Machine Learning

Clustering

Fabio Vandin

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A Model for Clustering

Let's formulate the clustering problem more formally:

- **Input**: set of elements $\mathcal{X}$ and *distance* function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, that is a function that
  - is symmetric: $d(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}', \mathbf{x})$ for all $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$
  - $d(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{X}$
  - d satisfies the triangle inequality: $d(\mathbf{x}, \mathbf{x}') \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{x}')$

- **Output**: a partition of $\mathcal{X}$ into *clusters*, that is $C = (C_1, C_2, \ldots, C_k)$ with
  - $\cup_{i=1}^{k} C_i = \mathcal{X}$
  - for all $i \neq j$: $C_i \cap C_j = \emptyset$

- **Notes:**
  - sometimes the input also includes the number k of clusters to produce in output
  - sometimes, the output is a **dendrogram** (from Greek *dendron* = tree, *gramma* = drawing), a tree diagram showing the arrangement of the clusters

A Model for Clustering (continue)

Sometimes instead of a distance function we have a similarity function $s : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, that is a function that:

- is symmetric: $s(\mathbf{x}, \mathbf{x}') = s(\mathbf{x}', \mathbf{x})$ for all $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$
- $s(\mathbf{x}, \mathbf{x}) = 1$ for all $\mathbf{x} \in \mathcal{X}$

Choice of distances/similarity:

- depends on the type of data
- different distances may be used for the same dataset
  $\Rightarrow$ choice of distances may have an impact on the results

Classes of Algorithms for Clustering

1. Cost minimization algorithms
2. Linkage-based algorithms

Cost Minimization Clustering

Common approach in clustering:

- define a cost function over possible partitions of the objects
- find the partition (=clustering) of minimal cost

Assumptions:

- data points $\mathbf{x} \in \mathcal{X}$ come from a larger space $\mathcal{X}'$, that is $\mathcal{X} \subseteq \mathcal{X}'$
- distance function $d(\mathbf{x}, \mathbf{x}')$ for $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$

input

For simplicity: assume $\mathcal{X}' = \mathbb{R}^d$ and $d(\mathbf{x}, \mathbf{x}') = ||\mathbf{x} - \mathbf{x}'||$

k-Means Clustering

$\in \mathbb{R}^d$ # clusters

Input: data points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$; $k \in \mathbb{N}^+$

Goal: find

- partition $C = (C_1, C_2, \ldots, C_k)$ of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$;
- *centers* $\mu_1, \mu_2, \ldots, \mu_k$ with $\mu_i \in \mathcal{X}'$ *center* for C_i, $1 \leq i \leq k$

