

Announcements

Upcoming Deadlines:

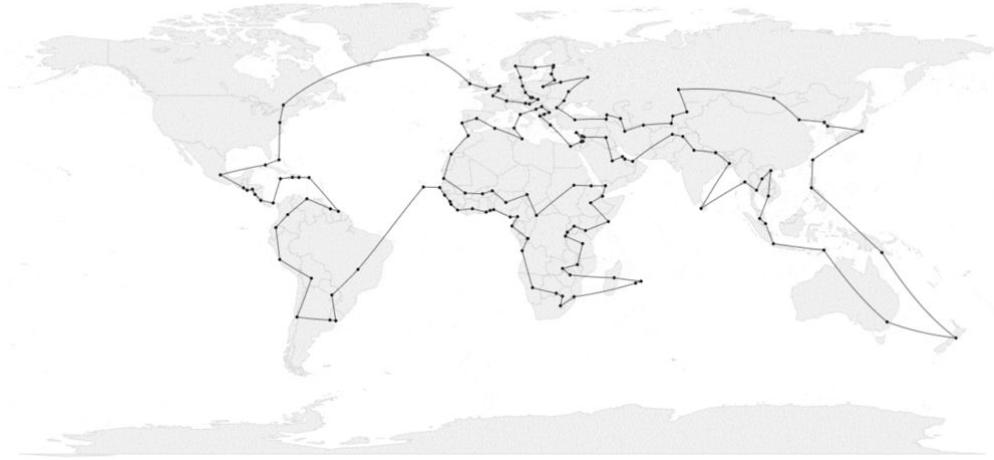
- Project 2 Phase 2 due March 5th.
- Hope it's been fun and not too stressful.

There's a live lecture questions thread today.

Lab:

- This week will be working on project 2 (free points).
- Labs next will be project 2 demos.
 - We will ask you to provide a gradescope link to the submission you want to demo.
 - If submitted after the project 2 phase 2 deadline, we'll deduct late points (but only from the demo part).

Distance: 80,652 miles
Temperature: 2
Iterations: 1,000,000



CS61B

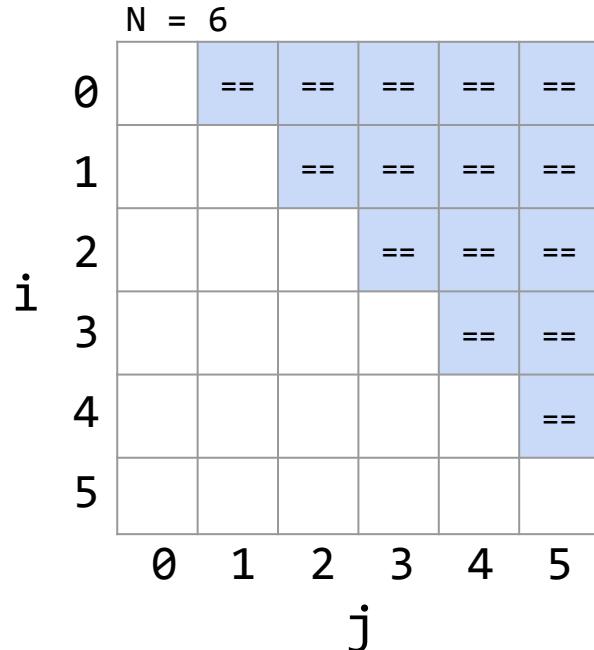
Lecture 18: Asymptotics II: Analysis of Algorithms

- Review of Asymptotic Notation
- Examples 1-2: For Loops
- Example 3: A Basic Recurrence
- Example 4: Binary Search
- Example 5: Mergesort

Example 1/2:For Loops

Loops Example 1: Based on Exact Count

Find the order of growth of the worst case runtime of dup1.



```
int N = A.length;
for (int i = 0; i < N; i += 1)
    for (int j = i + 1; j < N; j += 1)
        if (A[i] == A[j])
            return true;
return false;
```

Worst case number of `==` operations:

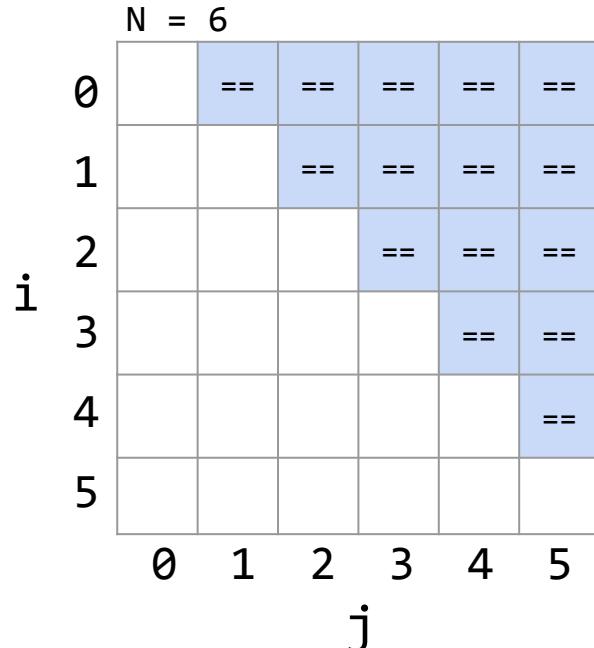
$$C = 1 + 2 + 3 + \dots + (N - 3) + (N - 2) + (N - 1) = N(N-1)/2$$

operation	worst case count
<code>==</code>	$\Theta(N^2)$

Worst case runtime: $\Theta(N^2)$

Loops Example 1: Simpler Geometric Argument

Find the order of growth of the worst case runtime of dup1.



```
int N = A.length;
for (int i = 0; i < N; i += 1)
    for (int j = i + 1; j < N; j += 1)
        if (A[i] == A[j])
            return true;
return false;
```

Worst case number of == operations:

- Given by area of right triangle of side length N-1.
- Area is $\Theta(N^2)$.

