CPE 345: Modeling and Simulation

Lecture 7

Today's topic

- Random Number Generation (chapter 7)
  - Properties of Random Numbers
  - Generating Pseudo-random Numbers
  - Techniques for Generating Random Numbers
  - Testing for Randomness

Properties of Random numbers

- What properties should random numbers satisfy?
- Uniformly distributed random numbers in (0,1)
  - A sequence of random numbers R_1, R_2, …. must have two important statistical properties:
    - **Uniformity**
      - Each random number R_i is a sample drawn from an uniform distribution

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & ow \end{cases}$$

$$E(X) = \frac{1}{2}$$

$$\operatorname{var}(X) = \frac{1}{12}$$

    - **Independence**
      - All samples (R_i, $i = 1,2,\ldots$) are independent

$$P(a \le R_i \le b) = \frac{1}{b-a}$$ for any sample R_i, $a, b \in (0,1),\ a \le b$

Important consequences

- If the interval (0,1) is divided into *n* classes, or subintervals of equal length, the number of observations in each interval is *N/n*; *N* = total number of observations
  - If *n* intervals of length $l_1, l_2, \ldots, l_n$; the number of observations in the interval *i* is N/l_i ($\sum_i l_i = 1$)
    - We have already used this property to generate arrivals and departures

- Observing a value in a particular interval is independent of the previous values drawn

Generating pseudo-random numbers

- Why pseudo-random?
  - Generating truly random numbers in a simulation is not possible
    - Natural processes generate truly random numbers
    - If random numbers are generated using a known method, then they can be replicated, so they are not truly random anymore
  - In practice: generate pseudo-random numbers (with uniform distribution) that satisfy the ideal properties of the random distribution.
  - Some errors/departure from randomness in generating pseudo - random numbers
    - The generated numbers may not be uniformly distributed
    - The generated numbers may be discrete valued instead of continuous valued
    - The mean may be too high or too low
    - The variance may be too high or too low
    - There may be dependence between the generated numbers
      - Autocorrelation
      - Numbers successively higher or lower
      - Several numbers above the mean, then several numbers below the mean

Pseudo-random number generator routines

- The routine should pass all tests for departure from uniformity and independence – will follow shortly
- Some other requirements:
  - Speed – usually very large number of random numbers should be generated in a typical simulation (hundreds of thousands)
  - Portability – between machines and languages – same results wherever it is executed
  - Long-cycle – the numbers are repeated after a certain cycle length.
  - Repeatability – You may want to run simulations in exactly the same conditions, or you may want to specifically choose different starting points
    - Seed selection

Techniques for generating random numbers

- ***Linear Congruential Method***
  - Produces a sequence of integers $X_1, X_2, \ldots X_{m-1}$, according to the following recursive relationship

$$X_{i+1} = (aX_i + c) \bmod m, \; i = 0,1,2,\ldots$$

$$\begin{aligned} X_0 &= \text{seed} \\ \text{a} &= \text{constant multiplier} \\ \text{c} &= \text{increment} \\ \text{m} &= \text{modulus} \end{aligned}$$

  - If c=0: multiplicative congruential method, otherwise, mixed congruential method
  - The choice of the parameters drastically affects the statistical properties and the cycle length
  - Variants of this technique used in the computer generation of random numbers (rand() functions)

Examples: ex. 7.1

Example 7.1: X0 = 27, a = 17, c = 43, m=100

X	Normalized X
27	0.27
2	0.2
77	0.77
52	0.52
27	0.27
2	0.02
77	0.77
52	0.52
27	0.27
2	0.02
77	0.77
52	0.52
27	0.27

Cycle length 4. Is this acceptable?
What happens for different seed selection?

