Comparison of Linear and Non-linear Method Towards the Mitigation of Torsional Resonance in Coupled Systems

Daniel M. Lofaro[1] and Tom Chmielewski[2]

***Abstract*— Connecting two rotary mechanical devices typically utilizes a flexible coupler to accommodate various shaft misalignments. These couplers, including belts and gear boxes, exhibit a spring constant and a viscous damping term. The spring constant causes the system to have a resonant frequency while the damping controls its amplitude. In the frequency domain this characteristic is called Torsional Resonance (TR). If the TR frequencies are allowed into the pass band of closed loop servo because it will cause instability. Some conventional solutions to obtain stable operation include: reduction of the servo's bandwidth below the TR frequencies; using stiffer, more expensive, components to increase the TR frequencies thus increasing the usable bandwidth; and using notch filters to reduce the resonant peak. The objective of this work is to explore control solutions to allow systems using elastic parts, including loose belt drives and plastic gears, achieve sufficient bandwidth to obtain their desired performance. A model of a commercial application exhibiting the TR characteristic has been made using Matlab and Simulink. Linear state feedback techniques described by Bigley et. al. and non-linear Sliding Mode Control (SMC) is modeled, implemented and compared. Conclusions and observations are discussed on the state of the art of torsional resonance effect reduction.**

I. Introduction

When connecting two rotary mechanical devices a coupler is used. These include the use of 1:1 flexible shaft couplers, to compensate for possible shaft misalignments, as well as belts, and gear boxes. In theory it is desired to have the given coupler be perfectly rigid. Every degree the actuator turns the connected load will turn to so that its precise location is denoted by the input and the gear ratio. For a coupler to be perfectly rigid the latter must be true for all time. In reality gear boxes and couplers are not perfectly rigid and have spring and damping terms associated with them. The spring constant causes the system to have two resonant frequencies collectively called the Torsional Resonance (TR). The TR frequencies, which are referred to as the resonant frequency, ω_r, and the anti-resonant frequency, ω_{ar}, can not fall within the pass-band of the closed loop servo because it will cause instability. Fig. 1 shows a diagram of a coupled system exhibiting TR. The ideal system can be reduced to a single acting inertia of $J_a + J_L$ while the system exhibiting TR is connected by a spring damper system thus that reduction can not be made. Our goal is to demonstrate a controller that uses commonly measured states (such as motor position) to reduce the effects of TR on coupled systems.

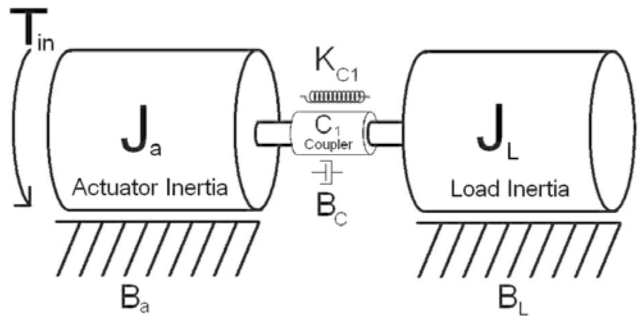

Fig. 1. System exhibiting TR. In an ideal system (without TR) the load can be represented as a single acting inertia $(J_a + J_L)$. The system exhibiting TR is connected by a spring damper system thus that reduction can not be made[1]

In this paper we are comparing linear state feedback and non-linear sliding mode control methods of reducing the effect of TR. Section II gives a brief overview of TR and the contemporary ways of reducing its effects. Section III models a system exhibiting TR. Section IV shows the effects of TR on the system. Section V goes through two method of reducing the effects of TR (one linear and one non-linear method) where Section V-A demonstrates a linear method based on state feedback by Bigley et. al. and Section V-B demonstrates our work in the form of a non-linear sliding-mode control based method. Section VI show and discusses the results of both methods and gives final thoughts.

II. Background

The ideal frequency response of a coupled system, with motor position θ_a as the output and motor torque T_{in} as the input, can be found in Fig. 2 (left). The magnitude plot exhibits a $-40\frac{dB}{dec}$ slope with corresponding phase of -180^o. The frequency response of the system exhibiting TR can be found in Fig. 2 (right). It can be noted that at resonance there is a peak. This peak reduces the gain margin of the system. The phase also changes around the resonance which reduces the phase margin. We define "*load switching*" as an offset in the $-40\frac{dB}{dec}$ slope that occurs between ω_{ar} and ω_r, see Fig. 4.

