# Proofs

**Theorem 1.1.1** *Let*

$$\mathrm{X} = \{\mathbf{x} \in \Re^n \ : \ g_i(\mathbf{x}) \leq 0 \ , \ i = 1, \ldots, m\}. \tag{1.2}$$

*If for all* $i \in \{1, \ldots, m\}$ *the functions* $g_i : \Re^n \to \Re$ *are convex, then the set* $\mathrm{X}$ *is convex.*

**Proof**: Clearly

$$\mathrm{X} = \bigcap_{i=1}^{m} X_i, \ \text{ where } \ X_i \ := \ \{\mathbf{x} \in \Re^n \ : \ g_i(\mathbf{x}) \leq 0\}.$$

By Proposition 1.1.1, it is then sufficient to prove that each set $X_i$ is convex. Indeed, given any two elements $\mathbf{x}$ and $\mathbf{y}$ of $X_i$ and a generic point $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ , $\lambda \in [0, 1]$, by the convexity hypothesis of the function $g_i$ we can write

$$g_i(\mathbf{z}) = g_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda \, g_i(\mathbf{x}) + (1-\lambda)g_i(\mathbf{y}) \leq 0,$$

where the latter inequality is valid since $g_i(\mathbf{x}) \leq 0$, $g_i(\mathbf{y}) \leq 0$, and $0 \leq \lambda \leq 1$. It follows that $g_i(\mathbf{z}) \leq 0$, hence $\mathbf{z} \in \mathrm{X}_i$. Given the arbitrariness of $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, one thus has that $\mathrm{X}_i$ is convex, as requested. $\square$

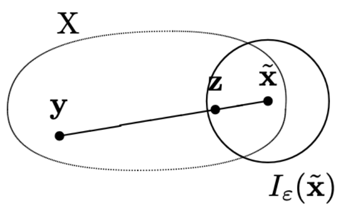
**Theorem 1.1.2** *Consider a* convex programming *problem, i.e., a problem* $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathrm{X}\}$ *where* $\mathrm{X} \subseteq \Re^n$ *is a convex set and* $f : \mathrm{X} \to \Re$ *is a convex function. Every locally optimal solution is also a globally optimal solution.*

**Proof**: Let $\tilde{\mathbf{x}}$ be any locally optimal solution. By the local optimum definition, there exists then $\varepsilon > 0$ such that $f(\tilde{\mathbf{x}}) \leq f(\mathbf{z})$ for all $\mathbf{z} \in I_\varepsilon(\tilde{\mathbf{x}}) := \{\mathbf{x} \in \mathrm{X} : ||\mathbf{x} - \tilde{\mathbf{x}}|| \leq \varepsilon\}$. We have to prove that $f(\tilde{\mathbf{x}}) \leq f(\mathbf{y})$ for all $\mathbf{y} \in \mathrm{X}$.

Given any $\mathbf{y} \in \mathrm{X}$, consider the point $\mathbf{z}$ belonging to the segment that connects $\tilde{\mathbf{x}}$ to $\mathbf{y}$ and defined as $\mathbf{z} := \lambda\tilde{\mathbf{x}} + (1-\lambda)\mathbf{y}$, where $\lambda < 1$ is chosen very close to the value 1 so that $\mathbf{z} \in I_\varepsilon(\tilde{\mathbf{x}})$ and hence $f(\tilde{\mathbf{x}}) \leq f(\mathbf{z})$. By the convexity hypothesis of $f$ it follows that

$$f(\tilde{\mathbf{x}}) \leq f(\mathbf{z}) = f(\lambda\tilde{\mathbf{x}} + (1-\lambda)\mathbf{y}) \leq \lambda f(\tilde{\mathbf{x}}) + (1-\lambda) f(\mathbf{y}),$$

from which, dividing by $1 - \lambda > 0$, we obtain $f(\tilde{\mathbf{x}}) \leq f(\mathbf{y})$, as requested. $\square$

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**Theorem 4.1.2** *If the set* $P$ *of the feasible solutions of the linear programming problem* $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P\}$ *is bounded, then there exists at least one optimal vertex of* $P$.

