Hyperbolic sine and cosine

The hyperbolic sine is defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{1}$$

where $x \in \mathbb{R}$ is a *real* variable.

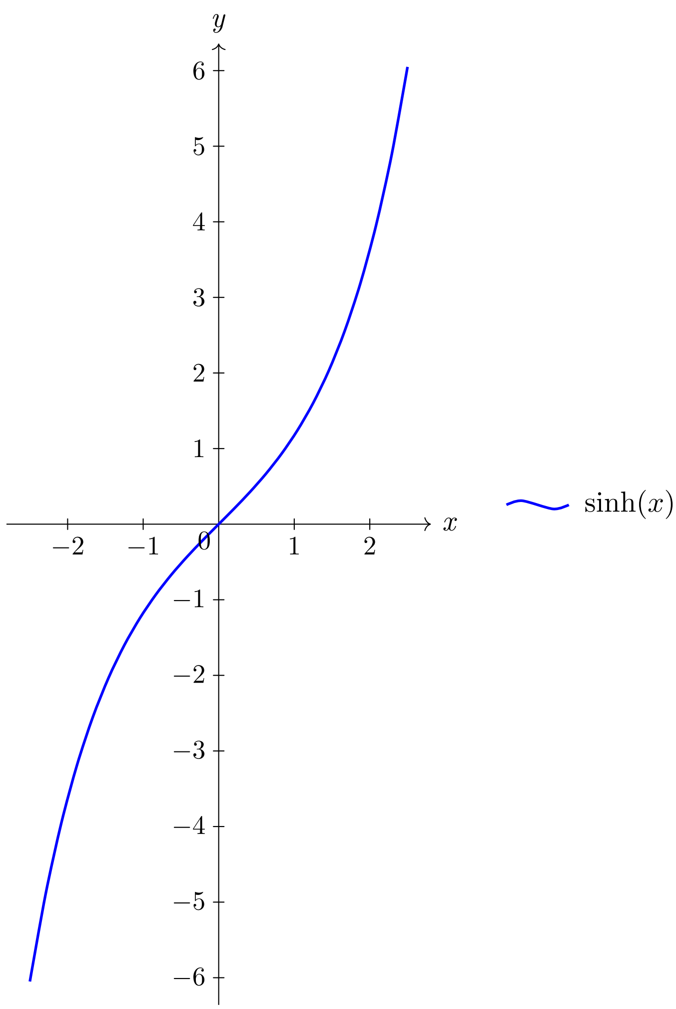

Figure 1: Hyperbolic sine as $x \in [-2.5, 2.5]$.

The hyperbolic cosine is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2}$$

$x \in \mathbb{R}$.

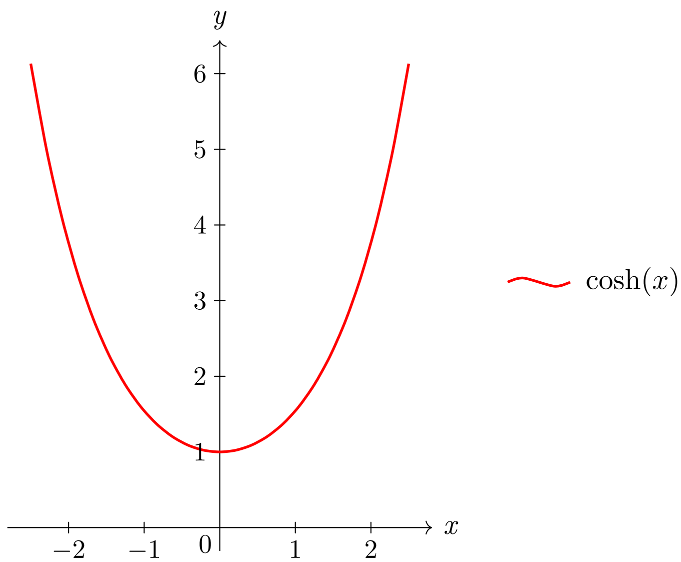

Figure 2: Hyperbolic cosine as $x \in [-2.5, 2.5]$.

