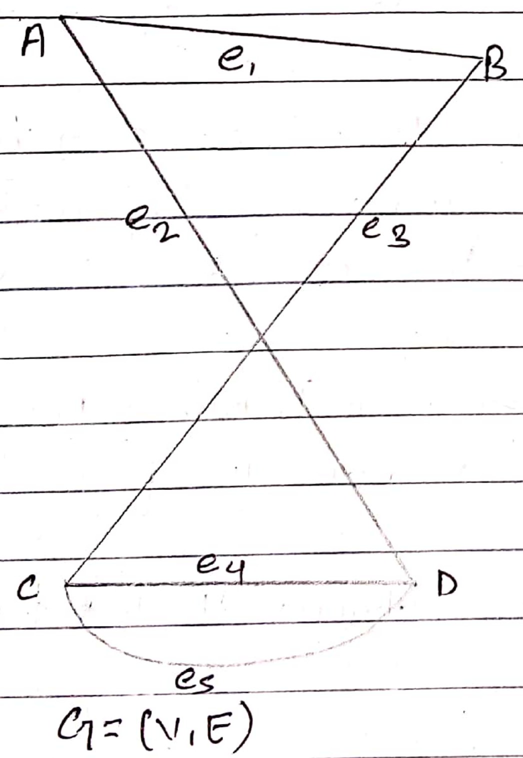
Introduction:

A Graph is a set of vertices and edges $G=(V,E)$ where 'V' is a non-empty set of vertices, 'E' is set of edges. And edge is said to connect it end points. Graph is a discret structure consisting of vertices. Graph can be used to represent almost any problem involving discret arrangement of object. where concern lies not with the internal properties of these object, but with relationship among them.



$G=(V,E)$

In the above graphs,

$G=(V,E)$

$V=\{A,B,C,D\}$ and set of edges $E=\{e_1,e_2,e_3,e_4,e_5\}$

where,

$e_1=\{A,B\}$, $e_2=\{A,D\}$, $e_3=\{B,C\}$, $e_4=\{C,D\}$, $e_5=\{C,D\}$

* Application of Graph:

Graphs are used to solve problems in many field, some of them are as follows:

(1) Graphs are used to model the geographic map of the cities in which each place in city can be represented

by edges.

(2) Graphs used to models the computer network in which each node is a machine (computer, hub, router, switch) and the link between them represents the edges.

(3) They are used to analyze the electric circuits, project planning, pipelining ~~planning~~ planning etc.

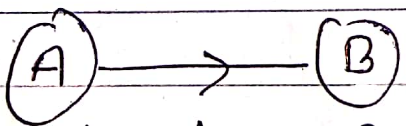
(4) Any structured problem can be moduled by graphs. Then can help to solve typical problems concerned with shortest path or most economical root between two vertices.

(5) Graphs are used to study the structure of www.

(6) It is used to find the number of different combination of flight between two cities in the airline network.

(7) It can be used to distinguish between two chemical compound with the same molecular formula but different structure.

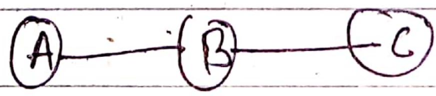
(8) It is also used to assign channels to the Television station. etc.



Adjacent A to B

~~Ajacent from A to~~

Adjacent B from A



A & C is not adjacent

* Finite & Infinite Loop :

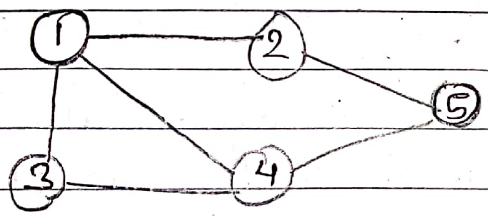
A graph with a finite set of vertex is called finite graph. A graph with a infinite set of vertex is infinite graph.