**Proof**: Let $\mathbf{x}^1, \ldots, \mathbf{x}^k$ be the vertices of $P$ and $z^* := \min\{\mathbf{c}^T\mathbf{x}^i : i = 1, \ldots, k\}$. Given any $\mathbf{y} \in P$, we need to prove that $\mathbf{c}^T\mathbf{y} \geq z^*$. Indeed, $\mathbf{y} \in P$ implies the existence of multipliers $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, such that $\mathbf{y} = \sum_{i=1}^k \lambda_i \mathbf{x}^i$. Hence we have

$$\mathbf{c}^T\mathbf{y} = \mathbf{c}^T \sum_{i=1}^k \lambda_i \mathbf{x}^i = \sum_{i=1}^k \lambda_i (\mathbf{c}^T\mathbf{x}^i) \geq \sum_{i=1}^k \lambda_i z^* = z^*.$$

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**Theorem 4.1.3** *A point* $\mathbf{x} \in P$ *is a vertex of the not empty polyhedron* $P := \{\mathbf{x} \geq 0 : \mathrm{A}\mathbf{x} = \mathbf{b}\ \}$ *if and only if* $\mathbf{x}$ *is a basic feasible solution of the system* $\mathrm{A}\mathbf{x} = \mathbf{b}$*.*

**Proof**: Let us first prove the implication "$\mathbf{x}$ is a basic feasible solution $\Rightarrow$ $\mathbf{x}$ is a vertex".

Let

$$\mathbf{x} = [\underbrace{x_1, \ldots, x_k}_{\text{positive}}, 0, \ldots, 0]^T$$

be any basic feasible solution associated with a basis B of A, where $k \geq 0$ is the number of non-zero (i.e., strictly positive) components of $\mathbf{x}$. It follows that columns $A_1, \ldots, A_k$ must be part of B, possibly together with other columns (in case of degenerate solution). Let us assume by contradiction that $\mathbf{x}$ is not a vertex. There exist thus

$$\mathbf{y} = [y_1, \ldots, y_k, 0, \ldots, 0]^T \ \in \ P$$
$$\mathbf{z} = [z_1, \ldots, z_k, 0, \ldots, 0]^T \ \in \ P$$

with $\mathbf{y} \neq \mathbf{z}$ such that $\mathbf{x} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z}$ for any $\lambda \in (0, 1)$, which implies that $k \geq 1$.

Note that both $\mathbf{y}$ and $\mathbf{z}$ must have the last components set to zero, otherwise their convex combination cannot give $\mathbf{x}$. For the hypotheses, we then have:

$$\mathbf{y} \in P \ \Rightarrow \ \mathrm{A}\mathbf{y} = \mathbf{b} \ \Rightarrow \ \mathrm{A}_1 y_1 + \ldots + \mathrm{A}_k y_k = \mathbf{b}$$
$$\mathbf{z} \in P \ \Rightarrow \ \mathrm{A}\mathbf{z} = \mathbf{b} \ \Rightarrow \ \mathrm{A}_1 z_1 + \ldots + \mathrm{A}_k z_k = \mathbf{b}.$$

By subtracting the second equation from the first we obtain

$$(y_1 - z_1)A_1 + \ldots + (y_k - z_k)A_k \ = \ \alpha_1 A_1 + \ldots + \alpha_k A_k \ = \ 0,$$

where $\alpha_i := y_i - z_i$ , $i = 1, \ldots, k$. Hence there exist $\alpha_1, \ldots, \alpha_k$ scalars not all zero (since $\mathbf{y} \neq \mathbf{z}$) such that $\sum_{i=1}^{k} \alpha_i A_i = 0$, thus columns $A_1, \ldots, A_k$ are linearly dependent and cannot be part of the basis B ($\Rightarrow$ contradiction).

