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KINEMATIC CONTROL EQUATIONS FOR SIMPLE MANIPULATORS

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Abstract

The basis for all advanced manipulator control is a relationship between the cartesian coordinates of the end-effector and the manipulator joint coordinates. A direct method for assigning link coordinate systems and obtaining the end effector position, and Jacobian, in terms of joint coordinates is reviewed. Techniques for obtaining the solution to these equations for kinematically simple manipulators, which includes all commercially available manipulators, is presented. Finally the inverse Jacobian is developed from the solution.

1. Introduction

Piper in his thesis at Stanford University [6], investigated the kinematics of manipulators and described an algorithm for obtaining the kinematic equations for any manipulator yielding the cartesian position and orientation of the end effector given the joint coordinates. He investigated the solution of these equations and demonstrated that if certain kinematic conditions were met a solution of the equations could be obtained. Solutions have subsequently been obtained for various manipulators employing various ad hoc methods. Solutions have been obtained for the Stanford arm by Paul [4], Lewis [2], Shimano [9] and Taylor [10]; and for the Unimate by Park and Nitzan [3]. A method of obtaining the Jacobian yielding the change in position and orientation of the end effector for a change in joint angles is described by [1] but in terms of an ad hoc coordinate system.

In this paper we review the rules for assigning coordinate frames to the manipulator links, including all the degenerate cases, and for prismatic links. The matrix formulation to obtain the kinematic equations is developed and demonstrated by obtaining the equations for the Stanford arm. The solution of these equations is then developed on a joint-by-joint basis and demonstrated for the Stanford arm. the solution is based on establishing an algorithm to position and orient the manipulator joint-by-joint and translating such an algorithm into the solution for the joint angles. A method for obtaining the Jacobian is then described based on the kinematic equations. Finally the inverse Jacobian is obtained by differentiating the solution.

We feel that the methods developed in this paper are generally applicable to all manipulators for which an algorithm can be established for positioning the joints, and forms a systematic method for obtaining both the kinematic equations, their solution, Jacobian and inverse Jacobian.

2. Coordinate Frames

A serial link manipulator consists of a sequence of links connected together by actuated joints. For an n-degree-of-freedom manipulator, there will be n links and n joints. The base of the manipulator is link 0 and is not considered one of the six links. Link 1 is connected to the base link by joint 1. There is no joint at the end of the final link. The only significance of links is that they maintain a fixed relationship between the manipulator joints at each end of the link [8]. Any link can be characterized by two dimensions: the common normal a_n and, α_n the angle between the axes is a plane perpendicular to a_n. It is customary to call a_n "the length" and α_n "the twist" of the link (see Figure 1).

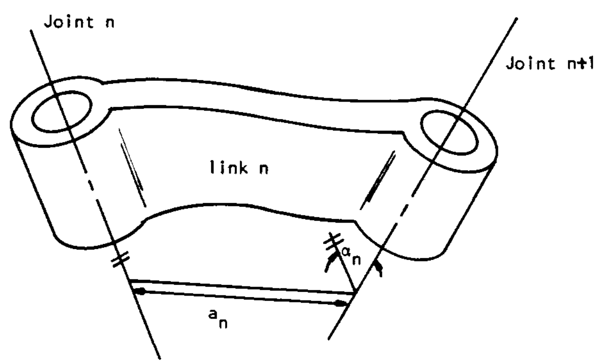

Figure 1. The length a, and twist α, of a link.

Generally, two links are connected at each joint axis (see Figure 2). The axis will have two normals to it, one for each link. The relative position of two such connected links is given by d_n, the distance between the normals along the joint n axis, and θ_n the angle between the normals meas-

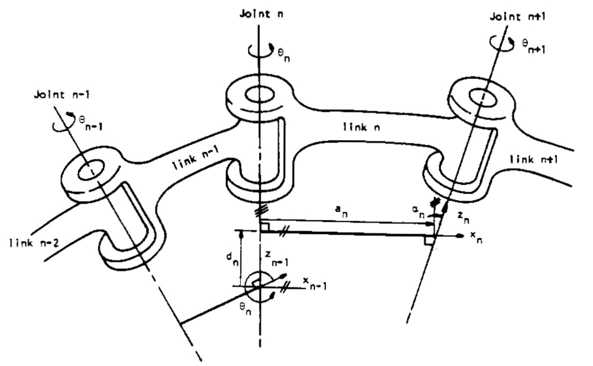

Figure 2. Link parameters θ, d, a, α

ured in a plane normal to the axis. d_n and θ_n are called "the distance" and "the angle" between the links.