The quantities they represent are the hyperbolic correspondants of the trigonometric quantities $\sin(x)$ and $\cos(x)$. There are some similarities and some differences between these functions.

As $\sin(x)$, $\sinh(x)$ is an odd function. As $\cos(x)$, $\cosh(x)$ is an even function.

The mutual behaviour of $\sin(x)$ and $\cos(x)$ remains unchanged for every real value of x (it is always $\sin(x + \pi/2) = \cos(x)$), but this is not the case for the hyperbolic functions.

$\sinh(x)$ and $\cosh(x)$ have different paths as $x \to 0$. Moreover,

$$\begin{aligned} \lim_{x\to+\infty} \sinh(x) = \lim_{x\to+\infty} \cosh(x) = \frac{e^x}{2} \\ \lim_{x\to-\infty} \sinh(x) = -\lim_{x\to+\infty} \cosh(x) = -\frac{e^{-x}}{2} \end{aligned} \tag{3}$$

They tend to coincide[1] for $x \to +\infty$; they tend instead to be symmetrical with respect to the x-axis as $x \to -\infty$, so that their *absolute values* still coincide[2], because the e^x terms tend to disappear.

Note that in both the limits the exponents of e are positive: in the first case, x is positive; in the second case, $-x$ is the result of a negative quantity x preceded by a negative sign.

	$\sinh(x)$	$\cosh(x)$
$\lim_{x\to+\infty}$	$\frac{e^x}{2}$	$\frac{e^x}{2}$
$\lim_{x\to-\infty}$	$-\frac{e^{-x}}{2}$	$\frac{e^{-x}}{2}$

Table 1: Asymptotic behaviours comparison.

The trigonometric functions are always bounded: $-1 \leq \sin(x) \leq 1$, $-1 \leq \cos(x) \leq 1$, $\forall x \in \mathbb{R}$. Their image is always the $[-1, 1]$ subset of $\mathbb{R}$.

The image of $\sinh(x)$ is instead the whole $\mathbb{R}$, and the image of $\cosh(x)$ is the subset $[1, +\infty)$ of $\mathbb{R}$. The hyperbolic cosine can never be smaller than 1.

Complex sine and cosine

When considering a *complex* variable $z \in \mathbb{C}$, the trigonometric sine is defined as:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{4}$$

while the trigonometric cosine is defined as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \tag{5}$$

There is a strong similarity between (4) and (1), as well as between (2) and (5).

[1]The terms $\pm e^{-x}$ tend to disappear and the two hyperbolic functions are overlapped in the limit case.

[2]The path of $\cosh(x)$ in fact overlaps $|\sinh(x)|$.

