CHAPTER -1

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

1.1 Numerical solution of Non-Linear equations

Introduction

The problem of solving the equation is of great importance in science and Engineering. In this section, we deal with the various methods which give a solution for the equation

Solution of Algebraic and transcendental equations

The equation of the form f(x) =0 are called algebraic equations if f(x) is purely a polynomial in x.For example: are algebraic equations .
If f(x) also contains trigonometric, logarithmic, exponential function etc. then the equation is known as *transcendental equation.*

Methods for solving the equation

The following result helps us to locate the interval in which the roots of

- Method of false position.
- Iteration method
- Newton-Raphson method

Method of False position (Or) Regula-Falsi method (Or) Linear interpolation method

In bisection method the interval is always divided into half. If a function changes sign over an interval, the function value at the mid-point is evaluated. In bisection method the interval from a to b into equal intervals ,no account is taken of the magnitude of .An alternative method that exploits this graphical insights is to join by a straight line.The intersection of this line with the X-axis represents an improved estimate of the root.The replacement of the curve by a straight line gives a “false position” of the root is the origin of the name ,method of false position ,or in Latin ,Regula falsi .It is also called the linear interpolation method.

Problems

1. Find a real root of that lies between 2 and 3 by the method of false position and correct to three decimal places.

Sol:

Let

The root lies between 2.5 and 3.
The approximations are given by

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Iteration(r)	a	b	x_r	$f(x_r)$

1	2.5	3	2.9273	-0.2664
2	2.927	3	2.9423	-0.0088
3	2.942	3	2.9425	-0.0054
4	2.942	3	2.9425	0.003

2. Obtain the real root of ,correct to four decimal places using the method of false position.

Sol:

Given

Taking logarithmic on both sides,

$$x\log_{10} x = 2$$

$$f(x) = x\log_{10} x - 2$$

$$f(3.5) = -0.0958 < 0 \text{ and } f(3.6) = 0.00027 > 0$$

The roots lies between 3.5 and 3.6

The approximation are given by $x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

Iteration(r)	a	b	x_r	$f(x_r)$
1	3.5	3.6	3.5973	0.000015
2	3.5	3.5973	3.5973	0

The required root is **3.5973**

Exercise:

1. Determine the real root of correct to four decimal places by Regula-Falsi method.
***Ans**:* 1.0499

2. Find the positive real root of correct to four decimals by the method of False position .
***Ans**:* 1.8955

3.Solve the equation by Regula-Falsi method,correct to 4 decimal places.
Ans: 2.7984

Newton's method (or) Newton-Raphson method (Or) Method of tangents

This method starts with an initial approximation to the root of an equation, a better and closer approximation to the root can be found by using an iterative process.

Derivation of Newton-Raphson formula

Let α be the root of $f(x)=0$ and x_0 be an approximation to α .If $h=\alpha-x_0$

Then by the Tayor's series

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},\text{i=0,1,2,3,………}$$

Note:

- The error at any stage is proportional to the square of the error in the previous stage.
- The order of convergence of the Newton-Raphson method is at least 2 or the convergence of N.R method is Quadratic.

Problems

1.Using Newton's method ,find the root between 0 and 1 of correct five decimal places.

Sol:

Given

$$f(x)=x^3-6x+4$$
$$f'(x)=3x^2-6$$

By Newton-Raphson formula, we have approximation

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},n=0,1,2,3.....$$

The initial approximation is $x_0=0.5$

First approximation:

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)},$$

$$=0.5-\frac{1.125}{-5.25},$$

=0.71429

Second approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

$$= 0.71429 - \frac{0.0787}{-4.4694},$$

=0.7319

Third approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)},$$

$$= 0.73205 - \frac{0.0006}{-4.3923},$$

=0.73205

Fourth approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)},$$

$$= 0.73205 - \frac{0.000003}{-4.3923},$$

=0.73205

The root is 0.73205, correct to five decimal places.

2. Find a real root of x=1/2+sinx near 1.5, correct to 3 decimal places by newton-Rapson method.

Sol:

Let

$$f(x) = x - \sin x - \frac{1}{2}$$

$$f'(x) = 1 - \cos x$$

By Newton-Raphson formula, we have approximation

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, n = 0,1,2,3.....$$

$$x_{n+1} = \frac{\sin x_n + 0.5 - x_n \cos x_n}{1 - \cos x_n}, n = 0,1,2,3,....$$

Let $x_0 = 1.5$ be the initial approximation

First approximation:

$$x_1 = 1.5 - \frac{(1.5 - \sin(1.5) - 0.5)}{1 - \cos 1.5}$$

$$= 1.5 - \frac{(0.0025)}{(0.9293)}$$
$=1.497$

Second approximation:

$$x_1 = 1.497 - \frac{(-0.00028)}{0.9263}$$

$=1.497.$

The required root is **1.497**

Exercise:

1. Find the real root of $e^x = 3x$,that lies between 1 and 2 by Newton's method ,correct to 4 decimal places.
Ans:1.5121

2.Use Newton-Raphson method to solve the equation $3x - \cos x - 1 = 0$
Ans:0.6071

3.Find the double root of the equation $x^3 - x^2 - x - 1 = 0$ by Newton's method.
Ans: 0.03226

Iteration method (Or) Method of successive approximations(Or)Fixed point method

For solving the equation $f(x) = 0$ by iteration method, we start with an approximation value of the root.The equation $f(x) = 0$ is expressed as $x = \phi(x)$.The equation $x = \phi(x)$ is called fixed point equation.The iteration formula is given by $x_{n+1} = \phi(x_n), n = 0,1,2,.....$ called fixed point iteration formula.

Theorem(Fixed point theorem)
Let $f(x) = 0$ be the given equation whose exact root is α.The equation $f(x) = 0$ be rewritten as $x = \phi(x)$.Let I be the interval containing the root $x = \alpha$. If $|\phi'(x) < 1|$ for all x in I, then the sequence of Approximation $x_1, x_2, x_3, ... x_n$ will converges to α.,If the initial starting value x_0 is chosen in I.

The order of convergence

Theorem

Let α. be a root of the equation $x = g(x)$ .If $g'(\alpha) = 0, g''(\alpha) = 0,....... g^{(p-1)}(\alpha) = 0 and g^{(p)}(\alpha) \neq 0$ .then the convergence of iteration $x_{i+1} = g(x_i)$ is of order p.

Note

The order of convergence in general is linear (i.e) =1

Problems

1.Solve the equation $x^3 + x^2 - 1 = 0$ or the positive root by iteration method ,correct to four decimal places.

Sol:

$$f(x) = x^3 + x^2 - 1$$

$$f(0) = -1 < 0 and f(1) = 1 > 0$$

The root lies between 0 and 1.

$$x^3 + x^2 - 1 = 0 \Rightarrow x^2(1+x) = 1$$

$$x = \frac{1}{\sqrt{1+x}}$$

The given equation can be expressed as $\phi(x) = \frac{1}{\sqrt{1+x}}$

$$\phi'(x) = -\frac{1}{2(1+x)^{3/2}}$$

$$|\phi'(x)| = -\frac{1}{2(1+x)^{3/2}}$$

$$|\phi'(0)| = \frac{1}{2} and |\phi'(1)| = \frac{1}{4\sqrt{2}} < 1$$

$$|\phi'(x)| = -\frac{1}{2(1+x)^{3/2}}$$

$$|\phi'(x)| < 1 \forall x \in (0,1)$$

Choosing $x_0 = 0.75$,the successive approximations are

$$x_1 = \frac{1}{\sqrt{1+0.75}}$$
$$=0.75593$$

$$x_2 = 0.75465$$
$$x_3 = 0.75463$$
$$x_4 = 0.75487$$
$$x_5 = 0.75488$$
$$x_6 = 0.75488$$

Hence the root is **0.7549**

Exercise:

1. Find the cube root of 15, correct to four decimal places, by iteration method

Ans: 2.4662

1.2 System of linear equation

Introduction

Many problems in Engineering and science needs the solution of a system of simultaneous linear equations .The solution of a system of simultaneous linear equations is obtained by the following two types of methods

- Direct methods (Gauss elimination and Gauss Jordan method)
- Indirect methods or iterative methods (Gauss Jacobi and Gauss Seidel method)

(a) Direct methods are those in which
- The computation can be completed in a finite number of steps resulting in the exact solution
- The amount of computation involved can be specified in advance.
- The method is independent of the accuracy desired.

(b) Iterative methods (self correcting methods) are which
- Begin with an approximate solution and
- Obtain an improved solution with each step of iteration
- But would require an infinite number of steps to obtain an exact solution without round-off errors
- The accuracy of the solution depends on the number of iterations performed.

Simultaneous linear equations

The system of equations in n unknowns $x_1, x_2, x_3, ... x_n$ is given by

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

....

....

....

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{mn}x_n = b_n$$

This can be written as $AX = B$ where $A = (a_{ij})_{n \times n}; X = (x_1, x_2, x_3, ...x_n)^T; B = (b_1, b_2, b_3, ...b_n)^T$

This system of equations can be solved by using determinants (Cramer's rule) or by means of matrices. These involve tedious calculations. These are other methods to solve such equations .In this chapter we will discuss four methods viz.

(i) Gauss –Elimination method
(ii) Gauss –Jordan method
(iii) Gauss-Jacobi method
(iv) Gauss seidel method

- **Gauss-Elimination method**

This is an Elimination method and it reduces the given system of equation to an equivalent upper triangular system which can be solved by Back substitution.
Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Gauss-algorithm is explained below:

Step 1. Elimination of x_1 from the second and third equations .If $a_{11} \neq 0,$ the first equation is used to eliminate x_1 from the second and third equation. After elimination, the reduced system is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3$$

Step 2:

Elimination of x_2 from the third equation. If $a'_{22} \neq 0,$ We eliminate x_2 from third equation and the reduced upper triangular system is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a''_{33}x_3 = b''_3$$

Step 3:

From third equation x_3 is known. Using x_3 in the second equation x_2 is obtained. using both x_2 And x_3 in the first equation, the value of x_1 is obtained.

Thus the elimination method, we start with the augmented matrix (A/B) of the given system and transform it to (U/K) by eliminatory row operations. Finally the solution is obtained by back substitution process.

Principle

$$(A/B) \xrightarrow{Gauss-e\lim ination} (U/K)$$

2. Gauss –Jordan method

This method is a modification of Gauss-Elimination method. Here the elimination of unknowns is performed not only in the equations below but also in the equations above. The co-efficient matrix A of the system AX=B is reduced into a diagonal or a unit matrix and the solution is obtained directly without back substitution process.

$$(A/B) \xrightarrow{Gauss-Jordan} (D/K) or (I/K)$$

Examples

1.Solve the equations $2x + y + 4z = 12, 8x - 3y + 2z = 20, 4x + 11y - z = 33$ **by (1)Gauss –Elimination method (2) Gauss –Jordan method**

Sol:

(i)Gauss-Elimination method

The given system is equivalent to $AX = B$ where

and

Principle. Reduce to

=

–

From this ,the equivalent upper triangular system of equations is

$$2x + y + 4z = 12$$

$$-7y - 14z = -28$$
$$-27z = -27$$

z=1,y=2,x=3 By back substitution.

The solution is x=3,y=2,z=1

(ii)Gauss –Jordan method

Principle. Reduce to

From this ,we have

$$y = 2$$
$$z=1$$

Exercise

1. Solve the system by Gauss –Elimination method $5x - y + 2z = 142, x - 3y - z = -30, 2x - y - 3z = 5$.
Ans: x= 39.3345;y=16.793;z=18.966

2. Solve the equations $x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$ by (1) Gauss –Elimination method (2) Gauss –Jordan method
Ans: x=1; y=3; z=5

Iterative method

These methods are used to solve a special of linear equations in which each equation must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation.Further in each equation, the absolute value of the large coefficient of the unknown is greater than the sum of the absolute values of the other coefficients of the other unknowns. Such type of simultaneous linear equations can be solved by the following iterative methods.

- Gauss-Jacobi method
- Gauss seidel method

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

i.e., the co-efficient matrix is diagonally dominant.

Solving the given system for x,y,z (whose diagonals are the largest values),we have

$$x = \frac{1}{a_1}[d_1 - b_1y - c_1z]$$
$$y = \frac{1}{b_2}[d_2 - a_2x - c_2z]$$
$$z = \frac{1}{c_3}[d_3 - a_3x - b_3y]$$

Gauss-Jacobi method

If the r[th] iterates are $x^{(r)}, y^{(r)}, z^{(r)}$,then the iteration scheme for this method is

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1y^{(r)} - c_1z^{(r)})$$
$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2x^{(r)} - c_2z^{(r)})$$
$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3x^{(r)} - b_3y^{(r)})$$

The iteration is stopped when the values x, y, z start repeating with the desired degree of accuracy.

Gauss-Seidel method

This method is only a refinement of Gauss-Jacobi method .In this method ,once a new value for a unknown is found ,it is used immediately for computing the new values of the unknowns.

If the r[th] iterates are, then the iteration scheme for this method is

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1y^{(r)} - c_1z^{(r)})$$
$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2x^{(r+1)} - c_2z^{(r)})$$
$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3x^{(r+1)} - b_3y^{(r+1)})$$

Hence finding the values of the unknowns, we use the latest available values on the R.H.S
The process of iteration is continued until the convergence is obtained with desired accuracy.

Conditions for convergence

Gauss-seidel method will converge if in each equation of the given system ,the absolute values o the largest coefficient is greater than the absolute values of all the remaining coefficients

$$|a_{ii}| > \sum_{j=1, j\neq 1}^{n} |a_{ij}| \forall i = 1,2,3......... n$$

This is the sufficient condition for convergence of both Gauss-Jacobi and Gauss-seidel iteration methods.

Rate of convergence

The rate of convergence of gauss-seidel method is roughly two times that of Gauss-Jacobi method. Further the convergences in Gauss-Seidel method is very fast in gauss-Jacobi .Since the current values of the unknowns are used immediately in each stage of iteration for getting the values of the unknowns.

Problems

1.Solve by Gauss-Jacobi method, the following system
$28x + 4y - z = 32, x + 3y + 10z = 24, 2x + 17y + 4z = 35$
Sol:

Rearranging the given system as

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

The coefficient matrix is is diagonally dominant.

Solving for x,y,z ,we have

—

—

—

We start with initial values =
The successive iteration values are tabulated as follows

Iteration	X	Y	Z
1	1.143	2.059	2.400
2	0.934	1.360	1.566
3	1.004	1.580	1.887
4	0.983	1.508	1.826
5	0.993	1.513	1.849
6	0.993	1.507	1.847
7	0.993	1.507	1.847

X=0.993;y=1.507;z=1.847

2.Solve by Gauss-Seidel method, the following system

$27x+6y-z=85; 6x+15y+2z=72, x+y+54z=110$

Sol:

The given system is diagonally dominant.

Solving for x,y,z, we get

We start with initial values =

The successive iteration values are tabulated as follows

Iteration	X	Y	Z
1	3.148	3.541	1.913
2	2.432	3.572	1.926
3	2.426	3.573	1.926
4	2.425	3.573	1.926
5	2.425	3.573	1.926

X=2.425;y=3.573;z=1.926

Exercise:

1.Solve by Gauss-Seidel method, the following system

$10x+2y+z=9; 2x+30y-2z=-44, -2x+3y+10z=22$

$Ans: x=0.89; y=-1.341; z=2.780$

2.Solve by Gauss-Seidel method and Gauss-Jacobi method, the following system

$30x-2y+3z=75;2x+2y+18z=30, x+17y-2z=48$

$Ans:(i)x=1.321; y=1.522; z=3.541(ii)x=2.580; y=2.798; z=1.069$

1.3 Matrix inversion

Introduction

A square matrix whose determinant value is not zero is called a non-singular matrix. Every non-singular square matrix has an inverse matrix. In this chapter we shall find the inverse of the non-singular square matrix A of order three. If X is the inverse of A,
Then

Inversion by Gauss-Jordan method

By Gauss Jordan method, the inverse matrix X is obtained by the following steps:

- Step 1: First consider the augmented matrix
- Step 2: Reduce the matrix A in to the identity matrix I by employing row transformations.

The row transformations used in step 2 transform I to A^{-1}
Finally write the inverse matrix A^{-1}.so the principle involved for finding A^{-1} is as shown below

Note

The answer can be checked using the result

Problems

1.Find the inverse of by Gauss-Jordan method.

Sol:

Thus

Hence

2.Find the inverse of by Gauss-Jordan method

Exercise:

1. Find the inverse of by Gauss-Jordan method.

2.By Gauss-Jordan method ,find A^{-1} if

1.4 Eigen value of a Matrix

Introduction

For every square matrix A, there is a scalar λ and a non-zero column vector X such that $AX = \lambda X$ .Then the scalar λ is called an Eigen value of A and X, the corresponding Eigen vector. We have studied earlier the computation of Eigen values and the Eigen vectors by means of analytical method. In this chapter, we will discuss an iterative method to determine the largest Eigen value and the corresponding Eigen vector.

