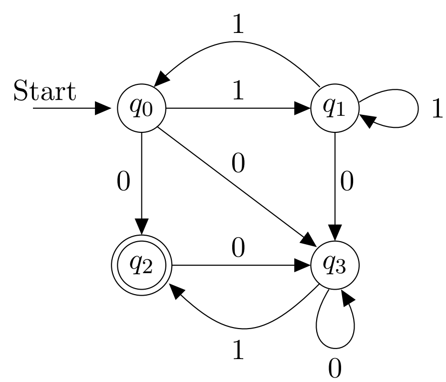
Master Degree in Computer Engineering

Final Exam for Automata, Languages and Computation

January 30th, 2024

1. **[6 points]** Assume the NFA A whose transition function is graphically represented below.



Consider the algorithm for transforming a FA into a regular expression, based on state elimination. Apply the following steps in the given order:

(a) eliminate state q_1 from A, and display the resulting automaton A';

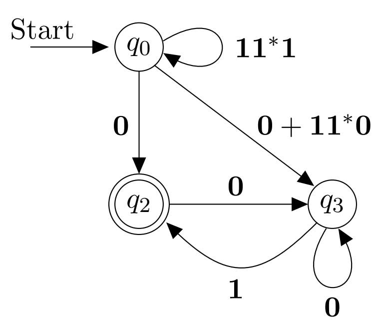
(b) eliminate state q_3 from A', and display the resulting automaton A'';

(c) convert A'' into the equivalent regular expression E_{q_2}.

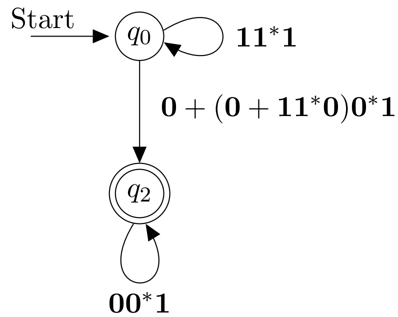
If you simplify any of the resulting regular expressions, **add some discussion**.

Solution Recall that, for every regular expression R, we have $\emptyset + R = R$, $\emptyset R = R\emptyset = \emptyset$, and $\epsilon R = R\epsilon = R$. We use these simplifications several times below.

(a) After the elimination of q_1 from A we obtain the automaton A', graphically represented as



(b) After the elimination of q_3 from A' we obtain the automaton A'', graphically represented as



(c) The automaton A'' has two states, with the initial and the final states representing distinct states. We need to apply the expression $E_q = (R + SU^*T)^*SU^*$, considering that in our case we have

$$\begin{aligned} R &= \mathbf{11^*1} \\ S &= \mathbf{0 + (0 + 11^*0)0^*1} \\ U &= \mathbf{00^*1} \\ T &= \emptyset \ . \end{aligned}$$

We then obtain the regular expression

$$\begin{aligned} E_{q_2} &= (\mathbf{11^*1} + (\mathbf{0} + (\mathbf{0} + \mathbf{11^*0})\mathbf{0^*1})(\mathbf{00^*1})^*\emptyset)^*(\mathbf{0} + (\mathbf{0} + \mathbf{11^*0})\mathbf{0^*1})(\mathbf{00^*1})^* \\ &= (\mathbf{11^*1} + \emptyset)^*(\mathbf{0} + (\mathbf{0} + \mathbf{11^*0})\mathbf{0^*1})(\mathbf{00^*1})^* \\ &= (\mathbf{11^*1})^* + (\mathbf{0} + (\mathbf{0} + \mathbf{11^*0})\mathbf{0^*1})(\mathbf{00^*1})^* \ . \end{aligned}$$

2. **[9 points]** Consider the following languages, defined over the alphabet $\Sigma = \{a, b\}$:

$$\begin{aligned} L_1 &= \{ba^m ba^n b \mid m, n \geq 1, \ m < n\} \\ L_2 &= \{ba^m a^n b \mid m, n \geq 1, \ m < n\} \\ L_3 &= L_2 L_1 \end{aligned}$$

For each of the above languages, state whether it belongs to REG, to CFL$\setminus$REG, or else whether it is outside of CFL. Provide a mathematical proof for all of your answers.

Solution

(a) L_1 belongs to the class CFL$\setminus$REG.

We first show that L_1 is not a regular language, by applying the pumping lemma for this class. Let N be the pumping lemma constant for L_1. We choose the string $w = ba^N ba^{N+1} b \in L_1$ with $|w| \geq N$, and consider all possible factorizations $w = xyz$ satisfying the conditions $|y| \geq 1$ and $|xy| \leq N$. We distinguish two cases.

Case 1: y spans the leftmost occurrence of b in w, and possibly more symbols from w. This means that $x = \epsilon$. We then choose $k = 0$ and obtain the string $w_0 = xy^0z = z$ which has fewer than 3 occurrences of symbol b, and therefore $w_0 \notin L_1$.

