CHAPTER

Recursive Definition:

→ Sometimes it is difficult to define a relation directly but may be easy to define these relation in term of it. This process is called recursion. The most common application is mathematic & computer science, in which it refers to a method of defining function in which the function being defined is applied within its own definition.

Let us suppose the domain of function is set of non-negative numbers then we use two steps to define a function.

(i) Basic step:

Specify the value of function generally for 0 or 1.

(ii) Recursive step:

Specify the rule for finding the value of function using the value of function already found.

Ex 1 Give a recursive definition of a sequence $\{a_n\}: n = 1, 2, 3, 4, \ldots, n$ if $\boxed{a_n = 10^n}$? ← explicit formula

Soln:

Basic Step: for $n = 1$,

$a_1 = 10^1 = 10$

~~$\therefore a_n = 10\, a_{n-1}$~~

Recursive steps: $a_n = 10\, a_{n-1}$

Thus, the recursive definition of sequence $a_n = 10^n$ is ~~$a_n =$~~
$a_n = 10 a_{n-1}$

Ex2 Give a recursive definition the sequence $\{a_n\}; n = 1, 2, 3 \ldots$ if
(i) $a_n = 6n$ (ii) $a_n = 5$ (iii) $a_n = 2n + 1$ (iv) $a_n = n^2$

(i) soln.

Basic step:

for $n = 1$,

$a_1 = 6 \times 1 = 6$

$a_1 = 6, \quad a_2 = 12 = a_1 + 6$

$a_3 = 18 = a_2 + 6$

$a_4 = \ldots\ a_3 + 6$

Recursive steps:

$a_n = 6 + a_{n-1}$

(ii) $a_n = 5$, $a_n = 5$

(iii) ~~$a_n = 2n + 1$~~ $a_n = 2n + 1$

$a_1 = 2 \times 1 + 1 = 3$

$a_2 = 2 \times 2 + 1 = 5 = a + 2$

$a_3 = 2 \times 3 + 1 = 7 = a_2 + 2$

$a_4 = 2 \times 4 + 1 = 9 = a_3 + 2$

$a_n = 2 + a_{n-1}$

iv) $a_n = n^2$

$a_1 = 1^2 = 1$

$a = a^2 = 2^2 = 3 + a$

$a_2 = 3^2 = 9 = 5 + 4$

$a_a = 16 = 7 + 9$

$a_5 = 25 = 9 + 16$

$\therefore$ ~~$a_n = (2n-1) + 2n + 1$~~ $\therefore a_n = (2n-1) + (2n-1)$

* Recurrence Relation:

A recurrence relation for the sequence $\{a_n\}$ is an equation that express a_n is term of 1 or more if the previous term of the sequence namely $a_0, a_1, a_2 \ldots a_j \ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-n e.g. active integer. A sequence is called a solution of recurrence relation if its term satisfy the recurrence relation. A recurrence relation is said to be re currencely defined a sequence.

Ex: let $\{a_n\}$ be a sequence that ~~solution~~ satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$ and suppose $n_0 = 2$ what are the values of a_1, a_2 and a_3?

Soln: Given that, $a_0 = 2$

$$a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8 \ldots$$

$$a_3 = a_{3-1} + 3 = a_2 + 3 = 8 + 3 = 11 \ldots$$

* General form of linear homogeneous recurrence relation:

A linear homogeneous recurrence relation of degree k with constant coefficient has the general form:

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_k a_{n-k}$ —— (I)

If $a_n = r^n$ is the solution of eq^n(I), then it must satisfy eq^n(I) i.e.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

Dividing by r^{n-k} on both sides, we get,

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$$

or, $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ —— (II)

This equation is known as characteristics equation of given reccurence relation and its provides which are used to give an explicit formula for all the solution of recurrence relation (RR).

* Theorem 1: (without proof):

let c_1 and c_2 be real numbers, suppose that ~~$r^2 = c$~~, $r^2 = c_1 r - c_2 = 0$ has two distinct roots r_1 & r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$ where α_1 and α_2 are constant.

Q. Solve the recurrence Relation $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, and $a_1 = 0$

Solution:

Given, recurrence relation is: $a_n = 5a_{n-1} - 6a_{n-2}$ — (1)

i.e. $a_n - 5a_{n-1} + 6a_{n-2} = 0$

∴ The characterstics eqn of given R.R is;

$r^2 - 5r + 6 = 0$

i.e. $r^2 - 3r - 2r + 6 = 0$

or, $r(r-3) - 2(r-3) = 0$

or. $(r-3)(r-2) = 0$

∴ $r = 3$ ∴ $r = 2$

Here, The roots are distinct so the ~~following~~ solution of R.R in the form,

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

i.e. $a_n = \alpha_1 3^n + \alpha_2 2^n$ — (11)

Now using initial value,

$a_0 = 1$

i.e. $\alpha_1 3^0 + \alpha_2 2^0 = 1$

or. $\alpha_1 + \alpha_2 = 1$ — (3)

and

$a_1 = 0$

i.e. $\alpha_1 3^1 + \alpha_2 2^1 = 0$

or. $3\alpha_1 + 2\alpha_2 = 0$ — (4)

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Solving eqn (3) and eqn (4)

$$3\alpha_1 + 3\alpha_2 = 3$$
$$3\alpha_1 + 2\alpha_2 = 0$$
$$- \quad - \quad -$$

$\therefore \alpha_2 = 3$

from eqn (3), $\alpha_1 = 1 - \alpha_2 = 1 - 3 = -2$,

Now,

putting the value of α_1 and α_2 in eqn (2) we get, the solution of given R.R as:

$$\boxed{a_n = -2.3^n + 3.2^n = 3.2^n - 2.3^n}$$

Q. Solve the Recurrence Relation $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$.

Soln.

