Short Papers

Optimal Extended Jacobian Inverse Kinematics Algorithms for Robotic Manipulators

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***Abstract*—Extended Jacobian inverse kinematics algorithms for redundant robotic manipulators are defined by combining the manipulator's kinematics with an augmenting kinematics map in such a way that the combination becomes a local diffeomorphism of the augmented taskspace. A specific choice of the augmentation relies on the optimal approximation by the extended Jacobian of the Jacobian pseudoinverse (the Moore–Penrose inverse of the Jacobian). In this paper, we propose a novel formulation of the approximation problem, rooted conceptually in the Riemannian geometry. The resulting optimality conditions assume the form of a Poisson equation involving the Laplace–Beltrami operator. Two computational examples illustrate the theory.**

***Index Terms*—Approximation, extended Jacobian, Jacobian pseudoinverse, robot kinematics.**

I. Introduction

The inverse kinematics problem for stationary or mobile manipulators consists in computing a manipulator's configuration corresponding to a prescribed location of the end-effector in the taskspace. Usually, this problem is solved numerically by means of Jacobian inverse kinematics algorithms, of which the most widely exploited is the Jacobian pseudoinverse algorithm. An alternative to the pseudoinverse is the extended Jacobian algorithm. The Jacobian pseudoinverse algorithm distinguishes by its speed of convergence, while the extended Jacobian algorithm has the desirable property of repeatability. Repeatable inverse kinematics algorithms transform closed paths in the taskspace into closed paths in the configuration space. The repeatability of stationary manipulators is a traditional subject of robotics [1]–[3]. Conditions for repeatability of inverse kinematics algorithms for mobile manipulators were given in [4]. The concept of the extended Jacobian inverse appeared in [5] and [6]. Its extension to mobile manipulators can be found in [7] and [8].

In this paper, we shall concentrate on inverse kinematics algorithms for stationary robotic manipulators. The idea of the optimal approximation of the Jacobian pseudoinverse inverse kinematics algorithm by a repeatable algorithm was introduced and developed in a series of papers by Roberts and Maciejewski [9]–[12]. One of the main threads of these papers can be reconstructed in the following way. Suppose that a coordinate representation of the manipulator's kinematics takes the form of a map

$$k : R^n \longrightarrow R^m, \qquad y = k(x)$$

from the jointspace into the taskspace. We assume that $n > m$, and let $J(x) = (\partial k/\partial x)(x)$ stand for the manipulator's Jacobian. Then, outside singular configurations of the manipulator, the Jacobian pseudoinverse $J^{P\#}(x) = J^T(x)M^{-1}(x)$, where $M(x) = J(x)J^T(x)$ denotes the manipulability matrix [13]. It is clear that $J(x)J^{P\#}(x) = I_m$. Now, setting $s = n - m$, we introduce an augmenting kinematics map

$$h : R^n \longrightarrow R^s, \qquad \tilde{y} = h(x) \tag{1}$$

and the extended kinematics

$$l = (k, h) : R^n \longrightarrow R^n, \qquad \bar{y} = (k(x), h(x)). \tag{2}$$

The extended Jacobian

$$\bar{J}(x) = \begin{bmatrix} \dfrac{\partial k(x)}{\partial x} \\ \dfrac{\partial h(x)}{\partial x} \end{bmatrix}$$

is represented by a square $n \times n$ matrix. At regular points of $\bar{J}(x)$, we define an extended Jacobian inverse

$$J^{E\#}(x) = \bar{J}^{-1}(x)|_{\text{first } m \text{ columns}}.$$

By definition, $J^{E\#}(x)$ is a right inverse of the Jacobian that is annihilated by the differential of the augmenting map

$$J(x)J^{E\#}(x) = I_m \qquad \text{and} \qquad \frac{\partial h}{\partial x}(x)J^{E\#}(x) = 0.$$

Given any right inverse $J^{\#}(x)$ of the Jacobian, and a desirable taskspace point $y_d \in R^r$, a solution of the inverse kinematics problem is obtained by taking a limit at $t \to +\infty$ of the trajectory of the dynamic system

$$\dot{x} = -\gamma J^{\#}(x(t))(k(x(t)) - y_d).$$

The positive number γ defines the convergence rate of the algorithm. Given the inverses $J^{P\#}(x)$ and $J^{E\#}(x)$, the approximation problem studied by Roberts and Maciejewski amounts to inventing an augmenting map (1) that minimizes the approximation error

$$\mathcal{E}(h) = \int_{\mathcal{M}} ||J^{P\#}(x) - J^{E\#}(x)||_F^2 \, dx \tag{3}$$

