Machine Learning

VC-Dimension

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PAC Learning

Question: which hypothesis classes $\mathcal{H}$ are PAC learnable?

Up to now: if $|\mathcal{H}| < +\infty \Rightarrow \mathcal{H}$ is PAC learnable.

What about $\mathcal{H}$: $|\mathcal{H}| = +\infty$? Not PAC learnable?

We focus on:

- *binary classification*:$\mathcal{Y} = \{0, 1\}$
- 0-1 loss

but similar results apply to other learning tasks and losses.

Restrictions

Definition (Restriction of $\mathcal{H}$ to $\mathcal{C}$)

Let $\mathcal{H}$ be a class of functions from $\mathcal{X}$ to $\{0,1\}$ and let $C = \{c_1, \ldots, c_m\} \subset \mathcal{X}$. The restriction $\mathcal{H}_C$ of $\mathcal{H}$ to C is:

$$\mathcal{H}_C = \{[h(c_1), \ldots, h(c_m)] : h \in \mathcal{H}\}$$

where we represent each function from C to $\{0,1\}$ as a vector in $\{0,1\}^{|C|}$.

if $|\mathcal{H}| \geq 1$

$\Rightarrow \quad 1 \leq |\mathcal{H}_C| \leq 2^m$

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Note: $\mathcal{H}_C$ is the set of functions from C to $\{0,1\}$ that can be derived from $\mathcal{H}$.