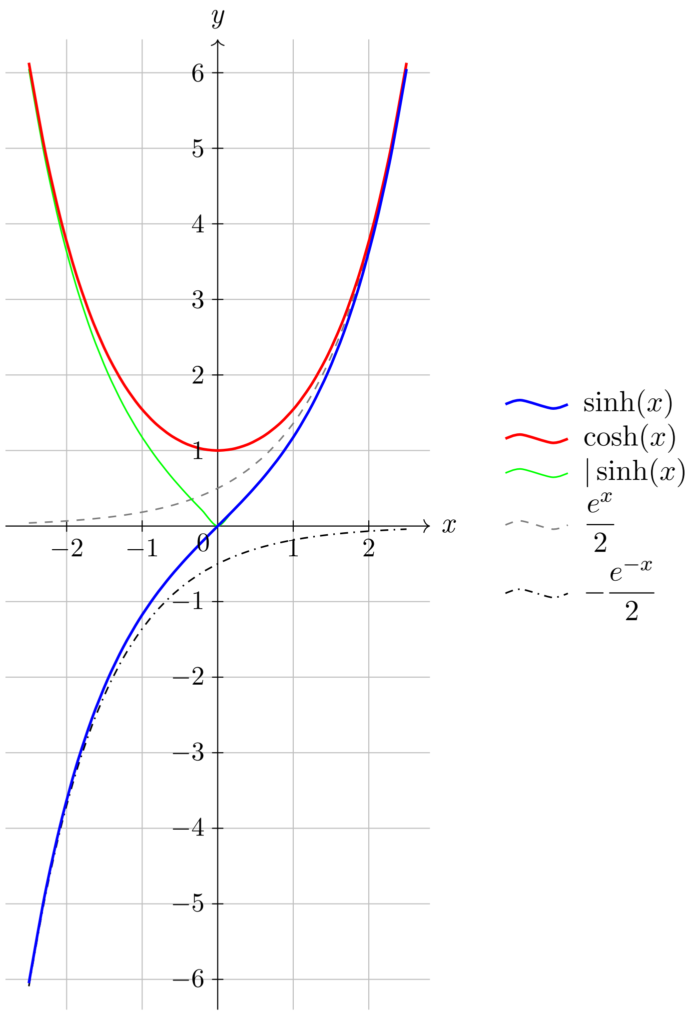

Figure 3: Graphical comparison of the asymptotic behaviours of $\sinh(x)$ and $\cosh(x)$, according to the limits and the Table 1.

In particular, if z is pure imaginary, so that $z = ix, x \in \mathbb{R}$, the obtained values are:

$$\begin{aligned}\sin(z) = \sin(ix) = \frac{e^{i\cdot(ix)} - e^{-i\cdot(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = -\frac{\sinh(x)}{i} = i\sinh(x) \\ \cos(z) = \cos(ix) = \frac{e^{i\cdot(ix)} + e^{-i\cdot(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x)\end{aligned} \tag{6}$$

The trigonometric sine of a *pure imaginary* quantity $z = ix$ is i times the hyperbolic sine of x: it is a pure imaginary quantity itself, being $\sinh(x) \in \mathbb{R}$.

The trigonometric cosine of a *pure imaginary* quantity $z = ix$ is the hyperbolic cosine of x. Note that $\cosh(x) \in \mathbb{R}$: so, given a pure imaginary angle ix, its cosine is a real quantity.

Real values greater than unity

While $\sinh(x)$ may assume any real value, included values such that $|\sinh(x)| \geq 1$, it is always $\cosh(x) \geq 1$. So, not only the $\cos(z)$ may exceed 1 when $z = ix$ is pure imaginary, but it can never be less than 1.

To obtain a $\sin(z) \geq 1$ as well, an appropriate complex[3] z can be chosen such that $\sin(z)$ equals the cosine of a pure imaginary quantity, which is – so far – the only suitable way to obtain a real quantity not smaller than 1.

Let S be the desired real quantity to be obtained from the sine. $S \geq 1$. This can be accomplished in two ways.

[3] The quantity z can not be real, neither pure imaginary, because it would respectively lead to $-1 \leq \sin(z) \leq 1$ and to (6).

Method 1

Choose

$$z = i \log\left[-i\left(S + \sqrt{S^2 - 1}\right)\right] \tag{7}$$

where log[...] is the natural logarithm. This value for z can be obtained by a simple inspection of the definition (4) of sine, such that its expression will finally be equal to S only.

To prove (7):

$$\begin{aligned}
\sin(z) &= \sin\left\{i \log\left[-i\left(S + \sqrt{S^2 - 1}\right)\right]\right\} = \\
&= \frac{e^{-\log\left[-i\left(S+\sqrt{S^2-1}\right)\right]} - e^{\log\left[-i\left(S+\sqrt{S^2-1}\right)\right]}}{2i} = \\
&= \frac{\dfrac{1}{-i\left(S + \sqrt{S^2-1}\right)} + i\left(S + \sqrt{S^2 - 1}\right)}{2i} = \\
&= \frac{\dfrac{1 + S^2 + 2S\sqrt{S^2-1} + S^2 - 1}{-i\left(S + \sqrt{S^2-1}\right)}}{2i} = \\
&= \frac{2S\left(S + \sqrt{S^2-1}\right)}{-i\left(S + \sqrt{S^2-1}\right)} \cdot \frac{1}{2i} = S
\end{aligned} \tag{8}$$