where $||M||_F = \sqrt{\mathrm{tr}(MM^T)} = \sqrt{\mathrm{tr}(M^TM)}$ denotes the Frobenius matrix norm and $\mathcal{M} \subset R^n$ is a singularity-free region of the jointspace. Using the fact that both $J^{P\#}(x)$ and $J^{E\#}(x)$ are right inverses of the Jacobian, we deduce that there exists an $s \times m$ matrix $W(x)$ such that the difference

$$J^{P\#}(x) - J^{E\#}(x) = K(x)W(x)$$

where $K(x)$ is a matrix with orthonormal columns, spanning the Jacobian kernel $J(x)K(x) = 0$, $K^T(x)K(x) = I_s$. After premultiplying the previous expression by $\partial h(x)/\partial x$, we compute

$$W(x) = \left(\frac{\partial h(x)}{\partial x}K(x)\right)^{-1} \frac{\partial h(x)}{\partial x} J^{P\#}(x)$$

and conclude that the error formula (3) becomes

$$\mathcal{E}(h) = \int_{\mathcal{M}} \mathrm{tr}\left(\left(\frac{\partial h(x)}{\partial x}K(x)\right)^{-1} \frac{\partial h(x)}{\partial x} J^{P\#}(x)J^{P\#\,T}(x) \right.$$
$$\left. \times \left(\frac{\partial h(x)}{\partial x}\right)^T \left(\frac{\partial h(x)}{\partial x}K(x)\right)^{-T}\right) dx. \tag{4}$$

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The optimality conditions for (4) lead to a system of nonlinear partial differential equations (PDEs), and moreover, a computation of the integrand in (4) becomes ill-conditioned close to singularities of the matrix $(\partial h(x)/\partial x)K(x)$.

The objective of this paper is to redefine the approximation problem in a way resulting in the optimality conditions that have a sound geometric interpretation and are tractable computationally. Some basic analogies with Riemannian geometry are used as the guidelines. Specifically, the new definition exploits two extended Jacobians: one associated with the augmenting kinematics map, the other based on complementing the range space of the Jacobian with its null space. A measure of the distance between the extended Jacobian inverse and the Jacobian pseudoinverse is induced by the distance between corresponding extensions. In this way, the error formula standing under the integral (3) gets embedded into the general linear group of matrices, and then integrated using the manipulability as the volume form. The main contribution of this paper consists in providing the optimality conditions in the form of a system of linear, elliptic PDEs involving the Laplace–Beltrami operator. The number of these equations equals the redundancy degree of the kinematics. All these equations include the same Laplace–Beltrami operator, and may differ only by the divergence term on the right-hand side. This means that, in fact, we need to solve a single equation with different substitutions on the right-hand side, and different boundary conditions. The benefits of this result are twofold: first, there exists an advanced theory of the Laplace–Beltrami operator [14], which may provide a theoretic insight into the approximation problem; second, numeric algorithms for solving linear, elliptic PDEs are offered by most commercial and scientific software packages dedicated to PDEs. Additionally, when the right-hand side of the equation is equal to zero, the optimal augmenting kinematics map becomes a harmonic map [15]; the object that has already been recognized in robotics as a potential function in the motion planning problem [16] or as a dexterous kinematics map minimizing the distortion [17]. If this is the case, all components of the augmenting map are obtained as a solution of the same PDE.

This paper is organized as follows. In Section II, we present our main result. Section III is devoted to computational examples. The paper is concluded with Section IV. A necessary background material from Riemannian geometry is sketched in the Appendix.

II. Main Result

The essence of the kinematics augmentation consists in extending the original kinematics to a local diffeomorphism (2) between the jointspace and the augmented taskspace. Suppose that $h(x)$ is an augmenting map. Wherever the matrix standing below on the left-hand side is nonsingular, we obtain

$$\begin{bmatrix} J(x) \\ \dfrac{\partial h(x)}{\partial x} \end{bmatrix}^{-1} = [\, J^{E\#}(x) \quad Q(x)\,] = A(x) \tag{5}$$

for a certain matrix $Q(x)$. At the same time, it is easily checked that at the regular points of $J(x)$

$$\begin{bmatrix} J(x) \\ K^T(x) \end{bmatrix}^{-1} = [\, J^{P\#}(x) \quad K(x)\,] = B(x) \tag{6}$$

where $J(x)K(x) = 0$ and $K^T(x)K(x) = I_s$. These regular configurations form an open set

$$X = \{x \in R^n \,|\, \det M(x) \neq 0\}$$

where $M(x)$ is the manipulability matrix. Suppose that the augmented taskspace $Y = R^n$ has been equipped with the Euclidean inner product defined by the unit matrix I_n (as a matter of fact, the following reasoning can be carried out for any Riemannian metric in the augmented taskspace). By analogy with the pullback operation [cf., (17)], we introduce into X a 2-form defined by a matrix

$$\begin{aligned} G(x) &= [\, J^T(x) \quad K(x)\,]\, I_n \begin{bmatrix} J(x) \\ K^T(x) \end{bmatrix} \\ &= J^T(x)J(x) + K(x)K^T(x). \end{aligned} \tag{7}$$