Case 2: y does not span the leftmost occurrence of b in w. Because of the condition $|xy| \leq N$, we have that y can only contain occurrences of symbol a, with these occurrences placed to the left of the second occurrence of symbol b in w. In this case, we choose $k = 2$ and obtain the string $w_2 = xy^2z$ which has the form $ba^{N+|y|}ba^{N+1}b$. Because of the condition $|y| \geq 1$, we have that $N + |y| \geq N + 1$, and therefore $w_2 \notin L_1$.

Since we have considered all possible factorizations for string w, we must conclude that L_1 is not a regular language.

As a second part of the answer, we need to show that L_1 belongs to the class CFL. Consider the CFG G_1 with productions:

$$
\begin{aligned}
S &\to bAb \\
A &\to aAa \mid aBa \\
B &\to Ba \mid ba
\end{aligned}
$$

It is not difficult to see that $L(G_1) = L_1$.

(b) L_2 belongs to the class REG.

To see this, we observe that we can rewrite the definition of this language as $L_2 = \{ba^n b \mid n \geq 3\}$. It is then easy to see that the regular expression $R = \boldsymbol{baaaa^*b}$ generates L_2.

(c) L_3 belongs to the class CFL$\smallsetminus$REG.

The easy part here is to show that L_3 is in CFL. We have already seen that L_2 is in REG and therefore in CFL, and we have already shown that L_1 is in CFL. Since $L_3 = L_2L_1$, and since the class CFL is closed under concatenation, we conclude that L_3 is in CFL.

We now prove that L_3 is not a regular language, again by applying the pumping lemma for this class. Let N be the pumping lemma constant for L_3. We choose the string $w = ba^3bba^Nba^{N+1}b \in L_3$ with $|w| \geq N$, and consider all possible factorizations $w = xyz$ satisfying the conditions $|y| \geq 1$ and $|xy| \leq N$. We observe that string w has three runs of symbols a: the first of length 3, the second of length N, and the third of length $N+1$. We call these three runs block 1, block 2, and block 3, respectively. We distinguish three cases.

Case 1: y spans at least one occurrence of b from w. We then choose $k = 0$ and obtain the string $w_0 = xy^0z = xz$ which has fewer than 5 occurrences of symbol b, and therefore $w_0 \notin L_3$.

Case 2: y spans zero occurrence of b and a few occurrences of symbol a from block 1 only. We choose $k = 0$ and obtain the string $w_0 = xy^0z = xz$ which has the form $ba^{3-|y|}ba^Nba^{N+1}b$. Because of the condition $|y| \geq 1$, we have $3 - |y| < 3$, and therefore $w_0 \notin L_3$.

Case 3: y spans zero occurrence of b and a few occurrences of symbol a from block 2 only. We choose $k = 2$ and obtain the string $w_2 = xy^2z$ which has the form $ba^3bba^{N+|y|}ba^{N+1}b$. Because of the condition $|y| \geq 1$, we have that $N + |y| \geq N + 1$, and therefore $w_2 \notin L_3$.

Since we have considered all possible factorizations for string w, we must conclude that L_3 is not a regular language.

We observe that the above proof showing that L_3 is not in REG is a little bit involved. There is an alternative, simpler way of proving that L_3 is not a regular language. Assume by now that L_3 is a regular language. From known properties of regular languages, it follows that L_3^R is also a regular language, where R is the string reversal operator, extended to languages as usual. Observing that we have $L_3^R = L_1^R L_2^R$, the language L_3^R can be rewritten as

$$L_3^R \quad = \quad \{ba^m ba^n bba^p b \mid m, n \geq 1,\ m > n, p \geq 3\}$$

We can now apply the pumping lemma to L_3^R, resulting in a proof that is very similar to the proof for L_1, consisting only of two cases. We then find that L_3^R is not a regular language, and we must therefore conclude that L_3 cannot be regular as well.

3. **[6 points]** With reference to the membership problem for context-free languages, answer the following two questions.

   (a) Specify the dynamic programming algorithm reported in the textbook for the solution of this problem.

   (b) Consider the CFG G in Chomsky normal form defined by the following rules:

$$\begin{aligned} S &\to CD \\ C &\to AC' \mid c \\ C' &\to CB \\ A &\to a \\ B &\to b \\ D &\to DD \mid d \end{aligned}$$

   Assuming as input the CFG G and the string $w = aacbbdddd$, trace the application of the algorithm in (a).

Solution

(a) The required dynamic programming algorithm is reported in Section 7.4.4 of the textbook.

(b) On input w and G, the algorithm constructs the table reported below.

$\{S\}$								
$\{S\}$								
$\{S\}$								
$\{S\}$								
$\{C\}$								
	$\{C'\}$				$\{D\}$			
	$\{C\}$				$\{D\}$	$\{D\}$		
		$\{C'\}$			$\{D\}$	$\{D\}$	$\{D\}$	
$\{A\}$	$\{A\}$	$\{C\}$	$\{B\}$	$\{B\}$	$\{D\}$	$\{D\}$	$\{D\}$	$\{D\}$
a	a	c	b	b	d	d	d	d

4. **[5 points]** Assess whether the following statements are true or false. Provide motivations for all of your answers.