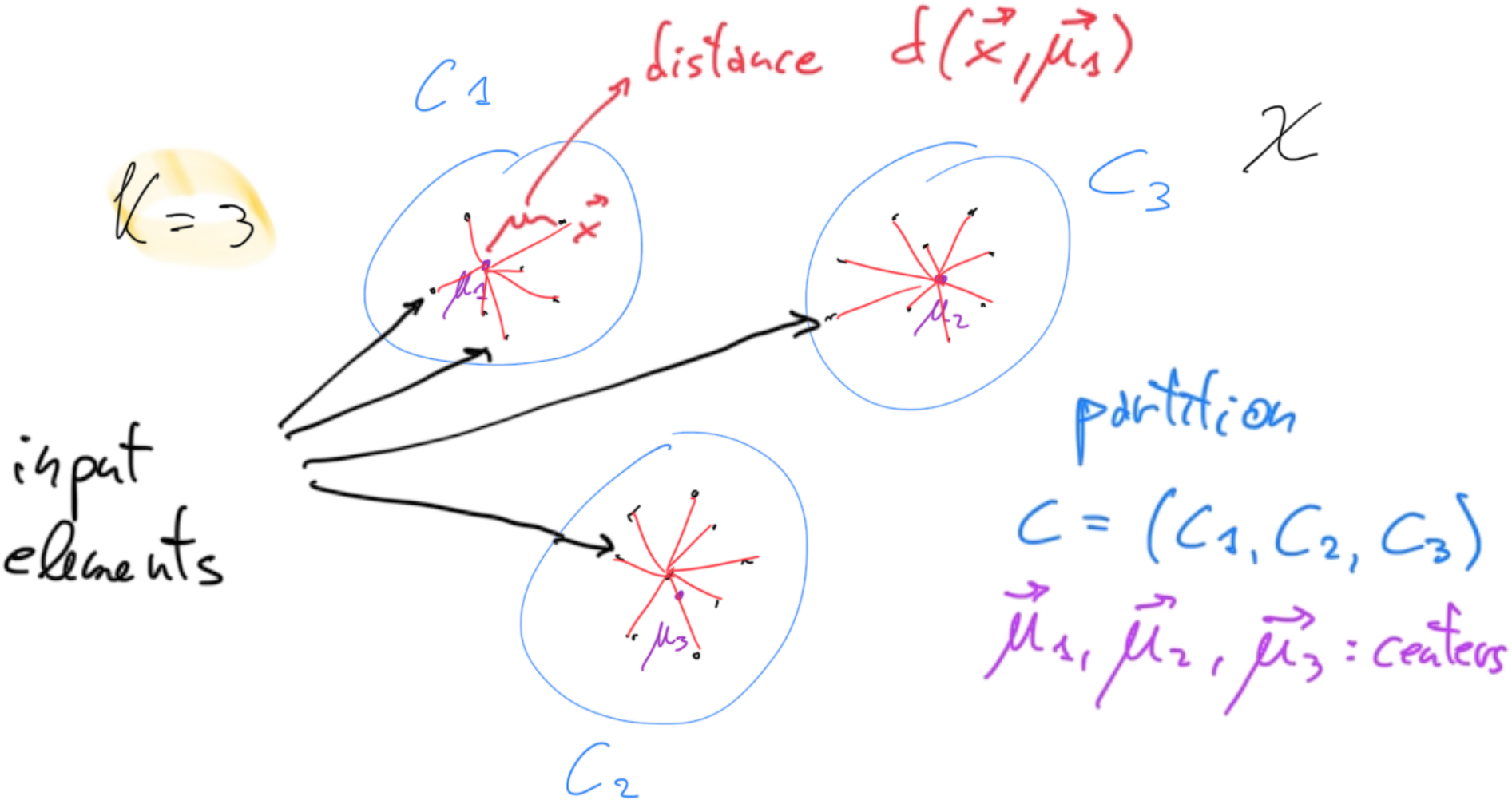
that minimizes the k-**means objective** (cost)

$$\sum_{i=1}^{k} \sum_{\mathbf{x} \in C_i} d(\mathbf{x}, \mu_i)^2$$

if C_i is fixed: find $\vec{\mu}_i$ that minimizes

$$\sum_{\vec{x} \in C_i} d(\vec{x}, \vec{\mu}_i)^2$$

Example


$k = 3$
C_1
distance $d(\vec{x}, \vec{\mu}_1)$
μ_1
$\vec{x}$
C_3
μ_2
$\mathcal{X}$
input elements
C_2
μ_3
partition
$C = (C_1, C_2, C_3)$
$\vec{\mu}_1, \vec{\mu}_2, \vec{\mu}_3$: centers

Other Objectives (Costs)

k-medoids objective:

$$\min_{\mu_1,\ldots,\mu_k \in \mathcal{X}} \sum_{i=1}^{k} \sum_{\mathbf{x} \in C_i} d(\mathbf{x}, \mu_i)^2$$

centers: are points in the input

k-median objective:

$$\min_{\mu_1,\ldots,\mu_k \in \mathcal{X}} \sum_{i=1}^{k} \sum_{\mathbf{x} \in C_i} d(\mathbf{x}, \mu_i)$$

Back to k-means clustering

What is more difficult: finding the clusters or finding the centers?