For the reason that (7) is symmetric and positive definite on X, we shall treat the pair (X, G) as a Riemannian manifold (although, formally $G(x)$ has not been defined by a pullback operation). Furthermore, from (6), we deduce that $G(x) = P^{-1}(x)$, where $P(x) = B(x)B^T(x) = J^{P\#}(x)J^{P\#\,T}(x) + K(x)K^T(x)$, compute $\det P(x) = \det B^T(x)B(x) = \det M^{-1}(x)$, and conclude that $V = \sqrt{\det M(x)}\, dx$ plays the role of the volume form (18) on (X, G). We recall that $m(x) = \sqrt{\det M(x)}$ is known as the manipulability of the configuration x [13].

In the region of regular configurations, the expressions (5) and (6) define an embedding of the inverses $J^{E\#}(x)$ and $J^{P\#}(x)$ into the linear group of invertible $n \times n$ matrices. These expressions include a pair of extended Jacobians: in (5), the extension is achieved by the augmenting map's Jacobian, whereas in (6), there is a natural extension by the transpose of $K(x)$. The main idea of this paper is to measure the distance between the inverses $J^{E\#}(x)$ and $J^{P\#}(x)$ by means of the distance of corresponding extended Jacobians. Actually, two equivalent error formulas can be derived. We begin with the left error

$$E_h^L(x) = \operatorname{tr}\left(A^{-1}(x)B(x) - I_n\right)^* \left(A^{-1}(x)B(x) - I_n\right)$$

where C^* denotes a map dual to C [cf., (19)]. Using the fact that the matrix $C(x) = A^{-1}(x)B(x) - I_n$ defines a linear map $C(x) : T_{l(x)}Y \longrightarrow T_{l(x)}Y$, we get $C^*(x) = C^T(x)$, and compute

$$E_h^L(x) = \operatorname{tr}\left(\frac{\partial h(x)}{\partial x}P(x)\left(\frac{\partial h(x)}{\partial x}\right)^T - 2\frac{\partial h(x)}{\partial x}K(x) + I_s\right). \tag{8}$$

Alternatively, we may consider the right error

$$E_h^R(x) = \operatorname{tr}\left(B(x)A^{-1}(x) - I_n\right)^* \left(B(x)A^{-1}(x) - I_n\right).$$

This time the matrix $C(x) = B(x)A^{-1}(x) - I_n$ corresponds to a linear map $C(x) : T_xX \longrightarrow T_xX$, so the dual map $C^*(x) = G^{-1}(x)C^T(x)G(x)$. Using the identity $G^{-1}(x) = P(x)$, we compute

$$\begin{aligned} E_h^R(x) = \operatorname{tr}\Bigg(& P(x)\left(J^{P\#}(x)J(x) + K(x)\frac{\partial h(x)}{\partial x} - I_n\right)^T \\ & \times G(x)\left(J^{P\#}(x)J(x) + K(x)\frac{\partial h(x)}{\partial x} - I_n\right)\Bigg) \end{aligned}$$

and after suitable developments, conclude that

$$E_h^R(x) = \operatorname{tr}\left(\frac{\partial h(x)}{\partial x}P(x)\left(\frac{\partial h(x)}{\partial x}\right)^T - 2\frac{\partial h(x)}{\partial x}K(x) + I_s\right)$$

i.e., the left and the right errors coincide. Having established this equivalence, we shall define the problem of optimal approximation of the Jacobian pseudoinverse by an extended Jacobian inverse as the minimization of the integrated left error (8) over a subset $\mathcal{M} \subset X$, using the volume form $V = m(x)\,dx$,

$$\mathcal{E}(h) = \int_{\mathcal{M}} \mathrm{tr}\left(\frac{\partial h(x)}{\partial x}P(x)\left(\frac{\partial h(x)}{\partial x}\right)^T - 2\frac{\partial h(x)}{\partial x}K(x) + I_s\right) m(x)\,dx. \tag{9}$$