Given, recurrence relation is: $a_n = 6a_{n-1} - 8a_{n-2}$ — (1)

i.e. $a_n - 6a_{n-1} + 8a_{n-2} = 0$

∴ The characteristics eqn of given R.R is,

$$r^2 - 6r + 8 = 0$$

or, $r^2 - 4r - 2r + 8 = 0$

or, $r(r-4) - 2(r-4) = 0$

or, $(r-2)(r-4) = 0$

∴ $r = 2$, ∴ $r = 4$

Here, the roots are distinct, so the solution of R.R in the form,

~~$a_n = \alpha_1 n' + \alpha_2 n$~~

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

i.e. $a_n = \alpha_1 2^n + \alpha_2 4^n$ ——— (11)

Now using initial value,

$a_0 = 4$

i.e. $\alpha_1 2^0 + \alpha_2 4^0 = 4$

i.e. $\alpha_1 + \alpha_2 = 4$ ——— (3)

and

$a_1 = 10$

i.e. $\alpha_1 2^1 + \alpha_2 4^1 = 10$

i.e. $2\alpha_1 + 4\alpha_2 = 10$ ——— (4)

~~Solving eqn (3) and eqn (4)~~

let, $\alpha_1 = 4 - \alpha_2$

putting value in eqn (4).

$2(4 - \alpha_2) + 4\alpha_2 = 10$

$8 - 2\alpha_2 + 4\alpha_2 = 10$

$8 + 2\alpha_2 = 10$

or, $2\alpha_2 = 2$

$\therefore \alpha_2 = 1$

from eqn (3), $\alpha_1 = 4 - \alpha_2$

$\alpha_1 = 4 - 1$

$\therefore \alpha_1 = 3$

Now,

putting the value of α_1 & α_2 in eqn (1) we get, the solution of given R.R as:

$\boxed{\therefore a_n = 3.2^n + 1.4^n = 3.2^n + 4^n}$

Exam

1.Q. Find the solution of R.R $f_n = f_{n-1} + f_{n-2}$ $n > 2$ and $f_0 = 0$, $f_1 = 1$, OR

- Find the explicit formula for fibonacci sequence.

2.Q. Solve the R.R $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

Q.No.1) Ans ⇒

Given, Recursive Recurrence Relation is:

~~$a_n = 5a$~~ $f_n = f_{n-1} + f_{n-2}$ —① i.e. $f_n - f_{n-1} - f_{n-2} = 0$ —②

The characteristics of given R.R is,

$r^2 - r - 1 = 0$

Hence, using quadratic equation,

~~$r = -(-1) \pm$~~

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1(-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

when,

taking positive value,

$\therefore r_1 = \frac{1+\sqrt{5}}{2}$

AND, Taking negative value,

$r_2 = \frac{1-\sqrt{5}}{2}$

∴ Here, ~~are~~ The roots are the distinct, so the solution of R.R in the form,

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

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i.e. $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ — (ii)

Now using initial value,

$f_0 = 0$

i.e. $\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$

or $\alpha_1 + \alpha_2 = 0$ — (iii)

i.e. $\alpha_1 = -\alpha_2$

AND,

$f_1 = 1$

i.e. $\alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

or $\left(\frac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_2 = 1$ — (iv)

~~let~~

putting value of α_1 in eqn (iv), we get

$\left(\frac{1+\sqrt{5}}{2}\right)(-\alpha_2) + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_2 = 1$

or, ~~$-\alpha_2 + $~~

or, $\frac{-\alpha_2 - \alpha_2\sqrt{5}}{2} + \frac{\alpha_2 - \alpha_2\sqrt{5}}{2} = 1$

or, $\frac{-\cancel{\alpha_2} - \alpha_2\sqrt{5} + \cancel{\alpha_2} - \alpha_2\sqrt{5}}{2} = 1$

or, $\frac{-2\alpha_2\sqrt{5}}{2} = 1$

or, $-\alpha_2\sqrt{5} = 1$

$\boxed{\text{i.e. } \alpha_2 = -\frac{1}{\sqrt{5}}}$

Again, $\alpha_1 = -\left(-\frac{1}{\sqrt{5}}\right)$

$$\therefore \alpha_1 = \frac{1}{\sqrt{5}}$$

Now, putting the value of α_1 & α_2 in eqn (2) we get the solution of given R.R as:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$\boxed{\therefore f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

* Find the explicit formula for fibonacci sequence.

Ans:

~~Given~~ According to question,

fibonacci sequence R.R relation is,

$$f_n = f_{n-1} + f_{n-2} \quad \text{—— (1)}$$

0	1	1	2, 3, 5
f_{n-2}	f_{n-1}	f_n	

i.e. $f_n - f_{n-1} - f_{n-2} = 0$

characteristic eqn of R.R is;

$$r^2 - r - 1 = 0$$

Note: solution of this question is same as last solved question.