Proposition

Given a cluster C_i, the center μ_i that minimizes $\sum_{\mathbf{x} \in C_i} d(\mathbf{x}, \mu_i)^2$ is

the points in C_i are known

$$\mu_i = \frac{1}{|C_i|} \sum_{\mathbf{x} \in C_i} \mathbf{x}$$

Proof: Exercise

Algorithm for k-means clustering

Naive (brute-force) algorithm to solve k-means clustering?

Try all possible partitions of the m points into k clusters, evaluate each partition, and find the best one.

Is it efficient?

Depends on the number of partitions of m points into k clusters:

- trivial upper bound: k^m
- exact count: number of ways in which we can partition a set of m objects into k subsets $\Rightarrow$ Stirling number of the second kind:

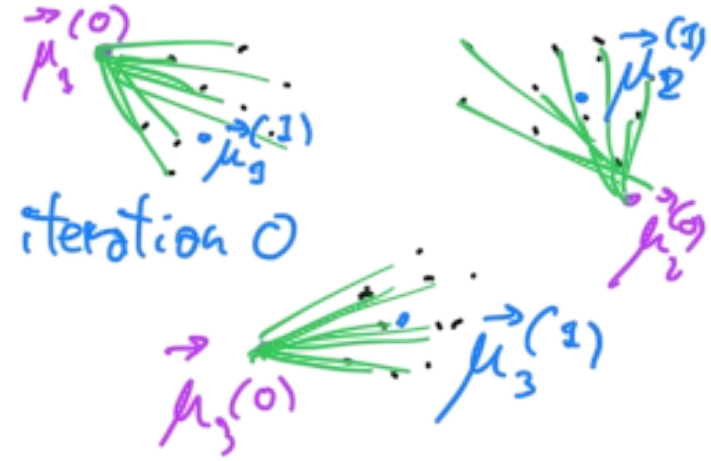
$$S(m,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^m$$

- simple bounds:
  - $S(m,k) \in O\left(\frac{k^m}{k!}\right)$
  - $S(m,k) \in \Omega\left(k^{m-k+1}\right)$

Fact

Finding the optimal solution for k-means clustering is computationally difficult (NP-hard). This is true for most optimization problems of cost minimization clusterings (including k-medoids and k-median)

Lloyd's Algorithm



A good practical heuristic to solve k-means

Input: data points $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$; $k \in \mathbb{N}^+$
Output: clustering $C = (C_1, C_2, \ldots, C_k)$ of $\mathcal{X}$; centers $\mu_1, \mu_2, \ldots, \mu_k$ with μ_i center for C_i, $1 \leq i \leq k$;

randomly choose $\mu_1^{(0)}, \ldots, \mu_k^{(0)}$;
for $t \leftarrow 0, 1, 2, \ldots$ **do** /* until convergence */
  for $i = 1, \ldots, k$: $C_i \leftarrow \{\mathbf{x} \in \mathcal{X} : i = \arg\min_j d(\mathbf{x}, \mu_j^{(t)})\}$;
  for $i = 1, \ldots, k$: $\mu_i^{(t+1)} \leftarrow \frac{1}{|C_i|} \sum_{\mathbf{x} \in C_i} \mathbf{x}$;
  if *convergence reached* **then**
    return $C = (C_1, \ldots, C_k)$ and $\mu_1^{(t+1)}, \mu_2^{(t+1)}, \ldots, \mu_k^{(t+1)}$

Notes

Convergence: commonly used criteria

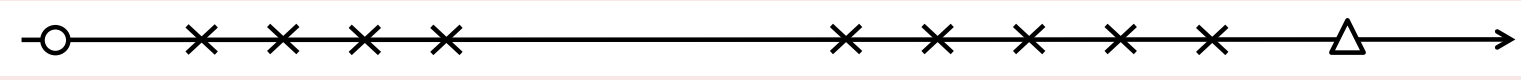
- the k-means objective for the cluster at iteration t is not lower than the k-means objective for the cluster at iteration $t-1$
- $\sum_{i=1}^{k} d(\mu_i^{(t+1)}, \mu_i^{(t)}) \leq \varepsilon$
- $\max_{1 \leq i \leq k} d(\mu_i^{(t+1)}, \mu_i^{(t)}) \leq \varepsilon$

Theorem

If the first convergence criteria above is used, then Lloyd's algorithm always terminates.

