ILP Models

Set Covering Problem

$$x_j = \begin{cases} 1 & \text{if column } j \text{ is selected in } S \\ 0 & \text{otherwise} \end{cases} \qquad (j = 1, \dots, n)$$

$$\min \sum_{j=1}^{n} c_j \, x_j \tag{3.35}$$

$$\sum_{j=1}^{n} a_{ij} \, x_j \geq 1 \qquad i = 1, \dots, m \tag{3.36}$$

$$x_j \in \{0, 1\} \qquad j = 1, \dots, n \tag{3.37}$$

Set Partitioning

This is a variant of the Set Covering Problem in which each row must be covered *exactly* once. The model is similar to that of the Set Covering but constraints (3.36) must be rewritten with the equality sign, which, in practice, makes the problem even more difficult to solve.

Steiner Tree

$$x_{ij} = \begin{cases} 1 & \text{if arc from } (i,j) \in A \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

we obtain the following ILP model:

$$\min \underbrace{\sum_{(i,j)\in A} c_{ij} x_{ij}}_{\text{solution cost}} \tag{8.3a}$$

$$\underbrace{\sum_{(i,j)\in\delta^-(j)} x_{ij}}_{\text{n. of arcs entering } j} \begin{cases} = 1 & \text{for all } j \in T \\ = 0 & \text{for } j = r \\ \leq 1 & \text{for all } j \in \mathrm{V} \setminus (T \cup \{r\}) \end{cases} \tag{8.3b}$$

$$\textstyle\sum_{(i,j)\in\delta^+(S)} x_{ij} \geq \sum_{(i,t)\in\delta^-(t)} x_{it}\ , \text{ for all } S \subset V : r \in S, \text{ and for all } t \in V \setminus S \tag{8.3c}$$

$$x_{ij} \geq 0 \text{ integer}\ ,\ (i,j) \in A. \tag{8.3d}$$

Plant Location

Introducing a variable for each edge $[i, j] \in E$

$$x_{ij} = \begin{cases} 1 & \text{if user } i \text{ connects to location } j \\ 0 & \text{otherwise} \end{cases}$$

and a variable for each location $j \in \{1, \ldots, m\}$

$$y_j = \begin{cases} 1 & \text{if location } j \text{ is activated} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the following ILP model:

$$\min \underbrace{\sum_{j=1}^{m} d_j y_j}_{\text{fixed cost}} + \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}}_{\text{connection cost}} \tag{8.4a}$$

$$\underbrace{\sum_{j=1}^{m} x_{ij} = 1\,,}_{\text{each user connects to exactly one location}} \qquad i \in \{1, \ldots, n\} \tag{8.4b}$$

$$\underbrace{x_{ij} \leq y_j\,,}_{\text{connection } i-j \;\Rightarrow\; \text{loc. } j \text{ activated}} \qquad i \in \{1, \ldots, n\}\,,\ j \in \{1, \ldots, m\} \tag{8.4c}$$

$$x_{ij} \geq 0 \text{ integer}\,,\ i \in \{1, \ldots, n\}\,,\ j \in \{1, \ldots, m\} \tag{8.4d}$$

$$0 \leq y_j \leq 1 \text{ integer}\,,\ j \in \{1, \ldots, m\}. \tag{8.4e}$$

Knapsack

Introducing the decision variables

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

we get the ILP model:

$$z^* := \max \sum_{j=1}^{n} p_j x_j \tag{8.1a}$$

$$\sum_{j=1}^{n} w_j x_j \leq W \tag{8.1b}$$

$$0 \leq x_j \leq 1 \text{ integer }, \; j \in \{1, \ldots, n\}. \tag{8.1c}$$

MST

$$x_e = \begin{cases} 1 & \text{if } e \text{ is chosen in the minimum spanning tree} \\ 0 & \text{otherwise.} \end{cases}$$

We then have the model:

$$\min \underbrace{\sum_{e \in E} c_e x_e}_{\text{total cost}} \tag{7.1a}$$

$$\underbrace{\sum_{e \in E} x_e = n - 1}_{n-1 \text{ chosen edges}} \tag{7.1b}$$

$$\underbrace{\sum_{e \in E(S)} x_e \leq |S| - 1}_{\text{cycle elimination condition}} \quad \forall S \subseteq V \; : \; S \neq \emptyset \tag{7.1c}$$

Shortest Path

$$\min \underbrace{\sum_{(i,j)\in A} c_{ij}x_{ij}}_{\text{path cost}} \tag{7.2a}$$

$$\underbrace{\sum_{(h,j)\in\delta^+(h)} x_{hj}}_{n.\text{ leaving arcs}} - \underbrace{\sum_{(i,h)\in\delta^-(h)} x_{ih}}_{n.\text{ entering arcs}} = \begin{cases} 1 & \text{if } h = s \\ -1 & \text{if } h = t \\ 0 & \text{for all } h \in \mathrm{V} \setminus \{s,t\} \end{cases} \tag{7.2b}$$

$$\underbrace{\sum_{(i,j)\,\in\, A(S)} x_{ij}}_{n.\text{ arcs in } S} \le |S| - 1 \,,\ \forall\, S \subseteq V \ :\ S \neq \emptyset \tag{7.2c}$$

$$0 \le x_{ij} \le 1 \quad \text{integer} \,,\ (i,j) \in A \tag{7.2d}$$

Traveling Sales Person

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ is chosen in the optimal circuit} \\ 0 & \text{otherwise.} \end{cases}$$

We thus obtain the following model:

$$\min \underbrace{\sum_{(i,j)\in A} c_{ij}x_{ij}}_{\text{circuit cost}} \tag{8.2a}$$

$$\underbrace{\sum_{(i,j)\in\delta^-(j)} x_{ij}}_{\text{one arc entering } j} = 1 \,,\ j \in V \tag{8.2b}$$

$$\underbrace{\sum_{(i,j)\in\delta^+(i)} x_{ij}}_{\text{one arc leaving } i} = 1 \,,\ i \in V \tag{8.2c}$$

$$\underbrace{\sum_{(i,j)\in\delta^+(S)} x_{ij}}_{\text{reachability from 1}} \ge 1 \,,\ S \subset V \ :\ 1 \in S \tag{8.2d}$$

$$x_{ij} \ge 0 \ \text{ integer} \,,\ (i,j) \in A. \tag{8.2e}$$

Max Flow

$$\mathbf{MAX-FLOW}: \quad \max\{\varphi_0 := \sum_{(s,j)\in\delta^+(s)} x_{sj} - \sum_{(i,s)\in\delta^-(s)} x_{is} \; : \; \text{constraints } (7.3)-(7.4)\}.$$

$$0 \le x_{ij} \le k_{ij}\ ,\ (i,j)\ \in A \tag{7.3}$$

$$\underbrace{\sum_{(h,j)\,\in\,\delta^+(h)} x_{hj}}_{\text{flow leaving from } h} - \underbrace{\sum_{(i,h)\,\in\,\delta^-(h)} x_{ih}}_{\text{flow entering into } h} = 0\ ,\ h\ \in\ V \setminus \{s,t\}. \tag{7.4}$$