V():                                number of vertices
  int E():                                number of edges
...
```

Graph API

Using a graph in Java:

```
public class Graph {
  public Graph(int V):                  Create empty graph with v vertices
  public void addEdge(int v, int w):    add an edge v-w
  Iterable<Integer> adj(int v):         vertices adjacent to v
  int V():                              number of vertices
  int E():                              number of edges
...
```

Example client:



degree(G, 2) = 2

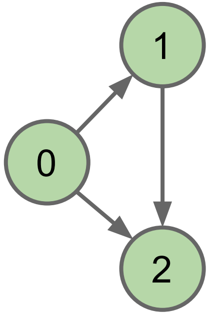
```
/** degree of vertex v in graph G */
public static int degree(Graph G, int v) {
    int degree = 0;
    for (int w : G.adj(v)) {
        degree += 1;
    }
    return degree; }
```

(degree = # edges)

Graph Representations

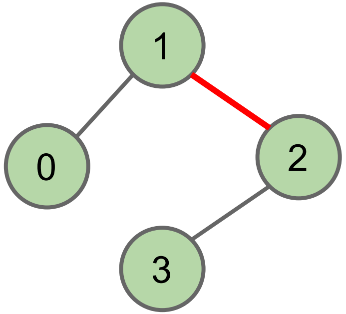
- Representation 1: Adjacency Matrix.

s \ t	0	1	2
0	0	1	1
1	0	0	1
2	0	0	0



For undirected graph: Each edge is represented twice in the matrix. Simplicity at the expense of space.

v \ w	0	1	2	3
0	0	1	0	0
1	1	0	**1**	0
2	0	**1**	0	1
3	0	0	1	0



Graph Printing Runtime: http://yellkey.com/paper

What is the order of growth of the running time of the following code if the graph uses an adjacency-matrix representation, where V is the number of vertices, and E is the total number of edges?

A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V*E)$

```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

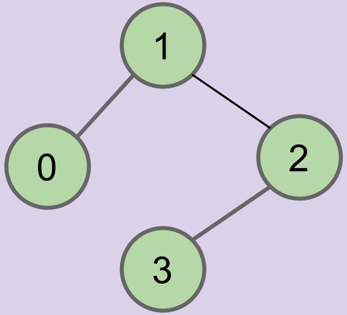
What is the runtime of the for-each?

-

How many times is the for-each run?

-

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0



Graph Printing Runtime: http://yellkey.com/paper

What is the order of growth of the running time of the following code if the graph uses an adjacency-matrix representation, where V is the number of vertices, and E is the total number of edges?

A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\boldsymbol{\Theta(V^2)}$
D. $\Theta(V*E)$

```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

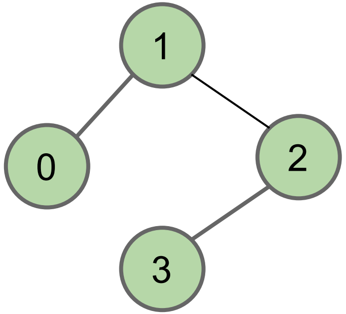
What is the runtime of the for-each?

- $\Theta(V)$.

How many times is the for-each run?

- V times.

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0



Graph Printing Runtime: http://yellkey.com/paper

What is the order of growth of the running time of the following code if the graph uses an adjacency-matrix representation, where V is the number of vertices, and E is the total number of edges?

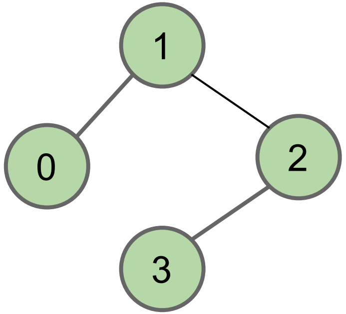
A. Θ(V)
B. Θ(V + E)
C. Θ(V^2)
D. Θ(V*E)

```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

What does G.adj(1) return?

- An iterator with next() = 0, then next() = 2.

	0	1	2	3
0	0	1	0	0
1	1	0	1	0
2	0	1	0	1
3	0	0	1	0

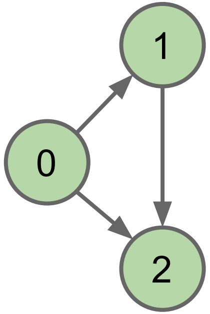


More Graph Representations

Representation 2: Edge Sets: Collection of all edges.

- Example: HashSet<Edge>, where each Edge is a pair of ints.