Power method (Von Mise's power method)

Power method is used to determine numerically largest Eigen value and the corresponding Eigen vector of a matrix A

Working Procedure

(For a square matrix)

- Assume the initial vector
- Then find
- Normalize the vector to get a new vector

(i.e) —— is the largest component in magnitude of

- Repeat steps (2) and (3) till convergence is achieved.
- The convergence of m_i and X_i will give the dominant Eigen value λ and the corresponding Eigen vector X Thus $\lambda_1 = \lim \frac{[X_{k+1}]_i}{[X_k]_i}, i = 0,1,2\ldots\ldots n$

And is the required Eigen vector.

Problems

1.Use the power method to find the dominant value and the corresponding Eigen vector of the matrix

Sol:

Let

$\lambda_1 \; X_1$

$\lambda_2 \; X_2$

$\lambda_3 \; X_3$

$\lambda_4 \; X_4$

$\lambda_5 \; X_5$

Hence the dominant Eigen value is 15.97 and the corresponding Eigen vector is

2. Determine the dominant Eigen value and the corresponding Eigen vector of

Using power method.

Exercise:

1. Find the dominant Eigen value and the corresponding Eigen vector of A Find also the other two Eigen values.

2. Find the dominant Eigen value of the corresponding Eigen vector of

By power method. Hence find the other Eigen value also.

Tutorial problems

Tutorial 1

1. Determine the real root of $xe^x = 3$ correct tot four decimal places by Regula falsi method.
Ans:1.0499
2. Solve the equation xtanx=-1 by Regula falsi method ,correct to 4 decimal places
Ans:2.7984
3. Find by Newton's method , the real root of $x\log_{10} x = 1.2$ correct to 4 decimal places
Ans:2.7406
4. Find the double root of the equation $x^3 - x^2 - x + 1 = 0$ by Newton's method.
Ans:x=1
5. Solve $x = 1 + \tan^{-1} x$ by iteration method ,starting with $x_0 = 2$
Ans:2.1323

Tutorial 2

1.Solve the following system by Gauss-elimination method

$5x_1+x_2+x_3+x_4=4; x_1+7x_2+x_3+x_4=12; x_1+x_2+6x_3+x_4=-5; x_1+x_2+x_3+4x_4=-6$

$Ans: x_1=1, x_2=2, x_3=-1, x_4=-2$

2.Solve the system of equation $x+2y+z=8, 2x+3y+4z=20, 4x+y+2z=12$ by Gauss-Jordan method

Ans:x=1,y=2,z=3.

3.Solve the following equations $10x+2y+z=9, 2x+30y-2z=-44, -2x+3y+10z=22$

By Gauss-Seidel method.

4.Solve by Gauss Jacobi method ,the following

$28x+4y-z=32, x+3y-10z=24, 2x+y+4z=35$

Ans:x=0.993,y=1.07,z=1.847.

5.Solve the following equations

$13x_1+5x_2-3x_3+x_4=18, 2x_1+12x_2-3x_3+4x_4=30, 3x_1-4x_2+10x_3+x_4=29, 2x_1+x_2-3x_3+9x_4=31$

By Gauss-Seidel method.

Ans: $x_1=0.58, x_2=3.482, x_3=3.703, x_4=4.163$

Tutorial 3

1.Find the inverse of using Gauss-Jordan method.

Ans:

2.By Gauss-Jordan method ,find A^{-1} if

Ans:

3.Determine the dominant Eigen value and the corresponding Eigen vector of

using power method.

Ans:$[0.062,1,0.062]^T$

4.Find the smallest Eigen value and the corresponding Eigen vector of

using power method.

Ans:$[0.707,1,0.707]^T$

5.Find the dominant Eigen value and the corresponding Eigen vector of by power method.Hence find the other Eigen value also .

Ans:λ=2.381

QUESTION BANK

PART A

1. What is the order of convergence of Newton-Raphson methods if the multiplicity of the root is one.

Sol:

Order of convergence of N.R method is 2

2. Derive Newton's algorithm for finding the p^{th} root of a number N.

Sol:

If $x = N^{1/p}$,
Then $x^p - N = 0$ is the equation to be solved.
Let $f(x) = x^p - N$, $f'(x) = px^{p-1}$
By N.R rule, if x_r is the r^{th} iterate

$X_{r+1} = x_r -$ ———

$= x_r -$ ———

$=$ ————

$=$ ————

3. What is the rate of convergence in N.R method?

Sol:

The rate of convergence in N.R method is of order 2

4. Define round off error.

Sol:

The round off error is the quantity R which must be added to the finite representation of a computed number in order to make it the true representation of that number.

5. State the principle used in Gauss-Jordan method.

Sol:

Coefficient matrix is transformed into diagonal matrix.

6. Compare Gaussian elimination method and Gauss- Jordan method.

Sol:

	Gaussian elimination method	**Gauss- Jordan method**
1	Coefficient matrix is transformed into upper triangular matrix	Coefficient matrix is transformed into diagonal matrix
2	Direct method	Direct method
3	We obtain the solution by back substitution method	No need of back substitution method

7. Determine the largest eigen value and the corresponding eigen value vector of the matrix correct to two decimal places using power method.

Sol:

$AX_1 =$ $=$ $= 2$ $= 2X_2$

$AX_2 =$ $=$ $= 2$ $= 2X_3$

This shows that the largest eigen value = 2

The corresponding eigen value =

8. Write the Descartes rule of signs

Sol:

1) An equation $f(x) = 0$ cannot have more number of positive roots than there are changes of sign in the terms of the polynomial $f(x)$.

2)An equation $f(x) = 0$ cannot have more number of positive roots than there are changes of sign in the terms of the polynomial $f(x)$.

9. Write a sufficient condition for Gauss seidel method to converge .(or)
State a sufficient condition for Gauss Jacobi method to converge.

Sol:

The process of iteration by Gauss seidel method will converge if in each equation of the system the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

10. State the order of convergence and convergence condition for NR method?

Sol:

The order of convergence is 2

Condition of convergence is

11. Compare Gauss Seidel and Gauss elimination method?

Sol:

	Gauss Jacobi method	**Gauss seidel method**
1.	Convergence method is slow	The rate of convergence of Gauss Seidel method is roughly twice that of Gauss Jacobi.
2.	Direct method	Indirect method
3.	Condition for convergence is the coefficient matrix diagonally dominant	Condition for convergence is the coefficient matrix diagonally dominant

12) Is the iteration method a self correcting method always?

Sol:

In general iteration is a self correcting method since the round off error is smaller.

13) If g(x) is continuous in [a , b] then under what condition the iterative method x = g(x) has a unique solution in [a , b].

Sol:

Let x = r be a root of x = g(x) .Let I = [a , b] be the given interval combining the point x = r. if $g'(x)$ for all x in I, the sequence of approximation x_0 , x_1,......x_n will converge to the root r, provided that the initial approximation x_0 is chosen in r.

14) When would we not use N-R method .

Sol:

If x_1 is the exact root and x_0 is its approximate value of the equation $f(x) = 0$.we know that $x_1 = x_0 -$

If is small, the error will be large and the computation of the root by this, method will be a slow process or may even be impossible.

Hence the method should not be used in cases where the graph of the function when it crosses the x axis is nearly horizontal.

15) Write the iterative formula of NR method.

Sol:

$X_{n+1} = x_n -$

Part B

1.Using Gauss Jordan method, find the inverse of the matrix

2.Apply Gauss-seidel method to solve the equations
20x+y-2z=17: 3x+20y-z=-18: 2x-3y+20z=25.

3.Find a positive root of 3x- $\log_{10}$ X =6, using fixed point iteration method.

4.Determine the largest eigen value and the corresponding eigen vector of the matrix

with $(1\ 0\ 0)^T$ as the initial vector by power method.

5.Find the smallest positive root of the equation x =− sin x correct to 3 decimal places using Newton-Raphson method.

6.Find all the eigen value and eigen vectors of the matrix using Jacobi method.

7.Solve by Gauss-seidel iterative procedure the system 8x-3y-2z=20: 6x+3y+12z=35: 4x+11y-z=33.

8.Find the largest eigen value of by using Power method.

9.Find a real root of the equation $x^3+x^2-1=0$ by iteration method.

10.Using Newton's method, find the real root of $x \log_{10} X=1.2$ correct to five decimal places.

11.Apply Gauss elimination method to find the solution of the following system :
$2x+3y-z=5$: $4x+4y-3z=3$: $2x-3y+2z=2$.

12.Find an iterative formula to find —, where N is a positive number and hence find –

13.Solve the following system of equations by Gauss-Jacobi method:
$27x+6y-z=85$:$x+y+54z=110$: $6x+15y+2z=35$.

14.Find the Newton's iterative formula to calculate the reciprocal of N and hence find the value of — .

15.Apply Gauss-Jordan method to find the solution of the following system:
$10x+y+z=12$: $2x+10y+z=13$: $x+y+5z=7$.

CHAPTER 2

INTERPOLATION AND APPROXIMATION

2.1 Interpolation with Unequal intervals

Introduction

Interpolation is a process of estimating the value of a function at ana intermediate point when its value are known only at certain specified points.It is based on the following assumptions:

(i) Given equation is a polynomial or it can be represented by a polynomial with a good degree of approximation.

(ii) Function should vary in such a way that either it its increasing or decreasing in the given range without sudden jumps or falls of functional values in the given interval.

We shall discuss the concept of interpolation from a set of tabulated values of when the values of x are given intervals or at unequal intervals .First we consider interpolation with unequal intervals.

Lagrange's Interpolation formula

This is called the Lagrange's formula for Interpolation.

Problems

1. **Using Lagrange's interpolation formula, find the value of y corresponding to x=10 from the following data**

X	**5**	**6**	**9**	**11**
Y	**12**	**13**	**14**	**16**

Sol:

Given

By Lagrange's interpolation formula

Using x=10 and the given data,

y (10) =14.67

2. Using Lagrange's interpolation formula, find the value of y corresponding to x=6 from the following data

X	3	7	9	10
Y	**168**	**120**	**72**	**63**

Sol:

Given

By Lagrange's interpolation formula

Using x=6 and the given data, y(6)=147

3.Apply Lagrange's formula to find f(5),given that f(1)=2,f(2)=4,f(3)=8 and f(7)=128

Sol:

Given

By Lagrange's interpolation formula

Using x=5 and the given data, y(5)=32.93

Exercise

1.Find the polynomial degree 3 fitting the following data

X	**-1**	**0**	**2**	**3**
Y	**-2**	**-1**	**1**	**4**

Ans: –

2.Given

Ans:

Inverse Interpolation by Lagrange's interpolating polynomial

Lagrange's interpolation formula can be used find a value of x corresponding to a given y which is not in the table. The process of finding such of x is called inverse interpolation.

If x is the dependent variable and y is the independent variable, we can write a formula for x as a function of y.

The Lagrange's interpolation formula for inverse interpolation is

Problems

1.Apply Lagrange's formula inversely to obtain the root of the equation

Given that

Sol:

Given that

To find x such that

Applying Lagrange's interpolation formula inversely ,we get

=0.8225

2.Given data

x	**3**	**5**	**7**	**9**	**11**
y	**6**	**24**	**58**	**108**	**174**

Find the value of x corresponding to y=100.

Sol:

Given that

Using the given data and y=100, we get

$\boxed{X=8.656}$

The value of x corresponding to y=100 is 8.656

2.2 Divided Difference –Newton Divided Difference Interpolation Formula

Introduction

If the values of x are given at unequal intervals, it is convenient to introduce the idea of divided differences. The divided difference are the differences of y=f(x) defined, taking into consideration the changes in the values of the argument .Using divided differences of the function y=f(x),we establish Newton's divided difference interpolation formula, which is used for interpolation which the values of x are at unequal intervals and also for fitting an approximate curve for the given data.

Divided difference

Let the function assume the values

Corresponding to the arguments $x_1, x_2, x_3, \ldots\ldots x_n$ respectively, where the intervals need to be equal.

Definitions

The first divided difference of f(x) for the arguments is defined by

It is also denoted by [ ]

Similarly for arguments and so on.

The second divided differences of f(x) for three arguments x_0, x_1, x_2

Is defined as

The third divided difference of f(x) for the four arguments x_0, x_1, x_2, x_3

Is defined as

And so on .

Representation by Divided difference table

Arguments	Entry	First Divided difference	Second D-D	Third D-D

Properties of divided difference

- The divided differences are symmetrical in all their arguments.
- The operator is linear.
- The n^{th} divided differences of a polynomial of n^{th} degree are constants.

Problems

1.If – find the divided difference –

Sol:

$-$

$-$

$= \frac{- \; -}{\quad}$

=

=

2.Find the third differences with arguments 2,4,9,10 for the function

Newton's Divided difference Formula (Or) Newton's Interpolation Formula for unequal intervals

Problems

1.Find the polynomial equation passing through(-1,3),(0,-6),(3,39),(6,822),(7,1611)
Sol:
Given data

x	-1	0	3	6	7
y	3	-6	39	822	1611

The divided difference table is given as follows

-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

By Newton's formula,

Is the required polynomial.

2. Given the data

x	0	1	2	5
f(x)	2	3	12	147

Find the form of the function .Hence find f(3).

Ans: f(3)=35

2.3 Interpolating with a cubic spline-cubic spline Interpolation

We consider the problem of interpolation between given data points (x_i, y_i), i=0, 1, 2, 3….n where a= $x_1 < x_2 < x_3 < ..... < x_n = b$ by means of a smooth polynomial curve. By means of method of least squares, we can fit a polynomial but it is appropriate."Spline fitting" is the new technique recently developed to fit a smooth curve passing he given set of points.

Definition of Cubic spline

A cubic spline s(x) is defined by the following properties.

- $S(x_i)=y_i$,i=0,1,2,…..n
- S(x),s'(x),s"(x) are continuous on [a,b]
- S(x) is a cubic polynomial in each sub-interval(x_i,x_{i+1}),i=0,1,2,3…..n=1

Conditions for fitting spline fit

The conditions for a cubic spline fit are that we pass a set of cubic through the points, using a new cubic in each interval. Further it is required both the slope and the curvature be the same for the pair of cubic that join at each point.

Natural cubic spline

A cubic spline s(x) such that s(x) is linear in the intervals $(-\infty, x_1) and (x_n, \infty)$ i.e. s_1=0 and s_n=0 is called a natural cubic spline

where s_1 =second derivative at (x_1, y_1)

s_n = Second derivative at (x_n, y_n)

Note

The three alternative choices used are

- $s_1 = 0$ and $s_n = 0$ i.e. the end cubics approach linearity at their extremities.
- $s_1 = s_2; s_n = s_{n-1}$ i.e. the end cubics approaches parabolas at their extremities
- Take s_1 as a linear extrapolation from s_2 and $s_3 ands_n$ is a linear extrapolation from $s_{n-1} ands_{n-2}$ .with the assumption, for a set of data that are fit by a single cubic equation their cubic splines will all be this same cubic
- For equal intervals ,we have $h_{i-1} = h_i = h$,the equation becomes

$$s_{i-1} + 4s_i + s_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \text{ for i=1,2,3,....n-1}$$

Problems

1.Fit a natural cubic spline to the following data

x	**1**	**2**	**3**
y	**-8**	**-1**	**18**

And compute (i) y(1.5) (ii) y'(1)

Solution

Here n=2 ,the given data $(x_1, y_1) = (1,-8), (x_2, y_2) = (2,-1), (x_3, y_3) = (3,18)$

For cubic natural spline, $s_1 = 0$ & $s_3 = 0$.The intervals are equally spaced.

For equally spaced intervals,the relation on s_1, s_2 & s_3 is given by

$$s_1 + 4s_2 + s_3 = \frac{6}{1^2}[y_1 - 2y_2 + y_3] \qquad [\because h = 1]$$

$$\therefore s_2 = 18$$

For the interval $1 \le x \le 2[x_1 \le x \le 2]$,the cubic spline is given by

$$y = a_1(x - x_1)^3 + b_1(x - x_1)^2 + c_1(x - x_1) + d_1$$

The values of a_1, b_1, c_1, d_1 are given by

$$a_1 = \frac{s_2 - s_1}{6}, b_1 = \frac{s_1}{2}, c_1 = (y_2 - y_1) - (\frac{2s_1 + s_2}{6}), d_1 = y_1 \qquad [\because h_i = 1]$$

$$a_1 = 3, b_1 = 0, c_1 = 4, d_1 = -8$$

The cubic spline for the interval $1 \le x \le 2$ is

$$s(x) = 3(x-1)^3 + 4(x-14) - 8$$

$$\therefore\ y(1.5) \approx s(1.5) = -\frac{45}{8}$$

The first derivative of s(x) is given by

$$s'(x_i) = -\frac{h_1}{6}(2s_i + s_{i+1}) + \frac{1}{h_i}(y_{i+1} - y_i)$$

Taking i=1, $s'(1) = -\frac{1}{6}(2s_1 + s_2) + (y_2 - y_1)$ $[\because h_i = 1]$

$$\therefore\ s'(1) = 4$$

$$\therefore\ y'(1) = 4$$

We note that the tabulated function is $y = x^3 - 9$ and hence the actual values of $y(1.5)$ and $y'(1)$

are respectively $-\frac{45}{8}$ and 3.