Contemporary ways of reducing the effects of TR include:

- Reducing the servo's bandwidth so it does not include the TR frequencies.
- Increasing the couplers spring constant K_c via using higher quality and anti backlash mechanical couplers

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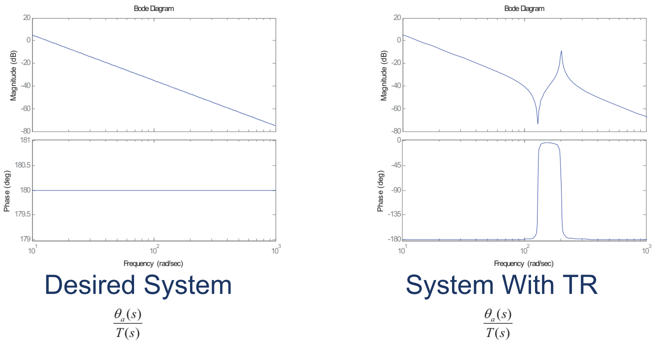

Fig. 2. Frequency Response of desired coupled system (Left), Frequency response of system exhibiting TR (Right)

and gearboxes to push the TR frequencies higher.

- Using notch filters to reduce the resonant peak. as well as active damping methods[2].

Reducing the bandwidth of the servo means that the designed system will not be able to respond as quickly to an input stimulus. Using higher quality and anti-backlash mechanical couplers and gearboxes will give you greater bandwidth but will also include a higher materials price tag. It is also important to note that the use of notch filters is only useful in systems where the TR does not change. Most systems that have TR present tend to have nonlinear, and time varying, elements, including dynamic loads, that will cause the TR frequency to change thus making the notch filter less effective in the ideal case. None of the latter methods address the problem of load switching.

Bigley et. al.[3] (circa 1978) presented a state feedback technique for, as they describe it, "*eliminating the destabilizing effect of mechanical resonance in feedback control systems*." The goal of their research was to make a wide band high performance closed loop servo system. In their research Bigley and Rizzo found a relationship between the torque, or current, on the DC motor and the velocity of the shaft. They found that when the current is at its maximum the velocity is at a minimum, no matter where the resonance or anti resonance occurs, see Fig. 7. Thus feeding back the velocity and torque states would compensate for the effects of TR no matter where it occurs even when the parameters of the system changes.

Sliding mode control (SMC) is a good candidate method to reduce the effects of TR as stated by Korondi et. al.[4]. This is because SMC is a robust control method that is able to properly control a system even when parameters are not precisely known. The main idea about SMC is that you are able to make a control that will guarantee the performance of a system in u and x, see Fig. 10, within a given error ψ and ϵ respectively. When using SMC one, or more, parameters are chosen to be unknown. The unknown parameters are given a range that they are able to be between and still have the system perform properly.

III. Torsional Resonance Model

A system with TR typically consists of three main items; An actuator, or motor, with a given inertia, J_a, and damping, B_a, associated with it. An inertial load, J_L, with a given damping, B_L, associated with it. A coupler between the actuator and the load with a spring constant, K_c, and a damping, B_c, associated with it.

It is assumed that the inertia of the coupler is included in the inertia of the actuator and the load. The physical diagram of the system with TR can be found in Fig.1. In order to find the transfer function for this system, where T_{in} is the torque input and the actuator angle θ_a is the output, the physical diagram is then converted into a mechanical network, see Fig. 3, so the system's dynamics can be written as

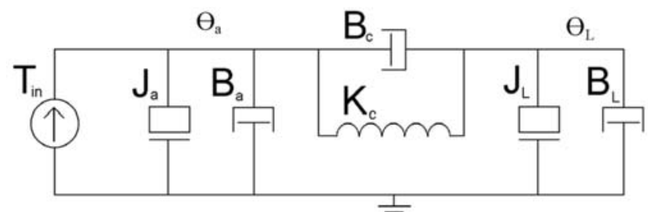

Fig. 3. Mechanical schematic drawling of system with TR

$$T(t) = J_a\ddot{\theta_a} + (B_a + B_c)\dot{\theta_a} = K_c\theta_a - (K_c\theta_L + B_c\dot{\theta_L}) \quad (1)$$

The relationship between θ_a and torque, T, is represented as

$$\frac{\theta_a(s)}{T(s)} = \frac{\frac{1}{J_aJ_L}\left(s^2 J_L + K_c + sB_c\right)}{s^2\left(s^2 + B_c\frac{J_a+J_L}{J_aJ_L}s + K_c\frac{J_a+J_L}{J_aJ_L}\right)} \quad (2)$$

In this system both B_a and B_L are assumed to be zero (Klafter et. al [5]).