An alternative proof can be made after some preliminary observations. It is worth noting in fact that, remembering the definition of the logarithm of a pure imaginary number:

$$i = e^{i\frac{\pi}{2}}$$

$$\log i = i\frac{\pi}{2} \tag{9}$$

and also the fact that $-i = 1/i$ and the properties of logarithms,

$$\begin{aligned} \log\left[-i\left(S+\sqrt{S^2-1}\right)\right] &= \log\left[\frac{\left(S+\sqrt{S^2-1}\right)}{i}\right] = \\ &= \log\left(S+\sqrt{S^2-1}\right) - \log i = \\ &= \log\left(S+\sqrt{S^2-1}\right) - i\frac{\pi}{2} \end{aligned} \tag{10}$$

the complex angle z in (7) can be rewritten as

$$\begin{aligned} z &= i\log\left[-i\left(S+\sqrt{S^2-1}\right)\right] = \\ &= i\cdot\left[\log\left(S+\sqrt{S^2-1}\right) - i\frac{\pi}{2}\right] = \\ &= \frac{\pi}{2} + i\log\left(S+\sqrt{S^2-1}\right) \end{aligned} \tag{11}$$

It has been stated that $\sin(z)$ in (8) can provide a real value $S \geq 1$ only if z is such that the sine can be rewritten as a cosine of a pure imaginary quantity. This *is* the case: in fact,

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha) \tag{12}$$

for any $\alpha \in \mathbb{C}$ and, in this case, in particular for $\alpha = ix$. Comparing (12) with (11):

$$\sin\left\{\frac{\pi}{2} + i\log\left[S+\sqrt{S^2-1}\right]\right\} = \cos\left[i\log\left(S+\sqrt{S^2-1}\right)\right] \tag{13}$$

Being $S \geq 1$, the square root is a real quantity, as well as the argument of the logarithm, and (therefore) its value. So, from (6),

$$\begin{aligned} &\cos\left[i\log\left(S+\sqrt{S^2-1}\right)\right] = \cos(ix) = \\ &= \cosh(x) = \cosh\left[\log\left(S+\sqrt{S^2-1}\right)\right] \end{aligned} \tag{14}$$

This seems not to immediately lead to the desired result S. However, remembering the definition of hyperbolic inverse cosine,

$$\operatorname{arccosh}(y) = \log\left(y+\sqrt{y^2-1}\right) \tag{15}$$

expression (14) is

$$\begin{aligned} &\cosh\left[\log\left(S+\sqrt{S^2-1}\right)\right] = \\ &= \cosh\left[\operatorname{arccosh}(S)\right] = S \end{aligned} \tag{16}$$

which is the same value already obtained in (8).

Summarizing, from equations (13)-(16):

$$\begin{aligned} &\sin\left\{\frac{\pi}{2} + i\log\left[S+\sqrt{S^2-1}\right]\right\} = \\ &= \cos\left[i\log\left(S+\sqrt{S^2-1}\right)\right] = \\ &= \cosh\left[\log\left(S+\sqrt{S^2-1}\right)\right] = \\ &= \cosh\left[\operatorname{arccosh}(S)\right] = S \end{aligned} \tag{17}$$

Equation (8) is a direct proof for method 1, by a simple substitution. Equation (17) is an alternative proof for method 1, which explicitly shows that this choice of z makes $\sin(z)$ equal to a cosine, which in turn provides the desired real value $S \geq 1$. The values of z (7) and (11)

$$
\begin{aligned}
z &= i \log\left[-i\left(S+\sqrt{S^2-1}\right)\right] \\
z &= \frac{\pi}{2} + i \log\left(S+\sqrt{S^2-1}\right)
\end{aligned} \tag{18}
$$

used in these proofs are, despite appearance, the same, as shown in (11).