Exercize

Draw (approximately) the solution (clusters and centers) found by Lloyd algorithm for the 2 clusters ($k = 2$) problem, when the data ($x_i \in \mathbb{R}$) are the crosses in the figure below and the algorithm is initialised with center values indicated with the circle ($\circ$, cluster 1) and triangle ($\triangle$, cluster 2) shown in the figure.



Complexity of Lloyd's Algorithm

Complexity:

- Assignment of points $\mathbf{x} \in \mathcal{X}$ to clusters C_i: time $O(kmd)$
- Computation of centers μ_i: time $O(md)$

If convergence after t iterations $\Rightarrow$ $O(tkmd)$

How many iterations are required for convergence?

Number of Iterations of Lloyd's Algorithm

- the number of iterations can be exponential in the input size: a trivial upper bound is $\approx k^m$ as before
- more sophisticated studies: upper bound $O\left(m^{kd}\right)$ ($\mathbf{x} \in \mathbb{R}^d$)
- recent studies: lower bound $2^{\Omega\left(\sqrt{m}\right)}$ in the worst-case
- in practice: much less than m iterations are required

Note: the convergence and the quality of the clustering depends on the initialization of the centers!

Effective Centers Initialization

Is there a way to choose the initial centers that is efficient but also provably leads to good clusters?

`k-means++`: simple but effective center initialization strategy proposed by D. Arthur and S. Vassilvitskii (article: D. Arthur and S. Vassilvitskii. *k-means++: the advantages of careful seeding.* Proc. of ACM-SIAM SODA 2007.)

Algorithm k-means++

input

$\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, with $\mathbf{x}_i \in \mathbb{R}^d$ for $1 \leq i \leq m$; $k \in \mathbb{N}^+$

Given a point $\mathbf{x} \in \mathcal{X}$ and a set F, let $d(\mathbf{x}, F) = \min_{\mathbf{f} \in F} d(\mathbf{x}, \mathbf{f})$

$F \subseteq \mathcal{X}$

The algorithm to compute the initial set F of centers is the following:

kmeans++

$\mu_1 \leftarrow$ random point from $\mathcal{X}$ chosen uniformly at random;
$F \leftarrow \{\mu_1\}$;
for $i \leftarrow 2$ **to** k **do**
  $\mu_i \leftarrow$ random point from $\mathcal{X} \setminus F$, choosing point $\mathbf{x}$ with probability $\frac{(d(\mathbf{x},\mathbf{F}))^2}{\sum_{\mathbf{x'} \in \mathcal{X} \setminus F} (d(\mathbf{x'},\mathbf{F}))^2}$;
  $F \leftarrow F \cup \{\mu_i\}$;
return F;

The following result is proved in the original paper by D. Arthur and S. Vassilvitskii.

Theorem

Let $\Phi^*_{k-means}(\mathcal{X}, k)$ be the cost of the optimal (i.e., minimum) k-means clustering of $\mathcal{X}$, and let $\Phi_{k-means}(\mathcal{X}, F_{k-means++})$ be the cost of the clustering $\mathcal{X}$ obtained by:

- using the points i $F_{k-means++}$ returned by `k-means++` as centers;
- assigning each point of $\mathcal{X}$ to its closest center.

(Note that $\Phi(\mathcal{X}, F_{k-means++})$ is a random variable.) Then

$$\mathbb{E}[\Phi_{k-means}(\mathcal{X}, F_{k-means++})] \leq 8(\ln k + 2)\Phi^*_{k-means}(\mathcal{X}, k).$$

Notes:

- the expectation $\mathbb{E}[\Phi_{k-means}(\mathcal{X}, F_{k-means++})]$ is over all possible sets $F_{k-means++}$ returned by `k-means++` (with input $\mathcal{X}$), which depends on the random choices in `k-means++`.
- `k-means++` already provides a good solution for k-means, but it makes sense to use it to initialize centers in Lloyd's algorithm (the solution can only improve in the next iterations, if the first convergence criteria is used)

Linkage-Based Clustering

General class of algorithms that follow the general scheme below.