Q. Solve the R.R $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution: The given R.R Relation is;

$$a_n = a_{n-1} + 6a_{n-2} \quad \text{—— (1)}$$

i.e; $a_n - a_{n-1} - 6a_{n-2} = 0$

The characteristics equation of R.R is,

$$r^2 - r - 6 = 0$$

or, $r^2 - 3r + 2r - 6 = 0$

or, $r(r-3) + 2(r-3) = 0$

or, $(r+2) + (r-3) = 0$

i.e, $r_1 = -2$, $r_2 = 3$

Here, the roots are distinct, so the solution of R.R in the form,

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

i.e, $a_n = \alpha_1 (-2)^n + \alpha_2 \cdot 3^n$ —— (11)

Now, using initial value,

$a_0 = 3$

$\alpha_1 \cdot (-2)^0 + \alpha_2 \cdot 3^0 = 3$

or, $\alpha_1 + \alpha_2 = 3$

or, $\alpha_1 = 3 - \alpha_2$ —— (3)

AND,

$a_1 = 6$

$\alpha_1 (-2)^1 + \alpha_2 (3)^1 = 6$

i.e. $-2\alpha_1 + 3\alpha_2 = 6$ —— (4)

Now,

putting α_1 in eqn (4) we get,

$$-2(3 - \alpha_2) + 3\alpha_2 = 6$$

Theorem 2 (without proof):

let c_1 and c_2 be real numbers with $c_2 \neq 0$. suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, 3, \dots$ where α_1 & α_2 are constants.

Q. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 1$ and $a_1 = 6$.

Given Recurrence Relation is. $a_n = 6a_{n-1} - 9a_{n-2}$ — (i)

i.e. $a_n - 6a_{n-1} + 9a_{n-2} = 0$

The characterstic of given R.R is,

$r^2 - 6r + 9 = 0$

or, $r^2 - 3r - 3r + 9 = 0$

or, $r(r-3) - 3(r-3) = 0$

or, $(r-3)(r-3) = 0$

we get,

$\therefore r_0 = 3 \qquad \therefore r_0 = 3$

Here, The roots are Same. And the solution of R.R in the form,

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$

i.e. $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$

i.e. $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ —— (ii)

Now, using initial value,

$a_0 = 1$

$\alpha_1 . 3^0 + \alpha_2 . 0 . 3^0 = 1$ i.e. $\alpha_1 = 1$ —— (iii)

AND,

$a_1 = 6$

$\alpha_1 . 3^1 + \alpha_2 . 1 . 3^1 = 6$ ——— (iv)

i.e. $3\alpha_1 + 3\alpha_2 = 6$

Now, putting value of α_1 in eqn (iv)

$3 \times (1) + 3\alpha_2 = 6$

$3 + 3\alpha_2 = 6$

$3\alpha_2 = 3$

$\therefore \alpha_2 = 1$

We get,

$\therefore \alpha_1 = 1$ & $\therefore \alpha_2 = 1$

Now,

putting the value of α_1 & α_2 in eqn (2) we get the solution of R.R is.

$\therefore a_n = 1.3^n + 1.n.3^n$

$\boxed{\text{i.e. } \therefore a_n = 3^n + n.3^n}$

Q.NO.2. $a_n = -6a_{n-1} - 9a_{n-2}$; $a_0 = 5$, $a_1 = -1$

Given,

R.R relation is, $a_n = -6a_{n-1} - 9a_{n-2}$ ——— (1)

i.e. $a_n + 6a_{n-1} + 9a_{n-2} = 0$

The characteristic of given R.R. is,

$r^2 + 6r + 9 = 0$

Q. $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$, $a_1 = 7$

Q. $a_n = 2a_{n-1} + 8a_{n-2}$, $a_0 = 4$, $a_1 = 10$

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or, $r^2 + 3r + 3r + 9 = 0$

or, $r(r+3) + 3(r+3) = 0$

or, $(r+3)(r+3) = 0$

we get,

$\therefore r_0 = -3$ & $\therefore r_0 = -3$

Here, The roots are same and the solution of R.R in the form,

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

i.e. $a_n = \alpha_1(-3)^n + \alpha_2 . n(-3)^n$ ——— (ii)

Now, using initial value,

$a_0 = 5$

or, $\alpha_1 (-3)^0 + \alpha_2 . 0.(-3)^0 = 5$

$\therefore \alpha_1 = 5$ ——— (iii)

AND,

$a_1 = -1$

or, $\alpha_1 . (-3)^1 + \alpha_2 . (-1) . (-3)^1 = 5$

or, $-3\alpha_1 + 3\alpha_2 = 5$ ——— (iv)

Here,

putting the value of α_1 in eqn (iv) we get,

$-3(5) + 3\alpha_2 = 5$

or, $-15 + 3\alpha_2 = 5$

or, $3\alpha_2 = 20$

$\therefore \alpha_2 = 20/3$

we get,

$\therefore \alpha_1 = 5$ & $\therefore \alpha_2 = 20/3$

Now,

putting the value of α_1 & α_2 in eqn (2) we get, the solution of R.R is.

$$\boxed{\therefore a_n = 5\cdot(-3)^n + \frac{20}{3}\cdot n\cdot(-3)^n}$$

Q. $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2, a_1 = 7$

Sol:

Given. R.R equation is: $a_n = a_{n-1} + 2a_{n-2}$ —— (i)

i.e. $a_n - a_{n-1} - 2a_{n-2} = 0$

The characteristics of R.R equation is:

$r^2 - r - 2 = 0$

or, $r^2 - 2r + r - 2 = 0$

or, $r(r-2) + 1(r-2) = 0$

or, $(r+1)(r-2) = 0$

$\therefore$ we get,

$r_1 = -1 \quad ; r_2 = 2$

Here, The roots are distinct, so the solution of R.R in the form,

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

i.e. $a_n = \alpha_1(-1)^n + \alpha_2\cdot(2)^n$ —— (ii)

Now, using initial value,

$a_0 = 2$

$\alpha_1(-1)^0 + \alpha_2(2)^0 = 2$

$\therefore \alpha_1 + \alpha_2 = 2$ —— (iii)

Then,

~~$a_1 = 7$~~

~~$\alpha_1 7^1 + \alpha_2 \cdot 7^1$~~

~~$\alpha_1(-1)^1 + \alpha_2(2)^1$~~

$a_1 = 7$

$\alpha_1\cdot(-1)^1 + \alpha_2\cdot(2)^1 = 7$

$-\alpha_1 + 2\alpha_2 = 7$ —— (iv)

~~putt~~ let, $\alpha_1 = 2 - \alpha_2$

putting the value of α_1 in eqn (iii), we get,

$$-(2-\alpha_2) + 2\alpha_2 = 7$$

$$-2 + \alpha_2 + 2\alpha_2 = 7$$

$$3\alpha_2 = 5$$

$$\therefore \alpha_2 = 5/2$$

Again, Taking the value of α_1,

$$\alpha_1 = 2 - 5/2$$

$$\therefore \alpha_1 = -1/2$$

we get,

$\therefore \alpha_1 = -1/2$ & $\therefore \alpha_2 = 5/2$

Now,

putting the value of α_1 & α_2 in eqn (ii), we get the solution of R.R is,

$$\therefore a_n = \left(-\frac{1}{2}\right)(-1)^n + \left(\frac{5}{2}\right) \cdot 2^n$$

$$\boxed{\therefore a_n = \left(-\frac{1}{2}\right) \cdot (-1)^n + \left(\frac{5}{2}\right) \cdot 2^n}$$

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Theorem 3 (without proof):

let, $c_1, c_2, \ldots c_k$ be real number, suppose that $r^k - c_1 r^{k-1} - c_k = 0$ has k distinct roots $r_1, r_2, \ldots r_k$. Then the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$

where $\alpha_1, \alpha_2, \ldots \alpha_k$ are distinct.

Theorem 4 (without proof):

If a recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$ has two similar roots r_1 and one distinct root r_2 then the solution of such recurrence relation is given by $a_n = \alpha_1 r_1^n + \alpha_2 n r_2^n + \alpha_3 r_2^n$, where $\alpha_1, \alpha_2, \alpha_3$ are constants.

Q. Find the solution of recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ for $n \geq 3$, $a_0 = 2$, $a_1 = 5$, $a_2 = 5$

Solution:

Recurrence relation eq^n is : $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

i.e. $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ ——(1)

The characteristics equation is,

$$r^3 - 6r^2 + 11r - 6 = 0$$

or, ~~$r^2(r-2)$~~

or, $r^2(r-1) - 5r(r-1) + 6(r-1) = 0$

or, $(r-1)(r^2 - 5r + 6) = 0$

or, $(r-1)(r-2)(r-3) = 0$

$\therefore r_1 = 1, r_2 = 2, r_3 = 3$

Note: we try to assume value 1 in r. let, we assume 1 and 1 is statisfy with 0=0 so we take $(r-1)$

$3\alpha_2 + 3 - \alpha_2 = 3$

Q. Find the solution of recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ for $n \geq 3$, $a_0 = 2$, $a_1 = 5$, $a_2 = 5$

Sol:- The Recurrence relation is $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ — (1)

i.e. $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

The characteristics eqn of the given RR is,

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$r^2(r-1) - 5(r-1) + 6(r-1) = 0$$

$$(r-1)(r^2 - 5r + 6) = 0$$

$$(r-1)(r^2 - 3r - 2r + 6) = 0$$

$$(r-1)\{r(r-3) - 2(r-3)\} = 0$$

$$(r-1)(r-3)(r-2) = 0$$

$\therefore r_1 = 1 \quad \therefore r_2 = 3 \quad \therefore r_3 = 2$

Since all three roots are different so the solution of given R.R. is:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

$$a_n = \alpha_1 (1)^n + \alpha_2 (3)^n + \alpha_3 (2)^n \quad \text{—— (2)}$$

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Q. Solve the following R.R.

① $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, for $n \geq 3$, $a_0 = 3, a_1 = 6$ & $a_2 = 0$

② $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$, for $n \geq 3$, $a_0 = 1$, $a_1 = 9$ & $a_2 = 15$

③ $a_n = -3a_{n-1} - 3a_{n-3}$, $a_0 = -5$, $a_1 = 4$, $a_2 = 15$.