   (a) Let L_1, L_3 be in REG (the class of regular languages) and let L_2 be in CFL. Then the language $L_1L_2L_3$ is always in REG.

   (b) Let L_1, L_3 be in REG and let L_2 be in CFL. Then the language $L_1L_2L_3$ is always in CFL.

   (c) The class RE defined over the alphabet $\Sigma = \{0, 1\}$ is closed under complementation.

   (d) The class $\mathcal{P}$ of languages over the alphabet $\Sigma = \{0, 1\}$ that can be recognized in polynomial time by a TM is closed under complementation.

Solution

(a) False. Consider as a counterexample the regular languages $L_1 = L_3 = \{\epsilon\}$ and the context-free language $L_2 = \{a^n b^n \mid n \geq 1\}$. Observe that $L_1L_2L_3 = L_2$, and we know that L_2 is not a regular language.

(b) True. We know that a language in REG is also a language in CFL. We also know that the class CFL is closed under concatenation. Therefore $L' = L_1L_2$ must be in CFL, and $L'L_3 = L_1L_2L_3$ must be in CFL.

(c) False. As a counterexample consider the language L_{ne} in RE, defined in the textbook. Consider also the language L_e, which is the complement of L_{ne} with respect to Σ^*. We now that L_e is not in RE.

(d) True. Consider an arbitrary language $L \in \mathcal{P}$. By the definition of the class $\mathcal{P}$, there exists a TM M such that $L(M) = L$, and M stops after a polynomial number of steps in the size of its input w. We can then construct a TM M' that, given as input a string w, simulates M on w. When the simulation stops in a state q, that is, when there is no next move for M, M' moves to a final state if q is not a final state for M, and M' moves to a non-final state if q is a final state for M. It is easy to see that $L(M') = \overline{L}$ and that M' runs in polynomial time. We therefore conclude that $\mathcal{P}$ is closed under complementation.

5. **[7 points]** Let R be the string reversal operator, extended to languages as usual. Consider the following property of the RE languages defined over the alphabet $\Sigma = \{0, 1\}$

$$\mathcal{P} \;=\; \{L \mid L \in \text{RE},\ L \cap L^R = \emptyset\}$$

where the condition $L \cap L^R = \emptyset$ means that for every string $w \in L$, w^R does not belong to L. Define $L_{\mathcal{P}} = \{\mathsf{enc}(M) \mid L(M) \in \mathcal{P}\}$.

(a) Use Rice's theorem to show that $L_{\mathcal{P}}$ is not in REC.

(b) State whether $L_{\mathcal{P}}$ is in RE$\smallsetminus$REC or else outside of RE.

Solution

(a) We have to show that property $\mathcal{P}$ is not trivial.

- $\mathcal{P} \neq \emptyset$. Consider the language $L = \{1100\}$. Since L is finite, L is also in RE. Observe that $L^R = \{0011\}$ and $L \cap L^R = \emptyset$. Therefore $L \in \mathcal{P}$.
- $\mathcal{P} \neq$ RE. Consider the language $L = \{1100, 0011\}$. Since L is finite, L is also in RE. Observe that $L \cap L^R = L \neq \emptyset$, and therefore $L \notin \mathcal{P}$.

(b) We now show that $L_{\mathcal{P}}$ is not in RE. The most convenient way to do this is to consider the complement language $\overline{L_{\mathcal{P}}} = L_{\overline{\mathcal{P}}}$, where $\overline{\mathcal{P}}$ is the complement of class $\mathcal{P}$ with respect to RE and can be specified as

$$\overline{\mathcal{P}} \;=\; \{L \mid L \in \text{RE},\ L \cap L^R \neq \emptyset\}$$

We specify a nondeterministic TM N such that $L(N) = L_{\overline{\mathcal{P}}}$. Since every nondeterministic TM can be converted into a standard TM, this shows that $L_{\overline{\mathcal{P}}}$ is in RE. Our nondeterministic TM N takes as input the encoding of a TM M and performs the following steps.

- N nondeterministically guesses a string $w \in \Sigma^*$ and checks that $w \in L(M)$ and $w^R \in L(M)$ are both satisfied.
- If the previous step terminates and is successful, N ends the computation in a final state. In all other cases, N ends the computation in a non-final state or runs for ever.

It is not difficult to see that $L(N) = L_{\overline{\mathcal{P}}}$.

Since $L_{\overline{\mathcal{P}}}$ is in RE, if its complement language $L_{\mathcal{P}}$ were in RE as well, then we would conclude that both languages are in REC, from a theorem in Chapter 9 of the textbook. But we have already shown in (a) that $L_{\mathcal{P}}$ is not in REC. We must therefore conclude that $L_{\mathcal{P}}$ is not in RE.