We will now prove the implication "$\mathbf{x}$ is a vertex $\Rightarrow$ $\mathbf{x}$ is a basic solution"; the fact that the basic solution is also feasible obviously derives from the hypothesis that $\mathbf{x} \in P$.

Writing, as before, $\mathbf{x} = [x_1, \ldots, x_k, 0, \ldots, 0]^T$ with $x_1, \ldots, x_k > 0$ and $k \geq 0$, we have that:

$$\mathbf{x} \in P \Rightarrow \mathrm{A}\mathbf{x} = \mathbf{b} \Rightarrow \mathrm{A}_1 x_1 + \ldots + \mathrm{A}_k x_k = \mathbf{b}. \tag{4.2}$$

Two cases can occur:

1. columns $A_1, \ldots, A_k$ are linearly independent (or $k = 0$): by arbitrarily selecting other $m - k$ linearly independent columns (which, as is well known, is always possible), we obtain a basis $\mathrm{B} = [A_1, \ldots, A_k, \ldots]$ whose basic associated solution is indeed $\mathbf{x}$ (which satisfies $\mathrm{A}\mathbf{x} = \mathbf{b}$ and has non-basic components all equal to zero), thus concluding the proof.

2. columns $A_1, \ldots, A_k$ are linearly dependent: we will prove that this case cannot actually happen. Indeed, if the columns were linearly dependent, then there would exist $\alpha_1, \ldots, \alpha_k$ not all zero and such that

   $$\alpha_1 A_1 + \ldots + \alpha_k A_k = 0, \tag{4.3}$$

   The sum of (4.2) and (4.3) multiplied by $\varepsilon > 0$ would give:

   $$(x_1 + \varepsilon\alpha_1)\mathrm{A}_1 + \ldots + (x_k + \varepsilon\alpha_k)\mathrm{A}_k = \mathbf{b}.$$

   Similarly, the subtraction of (4.3) from (4.2) multiplied by $\varepsilon$ would give:

   $$(x_1 - \varepsilon\alpha_1)\mathrm{A}_1 + \ldots + (x_k - \varepsilon\alpha_k)\mathrm{A}_k = \mathbf{b}.$$

   By defining

   $$\mathbf{y} := [x_1 - \varepsilon\alpha_1, \ldots, x_k - \varepsilon\alpha_k, 0, \ldots, 0]^T$$
   $$\mathbf{z} := [x_1 + \varepsilon\alpha_1, \ldots, x_k + \varepsilon\alpha_k, 0, \ldots, 0]^T,$$

   we would have $\mathrm{A}\mathbf{y} = \mathbf{b}$ and $\mathrm{A}\mathbf{z} = \mathbf{b}$, while choosing a sufficiently small $\varepsilon$ we would have $\mathbf{y}, \mathbf{z} \geq 0$ and thus $\mathbf{y}, \mathbf{z} \in P$ , $\mathbf{y} \neq \mathbf{z}$. But since by construction

   $$\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z},$$

   this would mean that vertex $\mathbf{x}$ can be expressed as the strict convex combination of two distinct points of $P$ ($\Rightarrow$ contradiction).

**Theorem 4.2.2** *Using Bland's rule, the simplex algorithm converges after, at most,* $\binom{n}{m}$ *iterations.* No loop

**Proof**: Let us suppose by contradiction that the thesis is false and let us consider as counterexample the *smallest* LP problem for which there is no convergence. As already seen, in this case the simplex algorithm has to "go through" a cyclic sequence $\mathrm{B}_1, \mathrm{B}_2, \dots, \mathrm{B}_k = \mathrm{B}_1$ of bases. During this sequence, pivot operations are performed on *all* rows and *all* columns of the tableau, otherwise eliminating the not involved rows/columns we would obtain a smaller counterexample. It follows that *all* variables enter and leave, in turn, the current basis. Moreover, we must have $\bar{b}_t = 0$ for all rows $t \in \{1, \dots, m\}$, otherwise in the iteration in which the pivot operation is performed on row $t$, we would

have $\vartheta > 0$, hence the value of the objective function would change—preventing cycling in the sequence of bases.