operation	worst case count
==	$\Theta(N^2)$

Worst case runtime: $\Theta(N^2)$

Loops Example 2 [attempt #1]: <http://yellkey.com/nature>

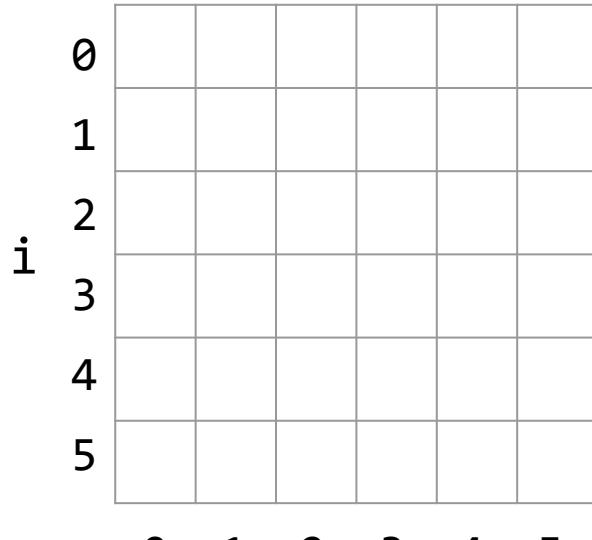
Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$. By simple, we mean there should be no unnecessary multiplicative constants or additive terms.

```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

- A. 1
- B. $\log N$
- C. N
- D. $N \log N$
- E. N^2
- F. Other

Note that there's only one case for this code and thus there's no distinction between "worst case" and otherwise.

Loops Example 2: Prelude to Attempt #2



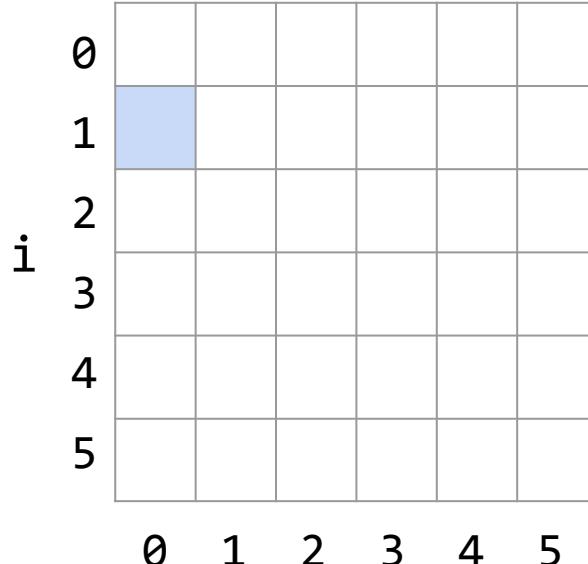
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?

Loops Example 2: Prelude to Attempt #2



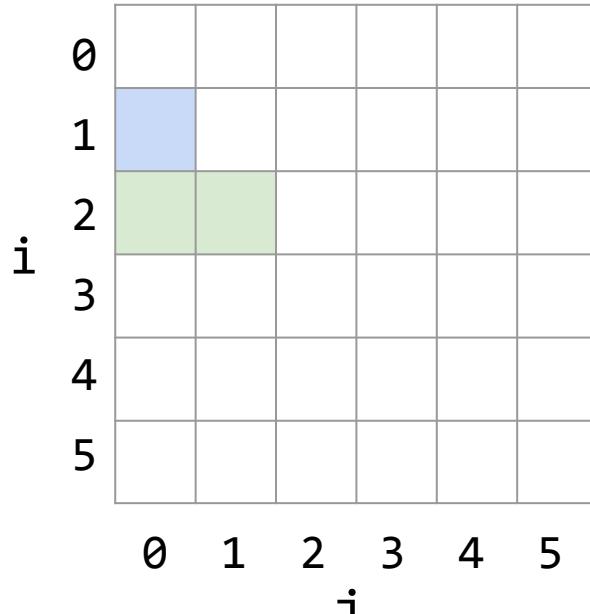
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1																	

Loops Example 2: Prelude to Attempt #2



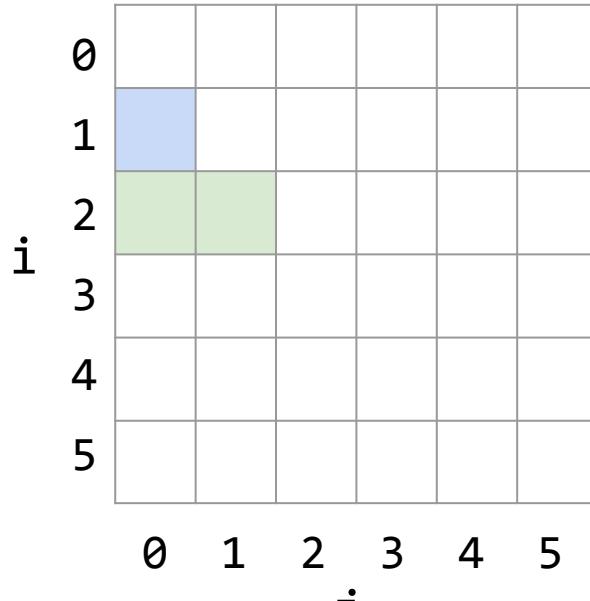
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3																

Loops Example 2: Prelude to Attempt #2



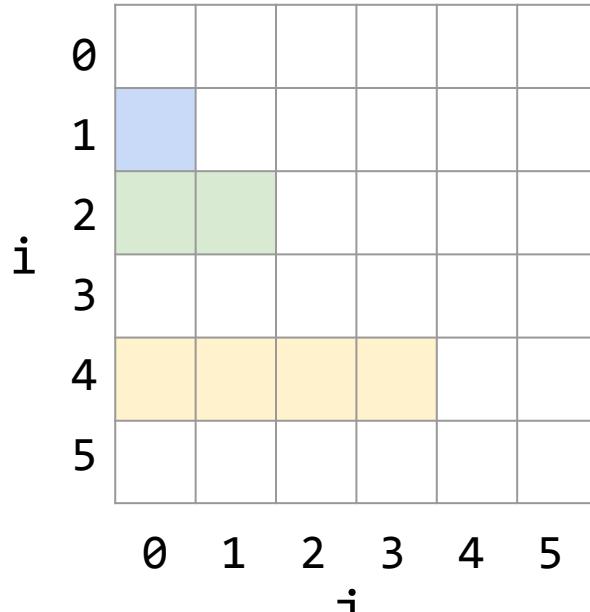
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3	3															
	N=2 and 3 both print 3 times																	

Loops Example 2: Prelude to Attempt #2



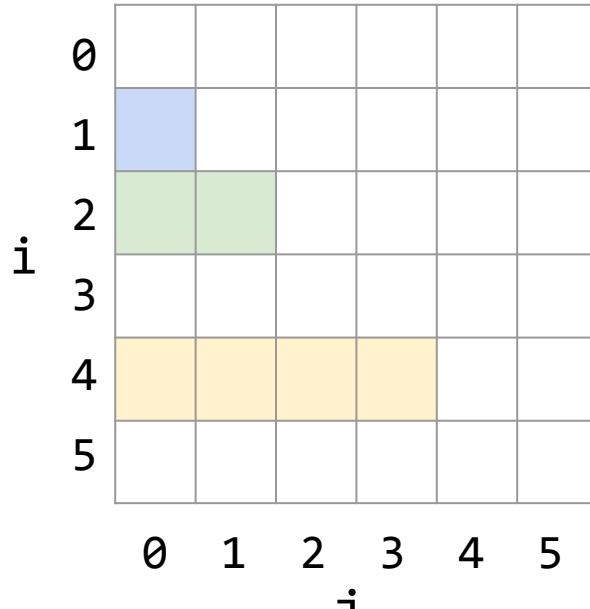
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3	3	7														

Loops Example 2: Prelude to Attempt #2



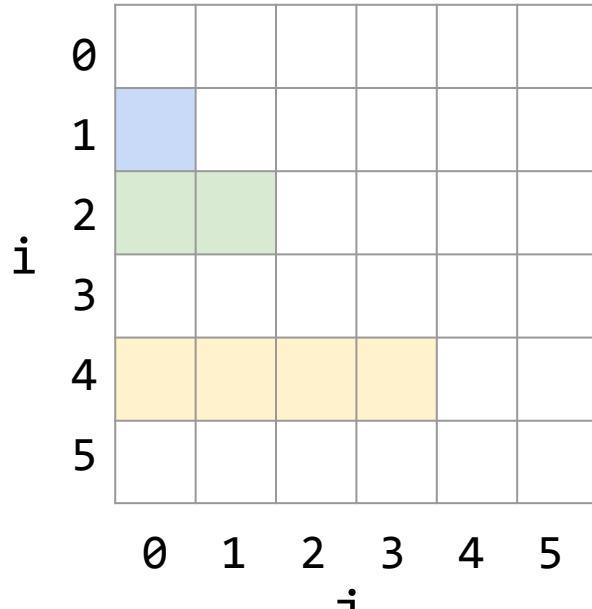
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3	3	7	7	7	7											
	$\underbrace{\hspace{14em}}$ $N=4,5,6,7$ all print 7 times																	

Loops Example 2: Prelude to Attempt #2



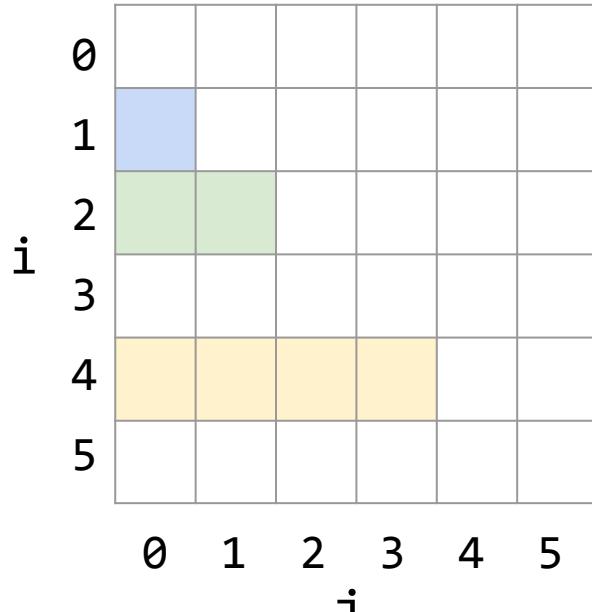
```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C(N)	1	3	3	7	7	7	7	15	15	15	15	15	15	15	15			
These N all print 15 times																		

