Math 217: The Proof of the spectral Theorem
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THE SPECTRAL THEOREM:
A square matrix is symmetric if and only if it has an orthonormal eigenbasis.

Equivalently, a square matrix is symmetric if and only if there exists an *orthogonal* matrix S such that S^TAS is diagonal.

That is, a matrix is orthogonally diagonalizable if and only if it is symmetric.

A. For each item, find an explicit example, or explain why none exists.

1. An orthonormal eigenbasis for an arbitrary 3×3 **diagonal** matrix;
2. A non-diagonal 2×2 matrix for which there exists an orthonormal eigenbasis (you do not have to find the eigenbasis, only the matrix)
3. A non-symmetric matrix which admits an orthonormal eigenbasis.
4. A non-diagonalizable 2×2 matrix
5. A non-symmetric but diagonalizable 2×2 matrix.
6. A square matrix Q such that Q^TQ has no real eigenvalues.
7. A 2×2 symmetric matrix with an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.
8. A 2×2 non-invertible matrix with an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.

Solution note:

1. The standard basis always works.
2. Any symmetric matrix, such as $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
3. No such thing, by spectral theorem.
4. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.
5. $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.
6. No such thing: Q^TQ is symmetric (check $(Q^TQ)^T = Q^TQ$), so the Spectral theorem says all its eigenvalues are real.
7. No such matrix by spectral theorem. Spectral theorem tells us a symmetric matrix is diagonalizable, but this would mean that the geometric multiplicities need to equal the algebraic multiplicities for all eigenvalues, in order to add up to 2.
8. $\begin{bmatrix} 0 & 0 \\ \pi & 0 \end{bmatrix}$.

B. The proof of the spectral theorem. Part I.

1. Show that if A can be orthogonally diagonalized, then A is symmetric. [Hint: write A as product of matrices you can easily transpose.] This proves one direction of the spectral theorem.

2. Suppose A is symmetric. Let λ_1 and λ_2 be *distinct* eigenvalues. Show that the corresponding eigenspaces are *orthogonal.* [Hint: Take v and w eigenvectors with different eigenvalues and compute $\vec{w} \cdot A\vec{v}$ using matrix multiplication[1].]

3. Prove that a symmetric matrix is diagonalizable, then it is orthogonally diagonalizable. (Hint: use Gram-Schmidt on each eigenspace).

(This not a complete proof of the Spectral Theorem—we still need to see why a symmetric matrix is diagonalizable).

Solution note:

1. Suppose A can be orthogonally diagonalized. This means $S^TAS = D$ where S is orthogonal (So $S^T = S^{-1}$) and D is diagonal. Transpose both sides: $(S^TAS)^T = D^T$. For a diagonal matrix D, we have $D = D^T$, so $(S^TAS)^T = S^TA^TS = D$. Rearranging, we see that both A and A^T are equal to SDS^T (using again that $S^T = S^{-1}$). So $A = A^T$.

2. Let v be a λ_1 eigenvector and w a λ_2 eigenvector. We need to show that $\vec{v} \cdot \vec{w} = 0$. Using the hint, compute $\vec{v} \cdot (A\vec{w}) = \vec{v}^T A\vec{w}$, which equals $(A^T\vec{v})^T\vec{w}$. Using $A = A^T$, we have $\vec{v} \cdot (A\vec{w}) = (A\vec{v}) \cdot \vec{w}$. Now using that $\vec{v}$ and $\vec{w}$ are eigenvectors, we have $\vec{v} \cdot \lambda_2\vec{w} = \lambda_1\vec{v} \cdot \vec{w}$, which means $\lambda_2\vec{v} \cdot \vec{w} = \lambda_1\vec{v} \cdot \vec{w}$. So either $\vec{v} \cdot \vec{w} = 0$, in which case the eigenvalues are orthogonal (so we are done!) or we can divide both sides by $\vec{v} \cdot \vec{w} = 0$ to get $\lambda_2 = \lambda_1$, which is a contradition.

3. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of A. Let $E_1, \ldots, E_r$ be the associated eigenspaces. We know their dimensions add up to n (where A is $n \times n$.) Each E_i has a basis, which we can assume is orthonormal by Gram Schmidt. But also, the vectors in E_i and E_j, if $i \not{j}$, are always orthogonal, so putting together these orthonormal eigenbases for the various E_i we have an orthonormal set (hence linearly independent set). The must span $\mathbb{R}^n$ since there are n of them. This is an orthonormal eigenbasis.

[1] If you still need a hint: figure out how to take advantage of the symmetry of A