Example 7.1. Different seed selection

X0 = 3
a = 17
c = 43
m=100

Cycle length 20

- Too small in both cases
- Depends on the choice of the seed

X	Normalized X
3	0.03
94	0.94
41	0.41
40	0.4
23	0.23
34	0.34
21	0.21
0	0
43	0.43
74	0.74
1	0.01
60	0.6
63	0.63
14	0.14
81	0.81
20	0.2
83	0.83
54	0.54
61	0.61
80	0.8
3	0.03
94	0.94

Choosing the parameters

- Some observations related to example 7.1
- Two more important properties
  - Maximum density
    - The numbers generated in example 7.1 are actually discrete, not continuous. The set of values

$$I = \{0, 1/m,\ 2/m, 3/m, \ldots (m-1)/m\}$$

    - The approximation for continuous is acceptable for m large
      - In practice, $m = 2^{31} - 1$ and $m = 2^{48}$ are commonly used
  - Maximum period
    - Maximal period can be achieved by proper choice of the parameters
      - For $m = 2^b$, and $c \neq 0$, the longest possible period is $P = m = 2^b$, achieved if *c* is relatively prime to *m* (the greatest common factor of c and m is 1), and $a = 1+4k$, *k* integer.

Maximum period – cont.

- For $m = 2^b$, and $c = 0$, the longest possible period is $P = m/4 = 2^{b-2}$, achieved if the seed X_0 is odd, and $a = 3+8k$ *or* $a = 5+8k$, $k=0,1, \ldots$
- For m a prime number and $c = 0$, the longest possible period is $P = m-1$, achieved if a has the property that the smallest integer k s.t. a^k-1 is divisible by m is $k = m-1$.
- Study example 7.4 in the book. The chosen parameters are actually used in generators:

$$a = 7^5 = 16807$$

$$m = 2^{31} - 1 = 2{,}147{,}483{,}647 \text{ (prime number)}$$

$$\mathrm{P} = \mathrm{m} - 1$$

Are these numbers sufficiently large?

Combining Linear Congruential Generators

- The period length for example 7.4. seems extremely large
- With PCs capable of hundreds of MIPS, simulations are getting bigger and 10^9 cycle period is too short.
- Solution: choose multiple generators and combine them
  - If $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$ are the *i-th* output for different generators, where the j-th generator has prime modulus m_j, and a_j is chosen such that $P_j = m_j - 1$

$$X_i = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \right) \operatorname{mod}(m_1 - 1)$$

$$R_i = \begin{cases} \dfrac{X_i}{m_1}, & \mathrm{X_i} > 0 \\ \dfrac{m_1 - 1}{m_1}, & \mathrm{X_i} = 0 \end{cases}$$

$$P = \frac{(m_1 - 1)(m_2 - 1)\ldots(m_k - 1)}{2^{k-1}}$$

Tests for Random Numbers

- Uniformity tests
  - Frequency test – Kolmogorov-Smirnov or χ^2 (chi-square) tests
    - Compare the distribution to the uniform distribution
- Independence tests
  - Runs test – looks for patterns of increasing/decreasing values
  - Autocorrelation test – tests for correlation between numbers
  - Gap test – counts the number of digits that appear between the repetition of a particular digit, than uses the Kolmogorov-Smirnov test to compare with the expected size of gaps
  - Poker test – Treats numbers grouped together as a poker hand. The hands obtained are compared to what is expected using a chi-square test.