Here we consider only finite graphs,

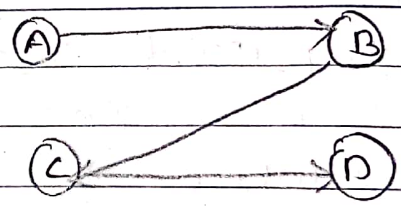
*Simple Graph:

A graph in which each edge connect two different vertices and where no two edges connect the same pair of vertices is called simple graph.

When a directed graph has no loops and has no multiple directed edges; it is called Simple directed graph.



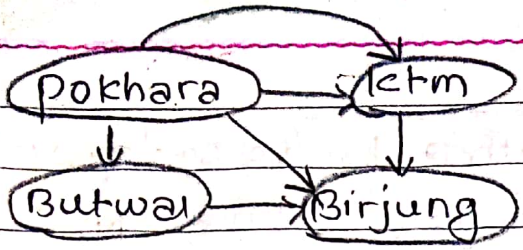
ⓐ simple graph



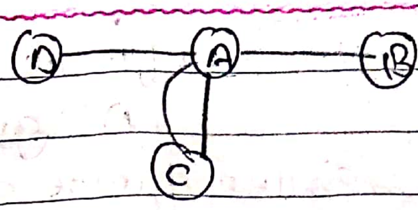
ⓑ simple directed graph

* Multiple or Multi graph:

Graphs may have multiple edges connecting the same vertices are called multigraphs. Directed graph that may have multiple directed edges from a vertex to a second vertex are called directed multi graph.



(a) Directed multigraph



(b) multi graph.

* Pseudo Graph:

Graph that may includes loop and possibly multiple edges connecting the same pair of vertices are sometimes called pseudo ~~code~~ graph.

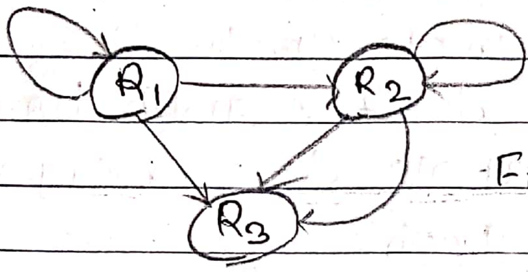


Fig. pseudo Graph

F

* Mixed Graph:

For some models we may need a graph where some edges are undirected while others are directed. A graph with both directed and undirected edges are called mixed graph.
For Eg - A mixed graph might be used to model to computer network consisting links that operate in both direction and other links that operate in one direction.

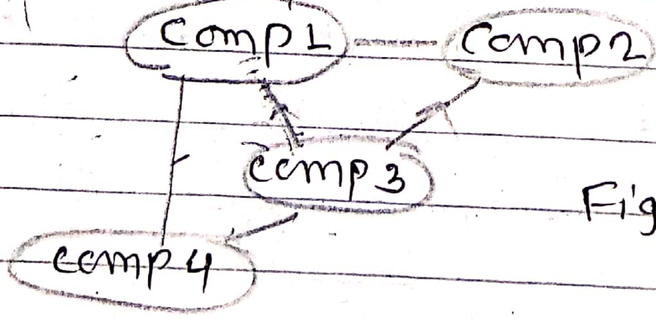
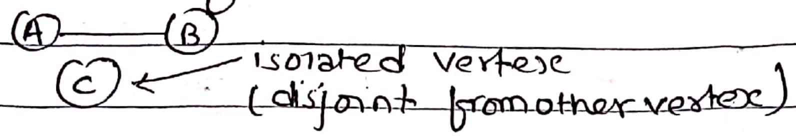


Fig. Mixed Graph

* Degree of Graph:

The degree of vertex in an undirected graph is the number of ~~degree~~ edge incident with it except that a loop at a vertex contribute ~~twice~~ twice to the degree of that vertex. The degree of vertex is denoted by deg( ).