The approximation error is a functional of the augmenting map. The Lagrangian appearing under the integral (9) equals

$$L\left(x, h, \frac{\partial h}{\partial x}\right) = \sum_{i=1}^{s}\left(\frac{\partial h_i}{\partial x}\right)^T m(x)P(x)\frac{\partial h_i}{\partial x} - 2\sum_{i=1}^{s}\left(\frac{\partial h_i}{\partial x}\right)^T m(x)K_i(x)$$

where $K_i(x)$ denotes the ith column of $K(x)$. Consequently, the optimality conditions assume the form of the Euler equations [18], i.e., for every $i = 1, \ldots, s$

$$\mathrm{tr}\,\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial(\partial h_i/\partial x)}\right) = \mathrm{tr}\frac{\partial}{\partial x}\left(m(x)P(x)\frac{\partial h_i}{\partial x}\right) - \mathrm{tr}\frac{\partial}{\partial x}\left(m(x)K_i(x)\right) = 0 \tag{10}$$

or after expansion

$$\sum_{r=1}^{n}\left(\sum_{k=1}^{n} m(x)P_{kr}(x)\frac{\partial^2 h_i(x)}{\partial x_k \partial x_r} + \sum_{k=1}^{n}\frac{\partial(m(x)P_{kr}(x))}{\partial x_r}\frac{\partial h_i(x)}{\partial x_k} - \frac{\partial(m(x)K_{ri}(x))}{\partial x_r}\right) = 0. \tag{11}$$

Using these definitions, we observe that the optimality conditions (10) require that the result of the action of the Laplace–Beltrami operator [cf., (20)] associated with $G(x)$ on every component of the augmenting map should be equal to the divergence (21) of a corresponding column of the Jacobian kernel, i.e., for every $i = 1, \ldots, s$

$$\Delta\, h_i = \mathrm{div}\, K_i. \tag{12}$$

This equation corresponds to the Poisson equation associated with the Laplace–Beltrami operator. When the right-hand side of (12) vanishes, the function $h_i(x)$ is called harmonic. It follows that in order to compute the augmenting map $h(x)$, we need to solve $s = n - m$ linear, elliptic PDEs (12) with the same operator on the left-hand side and suitably chosen boundary conditions.

III. Examples

In this section, we shall compute the optimal augmenting functions for two kinematics with degree of redundancy 1.

A. Example 1

As the first example, we shall study the kinematics of a 3-DOF planar manipulator recently built in the author's robotics laboratory for educational purposes (shown in Fig. 1).

The manipulator has three joint variables (x_1, x_2, x_3) and two task coordinates (y_1, y_2) describing the Cartesian position of the car $W2$ with respect to the inertial coordinate frame fixed to the base. The joint variable x_1 is moved directly by the motor $M1$; similarly, x_3 is driven by $M3$. The position of $W2$ along the runner $P2$ depends on x_3, and also on the revolution angle of the toothed wheel $z2$. This angle is coupled with the revolution angle of the toothed wheel $z1$ through a transmission gear whose gear ratio is adjusted by the joint variable x_2 moved by a motor $M2$ (not shown in the figure). A computation yields the kinematics $y_1 = c_1x_1$ and $y_2 = f_2(x_2)x_1 + c_3x_3$ for constants c_1, c_3, and a homographic function $f_2(x_2) = c_2(r_1 + ax_2)/(r_2 - ax_2)$, where c_2 denotes another constant, $a = (r_2 - r_1)/l$, and r_1, r_2, and l stand for the radii and the side edge of the truncated conical elements of the transmission gear. After a change of coordinates $q_1 = c_1x_1$, $q_2 = c_3x_3$, and $q_3 = (1/c_1)f_2(x_2)$, and restoring the original notations by setting $x = q$, these kinematics may be given the normal form

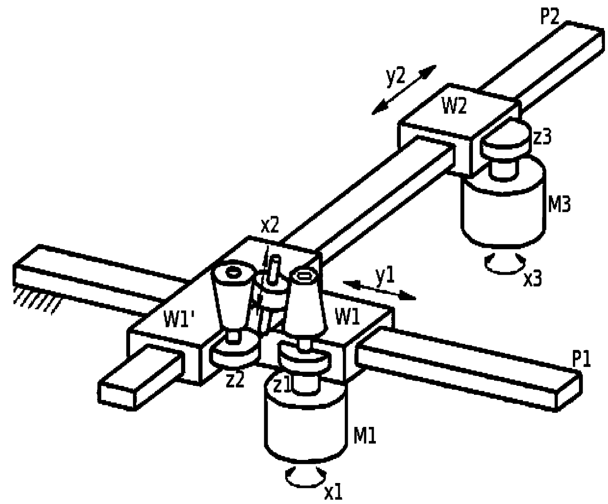

Fig. 1. Manipulator 1.

