1 Chapter 4

For two groups G_1, G_2 consider their direct product $G = G_1 \times G_2$. For two representations ρ_1, ρ_2 of G_1 and G_2, we can form their tensor representation

$$\rho_1 \otimes \rho_2(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$$

$$\rho_1 \otimes \rho_2 : G_1 \times G_2 \to GL(V_1 \otimes V_2)$$

over the $\mathbb{C}$, we have that

$$\chi_{V_1 \otimes V_2}(g_1, g_2) = \chi_{V_1}(g_1)\chi_{V_2}(g_2)$$

Lemma 1.1. If $k = \mathbb{C}$, V_1 and V_2 are irreducible representations of G_1 and G_2, respectively, then $V_1 \otimes V_2$ is an irreducible representation of $G_1 \times G_2$.

Proof. Easy way, we can check that its inner product of character with itself is 1, and indeed

$$\langle \chi_{V_1 \otimes V_2}(g_1, g_2) \rangle = \langle \chi_{V_1}(g_1), \chi_{V_1}(g_1) \rangle \langle \chi_{V_2}(g_2), \chi_{V_2}(g_2) \rangle = 1$$

$\square$

Example 1.2. The above Lemma does not hold over $\mathbb{R}$, take $G_1 = G_2 = C_3$.

$$\rho : C_2 \to GL_2(\mathbb{R})$$

sending

$$\rho(g) = \begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{bmatrix}$$

is irreducible as it has no eigenvalues, but note that

$$\rho \otimes \rho$$

has dimension 4, but we claim that every irreducible representation of a finite abelian group over $\mathbb{R}$ has dimension 1 or 2. Indeed, $\mathbb{R}G$ is a commutative algebra, from Artin-Wedderburn we have a decomposition

$$\mathbb{R}G = \bigoplus_{i=1}^{n} M_{n_1}(D_i) = \mathbb{R}^s \oplus \mathbb{C}^{r-s}$$

Corollary 1.3. For $k = \mathbb{C}$, if G_1 has irred reps $V_1, \ldots, V_r$ and G_2 has irred reps $W_1, \ldots, W_s$ then every irred rep of $G_1 \times G_2$ is isomorphic to $V_i \otimes W_j$. (Count conjugacy classes)

Let G is abelian, and $G = C_{l_1} \times \cdots \times C_{l_k}$ of cyclic groups of C_{l_i}. The 1 dimensional repr of C_i form a group of $G^\times$. The inverse of χ is the dual representation. in which case

$$G^* - C_{l_1}^* \times \cdots \times C_{l_k}^* \cong C_{l_1} \times \cdots \times C_{l_k}$$

but this decomposition is not unique.

1.1 Character table of $C_2 \times C_2$

Let $\langle g \rangle = C_2, \langle h \rangle = C_2$, we have

	1	$(g,1)$	$(1,h)$	(g,h)
χ_1	1	1	1	1
χ_2	-1	1	-1	1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

1.2 Duality

We now talk about duality (over $\mathbb{C}$), if V is an irreducible repr of G, our motivating question is how to determine of $V \cong V^*$ just by looking at χ_V. Well since

$$\chi_{V^*}(g) = \overline{\chi_V(g)}$$

they are isomorphic iff all character values are real.

Over an arbitrary ring R if V is an RG-module then $V^* = \mathrm{Hom}_R(V, R)$ is again an RG-module. So it is really a construction of a bilinear pairing

$$f : V \to V^*$$

$$\beta : V \otimes V \to R$$

$$\mathrm{Hom}_R(V, V^*)^G = \mathrm{Hom}_R(V \otimes V, R)^G$$

sending $\beta(gv, gw) \to \beta(v, w) \forall g \in G$. If V is a free R-module of finite rank n, then f is an isomorphism if β is nondegnerate.

We now show that the regular representation RG is self-dual. This implies a form $\beta : RG \otimes RG \to R$. $\beta(x, y) = \alpha(xy)$ for $\alpha : RG \to R$ sending each $g \to 1$ (take the sum of R coefficients). However, to make it invariant, we need

$$\beta(g, h) = \alpha(g^{-1}h)$$

same way, any permutation representation is self-dual.

We work over $\mathbb{C}$ again, suppose $V \cong V^*$ and V is irreducible. Then there exists a bilinear invariant form

$$\beta : V \otimes V \to \mathbb{C}$$

unique up to rescaling (already nondegenerate by simplicity of V)

$$V \otimes V = S^2V \oplus \wedge^2 V$$

from the representation $\rho : V \otimes V \to V \otimes V$ sending $v \otimes w \to w \otimes v$ so it is either symmetric or skew symetric.

How to compute $\chi_{S^2V}, \chi_{\wedge^2 V}$ For g, $\rho(g)$ is diagonalizable with eigenvalues λ_i in a suitable basis. Basis on S^2V is $(v_i, v_j), v \leq j = v_1 \otimes v_j + v_j \otimes v_i$ and basis on $\wedge^2 V$ is $v_i \wedge v_j - v_i \wedge v_j$. so trace of symmetric representation is TODO so at the end we have

$$\chi_{S^2V}(g) = \frac{\chi_V^2(g) + \chi_V(g^2)}{2}$$

and

$$\chi_{\wedge^2 V}(g) = \frac{\chi_V^2(g) - \chi_V(g^2)}{2}$$

The Schur index V is an irred rep

$$S_V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = \langle \chi_{S^2V} - \chi_{\wedge^2 V}, \chi_1 \rangle$$

which takes on 1 if V is symmetric, -1 if V is skew symmetric, and 0 if they are not dual. Useful for real representation, we start induction and restriction next time.