Algorithm

1. start from the trivial clustering: each data point is a (single-point) cluster
2. **until "termination condition"**: repeatedly merge the "closest" clusters of the previous clustering

We need to specify two "parameters":

- how to define distance between clusters
- termination condition

Linkage-Based Clustering (continue)

Different distances $D(A, B)$ between two clusters A and B can be used, resulting into different linkage methods:

- **single linkage**: $D(A, B) = \min\{d(\mathbf{x}, \mathbf{x}') : \mathbf{x} \in A, \mathbf{x}' \in B\}$
- **average linkage**: $D(A, B) = \frac{1}{|A||B|} \sum_{\mathbf{x} \in A, \mathbf{x}' \in B} d(\mathbf{x}, \mathbf{x}')$
- **max linkage**: $D(A, B) = \max\{d(\mathbf{x}, \mathbf{x}') : \mathbf{x} \in A, \mathbf{x}' \in B\}$

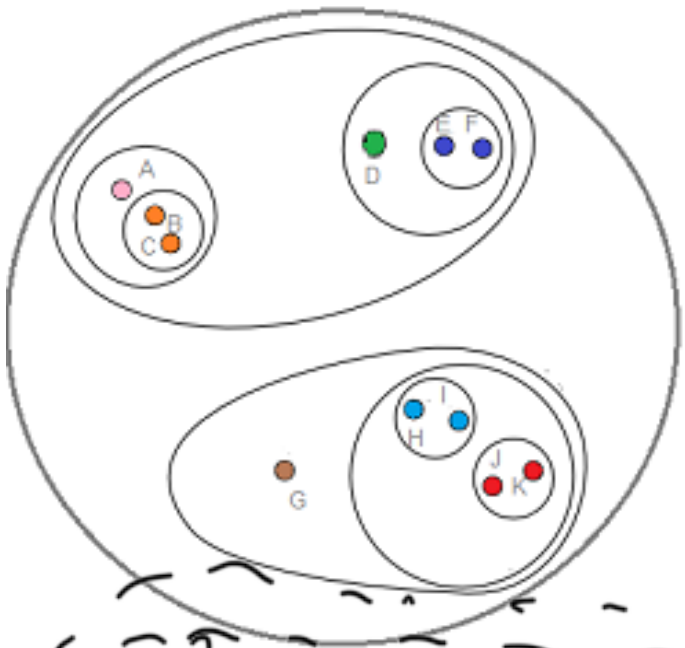
Common termination condition:

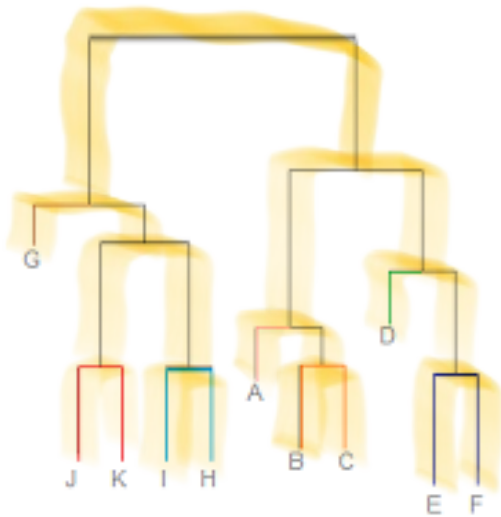
- data points are partitioned into k clusters
- minimum distance between pairs of clusters is $> r$, where r is a parameter provided in input
- all points are in a cluster $\Rightarrow$ output is a dendrogram