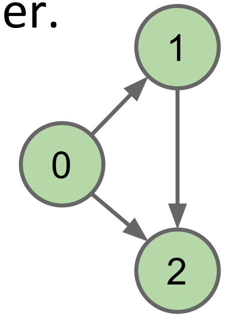
```
{(0, 1), (0, 2), (1, 2)}
```

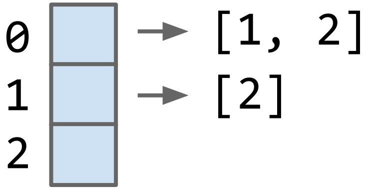


More Graph Representations

Representation 3: Adjacency lists.

- Common approach: Maintain array of lists indexed by vertex number.
- Most popular approach for representing graphs.





Graph Printing Runtime: http://shoutkey.com/laugh

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges?

A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V^*E)$

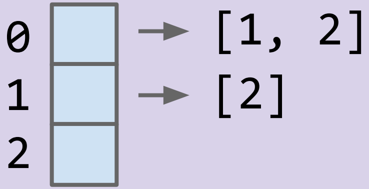
```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

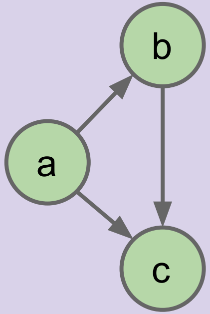
What is the runtime of the for-each?

-

How many times is the for-each run?

-





Graph Printing Runtime: http://shoutkey.com/laugh

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges? **Best case: Θ(V)** **Worst case: Θ(V²)**

A. Θ(V)
B. Θ(V + E)
C. Θ(V²)
D. Θ(V*E)

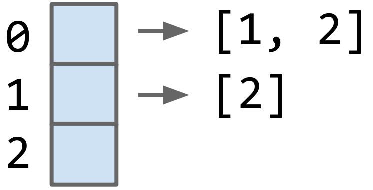
```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

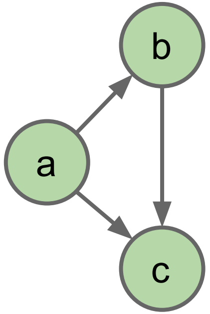
What is the runtime of the for-each? List can be between 1 and V items.

- Ω(1), O(V).

How many times is the for-each run?

- V.





Graph Printing Runtime: http://shoutkey.com/ready

What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges? **Best case: Θ(V) Worst case: Θ(V²)**

A. Θ(V)
B. Θ(V + E)
C. Θ(V²)
D. Θ(V*E)

?
?
?
?
?

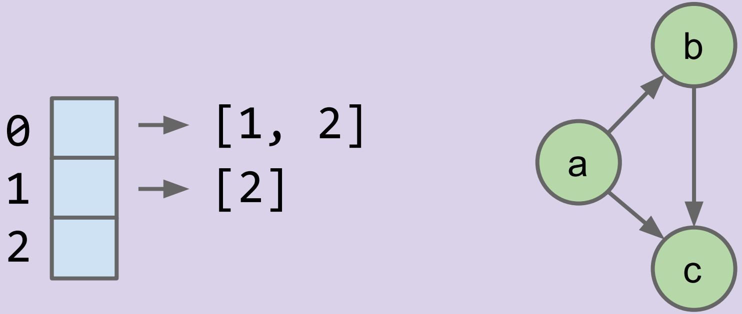
```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

What is the runtime of the for-each? List can be between 1 and V items.

- Ω(1), O(V).

How many times is the for-each run?

- V.



Graph Printing Runtime: http://shoutkey.com/ready

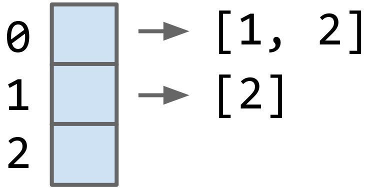
What is the order of growth of the running time of the following code if the graph uses an ***adjacency-list*** representation, where V is the number of vertices, and E is the total number of edges?

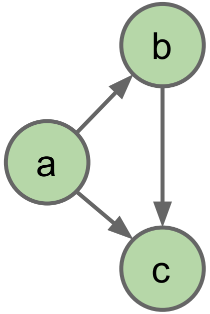
A. $\Theta(V)$
B. $\Theta(V + E)$
C. $\Theta(V^2)$
D. $\Theta(V*E)$

```java
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

Best case: $\Theta(V)$ Worst case: $\Theta(V^2)$

All cases: $\Theta(V + E)$





Graph Printing Runtime: http://shoutkey.com/ready

Runtime: $\Theta(V + E)$

V is total number of vertices.