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using initial value $a_0 = 2$

$\alpha_1(1)^0 + \alpha_2(3)^0 + \alpha_3(2)^0 = 2$

$\alpha_1 + \alpha_2 + \alpha_3 = 2$ — ③

also, $a_1 = 5$

$\alpha_1(1)^1 + \alpha_2(3)^1 + \alpha_3(2)^1 = 5$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 5$ — ④

And, $a_2 = 5$

$\alpha_1(1)^2 + \alpha_2(3)^2 + \alpha_3(2)^2 = 5$

$\alpha_1 + 9\alpha_2 + 4\alpha_3 = 5$ — ⑤

Eqn ③ and ⑤ we get,

$\alpha_1 + \alpha_2 + \alpha_3 = 2$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 5$

$-2\alpha_2 - \alpha_3 = -3$ — ⑥

eqn ④ and ⑤

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 5$

$\alpha_1 + 9\alpha_2 + 4\alpha_3 = 5$

$-6\alpha_2 - 2\alpha_3 = 0$ — ⑦

solving ⑥ & ⑦

$-6\alpha_2 - 3\alpha_3 = -9$

$-6\alpha_2 - 2\alpha_3 = 0$

$-\alpha_3 = -9$

$\boxed{\therefore \alpha_3 = 9}$

$-6\alpha_2 = 2\alpha_3$

$-6\alpha_2 = 2 \times 9$

$\alpha_2 = -18/6$

$\boxed{\therefore \alpha_2 = -3}$

$\alpha_1 + \alpha_2 + \alpha_3 = 2$

$9 - 3 + \alpha_1 = 2$

$\boxed{\alpha_1 = -4}$

putting the value of α_1, α_2 and α_3 in eqn (2) we get,

$a_n = -4(1)^n + (-3)(3)^n + 9(2)^n$

$= 9(2)^n - 4(1)^n - 3(3)^n$

Q. Solve the following R.R.

(1) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$. $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

Soln:. The given Recurrence Relation is $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ — (1)

i.e. $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$

The characteristics eqn of given RR is in form.

$r^3 - 2r^2 - r + 2 = 0$

$r^2(r-1) - r(r-1) - 2(r-1) = 0$

$(r-1)(r^2 - r - 2) = 0$

$(r-1)(r^2 - 2r + r - 2) = 0$

$(r-1)\{r(r-2) + 1(r-2)\} = 0$

$(r-1)(r-2)(r+1) = 0$

$\therefore r = 1 \quad r_2 = 2 \quad r_3 = -1$

Since all 3 roots are different ∴ So the solution of given R.R is in.

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$

$a_n = \alpha_1 (1)^n + \alpha_2 (2)^n + \alpha_3 (-1)^n$ —— (2)

g initial value $a_0 = 3$

$\alpha_1 (1)^0 + \alpha_2 (2)^0 + \alpha_3 (-1)^0 = 3$

$\alpha_1 + \alpha_2 + \alpha_3 = 3$ —— (3)

Also, $a_1 = 6$

$\alpha_1 (1)^1 + \alpha_2 (2)^1 + \alpha_3 (-1)^1 = 6$

$\alpha_1 + 2\alpha_2 - \alpha_3 = 6$ —— (4)

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Also, $a_2 = 0$

$\alpha_1(1)^2 + \alpha_2(2)^2 + \alpha_3(-1)^2 = 0$

$\alpha_1 + 4\alpha_2 + \alpha_3 = 0$ — (5)

solving (3) & (4)

$\alpha_1 + \alpha_2 + \alpha_3 = 3$

$\alpha_1 + 2\alpha_2 - \alpha_3 = 6$

$-\alpha_2 + 2\alpha_3 = -3$ — (6)

solving (4) & (5)

$\alpha_1 + 2\alpha_2 - \alpha_3 = 6$

$\alpha_1 + 4\alpha_2 + \alpha_3 = 0$

$-2\alpha_2 - 2\alpha_3 = 6$ — (7)

solving (6) & (7)

$-\alpha_2 + 2\alpha_3 = -3$

$-2\alpha_2 - 2\alpha_3 = 6$

$-3\alpha_2 = 3$

$\therefore \alpha_2 = -1$

$-2(-1) - 2\alpha_3 = 6$

$2 - 2\alpha_3 = 6$

$\alpha_3 = \frac{6-2}{-2}$

$\boxed{\therefore \alpha_3 = -2}$

$\alpha_1 + \alpha_2 + \alpha_3 = 3$

$\alpha_1 - 1 - 2 = 3$

$\boxed{\therefore \alpha_1 = 6}$

putting the value of α_1, α_2 and α_3 in eqn (2) we get. solution as.

$a_n = 6(1)^n - 1(2)^n - 2(-1)^n$

(2) $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$ $a_0 = 1$, $a_1 = 9$ & $a_2 = 15$

$\therefore$ R.R is $a_n - 5a_{n-1} + 7a_{n-2} - 3a_{n-3} = 0$ — (1)

The characteristic of RR eqn is,

$r^3 - 5r^2 + 7r - 3 = 0$

$r^2(r-1) - 4r(r-1) + 3(r-1) = 0$

$(r-1)(r^2 - 4r + 3) = 0$

$(r-1)(r^2 - 3r - r + 3) = 0$

$(r-1)\{r(r-3) - 1(r-3)\} = 0$

$(r-1)(r-3)(r-1) = 0$

$\therefore r_1 = 1 \quad r_2 = 3 \quad r_3 = 1$

Since two roots are same and one root is different. So the solution given eqn in the form.

$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 r_2^n$

$= \alpha_1 (1)^n + \alpha_2 . n . (1)^n + \alpha_3 (3)^n$ —— (2)

using initial value $a_0 = 1$.