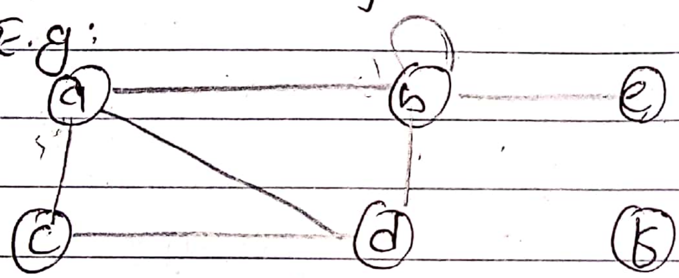
- A vertex of degree 0 is called isolated vertex.



It follows than an isolated vertex is not adjacent to any vertex.

A vertex is called pendent vertex if an only if it has degree 1, i.e. a pendent vertex is adjacent to exactly one other vertex.

E.g:

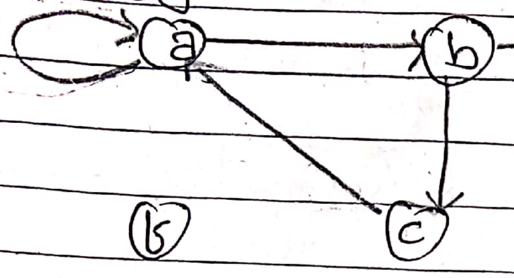


Deg (a) = 3
Deg (b) = 5
Deg (c) = 2
Deg (d) = 3
Deg (e) = 1
(Pendent Vertex)
Deg (f) = 0
(isolated vertex)

In a graph with a directed edges the indegree of vertex v denoted by $deg^-(v)$. And out degree of v denoted by $deg^+(v)$. A loop at a vertex v contributes degree one to both indegree and outdegree.

The undirected graph that results from ignoring direction of edges is called the underlying undirected graph. A graph with directed edges and its underlying number of edges.

E.g.



$Deg^-(a) = 2 \quad Deg^+(a) = 2$

$Deg^-(b) = 1 \quad Deg^+(b) = 2$

$Deg^-(c) = 1 \quad Deg^+(c) = 1$

$Deg^-(d) = 1 \quad Deg^+(d) = 0$

$Deg^-(f) = 0 \quad Deg^+(f) = 0$

Fig. Graph G

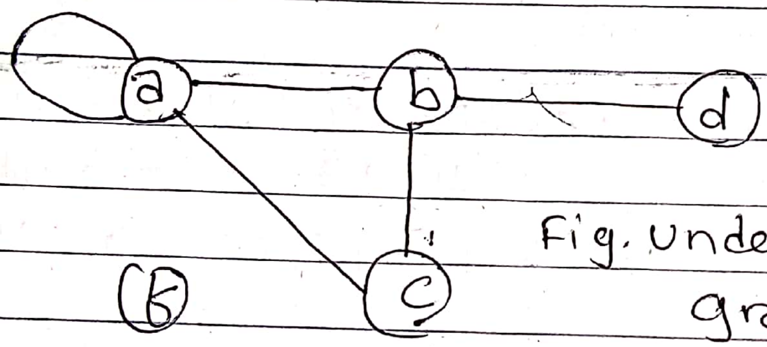


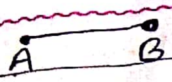
Fig. underlying undirected graph of G.

* The Handshaking Theorem:

let, $G = (V, E)$ be an undirected graph with e edges then,

$$2e = \sum_{\in V} deg(\,)$$

Note: This applies even if multiple edges and loops are present.



$$2 = \sum_{V} \deg() = \deg(A) + \deg(B)$$
$$= 1+1 = 2$$

Question:

Q.NO.1. How many edges are in a graph with 10 vertices, each of degree 6?

Soln:

No. of vertices = 10

Degree of each vertices = 6

we know that,

By H.S Theorem,

$$2e = \sum_{\in V} \deg()$$

i.e. $2e = \deg(V_1) + \deg(V_2) + \deg(V_3) + \deg(V_4) + \deg(V_5) + \deg(V_6)$

i.e. $2e = 6 \times 10$

i.e. $e = 30$

Q.NO.2. How many edges are there in a graph with 4 vertices where two vertices has degree 3 and two vertices has degree 2.