$$k(x) = (x_1, x_2 + x_1x_3). \tag{13}$$

A simple computation provides the Jacobian

$$J(x) = \begin{bmatrix} 1 & 0 & 0 \\ x_3 & 1 & x_1 \end{bmatrix}$$

and the manipulability matrix

$$M(x) = J(x)J^T(x) = \begin{bmatrix} 1 & x_3 \\ x_3 & 1 + x_1^2 + x_3^2 \end{bmatrix}$$

as well as the manipulability $m(x) = \sqrt{1 + x_1^2}$, and the Jacobian kernel

$$K(x) = \frac{1}{m(x)}(0, -x_1, 1)^T.$$

It is easily seen that the kinematics (13) is regular everywhere. The Riemannian metric $G(x) = P^{-1}(x)$, where

$$P(x) = \begin{bmatrix} 1 & \dfrac{-x_3}{1 + x_1^2} & \dfrac{-x_1x_3}{1 + x_1^2} \\ \dfrac{-x_3}{1 + x_1^2} & \dfrac{1 + x_1^2 + x_3^2 + x_1^4}{(1 + x_1^2)^2} & \dfrac{x_1x_3^2 - x_1^3}{(1 + x_1^2)^2} \\ \dfrac{-x_1x_3}{1 + x_1^2} & \dfrac{x_1x_3^2 - x_1^3}{(1 + x_1^2)^2} & \dfrac{1 + 2x_1^2 + x_1^2x_3^2}{(1 + x_1^2)^2} \end{bmatrix}.$$

Our objective consists of finding an augmenting function $h(x)$ that minimizes the error (9). Because $\mathrm{div}\, K(x) = 0$, the corresponding

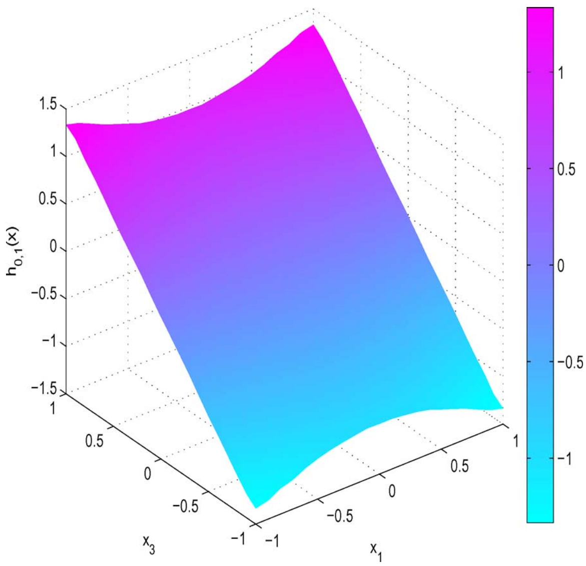

Fig. 2. Optimal augmenting function $h(x)$.

Euler equation (11) takes the form $\Delta h = 0$, and can be written as

$$\begin{aligned}&\sqrt{1+x_1^2}\frac{\partial^2 h(x)}{\partial x_1^2} + \frac{1+x_1^2+x_3^2+x_1^4}{(1+x_1^2)^{3/2}}\frac{\partial^2 h(x)}{\partial x_2^2}\\ &+\frac{1+2x_1^2+x_1^2x_3^2}{(1+x_1^2)^{3/2}}\frac{\partial^2 h(x)}{\partial x_3^2} - \frac{2x_3}{(1+x_1^2)^{1/2}}\frac{\partial^2 h(x)}{\partial x_1 \partial x_2}\\ &-\frac{2x_1x_3}{(1+x_1^2)^{1/2}}\frac{\partial^2 h(x)}{\partial x_1\partial x_3} + \frac{2x_1(x_3^2-x_1^2)}{(1+x_1^2)^{3/2}}\frac{\partial^2 h(x)}{\partial x_2 \partial x_3}\\ &+\frac{3x_1x_3}{(1+x_1^2)^{3/2}}\frac{\partial h(x)}{\partial x_2} + \frac{2x_1^2x_3 - x_3}{(1+x_1^2)^{3/2}}\frac{\partial h(x)}{\partial x_3} = 0. \qquad (14)\end{aligned}$$

We assume the boundary condition $h(x_1, x_2, 0) = 0$, and set $h(x) = x_3 f(x_1)$. Then, the PDE (14) reduces to a second-order linear ordinary differential equation

$$\frac{d^2 f(x_1)}{dx_1^2} - \frac{2x_1}{1+x_1^2}\frac{df(x_1)}{dx_1} + \frac{2x_1^2-1}{(1+x_1^2)^2}f(x_1) = 0. \qquad (15)$$