Consider now tableau $T$ in which variable $x_n$ leaves the current basis to let a given non-basic variable $x_h$ enter the basis. Let us indicate with $x_{\beta[i]}$ the basic variable in row $i \in \{1, \dots, m\}$, and with $t$ the row in which $x_n$ is in the basis (i.e., $\beta[t] = n$).

Tableau $T$ :

| | | $\dots$ | $x_{\beta[i]}$ | $\dots$ | $x_h$ | $\dots$ | $x_n$ | |
|---|---|---|---|---|---|---|---|---|
| $-z$ | | | 0 | | $-$ | | 0 | |
| $\dots$ | 0 | | 0 | | $-$ | | 0 | $\dots$ |
| $x_{\beta[i]}$ | 0 | | 1 | | $-$ | | 0 | $\mu_i \geq 0$ |
| $\dots$ | 0 | | 0 | | $-$ | | 0 | $\dots$ |
| $x_{\beta[t]}$ | 0 | | 0 | | $\oplus$ | | 1 | $\mu_t < 0$ |
| $\dots$ | 0 | | 0 | | $-$ | | 0 | $\dots$ |

Consider now tableau $\tilde{T}$ in which $x_n$ re-enters the basis:

Tableau $\tilde{T}$ :

| | $\dots$ | $x_{\beta[i]}$ | $\dots$ | $x_h$ | $\dots$ | $x_n$ |
|---|---|---|---|---|---|---|
| | $+$ | $+$ | $+$ | $+$ | $+$ | $-$ |
| | | | | | | |

Now, $\tilde{T}$ has been obtained from $T$ by means of a sequence of pivot operations, hence there exist appropriate multipliers $\mu_1, \dots, \mu_m$ such that:

$$[\text{row } 0 \text{ of } \tilde{T}] = [\text{row } 0 \text{ of } T] + \sum_{i=1}^{m} \mu_i [\text{row } i \text{ of } T].$$

But then:

- $\underbrace{\tilde{c}_{\beta[t]}}_{=\tilde{c}_n<0} = \underbrace{\overline{c}_{\beta[t]}}_{=0} + \mu_t \quad \Rightarrow \quad \mu_t < 0$
- $\underbrace{\tilde{c}_{\beta[i]}}_{\geq 0} = \underbrace{\overline{c}_{\beta[i]}}_{=0} + \mu_i \quad \Rightarrow \quad \mu_i \geq 0 \; \forall i \neq t$

hence there is a contradiction:

$$\underbrace{\tilde{c}_h}_{\geq 0} = \underbrace{\overline{c}_h}_{<0} + \sum_{i \neq t} \underbrace{\overline{a}_{ih}}_{\leq 0} \underbrace{\mu_i}_{\geq 0} + \underbrace{\overline{a}_{th}}_{>0} \underbrace{\mu_t}_{<0} < 0.$$

**Theorem 5.1.1 (Farkas' Lemma)** *The inequality $\mathbf{c}^T\mathbf{x} \geq c_0$ is valid for the non-empty polyhedron $P := \{\mathbf{x} \geq 0 \ : \ \mathrm{A}\mathbf{x} = \mathbf{b}\}$ if and only if $\mathbf{u} \in \Re^m$ exists such that*

$$\mathbf{c}^T \geq \mathbf{u}^T\mathrm{A} \tag{5.3}$$

$$c_0 \leq \mathbf{u}^T\mathbf{b}. \tag{5.4}$$

**Proof**: As already seen, the fact that the condition is sufficient is trivially true, given that for all $\mathbf{x} \geq 0$ such that $\mathrm{A}\mathbf{x} = \mathbf{b}$ we have:

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{u}^T\mathrm{A}\mathbf{x} = \mathbf{u}^T\mathbf{b} \geq c_0.$$

We will now prove that the condition is also necessary, i.e., that "$\mathbf{c}^T\mathbf{x} \geq c_0$ valid for $P \neq \emptyset \Rightarrow \exists\mathbf{u} \in \Re^m \ : \ \mathbf{c}^T \geq \mathbf{u}^T\mathrm{A}\,,\ c_0 \leq \mathbf{u}^T\mathbf{b}$". By the hypothesis of validity, we have that

$$c_0 \leq z^* := \min\{\mathbf{c}^T\mathbf{x} \ : \ \mathrm{A}\mathbf{x} = \mathbf{b}\ ,\ \mathbf{x} \geq 0\}, \tag{5.5}$$

which excludes $z^* = -\infty$. Let then $\mathbf{x}^*$ be an optimal basic feasible solution found by the simplex algorithm applied to problem (5.5). This solution exists by the convergence property of the simplex algorithm. In addition, let B be an optimal basis associated with $\mathbf{x}^*$, and let us partition as usual $\mathrm{A} = [\mathrm{B}, \mathrm{F}]$, $\mathbf{c}^T = [\mathbf{c}_\mathrm{B}^T, \mathbf{c}_\mathrm{F}^T]$, and $\mathbf{x}^* = (\mathbf{x}_\mathrm{B}^*, \mathbf{x}_\mathrm{F}^*)$ with $\mathbf{x}_\mathrm{B}^* = \mathrm{B}^{-1}\mathbf{b}$ and $\mathbf{x}_\mathrm{F}^* = 0$. We will prove that vector

$$\mathbf{u}^T := \mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}$$

verifies conditions (5.3) and (5.4), hence that the thesis is valid. Recalling the reduced cost expression computed in correspondence of the optimal basis B we have:

$$\overline{\mathbf{c}}^T \ := \ \mathbf{c}^T - \underbrace{\mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}}_{\mathbf{u}^T}\mathrm{A} \geq 0^T \ \Rightarrow \ \mathbf{c}^T \geq \mathbf{u}^T\mathrm{A}$$

and thus (5.3) is valid. In addition, for (5.5) we have that

$$c_0 \leq z^* = \mathbf{c}^T\mathbf{x}^* \ = \ \mathbf{c}_\mathrm{B}^T\mathbf{x}_\mathrm{B}^* + \mathbf{c}_\mathrm{F}^T\mathbf{x}_\mathrm{F}^* \ = \ \mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathbf{b} \ = \ \mathbf{u}^T\mathbf{b},$$

and thus also (5.4) is verified. □

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**Proposition 5.3.1** *The dual of the dual problem coincides with the primal problem.*

**Proof**: Considering without loss of generality a primal problem in canonical form and applying the known equivalence and transformation rules, we obtain:

$$\left\{\begin{array}{lccc} \min & \mathbf{c}^T\mathbf{x} & & \\ & \mathrm{A}\mathbf{x} & \geq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{lccc} \max & \mathbf{u}^T\mathbf{b} & & \\ & \mathbf{c}^T \geq & \mathbf{u}^T\mathrm{A} & \\ & \mathbf{u} & \geq & 0 \end{array}\right. \equiv \left\{\begin{array}{lccc} -\min & (-\mathbf{b}^T)\mathbf{u} & & \\ & (-\mathrm{A}^T)\mathbf{u} & \geq & -\mathbf{c} \\ & \mathbf{u} & \geq & 0 \end{array}\right.$$

## 5.8 Sensitivity Analysis

Once an optimal solution for the starting problem is found, it is often interesting to evaluate the "stability" of this solution with respect to changes in the problem data. In real applications, indeed, the mathematical model is often an approximation of the real situation. The model is usually deemed more reliable if its solutions are less sensitive to changes in the data (often obtained with measurements affected by error).

