## Exercise

Let

$$\mathcal{H}_d = \{ h_{\mathbf{w}}(\mathbf{x}) : h_{\mathbf{w}}(\mathbf{x}) = \operatorname{sign}(<\mathbf{w},\mathbf{x}>) \}$$

where  $\mathcal{X} = \mathbb{R}^d$ .

Prove that  $VCdim(\mathcal{H}_d) = d$ . Solution We need to prove that  $VCdim(\mathcal{H}_d) \gg d$  and that  $VCdim(\mathcal{H}_d) \leq d$ .

i) VCdim(HJ) 2d. We need to show a set of d vectors in IRd that is shattered by Hd.

Consider  $\{\vec{e}_3, \vec{e}_2, ..., \vec{e}_d\}$  with  $\vec{e}_i = \{\vec{e}_3, \vec{e}_2, ..., \vec{e}_d\}$  with  $\vec{e}_i = \{\vec{e}_3, \vec{e}_2, ..., \vec{e}_d\}$ 

This set is shattered by Hd: we need to show that far every labelly ys, ye, ..., yd, where y; is the lobel of Zi, with y; ef-1, 13, there is an hypothesis in Hd that

assigns such labels to the set.

Consider an arbitrary labeling  $y_3, y_2, ..., y_d$ : consider the hypothesis  $h_{\vec{w}}$  where  $\tilde{\vec{w}} = \begin{bmatrix} y_1 \\ y_2 \\ y_d \end{bmatrix}$ . We have that for every i, with 12i2d:  $h_{\overrightarrow{w}}(\overrightarrow{e_i}) = Sign(\overrightarrow{w}, \overrightarrow{e_i}) = Sign(\langle (\overrightarrow{y_i}) | (\overrightarrow{v_i}) \rangle) = Sign(\cancel{y_i}) = \cancel{y_i}$ ii) VCdim (tfd) Ed: we need to show that no set of des vectors in Rd can be shattened by Hd.

Consider an arbitrary set {x3, x2,..., xd41} with zelled They cannot be likewish independent  $\Rightarrow \exists \ a_1, a_2, ..., a_{d+1}$  with  $a_i \in \mathbb{R}$ ,  $1 \le i \le d+1$ , such that:

- Not all 
$$\delta_{i}$$
 is die  $\delta$  (A)

- Note and  $\delta_{i}$  is die  $\delta$  (A)

Define:  $I = \{i: \delta_{i} > 0\}$ . Note that it cannot be

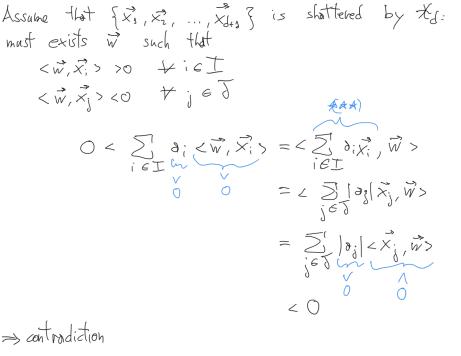
 $I = \{j: \delta_{i} < 0\}$ . That  $I = \emptyset = \delta$  (due to (A))

There die  $I = \{i: \delta_{i} > 0\}$ . Then

Case i) we see assuming  $I \neq \emptyset \neq J$ . Then

$$I = \{i: \delta_{i} > 0\}$$

$$I$$



Case ii):  $I \neq \emptyset = \mathcal{T}$ : so he steps lead to  $0 < ... < 0 \Rightarrow$  contradiction

Case iii):  $I = \emptyset \neq \mathcal{T}$ : some steps lead to  $0 < ... < 0 \Rightarrow$  contradiction