2.The following values of x and y are given ,obtain the natural cubic spline which agree with y(x) at the set of data points

x	**2**	**3**	**4**
y	**11**	**49**	**123**

Hence compute (i) y(2.5) and (ii) $y'(2)$

Exercise:

1.Fit the following data by a cubic spline curve

x	0	1	2	3	4
y	-8	-7	0	19	56

Using the end condition that s_1 & s_5 are linear extrapolations.

2.Fit a natural cubic spline to $f(x)=\frac{20}{1+5x^2}$ on the interval [-2,-1].Use five equispaced points of the function at x=-2(1)2.hence find y(1.5).

2.4 Interpolation with equal intervals

(Newton's Forward and Backward Difference formulas)

Introduction

If a function y=f(x) is not known explicitly the value of y can be obtained when s set of values of (x_i, y_i) i=1,2,3….n are known by using methods based on the principles of finite differences ,provided the function y=f(x) is continuous.
Assume that we have a table of values (x_i, y_i) i=1,2,3….n of any function, the values of x being equally spaced, $i.e., x_i = x_0 + ih, i = 0,1,2,....n$

Forward differences

If $y_0, y_1, y_2, ... y_n$ denote a set of values of y,then the first forward differences of y=f(x) are defined by $\Delta y_0 = y_1 - y_0; \Delta y_1 = y_2 - y_1; .... \Delta y_{n-1} = y_n - y_{n-1}$
Where Δ is called the *forward difference operator*.

Backward differences

The differences $y_1 - y_0, y_2 - y_1, ... y_n - y_{n-1}$ are called first backward differences and they are denoted by $\nabla y_n = 2.9\ \nabla y_1 = y_1 - y_0; \nabla y_2 = y_2 - y_1; .... \nabla y_n = y_n - y_{n-1}$
Where ∇ is called the backward difference operator.In similar way second, third and higher order backward differences are defined.
$\nabla^2 y_2 = \nabla y_2 - \nabla y_1; \nabla^2 y_3 = \nabla y_3 - \nabla y_2; .... \nabla^2 y_4 = \nabla y_4 - \nabla y_3 ....$
$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2; \nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$ and so on.

Fundamental Finite Difference operators

Forward Difference operator Δ ($\Delta y_0 = y_1 - y_0$)
If =f(x) then $\Delta y = f(x+h) - f(x)$
Where h is the interval of differencing

Backward Difference operator ∇ ($\nabla y_1 = y_1 - y_0$)
The operator ∇ is defined by $\nabla f(x) = f(x) - f(x-h)$

Shift operator

The shift operator is defined by $Ey_r = y_{r+1}$
i.e the effect of E is to shift functional value y_r to the next value y_{r+1} .Also $E^2 y_r = y_{r+2}$

In general $E^n y_r = y_{n+r}$

The relation between Δ and E is given by

$$\Delta = E - 1 (or) E \equiv 1 + \Delta$$

Also the relation between ∇ and E^{-1} is given by

$$\nabla \equiv 1 - E^{-1}$$

Newton's Forward Interpolation formula

Let y=f(x) be a function which takes the values $y_0, y_1, y_2, ... y_n$ corresponding to the values $x_0, x_1, x_2, ... x_n$ where the values of x are equally spaced.

i.e. $x_i = x_0 + ih, i = 0,1,2,......., n$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}. + ...$$

$$p = \frac{x - x_0}{h}$$

This formula is called Newton-Gregory forward interpolation formula.

Newton's Backward Interpolation formula

This formula is used for interpolating a value of y for given x near the end of a table of values .Let $y_0, y_1, y_2, ... y_n$ be the values of y =f(x) for $x = x_0, x_1, x_2, ... x_n$ where $x_i = x_0 + ih, i = 0,1,2,......., n$

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

$$p = \frac{x - x_n}{h}$$

This formula is called Newton Backward interpolation formula.

We can also use this formula to extrapolate the values of y, a short distance ahead of y_n.

Problems

1.Using Newton's Forward interpolation formula ,find f(1.02) from the following data

X	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**
F(x)	**0.841**	**0.891**	**0.932**	**0.964**	**0.985**

Sol:

Forward difference table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	0.841			
		0.050		
1.1	0.891		-0.009	
		0.041		0
1.2	0.932		-0.009	
		0.032		-0.002
1.3	0.964		-0.011	
		0.021		
1.4	0.985			

$y_0 = 0.841, \Delta y_0 = 0.050, \Delta^2 y_0 = -0.009$

Let $x_0 = 1.0$ & $x = 1.02$

$$p = \frac{x - x_0}{h}$$

$$\Rightarrow p = 0.2$$

By Newton's Forward interpolation formula,

$$y(1.02) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

=0.852

$y(1.02)$ =0.852

2.Using Newton's Forward interpolation formula, find the value of sin52 given that $\sin 45^0 = 0.7071, \sin 50^0 = 0.7660, \sin 55^0 = 0.8192, \sin 60^0 = 0.8660$. **Ans:0.7880**

3.Using Newton's Backward interpolation formula, find y when x=27,from the following data

x	**10**	**15**	**20**	**25**	**30**
y	**35.4**	**32.2**	**29.1**	**26.0**	**23.1**

Sol:

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	35.4				
		-3.2			
15	32.2		0.1		
		-3.1		-0.1	
20	29.1		0		0.3
		-3.1		0.2	
25	26.0		0.2		
		-2.9			
30	23.1				

Here $x_n = 30, y_n = 23.1, \nabla y_n = 2.9\ \nabla^2 y_n = 0.2, \nabla^3 y_n = 0.2, \nabla^4 y_n = 0.3$

By Newton's backward interpolation formula,

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots$$

Here x=27, $p = \frac{x - x_n}{h}$

$\Rightarrow p = -0.6$

y(27)=24.8

4.Find the cubic polynomial which takes the value y(0)=1,y(1)=0,y(2)=1,y(3)=10.Hence or otherwise ,obtain y(4).Ans:y(4)=33

Exercise:

1.Given the data

x	**0**	**1**	**2**	**3**	**4**
y	**2**	**3**	**12**	**35**	**78**

Find the cubic function of x,using Newton's backward interpolation formula.

Ans: $y(x) = x^3 + x^2 - x + 2$

2.Using Newton's Gregory backward formula, find $e^{-1.9}$ from the following data

x	**1.00**	**1.25**	**1.50**	**1.75**	**2.00**
e^{-x}	**0.3679**	**0.2865**	**0.2231**	**0.1738**	**0.1353**

Ans: $e^{-1.9}$ **=0.1496**

3.Estimate exp(1.85) from the following table using Newton's Forward interpolation formula

x	**1.7**	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**
e^{-x}	**5.474**	**6.050**	**6.686**	**7.839**	**8.166**	**9.025**	**9.974**

Ans:6.3601

4.Find the polynomial which passes through the points (7,3)(8,1)(9,1)(10,9)using Newton's interpolation formula.

Tutorial Problems

Tutorial 1

1.Apply Lagrange's formula inversely to obtain the root of the equation f(x)=0,given that f(0)=-4,f(1)=1,f(3)=29,f(4)=52.
Ans: 0.8225

2.Using Lagrange's formula of interpolation ,find y(9.5) given the data

X	7	8	9	10
Y	3	1	1	9

Ans: 3.625

3.Use Lagrange's formula to find the value of y at x=6 from the following data

x	3	7	9	10
y	168	120	72	63

Ans: 147

4.If log(300)=2.4771,log(304)=2.4829,log(305)=2.4843,log(307)=2.4871find log(301)
Ans: 2.8746

5. Given $u_0 = 6, u_1 = 9, u_3 = 33, u_7 = -15 .. Find\ u_2$
***Ans:*9**

Tutorial 2

1.Given the data

x	0	1	2	5
F(x)	2	3	12	147

Find the form of the function. Hence find f(3).Ans:35

2.Find f(x) as a polynomial in powers of(x-5) given the following table

X	0	2	3	4	7	9
Y	4	26	58	112	466	922

Ans: $f(x) = (x-5)^3 + 17(x-5)^2 + 98(x-5) + 194$

3.from the following table,find f(5)

X	0	1	3	6
F(x)	1	4	88	1309

Ans:676

4.Fit the following data by a cubic spline curve

X	0	1	2	3	4
Y	-8	-7	0	19	56

Using the end condition that s_1 and s_5 are linear extrapolations.
Ans:-8

5.Given the data

X	1	2	3	4
Y	0.5	0.333	0.25	0.20

Find y(2.5) using cubic spline function.
Ans:0.2829

Tutorial 3

1.from the following table,find the number of students who obtained less than 45 marks

Marks	30-40	40-50	50-60	60-70	70-80
No of students	31	42	51	35	31

Ans:48

2.Find tan(0.26) from the following valuesof tanx for $0.10 \le x \le 0.30$

X	0.10	0.15	0.20	0.25	0.30
Tanx	0.1003	0.1511	0.2027	0.2553	03093

Ans:0.2662

3.Using newton's interpolation formula find (i) y when x=48 (ii) y when x=84 from the following data

X	40	50	60	70	80	90
Y	184	204	226	250	276	304

Ans:199.84,286.96

4.Find the value of f(22) and f(42) from the following data

X	20	25	30	35	40	45
F(x)	354	332	291	260	231	204

Ans:352,219

5.Estimate $\sin 38^0$ from the following data given below

x	0	10	20	30	40
sinx	0	0.1736	0.3240	0.5000	0.6428

Ans:12.7696

Question Bank

Part A

1. State the Lagrange's interpolation formula.

***Sol*:**

Let y = f(x) be a function which takes the values $y_0, y_1, \ldots\ldots y_n$ corresponding to $x=x_0, x_1, \ldots\ldots x_n$

Then Lagrange's interpolation formula is

Y = f(x) = ——— ——— ——— y_o

+——— ——— ——— y_1

+ ………+——— ——— ——— y_n

2. What is the assumption we make when Lagrange's formula is used?

Sol:

Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not whether the difference of y become smaller or not.

3. When Newton's backward interpolation formula is used.

Sol:

The formula is used mainly to interpolate the values of y near the end of a set of tabular values and also for extrapolation the values of y a short distance ahead of y_0

4. What are the errors in Trapezoidal rule of numerical integration?

Sol:

The error in the Trapezoidal rule is

E<———y''

5. Why Simpson's one third rule is called a closed formula?

Sol:

Since the end point ordinates y_0 and y_n are included in the Simpson's 1/3 rule, it is called closed formula.

6. What are the advantages of Lagrange's formula over Newton's formula?

Sol:

The forward and backward interpolation formulae of Newton can be used only when the values of theindependent variable x are equally spaced and can also be used when the differences of the dependent variable y become smaller ultimately. But Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not and whether the difference of y become smaller or not.

7. When do we apply Lagrange's interpolation?

Sol:

Lagrange's interpolation formula can be used when the values of "x" are equally spaced or not. It is mainly used when the values are unevenly spaced.

8. When do we apply Lagrange's interpolation?

Sol:

Lagrange's interpolation formula can be used when the values of "x" are equally spaced or not. It is mainly used when the values are unevenly spaced.

9. What are the disadvantages in practice in applying Lagrange's interpolation formula?

Sol:

1. It takes time.
2. It is laborious

10. When Newton's backward interpolation formula is used.

Sol:

The formula is used mainly to interpolate the values of 'y' near the end of a set of tabular values.

11. When Newton's forward interpolation formula is used.

Sol:

The formula is used mainly to interpolate the values of 'y' near the beginnig of a set of tabular values.

12. When do we use Newton's divided differences formula?

Sol: This is used when the data are unequally spaced.

13. Write Forward difference operator.

Sol:

Let $y = f(x)$ be a function of x and let of the values of y. corresponding to of the values of x. Here, the independent variable (or argument), x proceeds at equally spaced intervals and h (constant),the difference between two consecutive values of x is called the interval of differencing. Now the forward difference operator is defined as

=

=

....

=

These are called first differences.

14.Write Backward difference operator.

Sol:

The backward difference operator is defined as

=

For n=0,1,2 …

=

=

=

………………….

These are called first differences

Part B

1.Using Newton's divided difference formula, find f(x) from the following data and hence find f(4).

x	0	1	2	5
f(x)	2	3	12	147

.

2.Find the cubic polynomial which takes the following values:

x	0	1	2	3
f(x)	1	2	1	10

3.The following values of x and y are given:

x	1	2	3	4
f(x)	1	2	5	11

Find the cubic splines and evaluate y(1.5) and y'(3).

4.Find the rate of growth of the population in 1941 and 1971 from the table below.

Year X	1931	1941	1951	1961	1971
Population Y	40.62	60.8	79.95	103.56	132.65

5.Derive Newton's backward difference formula by using operator method.

6.Using Lagrange's interpolation formula find a polynomial which passes the points (0,-12),(1,0),(3,6),(4,12).

7.Using Newton's divided difference formula determine f(3) from the data:

x	0	1	2	4	5
f(x)	1	14	15	5	6

8.Obtain the cubic spline approximation for the function y=f(x) from the following data, given that $y_0'' = y_3''=0$.

x	-1	0	1	2
y	-1	1	3	35

9.The following table gives the values of density of saturated water for various temperatures of saturated steam.

Temperature ^{0}C	100	150	200	250	300
Density hg/m^3	958	917	865	799	712

Find by interpolation, the density when the temperature is 275^0 .

10.Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 =2.8156$, $\log_{10} 658 =2.8182$, $\log_{10} 659 =2.8189$ and $\log_{10} 661 =2.8202$.

11.Find f'(x) at x=1.5 and x=4.0 from the following data using Newton's formulae for differentiation.

x	1.5	2.0	2.5	3.0	3.5	4.0
Y=f(x)	3.375	7.0	13.625	24.0	38.875	59.0

12.If f(0)=1,f(1)=2,f(2)=33 and f(3)=244. Find a cubic spline approximation, assuming M(0)=M(3)=0.Also find f(2.5).

13.Fit a set of 2 cubic splines to a half ellipse described by $f(x)=-[25-4x^2]^{1/2}$. Choose the three data points (n=2) as (-2.5,0), (0,1.67) and (2.5 , 0) and use the free boundary conditions.

14.Find the value of y at x=21 and x=28 from the data given below

x	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

15. The population of a town is as follows:

x year	1941	1951	1961	1971	1981	1991
y population (thousands)	20	24	29	36	46	51

Estimate the population increase during the period 1946 to1976.

CHAPTER 3

NUMERICAL DIFFERENTIATION AND INTEGRATION

Introduction

Numerical differentiation is the process of computing the value of — —

For some particular value of x from the given data (x_i,y_i), i=1, 2, 3…..n where y=f(x) is not known explicitly. The interpolation to be used depends on the particular value of x which derivatives are required. If the values of x are not equally spaced, we represent the function by Newton's divided difference formula and the derivatives are obtained. If the values of x are equally spaced, the derivatives are calculated by using Newton's Forward or backward interpolation formula. If the derivatives are required at a point near the beginning of the table, we use Newton's Forward interpolation formula and if the derivatives are required at a point near the end of table. We use backward interpolation formula.

3.1 Derivatives using divided differences

Principle

First fit a polynomial for the given data using Newton's divided difference interpolation formula and compute the derivatives for a given x.

Problems

1.Compute $f'(3.5) and f''(4)$ **given that f(0)=2,f(1)=3;f(2)=12 and f(5)=147**

Sol:

Divided difference table

x	**f**		$\Delta_1^2 f$	$\Delta_1^3 f$
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

Newton's divided difference formula is

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + ...$

Here $x_0 = 0; x_1 = 1,; x_2 = 2, f(x_0) = 2, \; f(x_0, x_1) = 1, \; f(x_0, x_1, x_2) = 4; f(x_0, x_1, x_2, x_3) = 1$

$f(x) = x^3 + x^2 - x + 2$

$f'(x) = 3x^2 + 2x - 1$
$f''(x) = 6x + 2$

$f'(3.5) = 42.75, \; f''(4) = 26.$

2.Find the values $f'(5) and f''(5)$ using the following data

x	**0**	**2**	**3**	**4**	**7**	**9**
F(x)	**4**	**26**	**58**	**112**	**466**	**22**

Ans: 98 and 34.