$\alpha_1 (1)^0 + \alpha_2 (1)^0 . 0 + \alpha_3 (3)^0 = 1$

$\alpha_1 + \alpha_3 = 1$ —— (3)

Also, $a_1 = 9$

$\alpha_1 (1)^1 + \alpha_2 . 1 . (1)^1 + \alpha_3 . (3)^1 = 9$

$\alpha_1 + \alpha_2 + 3\alpha_3 = 9$ —— (4)

Also, $a_2 = 15$

$\alpha_1 (1)^2 + \alpha_2 . 2 . (1)^2 + \alpha_3 . (3)^2 = 15$

$\alpha_1 + 2\alpha_2 + 9\alpha_3 = 15$ —— (5)

solving (3) & (4)

$\alpha_1 + \alpha_3 = 1$

$\alpha_2 + \alpha_1 + 3\alpha_3 = 9$

$-\alpha_2 - 2\alpha_3 = -8$ —— (6)

solving (4) & (5)

$\alpha_1 + \alpha_2 + 3\alpha_3 = 9$

$\alpha_1 + 2\alpha_2 + 9\alpha_3 = 15$

$-\alpha_2 - 6\alpha_3 = -6$ —— (7)

solving (6) & (7)

$-\alpha_2 - 2\alpha_3 = -8$

$-\alpha_2 - 6\alpha_3 = -6$

$4\alpha_3 = -2$

$\boxed{\therefore \alpha_3 = -\frac{1}{2}}$

$\alpha_1 + \alpha_3 = 1$

$\alpha_1 - \frac{1}{2} = 1$

$\alpha_1 = 1 + \frac{1}{2}$

$\boxed{\therefore \alpha_1 = \frac{3}{2}}$

$\alpha_1 + \alpha_2 + 3\alpha_3 = 9$

$\frac{3}{2} + \alpha_2 + 3(-\frac{1}{2}) = 9$

$\alpha_2 = 9 - \frac{3}{2} + \frac{3}{2}$

$\therefore \alpha_2 = 9$

∴ The required solution is using value of α_1, α_2 & α_3

~~$a_n = \frac{3}{2}(1)^n + 9n(1)^n - \frac{1}{2}(9)^n$~~

$a_n = \frac{3}{2}(1)^n + 9n(1)^n - \frac{1}{2}(9)^n$ //

(3) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, $a_0 = -5$, $a_1 = 4$, $a_2 = 15$

soln:

Given R-R is $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ — (1)

The characteristics.

$r^3 + 3r^2 + 3r + 1 = 0$

$r^2(r+1) + 2r(r+1) + 1(r+1) = 0$

$(r+1)(r^2+2r+1) = 0$

comparing eqn (r^2+2r+1) with $ar^2+br+c = 0$

$a = 1$, $b = 2$, $c = 1$

Then root $= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-2 \pm \sqrt{2^2-4.1.1}}{2 \times 1}$

$= \frac{-2 \pm \sqrt{4-4}}{2}$

$= -1$

∴ $r_1 = -1$ ∴ $r_2 = -1$ $r_3 = -1$

Since all roots are same so the given eqn is in form

$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n + \alpha_3 n^2 r_0^n$

$= \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n$ — (2)

using initial value $a_0 = -5$

$\alpha_1(-1)^0 + \alpha_2 . 0 (-1)^0 + \alpha_3 . 0^2 . (-1)^0 = -5$

$\boxed{\alpha_1 = -5}$

Also,
$\alpha_1(-1)^1+\alpha_2.1.(-1)^1+\alpha_3.1^2(-1)=4$
$-\alpha_1-\alpha_2-\alpha_3=4$
$-(-5)-\alpha_2-\alpha_3=4$
$-\alpha_2-\alpha_3=4-5$
$\therefore -\alpha_3=-1+\alpha_2$ —— (3)

Also, $\alpha_2=15$
$\alpha_1(-1)^2+\alpha_2(-1).2+\alpha_3 2^2(-1)^2=15$
$\alpha_1+2\alpha_2+4\alpha_3=15$
$-5+2\alpha_2+4\alpha_3=15$
$2\alpha_2+4\alpha_3=20$
$\alpha_2+2\alpha_3=10$ —— (4)

solving (3) & (4)
$\alpha_2+2(-1-\alpha_2)=10$
$\alpha_2-2-2\alpha_2=10$
$-\alpha_2=12$
$\boxed{\alpha_2=-12}$

$-\alpha_3=-1+\alpha_2$
$=\alpha_3=-1+(-12)$
$-\alpha_3=-13$
$\boxed{\therefore \alpha_3=13}$

$\therefore$ The required solution of given recurrence relation
$a_n=-5(-1)^n-12n(-1)^n+13n^2(-1)^n$ #

Q Model the following problem using recurrence relation
a) suppose that a person deposits \$10000 in a saving account at a ~~result~~ interest rate 11% per year computing actually. How much will be in the account after 30 years?