Loops Example 2: Prelude to Attempt #2



```
public static void printParty(int N) {  
    for (int i = 1; i <= N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3	3	7	7	7	7	15	15	15	15	15	15	15	15	31	31	31

$$C(N) = 1 + 2 + 4 + \dots + N, \text{ if } N \text{ is a power of 2}$$

Loops Example 2 [attempt #2]: <http://yellkey.com/question>

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

- A. 1
- B. $\log N$
- C. N
- D. $N \log N$
- E. N^2
- F. Other

```
public static void printParty(int N) {  
    for (int i = 1; i<=N; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;
```

Cost model $C(N)$, `println("hello")` calls:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$C(N)$	1	3	3	7	7	7	7	15	15	15	15	15	15	15	15	31	31	31

$$C(N) = 1 + 2 + 4 + \dots + N, \text{ if } N \text{ is a power of 2}$$

Loops Example 2: Prelude to Attempt #3

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

N	C(N)	0.5 N	2N
1	1	0.5	2
4	$1 + 2 + 4 = 7$	2	14
7	$1 + 2 + 4 = 7$	3.5	14
8	$1 + 2 + 4 + 8 = 15$	4	16
27	$1 + 2 + 4 + 8 + 16 = 31$	13.5	54
185	$\dots + 64 + 128 = 255$	92.5	370
715	$\dots + 256 + 512 = 1023$	357.5	1430

Loops Example 2 [attempt #3]: <http://yellkey.com/article>

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

N	C(N)	$0.5 N$	$2N$
1	1	0.5	2
4	7	2	14
7	7	3.5	14
8	15	4	16
27	31	13.5	54
185	255	92.5	370
715	1023	357.5	1430

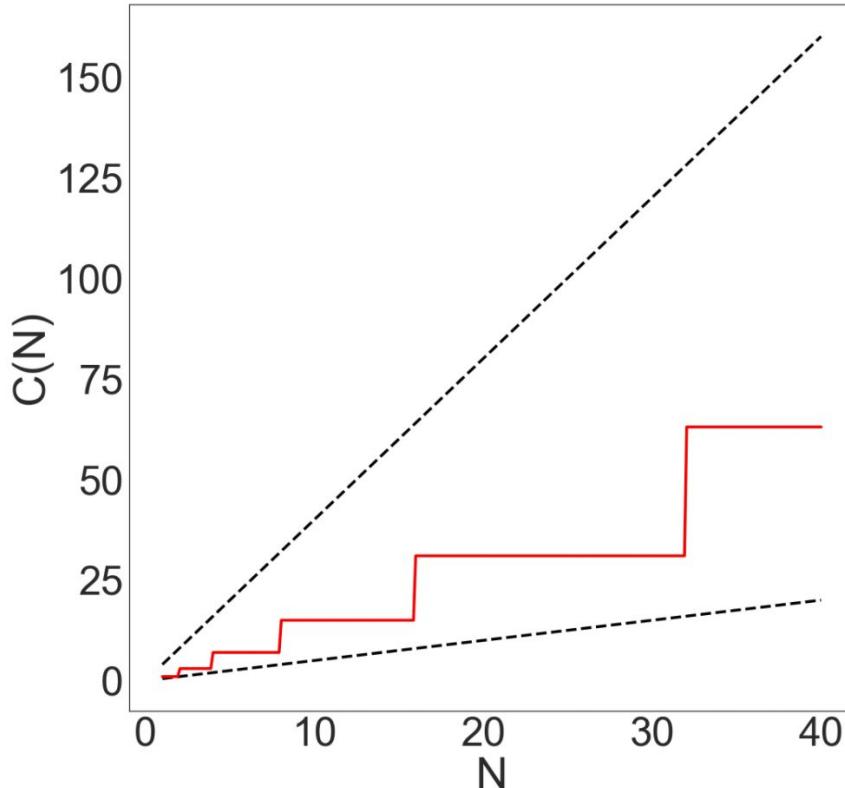
```
public static void printParty(int n) {  
    for (int i = 1; i<=n; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;  
        }  
    }  
}
```

Cost model $C(N)$, `println("hello")` calls:

- $R(N) = \Theta(1 + 2 + 4 + 8 + \dots + N)$ if N is power of 2.
- | | |
|-------------|-------------------|
| A. 1 | D. $N \log N$ |
| B. $\log N$ | E. N^2 |
| C. N | F. Something else |

Loops Example 2 [attempt #3]: <http://shoutkey.com/TBA>

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.



$$\begin{aligned}R(N) &= \Theta(1 + 2 + 4 + 8 + \dots + N) \\&= \Theta(N)\end{aligned}$$

- | | |
|-------------|-------------------|
| A. 1 | D. $N \log N$ |
| B. $\log N$ | E. N^2 |
| C. N | F. Something else |

Can also compute exactly:

- $1 + 2 + 4 + \dots + N = 2N - 1$
- Ex: If $N = 8$
 - LHS: $1 + 2 + 4 + 8 = 15$
 - RHS: $2 \cdot 8 - 1 = 15$

Repeat After Me...

There is no magic shortcut for these problems (well... usually)

- Runtime analysis often requires careful thought.
- CS70 and especially CS170 will cover this in much more detail.
- This is not a math class, though we'll expect you to know these:
 - $1 + 2 + 3 + \dots + Q = Q(Q+1)/2 = \Theta(Q^2)$ ← Sum of First Natural Numbers ([Link](#))
 - $1 + 2 + 4 + 8 + \dots + Q = 2Q - 1 = \Theta(Q)$ ← Sum of First Powers of 2 ([Link](#))

Where Q is a power of 2.

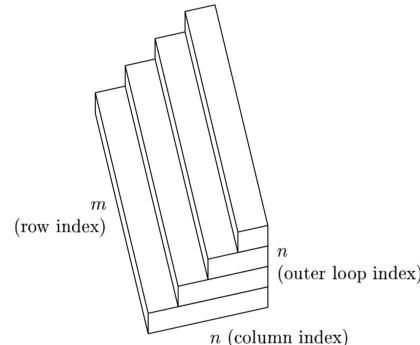
```
public static void printParty(int n) {  
    for (int i = 1; i <= n; i = i * 2) {  
        for (int j = 0; j < i; j += 1) {  
            System.out.println("hello");  
            int ZUG = 1 + 1;
```

Repeat After Me...

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- Runtime analysis often requires careful thought.
- CS70 and especially CS170 will cover this in much more detail.
- This is not a math class, though we'll expect you to know these:
 - $1 + 2 + 3 + \dots + Q = Q(Q+1)/2 = \Theta(Q^2)$ ← Sum of First Natural Numbers ([Link](#))
 - $1 + 2 + 4 + 8 + \dots + Q = 2Q - 1 = \Theta(Q)$ ← Sum of First Powers of 2 ([Link](#))
- Strategies:
 - Find exact sum.
 - Write out examples.
 - Draw pictures.

QR decomposition runtime,
from “Numerical Linear
Algebra” by Trefethen.



The $m \times n$ rectangle at the bottom corresponds to the first pass through the outer loop, the $m \times (n - 1)$ rectangle above it to the second pass, and so on.

Example 3: Recursion

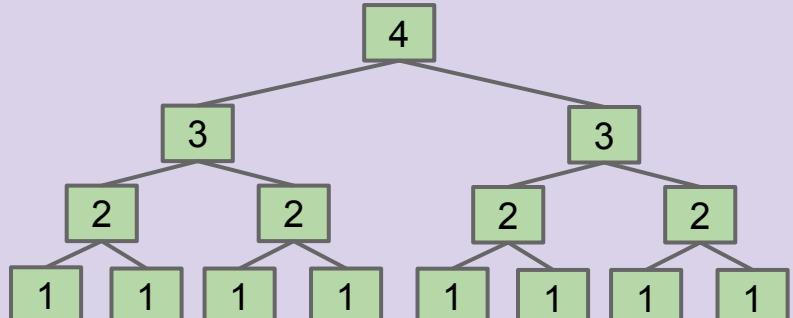
Recursion (Intuitive): <http://yellkey.com/sound>

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Using your intuition, give the order of growth of the runtime of this code as a function of N ?

- A. 1
- B. $\log N$
- C. N
- D. N^2
- E. 2^N



Recursion (Intuitive)

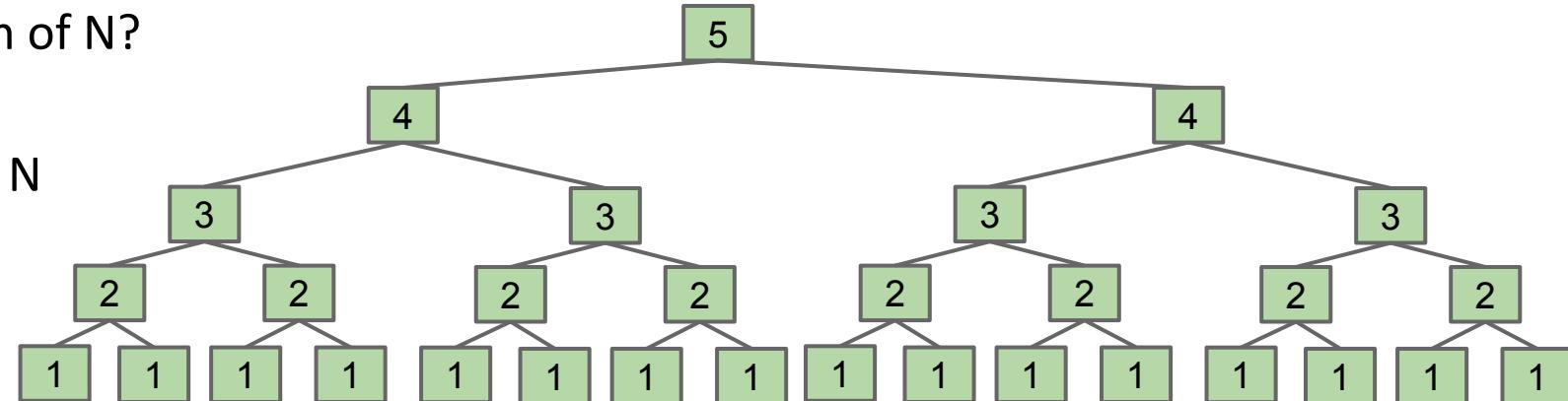
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```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

2^N : Every time we increase N by 1, we double the work!

Using your intuition, give the order of growth of the runtime of this code as a function of N ?

- A. 1
- B. $\log N$
- C. N
- D. N^2
- E. 2^N



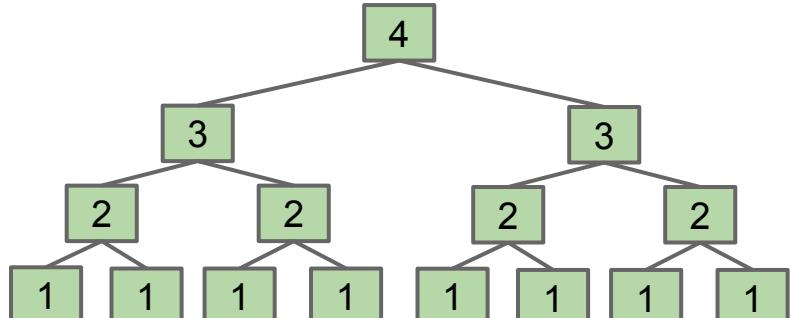
Recursion and Exact Counting

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(1) = 1$
- $C(2) = 1 + 2$
- $C(3) = 1 + 2 + 4$



Recursion and Exact Counting: <http://yellkey.com/full>

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

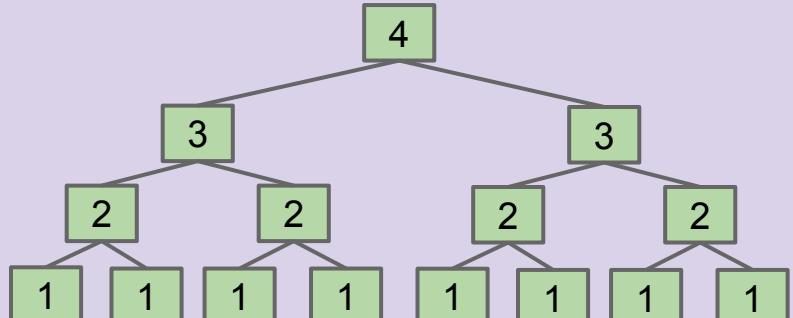
```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(3) = 1 + 2 + 4$
- $C(N) = 1 + 2 + 4 + \dots + ???$