3.1.1 Derivatives using Finite differences

Newton's Forward difference Formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + ...]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + ...]$$

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + ...]$$

Newton's Backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + ...]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...]$$

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...]$$

Problems

1.Find the first two derivatives of y at x=54 from the following data

x	50	51	52	53	54
y	3.6840	3.7083	3.7325	3.7563	3.7798

Sol:

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	3.6840				
		0.0244			
51	3.7083		-0.0003		
		0.0241		0	
52	3.7325		-0.0003		0
		0.0238		0	
53	3.7563		-0.0003		
		0.0235			
54	3.7798				

By Newton's Backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \ldots\ldots\ldots]$$

Here h=1; ∇y_n=0.0235; $\nabla^2 y_n = -0.0003$

$$\left(\frac{dy}{dx}\right)_{x=54} = 0.02335$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=54} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\ldots\ldots$$

=-0.0003

2.Find first and second derivatives of the function at the point x=12 from the following data

x	**1**	**2**	**3**	**4**	**5**
y	**0**	**1**	**5**	**6**	**8**

Sol:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		1			
2	1		3		
		4		-6	
3	5		-3		10
		1		4	
4	6		1		
		2			
5	8				

We shall use Newton's forward formula to compute the derivatives since x=1.2 is at the beginning.

For non-tabular value $x = x_0 + ph$,we have

$$\frac{dy}{dx} = \frac{1}{h}[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12}\Delta^4 y_0 + ...$$

Here x_0=1, h=1, x=1.2

$$\therefore\ p = \frac{x - x_0}{h}$$

$$p = 0.2$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = 1.773$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = 14.17$$

Exercise:

1.Find the value of $\sec 31^0$ using the following data

x	31	32	33	34
Tanx	0.6008	0.6249	0.6494	0.6745

Ans:1.1718

2.Find the minimum value of y from the following table using numerical differentiation

x	0.60	0.65	0.70	0.75
y	0.6221	0.6155	0.6138	0.6170

Ans: minimum y=0.6137

3.2 Numerical integration
(Trapezoidal rule & Simpson's rules, Romberg integration)

Introduction

The process of computing the value of a definite integral from a set of values (x_i,y_i),i=0,1,2,….. Where $x_0 = a; x_n = b$ of the function y=f(x) is called Numerical integration. Here the function y is replaced by an interpolation formula involving finite differences and then integrated between the limits a and b, the value is found.

General Quadrature formula for equidistant ordinates
(Newton cote's formula)

On simplification we obtain

$$\int_{x_0}^{x_n} y\,dx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots$$

This is the general Quadrature formula
By putting n=1, Trapezoidal rule is obtained
By putting n=2, Simpson's 1/3 rule is derived
By putting n=3, Simpson's 3/8 rule is derived.

Note

The error in Trapezoidal rule is of order h^2 and the total error E is given by y''( )
is the largest of y.

Simpson's 1/3 rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots\ldots + y_{n-2})]$$

Error in Simpson's 1/3 rule
The Truncation error in Simpson's 13 rule is of order h^4 and the total error is given by
$E = \frac{(b-a)}{180} h^4 y^{iv}(\bar{x}) where y^{iv}(\bar{x})$ is the largest of the fourth derivatives.

Simpson's 3/8 rule

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \ldots\ldots) + 2(y_3 + y_6 + y_9 + \ldots\ldots)]$$

Note
The Error of this formula is of order h^5 and the dominant term in the error is given by
$-\frac{3}{80} h^5 y^{iv}(\bar{x})$

Romberg's integration
A simple modification of the Trapezoidal rule can be used to find a better approximation to the value of an integral. This is based on the fact that the truncation error of the Trapezoidal rule is nearly proportional to h^2.

$$I = \frac{I_1 h_2^{\ 2} - I_2 h_1^{\ 2}}{h_2^{\ 2} - h_1^{\ 2}}$$

This value of I will be a better approximation than I_1 or I_2.This method is called Richardson's deferred approach to the limit.
If h_1=h and h_2=h/2,then we get

$$I(h, \frac{h}{2}) = \frac{4I_2 - I_1}{3}$$

Problems

1.Evaluate $\int_0^{\pi} \sin x dx$ by dividing the interval into 8 strips using (i) Trapezoidal rule

(ii) Simpson's 1/3 rule
Sol:
For 8 strips ,the values of y=sinx are tabulated as follows :

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
sinx	0	0.3827	0.7071	0.9239	1	0.9239	0.7071	0.3827	0

(i)Trapezoidal rule

$$\int_0^{\pi} \sin x dx = \frac{\pi}{16}[4(0.3827+0.7071+0.9239)+2]$$

$$=1.97425$$

(ii)Simpson's 1/3 rule

$$\int_0^{\pi} \sin x dx = \frac{\pi}{24}[4(0.3827+0.23+0.923+03827)+2(0.7071+1+0.7071)]$$

$$=2.0003$$

2..Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpsons 1/3 and 3/8 rule. Hence obtain the approximate value of π in each case.

Sol:

We divide the range (0,1) into six equal parts, each of size h=1/6.The values of $y = \frac{1}{1+x^2}$ at each point of subdivision are as follows.

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	0.9730	0.9	0.8	0.623	0.5902	0.5

By Simpson's 1/3 rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}[1.5+4(2.3632)+2(1.5923)]$$

$$=0.7854 \quad \textbf{(1)}$$

By Simpson's 3/8 rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{1}{16}[1.5+3(3.1555)+1.6]$$

$$=0.7854 \quad \textbf{(2)}$$

But $\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

From equation (1) and (2), we have – ,

π=3.1416

Exercise

1.Evaluate $\int_0^{1.2} e^{-x^2}dx$ using (i) Simpson's 1/3 rule (ii) Simpson's 3/8 rule ,taking h=0.2

Ans:

(i) $\int_0^{1.2} e^{-x^2}dx$ =0.80675 (ii) $\int_0^{1.2} e^{-x^2}dx$ =0.80674

2. Evaluate $\int_0^{\pi} \cos x dx$ by dividing the interval into 8 strips using (i) Trapezoidal rule (ii) Simpson's 1/3 rule

Romberg's method
Problems

1. Use Romberg's method, to compute $I = \int_0^1 \frac{dx}{1+x}$ correct to 4 decimal places. Hence find

$\log_e 2$

Sol:

The value of I can be found by using Trapezoidal rule with=0.5, 0.25, 0.125

(i) h=0.5 the values of x and y are tabulated as below:

x	0	0.5	1
y	1	0.6667	0.5

Trapezoidal rule gives

$$I = \frac{1}{4}[1.5 + 2(0.6667)]$$

=0.7084

(i) h=0.25, the tabulated values of x and y are as given below:

x	0	0.25	0.5	0.75	1
y	1	0.8	0.6667	0.5714	0.5

Trapezoidal rule gives

$$I = \frac{1}{8}[1.5 + 2(0.8 + 0.6667 + 0.5714)]$$

=0.6970

(ii) h=0.125 **,** , the tabulated values of x and y are as given below:

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1
y	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

Trapezoidal rule gives,

$$I = \frac{1}{16}[1.5 + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333)]$$

=0.6941

We have approximations for I as I_1=0.7084,I_2=0.6970,I_3=0.6941

Use Romberg's method,the better approximation are caluculated as follws

Using $I = I_2 + \frac{I_2 - I_1}{3}$

$$I(h, \frac{h}{2}) = 0.6932$$

$$I(\frac{h}{2}, \frac{h}{4}) = 0.6931$$

$$I(h, \frac{h}{2}, \frac{h}{4}) = 0.6931$$

By actual integration,

$$I = \int_0^1 \frac{\mathrm{dx}}{1+\mathrm{x}} = \log_e (1+x)_0^1 = \log_e 2$$

$\log_e 2$=0.6931

2.Use Romberg's method ,evaluate $\int_0^{\pi} \sin x dx$,correct to four decimal places.

Ans:I=1.9990

Exercise:

1.Use Romberg's method to compute —— correct to 4 decimal places.Hence find an approximate value of

Ans:3.1416

3.3 Gaussian Quadrature

For evaluating the integral $I = \int_a^b f(x)dx$,we derived some integration rules which require the values of the function at equally spaced points of the interval. Gauss derived formula which uses the same number of function values but with the different spacing gives better accuracy.

Gauss's formula is expressed in the form $\int_{-1}^{1} F(u)du = w_1 F(u_1) + w_2 F(u_2) + \ldots\ldots + w_n F(u_n)$

$$= \sum_{i=1}^{n} w_i F(u_i)$$

Where w_i and u_i are called the weights and abscissa respectively. In this formula, the abscissa and weights are symmetrical with respect to the middle of the interval.
The one-point Gaussian Quadrature formula is given by

$\int_{-1}^{1} f(x)dx = 2f(0)$, which is exact for polynomials of degree upto 1.

Two point Gaussian formula

$$\int_{-1}^{1} f(x)dx = [f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}})]$$

And this is exact for polynomials of degree upto 3.

Three point Gaussian formula

$$\int_{-1}^{1} f(x)dx = \frac{8}{9}f(0) + \frac{5}{9}[f(-\sqrt{\frac{3}{5}}) + f(\sqrt{\frac{3}{5}})]$$

Which is exact for polynomials of degree upto 5

Note:

Number of terms	**Values of t**	**Weighting factor**	**Valid upto degree**
2	T= $-\frac{1}{\sqrt{3}}$ & $\frac{1}{\sqrt{3}}$ -0.57735& 0.57735	1&1	3
3	$-\sqrt{\frac{3}{5}} = -0.7746$ $\sqrt{\frac{3}{5}} = 0.7746$	$\frac{5}{9} = 0.5555$ $\frac{8}{9} = 0.8889$ $\frac{5}{9} = 0.5555$	5

Error terms

The error in two –point Gaussian formula= $\frac{1}{135} f^{iv}(\xi)$ and the error in three point Gaussian formula is= $\frac{1}{15750} f^{vi}(\xi)$

Note

The integral $\int_a^b F(t)dt$,can be transformed into $\int_{-1}^1 f(x)dx$ by h line transformation

$$t = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a)$$

Problems

1.Evaluate $\int_{-1}^1 \frac{dx}{1+x^2}$ by two point and three point Gaussian formula and compare with the exact value.

Sol:

By two –point Gaussian formula,

$$\int_{-1}^1 f(x)dx = [f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}})]$$

$$\int_{-1}^1 \frac{dx}{1+x^2} = \mathbf{1.5}$$

By three-point Gaussian formula

$$\int_{-1}^1 f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}[f(-\sqrt{\frac{3}{5}}) + f(\sqrt{\frac{3}{5}})]$$

$$\int_{-1}^1 \frac{dx}{1+x^2} = \mathbf{1.5833}$$

But exact value= $2\int_0^1 \frac{dx}{1+x^2} = 2[\tan^{-1} x]_0^1 = \frac{\pi}{2} = 1.5708$

2.Evaluate $\int_{-1}^1 \frac{x^2 dx}{1+x^4}$ by using Gaussian three point formula

Sol:

Here $f(x) = \frac{x^2 dx}{1+x^4}$

By three-point Gaussian formula

$$\int_{-1}^1 f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}[f(-\sqrt{\frac{3}{5}}) + f(\sqrt{\frac{3}{5}})] \qquad \text{…………(1)}$$

$$f(-\sqrt{\frac{3}{5}}) = \frac{15}{34}$$

$$f(-\sqrt{\frac{3}{5}}) = \frac{15}{34}$$

By equation (1),

$$\int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}[f(-\sqrt{\frac{3}{5}}) + f(\sqrt{\frac{3}{5}})]$$

$$= \frac{25}{21}$$

$$\int_{-1}^{1} \frac{x^2 dx}{1+x^4} = \mathbf{0.4902}$$

3.Use Gaussian two point formula ,to evaluate $\int_{1}^{2} \frac{dx}{x}$

Sol:
a=1,b=2.

The transformation is $x = \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$

$$x = \frac{t+3}{2}$$

$$\int_{1}^{2} \frac{dx}{x} = \int_{-1}^{1} \frac{2dt}{(t+3)2} = \int_{-1}^{1} \frac{dt}{(t+3)}$$

By two –point Gaussian formula,

$$\int_{-1}^{1} f(x)dx = [f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}})]$$

$$= \frac{1}{3 - \frac{1}{\sqrt{3}}} + \frac{1}{3 + \frac{1}{\sqrt{3}}}$$

=0.6923

Exercise

1. Obtain two point and three point Gaussian formula for the gauss-chebyshev Quadrature formula, given by $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$

   Ans: $\lambda_1 = \lambda_0 = \frac{\pi}{3}; x_1 = \pm \frac{\sqrt{3}}{2}$

2. Find $I = \int_0^1 x dx$,by Gaussian three point formula, correct to 4 decimal places.

   Ans: 0.5

3. Evaluate $I = \int_{-1}^{1} \cos x dx$ using Gaussian two point and three point formula

   Ans: 1.6831

4. Evaluate $I = \int_{-1}^{1} (1 + x + x^2) dx$ using Gaussian here point formula. Ans: 8/3

3.4 Double integrals

In this chapter we shall discuss the evaluation of $\int_a^b \int_c^d f(x, y) dxdy$ using Trapezoidal and Simpson's rule.

The formulae for the evaluation of a double integral can be obtained by repeatedly applying the Trapezoidal and simpson's rules.

Consider the integral $I = \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x, y) dxdy$ (1)

The integration in (1) can be obtained by successive application of any numerical integration formula with respect to different variables.

Trapezoidal Rule for double integral

$$I = \frac{hk}{4} \begin{Bmatrix} [f(x_0, y_0) + 2f(x_0, y_1) + f(x_0, y_2)] + \\ 2[f(x_1, y_0) + 2f(x_1, y_1) + f(x_1, y_2)] + \\ f(x_2, y_0) + 2f(x_2, y_1) + f(x_2, y_2) \end{Bmatrix}$$

Simpson's rule for double integral

The general expression for Simpson's rule for the double integral (1) can also be derived.

In particular, Simpson's rule for the evaluation of

$$I = \frac{hk}{9} \begin{Bmatrix} [f(x_{i-1}, y_{j-1}) + 4f(x_{i-1}, y_j) + f(x_{0i-1}, y_{j+1})] + 4[f(x_i, y_{ij-1}) + 4f(x_j, y_j) + f(x_i, y_{j+1})] + \\ f(x_{i+1}, y_{j-1}) + 4f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) \end{Bmatrix}$$

Problems:

1. Evaluate $I = \int_0^1\int_0^1 e^{x+y}\,dxdy$ using trapezoidal and Simpson's rule

Sol:

Taking h=k=0.5

The table values of e^{x+y} are given a follows

	(x_0)	(x_1)	(x_2)
y\x	0	0.5	1
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1	2.7183	4.4817	7.3891

(i) Using Trapezoidal rule,we obtain

$$I = \int_{y_0}^{y_2}\int_{x}^{x_2} f(x,y)\,dxdy \qquad \text{where } f(x,y)=e^{x+y}$$

$$I = \frac{hk}{4}\begin{Bmatrix}[f(x_0,y_0)+2f(x_0,y_1)+f(x_0,y_2)]+\\ 2[f(x_1,y_0)+2f(x_1,y_1)+f(x_1,y_2)]+\\ f(x_2,y_0)+2f(x_2,y_1)+f(x_2,y_2)\end{Bmatrix}$$

=3.0762

(ii) Using Simpson's rule,we obtain

$$I = \frac{hk}{9}\begin{Bmatrix}[f(x_0,y_0)+4f(x_0,y_1)+f(x_0,y_2)]+4f(x_1,y_0)+16f(x_1,y_1)+4(x_1,y_2)]+\\ f(x_2,y_0)+4f(x_2,y_1)+f(x_2,y_2)]\end{Bmatrix}$$

=2.9543

2.Evaluate $I = \int_0^1\int_0^1 \frac{dxdy}{1+x+y}$ by (i) Trapezoidal rule and (ii) Simpson's rule with step sizes h=k=0.5

Sol:

Taking h=k=0.5

The table values of $\frac{1}{1+x+y}$ are given a follows

	(x_0)	(x_1)	(x_2)
y\x	0	0.5	1
0	1	0.67	0.5
0.5	0.67	0.5	0.4
1	0.5	0.4	0.33

(i) Using Trapezoidal rule,we obtain

$$I = \int_{y_0}^{y_2}\int_{x}^{x_2} f(x,y)dxdy \qquad \text{where } f(x,y) = \frac{1}{1+x+y}$$

$$I = \frac{hk}{4}\begin{Bmatrix} [f(x_0,y_0)+2f(x_0,y_1)+f(x_0,y_2)]+ \\ 2[f(x_1,y_0)+2f(x_1,y_1)+f(x_1,y_2)]+ \\ f(x_2,y_0)+2f(x_2,y_1)+f(x_2,y_2) \end{Bmatrix}$$

=0.5381

(i) Using Simpson's rule,we obtain

$$I = \frac{hk}{9}\begin{Bmatrix} [f(x_0,y_0)+4f(x_0,y_1)+f(x_0,y_2)]+4f(x_1,y_0)+16f(x_1,y_1)+4(x_1,y_2)]+ \\ f(x_2,y_0)+4f(x_2,y_1)+f(x_2,y_2)] \end{Bmatrix}$$

=0.5247

Exercise

1. Compute $I = \int_0^1\int_0^1 xydxdy$ with h=0.25 and k=0.5 using Simpson's rule

   Ans:I=0.25

2. Using Trapezoidal rule ,evaluate $I = \int_1^2\int_1^2 \frac{dxdy}{x+y}$ taking four sub intervlas

   Ans:I=0.34065

Tutorial problems

Tutorial -1

1.Find the value of $f''(3)$ using divided differences, given data

x	0	1	4	5
F(x)	8	11	68	123

Ans:16

2.Find the first and second derivatives of the function at the point x=1.2 from the following data

X	1	2	3	4	5
Y	0	1	5	6	8

Ans:14.17

3.A rod is rotating in a plane .The angle θ (in radians) through which the rod has turned for various values of time t (seconds) are given below

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.122	0.493	1.123	2.022	3.220	4.666

Find the angular velocity and angular acceleration of the rod when t=06 seconds.