Solution: Initial condition $a_0=\$10000$
Annual Interest rate $=11\%$.
Given, that interest is compounding annually,
After 1 year, $a_1=a_0+a_0\times 11\%$
$=a_0+0.11a_0 \quad =1.11a_0$

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$a_2 = a_1 + a_1 \times 0.11\% = a_1 + 0.11 a_1 = 1.11 a_1$

$a_3 = a_2 + a_2 \times 11\% = a_2 + 0.11 a_2 = 1.11 a_2$

$\vdots$

$a_n = a_{n-1} + a_{n-1} \times 0.11 = 1.11\, a_{n-1}$

$\therefore$ The recurrence relation for given case is $a_n = 1.11 a_{n-1}$

i.e. $a_n - 1.11 a_{n-1} = 0$

Therefore, The characterstics equation.

$r - 1.11 = 0$

$\therefore r = 1.11$

Now, The solution of reccurence relation is in the form,

$a_n = \alpha r^n$

i.e. $a_n = \alpha (1.11)^n \longrightarrow (1)$

Using initial condition,

$a_0 = \$10000$

$\alpha \cdot (1.11)^0 = \$10000$

$\alpha = \$10000$

Hence, The explicit formula for given case is,

$a_n = 10000 \cdot (1.11)^n$

Now, we can calculate an amount in account of 30 years.

$a_n = 10000 (1.11)^n$

$a_{30} = 10000 (1.11)^{30}$

$= \$228,922.96$

b) A factory makes custom parts of car at increasing rate. In the first month only one car is made in the second month two cars are made, and so on with 'n' cars made at 'n' months.

i) set up a reccurence relation for the number of cars produced in the first n-months by this factory.

ii) How many cars are produced in first year.

iii) Find an explicit formula for the no. of cars produced in the first n months by this factory.

Soln:

According to question, the car are produced in ~~inea~~ increasing rate as $a_1 = 1$

$$a_2 = a_1 + 2 = 1 + 2 = 3$$

$$a_3 = a_2 + 3 = 3 + 3 = 6$$

$$a_4 = a_3 + 4 = 6 + 4 = 10$$

$$\vdots$$

$$a_n = a_{n-1} + n$$

This is required R.R i.e. $a_n = a_{n-1} + n$ —— (1)

(iii) we have,

$a_n = 1 + 2 + 3 + 4 + \dots + n$ (sum of natural number)

$$a_n = \frac{n(n+1)}{2}$$

This is the explicit formula for production of car, for given factory.

(ii) The number of car produced in first year is

$$a_{12} = \frac{12(12+1)}{2}$$

$$\boxed{\therefore a_{12} = 78}$$

8) Suppose that number of bacteria in a colony triples every hours.

i) set up recurrence relation for number of bacteria after n hrs have elapsed.

ii) If 100 bacteria are used to begin a new colony: how many bacteria will be in colony in 10 hours.

soln:

According to question, bacteria are triples in every hours as

After 1 hours, $a_1 = 3$

$a_2 = 3a_1 = 3 \times 3 = 9$

$a_3 = 3a_2 = 3 \times 9 = 27$

$\vdots$

$a_n = 3a_{n-1}$

$\therefore$ The required R.R is $a_n = 3a_{n-1}$ ——— (1)

The characteristics eqn of R.R is:

$r - 3 = 0$

$\boxed{\therefore r = 3}$

The solution of recurrence relation is in form,

$a_n = \alpha_1 r^n$

$= \alpha . 3^n$ ——— (2)

Using initial value, ~~$a_0 = B$~~ $a_0 = 100$

$\alpha . 3^0 = 100$

$\boxed{\therefore \alpha = 100}$

Eq(2) becomes $a_n = 100 . 3^n$

After 10 hours,

$\therefore a_{10} = 100.(3)^{10}$

$= 5904800.$

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Q. An employee joined a company in 1999 with a starting salary of \$50000. Every year this employee receives a raise of \$1000 + 5% of salary of previous year.

a) Setup a R.R for salary of these employee in years after 1999.

b) What will be the salary of employee in 2007.

c) Find the explicit formula for the salary of this employee n years after 1999.

Soln:

Initial condition $(a_0) = \$50000$

$a_1 = a_0 + a_0 \times 5\% + 1000 = 1.05 a_0 + 1000$

$a_2 = a_1 + a_1 \times 5\% + 1000 = 1.05 a_1 + 1000$

$\vdots$

$a_n = 1.05 a_{n-1} + 1000$

(a) $\therefore$ The required eqn of RR is $a_n = 1.05 a_{n-1} + 1000$ — (1)

(c) $a_0 = \$50000$

$a_1 = 1.05 a_0 + 1000$

$a_2 = 1.05 a_1 + 1000$

$= 1.05(1.05 a_0 + 1000) + 1000$

$= 1.05^2 a_0 + 1000 \times 1.05 + 1000$

$a_3 = 1.05 a_2 + 1000$

$= 1.05(1.05^2 a_0 + 1000 \times 1.05 + 1000) + 1000$

$= 1.05^3 a_0 + 1000 \times 1.05^2 + 1000 \times 1.05 + 1000$

~~$\therefore a_n = 1.05^n a_0 + 1000 \times 1.05^{n-1} + 1000 \times 1.05^{n-2} + 1000$~~

$a_n = 1.05^n a_0 + (1 + 1.05^2 + 1.05^3 + \cdots + 1.05^{n-1}) 1000$

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$$(1 + r^2 + r^3 + r^4 \dots + r^{n-1}) = \frac{1-r^n}{1-r}$$

$$= 1.05^n a_0 + \left(\frac{1-(1.05)^n}{1-1.05}\right) 1000$$

∴ This explicit formula of this employee in 1999.

ii) In 2007:

$$a_8 = 1.05^8 \times 50000 + \left(\frac{1-(1.05)^8}{1-1.05}\right) 1000$$

$$= 83,412.772$$

∴ Hence, in 2007 or after 8 years the employee salary will be $83,412.772 #.