In order to describe the relationship between links, we will assign coordinate frames to each link. We will first consider revolute joints in which θ_n is the joint variable. The origin of the coordinate frame of link n is set to be at the intersection of the common normal between joints n and n+1 and the axis of joint n+1. In the case of intersecting joint axes, the origin is at the point of intersection of the joint axes. If the axes are parallel, the origin is chosen to make the joint distance zero for the next link whose coordinate origin is defined. The z axis for link n shall be aligned with the axis of joint n+1. The x axis will be aligned with any common normal which exists and is directed along the normal from joint n to joint n+1. In the case of intersecting joints, the direction of the x axis is parallel or antiparallel to the vector cross product $\underline{z}_{n-1} \times \underline{z}_n$. Notice this condition is also satisfied for the x axis directed along the normal between joints n and n+1.

In the case of a prismatic joint the distance d_n is the joint variable. The direction of the joint axis is the direction in which the joint moves. Although the direction of the axis is defined, unlike a revolute joint, its position in space is not defined. In the case of a prismatic joint the angle θ_n and the length a_n have no meaning and are set to zero. The origin of the coordinate frame for a prismatic joint is coincident with the next defined link origin. The z axis of the prismatic link is aligned with the axis of joint n+1. The x axis is parallel to the x_{n-1} axis (see Figure 3).

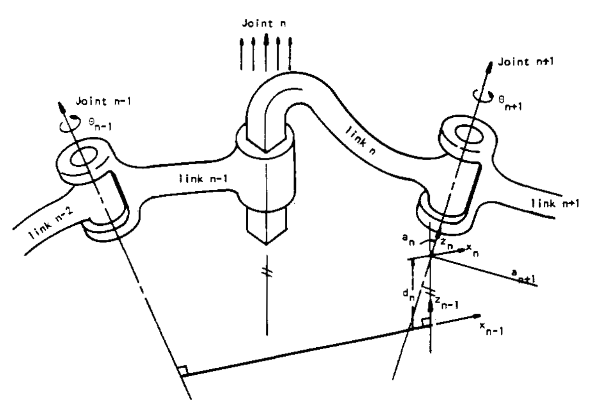

Figure 3. Link parameters d, α for a prismatic joint.

For a sequence of joints, both revolute and prismatic, with all joint variables zero the x axes all have the same direction in space. We will call this zero position with $\theta_i = 0$ for all revolute joints and $d_i = 0$ for all prismatic joints.

With the manipulator in its zero position the positive sense of rotation for revolute joints or displacement for prismatic joints can be decided and the sense of the direction of the z axes determined.

The origin of the base link (zero) will be coincident with the origin of link 1. If it is desired to define a different reference coordinate system then the relationship between the reference and base coordinate systems can be described by a fixed homogeneous transformation [7]. At the end of the manipulator the final displacement d_6 or rotation θ_6 occurs with respect to z_5. The z_6 coordinate system for link 6 is chosen to be coincident with the link 5 coordinate system. If a tool or end effector is used whose origin and axes do not coincide with the coordinate system of link 6 the tool can be related by a fixed homogeneous transformation to link 6.

Having assigned coordinate frames to all links according to the preceding scheme, we can establish the relationship between successive frames n-1, n by the following rotations and translations:

Rotate about z_{n-1}, an angle, θ_n

Translate along z_{n-1}, a distance d_n

Translate along rotated $x_{n-1} = x_n$ a length a_n

Rotate about x_n, the twist angle α_n

This may be expressed as the product of four homogeneous transformations relating the coordinate frame of link n to the coordinate frame of link n-1 and is called A matrix.

$$A_n = \begin{bmatrix} C\theta & -S\theta C\alpha & S\theta S\alpha & aC\theta \\ S\theta & C\theta C\alpha & -C\theta S\alpha & aS\theta \\ 0 & S\alpha & C\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

where C and S refer to sine and cosine respectively. For a prismatic joint the A matrix reduces to:

$$A_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2)$$

Once the link coordinate frames have been assigned to the manipulator the various constant link parameters can be tabulated, d, a, and α for a link following a revolute joint and, a and α for a link following a prismatic joint. Based on these parameters the constant sine and cosine values of the α's may be evaluated and the values for the six A_i transformation matrices determined.

3. Kinematic Equations

The description of link coordinate frame n with respect to the base coordinate system is given by T_n where

$$T_n = A_1 + A_2 + \dots + A_n \tag{3}$$

where the + sign indicates matrix multiplication of homogeneous link transformations. The end of the manipulator with respect to the base is given by T_6:

$$T_6 = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \tag{4}$$

If the manipulator is related to a reference coordinate frame by a transformation Z, and has a tool attached to its end described by E, we have the description of the end of the tool with respect to the reference coordinate system described by X as follows [5]:

$$X = Z + T_6 + E \tag{5}$$

which may be solved for T_6 as:

$$T_6 = -Z + X - E \tag{6}$$

where the - sign indicates multiplication by the matrix inverse.