Ans:6.7275 radians/sec^2.

4.From the table given below ,for what value of x, y is minimum? Also find this value of y.

X	3	4	5	6	7	8
Y	0.205	0.240	0.25	0.262	0.250	0.224

Ans:5.6875,0.2628

5.Find the first and second derivatives of y at x=500 from the following dat

x	500	510	520	530	540	550
y	6.2146	6.2344	6.2538	6.2792	6.2196	6.3099

Ans:0.002000,-0.0000040

Tutorial -2

1.Evaluate $\int_0^1 \frac{xdx}{1+x}$ by Simpson's 1/3 rule with h=0.1

2.Evalute $\int_0^1 \cos^2 xdx$ with h=0.1 by Simpson's 1/3 rule.

3.Apply Trapezoidal and Simpson's rules ,to find $\int_0^1 \sqrt{1-x^2}\,dx$,taking h=0.1.

4.Evaluate $\int_0^6 \frac{dx}{1+x}$ by dividing the range into eight equal parts.

5.Using Simpson's rule ,find $\log_e 7$ approximately from the integral $I = \int_1^7 \frac{dx}{x}$

Tutorial -3

1.Evaluate $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ using Gaussian three point formula.

2.Evaluate $\int_0^1 e^x dx$ using three-term Gaussian Two-point Quadrature formula.

3.Evaluate $\int_{-1}^{1} \frac{dx}{1+x^4}$ using Gaussian three point quadrature formula.

4. Evaluate $\int_{-1}^{1} \frac{\cos 2x dx}{1+\sin x}$ using Gaussian two point quadrature formula.

5.Evaluate $\int_2^4 (x^2 + 2x)dx$ using Gaussian two point quadrature formula.

Question Bank

Part A

1. State the disadvantages of Taylor series method.

Sol:

In the differential equation — = f(x,y) the function f(x,y) may have a complicated algebraical structure. Then the evaluation of higher order derivatives may become tedious. This is the demerit of this method.

2. Which is better Taylor's method or R.K method?

Sol:

R.K methods do not require prior calculation of higher derivatives of y(x), as the Taylor method does. Since the differential equations using in application are often complicated, the

calculation of derivatives may be difficult. Also R.K formulas involve the computations of f(x,y) at various positions instead of derivatives and this function occurs in the given equation.

3. What is a predictor- collector method of solving a differential equation?

Sol:

predictor- collector methods are methods which require the values of y at x_n, x_{n-1}, x_{n-2},… for computing the values of y at x_{n+1}. We first use a formula to find the values of y at x_{n+1} and this is known as a predictor formula. The value of y so get is improved or corrected by another formula known as corrector formula.

4. Define a difference Quotient.

Sol:

A difference quotient is the quotient obtained by dividing the difference between two values of a function, by the difference between the two corresponding values of the independent.

5. Write down the expression for $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ $atx = x_n$ by Newton's backward difference formula.

6. State the formula for Simpson's 3/8th rule.

7. Write Newton's forward derivative formula.

8.State the Romberg's method integration formula fin the value of $I = \int_a^b f(x)dx$ using h and h/2.

9.Write down the Simpson's 3/8 rule of integration given (n+1) data.

10.Evaluate $\int_1^4 f(x)dx$ from the table by Simpson's 3/8 rule.

<u>Part B</u>

1.Apply three point Gaussian quadrature formula to evaluate —— .

2.Using Trapezoidal rule, evaluate —— numerically with h=0.2 along x-direction and k=0.25 along y-direction.

3.Find the first and second derivative of the function tabulated below at x=0.6

x	0.4	0.5	0.6	0.7	0.8
y	1.5836	1.7974	2.0442	2.3275	2.6511

4. Using Romberg's method to compute —— dx correct to 4 decimal places.Also evaluate the same integral using three –point Gauss quadrature formula. Comment on the obtained values by comparing with exact value of the integral which is equal to – .

5.Evaluate using Simpson's rule by taking h=– and k = –.

6.Find — and — at x=51 from the following data.

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

7.Evaluate I= by dividing the range into ten equal parts,using
(i)Trapezoidal rule
(ii)Simpson's one-third rule. Verify your answer with actual integration.

8. Find the first two derivatives of ‾ at x=50 and x=56, for the given table :

x	50	51	52	53	54	55	56
$Y=x^{1/3}$	3.6840	3.7084	3.7325	3.7325	3.7798	3.8030	3.8259

9.The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time(min)	0	2	4	6	8	10	12
Velocity(km/hr)	0	22	30	27	18	7	0

Using Simpson's $1/3^{rd}$- rule find the distance covered by the car.

10. Given the following data, find y'(6) and the maximum value of y (if it exists)

x	0	2	3	4	7	9
Y	4	26	58	112	466	922

Chapter 4

Numerical Solution of ordinary Differential equations

Taylor's series method

Introduction

An ordinary differential equation of order n is a relation of the form $F(x, y, y', y'', \ldots\ldots y^{(n)}) = 0$ where y=y(x) and $y^{(r)} = \dfrac{d^r y}{dx^r}$.The solution of this differential equation involves n constants and these constants are determined with help of n conditions.

Single step methods

In these methods we use information about the curve at one point and we do not iterate the solution. The method involves more evaluation of the function. We will discuss the Numerical solution by Taylor series method, Euler methods and Runge - Kutta methods all require the information at a single point $x=x_0$.

Multi step methods

These methods required fewer evaluations of the functions to estimate the solution at a point and iteration are performed till sufficient accuracy is achieved. Estimation of error is possible and the methods are called Predictor-corrector methods. In this type we mainly discuss Milne's and Adams-Bashforth method.

In the multi step method , to compute y_{n+1},we need the functional values y_n,y_{n-1}, y_{n-2} and y_{n-3}.

4.1 Taylor Series method

Consider the first order differential equation $\dfrac{dy}{dx} = f(x, y), y(x_0) = y_0$ ……(1)

The solution of the above initial value problem is obtained in two types

- Power series solution
- Point wise solution

(i) Power series solution

$$\frac{}{} \qquad \frac{}{} \qquad \frac{}{} + \ldots\ldots. \qquad (2)$$

Where $y_0^{(r)} = \frac{d^r y}{dx^r}$ at(

Using equation (1) the derivatives can be found by means of successive differentiations .Expressions (2) gives the value for every value of x for which (2) converges.

(ii)Point wise solution

$= \; + \frac{h}{1!} \; +— \; +— \; +\dots$

Where $y_m^{(r)} = \frac{d^r y}{dx^r}$ at (

Where m=0,1,2,…….

Problems:

1. **Using Taylor series method find y at x=0.1 if $\frac{dy}{dx}$ = y-1,y(0)=1.**

Solution:

Given = y-1 and =0, =1,h=.1

Taylor series formula for is

$= \; + \frac{h}{1!} \; +— \; +— \; +\dots$

= y-1	= =0-1=-1
=2xy+	+ =0+0=0
= + +2x =2x +2y+ +2x =2y+ 4x +	=2 +4 + =2(1)+4(0)(-1)+( )(0) =2
= +4 + + = +4x +4 + + 2x = + 6x +	= +6 + =6(-1)+6(0)(0)+(0)(2) =-6

(1) $\Rightarrow$ =1+—(-1)+——(0)+——(2)+—— +…

Y(0.1)=1-(0.1)+—— ——+…

=1-0.1+0.00033-0.000025=0.900305

2. Find the Taylor series solution with three terms for the initial value problem.—= +y,y(1)=1.

Solution:

Given = **+y,** =1, =1.

= **+y**	= + =1+1=2
=3 **+**	+ =3(1)+2 =5
=6x+	= + =6(1)+5 =11

Taylor series formula

Y= —— ——— ——— +…….

=1+(x-1)(2)+ ———(5)+ ———(11)(app)

=1+2(x-1)+– +—

Exercise*:*

1.Find y(0.2),y(0.4) given—=x +1,y(0)=1

[***Ans:-1.226,1.5421]***

2.Find y(0.1) given = y-1,y(0)=1

[Ans:- 0.9003]

EULER'S METHOD:

Introduction

In Taylor's series method, we obtain approximate solutions of the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ as a power series in x , and the solution can be used to compare y numerically specified value x near x_0.

In Euler's methods,we compute the values of y for $x_i = x_0 + ih, i = 1,2.....$ with a step h>0

i.e., $y_i = y(x_i)$ where $x = x_0 + ih, i = 1,2.....$

Euler's method

= +*f* *),n=0,1,2,3,……*

Modified Euler's method

To compute *for*

(i) $y_{n+1}^{(1)} = y_n + hf(x_n, y_n)$

(ii) $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_{n+1}^{(1)})$ error n=0,1,2…..

Error=O(h^3)

Note

In Euler method $y_{n+1} = y_n + \Delta y$

Where $\Delta y = hf(x_0, y_0)$

Where $f(x_0, y_0) = slope$ at $f(x_0, y_0)$.

In modified Euler's method $y_{n+1} = y_n + \Delta y$,

Where Δy =h[average of the slopes at x_0 and x_1](Or)

=h[average of the values of $\frac{dy}{dx}$ at the ends of the interval x_0 to x_1]

1.Using Euler's method compute y in the range 0 .5 if y satisfies — 3x+ ,y(0)=1.

Solution:

Here f(x,y) = *3x+ , =0, =1*

By Euler's method

= +f),n=0,1,2,3,…… *(1)*

Choosing h=0.1, we compute the value of y using (1)

= +f),n=0,1,2,3,……

= +hf( )=1+(0.1)[3(0)+] =1.1

= +hf( )=1.1+(0.1)[0.3+] =1.251

= +hf( )=1.251+(0.1)[0.6+] =1.4675

= +hf( )=1.4675+(0.1)[0.9+] =1.7728

= +hf( )=1.7728+(0.1)[1.2+] =2.2071

Exercise:

1.Given that $\frac{dy}{dx} = x + y$ **and y=1 when x=0,Find y when x=0.05,0.10 and 0.15 using modified Euler method.**

Ans:

1.0525,1.1103,1.1736

2.Using Euler's method,solve $y' = x + y + xy, y(0) = 1.$ **Compute y(1.0) with h= 0.2**

Ans:3.9778

MODIFIED EULER'S METHOD:

1. Given —=-x ***, y(0)=2. Using Euler's modified method, find y(0.2)in two steps of 0.1 each.***

Solution:

Let *= +h. here , and h=0.1*

The value of y at x= is

By modified Euler method,

= *hf(*

− *f(* *f(*]

Here f(x,y) = -x

f(= - *=0*

2+(0.1)(0)=2

2+—[f(0,2)+f(0.1,2)]

=2+0.05[0+(-0.1)(4)]

1.98

Y(0.1)=1.98

To find y(0.2)

=1.98 –(0.1)(0.1)(1.98)=1.9602

+− *f(* *f(*]

=1.9602-(0.05)[(0.198)+(0.39204)]

Hence y(0.2)=1.9307

Runge-kutta method

The Taylor's series method of solving differential equations numerically is restricted because of the evaluation of the higher order derivatives.Runge-kutta methods of solving intial value problems do not require the calculations of higher order derivatives and give greater accuracy.The Runge-Kutta formula posses the advantage of requiring only the function values at some selected points.These methods agree with Taylor series solutions upto the term in h^r where r is called the order of that method.

Fourth-order Runge-Kutta method

This only is commonly used for solving initial values problem

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

Working rule

The value of $y_1 = y(x_1)$ where $x_1 = x_0 + h$ where h is the step-size is obtained as follows.We calculate successively.

$$k_1 = hf(x_m, y_m)$$
$$k_2 = hf(x_m + \frac{h}{2}, y_m + \frac{k_1}{2})$$
$$k_3 = hf(x_m + \frac{h}{2}, y_m + \frac{k_2}{2})$$
$$k_4 = hf(x_m + h, y_m + k_3)$$

Finally compute the increment

$$y_{m+1} = y_m + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{where } m=0,1,2\ldots\ldots$$

Problems

1. Obtain the value of y at x =0.2 if satisfies $\frac{dy}{dx} = x^2 y + x, x_0 = 0, y_0 = 1$ **using Runge-kutta method of fourth order.**

Sol:

Here $f(x,y) = x^2 y + x, x_0 = 0, y_0 = 1$

Let $x_1 = x_0 + h$

Choosing h=0.1,x_1=0.1

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0$$
$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.00525$$
$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.00525$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.0110050$$

y(0.1)=1.0053

To find $y_2 = y(x_2)$ where $x_2 = x_1 + h$ Taking x_2=0.2

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.0110$$
$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.01727$$
$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.01728$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.02409$$

y(0.2)=1.0227

2.Apply Runge –kutta method to find an approximate value of y for x=0.2 in steps of 0.1 if

$\frac{dy}{dx} = x + y^2, y(0) = 1$**,correct to four decimal places.**

Sol:

Here f(x,y)=x+y^2,x_0=0,y(0)=1

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1152$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1347$

y(0.1)=1.1165

To find $y_2 = y(x_2)$

where $x_2 = x_1 + h$ Taking x_2=0.2

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$k_1 = hf(x_1, y_1) = 0.1347$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1151$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1576$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1823$

y(0.2)=1.2736

Exercise:

1.Use Runge-kutta method to find when x=1.2 in steps of 0.1 ,given that $\frac{dy}{dx} = x^2 + y^2, y(1) = 1.5$

Ans:1.8955, 2.5043

2.Solve $y' = xy + 1$ for x=0.2(0.2)0.6 by Fourht order method,given that y=2 when x=0

Ans: 2.2431,2.5886, 3.0719

Multi step methods(Predictor-Corrector Methods)

Introduction

Predictor-Corrector Methods are methods which require function values at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ for the compulation of the function value at x_{n+1}.A predictor is used to find the value of y at x_{n+1} and then a corrector formula to improve the value of y_{n+1}.

The following two methods are discussed in this chapter.

- Milne's method
- Adam's method.

Milne's Predictor-Corrector method

Milne's predictor formula is

And the error $= \frac{14h^5}{45} y^{(v)}(\xi)$

Where $x_{n-3} < \xi < x_{n+1}$

Milne's corrector formula is

And the error $= -\frac{h^5}{90} y^{(v)}(\xi)$

Where $x_{n-1} < \xi < x_{n+1}$

Problems

1.Using Milne's method ,compute y(0.8) given that

$$\frac{dy}{dx} = 1 + y^2, y(0) = 1, y(0.2) = 0.2027, y(0.4) = 0.4228, y(0.6) = 0.6841$$

Sol:

We have the following table of values

X	y	$Y' = 1+y^2$
0 0.2 0.4 0.6	0 0.2027 0.4228 0.6841	1.0 1.0411 1.1787 1.4681

$$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$$
$$y_0 = 0, y_1 = 0.2027, y_2 = 0.4228, y_3 = 0.6841$$
$$y_0' = 1, y_1' = 1.0411, y_2' = 1.1787, y_3' = 1.4681$$

To find y(0.8)

$x_8 = 0.8$,here h=0.2

Milne's predictor formula is

— where).

=1.0239

=2.0480

Milne's corrector formula is

– where).

=1.0294

y(0.8)=1.0294

2. Compute y(0.4) by Milne's method ,given that $y' = x + y, y(0) = 1$ with h=0.1 use Taylor's method to find the starting values.

Ans:1.9816, 1.5774

Adams-bashforth Predictor and Adams-bashforth corrector formula.

Adams-bash forth predictor formula is

—

Adams –bash forth corrector formula is

— where).

The error in these formulas are $\frac{251}{720}h^5 f^{iv}(\xi)$ and $-\frac{19}{720}h^5 f^{iv}(\xi)$

—

—

Problems

1.Given $y' = y - x^2$, $y(0) = 1, y(0.2) = 1.2186$, $y(0.4) = 1.4682$, $y(0.6) = 1.7379$ **estimate y(0.8) by Adam's-Bashforth method.**

Sol:

$$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$$
$$y_0 = 0, y_1 = 1.2186, y_2 = 1.4682, y_3 = 1.7379$$
$$y_0' = 1, y_1' = 1.1782, y_2' = 1.3082, y_3' = 1.3779$$

To find y_4 for $x_4 = 0.8$ here h=0.2

By Predictor formula,

—

=2.0146

$$y_4' = 1.3746$$

—

=2.0145

y(0.8)=2.0145

2.Given $y' = 1 + y^2, y(0) = 0, y(0.2) = 0.2027$, $y(0.4) = 1.4228$, $y(0.6) = 0.6841$ **,**Estimate y(0.8) using Adam's method.