What is the final term of the sum?

- | | |
|------------|----------------|
| A. N | D. 2^{N-1} |
| B. 2^N | E. $2^{N-1}-1$ |
| C. 2^N-1 | |



Recursion and Exact Counting

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

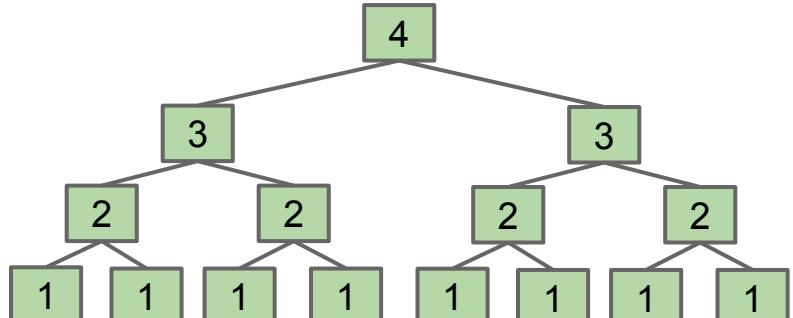
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    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(3) = 1 + 2 + 4$
- $C(N) = 1 + 2 + 4 + \dots + ???$

What is the final term of the sum?

D. 2^{N-1}



Recursion and Exact Counting

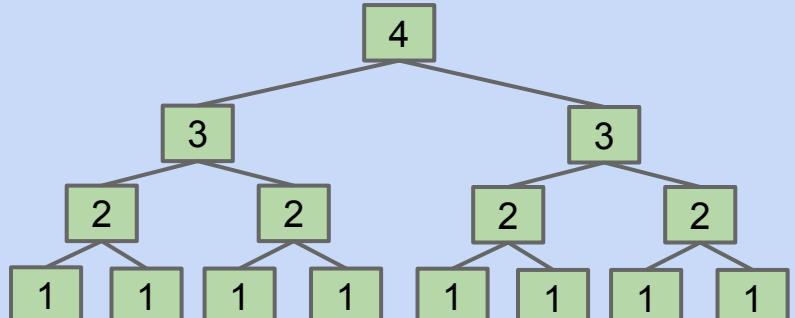
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```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(N) = 1 + 2 + 4 + \dots + 2^{N-1}$

Give a simple expression for $C(N)$.



