Machine Learning

Linear Models

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Linear Predictors and Affine Functions

Consider $\mathcal{X} = \mathbb{R}^d$

"Linear" (affine) functions:

$$L_d = \{h_{\mathbf{w},b} : \mathbf{w} \in \mathbb{R}^d, b \in \mathbb{R}\}$$

parameters

where

$$h_{\mathbf{w},b}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle + b = \left(\sum_{i=1}^{d} w_i x_i \right) + b$$

"

features

$|L_d| = +\infty$

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Note:

- each member of L_d is a function $\mathbf{x} \to \underbrace{\langle \mathbf{w}, \mathbf{x} \rangle + b}_{\in \mathbb{R}}$
- b: *bias*

Linear Models

Hypothesis class $\mathcal{H}$: $\phi \circ L_d$, where $\phi : \mathbb{R} \to \mathcal{Y}$

- $h \in \mathcal{H}$ is $h : \mathbb{R}^d \to \mathcal{Y}$

ϕ depends on the learning problem

binary classification: $\phi = \text{sign}$

regression: $\phi(x) = x$

Linear Models

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ϕ depends on the learning problem

Example

- binary classification, $\mathcal{Y} = \{-1, 1\} \Rightarrow \phi(z) = \text{sign}(z)$
- regression, $\mathcal{Y} = \mathbb{R} \Rightarrow \phi(z) = z$

Equivalent Notation

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix} \in \mathbb{R}^d$

Given $\mathbf{x} \in \mathcal{X}$, $\mathbf{w} \in \mathbb{R}^d$, $b \in \mathbb{R}$, define:

- $\mathbf{w}' = (b, w_1, w_2, \ldots, w_d) \in \mathbb{R}^{d+1}$

Equivalent Notation

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Given $\mathbf{x} \in \mathcal{X}$, $\mathbf{w} \in \mathbb{R}^d$, $b \in \mathbb{R}$, define:

- $\mathbf{w}' = (b, w_1, w_2, \ldots, w_d) \in \mathbb{R}^{d+1}$
- $\mathbf{x}' = (1, x_1, x_2, \ldots, x_d) \in \mathbb{R}^{d+1}$

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Then:

$$h_{\mathbf{w},b}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle + b = \langle \mathbf{w}', \mathbf{x}' \rangle \qquad (1)$$

$\Rightarrow$ we will consider bias term as part of $\mathbf{w}$ and assume $\mathbf{x} = (1, x_1, x_2, \ldots, x_d)$ *when needed*, with $h_{\mathbf{w}}(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle$

Linear Classification

$\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{-1, 1\}$, 0-1 loss

Hypothesis class = *halfspaces*

$$HS_d = \text{sign} \circ L_d = \{\mathbf{x} \to \text{sign}(h_{\mathbf{w},b}(\mathbf{x})) : h_{\mathbf{w},b} \in L_d\}$$

Example: $\mathcal{X} = \mathbb{R}^2$