Soln: No. of ~~vertices~~ vertex = 4

Degree of two vertex = 3

another degree of two vertices = 2

we know that,

By H.S Theorem,

$$2e = \sum_{\in V} \deg()$$

$$2e = 3 \times 2 + 2 \times 2$$

i.e. $2e = 10$ $\therefore e = 5$ //

Imp

Th **Theorem:** An undirected graph has even number of odd degree vertices.

proof:

let, V_1 & V_2 be the set of vertices of even degree and the set of vertices of odd degree respectively.

In an undirected graph $G=(V,E)$, using hanshaking theorem.

$$2e = \sum_{\in V} \deg() = \sum_{\in V_1} \deg() + \sum_{\in V_2} \deg()$$

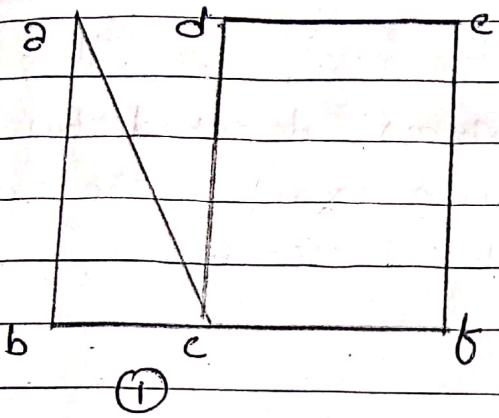
~~Because~~ $\cancel{\Sigma}$

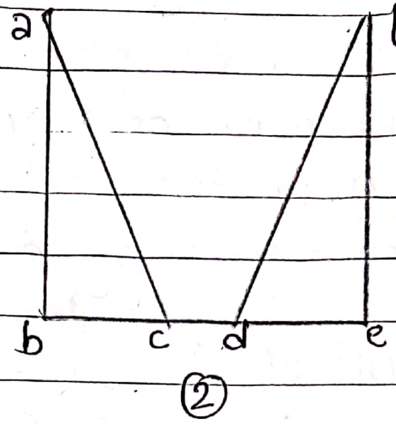
Because $\sum_{\in V_1} \deg()$ is even furthermore the sum of two terms on the right hand side is even. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are even no. of vertices of odd degree.

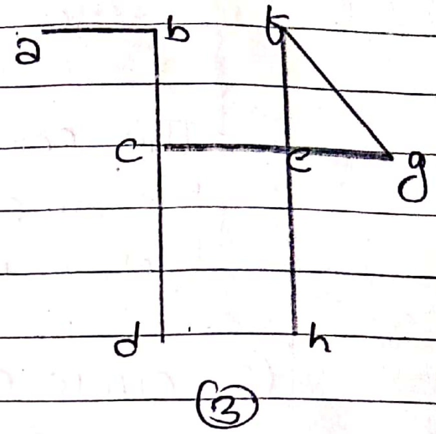
* Cut-in vertices & Cut-in Edges:

A cut-in Vertex ~~(c~~ (cut vertex) is a vertex that when removed from a graph creates more components than previously in the graph i.e. removing a cut vertex from a graph breaks it in two or more graphs.

A cut edge is a edge that when removed from a graph creates more components that previously in the graph i.e. removing ~~a~~ cut edges from a graph, breaks it in two or more graphs.







Graph ②

cut vertices : c & d

cut edges : (c,d)

Note: cut edges are also called cut bridges.

