Machine Learning

Learning Model

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Realizability Assumption: there exists $h^* \in \mathcal{H}$ such that $L_{\mathcal{D},f}(h^*) = 0$

Informally: the label is fully determined by the instance x

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Relaxation: \mathcal{D} is a probability distribution over $\mathcal{X} \times \mathcal{Y}$ $\Rightarrow \mathcal{D}$ is the *joint distribution* over domain points and labels.

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$$\forall \vec{z} \in \mathcal{X}$$
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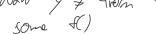
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- D: (marginal) distribution over domain points
- $\mathcal{D}((x,y)|x)$: conditional distribution over labels for each domain point way of from "f(x)" for



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For example, two components of \mathcal{D} :

- \mathcal{D}_{x} : (marginal) distribution over domain points
- $\mathcal{D}((x,y)|x)$: conditional distribution over labels for each domain point

Given x, label y is obtained according to a conditional probability $\mathbb{P}[y|x]$.

The Empirical and True Error

With \mathcal{D} that is a probability distribution over $\mathcal{X} \times \mathcal{Y}$ the *true error* (or risk) is:

$$L_{\mathcal{D}}(h) \stackrel{\text{def}}{=} \mathbb{P}_{(\mathbf{x},\mathbf{y}) \sim \mathcal{D}}[h(\mathbf{x}) \neq \mathbf{y}]$$

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As before $\mathcal D$ is not known to the learner; the learner only knows the training data $\mathcal S$

Empirical risk: as before, that is

$$L_{\mathcal{S}}(h) \stackrel{\text{def}}{=} \frac{|\{i, 0 < i \leq m : h(x_i) \neq y_i\}|}{m}$$

Note: $L_S(h) =$ probability that for a pair (x_i, y_i) taken uniformly at random from S the event " $h(x_i) \neq y_i$ " holds.

An Optimal Predictor

Learner's goal: find $h: \mathcal{X} \to \mathcal{Y}$ minimizing $L_{\mathcal{D}}(h)$

Is there a best predictor?

Given a probability distribution \mathcal{D} over $\mathcal{X} \times \{0,1\}$, the best predictor is the Bayes Optimal Predictor

$$f_{\mathcal{D}}(x) = \left\{ egin{array}{ll} 1 & ext{if } \mathbb{P}[y=1|x] \geq 1/2 \\ 0 & ext{otherwise} \end{array} \right.$$

Proposition

For any classifier $g: \mathcal{X} \to \{0,1\}$, it holds $L_{\mathcal{D}}(f_{\mathcal{D}}) \leq L_{\mathcal{D}}(g)$.

PROOF: Exercize

Agnostic PAC Learnability

Consider only predictors from a hypothesis class \mathcal{H} .

We are going to be ok with not finding the best predictor, but not being too far off.

Definition

A hypothesis class \mathcal{H} is agnostic PAC learnable if there exist a function $m_{\mathcal{H}}$: $(0,1)^2 \to \mathbb{N}$ and a learning algorithm such that for every $\delta, \varepsilon \in (0,1)$, for every distribution \mathcal{D} over $\mathcal{X} \times \mathcal{Y}$, when running the learning algorithm on $m \geq m_{\mathcal{H}}(\varepsilon, \delta)$ i.i.d. examples generated by \mathcal{D} the algorithm returns a hypothesis h such that, with probability $\geq 1 - \delta$ (over the choice of the m training examples):

$$L_{\mathcal{D}}(h) \leq \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \varepsilon.$$

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Note: this is a generalization of the previous learning model.

A More General Learning Model: Beyond Binary Classification

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Binary classification: \mathcal{Y} = \{0, 1\}
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Other learning problems:

- multiclass classification: classification with > 2 labels
- regression: $\mathcal{Y} = \mathbb{R}$

Multiclass classification: same as before!

Regression

Domain set: \mathcal{X} is usually \mathbb{R}^p for some p.

Target set: \mathcal{Y} is \mathbb{R}

Training data: (as before) $S = ((x_1, y_1), \dots, (x_m, y_m))$

Learner's output: (as before) $h: \mathcal{X} \to \mathcal{Y}$

Loss: the previous one does not make much sense...

(Generalized) Loss Functions

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Risk function = expected loss of a hypothesis $h \in \mathcal{H}$ with respect to \mathcal{D} over Z:

$$L_{\mathcal{D}}(h) \stackrel{\text{def}}{=} \mathbb{E}_{z \sim \mathcal{D}}[\ell(h, z)]$$

Empirical risk = expected loss over a given sample
$$S = (z_1, \ldots, z_m) \in Z^m$$
:

$$(x_3, y_3) \qquad (x_k, y_k) \qquad L_S(h) \stackrel{def}{=} \frac{1}{m} \sum_{i=1}^m \ell(h, z_i)$$

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Some Common Loss Functions

0-1 loss:
$$Z = \mathcal{X} \times \mathcal{Y}$$

$$\ell_{0-1}(h,(x,y)) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} 0 & \text{if } h(x) = y \\ 1 & \text{if } h(x) \neq y \end{array} \right.$$

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Commonly used in binary or multiclass classification.

Squared loss:
$$Z = \mathcal{X} \times \mathcal{Y}$$

$$\ell_{sq}(h, (x, y)) \stackrel{def}{=} (h(x) - y)^{2}$$

$$\ell(h, (x, y)) = [h(x) - y]$$

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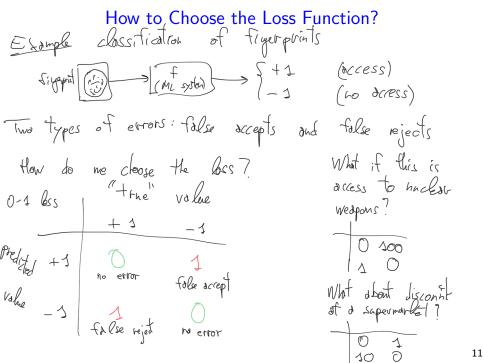
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Commonly used in regression.

Note: in general, the loss function may depend on the application! But computational considerations play a role...



Agnostic PAC Learnability for General Loss Functions

Definition

A hypothesis class $\mathcal H$ is agnostic PAC learnable with respect to a set Z and a loss function $\ell:\mathcal H\times Z\to\mathbb R_+$ if there exist a function $m_{\mathcal H}\colon (0,1)^2\to\mathbb N$ and a learning algorithm such that for every $\delta,\varepsilon\in(0,1)$, for every distribution $\mathcal D$ over Z, when running the learning algorithm on $m\geq m_{\mathcal H}(\varepsilon,\delta)$ i.i.d. examples generated by $\mathcal D$ the algorithm returns a hypothesis h such that, with probability $\geq 1-\delta$ (over the choice of the m training examples):

$$L_{\mathcal{D}}(h) \leq \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \varepsilon$$

where
$$L_{\mathcal{D}}(h) = \mathbb{E}_{z \sim \mathcal{D}}[\ell(h, z)]$$

Leslie Valiant, Turing award 2010

For transformative contributions to the theory of computation, including the theory of probably approximately correct (PAC) learning, the complexity of enumeration and of algebraic computation, and the theory of parallel and distributed computing.



Bibliography

Up to now: [UML] Chapter 2 and Chapter 3