Recursion and Exact Counting

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

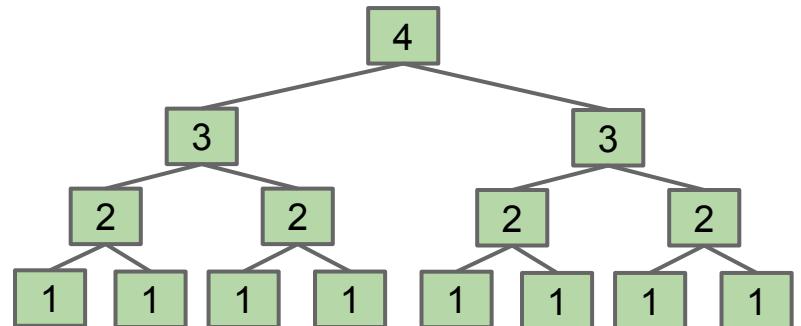
```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(N) = 1 + 2 + 4 + \dots + 2^{N-1}$

Give a simple expression for $C(N)$.

- $C(N) = 2^N - 1$
- Why? It's the **Sum of First Powers of 2**.
 - See next slide for details.



Recursion and Exact Counting, Solving for C(N)

$$C(N) = 1 + 2 + 4 + 8 + \dots + 2^{N-1}$$

We know that the **Sum of the First Powers of 2** from before, i.e. as long as Q is a power of 2, then:

$$1 + 2 + 4 + 8 + \dots + Q = 2Q - 1$$

Thus, since $Q = 2^{N-1}$, we have that:

$$C(N) = 2Q - 1 = 2(2^{N-1}) - 1 = 2^N - 1$$

Recursion and Exact Counting

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

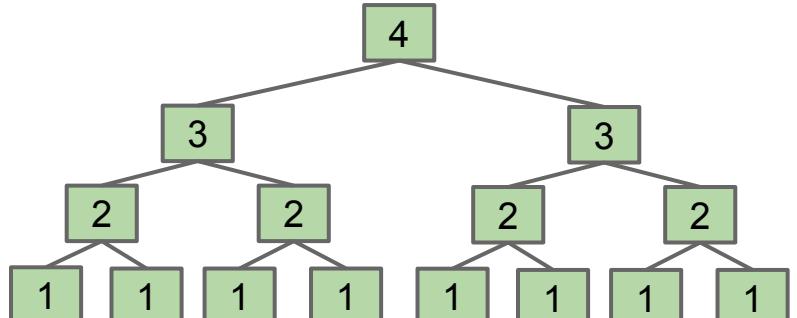
```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1);  
}
```

Another approach: Count number of calls to $f3$, given by $C(N)$.

- $C(N) = 1 + 2 + 4 + \dots + 2^{N-1}$
- Solving, we get $C(N) = 2^N - 1$

Since work during each call is constant:

- $R(N) = \Theta(2^N)$



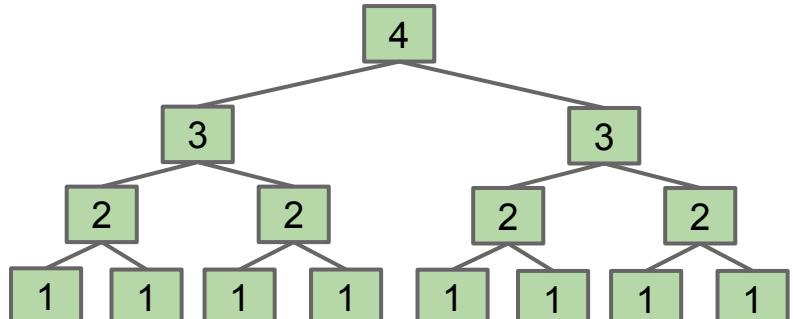
Recursion and Recurrence Relations

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1)  
}
```

A third approach: Count number of calls to $f3$, given by a “recurrence relation” for $C(N)$.

- $C(1) = 1$
- $C(N) = 2C(N-1) + 1$



Recursion and Recurrence Relations (Extra, Outside 61B Scope)

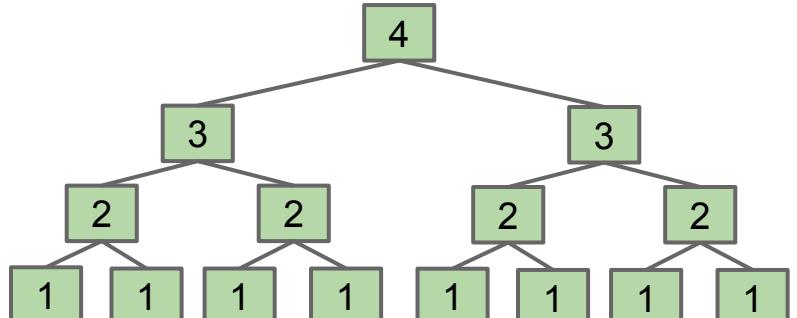
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```

A third approach: Count number of calls to $f3$, given by a “recurrence relation” for $C(N)$.

- $C(1) = 1$
- $C(N) = 2C(N-1) + 1$

More technical to solve. Won’t do this in our course. See next slide for solution.



Recursion and Recurrence Relations (Extra, Outside 61B Scope)

Find a simple $f(N)$ such that the runtime $R(N) \in \Theta(f(N))$.

```
public static int f3(int n) {  
    if (n <= 1)  
        return 1;  
    return f3(n-1) + f3(n-1)  
}
```

This approach not covered in class. Provided for those of you who want to see a recurrence relation solution.

One approach: Count number of calls to $f3$, given by $C(N)$.

$$C(1) = 1$$

$$C(N) = 2C(N - 1) + 1$$

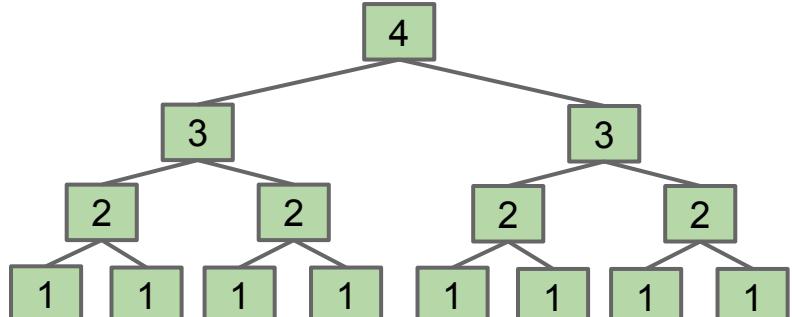
$$= 2(2C(N - 2) + 1) + 1$$

$$= 2(2(2C(N - 2) + 1) + 1) + 1$$

$$= 2(\dots 2 \cdot 1 + 1) + 1 + \dots 1$$

$$= \underbrace{2(\dots 2)}_{N-1} \cdot 1 + 1 + \dots 1$$

$$= 2^{N-1} + 2^{N-2} + \dots + 1 = 2^N - 1 \in \Theta(2^N)$$



Example 4: Binary Search

Binary Search (demo: <https://goo.gl/3VvJNw>)

Trivial to implement?

- Idea published in 1946.
- First correct implementation in 1962.
 - Bug in Java's binary search discovered in 2006.

See Jon Bentley's book
Programming Pearls.

See
<http://goo.gl/gQI0FN>

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Binary Search (Intuitive): <http://yellkey.com/car>

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find runtime in terms of $N = hi - lo + 1$ [i.e. # of items being considered]

- Intuitively, what is the order of growth of the worst case runtime?
 - 1
 - $\log_2 N$
 - N
 - $N \log_2 N$
 - 2^N



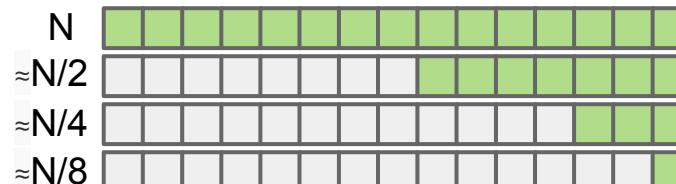
Binary Search (Intuitive)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find runtime in terms of $N = hi - lo + 1$ [i.e. # of items being considered]

- Intuitively, what is the order of growth of the worst case runtime?

B. $\log_2 N$



Why? Problem size halves over and over until it gets down to 1.

- If C is number of calls to binarySearch, solve for $1 = N/2^C \rightarrow C = \log_2(N)$

Example 4: Binary Search Exact (Optional) (see web video)

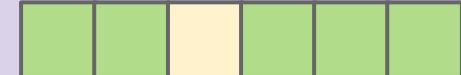
Binary Search (Exact Count): <http://yellkey.com/allow>

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

Cost model: Number of `binarySearch` calls.

N=6



- What is $C(6)$, number of total calls for $N = 6$?
 - A. 6
 - B. 3
 - C. $\log_2(6) = 2.568$
 - D. 2
 - E. 1

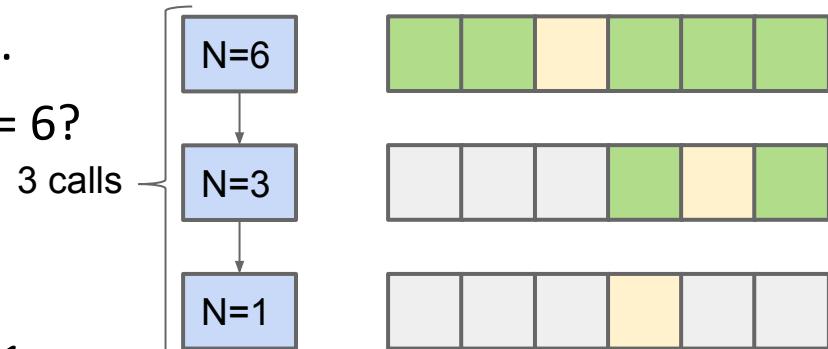
Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

Cost model: Number of `binarySearch` calls.