The poles of a system with TR (2) consists of a double pole at the origin and complex conjugate (CC) poles. The CC poles are in the left half plane (LHP) as long as the following inequality is satisfied.

$$4K_c > B_c^2\left(\frac{J_c + J_L}{J_aJ_L}\right) \quad (3)$$

The system can represented in state space as

$$\begin{aligned}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t)\\ y(t) &= \mathrm{C}x(t) + \mathrm{D}u(t)\end{aligned} \quad (4)$$

$$x(t) = \begin{bmatrix}\dot{\theta_a}\\ \theta_a\\ \dot{\theta_L}\\ \theta_L\end{bmatrix}, \quad u(t) = T(t) \quad (5)$$

and

$$A = \begin{bmatrix} \frac{-(B_a+B_c)}{J_a} & \frac{-K_c}{J_a} & \frac{B_c}{J_a} & \frac{K_c}{J_a} \\ 1 & 0 & 0 & 0 \\ \frac{B_c}{J_L} & \frac{K_c}{J_L} & \frac{-(B_c+B_L)}{J_L} & \frac{-K_c}{J_L} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$B = \begin{bmatrix} J_a^{-1} & 0 & 0 & 0 \end{bmatrix}^{-1} \quad (7)$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad , \quad D = 0 \quad (8)$$

standard techniques show that the system is fully controllable and with the angular position θ_a being the only output is also fully observable. The controllability and observability matrix are both full rank.

The resonance and anti-resonance frequencies, ω_r and ω_{ar}, are defined by the inertial load of coupled system as well as the spring constant of the coupler K_c. Fig. 4 shows the frequency response of a system that is:

- exhibiting TR
- not exhibiting TR with a total inertial load of J_a
- not exhibiting TR with a total inertial load of $J_a + J_L$.

The dominant load seen by the actuator before resonance is the total inertial load $(J_a + J_L)$ where after resonance it is only the actuator's inertia. The resonance ω_r and anti-resonance ω_{ar} can be calculated by

$$\omega_{ar} = \left(\frac{K_c}{J_L}\right)^{\frac{1}{2}} \quad \omega_r = \sqrt{\frac{K_c(J_a+J_L)}{J_a J_L}} \quad (9)$$

and the gain separation when going from an acting load of $J_L + J_a$ to J_a, as seen in Fig. 4, is calculated by Chmielewski [6] and is

$$\Delta dB = 40\log_{10}\left(\frac{\omega_r}{\omega_{ar}}\right) \quad (10)$$

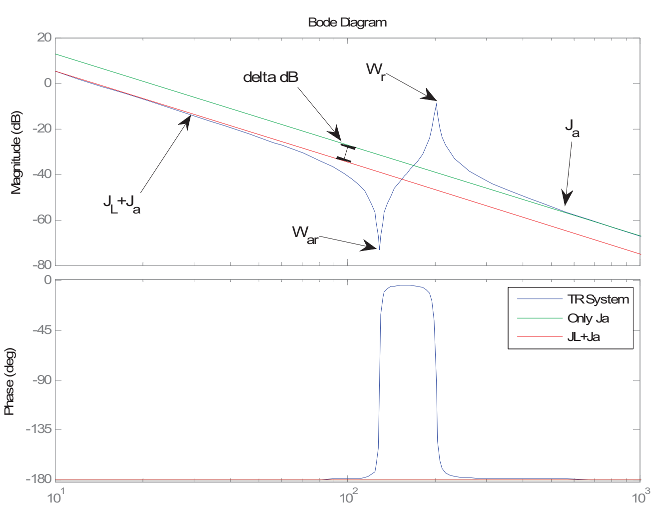

Fig. 4. Frequency Response Plot of System with TR (from Eq. 5), System with no TR with inertial load of J_a, and System with no TR and inertial load of $J_a + J_L$

TABLE I
VALUES USED FOR TR MODEL BASED ON THE PITTMAN™ N2314 SERIES BRUSHLESS DC SERVO MOTOR AND A CUSTOM INERTIAL LOAD/COUPLER

	Inertia 'J' $(oz-in-sec^2)$	Damping 'B' $(\frac{oz-in}{krpm})$	Spring 'K' $(\frac{oz}{in})$
Actuator	0.0023	0	0
Coupler	0	0.005	55
Load	0.0033	0	0

The anti-resonance ω_{ar} is due to the complex conjugate zeros and the resonance ω_r is due to the complex conjugate poles. The anti-resonance will always precede the resonance.