By inspection, we discover a specific solution $f(x_1) = \sqrt{1+x_1^2}$ of (15). Since this equation is linear, a standard reduction of order procedure [19] provides the general solution $f(x_1) = (ax_1 + b)\sqrt{1+x_1^2}$, for some constants a and b. Thus, we have obtained a family of harmonic optimal augmenting functions $h_{a,b}(x) = x_3(ax_1+b)\sqrt{1+x_1^2}$. Let us choose the constants $a = 0$, $b = 1$. The corresponding augmenting function $h_{0,1}(x) = x_3\sqrt{1+x_1^2}$. The same solution obtained numerically with the help of the MATLAB PDE toolbox is shown in Fig. 2. It has been demonstrated that for the normal-form kinematics (13), the optimal augmenting function can be found analytically. Since the normal form and the original kinematics of the manipulator shown in Fig. 1 are diffeomorphic, the extended Jacobian inverse for (13) will produce an extended Jacobian inverse (although, in general, not optimal) for the original kinematics.

B. Example 2

The second example involves a 3-DOF planar manipulator presented schematically in Fig. 3, whose kinematics

$$k(q) = (q_1 + l_2\cos q_2 + l_3\cos q_3, l_2\sin q_2 + l_3 \sin q_3). \qquad (16)$$

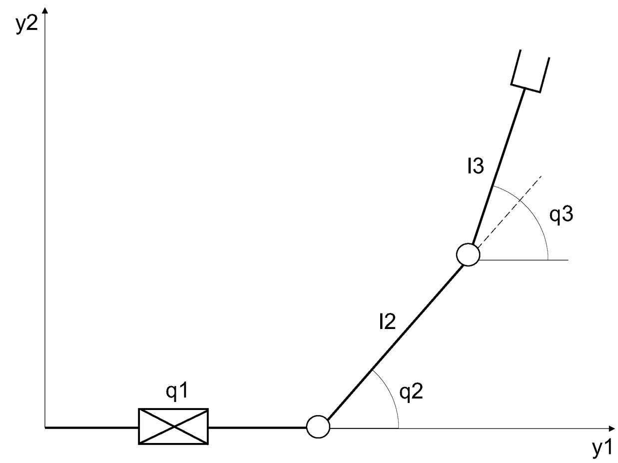

Fig. 3. Schematic of manipulator 2.

The manipulator's Jacobian, the manipulability matrix, the manipulability, and the Jacobian kernel have been computed as follows:

$$\begin{aligned}J(q) &= \begin{bmatrix} 1 & -l_2\sin q_2 & -l_3 \sin q_3\\ 0 & l_2\cos q_2 & l_3\cos q_3\end{bmatrix}\\ M(q) &= \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\\ m(q) &= ||K(q)|| = l_2^2\cos^2 q_2 + l_3^2\cos^2 q_3 + l_2^2 l_3^2 \sin^2(q_2-q_3)\\ K(q) &= \frac{1}{m(q)}(l_2l_3\sin(q_2-q_3), l_3\cos q_3, -l_2\cos q_2)^T\end{aligned}$$

where $M_{11} = 1 + l_2^2\sin^2 q_2 + l_3^2 \sin^2 q_3$, $M_{12} = M_{21} = -l_2^2\sin q_2\cos q_2 - l_3^2 \sin q_3 \cos q_3$, and $M_{22} = l_2^2\cos^2 q_2 + l_3^2\cos^2 q_3$. It is easily checked that the kinematics (16) become singular at $q_2, q_3 = \pm\pi/2$. We are looking for an augmenting function $h(q)$, satisfying the optimality conditions (10). Since, by (21), again $\operatorname{div} K(q) = 0$, we obtain

$$\operatorname{tr}\frac{\partial}{\partial q}\left(R(q)\frac{\partial h(q)}{\partial q}\right) = 0$$

where $R(q) = m(q)P(q)$. The entries of this matrix $R_{ij}(q)$, computed for the unit arm lengths $l_2 = l_3 = 1$ of the manipulator, are the following:

$$\begin{aligned}R_{11}(q) &= \frac{2}{m(q)}(\cos^2 q_2 + \cos^2 q_3)\\ R_{12}(q) &= R_{21}(q) = \frac{1}{m(q)}\cos q_2(\sin q_2\cos q_2 + \sin q_3\cos q_3\\ &\quad - \sin(q_2-q_3)\cos(q_2-q_3))\\ R_{13}(q) &= R_{31}(q) = \frac{1}{m(q)}\cos q_3(\sin q_2 \cos q_2 + \sin q_3\cos q_3\\ &\quad + \sin(q_2-q_3)\cos(q_2-q_3))\\ R_{22}(q) &= \frac{1}{m(q)}(\cos^2 q_2 + \cos^2 q_3 + \sin^2(q_2-q_3)\\ &\quad - 2\cos q_2 \sin q_3 \sin(q_2-q_3))\\ R_{23}(q) &= R_{32}(q) = \frac{1}{m(q)}\sin(q_2-q_3)(\sin q_2\cos q_2\\ &\quad - \sin q_3\cos q_3 - \sin(q_2-q_3)\cos(q_2-q_3))\end{aligned}$$