Consider a problem in standard form of the type $\min\{\mathbf{c}^T\mathbf{x} \,:\, \mathrm{A}\mathbf{x} = \mathbf{b} \,,\, \mathbf{x} \geq 0\}$ and let B be an optimal basis—e.g., the one identified by means of the simplex algorithm. Under the usual notation, the corresponding optimal basic solution is $\overline{\mathbf{x}} = (\overline{\mathbf{x}}_{\mathrm{B}}, \overline{\mathbf{x}}_{\mathrm{F}})$, with $\overline{\mathbf{x}}_{\mathrm{F}} = 0$ and $\overline{\mathbf{x}}_{\mathrm{B}} = \mathrm{B}^{-1}\mathbf{b}$. By theorem 5.5.1, $\overline{\mathbf{x}}$ is optimal if and only if there exists $\overline{\mathbf{u}} \in \Re^m$ such that the optimality conditions $(1') - (3')$ are satisfied, for instance $\overline{\mathbf{u}} = \mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}$. Note that, in case of degeneracy, $\overline{\mathbf{u}}$ is not necessarily unique, but it depends on the basis B associated with $\overline{\mathbf{x}}$. The optimality conditions written for basis B become:

$$
\begin{array}{lll}
(c.1) & \mathrm{B}^{-1}\mathbf{b} \geq 0 & \text{(primal feasibility for } \overline{\mathbf{x}}\text{)} \\
(c.2) & \overline{\mathbf{c}}^T := \mathbf{c}^T - \underbrace{\mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}}_{\overline{\mathbf{u}}^T}\mathrm{A} \geq 0^T & \text{(dual feasibility for } \overline{\mathbf{u}}\text{)}
\end{array}
$$

while, as is well known, the complementary condition $(3')$ derives from the choice $\overline{\mathbf{u}} = \mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}$.

Sensitivity analysis is the study of the perturbations to the initial data whereby conditions $(c.1)$ and $(c.2)$ remain verified. This study makes it possible to define the necessary and sufficient conditions such that basis B remains feasible and optimal when changing data: i.e., the conditions refer to *basis* B and not to the corresponding *basic solution* $\overline{\mathbf{x}}$. For the sake of simplicity, we will only consider the following possibilities:

- Changes in the right-hand sides: $\mathbf{b} \;\rightarrow\; \mathbf{b} + \Delta\mathbf{b}$
- Changes in the costs of basic variables: $\mathbf{c}_{\mathrm{B}}^T \;\rightarrow\; \mathbf{c}_{\mathrm{B}}^T + \Delta\mathbf{c}_{\mathrm{B}}^T$
- Changes in the costs of non-basic variables: $\mathbf{c}_{\mathrm{F}}^T \;\rightarrow\; \mathbf{c}_{\mathrm{F}}^T + \Delta\mathbf{c}_{\mathrm{F}}^T$.

**Changes in the right-hand sides**

Assuming a change $\Delta\mathbf{b}$ of vector $\mathbf{b}$ of the right-hand sides, the optimality conditions $(c.1)$ - $(c.2)$ for basis B become

$$\begin{aligned} &(c.1) \quad \mathrm{B}^{-1}(\mathbf{b}+\Delta\mathbf{b}) \geq 0 \\ &(c.2) \quad \overline{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathrm{A} \geq 0^T \text{ (unchanged).} \end{aligned}$$

Thus, basis B remains feasible and optimal if and only if:

$$\mathrm{B}^{-1}\mathbf{b} \geq -\mathrm{B}^{-1}\Delta\mathbf{b}.$$

This system of $m$ inequalities in the $m$ variables $\Delta b_i$ defines a polyhedron in $\Re^m$ containing vectors $\Delta\mathbf{b}$ for which the optimal basis does not change. Note that when $\mathbf{b}$ changes, the coordinates of the corresponding basic solution $\overline{\mathbf{x}}$ and the optimal value of the objective function change. Indeed, the optimal value changes from $\mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}\mathbf{b}$ to $\mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1}(\mathbf{b}+\Delta\mathbf{b})$, with a change of

$$\Delta z \;:=\; (\mathbf{c}_\mathrm{B}^T\mathrm{B}^{-1})\Delta\mathbf{b} = \overline{\mathbf{u}}^T\Delta\mathbf{b} = \sum_{i=1}^{m} \overline{u}_i \Delta b_i.$$

The dual variables $\overline{u}_i$ measure thus the “sensitivity” of the optimal value of the objective function with respect to changes $\Delta b_i$ of the right-hand sides.

