The letter A will denote an arbitrary ring, then a nonzero module M is a simple module if it does not have any proper non-zero submodules. Note that

$$M \text{ is simple} \implies M \text{ is generated by 1 element}$$

hence it is cyclic so we must have

$$M \cong A/I$$

for I a maximal ideal.

Proposition 0.1. *Suppose that V is a simple representation of a finite group G over a field F. Then $dim(V) \leq |G|$.*

Proof. Let $A = FG$, then $\dim_F A = |G|$, so $V = FG/I$ so $\dim_F V \leq |G|$. □

Now suppose that $G = C_{p^n}$ a cyclic field group and char$F = p$. Then any irreducible representation of G over F is trivial. Indeed, if $C_{p^n} = \langle g \rangle$, then $FG = F[g]/(g^{p^n} - 1)$. For $x = g - 1$ this is $F[x]/(x^{p^n})$ is a local ring with unique maixmal ideal (x), so the only representation is $F[x]/(x) = F$.

0.1 Semisimple modules in general

Proposition 0.2. *The following conditions on an A-module V are equivalent.*
(1) V is a sum of all its simple submodules.
(2) $V = \bigoplus_i S_i$ for simple submodules S_i.
(3) Every submodule $W \subset V$ is a direct summand of V, i.e. $V = W \oplus W'$ for some submodule W'.
An A-module is called semisimple if it satisfies any of the above equivalent conditions.

Proof. This is proved in the textbook in the finite case, it holds in general by an application of Zorn's Lemma. We give a sketch of the proof.

(1) $\implies$ (2) : If $\{S_j\}_{j \in J}$ is all simple submodules, then by Zorn's there exists some $I \subset J$ such that

$$\sum_{i \in I} S_i = \bigoplus_{i \in I} S_i$$

we can check that $V = \bigoplus_{i \in I} S_i$.

(2) $\implies$ (3) : For a submodule W, choose a maximal subset $K \subset I$ such that $W \cap (\bigoplus_{i \in I} S_i) = \{0\}$ then $\bigoplus_{j \in I/K} S_j = W'$ which is readily checked using properties of simple submodules.

(3) $\implies$ (1): See the textbook.

We first give a definition.

Definition 0.3. The *socle* of V is the sum of all simple submodules of V, we will also denote this as $\mathbf{Soc}(V)$.

Assuming (3) holds, we first claim that Soc(V) is nonzero. For $v \in V, v \neq 0$, there exists a maximal $M \subset Av$. Then $V = Av \oplus N$ then $M \oplus N \subset V$ is maximal, hence $S = W/M \oplus N$ is simple. Therefore $V = M \oplus N \oplus S$ by property 3, and another use gives us that Soc$(V) = V$ as we wanted to show. □

Proposition 0.4. *A quotient and submodule of a semisimple module is semi-simple.*

Proof. The first is readily verified using the equivalent properties proven above. □

As an example, if D is a division ring then any D-module is semisimple over itself.

Go to a representation of G, a semisimple FG-module is called a *completely reducible* representation of G. We shall now prove the firsrt main theorem of representation theorem

Theorem 0.5 (Maschke's Theorem). *Let G be a finite group and F be a field such that the characteristic of F does not divide $|G|$. Then any representation of G over F is completely irreducible.*

Proof. We verify property (3). Let V be a FG module, and W any invariant subspace (subrepresentation): we wish to find a complementary subspace W' such that $V = W \oplus W'$. There always exists such a subspace W'' (take the kernel of a projection π of V onto W). However W'' is not necessarily G-invariant. Define

$$\overline{\pi} = \frac{1}{G} \sum_{g \in G} g \pi g^{-1}$$

which is well defined as $|G|$ is assumed invertible in F.

We first show that $\overline{\pi} \in \mathrm{End}_{FG}(V)$. Indeed, for any $h \in G$ $h\overline{\pi}h^{-1} = \overline{\pi}$ by a simple computation. Also note that $\overline{\pi}(V)$ is also a projection onto W since W is invariant under FG and π is the identity rericteded to W. We shall show that $W'' = \ker(\overline{\pi})$ is our desired subspace. Indeed, if $\overline{\pi}w = 0$ then $\overline{\pi}(gw) = g\overline{\pi}w = 0$. □

Example 0.6. Take $G = S_3$, consider the permutation representation in $F^3 = V$. Take $W = \{a_1e_1 + a_2e_2 + a_3e_3\}$ such that $a_1 + a_2 + a_3 = 0$. If $\mathrm{char}F \neq 3$, then $W' = a(e_1 + e_2 + e_3)$ is complementary. However, if $\mathrm{char}F = 3$ then $W' \subset W$ so it doesn't hold. Hence $\mathrm{Soc}(V) = V$ if $\mathrm{char}(F) \neq 3$ and W' if $\mathrm{char}(F) = 3$.

We introduce the next result but don't prove it, you may recognize it as a general form of a theorem often encountered in group theory.

Theorem 0.7 (Jordan Holder)**.** *Let V be an A-module. V is of finite length if there exists a finite filtration of submodules*

$$0 = V_1 \subset V_1 \subset \cdots \subset V_e = V$$

of submodules such that V_i/V_{i-1} is simple. If V is of finite length, then any other such filtration has the same length and simple quotients, though not necessarily in the same order. These S_i's are called simple constituents of V.