In Figure 4 the Stanford arm is shown with coordinate frames assigned to the links. The parameters are shown in Table 1.

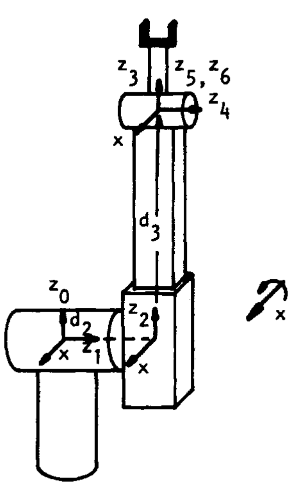

Figure 4. Coordinate frames for the Stanford Arm.

Link	Variable	α	a	d	$C\alpha$	$S\alpha$
1	θ_1	-90°	0	0	0	-1
2	θ_2	-90°	0	d_2	0	1
3	d_3	0°	0	d_3	1	0
4	θ_4	-90°	0	0	0	-1
5	θ_5	90°	0	0	0	1
6	θ_6	0°	0	0	1	0

Table 1 Link parameters for the Stanford Arm

The A matrices for the Stanford Arm are as follows:

$$A_1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$A_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$A_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$A_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

The T_i matrices for the Stanford arm are:

$$T_1 = A_1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & C_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$T_2 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & -S_1d_2 \\ S_1C_2 & C_1 & S_1S_2 & C_1d_2 \\ -S_2 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

$$T_3 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & -S_1d_2 + C_2S_2d_3 \\ S_2C_2 & C_1 & S_1S_2 & C_1d_2 + S_1S_2d_3 \\ -S_2 & 0 & C_2 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$T_4 = \begin{bmatrix} C_2C_2C_4 - S_1S_4 & -C_1S_2 & -C_1C_2S_4 - S_1C_4 & -S_1d_2 \bullet C_1S_2d_3 \\ S_1C_2C_4 + C_1S_4 & -S_1S_2 & -S_1C_2S_4 \bullet C_1C_4 & C_1d_2 \bullet S_1S_2d_3 \\ -S_2C_4 & -C_2 & +S_2S_4 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

$$T_5 = \begin{bmatrix} (C_1C_2C_4-S_1S_4)C_5-C_1S_2S_5 & -C_1C_2S_4-S_1C_4 & (C_1C_2S_4-S_1S_4)S_5+C_1S_2C_5 & -S_1d_2+C_1S_2d_3 \\ (S_1C_2C_4+C_1S_4)C_5-S_1S_2S_5 & -S_1C_2S_4+C_1C_4 & (S_1C_2C_4+C_1S_4)S_5+S_1S_2C_5 & C_1d_2+S_1S_2d_3 \\ -S_2C_4C_5-C_2S_5 & S_2S_4 & -S_2C_4S_5+C_2C_5 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

where:

$$o_x = -[(C_1S_2C_4-S_1S_4)C_5-C_1S_2S_5]S_6-[C_1C_2S_4+S_1S_4]C_6 \tag{19}$$

$$o_y = -[(S_1C_2C_4+C_1S_4)C_5-S_1S_2S_5]S_6-[S_1C_2S_4-C_1C_4]C_6 \tag{20}$$

$$o_z = [S_2C_4C_5+C_2S_5]S_6+S_2S_4C_6 \tag{21}$$

$$a_x = (C_1C_2C_4-S_1S_4)S_5+C_1S_2C_5 \tag{22}$$

$$a_y = (S_1C_2C_4+C_1S_4)S_5+S_1S_2C_5 \tag{23}$$

$$a_z = -S_2C_4S_5+C_2C_5 \tag{24}$$

$$p_x = -S_1d_2+C_1S_2d_3 \tag{25}$$

$$p_y = C_1d_2+S_1S_2d_3 \tag{26}$$

$$p_z = C_2d_3 \tag{27}$$

The first column of T_6 can be obtained as the vector cross product of the second and third columns.

If the joint coordinates are given, the position and orientation of the hand is obtained by evaluating T_6. Taking advantage of common subexpressions, this corresponds to 38 multiplies, 17 additions and 10 transcendental function calls.

4. Solution

In order to control the manipulation, we are interested in the reverse problem, that is, given T_6 what are the corresponding joint coordinates?

We could attempt to solve equations (19-27) for θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 directly but this approach is difficult for the following reasons:

- The equations are transcendental
- We will need both the sine and cosine in order to determine angles uniquely and accurately.
- The manipulator exhibits more than one solution for a given position.
- We have twelve equations in six unknowns.
- The equations are not independent.