IV. Effects of TR

A. Effect as Seen at the Actuator

The dipping and peaking of the magnitude of the frequency response plot of a system with TR is not the only effect that TR has on the system. Another effect is a change in the effective load seen by the system before ω_{ar} to after ω_r, see Fig. 4. Before the ω_{ar} the acting inertial load on the system is the sum of the actuators inertial load, J_a, and the loads inertial load, J_L. This is a second order system with a slope of negative $40\frac{dB}{dec}$ on the frequency response plot. After the ω_r the acting inertial load is only the actuators inertia, J_a, which also defines a second order system with a slope of negative $40\frac{dB}{dec}$ on the frequency response plot. The latter characteristics can be seen in the frequency response plot in Fig. 4.

Transfer function $\frac{\theta_{a1}(s)}{T(s)}$ is for a mass spring damper system with effective load of $J_a + J_L$, ideal coupler (pre-ω_{ar})

$$\frac{\theta_{a1}(s)}{T(s)} = \frac{1}{(J_a + J_L)s^2} \quad (11)$$

Transfer function $\frac{\theta_{a2}(s)}{T(s)}$ is for a mass spring damper system with effective load of J_a, ideal coupler (post-ω_r)

$$\frac{\theta_{a2}(s)}{T(s)} = \frac{1}{J_a s^2} \quad (12)$$

It can be noted that there is an offset between the frequency responses of the second order estimations of the system with TR when going from before the ω_{ar} to after the ω_r when looking at the magnitude in dB of the response. This offset is known as gain separation Gain separation is the resulting effect of the load switching around ω_{ar} and ω_r. Fig. 4 shows an example of gain separation in the frequency response of the system with TR. System with TR is Eq. 2 using values from Table I. In addition the frequency responses of systems without TR and having the effective loads of $J_L + J_a$ and J_a are also shown. This clearly shows gain separation due to load switching.

B. Effect on Velocity

When there is a step torque command input to the system, like that seen in Fig. 5 (top), the output velocity exhibits an oscillatory behavior. The frequency of oscillation is expected to be equal to ω_r of the system from Eq. 2. Therefore this should have a ripple frequency of 201.4 rad/sec, 32.1 Hz.

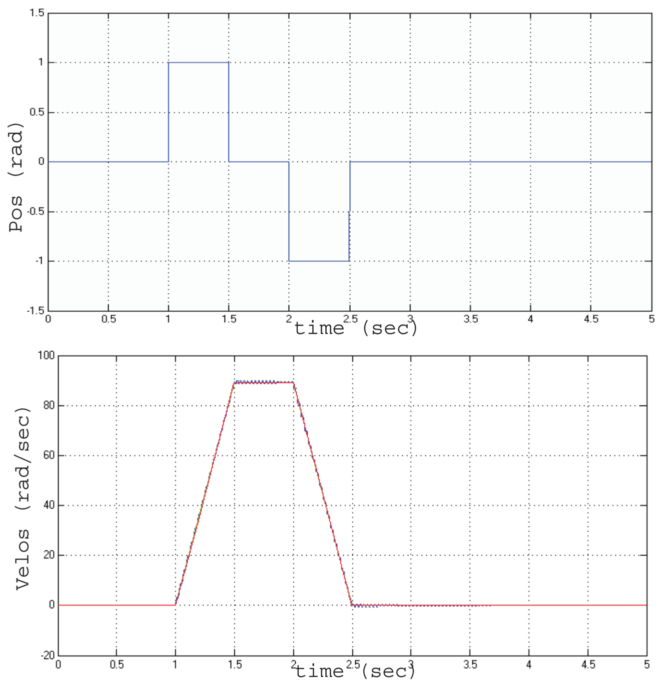

Fig. 5. (TOP) Step Torque Input to TR System. Vertical Axis is Magnitude, Horizontal Axis is Time in sec. (BOTTOM) Angular Velocity of the Actuator Shaft of the system with TR (Yellow), Angular Velocity of the Actuator Shaft of the system without TR (Pink) due to the torque input. Vertical Axis is Magnitude, Horizontal Axis is Time in sec.