Graph ①

cut vertices : c

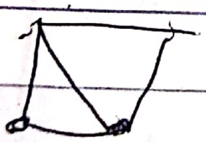
cut edges : No cut edges

Graph ③

cut vertices : c, e, b, i

cut edges: (a,b), (b,c), (c,d), (c,e), (e,i)

(i,h)

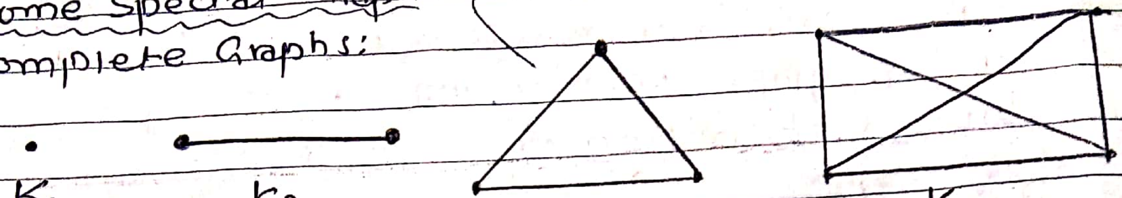


$K_{10} = \frac{10\times(10-1)}{2} = 45$

Graphs:

Some Special Graphs: $\frac{n\times(n-1)}{2}$

① Complete Graphs:

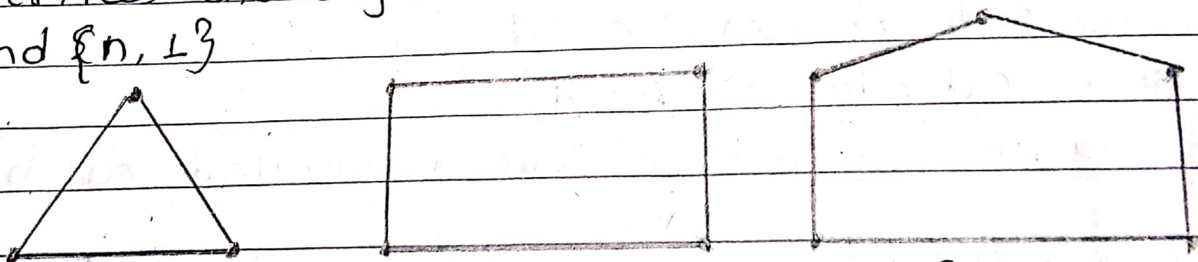


K_1 $\quad$ K_2 $\quad$ K_3 $\quad$ K_4

The complete graph on n vertices, denoted by K_n is the simple graph that contains exactly one edges between each pair of distinct vertices.

② Cycle Graphs: (C)

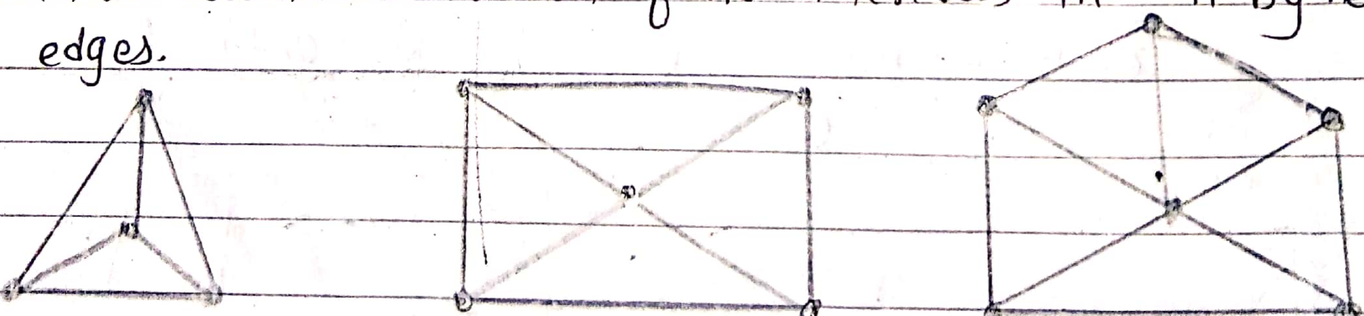
The cycle graph C_n, $n \geq 3$ consist of n vertices and edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, ... $\{n-1, n\}$ and $\{n, 1\}$



C_3 $\quad$ C_4 $\quad$ C_5

③ Wheel Graphs:

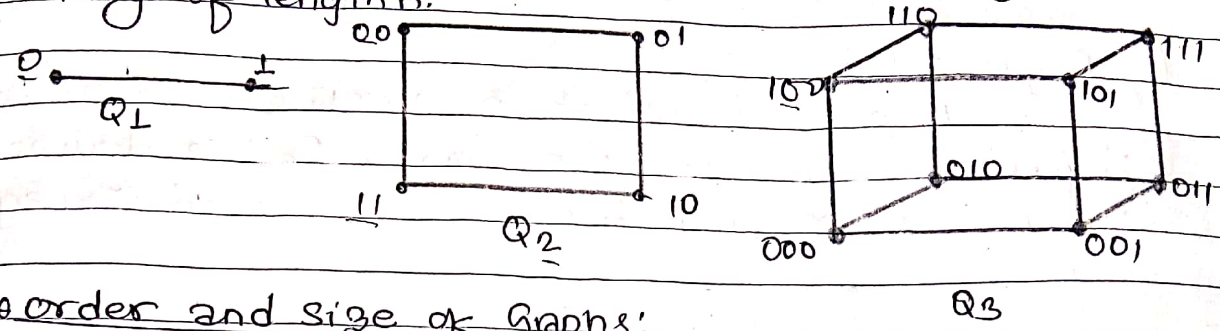
we optain the wheel graph 'W_n', when we add additional vertex to the cycle 'C_n' for $n \geq 3$ and Connect these new vertex to each of the n vertices in 'C_n' by new edges.



W_3 $\quad$ W_4 $\quad$ W_5

④ Cube Graph:

The n-dimensional hyper cube, or n-cube denoted by Q_n is the graph that has vertices representing the 2^n bit string of length n.



Order and size of Graphs:

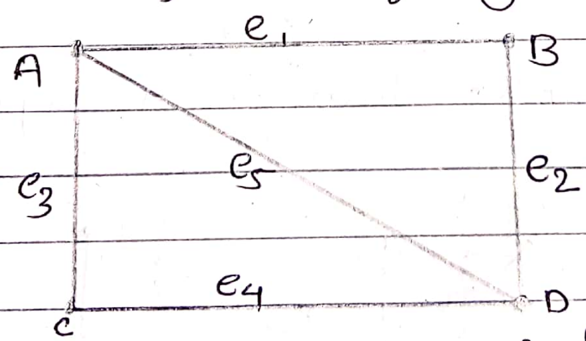
If $G=(V,E)$ is a finite graph then the number of vertices in a graph is called order of graph and the number of ~~ar~~ edges in G is called size of Graph G.

Adjacent Vertices and edges:

The ~~vertex~~ vertices U and V in an undirected graph and called Adjacent if there is an edge between U and V.

Two edges e_1 & e_2 in an undirected graph are said to be adjacent, if they shared a common vertex.

extra



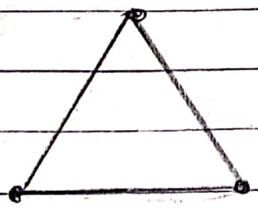
A & B are adjacent while B & C are not.
e_1 and e_2 are adjacent while e_1 and e_4 are not.

⑤ Trivial Graph:

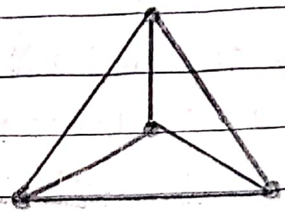
A graph with one vertex and no edges is called trivial graph.

⑥ Regular Graph:

A graph in which all vertices are of same degree is called regular graph. If the degree of vertex is 'r' then the graph is called r-Regular graph.