The solution of these equations are in fact usually guided by geometric insights. Instead of attempting to solve the equations, we will employ these geometric insights directly in order to obtain the solution as follows. We will develop an algorithm to position the joints one by one in order to obtain the correct position and orientation of the last link. Using vector methods, and the expressions represented by the A and T matrices, we will translate this algorithm into equations for the joint coordinates. While this algorithm will vary from manipulator to manipulator we will outline the method by developing the solution for the Stanford arm.

An algorithm to position and orient the Stanford arm starting from the zero position is as follows:

1. Set joint 1 such that a rotation about joint 2 will point the axis of link 3 at the given position.

2. Set the joint 2 to point the axis of link 3 at the given position.

3. Set joint 3 such that the origin of link 3 is coincident with the given position.

4. Set joint 4 such that a rotation about joint 5 will align the axis of joint 6 with the given approach vector.

5. Set joint 5 to align the axis of joint 6 with the approach vector.

6. Set joint 6 to align the given orientation and normal vectors.

We then proceed to translate the algorithm into equations as follows. To solve for θ_1 consider the plan view of the arm shown in Figure 5.

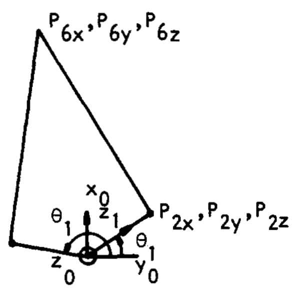

Figure 5. Plan view of Stanford arm.

The end of the manipulator is shown at a position P_{6x}, P_{6y}, P_{6z}. The origin of link 2 coordinates are at P_{2x}, P_{2y}, P_{2z}. The rotation θ_1 must be such as to make the unit vector z_1 perpendicular to a vector from $\underline{P}_2$ to $\underline{P}_6$. The vector $\underline{P}_2$ is given by the right hand column of T_2, the unit vector z_1 is given by the third column of T_1. We have:

$$\underline{z}_1 \cdot (\underline{P}_6 - \underline{P}_2) = 0 \tag{28}$$

or: $$-S_1(P_{6x}+S_1d_2) + C_1(P_{6y}-C_1d_2) = 0 \tag{29}$$

and: $$C_1 P_{6y} - S_1 P_{6x} = d_2 \tag{30}$$

Substituting $C_1 = \frac{1-t^2}{1+t^2}$ and $S_1 = \frac{2t}{1+t^2}$

where $t = \tan \theta_1/2$

we obtain: $$(P_{6y}+d_2)t^2 + 2P_{6x}\, t+(d_2-P_{6y}) = 0 \tag{31}$$

and solving for t:

$$\theta_1 = 2 \times \tan^{-1}\left[\frac{-P_x \pm \sqrt{P_x^2+P_y^2-d_2^2}}{P_y+d_2}\right] \tag{32}$$

or:

$$\theta_1 = \tan^{-1}\left[\frac{2(P_{6y}+d_2)(-P_{6x} \pm \sqrt{P_{6x}^2+P_{6y}^2-d_2^2})}{(P_{6y}+d_2)^2 - (-P_{6x} \pm \sqrt{P_{6x}^2+P_{6y}^2-d_2^2})^2}\right] \tag{33}$$

The − sign corresponds to a right hand shoulder θ_1 and

the + sign corresponds to a left hand shoulder θ_1.

If the discriminant is negative, no solution exists.

Note the arctangent functions of two arguments must be used as $-180° \leq \theta < 180°$.

The rotation θ_2 is required to align the $\underline{z}_2$ vector with the vector $\underline{P}_6-\underline{P}_2$. If we normalize $\underline{P}_6-\underline{P}_2$, we can equate components of $\underline{z}_2$, given by the third column at $\underline{T}_2$. If the magnitude of $\underline{P}_6-\underline{P}_2 = N$, then the vector has the following components:

$$1/N(P_{6x} + s_1d_2)\ \underline{i} + 1/N(P_{6y} - C_1d_2)\ \underline{j} + 1/N\ P_{6z}\ \underline{k}$$

!!! equation !!! (34)

we have the relationship:

$$T_1 + A_2 = \begin{bmatrix} ? & ? & 1/N(P_{6x}+S_1d_2) & ? \\ ? & ? & 1/N(P_{6y}-C_1d_2) & ? \\ ? & ? & 1/NP_{6z} & ? \\ ? & ? & 0 & ? \end{bmatrix} \tag{35}$$

where the ? represents a row or column of the matrix we either do not know or do not require. Solving for A_2

$$A_2 = T_1^{-1} + \begin{bmatrix} ? & ? & 1/N(P_{6x}+S_1d_2) & ? \\ ? & ? & 1/N(P_{6y}-C_1d_2) & ? \\ ? & ? & 1/NP_{6z} & ? \\ ? & ? & 0 & ? \end{bmatrix} \tag{36}$$