Fig.6 shows Fig. 5 magnified where the ideal velocity is constant (1.48 sec to 0.62 sec). The angular velocity of the actuator shaft has a decaying sinusoid on it due to the resonance and the spring/damping. The period of is 0.031sec (32.2 Hz, 202.3 rad/sec). This frequency is the same as the ω_r computed in Eq.2, shown in Fig. 4 and uses the values in Table I. This frequency is calculated by ω_r in Eq. 9.

V. Methodology

A. Resonance Equalization

State-variable-feedback (SVF) is a control method that uses different states of a system, such as position, velocity, and acceleration, to create an input to achieve a desired performance. In this case the states of the system consist of the angular position and velocity of the actuator and load with the input being torque. At this point it is pertinent to note the torque applied by a direct current (DC) motor is directly proportional to the current through the windings of the motor. It is also pertinent to note that it is widely known that the DC motor is a fully observable system when the output is the angular position, thus a full or partial state observer can be made for this system. With the latter being said Bigley et. al.[3] analyzed what the effects of TR had on various parameters of the system including shaft velocity and input current, Fig. 7 is an example of some of these plots. A key point noted is that the input current and output velocity are almost exactly 180^o out of phase from one

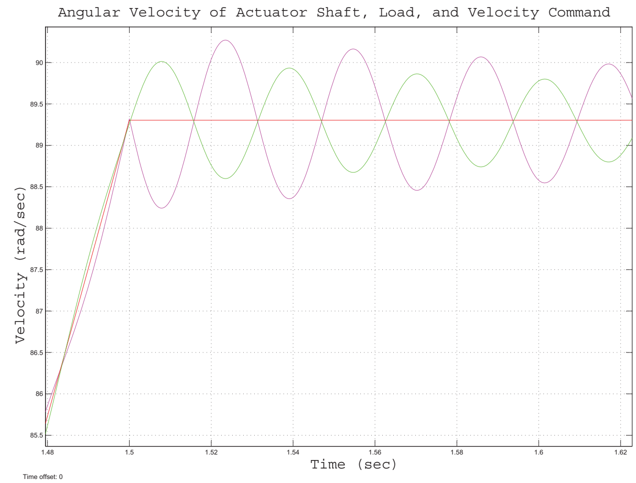

Fig. 6. Angular velocity of the actuator shaft $\dot{\theta}_a$ for the system with TR (PURPLE), Angular velocity of the load shaft $\dot{\theta}_L$ for the system with TR (GREEN), Angular Velocity of the actuator shaft for the system without TR (RED) due to the torque input. It is important to note that $\dot{\theta}_a$ and $\dot{\theta}_L$ are 180^o out of phase.

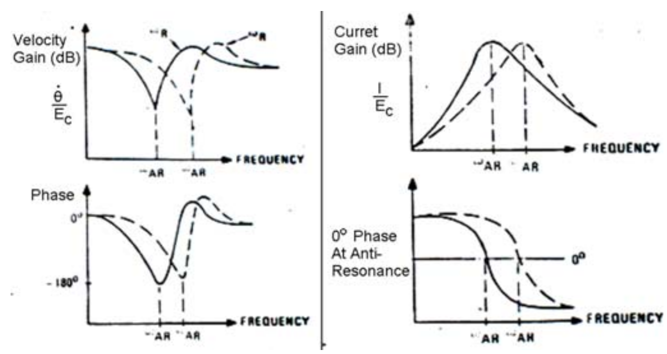

Fig. 7. Plot of the Actuator Shaft Velocity and Phase (Left), Plot of the Actuator Current and Phase (Right) around ω_r and ω_{ar}[3]

another. The importance of this is that the actuator shaft velocity can be used as feedback in combination with the input current to compensate for the effects of the resonance. The latter relationship will ensure that the control is effective for changes in the resonant frequencies.

Using the knowledge that the current, or torque, is about 180^o out of phase with the angular velocity of the actuator Bigley showed that "*the resonance peaking factor F is completely eliminated*." Bigleys method is implemented in Fig. 8 with the SVF gains of -0.2 for the current feedback gain and 20 for the angular velocity feedback. Table III shows the maximum change in magnitude and phase of the frequency response plot of the system with TR and the addition of Bigleys resonance equalization (RE)[3].