Hypothesis Testing

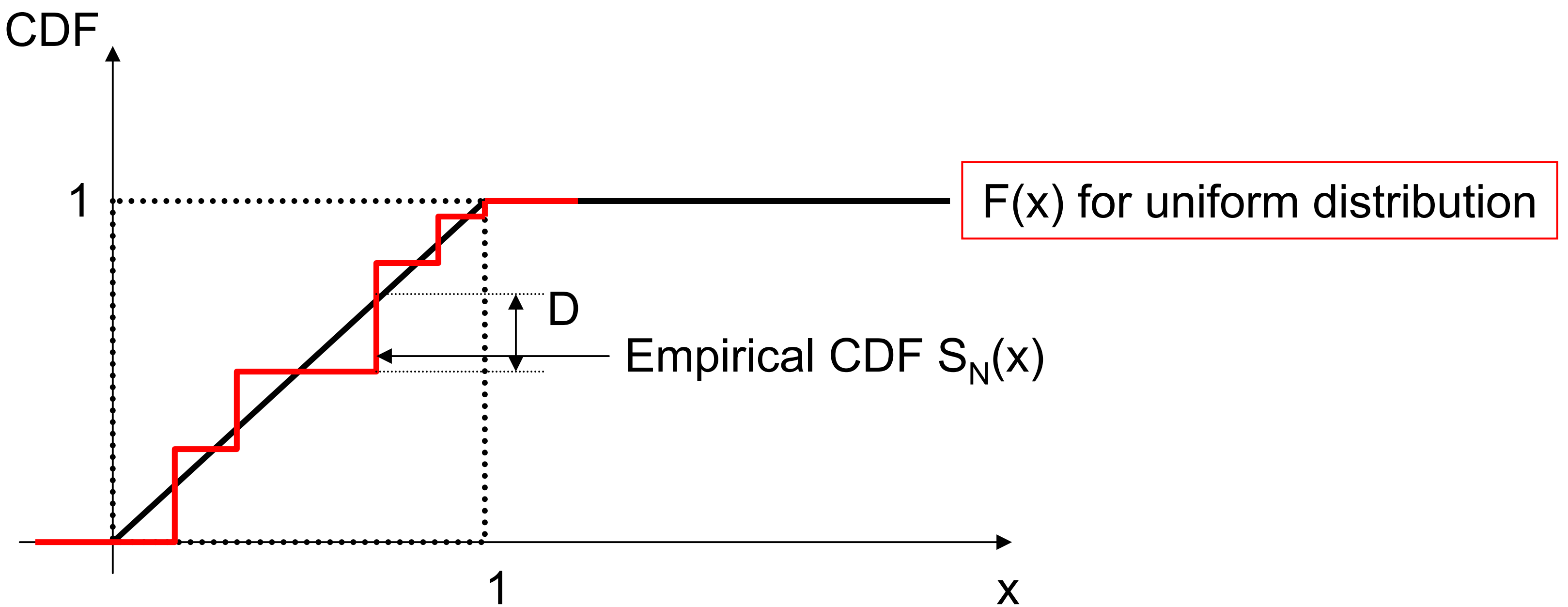
- Uniformity test
  - H0 = numbers are uniformly distributed
  - H1 = numbers are not uniformly distributed
    - Failure to reject the null hypothesis means that no evidence of non-uniformity has been found on the basis of this test. It does not mean that further testing is unnecessary.
- Independence test
  - H0 = numbers are independently generated
  - H1 = numbers are not independently generated
    - Failure to reject the null hypothesis means that no evidence of non-independence has been found on the basis of this test. It does not mean that further testing is unnecessary.
- For each test, a level of significance must be stated

$$\alpha = P(\text{reject } H0 \mid H0 \text{ true})$$

  - frequently, α is set to 0.01 or 0.05 (1%, 5% failure by chance)

Frequency Tests

- Kolmogorov-Smirnov test:
  - Compare empirical CDF (for N samples) with the expected CDF



$$D = \max \left| F(x) - S_N(x) \right|$$ maximum deviation

For a given α and N random samples, find the critical value from Table A.8

Frequency Test – cont.

- Critical values for large number of samples:
  - For N>35:

$D_{0.10}$	$D_{0.05}$	$D_{0.01}$
$1.22/\sqrt{N}$	$1.36/\sqrt{N}$	$1.63/\sqrt{N}$

- How to compute empirical CDF

$$S_N(x) = \frac{\text{number of } R_1, R_2, \ldots, R_N \text{ which are} \leq x}{N}$$

Kolmogorov-Smirnov test: step by step

- Rank the data from smallest to largest. *R(i)* = the smallest *i*-th observation

$$R(1) \le R(2) \le \ldots \le R(N)$$

- Compute $$D^+ = \max_{1 \le i \le N} \left\{ \frac{i}{N} - R(i) \right\}$$

$$D^- = \max_{1 \le i \le N} \left\{ R(i) - \frac{i-1}{N} \right\}$$

- Compute $D = \max(D^+, D^-)$
- Determine a critical value D_α from Table A.8 for the specified significance level (α) and the given sample size *N*
- If D $\ge D_\alpha$ the null hypothesis is rejected;
- If D $\le D_\alpha$ no difference is detected between the true distribution and the uniform distribution