Sol:

$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$
$y_0 = 0, y_1 = 0.2027, y_2 = 0.4228, y_3 = 0.6841$
$y_0' = 1, y_1' = 1.0411, y_2' = 1.1786, y_3' = 1.4680$

To find y_4 for $x_4 = 0.8$ here h=0.2

By Predictor formula,

—

=1.0235

$y_4' = 2.0475$

—

=1.0297

y(0.8)=1.0297

Tutorial problems

Tutorial-1

1.Using Taylor\s series method ,find y(0.2) approximately ,given that $\frac{dy}{dx} = 2y + 3e^x$ y(0)=0 correct to four decimal places.

2.Compute y for x=0.1,0.2 correct to four decimal places given $\frac{dy}{dx} = y - x, y(0) = 2$ using taylor's series method.

3.Using Euler's Modified method ,find the values of y at x=0.1,0.2,0.3,0.4 given $\frac{dy}{dx} = 1 - y, y(0) = 0$ correct to four decimal places.

4. Using Euler's method ,find y(0.1),y(0.2) and y(0.3)given that $\frac{dy}{dx} = 1 + y^2, y(0) = 0$

5.Using Euler's modified method, find a solution of the equation $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$ with the initial condition y(0)=1 for the range $0 \le x \le 0.6$ in steps of 0.2.

Tutorial-2

1.Use R-K fourth order method, of find y at x=0.1(0.1)0.3 if $y' = \frac{xy}{1+x^2}, (0) = 1$

2.Apply R-K method of order 4 to solve $\frac{dy}{dx} = xy^{1/3}, y(1) = 1$ with h=0.1

3.Using R-K method to solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y^2, y(0) = 1, y'(0) = 0$ to find y(0.2) and $y'(0.2)$

4.Use the R-K method of fourth order ,find y(0.1)=1 given that

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, y(0) = 1, y'(0) = 0$$

5.Compute y and z for x=0.1 given $\frac{dy}{dx} = 1 + z; \frac{dz}{dx} = x - y, y(0) = 0, z(0) = 1$

Tutorial-3

1.Apply Milne's method ,to find y(0.4) given that $y' = xy + y^2, y(0) = 1.$ Use Taylor's series method to compute y(0.1),y(0.2),y(0.3).

2.using Adam-Bashforth method, find the solution of $\frac{dy}{dx} = x + y$ at x=0.4,given the values y(0)=1,,y(0.1)=1.1103,y(0.2)=1.2428,y(0.3)=1.3997

3.Using adam's method ,determine y(0.4) given that
$\frac{dy}{dx} = \frac{1}{2}xy, y(0) = 1, y(0.1) = 1.0025, y(0.2) = 1.0101, y(0.3) = 1.0228$

4.Given $y' = \frac{1}{2} - x + 2y, y(0) = 1, compute y(0.1), y(0.2), y(0.3)$ by the fourth order R-K method and y(0.4) by Adam's method.

5.Compute y for x=0.1(0.1)0.3 by the fourth order R-K method and y(0.4) by Adam's method ,if
$\frac{dy}{dx} = 1 - x + 4y, y(0) = 1$

Question Bank

Part A

1.State Modified Euler algorithm to solve y'= ,y(x_0)=y) at x=x_0 +h

Sol:

– –

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2. State the disadvantage of Taylor series method.

***Sol*:**

In the differential equation $f(x, y)$,— $= f(x, y)$, the function $f(x, y)$,may have a complicated algebraical structure. Then the evaluation of higher order derivatives may become tedious. This is the demerit of this method.

3. Write the merits and demerits of the Taylor method of solution.

Sol:

The method gives a straight forward adaptation of classic to develop the solution as an infinite series. It is a powerful single step method if we are able to find the successive derivatives easily. If $f(x.y)$ involves some complicated algebraic structures then the calculation of higher derivatives becomes tedious and the method fails.This is the major drawback of this method. However the method will be very useful for finding the starting values for powerful methods like Runge - Kutta method, Milne's method etc.

4.Which is better Taylor's method or R. K. Method?(or) State the special advantage of Runge-Kutta method over taylor series method

Sol:

R.K Methods do not require prior calculation of higher derivatives of $y(x)$,as the Taylor method does. Since the differential equations using in applications are often complicated, the calculation of derivatives may be difficult.

Also the R.K formulas involve the computation of $f(x, y)$ at various positions, instead of derivatives and this function occurs in the given equation.

5.Compare Runge-Kutta methods and predictor – corrector methods for solution of initial value problem.

Sol:Runge-Kutta methods

1.Runge-methods are self starting,since they do not use information from previously calculated points.

2.As mesne are self starting,an easy change in the step size can be made at any stage.
3.Since these methods require several evaluations of the function $f(x, y)$, they are time consuming.
4.In these methods,it is not possible to get any information about truncation error.
Predictor Corrector methods:

1.These methods require information about prior points and so they are not self starting.
2.In these methods it is not possible to get easily a good estimate of the truncation error.

6. What is a Predictor-collector method of solving a differential equation?

Sol:

Predictor-collector methods are methods which require the values of y at $x_n, x_{n-1}, x_{n-2}, \ldots$ for computing the value of y at . x_{n+1} We first use a formula to find the value of y at . x_{n+1} and this is known as a predictor formula.The value of y so got is improved or corrected by another formula known as corrector formula.

7. State the third order R.K method algorithm to find the numerical solution of the first order differential equation.

Sol:

To solve the differential equation $y' = f(x, y)$ by the third order R.K method, we use the following algorithm.

– —

and –

8.Write Milne's predictor formula and Milne's corrector formula.

Sol:

Milne's predictor formula is

— where).

Milne's corrector formula is

– where).

9.Write down Adams-bashforth Predictor and Adams-bashforth corrector formula.

Sol:

Adams-bashforth predictor formula is

—

Adams –bashforth corrector formula is

— where).

10.By Taylor's series method ,find y(1.1) given $y' = x + y, y(1) = 1$

Ans:1.2053

Part B

1.Using Runge-Kutta method find an approximate value of y for x=0.20 if — =x + given that y=1 when x=0.

2.Given that y" +xy'+y =0,y(0)=1,y'(0)=0 obtain y for x=0,1,0.2,0.3 by Taylor's series method and find the solution for y(0.4) by Milne's method.

3.Obtain y by Taylor series method,given that y'=xy+1,y(0)=1for x=0.1 and 0.2 correct to four decimal places.4.Solve for y(0.1) and z(0.1) from the simultaneous differential equations — =2y +z: — =y-3z; y(0)=0,z(0)=0.5 using Runge-Kutta method of the fourth order.

5.Using Adams method find y(1.4) given y'=x^2(1+y) , y(1)=1, y(1.1)=1.233, y(1.2)=1.548 and y(1.3)=1.979.

6.Using Milne's Predictor-Corrector formula to find y(0.4) given — =———, y(0)=1,y(0.1)=1.06, y(0.2)=1.12 and y(0.3)=1.21.

7.Using Modified Euler's method , find y(4.1) and y(4.2) if 5x— + -2 =0: y(4)=1.

8.Given that — =1+ ; y(0.6)=0.6841, y(0.4)=0.4228, y(0.2)=0.2027, y(0)=0, find y(-0.2) using Milne's method.

9.Given that y'=y-x^2 ; y(0)=1: y(0.2)=1.1218; y(0.4)=1.4682 and y(0.6)= 1.7379 , evaluate y(0.8) by Adam's predictor –Corrector method.10.Solve by Euler's method the following differential equation x=0.1, correct to four decimal places, — =── with initial condition y(0)=1.

Chapter 5

Boundary value Problems in ODE&PDE

5.1 Solution of Boundary value problems in ODE

Introduction

The solution of a differential equation of second order of the form $F(x, y, y', y'') = 0$ contains two arbitrary constants. These constants are determined by means of two conditions. The conditions on y and y' or their combination are prescribed at two different values of x are called ***boundary conditions***.

The differential equation together with the boundary conditions is called a ***boundary value problem***.

In this chapter ,we consider the finite difference method of solving linear boundary value problems of the form.

Finite difference approximations to derivatives

First derivative approximations

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \qquad \text{(Forward difference)}$$

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \qquad \text{(Backward difference)}$$

$$y'(x) = \frac{y(x+h) - y(xh)}{2h} + O(h^2) \qquad \text{(Central difference)}$$

Second derivative approximations

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2) \qquad \text{(Central difference)}$$

Third derivative approximations

$$y^{iii}(x) = \frac{1}{2h^3}[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}]$$

Fourth derivative approximations

$$y^{iv}(x) = \frac{1}{h^4}[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}]$$

Solution of ordinary differential equations of Second order

$$y_i{}' = \frac{y_{i+1} - y_{i-1}}{2h}$$

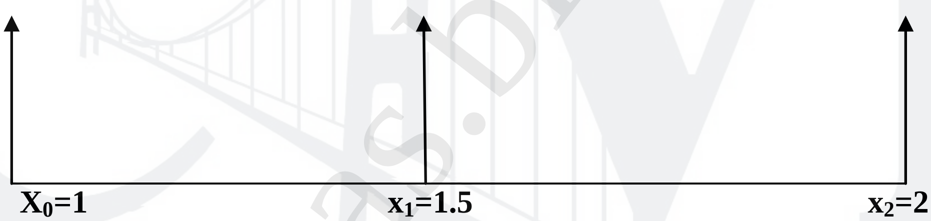
$$y_i{}'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Problems

1. Solve $xy'' + y = 0, y(1) = 1, y(2) = 2$ **with h=0. And h=0.25 by using finite difference method.**

Sol:

(i) Divide the interval[1,2] into two sub-intervals with h=(2-1)/2=0.5



Let $x_{i+1} = x_i + h$,

Here $x_0 = 1, x_1 = 1.5, x_2 = 2$

By Boundary conditions are $y(1) = 1, y(2) = 2$

$\Rightarrow y_0 = 1, y_2 = 2$

We have to determine y_1 for x_1=1.5

Replacing the derivative y'' by $y'' = \frac{y_{i+1} - 2y_i + y_{i+1}}{h^2}$

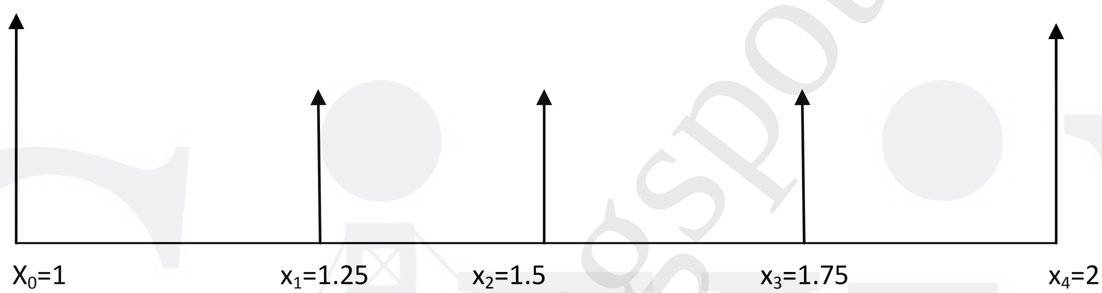
$$\Rightarrow \frac{x_i}{h^2}[y_{i+1} - 2y_i + y_{i+1}] + y_i = 0 \qquad (1)$$

Put i=1 we get $4x_1[y_0 - 2y_1 + y_2] + y_1 = 0$

Using x_1=1.5, $y_0 = 1, y_2 = 2$ we get, $y_1 = \frac{18}{11}$

$\Rightarrow$y(1.5)=1.6364

(ii) With h=0.25 ,we have the number of intervals =4



By boundary condition, $y_0 = 1, y_4 = 2$

We find the unknown from the relation,

$16x_i(y_{i-1} - 2y_i + y_{i+1}) + y_i = 0, i = 1,2,3....$ (2)

Put i=1 in (2), $39y_1 - 20y_2 = 20$

Put i=2 in (2), $28y_1 - 47y_2 + 24y_3 = 0$

Put i=3 in (2), $28y_2 - 55y_3 + 56 = 0$

We have the system of equations ,

$$39y_1 - 20y_2 = 20$$

$$28y_1 - 47y_2 + 24y_3 = 0$$

$$28y_2 - 55y_3 = -56$$

Solution of Gauss-Elimination method

=

The equivalent system is

$$y_1 - 0.513y_2 = 0.513$$
$$-1.445y_2 + y_3 = 0.513$$
$$-1.838y_3 = -3.403$$

By back substitution, the solution is $y_3 = 1.851$, $y_2 = 1.635$, $y_1 = 1.351$

Hence y(1.25)=1.351,y(1.5)=1.635,y(1.75)=1.851

Exercise

1. Solve by finite difference method, the boundary value problem $\frac{d^2y}{dx^2} - y = 0$ with y (0) =0 and y (2) =4.
   Ans: $y_3 = 2.3583$, $y_2 = 1.3061$, $y_1 = 0.5805$
2. Solve $xy'' + y = 0, y'(1) = 0 and y(2) = 1$ with h=0.5

5.2 Solution of Laplace Equation and Poisson equation

Partial differential equations with boundary conditions can be solved in a region by replacing the partial derivative by their finite difference approximations. The finite difference approximations to partial derivatives at a point (x_i,y_i) are given below. The xy-plane is divided into a network of rectangle of lengh $\Delta x = h$ and breadth $\Delta y = k$ by drawing the lines x=ih and y=jk, parallel to x and y axes.The points of intersection of these lines are called grid points or mesh points or lattics points.The grid points (x_i, y_j) is denoted by (i,j) and is surrounded by the neighbouring grid points (i-1,j),(i+1,j),(i,j-1),(i,j+1) etc.,

Note

The most general linear P.D.E of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x, y) \qquad (1)$$

Where A,B,C,D,E,F are in general functions of x and y.

The equation (1) is said to be

- Elliptic if B^2-4AC<0
- Parabolic if B^2-4AC=0
- Hyperbolic if B^2-4AC>0

Solution of Laplace equation $u_{xx}+u_{yy}=0$

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$$

This formula is called *Standard five point formula*

$$u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}]$$

This expression is called *diagonal five point formula.*

Leibmann's Iteration Process

We compute the initial values of $u_1, u_2, ...u_9$ by using standard five point formula and diagonal five point formula .First we compute u_5 by standard five point formula (SFPF).

$$u_5 = \frac{1}{4}[b_7 + b_{15} + b_{11} + b_3]$$

We compute $u_1, u_3, u_7 . u_9$ by using diagonal five point formula (DFPF)

$$u_1 = \frac{1}{4}[b_1 + u_5 + b_3 + b_{15}]$$

$$u_3 = \frac{1}{4}[u_5 + b_5 + b_3 + b_7]$$

$$u_7 = \frac{1}{4}[b_{13} + u_5 + b_{15} + b_{11}]$$

$$u_9 = \frac{1}{4}[b_7 + b_{11} + b_9 + u_5]$$

Finally we compute u_2, u_4, u_6, u_8 by using standard five point formula.

$$u_2 = \frac{1}{4}[u_5 + b_3 + u_1 + u_3]$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + b_{15} + u_7]$$

$$u_6 = \frac{1}{4}[u_3 + u_9 + u_5 + b_7]$$

$$u_8 = \frac{1}{4}[u_7 + b_{11} + u_9 + u_5]$$

The use of Gauss-seidel iteration method to solve the system of equations obtained by finite difference method is called *Leibmann's method.*

Problems

1.Solve the equation $\nabla^2 u = 0$ for the following mesh,with boundary vaues as shown using Leibmann's iteration process.

u_1	u_2	u_3	
u_4	u_5	u_6	
u_7	u_8	u_9	

0 500 1000 500 0

Sol:

Let $u_1, u_2 \ldots\ldots u_9$ be the values of u at the interior mesh points of the given region.By symmetry about the lines AB and the line CD,we observe

$u_1 = u_3$ $\quad$ $u_1 = u_7$

$u_2 = u_8$ $\quad$ $u_4 = u_6$

$u_3 = u_9 \qquad u_7 = u_9$

$u_1 = u_3 = u_7 = u_9, u_2 = u_8, u_4 = u_6$

Hence it is enough to find u_1, u_2, u_4, u_5

Calculation of rough values

u_5=1500

u_1=1125

u_2=1187.5

u_4=1437.5

Gauss-seidel scheme

$$u_1 = \frac{1}{4}[1500 + u_2 + u_4]$$

$$u_2 = \frac{1}{4}[2u_1 + u_5 + 1000]$$

$$u_4 = \frac{1}{4}[2000 + u_5 + u_4]$$

$$u_5 = \frac{1}{4}[2u_2 + 2u_4]$$

The iteration values are tabulated as follows

Iteration No k	u_1	u_2	u_4	u_5
0	**1500**	**1125**	**1187.5**	**1437.5**
1	**1031.25**	**1125**	**1375**	**1250**
2	**1000**	**1062.5**	**1312.5**	**1187.5**
3	**968.75**	**1031.25**	**1281.25**	**1156.25**
4	**953.1**	**1015.3**	**1265.6**	**1140.6**
5	**945.3**	**1007.8**	**1257.8**	**1132.8**
6	**941.4**	**1003.9**	**1253.9**	**1128.9**
7	**939.4**	**1001.9**	**1251.9**	**1126.9**
8	**938.4**	**1000.9**	**1250.9**	**1125.9**
9	**937.9**	**1000.4**	**1250.4**	**1125.4**
10	**937.7**	**1000.2**	**1250.2**	**1125.2**
11	**937.6**	**1000.1**	**1250.1**	**1125.1**
12	**937.6**	**1000.1**	**1250.1**	**1125.1**

$u_1 = u_3 = u_7 = u_9$=937.6, $u_2 = u_8$=1000.1, $u_4 = u_6$=1250.1, $u_5 = 1125.1$

2.When steady state condition prevail,the temperature distribution of the plate is represented by Laplace equation $u_{xx}+u_{yy}=0$.The temperature along the edges of the square plate of side 4 are given by along x=y=0,$u=x^3$ along y=4 and u=16y along x=4,divide the square plate into 16 square meshes of side h=1 ,compute the temperature at all of the 9 interior grid points by Leibmann's iteration process.