E is total number of edges in the entire graph.

```
for (int v = 0; v < G.V(); v++) {
    for (int w : G.adj(v)) {
        System.out.println(v + "-" + w);
    }
}
```

How to interpret: No matter what "shape" of increasingly complex graphs we generate, as V and E grow, the runtime will always grow exactly as $\Theta(V + E)$.

- Example shape 1: Very sparse graph where E grows very slowly, e.g. every vertex is connected to its square: 2 - 4, 3 - 9, 4 - 16, 5 - 25, etc.
  - E is $\Theta(\text{sqrt}(V))$. Runtime is $\Theta(V + \text{sqrt}(V))$, which is just $\Theta(V)$.
- Example shape 2: Very dense graph where E grows very quickly, e.g. every vertex connected to every other.
  - E is $\Theta(V^2)$. Runtime is $\Theta(V + V^2)$, which is just $\Theta(V^2)$.

Graph Representations

Runtime of some basic operations for each representation:

idea	addEdge(s, t)	for(w : adj(v))	printgraph()	hasEdge(s, t)	space used
adjacency matrix	Θ(1)	Θ(V)	$\Theta(V^2)$	Θ(1)	$\Theta(V^2)$
list of edges	Θ(1)	Θ(E)	Θ(E)	Θ(E)	Θ(E)
adjacency list	Θ(1)	Θ(1) to Θ(V)	Θ(V+E)	Θ(degree(v))	Θ(E+V)

Note: These operations are not part of the `Graph` class's API.

In practice, adjacency lists are most common.

- Many graph algorithms rely heavily on `adj(s)`.
- Most graphs are sparse (not many edges in each bucket).

Bare-Bones Undirected Graph Implementation

```
public class Graph {
    private final int V;  private List<Integer>[] adj;

    public Graph(int V) {
        this.V = V;
        adj = (List<Integer>[]) new ArrayList[V];
        for (int v = 0; v < V; v++) {
            adj[v] = new ArrayList<Integer>();
        }
    }

    public void addEdge(int v, int w) {
        adj[v].add(w);   adj[w].add(v);
    }

    public Iterable<Integer> adj(int v) {
        return adj[v];
    }
}
```

Cannot create array of anything involving generics, so have to use weird cast as with project 1A.

Depth-First Traversal

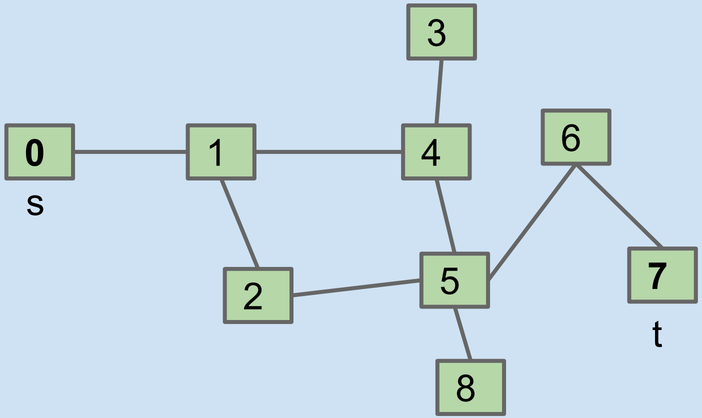
Maze Traversal / s-t Path

Suppose we want to know if there exists a path from vertex s=0 to vertex t=7. What is wrong with the following recursive algorithm for connected(s, t)?

- Does s == t? If so, return true.
- Otherwise, check all of s's children for connectivity to t.

Example:

- connected(0, 7):
  - Does 0 == 7? No, so...
  - if (connected(1, 7)) return true;
  - return false;



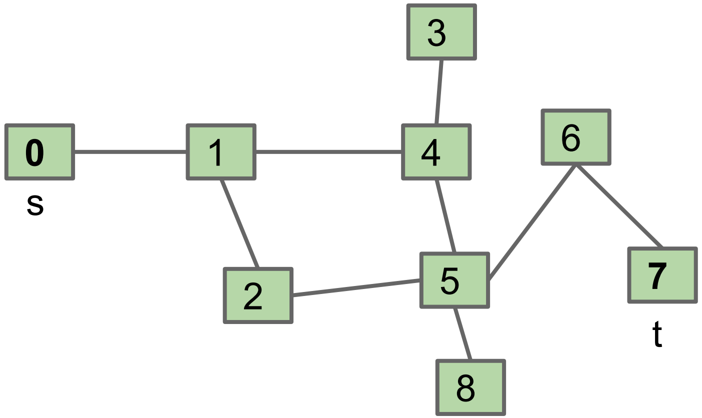
- connected(1, 7): ...