2-Regular graph

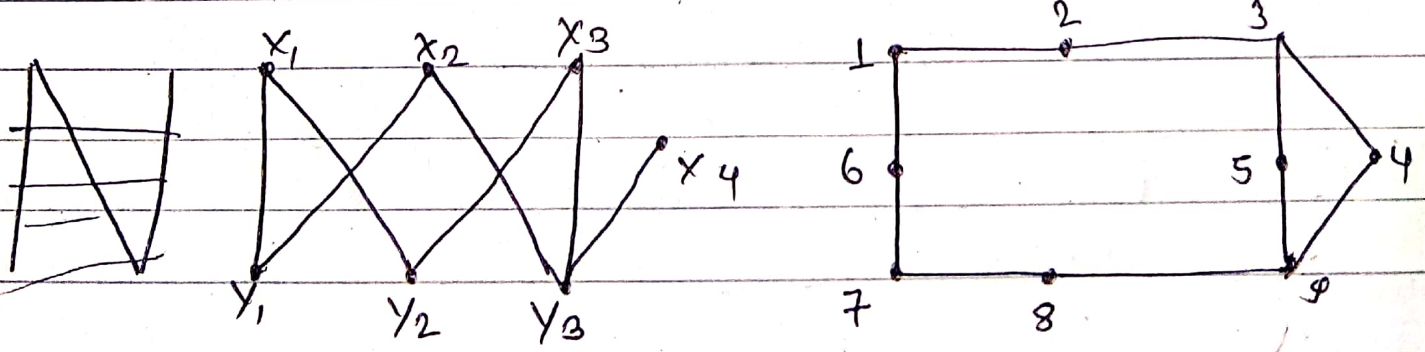


3-Regular graph

⑦ Bipartite Graph:

A graph $G=(V,E)$ is said to be bipartite of its vertex set V can be partition into two ~~subject se~~ subset V_1 & V_2 such that each edges of G connects the vertex of V_1 to vertex of V_2 so that no edges in G connect two vertices in V_1 or two vertices in V_2.

The graphs shown in the figure are bipartite.



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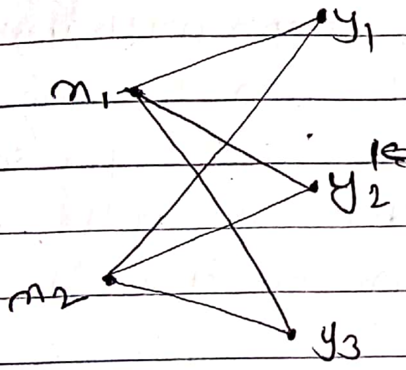
second second Sec Second

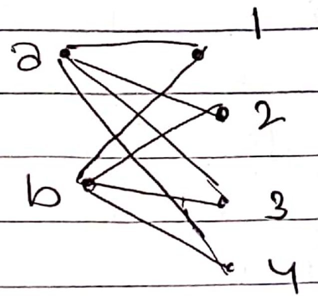
a) Bipartite sets are: $\{x_1, x_2, x_3, x_4\}$ & $\{y_1, y_2, y_3\}$
b) Bipartite sets are: $\{1, 3, 7, 9\}$ & $\{2, 4, 6, 8, 5\}$

8. Complete Bipartite Graph (K_{mn}):

A bipartite graph G is said to be complete if each vertex of first bipartite set is connected to every vertices of another bipartite set. A complete bipartite graph with m no. of vertices in first bipartite set & n no. of vertices in second bipartite set is denoted by K_{mn}.