Solution of Poisson equation

An equation of the type $\nabla^2 u = f(x,y)$ i.e., is called $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f(x,y)$ poisson's equation where f(x,y) is a function of x and y.

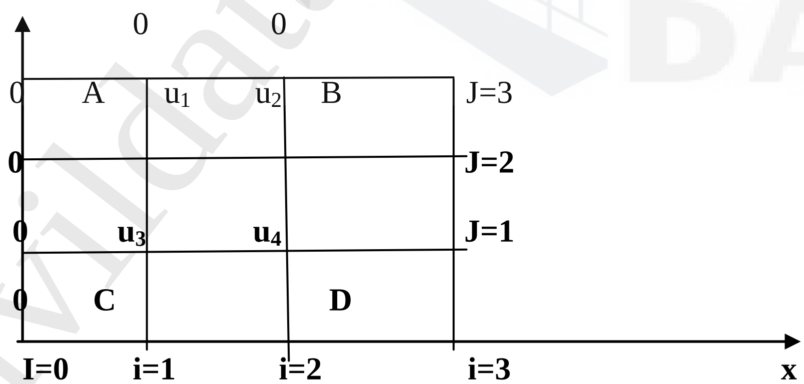
$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}-4u_{i,j}=h^2 f(ih,jh)$$

This expression is called the replacement formula.applying this equation at each internal mesh point ,we get a system of linear equations in u_i,where u_i are the values of u at the internal mesh points.Solving the equations,the values u_i are known.

Problems

1.Solve the poisson equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides x=0,y=0,x=3,y=3 and u=0 on the boundary .assume mesh length h=1 unit.

Sol:



Here the mesh length $\Delta x = h = 1$

Replacement formula at the mesh point (i,j)

$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=-10(i^2+j^2+10) \qquad (1)$$

$$u_2 + u_3 - 4u_1 = -150$$
$$u_1 + u_4 - 4u_3 = -120$$
$$u_2 + u_3 - 4u_4 - 150$$
$$u_1 = u_4 = 75, u_2 = 82.5, u_3 = 67.5$$

2.Solve the poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -81xy, 0 < x < 1; 0 < y < 1$ and

u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshes ,each of length h=1/3.

5.3 Solution of One dimensional heat equation

In this chapter, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit and implicit method

Explicit Method(Bender-Schmidt method

Consider the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ .This equation is an example of parabolic equation.

$$u_{i,j+1} = \lambda u_{i+1,j} + 1 - 2\lambda u_{i,j} + \lambda u_{i-1,j} \qquad (1)$$

Where $\lambda = \frac{k}{ah^2}$

Expression (1) is called the explicit formula and it valid for $0 < \lambda \le \frac{1}{2}$

If λ=1/2 then (1) is reduced into

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (2)$$

This formula is called Bender-Schmidt formula.

Implicit method (Crank-Nicholson method)

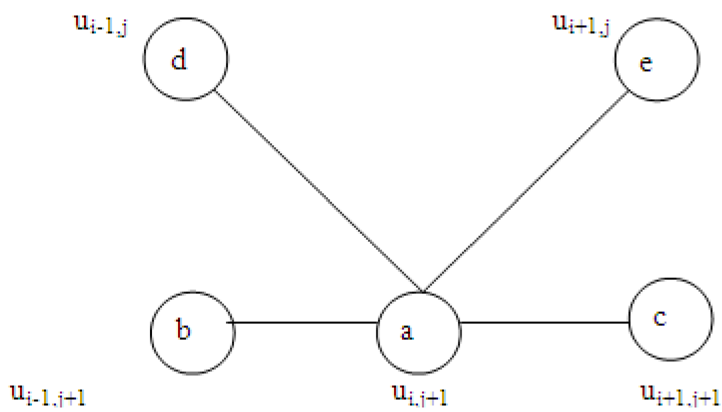
$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

This expression is called Crank-Nicholson's implicit scheme. We note that Crank Nicholson's scheme converges for all values of λ

When λ=1, i.e., $k=ah^2$ the simplest form of the formula is given by

$$\Rightarrow u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

The use of the above simplest scheme is given below.



The value of u at A=Average of the values of u at B, C, D, E

Note

In this scheme, the values of u at a time step are obtained by solving a system of linear equations in the unknowns u_i.

Solved Examples

1.Solve $u_{xx} = 2u_t$ when u(0,t)=0,u(4,t)=0 and with initial condition u(x,0)=x(4-x) upto t=sec assuming $\Delta x = h = 1$

Sol:

By Bender-Schmidt recurrence relation ,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \qquad (1)$$

For applying eqn(1) ,we choose $k = \frac{ah^2}{2}$

Here a=2,h=1.Then k=1

By initial conditions, u(x,0)=x(4-x) ,we have

$u_{i,0} = i(4-i)i = 1,2,3$
$, u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3$

By boundary conditions, u(0,t)=0,u_0=0,u(4,0)=0 $\Rightarrow u_{4,j} = 0 \forall j$

Values of u at t=1

$$u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = 2$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = 3$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = 2$$

The values of u up to t=5 are tabulated below.

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0	0.75	0.5	0

2.Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions u(0,t)=u(5,t)=0 and $u(x,0) = x^2(25 - x^2)$ taking h=1 and k=1/2,tabulate the values of u upto t=4 sec.

Sol:

Here a=1,h=1

For λ=1/2,we must choose k=ah^2/2

K=1/2

By boundary conditions

$$u(0,t) = 0 \Rightarrow u_{0,j} = 0 \forall j$$
$$u(5,t) = 0 \Rightarrow u_{5,j} = 0 \forall j$$
$$u(x,0) = x^2(25 - x^2)$$
$$\Rightarrow u_{i,0} = i^2(25 - i^2), i = 0,1,2,3,4,5$$
$$u_{1,0} = 24, u_{2,0} = 84, u_{3,0} = 144, u_{4,0} = 144, u_{5,0} = 0$$

By Bender-schmidt realtion,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$$

The values of u upto 4 sec are tabulated as follows

j\i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0

5.4 Solution of One dimensional wave equation

Introduction

The one dimensional wave equation is of hyperbolic type. In this chapter, we discuss the finite difference solution of the one dimensional wave equation $u_{tt} = a^2 u_{xx}$.

Explicit method to solve $u_{tt} = a^2 u_{xx}$

$$u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad \textbf{(1)}$$

Where λ=k/h

Formula (1) is the explicit scheme for solving the wave equation.

Problems

1.Solve numerically , $4u_{xx} = u_{tt}$ **with the boundary conditions u(0,t)=0,u(4,t)=0 and the initial conditions** $u_t(x,0) = 0 \ \& \ u(x,0) = x(4-x),$ **taking h=1.Compute u upto t=3sec.**

Sol:

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$$u(0,t) = 0 \Rightarrow u_{0,j} \ \& \ u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$

$$u(x,0) = x(4-x) \Rightarrow u_{i,0} = i(4-i), i = 0,1,2,3,4$$

$$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 3, u_{4,0} = 0$$

$$u_{1,1} = 4 + 0/2 = 2$$

$$u_{2,1} = 3, u_{3,1} = 2$$

The values of u for steps t=1,1.5,2,2.5,3 are calculated using (1) and tabulated below.

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	3	-2	0
4	0	-3	-4	3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0

2.Solve $u_{xx} = u_{tt}$ given u(0,t)=0,u(4,t)=0,u(x,0)= $u(x,0) = \frac{x(4-x)}{2}$ & $u_t(x,0) = 0.$ **Take h=1.Find the solution upto 5 steps in t-direction.**

Sol:

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \; \& \; u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x)/2 \Rightarrow u_{i,0} = i(4-i)/2, i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 1.5, u_{2,0} = 2, u_{3,0} = 1.5, u_{4,0} = 0$$
$$u_{1,1} = 1$$
$$u_{2,1} = 1.5, u_{3,1} = 1$$

The values of u upto t=5 are tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	1.5	2	1.5	0
1	0	1	1.5	1	0
2	0	0	0	0	0
3	0	-1	-1.5	-1	0
4	0	-1.5	-2	-1.5	0
5	0	-1	-1.5	-1	0

Exercise

1.Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$,0<x<1,t>0 given that u(x,0)=100sinπx, $\frac{\partial u}{\partial t}(x,0) = 0, u(0,t) = u(1,t) = 0$

Taking h=1/4 compute u for 4 time steps.

2. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$,0<x<1,t>0 given that u(x,0)=100(x-x^2), $\frac{\partial u}{\partial t}(x,0) = 0, u(0,t) = u(1,t) = 0$

compute u for 4 time steps taking h=0.25.

Tutorial problems

Tutorial-1

1.Solve the boundary value problem $y''-64y+10=0$ with y(0)=y(1)=0 by the finite difference method.

Ans:0.129,0.147

2.Determine the values of y at the pivotal pointa of the interval (0,1) if y satisfies the boundary value problem $y^{iv}+81y=81x^2, y(0)=y(1)=y''(0)=y''(1)=0$ take n=3.

Ans:0.1222,0.1556

3.Write down the finite difference analogue of the equation $u_{xx}+u_{yy}=0$ solve it for the region bounded by the squre $0\le x\le 4, 0\le y\le 4$ and the boundary conditions being given as,u=0 at x=0;u=8+2y at x=4 ; $u=x^2/2$ at y=0; $u=x^2$ at y=4.with h=k=1,use Gauss-seidel method to compute the values at the initial at the internal mesh points.

Ans:2,4.9,9,2.1,4.7,8.1,1.6,3.7,6.6

4.Solve the poisson equation $u_{xx}+u_{yy}=-81xy, 0<x<1; 0<y<1$ and u(0,y)=u(x,0)=0,u(x,1)=u(1,y)=100 with the square meshs,each of length h=1/3

Ans:51.08,76.54,25.79

5.Solve by Crank- Nicholson's implicit method, $u_t=u_{xx}, 0<x<1, t>0$ with $u(x,0)=100(x-x^2), u(0,t)=0, u(1,t)=0$. compute u for one time step with h=0.25.

Ans:9.82,14.29

Tutorial -2

1.Solve the boundary value problem $y''-64y+10=0$ with y(0)=y(1)=0by the finite difference method.

Ans: ***y***(0.5)=0.147,y(0.25)=y(0.75)=0.129

2.Solve $xy''+y=0, y'(1)=0, y(2)=1$ with h=0.5

Ans: y (0)=1.6552 and y(0.5)=1.4483

3.When steady condition prevail,the temperature distribution of the plate is representd by Laplce equation $u_{xx}+u_{yy}=0$ .The temperature along the edges of the square plate of side 4 are given by u=0 along x=y=0;u=x^3 along y=4 and u=16y alon x=4.Divide the square plate into 16 square meshes of side h=1,compute the temperatures at all of the 9 interior grid points by Leibmann's iteration process.

Ans: $u_1 = 4.5, u_2 = 12.5, u_3 = 26.9, u_4 = 4.4, u_5 = 10.7, u_6 = 20.0, u_7 = 2.6, u_8 = 6, u_9 = 10.5$

4.Solve the Poisson's equation $\nabla^2 u = 8x^2 y^2$ inside a square region bounded by the lines $x = \pm 2, y = \pm 2, u = 0$ on the boundary .Assume the origin at the centre of the square and divide the square into 16 equal parts.

Ans: $$\begin{aligned} u_1 &= u_3 = u_7 = u_9 = -3 \\ u_2 &= u_4 = u_6 = u_8 = u_5 = -2 \end{aligned}$$

5.Solve the Laplace equation $\nabla^2 u = 0$ inside the square region bounded by the lines x=0,x=4,y=0,and y=4 given that $u = x^2 y^2$ on the boundary..

Ans: $u_1 = 22, u_2 = 55.5, u_3 = 99.7, u_4 = 16.6, u_5 = 36, u_6 = 55, u_7 = 8.3, u_8 = 16.6, u_9 = 22$

<u>Tutorial -3</u>

1.Find the values of the function u(x,y) satisfying the differential equation $u_t = 4u_{xx}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x - \frac{x^2}{2}$ for points x=0,1,2,3,4,5,6,7,8, $t = \frac{j}{2}$,j=0,1,2,3,4,5.

2. Solve by the Crank-Nicholson's method, $\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0,$ given that $u(x,0) = 100\sin \pi x, u(0,t) = u(1,t) = 0.$ Compute u for one time step, taking h=1/4.

Ans: $u(\frac{1}{4},1) = 38.67; u(\frac{1}{2},1) = 54.69, u(\frac{3}{4},1) = 38.67$

3. Using Crank-Nicholson's method solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$,subject to $u(x,0) = 0, u(0,t) = 0, u(1,t) = 16t.$ taking h=1/4 and k=1/8

Ans: $u(\frac{1}{4},\frac{1}{8}) = 0.0952; u(\frac{1}{2},\frac{1}{8}) = 0.2856, u(\frac{3}{4},\frac{1}{8}) = 0.7619.$

4.339,5.341

Question Bank

Part A

1.What is the error for solving Laplace and Poisson's equations by finite difference method?

Sol:

The error in replacing — by the difference expression is of the order . Since h=k, the error in replaing — by the difference expression is of the order .

2. Define a difference quotient.

Sol:

A difference quotient is the quotient obtained by dividing the difference between two values of a function by the difference between two corresponding values of the independent variable.

3. Why is Crank Nicholson's scheme called an implicit scheme?

Sol:

The Schematic representation of crank Nicholson method is shown below.
The solution value at any point (i,j+1) on the $(j+1)^{th}$ level is dependent on the solution values at the neighboring points on the same level and on three values on the j^{th} level. Hence it is an implicit method.

4. What are the methods to solve second order boundary-value problems?

Sol:

(i)Finite difference method
(ii)Shooting method.

5. What is the classification of one dimensional heat flow equation.

Sol:

One dimensional heat flow equation is — ——

Here A=1,B=0,C=0
$B^2 - 4AC = 0$
Hence the one dimensional heat flow equation is parabolic.

6. **State Schmidt's explicit formula for solving heat flow equation**

Sol:

+(1-2) if − , =− .

7. Write an explicit formula to solve numerically the heat equation (parabolic equation)

Sol:

+(1-2) Where — (h is the space for the variable x and k is the space in the time direction).
The above formula is a relation between the function values at the two levels j+1 and j and is called a two level formula. The solution value at any point (i,j+1) on the $(j+1)^{th}$ level is expressed in terms of the solution values at the points (i-1,j),(i,j) and (i+1,j) on the j th level.Such a method is called explicit formula. the formula is geometrically represented below.

8. **State the condition for the equation to be (i) elliptic,(ii)parabolic(iii)hyperbolic when A,B,C are functions of** x **and** y

Sol:

The equation is elliptic if $(2B^2) - 4AC < 0$
(i.e) $B^2 - AC < 0$. It is parabolic if $B^2 - AC = 0$ and hyperbolic if $B^2 - 4AC > 0$

9. **Write a note on the stability and convergence of the solution of the difference equation corresponding to the hyperbolic equation** .

Sol:

For ,$\lambda = -$the solution of the difference equation is stable and coincides with the solution of the differential equation. For $\lambda >-$,the solution is unstable.

For $\lambda < -$,the solution is stable but not convergent.