Improving Our Connectivity Algorithm

Goal: Search for a path from s to t, but visit each vertex at most once. To do this, we can mark each vertex as we search. Resulting algorithm for connected(s, t) is as follows:

- Mark s.
- Does s == t? If so, return true.
- Check all of s's unmarked neighbors for connectivity to t.

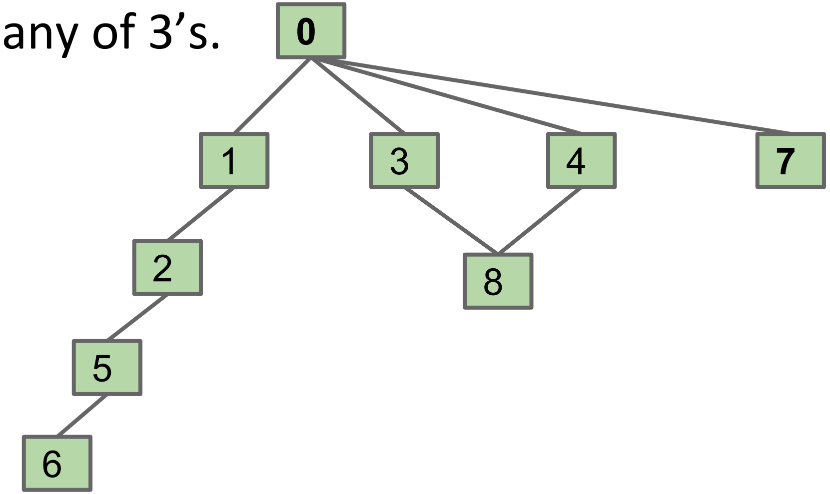
Recursive connectivity demo.

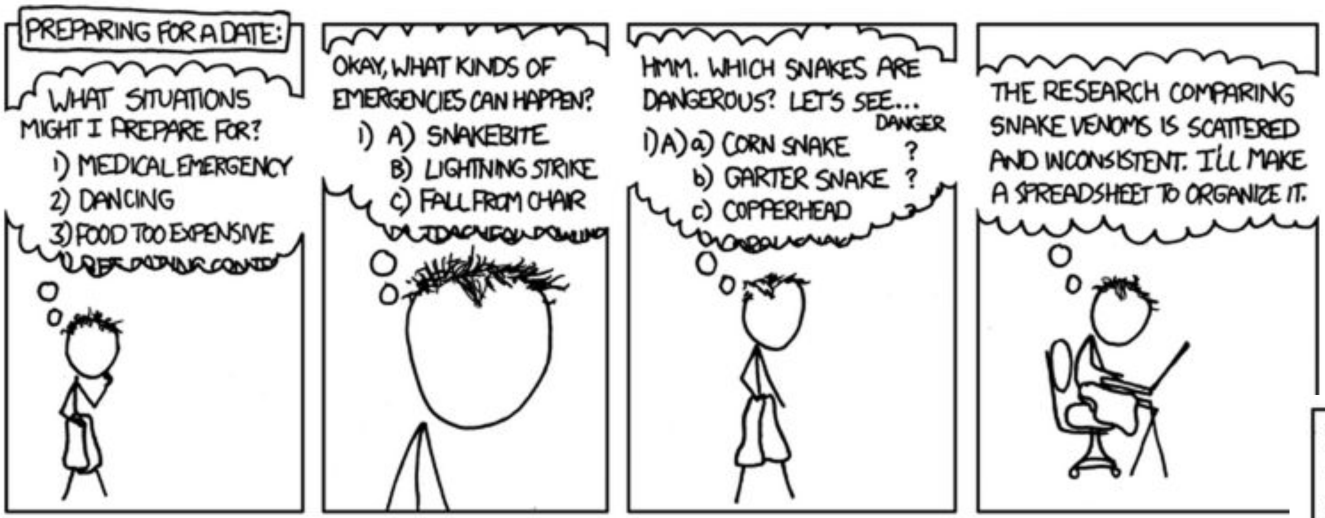


Depth First Traversal

This idea of exploring the entire subgraph for each child is known as Depth First Traversal.