This is a vector equation for the third column of A_2 or:

$$\begin{bmatrix} S_2 \\ -C_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 & S_1 & 0 & ? \\ 0 & 0 & -1 & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} + \begin{bmatrix} 1/N(P_{6x}+S_1 d_2) \\ 1/N(P_{6y}-C_1 d_2) \\ 1/NP_{6z} \\ 0 \end{bmatrix} \quad (37)$$

We have two equations one for NS_2 and one for NC_2.

$$NS_2 = (C_1 P_{6x} + S_1 P_{6y}) \quad (38)$$

$$NC_2 = P_{6z} \quad (39)$$

and:

$$\theta_2 = \tan^{-1} \frac{(C_1 P_{6x} + S_1 P_{6y})}{P_{6z}} \quad (40)$$

In the case of joint 3, the prismatic joint, d_3 must cause the origin of link coordinate frame 3 to coincide with $\underline{P}_6$ or:

$$[T_2] * [A_3] = [\ ?\ ?\ ?\ \underline{P}] \quad (41)$$

and: $$[A_3] = [T_2]^{-1} * [\ ?\ ?\ ?\ \underline{P}] \quad (42)$$

or: $$\begin{bmatrix} 0 \\ 0 \\ d_3 \\ 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ C_1 S_2 & S_1 S_2 & C_2 & 0 \\ ? & ? & ? & ? \end{bmatrix} \times \begin{bmatrix} P_{6x} \\ P_{6y} \\ P_{6z} \\ 1 \end{bmatrix} \quad (43)$$

and: $$d_3 = S_2(C_1\ P_x + S_1\ P_y) + C_2 P_2 \quad (44)$$

$$d_3 = S_2(NS_2) + C_2 P_2 \quad (45)$$

In the case of joint 4, its axis is perpendicular to the joint 3 axis and also to the joint 5 axis. However, the joint 5 axis must be aligned with the approach vector of T_6, $\underline{a}$. This relationship is expressed by the vector cross product:

$$\underline{z}_4 = \pm N(\underline{z}_3 \times \underline{a}) \quad (46)$$

If $\underline{z3} \underline{xa} = 0$ the manipulator is degenerate and any value may be used for θ_4. Forming the cross product with $\underline{z}_3$, obtained from the third column of $\underline{T}_3$, and $\underline{a}$ we have:

$$\underline{z}_4 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ C_1 S_2 & S_1 S_2 & C_2 \\ a_x & a_y & a_z \end{vmatrix} \quad (47)$$

and:

$$\underline{z}_4 = \pm N(S_1 S_2 a_z - C_2 a_y)\underline{i} \pm N(C_2 a_x - C_1 S_2 a_z)\underline{j} \pm N(C_1 S_2 a_y - S_1 S_2 a_x$$

!!! Equation !!! (48)

The rotation θ_4 is obtained from the relation:

$$[T_3] * [A_4] = [\ ?\ ?\ \underline{z}_4\ ?\ ] \quad (49)$$

and:

$$\begin{bmatrix} -S_4 \\ C_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 C_2 & S_1 C_2 & -S_2 & 0 \\ -S_1 & C_1 & 0 & 0 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \times \begin{bmatrix} \pm N(S_1 S_2 a_z - C_2 a_y) \\ \pm N(C_2 a_x - C_1 S_2 a_z) \\ \pm N(C_1 S_2 a_y - S_1 S_2 a_x) \\ 0 \end{bmatrix} \quad (50)$$

and solving for S_4 and C_4 and simplifying:

$$S_4 = \pm N(C_1 a_y - S_1 a_x) \quad (51)$$

$$C_4 = \pm N[C_1 C_2 a_x + S_1 C_2 a_y - S_2 a_z] \quad (52)$$

and

$$\theta_4 = \tan^{-1} \frac{C_1 a_y - S_1 a_x}{C_1 C_2 a_x + S_1 C_2 a_y - S_2 a_z} \quad \text{if } N > 0 \quad (53)$$

and $\theta_4 = \theta_4 + 180°$ if $N < 0$. (54)

we have two solutions corresponding to the two choices for the sign of k. The rotation θ_5 about z_4 must align z_5 with the approach vector $\underline{a}$

$$[T_4] * [A_4] = [\ ?\ ?\ \underline{a}\ ?\ ] \quad (55)$$

and:

$$\begin{bmatrix} S_5 \\ -C_5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{4xx} & T_{4xy} & T_{4xz} & ? \\ T_{4yx} & T_{4yy} & T_{4yz} & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \times \begin{bmatrix} a_x \\ a_y \\ a_z \\ 0 \end{bmatrix} \quad (56)$$

thus:

$$S_5 = \underline{T_{4x}} \cdot \underline{a} \quad (63)$$

$$C_5 = -\underline{T_{4y}} \cdot \underline{a} \quad (64)$$

and:

$$\theta_5 = \tan^{-1} \frac{S_5}{C_5} \quad (65)$$

In this case, we obtain values for S_5 and C_5 directly and do not need to call the sine and cosine routines. Finally, T_6 is given by:

$$[T_5] * [A_6] = [n\ o\ ?\ ?\ ] \tag{66}$$

and:

$$\begin{bmatrix} C_6 & -S_6 \\ S_6 & C_6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{5xx} & T_{5xy} & T_{5xz} & ? \\ T_{5yx} & T_{5yy} & T_{5yz} & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \times \begin{bmatrix} n_x & o_x \\ n_y & o_y \\ n_z & o_z \\ 0 & 0 \end{bmatrix}$$