## Changes in the costs of non-basic variables

Consider now a change $\Delta\mathbf{c}_{\mathrm{F}}^T$ of vector $\mathbf{c}_{\mathrm{F}}^T$, and let $\bar{\mathbf{c}}$ and $\tilde{\mathbf{c}}$ be the reduced cost vectors before and after change $\Delta\mathbf{c}_{\mathrm{F}}$, respectively. Conditions $(c.1)$ and $(c.2)$ become:

$$\begin{aligned} &(c.1) \quad \mathrm{B}^{-1}\mathbf{b} \geq 0 \text{ (unchanged)} \\ &(c.2) \quad \tilde{\mathbf{c}}^T := [\tilde{\mathbf{c}}_{\mathrm{B}}^T, \tilde{\mathbf{c}}_{\mathrm{F}}^T] \;=\; [0^T, (\mathbf{c}_{\mathrm{F}}^T + \Delta\mathbf{c}_{\mathrm{F}}^T) - \mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}\mathrm{F}] \geq 0^T \end{aligned}$$

hence basis B remains optimal if and only if

$$\tilde{\mathbf{c}}_F^T = \underbrace{\mathbf{c}_{\mathrm{F}}^T - \mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}\mathrm{F}}_{\bar{\mathbf{c}}_{\mathrm{F}}^T} + \Delta\mathbf{c}_{\mathrm{F}}^T \;=\; \bar{\mathbf{c}}_{\mathrm{F}}^T + \Delta\mathbf{c}_{\mathrm{F}}^T \geq 0 \;\;\Leftrightarrow\;\; \Delta\mathbf{c}_{\mathrm{F}} \geq -\bar{\mathbf{c}}_{\mathrm{F}}.$$

In this way, we obtain the $n - m$ inequalities, independent from each other,

$$\Delta c_j \geq -\bar{c}_j \;\forall x_j \;\text{ non-basic.}$$

It follows that the reduced cost $\bar{c}_j \geq 0$ can be interpreted as the maximum *decrease* in cost $c_j$ under which basis B remains optimal: greater decreases would produce a reduced cost $\tilde{c}_j < 0$, hence *basis* B would be no longer optimal (in case of degeneracy, however, the *basic solution* $\bar{\mathbf{x}}$ could still remain optimal).

**Changes in the costs of basic variables**

Consider now a change $\Delta\mathbf{c}_{\mathrm{B}}^T$ of vector $\mathbf{c}_{\mathrm{B}}^T$. Indicating as before with $\bar{\mathbf{c}}$ and $\tilde{\mathbf{c}}$ the reduced cost vectors before and after the change, respectively, we obtain the following conditions:

$$\begin{array}{ll} (c.1) & \mathrm{B}^{-1}\mathbf{b} \geq 0 \text{ (unchanged)} \\ (c.2) & \tilde{\mathbf{c}}^T := [\tilde{\mathbf{c}}_{\mathrm{B}}^T, \tilde{\mathbf{c}}_{\mathrm{F}}^T] \;=\; [0^T, \mathbf{c}_{\mathrm{F}}^T - (\mathbf{c}_{\mathrm{B}}^T + \Delta\mathbf{c}_{\mathrm{B}}^T)\mathrm{B}^{-1}\mathrm{F}] \geq 0^T, \end{array}$$

from which we obtain:

$$\tilde{\mathbf{c}}_{\mathrm{F}}^T := \underbrace{\mathbf{c}_{\mathrm{F}}^T - \mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}\mathrm{F}}_{\bar{\mathbf{c}}_{\mathrm{F}}^T} - \Delta\mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}\mathrm{F} \;\geq\; 0^T,$$

i.e.