- Ex. Visit all of 1's children before we visit any of 3's.



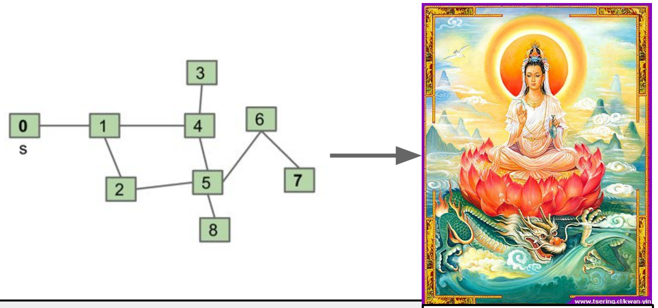


Or a more visceral example: https://xkcd.com/761/

Depth First Search Implementation

Common design pattern in graph algorithms: Decouple type from processing algorithm.

- Create a graph object.
- Pass the graph to a graph-processing method (or constructor) in a client class.
- Query the client class for information.



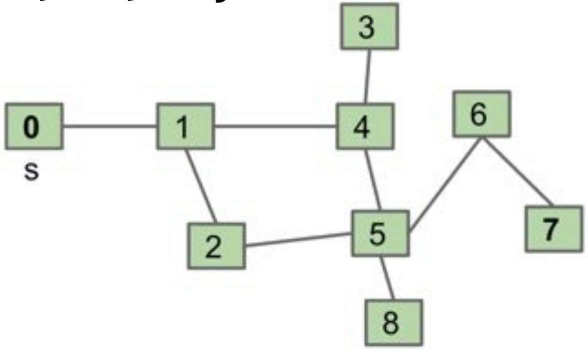
Paths.java

```
public class Paths {
    public Paths(Graph G, int s):    Find all paths from G
    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

Example Usage

Start by calling: `Paths P = new Paths(G, 0);`

- `P.hasPathTo(3); //returns true`
- `P.pathTo(3); //returns {0, 1, 4, 3}`



Paths.java

```
public class Paths {
    public Paths(Graph G, int s):    Find all paths from G
    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

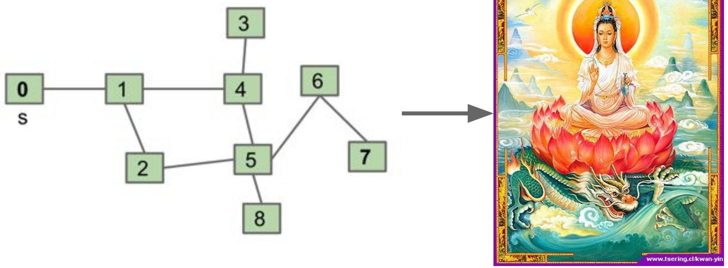
Implementing Paths With Depth First Search

To visit a vertex v:

- Mark vertex v.
- Recursively visit all unmarked vertices adjacent to v.

Data Structures:

- `boolean[] marked`
- `int[] edgeTo`
    - `edgeTo[4] = 1`, means we went from 1 to 4.



Paths.java

```
public class Paths {
    public Paths(Graph G, int s):    Find all paths from G
    boolean hasPathTo(int v):        is there a path from s to v?
    Iterable<Integer> pathTo(int v): path from s to v (if any)
}
```

DepthFirstPaths

Demo: DepthFirstPaths

DepthFirstPaths, Recursive Implementation

```
public class DepthFirstPaths {
    private boolean[] marked;
    private int[] edgeTo;
    private int s;

    public DepthFirstPaths(Graph G, int s) {
        ...
        dfs(G, s);
    }

    private void dfs(Graph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v)) {
        if (!marked[w]) {
            edgeTo[w] = v;
            dfs(G, w);
        }
    }
}
```

marked[v] is true iff v connected to s

edgeTo[v] is previous vertex on path from s to v

not shown: data structure initialization

find vertices connected to s.

recursive routine does the work and stores results in an easy to query manner!

Question: How would we write hasPathTo(v)?

DepthFirstPaths Summary

Demo: [DepthFirstPaths]

Properties of Depth First Search:

- Guaranteed to reach every node.
- Runs in $O(V + E)$ time.
  - Analysis next time, but basic idea is that every edge is used at most once, and total number of vertex considerations is equal to number of edges.
  - Runtime may be faster than $\Theta(V+E)$ for problems which quit early on some stopping condition (for example connectivity).