This results in a reduction of the resonance and anti-resonance and the corresponding phase shift. Fig. 9 shows the frequency response of RE on the TR system. RE eliminates the load switching (ΔdB) as shown in Fig. 4. However the motor velocity $\dot{\theta}_a$ and motor current i_a states are required to implement RE.

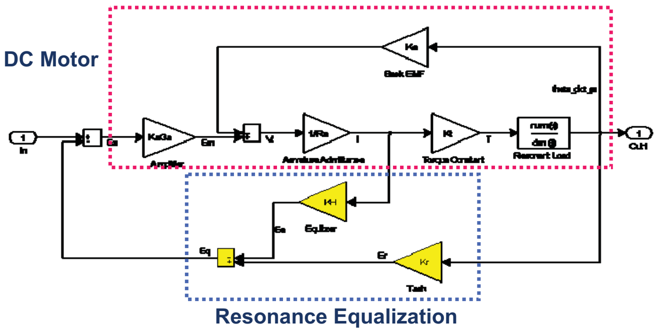

Fig. 8. W. Bigleys resonance equalized rate loop diagram using SVF[3]

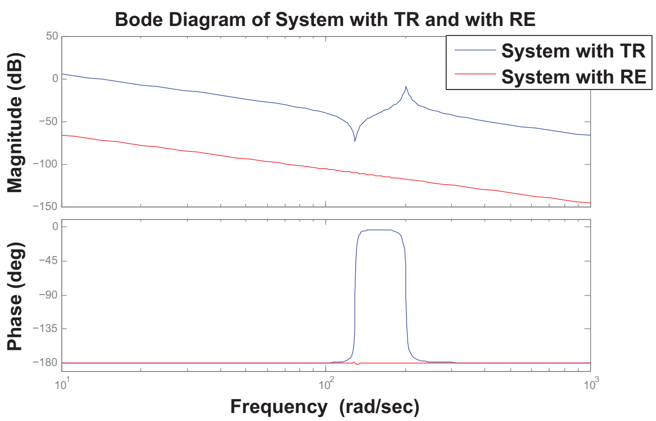

Fig. 9. Frequency response of W. Bigleys resonance equalization (RED) plotted over a system exhibiting TR system (BLUE). The resonance equalized system is using an ideal model of a DC motor for use in the state variable feedback.

B. Sliding Mode Control

In our work we used the Sliding Mode Control (SMC) to reduce the effects of TR. SMC is a form of non-linear control that is able deal with systems that are not linear as well as systems that have parameters which are not precisely known but are bounded. We are using sliding mode with boundary layer as defined in Slotine et. al[7]. This leads to tracking within a guaranteed precision ϵ (rather than perfect tracking) and generally guarantees that for all trajectories starting inside boundary layer B (see Fig 10) where

$$\text{B} = \{x, |s(x;t)| <= \Phi\} \quad \text{where} \quad \Phi > 0 \tag{13}$$

When using SMC one, or multiple, parameters are chosen to be unknown. These unknown parameters are specified a range at which the given parameters are able to vary between and retain system perform within its given bounds ψ and ϵ. It is important to note that this range affects the boundary condition ψ and ϵ. Once the system has entered within the boundary layer, or sliding boundary, it will not leave it. The latter will hold true as long as the parameters which are able to vary do not go out of the designed bounds [8][9].

The fact that a parameter can be defined to be between a given bound and still have stable control is one of SMCs strongest arguments for its use for the the reduction of TR.

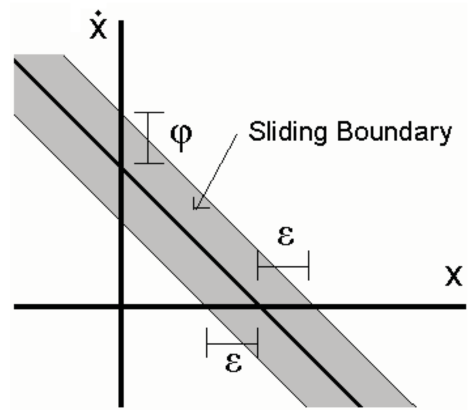

Fig. 10. SMC Sliding Boundary/Sliding Layer B

This is also because an exact model of the system does not have to be made for SMC to be effective. SMC is able to take care of some of the unmodeled dynamics including TR. Fig. 4 shows the effective inertial load on a system with TR before ω_{ar} and after ω_r. As stated previously in Fig. 4 and Eq. 10 there is an offset in the magnitude of the frequency response due to the change in the effective load. Due to the latter the inertia has been chosen to be the parameter to be unknown within a given bound. The inertial load will vary from $0.9J_a$ to $1.1(J_a + J_L)$ to ensure the system will always fall within the bounds of the SMC and thus stay within the sliding boundary.