- What is $C(6)$, number of total calls for $N = 6$?
B. 3



Three total calls, where $N = 6$, $N = 3$, and $N = 1$.

Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.

N	1	2	3	4	5	6	7	8	9	10	11	12	13
C(N)	1					3							

N=1

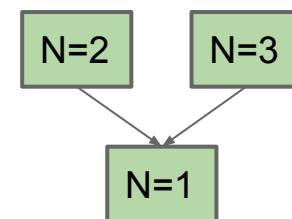
Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.

N	1	2	3	4	5	6	7	8	9	10	11	12	13
C(N)	1	2	2			3							



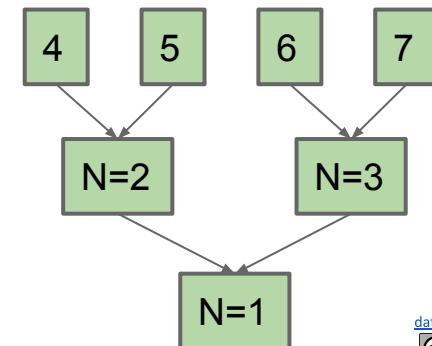
Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.

N	1	2	3	4	5	6	7	8	9	10	11	12	13
C(N)	1	2	2	3	3	3	3						



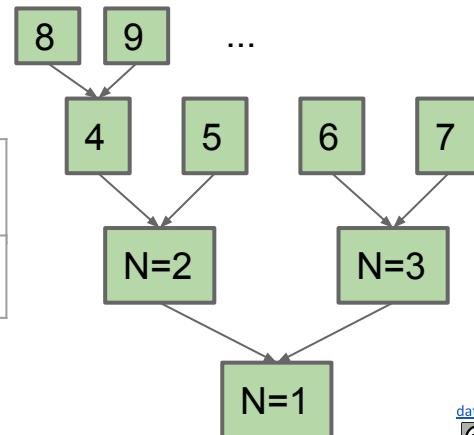
Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.

N	1	2	3	4	5	6	7	8	9	10	11	12	13
C(N)	1	2	2	3	3	3	3	4	4	4	4	4	4



Binary Search (Exact Count)

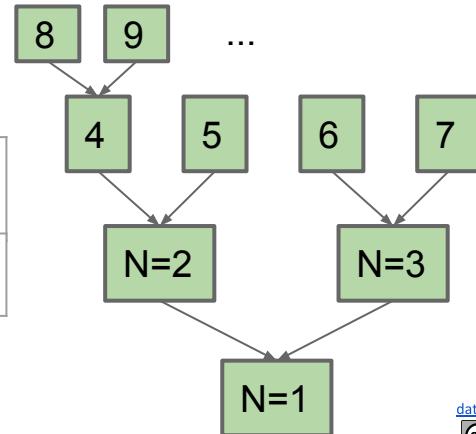
```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.

N	1	2	3	4	5	6	7	8	9	10	11	12	13
C(N)	1	2	2	3	3	3	3	4	4	4	4	4	4

$$C(N) = \lfloor \log_2(N) \rfloor + 1$$

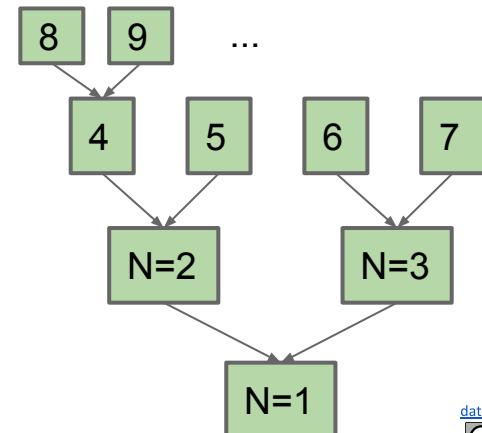


Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.
- $C(N) = \lfloor \log_2(N) \rfloor + 1$
- Since each call takes constant time, $R(N) = \Theta(\lfloor \log_2(N) \rfloor)$
 - This $f(N)$ is way too complicated. Let's simplify.



Handy Big Theta Properties

Goal: Simplify $\Theta(\lfloor \log_2(N) \rfloor)$

For proof:
See online textbook exercises.

- Three handy properties to help us simplify:
 - $\lfloor f(N) \rfloor = \Theta(f(N))$ [the floor of f has same order of growth as f]
 - $\lceil f(N) \rceil = \Theta(f(N))$ [the ceiling of f has same order of growth as f]
 - $\log_p(N) = \Theta(\log_q(N))$ [logarithm base does not affect order of growth]

$$\lfloor \log_2(N) \rfloor = \Theta(\log N)$$

Since base is irrelevant, we omit from our big theta expression. We also omit the parenthesis around N for aesthetic reasons.

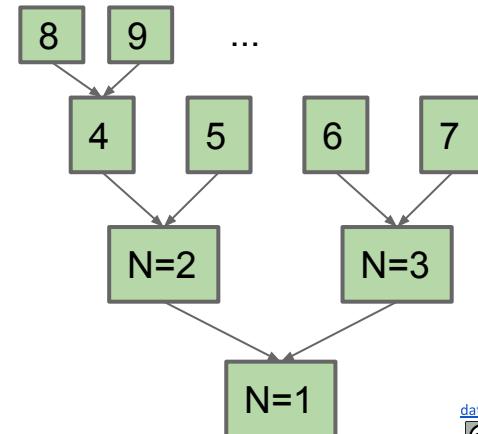
Binary Search (Exact Count)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Goal: Find worst case runtime in terms of $N = hi - lo + 1$ [i.e. # of items]

- Cost model: Number of `binarySearch` calls.
- $C(N) = \lfloor \log_2(N) \rfloor + 1 = \Theta(\log N)$
- Since each call takes constant time, $R(N) = \Theta(\log N)$

... and we're done!



Binary Search (using Recurrence Relations)