10. **State the explicit scheme formula for the solution of the wave equation.**

Sol:

The formula to solve numerically the wave equation =0 is

The schematic representation is shown below.
The solution value at any point (i,j+1) on the $(j+1)^{th}$ level is expressed in terms of solution values on the previous j and (j-1) levels (and not interms of values on the same level).Hence this is an explicit difference formula.

Part B

1.Reduce the following elliptic partial differential equations using orthogonal collocation in both the x and y directions:

—+—=f(x,y);0 x,y 1, X=0; T= ; X=1; T= ; y=0; T= ; y=1;— =0.

2. Solve — =— , subject to u(0,t)=u(1,t)=0 and u(x,0)=sin , 0<x<1, using Bender-Schmidt method.

3.Obtain the finite difference from the first order partial differential equation

— + — =0: - <x < , T(0,x)=T*(x).

4. Solve $U_{xx} = U_{tt}$ with boundary condition u(0,t) = u(4,t) and the initial condition $u_t(x,0) = 0$, u(x,0)=x(4-x) taking h =1, k = ½ (solve one period)
.

5. Solve $y_{tt} = 4y_{xx}$ subject to the condition y(0,t) =0, y(2,t)=0, y(x,0) = x(2-x),

— (x, 0)= 0. Do 4steps and find the values upto 2 decimal accuracy.

6.By iteration method solve the elliptic equation $u_{xx}+u_{yy}=0$ over the square region of side 4,satisfying the boundary conditions.

(i) u(0,y)=0,0 y 4

(ii) u(4,y)=8+2y, 0 y 4

(iii) u(x,0)=— , 0 x 4

(iv) $u(x,4)=x^2$, 0 x 4.

7.Using Crank-Nicolson's scheme, solve 16— =— , 0 x 1,t>0 subject to u(x,0)=0, u(0,t)=0, u(1,t)=100t. Compare u for one step in t direction taking h=1/4.

8.Solve $u_{tt}=u_{xx}$; 0<x<2, t>0 subject to u(x,0)=0,$u_t(x,0)=100(2x-x^2)$, u(0,t)=0,u(2,t)=0,choosing h=1/2 compute u for four time steps.

9.Solve by Bender Schmidt formula upto t=5 for the equation $u_{xx}=u_t$, subject to

u(0,t)=0, u(5,t)=0 and $u(x,0)=x^2(25-x^2)$, taking h=1.

10. Find the pivotal values of the equation – — =— with given conditions u(0,t)=0=u(4,t), u(x,0)=xu(4-x) and — by taking h=1 for 4 time steps.

11. Obtain the Crank-Nicholson finite difference method by taking — = 1. Hence find u(x ,t) in the rod for two time steps for the heat equation — — , given u (x ,0) =sin , u(0, t) =0, u(1,t,)= 0. Taking h= 0.2.

12. .Solve $8x^2y^2$ for square mesh given u= 0 on the four boundaries

dividing the square into 16 sub-squares of length 1 unit.

R3421

B.E/B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2007

Sixth Semester

Computer Science and Engineering

MA 1011/MA 1251 - NUMERICAL METHODS

(Common to Chemical Engineering, Information Technology, Electronics and Communication Engineering and Mechanical Engineering)

(Regulation 2004)

Time : Three hours **Maximum : 100 marks**

Answer ALL Questions

Part – A (10 x 2 = 20 marks)

1. On what type of equations Newton's method can be applicable?

2. By Gauss elimination method solve $x + y = 2$ and $2x + 3y = 5$.

3. What is the nature of n^{th} divided differences of a polynomial of n^{th} degree?

4. Find the second divided differences with arguments a, b, c if $f(x) = \frac{1}{3}$.

5. State Simpson's $\frac{1}{3}^{rd}$ rule formula to evaluate $\int_0^b f(x)dx$.

6. Write down the formula to calculate errors in quadrature formulae.

7. Explain the terms initial and boundary value problems.

8. Using Euler's method find y(0.2) given that $y' = x + y$, $y(0) = 1$.

9. Write down the diagonal five point formula in Laplace equation.

10. What is the order of error in solving Laplace and Poisson's equations by using finite difference method?

Part – B (5 x 16 = 80 marks)

11. a) (i) Use Newton's method to find the real root of $3x - \cos x - 1 = 0$. (8)

(ii) Apply Gauss Jordan method to solve the equations $x + y + z = 9$, $2x - 3y + 4z = 13$, $3x + 4y + 5z = 40$. (8)

(or)

b) i) Solve by Jacobi iteration method correct to two decimal places.

$10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$. (8)

(ii) Obtain by power method the numerically largest eigenvalue of the matrix

$$\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$ (8)

12. (a) (i) Using Newton's divided difference interpolation, find the polynomial of the given data

x:	-1	1	0	3
f:	2	1	0	-1

(ii) Find the cubic spline interpolation. (8)

X:	1	2	3	4	5
f:	1	0	1	0	1

(or)

(b) (i) From the given table, the value of y are consecutive terms of a series of which 23.6 is the 6^{th} term.

Find the first and tenth terms of the series. (8)

x:	3	4	5	6	7	8	9
y:	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(ii) Given the values. (8)

x:	5	7	11	13	17
f(x):	150	392	1452	2366	5202

13. (a) (i) Using the given data find f'(5) (8)

X:	0	2	3	4	7	9
f(x):	4	26	58	112	466	922

(ii) Evaluate $I = \int_0^1 \frac{dx}{1+x}$ by two and three point Gaussian formulae. (8)

(or)

(b) (i) use Romberg's method to compute $\int_0^1 \frac{1}{1+x^2} dx$ correct to 4 decimal places by taking h = 0.5, 0.25 and 0.125. (8)

(ii) Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{1+x+y}$ by using Trapezoidal rule, with step sizes h = k = 0.5. (8)

14. (a) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given that y(0) =1 at x = 0.2 and x = 0.4. (16)

(or)

(b) Given $y' = x(x^2 + y^2)e^{-x}$, y(0) = 1 find y at x = 0.1, 0.2 and 0.3 by Taylor's series method and compute y(0.4) by Milne's method. (16)

15. (a) Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides x = 0, y = 0, x = 3, y = 3 with u = 0 on the boundary and mesh length 1 unit. (16)

(b) Solve by Crank-Nicholson method the equation $u_{xx} = u_t$ subject to u(x, 0) = 0, u (0, t) and u(1, t) = t for two time steps.

C 3276

B.E/B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2007

Sixth Semester

(Regulation 2004)

Computer Science and Engineering

MA 1251/MA 1011 - NUMERICAL METHODS

(Common to Chemical Engineering/Information Technology/Electronics and Communication Engineering/Mechanical Engineering)

Time : Three hours **Maximum : 100 marks**

Answer ALL Questions

Part – A (10 x 2 = 20 marks)

1. State the formula for the method of false position to determine a root of f(x) = 0.

2. State the sufficient condition on $\phi(x)$ for the convergence of an iterative method for f(x) = 0 written as $x = \phi(x)$.

3. Show that the divided differences are symmetrical in their arguments.

4. State Newton's backward difference interpolation formula.

5. State Romberg's method integration formula to find the value of $I = \int_a^b f(x)dy$, using h and $\frac{h}{2}$.

6. Write down the Simpson's $-\frac{3}{8}$ rule of integration given (n + 1) data.

7. State the Taylor series formula to find $y(x_1)$ for solving $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

8. State Milne's predictor-corrector formula.

9. Name at least two numerical methods that are used to solve one dimensional diffusion equation.

10. Write down Leplace equation and its finite difference analogue and the standard five-point formula.

Part – B (5 x 16 = 80 marks)

11. a) (i) Prove the quadratic convergence of Newton-Raphson method. Find a positive root of $f(x) = x^3 - 5x + 3 = 0$, using this method.

(ii) Solve the following system by Gauss-Seidal method:

$28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35.$

(Or)

b) i) Find the inverse of the matrix by Gauss-Jordan method:

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$$

(ii) Find the dominant Eigen value and the corresponding Eigen vector of the matrix

$$A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

12. a) (i) If f(0) = 0, f(1) = 0, f(2) = -12, f(4) = 0, f(5) = 600 and f(7) = 7308, find a polynomial that satisfies that data using Newton's divided difference interpolation formula. Hence, find f(6).

(ii) Given the following table, find f(2.5) using cubic spline functions:

i:	0	1	2	3
x_1:	1	2	3	4
$f(x_i)$	0.5	0.3333	0.25	0.2

(Or)

(b) (i) Find the Lagrange's polynomial of degree 3 to fit the data:

y(0) = -12, y(1) = 0, y(3) = 6 and y(4) = 12. Hence, find y(2).

(ii) Find a polynomial of degree two for the data by Newton's forward difference method:

x:	0	1	2	3	4	5	6	7
y:	1	2	4	7	11	16	22	29

13. a) (i) Find f'(6) and the maximum value of y = f(x) given the data:

x:	0	2	3	4	7	9
f(x):	4	26	58	112	466	922

(ii) Evalaute $\int_0^1 \frac{dx}{1+x^2}$ using Romberg method by taking h = 0.5, 0.25 and 0.125 successively.

(Or)

b) (i) Evaluate $I = \int_0^1 \frac{dx}{1+x}$ using three point Gauss-quadrature formula.

(ii) Use Trapezoidal rule of evaluate $I = \int_1^2\int_1^2 \frac{dx\,dy}{x+y}$ using 4 subintervals.

14. a) (i) Solve $\frac{dy}{dx} = \log_{10}(x+y)$, y(0) = 2 by Euler's modified method and find the values of y(0.2),

(ii) Given $y''+xy'+y=0$, $y(0)=1$, $y'(0)=0$, find the value of y(0.1) by Runge-Kutta Method of fourth order.

(Or)

b) (i) Solve $\frac{dy}{dx} = xy + y^2$, y(0) = 1, using Milne's predictor-corrector formulae and find y(0.4). Use Taylor series method to find y(0.1), y(0.2) and y(0.3).

(ii) Compute y(0.2), given $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, y(0) = 1, by Runge-Kutta method of fourth order, taking h = 0.2

15. (a) (i) Solve the boundary value problem $y'' = xy$, subject to the conditions $y(0) + y'(0) = 1$, $y(1) = 1$, taking $h = \frac{1}{3}$, by finite difference method.

(ii) Using Bender-Schmitt formula, solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 0$, $u(5,t) = 0$, $u(x, 0) = x^2(25 - x^2)$. Assume $\Delta x = 1$. Find $u(x,t)$ up to $t = 5$.

(Or)

b) (i) Use Crank-Nicholson Scheme to solve $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$, $0 < x < 1$ and $t > 0$ given $u(x, 0) = 0$, $u(0, t) = 0$ and $u(1, t) = 100t$. Compute $u(x, t)$ for one time step, taking $\Delta x = \frac{1}{4}$.

(ii) Evaluate $u(x, t)$ at the pivotal points of the equation $16\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0, t) = 0$, $u(5, t) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x, 0) = x^2(5 - x)$ taking $\Delta x = 1$ and upto $t = 1.25$.

T 3321

B.E/B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2007

Sixth Semester

(Regulation 2004)

Computer Science and Engineering

MA 1011/MA 1251 - NUMERICAL METHODS

(Common to Chemical Engineering, Information Technology, Electronics and Communication Engineering and Mechanical Engineering)

Time : Three hours **Maximum : 100 marks**

Answer ALL Questions

Part – A (10 x 2 = 20 marks)

1. By Newton's method, find an iterative formula to find $\sqrt{N}$ (Where N is a positive number).

2. Why Gauss Seidel iteration is a method of successive corrections?

2. Why Gauss Seidel iteration is a method of successive corrections?

3. Define forward, backward, central differences and divided differences.

4. Evaluate $\Delta^{10}(1-x)(1-2x)(1-3x)\ldots(1-10x)$, by taking h = 1.

5. Show that the divided differences operator ****** is linear.

6. In numerical integration, what should be the number of intervals to apply Simpson's one-third rule and Simpson's three-eights rule.

7. Explain one-step methods and multi step methods.

8. State the Adams-Bashforth predictor-corrector formula.

9. For what points of x and y, the equation $x.f_{xx} + y.f_{yy} = 0$, x > 0, y > 0 is elliptic.

10. Define difference quotient of a function y(x).

Part – B (5 x 16 = 80 marks)

11. (a) (i) Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units; satisfying the following conditions.

u(x, 0) = 3x for $0 \le x \le 4$

u(x, 4) = x^2 for $0 \le x \le 4$

u(0, y) = 0 for $0 \le y \le 4$; u(4, y) = 12 + y for $0 \le y \le 4$. (10)

(ii) Solve $\frac{\partial^2 u}{\partial x^2} - 2.\frac{\partial u}{\partial t} = 0$ given u(0, t) = 0, u(4, t) = 0, u(x, 0) = x(4 – x). Assume h = 1. Find the values of u upto t = 5. (6)

(Or)

(b) Solve $y_{tt} = 4y_{xx}$ subject to the conditions y(0, t) = 0, y(2, t) =, y(x, 0) = x(2 – x), $\frac{\partial y}{\partial x}(x,0) = 0$. Do 4 steps. Find values upto 2 decimal accuracy.

12. (a) (i) Apply Guass-Jordan method to find the solution of the following system.

10x + y + z = 12; 2x + 10y + z = 13; x + y + 5z = 7 (10)

(ii) Solve for a positive root of x – cos x = 0 by Regula Falsi Method. (6)

b) (i) Find the numerically largest eigen value of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector by power method. (10)

(ii) Find the root between 1 and 2 of $2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to five decimal places. (6)

13. a) (i) Using Lagrange's interpolation formula, find y(10) from the following table: (10)

x:	5	6	9	11
y:	12	13	14	16

(ii) Find the sixth term of the sequence 8, 12, 19, 29, 42. (6)

(Or)

$u(x, 0) = 3x$ for $0 \le x \le 4$

$u(x, 4) = x^2$ for $0 \le x \le 4$

$u(0, y) = 0$ for $0 \le y \le 4$; $u(4, y) = 12 + y$ for $0 \le y \le 4$. (10)

(ii) Solve $\frac{\partial^2 u}{\partial x^2} - 2.\frac{\partial u}{\partial t} = 0$ given $u(0, t) = 0$, $u(4, t) = 0$, $u(x, 0) = x(4 - x)$. Assume $h = 1$. Find the values of u upto $t = 5$. (6)

(Or)

(b) Solve $y_{tt} = 4y_{xx}$ subject to the conditions $y(0, t) = 0$, $y(2, t) =$, $y(x, 0) = x(2 - x)$, $\frac{\partial y}{\partial x}(x,0) = 0$. Do 4 steps. Find values upto 2 decimal accuracy.

12. (a) (i) Apply Guass-Jordan method to find the solution of the following system.

$10x + y + z = 12$; $2x + 10y + z = 13$; $x + y + 5z = 7$ (10)

(ii) Solve for a positive root of $x - \cos x = 0$ by Regula Falsi Method. (6)

(Or)

(Or)

b) (i) Find the numerically largest eigen value of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector by power method. (10)

(ii) Find the root between 1 and 2 of $2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to five decimal places. (6)

13. a) (i) Using Lagrange's interpolation formula, find y(10) from the following table: (10)

x:	5	6	9	11
y:	12	13	14	16

(ii) Find the sixth term of the sequence 8, 12, 19, 29, 42. (6)

(Or)

b) (i) From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46. (10)

Age x	45	50	55	60	65
Premium y	114.84	96.16	83.32	74.48	68.48

(ii) Form the divided difference table for the following data: (6)

x:	-2	0	3	5	7	8
y=f(x)	-792	108	-72	48	-144	-252

14. a) (i) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg's method. Hence obtain an approximate value of π.

(ii) A river is 80 meters wide. The depth 'd' in meters at a distance x meters from one bank is given by the following table. Calculate the area of cross-section of the river using Simpson's $\frac{1}{3}$ rule.

(6)

(6)

x:	0	10	20	30	40	50	60	70	80
d:	0	4	7	9	12	15	14	8	3

b) (i) The table given below reveals the velocity v of a body during the time 't' specified. Find its acceleration at t = 1.1. (1)

t:	1.0	1.1	1.2	1.3	1.4
v:	43.1	47.7	52.1	56.4	60.8

(ii) Evaluate the integral of

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ between the limits x = 0 to 0.8 using two point Gaussian quadrature formula. (6)

15. a) (i) Given $y' + xy^2 + y = 0$, y(0) = 1, find the value of y(0.2) by using Runge-Kutta method of fourth order. (10)

(ii) Using Taylor series method, find correct to four decimal places, the value of y(0, 1), given

$\frac{dy}{dx} = x^2 - y^2$ and y(0) = 1. (6)

(Or)

(b) (i) Using Milne's method find y(4.4) given $5xy' + y^2 - 2 = 0$ given y(4) =1; y(4.1) =1.0049; y(4.2) = 1.0097 and y(4.3) = 1.0143. (10)

(ii) Using Euler's method, solve numerically the equation $y' = x + y$, y(0) = 1, for x = 0.0 (0.2) (0.1). Check your answer with the exact solution. (6)