Since the effective load has been taken care of by letting it vary; thus the model of the TR system from Fig. 1 can be reduced and assumed to be a system with out TR, see Fig. 11.

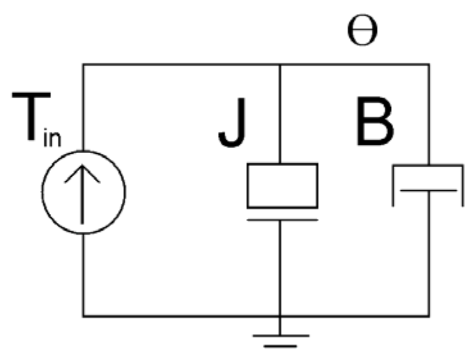

Fig. 11. Mechanical Schematic Drawling of Reduced System for SMC based on Fig 1

The dynamic equations for the system in Fig. 10 are written as

$$T(t) = J\ddot{\theta} + B\dot{\theta} \quad \rightarrow \quad \ddot{\theta} = \tfrac{1}{J}T(t) - \tfrac{B}{J}\dot{\theta} \tag{14}$$

to implement the SMC u that consists of the approximation of continuous control $\hat{u}$ combined with a switching mode $Q\text{sgn}(r)$ that is designed to keep the system on the sliding surface

$$u = \hat{u} - Q\text{sgn}(r) \tag{15}$$

where r is the sliding surface

$$r = \dot{\tilde{\theta}} + \lambda\tilde{\theta} \tag{16}$$

and

$$\hat{u} = \hat{b}^{-1}\left[\ddot{\theta}_d - \hat{f} - \lambda\dot{\tilde{\theta}}\right] \tag{17}$$

where d denotes the desired state, hat (ˆ) denotes the estimate, tilde (˜) denotes the error between the observed state and the desired state, λ is a gain chosen on a per-system basis, and the following parameters are defined as

$$b = \tfrac{1}{J}, \quad R = \tfrac{B}{J} \tag{18}$$

and

$$f = -R\dot{\theta}, \quad \hat{f} = -R_{ave}\dot{\theta} \tag{19}$$

where bounds on J and R are set as

$$R_{min} \leq R \leq R_{max}, \quad R_{min} = \tfrac{B}{J_a + J_L}, \quad R_{max} = \tfrac{B}{J_a} \tag{20}$$

The maximum deviation and average value of R is defined by R_x and R_{ave} respectively

$$R_x = \left|R_{max} - R_{min}\right|, \quad R_{ave} = \tfrac{1}{2}\left(R_{max} + R_{min}\right) \tag{21}$$

values of Q mus be sufficiently large so that it stays on the sliding surface

$$Q \geq \hat{b}^{-1}\left(\eta - F\right) + \left(\beta - 1\right)\left|\hat{u}\right| \tag{22}$$

where

$$F \geq \left|f - \hat{f}\right| \quad \rightarrow \quad F = R_x\left|\dot{\theta}\right| \tag{23}$$

and

$$\beta = \left(\tfrac{b_{max}}{b_{min}}\right)^{\frac{1}{2}}, \quad b_{min} = \tfrac{1}{J_a + J_L}, \quad b_{max} = \tfrac{1}{J_a} \tag{24}$$

and $\hat{b}$ (the estimate of b) is the geometric mean of b. The values η and λ are restricted by the sampling rate of the controller. Both of the latter values are not to exceed a numerical value of half the sampling frequency in Hz. Fig. 12 shows the block diagram of the SMC on the TR system. Fig. 13 shows the frequency response of SMC on the TR system. The frequency response of this non-linear SMC controller applied to the system exhibiting TR was achieved by applying a frequency sweet of sinusoids and recording the output. The resulting magnitudes and phase shifts were compared with the input. This gives us a comparison to the frequency response of the linear RE controller on the system exhibiting TR, Fig. 9.