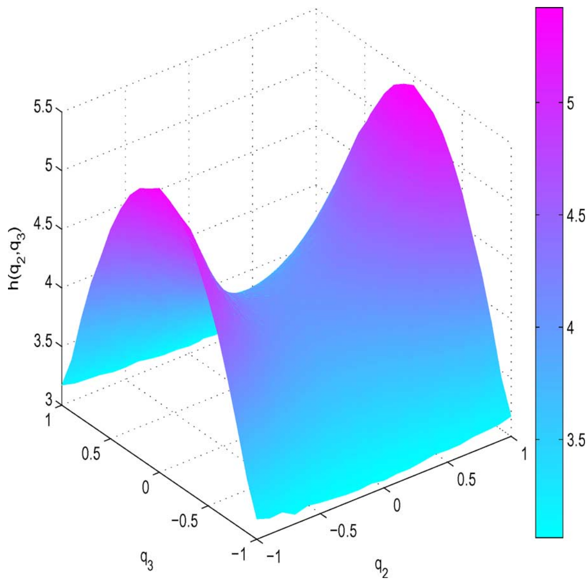

Fig. 4. Optimal augmenting function $h(q_2, q_3)$.

$$R_{33}(q) = \frac{1}{m(q)}(\cos^2 q_2 + \cos^2 q_3 + \sin^2(q_2 - q_3) - 2\sin q_2 \cos q_3 \sin(q_2 - q_3)).$$

It follows that the optimal augmenting function will be harmonic. The computation has been accomplished under an additional assumption that $h(q) = h(q_2, q_3)$. The result provided by the MATLAB PDE toolbox is shown in Fig. 4.

IV. Conclusion

Using some instruments of Riemannian geometry, we have addressed the optimal synthesis problem of extended Jacobian inverse kinematics algorithms for stationary robotic manipulators. The proposed problem formulation leads to explicit optimality conditions involving the Laplace–Beltrami operator acting on the augmenting kinematics map, tantamount with solving a linear, elliptic PDE with functional coefficients. The theoretical concepts have been illustrated with elementary computations. In the general case, solving the optimization problem may encounter two types of computational complications: first, a symbolic generation of the basic PDE (10) or (11) will likely lead to a complicated (so hardly checkable) formula; second, a numerical solution of this equation by means of available software packages may need a prior decomposition of the problem. These computational aspects as well as an application of the proposed approach to inverse kinematics algorithms for mobile manipulators will be a subject of our future research.

Appendix

The definitions of the approximation error introduced in Section II are built on an analogy with some basic concepts of Riemannian geometry. Next, we recall them briefly referring the reader for more details, e.g., to [20]. Let (X, G) and (Y, H) denote a pair of Riemannian manifolds. Assume that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ are local coordinate systems in X and Y. In these coordinates, the Riemannian metrics can be identified with symmetric, positive-definite matrices of suitable size. Choose a pair of vectors v, w tangent to X at the point x. Then, the Riemannian metric G defines the inner product $\langle v, w\rangle_X = v^T G(x) w$, similarly for H. A local diffeomorphism $\varphi : X \longrightarrow Y$ transfers the metric H from Y to X by the pullback

$$(\varphi^* H)(x) = \left(\frac{\partial\varphi(x)}{\partial x}\right)^T H(\varphi(x))\frac{\partial\varphi(x)}{\partial x}. \tag{17}$$

Given the Riemannian metric $G(x)$, the volume form on X is represented by

$$V = \sqrt{\det G(x)}\, dx. \tag{18}$$

Now, let for some points x, y, the map $C(x, y) : T_x X \longrightarrow T_y Y$ be a linear transformation of tangent spaces, represented in coordinates by an $n \times m$ matrix. Then, it can be shown that the dual map $C^*(x, y) : T_y^* \longrightarrow T_x^*$, transforming suitable dual spaces, can be defined as

$$C^*(x, y) = G^{-1}(x) C^T(x, y) H(y). \tag{19}$$

On the Riemannian manifold (X, G), the operator

$$\Delta f = \frac{1}{\sqrt{\det G}} \operatorname{tr} \frac{\partial}{\partial x}\left(\sqrt{\det G} G^{-1}\frac{\partial f}{\partial x}\right) \tag{20}$$

acting on a function f is called the Laplace–Beltrami operator [15], [20], whereas the divergence of a vector field Z on X is understood as

$$\operatorname{div} Z = \frac{1}{\sqrt{\det G}} \operatorname{tr} \frac{\partial}{\partial x}\left(\sqrt{\det G} Z\right). \tag{21}$$