Q. suppose that each pair of genetically engineered species of rabbits left on an island produces two new pair of rabbits at the age of one month and 6 new pairs of rabbit at the age of two month and every 2 month afterward.

~~So~~

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* Solving linear non-homogeneous Recurrence Relation with constant coefficient:

The recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_k a_{n-k} + F(n)$ where c_1, c_2, c_3 are real numbers and $F(n)$ is function depending upon n is called linear non-homogeneous recurrence relation.

The recurrence relation proceeding $F(n)$ is called associated homogeneous recurrence relation.

For Ex: $a_n = 7a_{n-1} + 3a_{n-2} + 6n$ is a linear non-homogeneous recurrence relation with constant coefficient.

* Theory (without proof): If $a_n(p)$ is a particular solution of non-homogeneous linear recurrence relation with constant coefficient $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$. the every solution of the form $a_n = a_n(h) + a_n(p)$ where $a_n(h)$ is solution of linear non-homogeneous reccurence relation.

* Solve the Recurrence Relation:

1) $a_n = 2a_{n-1}, \quad n \geq 1, \quad a_0 = 3$

So: The characteristics of given RR is in the form:

$$r - 2 = 0$$

$$\therefore r = 2$$

$$a_n = \alpha r^n$$

$$= \alpha . 2^n \quad \text{——— (1)}$$

Using initial value $a_0 = 3$

$$\alpha . 2^0 = 3$$

$$\boxed{\therefore \alpha = 3}$$

Hence eqn (1) become

$$\boxed{a_n = 3.2^n}$$

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2) ~~$a_n = a_{n-1} + 2a_{n-2}$~~

Q. Find all the solution of RR $a_n = 4a_{n-1} + n^2$. Also find solution of RR with initial condition $a_1 = 1$.

Soln:

Given RR is $a_n = 4a_{n-1} + n^2$

The associated homogeneous recurrence relation is $a_n = 4a_{n-1}$ (i)

The characterstic eqn of homogeneous R.R is.

$r - 4 = 0$ or, $r = 4$

∴ The solution of this R.R is in the form,

$a_n = \alpha r^n$ i.e. $a_n = \alpha . 4^n$ —— (ii)

Here,

The particular function $F(n) = n^2$

A trail solution of quadratic function in n is.

$p(n) = an^2 + bn + c$ where, a, b, & c are constant.

then the eqn becomes

$a_n = 4[a(n-1)^2 + b(n-1) + c] + n^2$

or, $an^2 + bn + c = 4a(n-1)^2 + 4b(n-1) + 4c + n^2$

or, $an^2 + bn + c = 4a(n^2 - 2n + 1) + 4b(n-1) + 4c + n^2$

or, $an^2 + bn + c = 4an^2 + n^2 - 8an + 4bn + 4a - 4b + 4c$

i.e. $an^2 + bn + c = (4a+1)n^2 + (-8a+4b)n + (4a - 4b + 4c)$

Comparing the coefficient we get as:

$a = 4a + 1$	$-8a + 4b = b$	$4a - 4b + 4c = c$
i.e. $3a = -1$	i.e. $3b = 8a$	i.e. $3c = -4a + 4b$
∴ $a = -1/3$	i.e. $b = 8a/3$	∴ $c = \frac{-4a + 4b}{3}$
	$= 8/3(-1/3)$	
	$= -8/9$	

$$= \frac{-4\left(-\frac{1}{3}\right) + 4\left(-\frac{8}{9}\right)}{3} = \frac{\frac{4}{3} - \frac{32}{9}}{3} = \frac{-20}{27}$$

So, the particular solution is,

$$a_n(p) = an^2 + bn + c$$

$$= -\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$$

$$= -\frac{1}{3}\left(n^2 + \frac{8}{3}n + \frac{20}{9}\right)$$

∴ The solution of given R.R. is:

$$a_n = a_n(h) + a_n(p)$$

$$\boxed{\therefore a_n = \alpha . 4^n - \frac{1}{3}\left(n^2 + \frac{8}{3}n + \frac{20}{9}\right)}$$

Using initial condition we have,

$$a_1 = 1$$

$$\alpha . 4^1 - \frac{1}{3}\left(1^2 + \frac{8 . 1}{3} + \frac{20}{9}\right) = 1$$

$$4\alpha - \frac{1}{3}\left(1 - \frac{8}{3} + \frac{20}{9}\right) = 1$$

$$4\alpha - \frac{1}{3}\left(\frac{9 + 24 + 20}{9}\right) = 1$$

$$4\alpha - \frac{53}{27} = 1$$

$$4\alpha = 1 + \frac{53}{27}$$

$$4\alpha = \frac{27 + 53}{27}$$

$$\therefore 4\alpha = \frac{80}{27} \quad \boxed{\therefore \alpha = \frac{20}{27}}$$

∴ Hence, using initial condition the ~~relation~~ solution of given RR is

$$a_n = \frac{20}{27} . 4^n - \frac{1}{3}\left(n^2 + \frac{8n}{3} + \frac{20}{9}\right)$$

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