```
static int binarySearch(String[] sorted, String x, int lo, int hi)
    if (lo > hi) return -1;
    int m = (lo + hi) / 2;
    int cmp = x.compareTo(sorted[m]);
    if (cmp < 0) return binarySearch(sorted, x, lo, m - 1);
    else if (cmp > 0) return binarySearch(sorted, x, m + 1, hi);
    else return m;
}
```

Approach: Measure number of string comparisons for $N = hi - lo + 1$.

- $C(0) = 0$
- $C(1) = 1$
- $C(N) = 1 + C((N-1)/2)$

Can show that $C(N) = \Theta(\log N)$. Beyond scope of class, so won't solve in slides.

Log Time Is Really Terribly Fast

In practice, logarithmic time algorithms have almost constant runtimes.

- Even for incredibly huge datasets, practically equivalent to constant time.

N	$\log_2 N$	Typical runtime (seconds)
100	6.6	1 nanosecond
100,000	16.6	2.5 nanoseconds
100,000,000	26.5	4 nanoseconds
100,000,000,000	36.5	5.5 nanoseconds
100,000,000,000,000	46.5	7 nanoseconds

Example 5: Mergesort

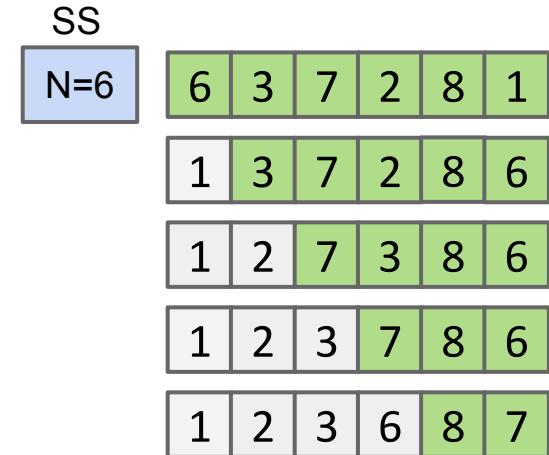
Selection Sort: A Prelude to Mergesort/Example 5

Earlier in class we discussed a sort called selection sort:

- Find the smallest unfixed item, move it to the front, and ‘fix’ it.
- Sort the remaining unfixed items using selection sort.

Runtime of selection sort is $\Theta(N^2)$:

- Look at all N unfixed items to find smallest.
- Then look at $N-1$ remaining unfixed.
- ...
- Look at last two unfixed items.
- Done, sum is $2+3+4+5+\dots+N = \Theta(N^2)$



Selection Sort: A Prelude to Mergesort/Example 5

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- Then look at $N-1$ remaining unfixed.
- ...
- Look at last two unfixed items.
- Done, sum is $2+3+4+5+\dots+N = \Theta(N^2)$

~36 AU
SS
N=6

~2048 AU
SS
N=64

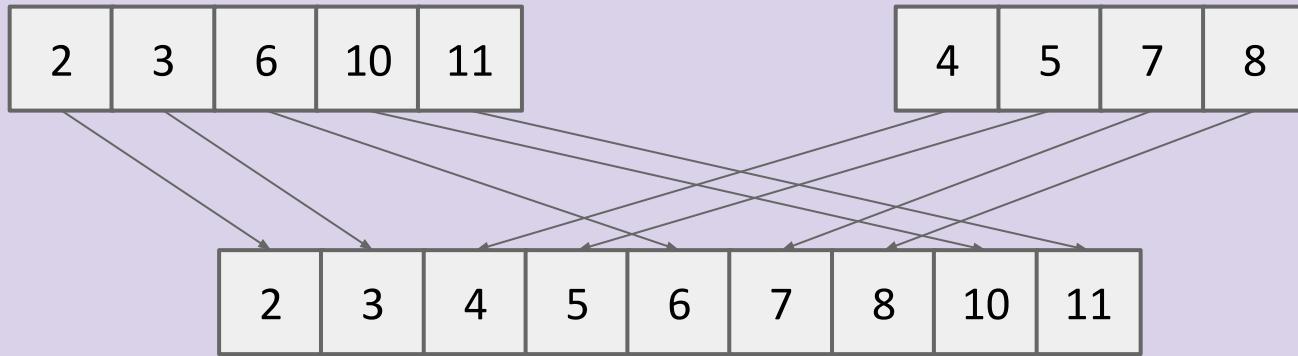
Given that runtime is quadratic, for $N = 64$, we might say the runtime for selection sort is 2,048 arbitrary units of time (AU).

The Merge Operation: Another Prelude to Mergesort/Example 5

Given two sorted arrays, the merge operation combines them into a single sorted array by successively copying the smallest item from the two arrays into a target array.

Merging Demo ([Link](#))

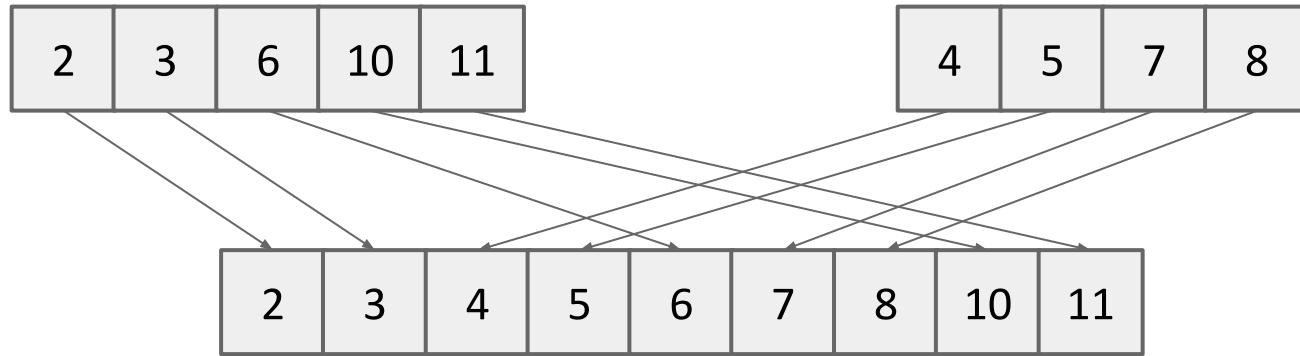
Merge Runtime: <http://yellkey.com/west>



How does the runtime of merge grow with N, the total number of items?

- A. $\Theta(1)$
- B. $\Theta(\log N)$
- C. $\Theta(N)$
- D. $\Theta(N^2)$

Merge Runtime: <http://shoutkey.com/TBA>



How does the runtime of merge grow with N , the total number of items?

C. $\Theta(N)$. Why? Use array writes as cost model, merge does exactly N writes.

Using Merge to Speed Up the Sorting Process

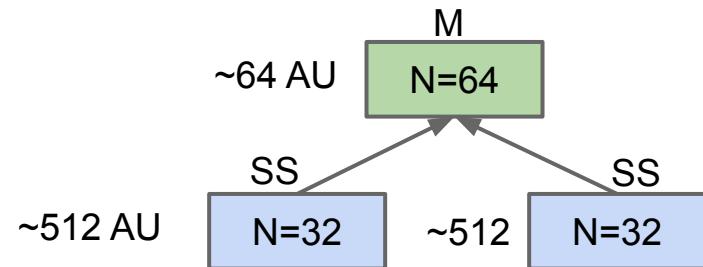
Merging can give us an improvement over vanilla selection sort:

- Selection sort the left half: $\Theta(N^2)$.
- Selection sort the right half: $\Theta(N^2)$.
- Merge the results: $\Theta(N)$.



N=64: ≈1088 AU.

- Merge: ≈64 AU.
- Selection sort: $\approx 2 \cdot 512 = \approx 1024$ AU.



Still $\Theta(N^2)$, but faster since $N+2 \cdot (N/2)^2 < N^2$

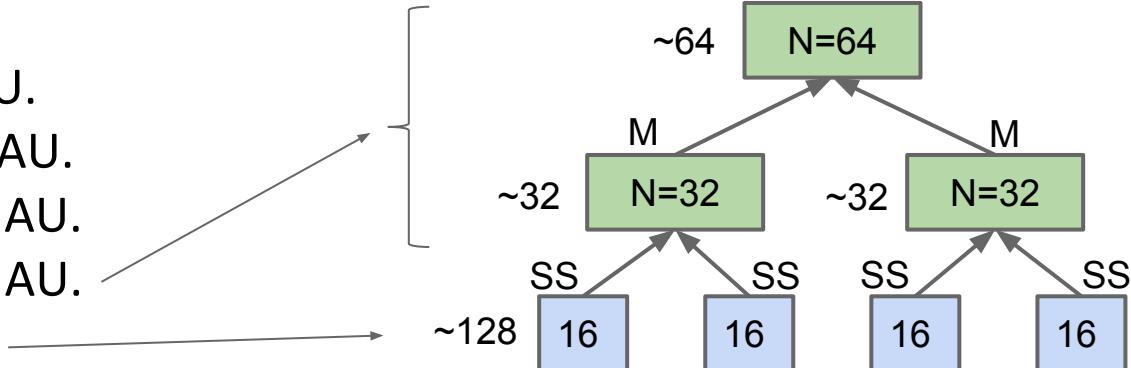
- ≈1088 vs. ≈2048 AU for N=64.

Two Merge Layers

Can do even better by adding a second layer of merges.

Runtime for each sort:

- Selection sort only: ~ 2048 AU.