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Differentially Flat Designs of Underactuated Open-Chain Planar Robots

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***Abstract*—A fully actuated system can execute any joint trajectory. However, if the system is underactuated, not all joint trajectories are attainable. For such systems, it is difficult to characterize attainable joint trajectories analytically. Numerical methods are generally used to characterize these. This paper investigates the property of *differential flatness* for underactuated planar open-chain robots and studies dependence on inertia distribution within the system. It is shown that certain choices of inertia distributions make an underactuated open-chain planar robot with revolute joints feedback linearizable, i.e., also differentially flat. Once this property is established, trajectory between any two points in the state space can be planned, and a controller can be developed to correct for errors. To demonstrate the proposed methodology in hardware, experiments with an underactuated 3-DOF planar robot are also presented.**

***Index Terms*—Differential flatness, manipulator, underactuated.**

I. Introduction

Underactuation is encountered in numerous situations in robotics such as during walking [1]. Cost and weight considerations can make underactuation a design choice for space and industrial applications [2], [3]. Some interesting underactuated robots that have been studied include the Acrobot [4], brachiating robots [5], and surgical robots [6]. Another important reason to study underactuation is to restore operation during actuator failure [7].

For an n-DOF open-chain robot described by the coordinates $\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$, where $\mathbf{q} \in \mathcal{R}^n$, the structure of the dynamic equations [8] is given as

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{u}. \tag{1}$$

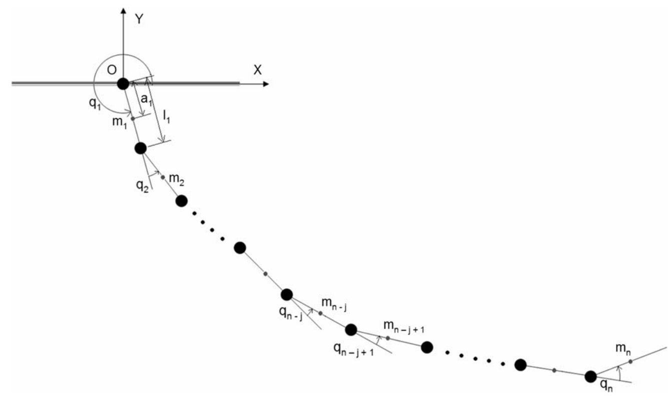

Fig. 1. n-DOF planar robot. The mass and length of the ith link are denoted by m_i and l_i, respectively, and a_i is the displacement from joint i to COM of link i, positive from joint i to $i+1$.

The open-chain robots considered in this paper comprise rigid links connected via revolute joints in a serial fashion, where q_i denotes the angular displacement of body i with respect to the body $i-1$, with q_1 as the absolute orientation of the first body with respect to the ground. In the input vector $\mathbf{u}$, u_i represents the ith joint actuator torque. The underactuated manipulators studied in this paper are those where some of the $u_i = 0$. An actuator at every joint will result in a fully actuated robot. For an underactuated robot, only those joint trajectories are valid that do not require inputs at the joints with missing actuators. This implies that the system might not be able to traverse between two arbitrary states; and hence, may not be controllable.

Arai *et al*. [9] have shown that a 3-DOF planar manipulator with a passive last joint is controllable using a constructive method. A proof of controllability for an n-link manipulator, having only one passive joint, has been presented by Kobayashi *et al*. [10]. De Luca *et al*. [11] have reported another study with single underactuation at the last joint for planar manipulators. Controllability properties of 3-DOF RRR and PPR manipulators, with a single passive joint, have been studied by Mahindrakar *et al*. [12]. Control paradigm for n-link planar manipulators with a passive first joint has been reported by Grizzle *et al*. [13]. To the best of authors knowledge, *the current literature for underactuated planar manipulators is focussed on systems with a single unactuated joint. Highly nonlinear and coupled nature of governing differential equations render planar manipulators with multiple unactuated joints nearly intractable*.

The technical approach adopted in this paper is to investigate the property of *differential flatness* [14], [15] for underactuated planar open-chain robots, and study its dependence on inertia distribution within the system. In general, for an underactuated system with n states and m inputs, any arbitrary set of m output functions do not posses a full relative degree. The paradigm of differential flatness allows the determination of those m outputs, called the flat outputs, that have a full relative degree. Once this property is established, trajectory between any two points in state space can be shown to be consistent with the dynamics. *Input-state feedback linearizable* systems are a subclass of differentially flat systems. In this paper, we show that certain choices of inertia distributions make an underactuated open-chain planar robot with revolute joints static feedback linearizable, i.e., also differentially flat. Hence, point-to-point and cyclic trajectories can be

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