!!! Equation !!! (67)

then:

$$S_6 = \underline{T_{5y}} \cdot \underline{n} \tag{71}$$

$$C_6 = \underline{T_{5y}} \cdot \underline{o} \tag{72}$$

and

$$\theta_6 = \tan^{-1} \frac{S_5}{C_5} \tag{73}$$

The solution program requires 12 transcendental function calls, 40 multiplies and 22 additions, taking advantage of common sub-expressions.

5. The Jacobian

We have developed both the kinematic equations and their solution and will now consider the differential relationship $dT_6 = d\theta$. We will develop $dT_6/d\theta$ in terms of a vector differential rotation $\underline{\delta}$ and differential translation $\underline{d}$ of a coordinate frame T, parallel to the base coordinate frame but situated at the origin of T_6. This appears to be the simplest coordinate frame in which to develop $dT_6/d\theta$. We will further assume that the solution has been obtained and that $\underline{T_6}$, $\underline{\theta}$ and all the sines and cosines of angles developed in the solution are available.

If a joint is revolute, a change of joint coordinate $d\theta_i$ corresponds to a differential rotation about z_{i-1}, the z axis of link n-1. If the joint is prismatic, then the change corresponds to a differential translation dd_i along z_{i-1}. It can be seen from Figure 6 that the change in position of T for a revolute joint is given by:

$$\underline{d}_i = \underline{z}_{i-1} \times (\underline{P}_6 - \underline{P}_{i-1}) d\theta_i \tag{74}$$

and the change of orientation of T:

$$\underline{\delta}_i = \underline{z}_{i-1} d\theta_1 \tag{75}$$

z_{i-1} is the axis of joint i and is obtained from the third column of T_{i-1}. $\underline{P}_6 - \underline{P}_{i-1}$ represents a vector from the origin of T_6 to the origin of T_{i-1}. The vector $\underline{P}_{i-1}$ is obtained from the right hand column of T_{i-1}.

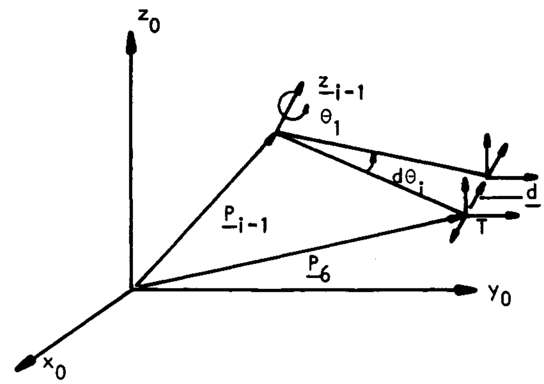

Figure 6. Change in Position of frame T

For a prismatic joint we obtain:

$$d_i = \underline{z}_{i-1}\, dd_i \tag{76}$$

$$\underline{\delta}_i = 0 \tag{77}$$

The total change in position and orientation is obtained from:

$$\begin{bmatrix} dT_x \\ dT_y \\ dT_z \\ \delta T_x \\ \delta T_y \\ \delta T_z \end{bmatrix} = \begin{bmatrix} J \end{bmatrix} \bullet \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \\ dq_4 \\ dq_5 \\ dq_6 \end{bmatrix} \tag{78}$$

where $dq_i = d\theta_i$ if the joint is revolute and dd_i if the joint is prismatic.

[J] The Jacobian is a six by six matrix and is composed of columns of the following form:

Revolute	Prismatic
$[z_{i-1} \times (P_6 - P_{i-1})]_x$	z_{ix}
$[z_{i-1} \times (P_6 - P_{i-1})]_y$	z_{iy}
$[z_{i-1} \times (P_6 - P_{i-1})]_z$	z_{iz}
z_{i-1x}	0
z_{i-1y}	0
z_{i-1z}	0

The first column of J is given below for either type of joint:

Revolute	Prismatic
$-P_{6y}$	0
P_{6x}	0
0	1
0	0
0	0
1	0

where $\underline{P}_6$ is the position vector of the hand The last column of J is:

Revolute	Prismatic
0	a_x
0	a_y
0	a_z
a_x	0
a_y	0
a_z	0

where $\underline{a}$ is the approach vector of T_6.