Example (7.6)

- Example (7.6) Kolmogorov – Smirnov Test
  - Five numbers are generated: 0.44; 0.81; 0.14; 0.05; 0.93
  - Significance level $\alpha = 0.05$

R(i)	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
i/N-R(i)	0.15	0.26	0.16	-	0.07
R(i)-(i-1)/N	0.05	-	0.04	0.21	0.13

$D_{0.05} = 0.565 \Rightarrow$ The distribution cannot be distinguished from uniform

Chi-square test

- Uses the statistic

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$ Chi-square distributed with n-1 degrees of freedom

  - O_i = the observed number in the *i*-th class
  - E_i = the expected number in the *i*-th class
  - n = the number of classes

- Compare with critical value $\chi^2_{\alpha,n-1}$ (critical value table)
  - If $\chi_0^2 > \chi^2_{\alpha,n-1}$, reject null hypothesis
  - If $\chi_0^2 \le \chi^2_{\alpha,n-1}$, indistinguishable from the uniform distribution
- For equally spaced classes and *N* samples $E_i = \dfrac{N}{n}$
- The test is valid for large number of samples N $\geq$ 50
  - Recommended that N, and n are chosen such that each $E_i \geq 5$

Study Example 7.7 in your text book

Independence Tests

- Sometimes the generators pass the Kolmogorov-Smirnov and chi-square tests for uniformity, but the numbers generated are not independent → need independence tests

- Run Tests
  - Run = succession of similar events
  - Example: coin flipping: H T T H H T T T H T
    - Six runs: length 1, 2, 2, 3, 1, 1
  - For sequence of random numbers, we can define up runs and down runs (successive numbers are increasing or decreasing)

$$^{-}0.87,\ ^{+}0.15,\ ^{+}0.23,\ ^{+}0.45,\ ^{-}0.69,\ ^{-}0.32,\ ^{-}0.30,\ ^{+}0.19,\ 0.24.$$

$$- \ + \ + \ + - \ - \ - \ +$$

$$^{-}0.87,\ ^{+}0.15,\ ^{-}0.23,\ ^{+}0.05,\ ^{-}0.69,\ ^{+}0.32,\ ^{-}0.40,\ ^{+}0.19,\ 0.24.$$

$$- \ + \ - \ + - \ + - \ +$$

Are these reasonable generators?

Runs test

- The number of runs
  - *a* = total number of runs in a truly random sequence
  - For $N > 20$, distribution of *a* assumed Gaussian (normal) with

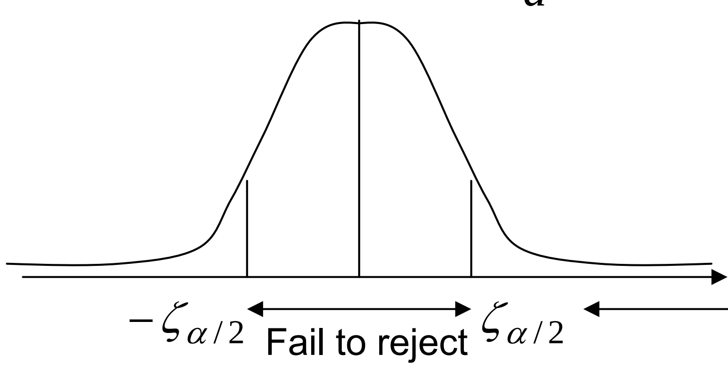
$$\mu_a = \frac{2N-1}{3}$$

$$\sigma_a^2 = \frac{16N-29}{90}$$

  - Standardized normal test statistic: test for independence

$$Z_0 = \frac{a-\mu_a}{\sigma_a}$$

→ Gaussian with zero mean and unit variance $Z_0 \sim N(0,1)$



Find from Gaussian distribution tables
For significance $\alpha/2$

Determine critical values for significance $\alpha/2$

- Significance level $\alpha = P(\text{reject } H0 \,|\, H0 \text{ true})$

$$\alpha = P(Z_0 \geq \zeta_{\alpha/2}) + P(Z_0 \leq -\zeta_{\alpha/2}) = 2P(Z_0 \geq \zeta_{\alpha/2})$$

$$\frac{\alpha}{2} = P(Z_0 \geq \zeta_{\alpha/2}) = 1 - \phi(\zeta_{\alpha/2}) \Rightarrow \text{determine } \zeta_{\alpha/2}$$

tabulated

Runs above and below the mean

- For example, if mean is 0.5

0.1 0.2 0.22 0.14 0.33 0.18 0.63 0.58 0.53 0.76 0.9 0.82 0.27 …

- In terms of runs up and down:

  + + - + - + - - + + - (*)

- If we define the runs as above and below the mean:

  - - - - - - + + + + + + - (**)

- Same test can be run with the runs defined as in (**)
  - n1 = number above of mean
  - n2 = number below the mean
  - b = number of runs – Gaussian r.v.

$$\mu_b = \frac{2n_1 n_2}{N} + \frac{1}{2}$$

$$\sigma_b^2 = \frac{2n_1 n_2 (2n_1 n_2 - N)}{N^2 (N-1)}$$

Auto-correlation test

- Computation of the auto-correlation between every m (the lag) numbers starting with *i*
  - autocorrelation ρ_{im} -> between R_i, R_{i+m}, R_{i+2m}, $R_{i+(M+1)m}$, M is the largest value s.t. $i+(M+1)m \leq N$
- Nonzero autocorrelation → lack of independence
  - H0: $\rho_{im} = 0$
  - H1: $\rho_{im} \neq 0$
- For large values of M, the distribution of the estimator $\hat{\rho}_{i,m}$ is approx. normal (Gaussian).

The statistic $z_0 = \dfrac{\hat{\rho}_{i,m}}{\sigma_{\hat{\rho}_{i,m}}}$ is normally distributed with mean 0 and unit variance, for large M

Auto-correlation test: step-by-step

- Compute

$$\hat{\rho}_{i,m} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

[Schmidt & Taylor – 1970]

$$\sigma_{\hat{\rho}_{i,m}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

- Compute statistic

$$z_0 = \frac{\hat{\rho}_{i,m}}{\sigma_{\hat{\rho}_{i,m}}}$$

- Do not reject the null hypothesis (independence) if

$$-\zeta_{\alpha/2} \leq z_0 \leq \zeta_{\alpha/2}$$

α = level of significance

Gap test

- Gap = interval between the recurrences of the same digit
- Frequency of the gaps
  - The observed frequencies of various gap sizes compared to the theoretical frequency – Kolmogorov-Smirnov test
  - Theoretical frequency distribution

$$P(gap \le x) = F(x) = 0.1\sum_{n=0}^{x}(0.9)^n$$

Note: Probability of occurrence for certain digit is 0.1.

Poker test

- Frequency with which certain digits are repeated in a series of numbers
- Example

  0.255, 0.577, 0.414, 0.828, 0.909, 0.303, 0.001
- Pair of like digits generated
- For three digits: three possibilities
  - All different
  - All equal
  - One pair of like digits

Given a fixed digit, this digit different

$$P(\text{exactly one pair}) = \binom{3}{2}(0.1)(0.9) = 0.27$$

no. of possibilities

Given a fixed digit, this digit is the same

Poker test: cont.

$$P(\text{three different digits}) = P(\text{second different from first})P(\text{third different from first and second}) =$$
$$= (0.9)(0.8) = 0.72$$

$$P(\text{three like digits}) = P(\text{second digit same as first})P(\text{third digit same as first and second}) =$$
$$= (0.1)(0.1) = 0.01$$

Poker test:

Measure observed frequency for the three cases
Compute expected frequency E_i (probabilities*1000)
Perform chi-square test

Example 7.14.

Combination 1	Obs. Freq.	Expected Freq.	(Oi-Ei)2/Ei
3 different digits	680	720	2.22
3 like digits	31	10	44.10
Exactly 1 pair	289	270	1.33

$$47.65 > \chi^2_{0.05,2} = 5.99$$

Homework

- Problem 7 page 286, chapter 7