$$\Delta\mathbf{c}_{\mathrm{B}}^T\mathrm{B}^{-1}\mathrm{F} \leq \bar{\mathbf{c}}_{\mathrm{F}}^T.$$

This system defines a polyhedron in $\Re^m$, whose points correspond to the vectors $\Delta\mathbf{c}_{\mathrm{B}}$ for which the optimal basis does not change.

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**Theorem 6.2.3** *Let* $\mathrm{A}$ *be a matrix with* $a_{ij} \in \{-1, 0, 1\}\ \forall i, j$. $\mathrm{A}$ *is totally unimodular if the following conditions hold (see Figure 6.2):*

*(1)* *every column of* $\mathrm{A}$ *has no more that two non-zero elements;*

*(2)* *there exists a partition* $(I_1, I_2)$ *of the rows of* $\mathrm{A}$ *such that each column with two non-zero elements has these two elements belonging to rows on different sets if and only if the two elements have the same sign.*

**Proof**: We have to prove that $det(\mathrm{Q}) \in \{-1, 0 - 1\}$ for any submatrix Q of A of order $k$ $(k = 1, \ldots, m)$. The proof is by induction on $k$.

If $k$=1 then $\mathrm{Q} = [a_{ij}]$ and hence $det(\mathrm{Q}) = a_{ij} \in \{-1, 0, 1\}$, as requested.

Let us suppose now that $det(\mathrm{Q}') \in \{-1, 0, 1\}$ for any submatrix Q′ of order $k'$, where $k' \geq 1$ is a fixed value. Let us consider any submatrix Q of order $k := k'+1$. By condition (1) only three cases can occur:

- Q has one column of zeros: in this case $det(\mathrm{Q}) = 0$.
- Q has a column with only one element different from zero: in this case, barring permutations of rows and/or columns, Q is of the type

$$\mathrm{Q} = \left[\begin{array}{c|ccc} \pm 1 & * & \ldots & * \\ \hline 0 & & & \\ \vdots & & \mathrm{Q}' & \\ 0 & & & \end{array}\right] \quad \Rightarrow \quad det(\mathrm{Q}) = \pm 1 \cdot det(\mathrm{Q}')$$

  where $det(\mathrm{Q}') \in \{-1, 0, 1\}$ given that Q′ has order $k - 1 = k'$.
- Each column of Q has exactly two non-zero elements: in this case, let $I(\mathrm{Q})$ be the set of rows of Q. By hypothesis (2), Q has the following form

$$\mathrm{Q} = \begin{array}{|cccc|l} \hline 1 & & 1 & & \multirow{2}{*}{$I_1 \cap I(\mathrm{Q})$} \\ & -1 & -1 & & \\ \hline 1 & & & -1 & \multirow{2}{*}{$I_2 \cap I(\mathrm{Q})$} \\ & -1 & & 1 & \\ \hline \end{array}$$

  But then we have

$$\sum_{i \in I_1 \cap I(\mathrm{Q})} [\text{row } i \text{ of Q}] - \sum_{i \in I_2 \cap I(\mathrm{Q})} [\text{row } i \text{ of Q}] = [\text{null row}]$$

  given that in every column the two non-zero elements cancel out. It follows that the rows of Q are linearly dependent, hence $det(\mathrm{Q}) = 0$.

In each of the three cases above we therefore have $det(\mathrm{Q}) \in \{-1, 0, 1\}$, hence this property applies to all submatrices Q of order $k = k' + 1$. Applying the same reasoning, the result can be inductively extended to matrices Q of any order. $\square$

The proof of the following properties is straightforward, and is left as an exercise to the reader.