Method 2

This procedure is similar to the second proof shown for Method 1. Let the desired value $S \geq 1$ be $S = \cosh(C)$. It follows that

$$C = \operatorname{arccosh}(S) \tag{19}$$

Note that C is real and non-negative. This value will be used to determine the angle z. Choose:

$$z = \frac{\pi}{2} - iC \tag{20}$$

Then,

$$
\begin{aligned}
&\sin(z) = \sin\left(\frac{\pi}{2} - iC\right) = \\
&= \frac{e^{i\left(\frac{\pi}{2}-iC\right)} - e^{-i\left(\frac{\pi}{2}-iC\right)}}{2i} = \\
&= \frac{e^{i\frac{\pi}{2}}e^{-iC} - e^{-i\frac{\pi}{2}}e^{iC}}{2i} = \\
&= \frac{ie^{-iC} + ie^{iC}}{2i} = \cosh(C) = \\
&= \cosh\left[\operatorname{arccosh}(S)\right] = S
\end{aligned} \tag{21}
$$

In this case, $\sin(z)$ provided not immediately a number, but the cosh of a quantity C. Relation (19) has then been used in order to directly express it as the desired number S.

Note that the value of the angle (20) used here is not the same as the one used in the second proof of Method 1, which was of the form

$$z = \frac{\pi}{2} + i\theta, \ \theta \in \mathbb{R} \tag{22}$$

while (20) is of the form

$$z = \frac{\pi}{2} - i\theta, \ \theta \in \mathbb{R} \tag{23}$$

Method 1 is however equivalent to Method 2. Remembering the properties of the sine function and completing relation (12):

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha) \tag{24}$$

for any $\alpha \in \mathbb{C}$, and so particularly when $\alpha = i\theta$. Then, both the Methods lead to $\cos(\alpha) = \cosh(\theta)$, and S is obtained from θ through the arccosh in the second proof of Method 1 and in this Method 2.

Pure imaginary values

With an appropriate choice ((22) or (23)) for the complex angle z, its sine has become equal to the cosine of a pure imaginary angle:

$$\begin{aligned} \sin(z) = \sin\left(\frac{\pi}{2} \pm iC\right) = \cos\left(iC\right) = \\ = \cosh(C) \geq 1, \ C \in \mathbb{R} \end{aligned} \tag{25}$$

Unlike the trigonometric sine of a real angle, it will thus assume **real** values not smaller than 1.

The drawback is that the cosine of the same complex angle will become equal to the sine of a pure imaginary angle, thus assuming only **pure imaginary** values.

Using the form (23) for z:

$$z = \frac{\pi}{2} - iC \tag{26}$$

The value of the $\cos(z)$ can be obtained from the sine itself, with an argument $\pi/2 - z$ instead of z:

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) = \sin\left(iC\right) \tag{27}$$

Or, alternatively,

$$\cos(z) = \sin\left(\frac{\pi}{2} + z\right) = \sin\left(\pi - iC\right) = \sin\left(iC\right) \tag{28}$$

being $\sin(\pi - \alpha) = \sin\alpha$, $\forall\alpha \in \mathbb{C}$. So, either way,

$$\cos(z) = \sin\left(iC\right) = i\sinh(C) \tag{29}$$

according to the result already obtained in (6). With the choice (26) for z, while the sine is forced to be a real quantity not less than 1, the cosine is forced to be a pure imaginary quantity: **their roles**, compared to the result in (6), **have been exchanged**. Due to (19), C is a real, non-negative value and consequently the imaginary part of $\cos(z)$ is non-negative, too.

These considerations hold also when the form (22) is chosen for the complex angle z:

$$z = \frac{\pi}{2} + iC \tag{30}$$

$$\cos(z) = \sin\left(\frac{\pi}{2} + z\right) = \sin\left(\pi + iC\right) = -\sin(iC) = -i\sinh(C) \tag{31}$$

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) = \sin\left(-iC\right) = -\sin(iC) = -i\sinh(C) \tag{32}$$

While both (22) and (23) lead to the same $\sin(z) = \cosh(C)$, the corresponding z values represent *different* angles. When different angles generate the same sine values, their cosines will necessary have opposite signs[4], due to their position in the unit circle and the definitions themselves of sine and cosine.

Regardless of this, also the second example, just presented, showed that $\cos(z)$ is a pure imaginary number. The only difference is the sign of the imaginary part: here, it is non-positive.

[4] And vice-versa, when two different angles are related to the same cosine values.