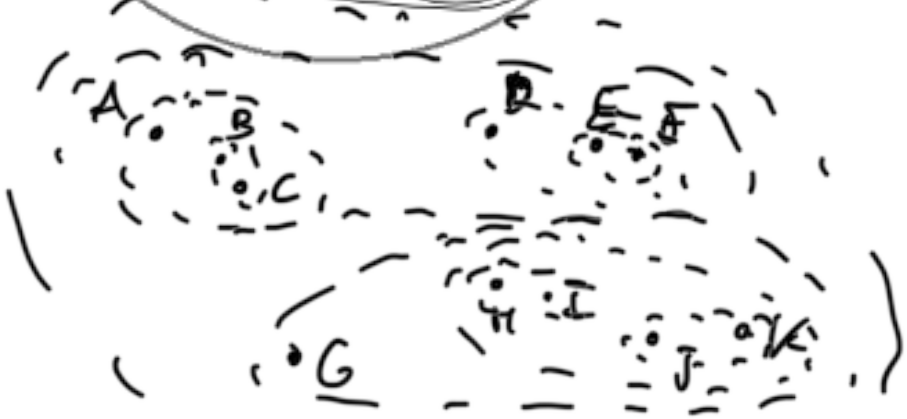
Dendrogram: Example

Dendrogram: tree, with input points $\mathbf{x} \in \mathcal{X}$ as leaves, that shows the arrangement/relation between clusters.

$\mathbb{R}^2$





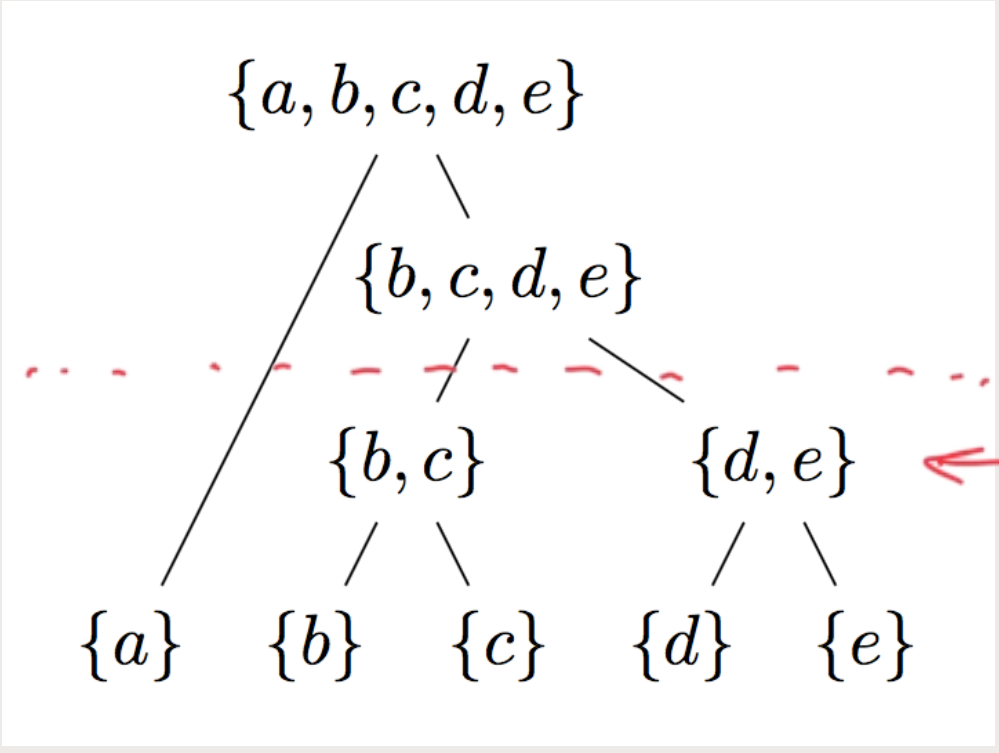


Exercize

Let the dataset $\mathcal{X}$ be as in figure below. Show the output of running the single linkage clustering algorithm when the termination condition is given by having all points in a cluster.

Solution

The output is a dendrogram:



k=3

k=3

solution:

{a},

{b,c},

{d,e}