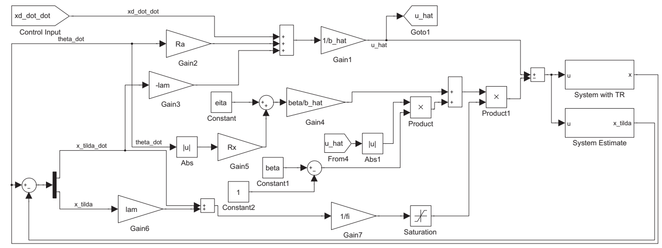

Fig. 12. Frequency response plot for SMC on system with TR with $\eta = 0.3$ and $\lambda = 30,000$

TABLE II

EFFECTS OF CHANGING THE RATIO OF $J_L : J_a$ (OPEN LOOP) $K_c = 55.0\frac{oz}{in}$

Ratio $(J_L : J_a)$	J_L $(oz \cdot in \cdot sec^2)$	ΔdB (dB)	Δ TR Phase (deg)
2.000	0.00460	9.542	180.0
1.435	0.00330	7.729	180.0
0.500	0.00115	3.522	180.0

Multiple simulations were made on the SMC control method to reduce the effects of TR. Table III shows the results of these test for various ratios of $J_L : J_a$. It is apparent from the simulations that the system will track the desired waveform. Load switching has no effect on the system as long as the apparent load stays between J_{min} and J_{max}. As the ration $\frac{J_L}{J_a} \rightarrow 0$ the effect of TR on both the magnitude and the phase goes to zero. Motor position θ_a and motor velocity $\dot{\theta}_a$ are needed to preform proper SMC in a TR system.

VI. RESULTS & CONCLUSION

The two different control designs for reducing the effect of TR in coupled systems, RE and SMC, both reduce the effect greatly. Table III shows the results from the control using RE and the control using SMC. These results show that RE reduces the Δ TR Mag over a greater range than SMC does. Both RE and SMC significantly reduce the ΔdB. However SMC reduces the ΔdB to $0dB$ over the defined ratios of $J_L : J_a$ while RE continues to have the switching effect causing a ΔdB around $0.5dB$. It has also been shown that the SMC is effective for dealing with the load switching over

TABLE III

EFFECTS OF CHANGING THE RATIO OF $J_L : J_a$ WHEN USING RE AND SMC (CLOSED LOOP) $K_c = 55.0\frac{oz}{in}$

Control Method	Ratio $(J_L : J_a)$	J_L $(oz \cdot in \cdot sec^2)$	ΔdB (dB)	Δ TR Mag (dB)	Δ TR Phase (deg)
RE	2.000	0.00460	0.500	0.400	4.600
RE	1.435	0.00330	0.300	0.200	3.500
RE	0.500	0.00115	0.500	0.100	1.600
SMC	2.000	0.00460	0.000	24.652	130.840
SMC	1.435	0.00330	0.000	10.200	23.770
SMC	0.500	0.00115	0.000	0.012	1.815

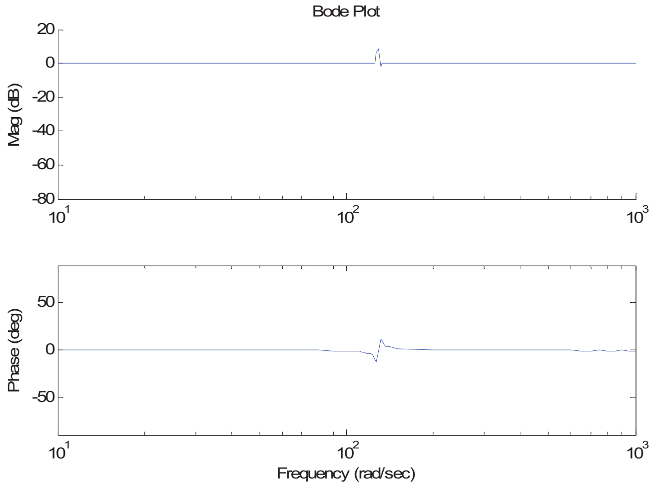

Fig. 13. Frequency response plot for SMC on system with TR with $\eta = 0.3$ and $\lambda = 30,000$

a range of $J_L : J_a$ ratios. The performance and simplicity of RE makes it a good choice if you have access to the current i_a and velocity $\dot{\theta}_a$ states. If do not have a current sensor but you do have access to the position θ_a and velocity $\dot{\theta}_a$ states then SMC is a valid choice.

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