- One layer of merges: ~ 1088 AU.
- Two layers of merges: ~ 640 AU.
 - Merge: ~ 64 AU + $2 * \sim 32$ AU.
 - Selection sort: $4 * \sim 128$.



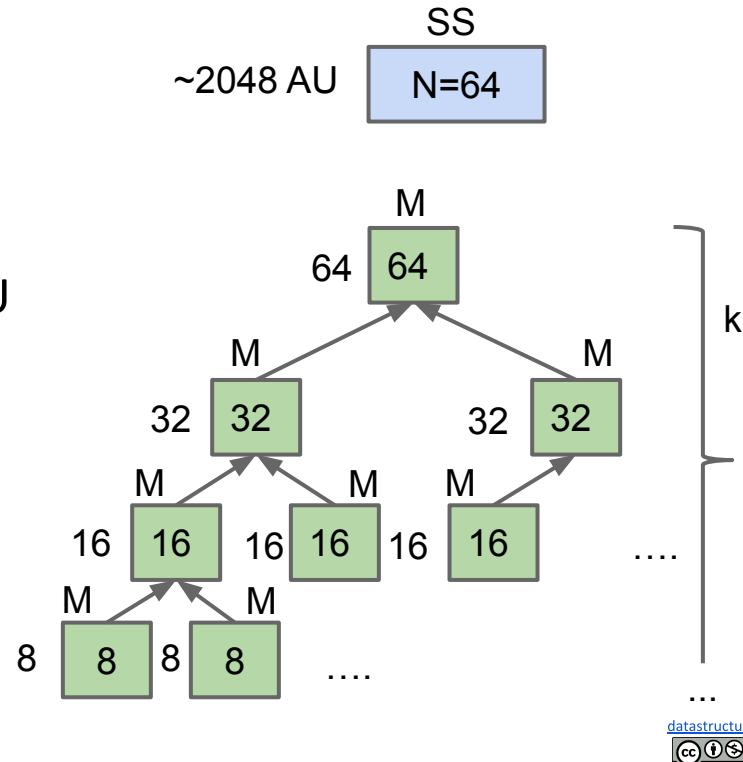
Example 5: Mergesort

Mergesort does merges all the way down (no selection sort):

- If array is of size 1, return.
- Mergesort the left half: $\Theta(\text{??})$.
- Mergesort the right half: $\Theta(\text{??})$.
- Merge the results: $\Theta(N)$.

Total runtime to merge all the way down: ~ 384 AU

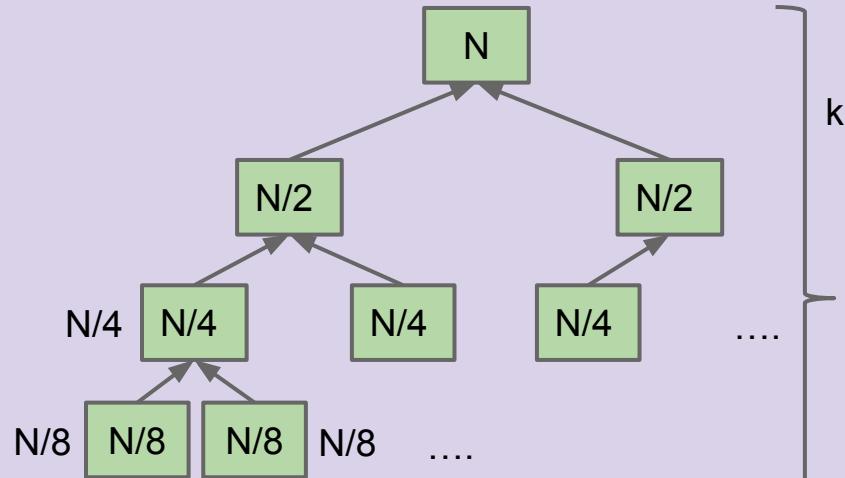
- Top layer: $\sim 64 = 64$ AU
- Second layer: $\sim 32 \times 2 = 64$ AU
- Third layer: $\sim 16 \times 4 = 64$ AU
- Overall runtime in AU is $\sim 64k$, where k is the number of layers.
- $k = \log_2(64) = 6$, so ~ 384 total AU.



Example 5: Mergesort Order of Growth, yellkey.com/friend

For an array of size N , what is the worst case runtime of Mergesort?

- A. $\Theta(1)$
- B. $\Theta(\log N)$
- C. $\Theta(N)$
- D. $\Theta(N \log N)$
- E. $\Theta(N^2)$



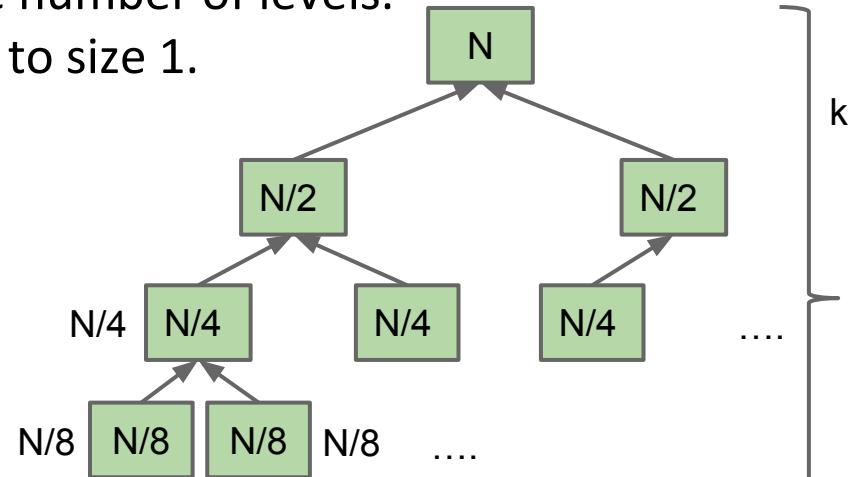
Example 5: Mergesort Order of Growth

Mergesort has worst case runtime = $\Theta(N \log N)$.

- Every level takes $\sim N$ AU.
 - Top level takes $\sim N$ AU.
 - Next level takes $\sim N/2 + \sim N/2 = \sim N$.
 - One more level down: $\sim N/4 + \sim N/4 + \sim N/4 + \sim N/4 = \sim N$.
- Thus, total runtime is $\sim Nk$, where k is the number of levels.
 - How many levels? Goes until we get to size 1.
 - $k = \log_2(N)$.
- Overall runtime is $\Theta(N \log N)$.

Exact count explanation is tedious.

- Omitted here. See textbook exercises.



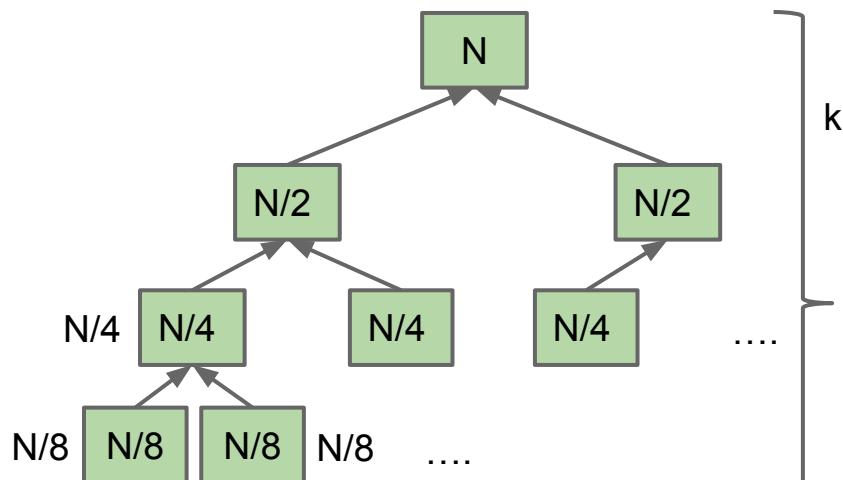
Mergesort using Recurrence Relations (Extra)

$C(N)$: Number of calls to mergesort + number of array writes.

$$C(N) = \begin{cases} 1 & : N < 2 \\ 2C(N/2) + N & : N \geq 2 \end{cases}$$

$$\begin{aligned} C(N) &= 2(2C(N/4) + N/2) + N \\ &= 4C(N/4) + N + N \\ &= 8C(N/8) + N + N + N \\ &= N \cdot 1 + \underbrace{N + N + \dots + N}_{k=\lg N} \\ &= N + N \lg N \in \Theta(N \lg N) \end{aligned}$$

Only works for $N=2^k$. Can be generalized at the expense of some tedium by separately finding Big O and Big Omega bounds (see next lecture).



Linear vs. Linearithmic ($N \log N$) vs. Quadratic

$N \log N$ is basically as good as N , and is vastly better than N^2 .

- For $N = 1,000,000$, the $\log N$ is only 20.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

(from Algorithm Design: Tardos, Kleinberg)

Summary

Theoretical analysis of algorithm performance requires careful thought.

- There are no magic shortcuts for analyzing code.
- In our course, it's OK to do exact counting or intuitive analysis.
 - Know how to sum $1 + 2 + 3 \dots + N$ and $1 + 2 + 4 + \dots + N$.
 - We won't be writing mathematical proofs in this class.
- Many runtime problems you'll do in this class resemble one of the five problems from today. See textbook, study guide, and discussion for more practice.
- This topic has one of the highest skill ceilings of all topics in the course.

Different solutions to the same problem, e.g. sorting, may have different runtimes.

- N^2 vs. $N \log N$ is an enormous difference.
- Going from $N \log N$ to N is nice, but not a radical change.