Evaluating the expressions given for the columns using the values of $\underline{P}_i$ and $\underline{z}_i$ obtained from the T matrices we obtain the Jacobian matrix for the Stanford arm:

$$\begin{bmatrix} dT_x \\ dT_y \\ dT_z \\ \delta T_x \\ \delta T_y \\ \delta T_2 \end{bmatrix} = \begin{bmatrix} -P_{6y} & C_1C_2d_3 & C_1S_2 & 0 & 0 & 0 \\ P_{6x} & S_1C_2d_3 & S_1S_2 & 0 & 0 & 0 \\ 0 & -S_2d_3 & C_2 & 0 & 0 & 0 \\ 0 & -S_1 & 0 & C_1S_2 & -C_1C_2S_4-S_1C_4 & a_x \\ 0 & C_1 & 0 & S_1S_2 & -S_1C_2S_4+C_1C_4 & a_y \\ 1 & 0 & 0 & C_2 & S_2S_4 & a_2 \end{bmatrix} \times \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dd_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \tag{79}$$

Having obtained the change in position and orientation of T represented by $\underline{dT}$ and the small rotation $\underline{\delta T}$. We can obtain the change in T_6 as:

$$dT_6 = \begin{bmatrix} 0 & -\underline{\delta T}\cdot\underline{a} & \underline{\delta T}\cdot\underline{o} & \underline{dT}\cdot\underline{n} \\ \underline{\delta T}\cdot\underline{a} & 0 & -\underline{\delta T}\cdot\underline{n} & \underline{dT}\cdot\underline{o} \\ -\underline{\delta T}\cdot\underline{o} & \underline{\delta T}\cdot\underline{n} & 0 & \underline{dT}\cdot\underline{a} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{80}$$

where $\underline{\delta T}$ is the vector represented by the first three elements of the left hand column matrix of equation (79) and $\underline{dT}$ is represented by the last three elements. $\underline{n}$, $\underline{o}$, and $\underline{a}$ are from T_6.

In order to evaluate $\underline{dT}_6$, we need 40 multiplies and 36 additions, no transcendental functions.

6. Differential Change in Position

In order to obtain the reverse differential solution $d\theta/dT_6$ we will differentiate the solution in the same order in which we obtained the joint coordinates. Each subsequent change in joint angle will thus be a function of dT_6 and dq_i. Where dq_i represents the differential changes in joint angle already determined.

Differentiating equation 30 we obtain:

$$-S_1P_yd\theta_1 + C_1dP_y - C_1P_xd\theta_1 - S_1dP_x = 0 \tag{81}$$

and solving for $d\theta_1$

$$d\theta_1 = \frac{C_1dP_y - S_1dP_x}{NS_2} \tag{83}$$

NS_2 is obtained from equations 38.

The equation for θ_2 is in terms of the arctangent and in general if

$$\tan\theta = \frac{N\sin\theta}{N\cos\theta} \tag{84}$$

then:

$$d\theta = \frac{N\cos\theta\ d(N\sin\theta) - n\sin\theta\ d(N\cos\theta)}{(N\sin\theta)^2 + (N\cos\theta)^2} \tag{85}$$

For θ_2 we have NS_2 and NC_2 from equations 38 and 39 Thus:

$$dNS_2 = d(C_1)P_x + C_1d(P_x) + d(S_1)P_y + S_1d(P_y) \tag{86}$$

$$dNC_2 = dP_z \tag{87}$$

and:

$$d\theta_2 = \frac{P_zd(NS_2) - NS_2dP_z}{NS_2^2 + P_z^2} \tag{88}$$

we shall also need:

$$d(C_1C_2) = d(C_1)C_2 + C_1d(C_2) \tag{89}$$

$$d(S_1C_2) = d(S_1)C_2 + S_1d(C_2) \tag{90}$$

$$d(C_1S_2) = d(C_1)S_2 + C_1d(S_2) \tag{91}$$

$$d(S_1S_2) = d(S_1)S_2 + S_1d(S_2) \tag{92}$$

We have d_3 given by equation 45 and thus:

$$dd_3 = dS_2NS_2 + S_2d(NS_2) + d(C_2)P_z + C_2(dP_z) \tag{93}$$

For θ_4 we have:

$$NS_4 = C_1a_y - S_1a_x \tag{94}$$

$$NC_4 = \underline{T_{3x}} \cdot \underline{a} \tag{95}$$

where $\underline{T_{3x}}$ is the first column of T_3 and:

$$dN_{S4} = d(C_1)a_y + C_1d(ay) - d(S_1)ax - S_1d(a_x) \tag{96}$$

$$dT_{3xx} = d(C_2C_2) \tag{97}$$

$$dT_{3xy} = d(S_1C_2) \tag{98}$$

$$dT_{3xz} = -d(S_2) \tag{99}$$

Thus:

$$d\theta_4 = \frac{NC_4[d(NS_4)-NS_4\cdot(d\underline{T_{3x}})\cdot\underline{a}+\underline{T_{3x}}\cdot\underline{d(a)}]}{NS_4^2+NC_4^2} \tag{100}$$

for θ_5 we have expressions for S_5 and C_5, equations 63 and 64 thus:

$$d(T_{4xx}) = d(C_1C_2)C_4+C_1C_2d(C_4)-d(S_1)S_4-S_1(dS_4) \tag{101}$$

$$d(T_{4xy}) = d(S_1C_2)C_4+S_1C_2d(C_4)+d(C_1)S_4+C_1d(S_4) \tag{102}$$

$$d(T_{4xz}) = -d(S_2)S_4-S_2d(S_4) \tag{103}$$

$$d(T_{4yx}) = -d(C_1S_2) \tag{104}$$

$$d(T_{4yy}) = -d(S_1S_2) \tag{105}$$

$$d(T_{4yz}) = -d(C_2) \tag{106}$$

and:

$$d\theta_5 = C_5[d\underline{(T_{4x}})\cdot\underline{a} + \underline{T_{4x}}\cdot\underline{d(a)}] + S_5[d\underline{(T_{4o}})\cdot\underline{a} + \underline{T_{4o}}\cdot d\underline{(a)}] \tag{107}$$

Similarly for θ_6 we have expressions for S_6 and C_6 equations 71 and 71 thus:

$$d(T_{5yx}) = -d(C_1C_2)S_4 - C_1C_2d(S_4)-d(S_1)C_4-S_1d(C_4)$$

!!! Equation !!! (108)

$$d(T_{5yy}) = -d(S_1C_2)S_4-S_1C_2d(S_4)+d(C_1)C_4+C_1d(C_4)$$

!!! Equation !!! (109)

$$d(T_{5yx}) = d(S_2)S_4+S_2d(S_4) \tag{110}$$

$$d\theta_6 = C_6[d\underline{(T_{5y}})\cdot\underline{n} + \underline{T_{5y}}\cdot d\underline{(n)}]-S_5[d(\underline{T_{5y}})\cdot\underline{o}+\underline{T_{5o}}\cdot\underline{d(o)}] \tag{111}$$

The differential solution $d\theta/dT_6$ represents 89 multiplies and 56 additions and no transcendental function calls.

7. Summary

We have reviewed the method of assigning coordinate frames to the links of a manipulator. In terms of these coordinate frames the kinematic equations and the Jacobian can be developed in a straight forward manner. These equations and the Jacobian can be obtained for any manipulator. If a positioning algorithm can be developed for a manipulator, then a method of translating such an algorithm into the solution of the kinematic equations is presented. From this solution, the change in joint coordinates for a change in position and orientation is developed.

8. Acknowledgement

This material is based upon research supported by the National Science Foundation under Grants Numbers APR77-14533, APR75-13074, and APR74-01390. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation. The solution for a differential change in manipulator position was based on a suggestion by T. Binford. Mickey Krebs was responsible for the document preparation.

9. References

[6] D. L. Piper, "The Kinematics of Manipulators Under Computer Control," Stanford Artificial Intelligence Project, Memo AIM-72, October 1968.

[4] R. Paul, "Modelling, Trajectory Calculation, and Servoing of a Computer Controlled Arm," Stanford Artificial Intelligence Laboratory Memo AIM-77, November 1972.

[2] R. A. Lewis, "Autonomous Manipulations on a Robot: Summary of Manipulator Software Functions," JPL TM 33-679, 1974.

[9] t. Binford, et al. "Exploratory Studies of Computer Integrated Assembly Systems," NSF Progress Report. Stanford Artificial Intelligence Laboratory Memo AIM-285, July 1976.

[3] C. Rosen, D. Nitzan, et al. "Exploratory Research in Advanced Automation," Second Report, Stanford Research Institute, August 1974.

[1] R. C. Groome, Jr., "Force Feedback Steering of a Teleoperator System," S. M. Thesis, MIT 1972.

[8] B. Roth, "Performance Evaluation of Manipulators from a Kinematic Viewpoint," V.B.S. Report SP-459, 1976.

[7] L. G. Roberts, "Homogeneous Matrix Representation and Manipulation of N-Dimensional Constructs," Lincoln Las Document MS-1045, May 1965.

[5] R. Paul, "Advanced Industrial Robot Control Systems," First Report Purdue University Memo EE 78-25, May 1978.

[10] R. H. Taylor, "Planning and Execution of Straight-Line Manipulator Trajectories," IBM Research Report RC 6657, July 1977.