e.g. K_{23} K_{24}



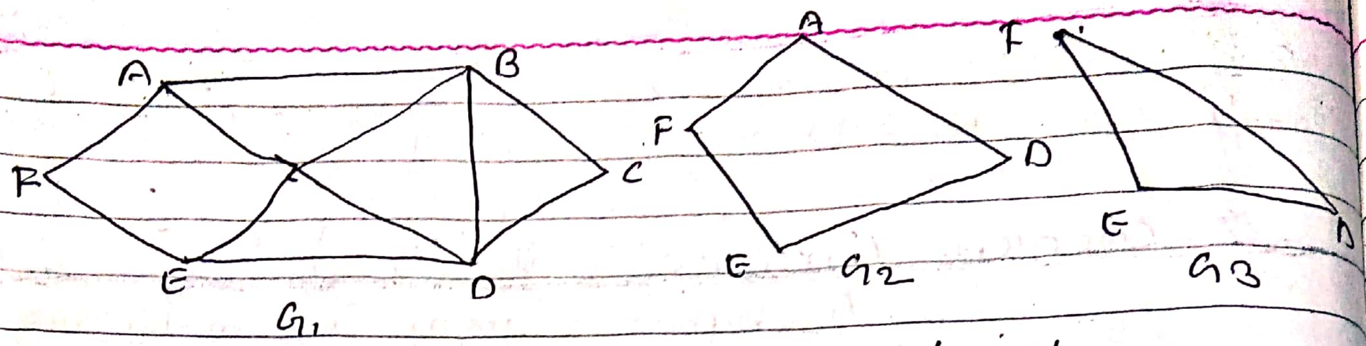


Sub Graphs:

lets $G = (V, E)$ be a graph with vertex set $V(G)$ & edge set $E(G)$ then $H = (V, E)$ be a graph with vertex set $V(H)$ & edge set $E(H)$ if H is the sub graph of G. Then

1. All the vertices of H are in G.
2. All the edges of H are in G.
3. Each edge of H has the same end points in H as in G

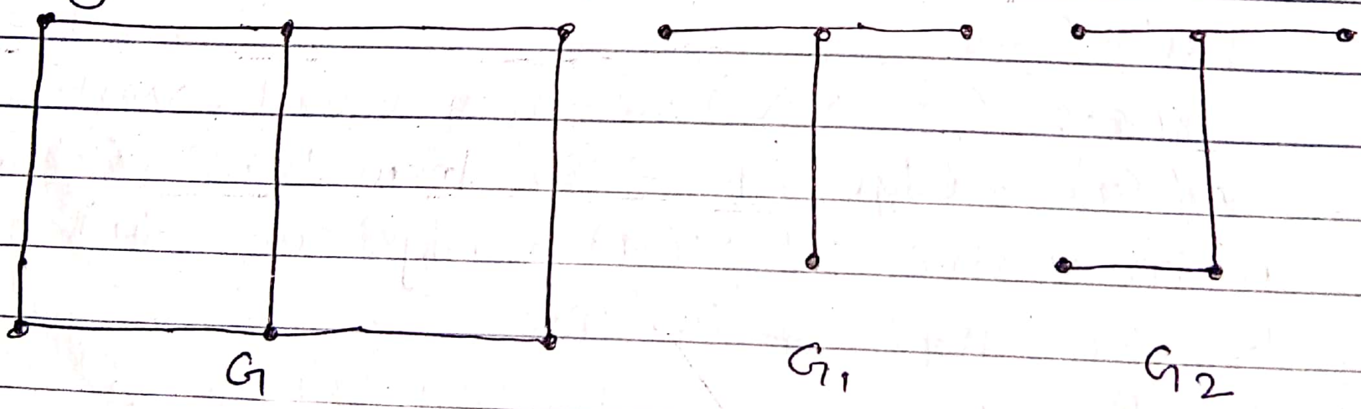
e.g.



Here, G_2 is sub Graph of G_1 but G_3 is not.

* Spanning sub Graphs: The sub graph H of graph G is said to be spanning sub graph if $V(H) = V(G)$ i.e. H & G has exactly same vertex set.

Induced sub graph: If G is a graph with vertex set V, U is a sub-set of V then the sub graph G(U) with the vertex set U and edges set consisting of those edge of G that have both ends in U is called induced sub graph of G induced by vertices in U.



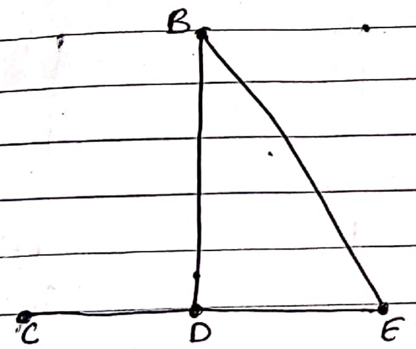
Here G_1 is induced Subgraph of G but G_2 is not.

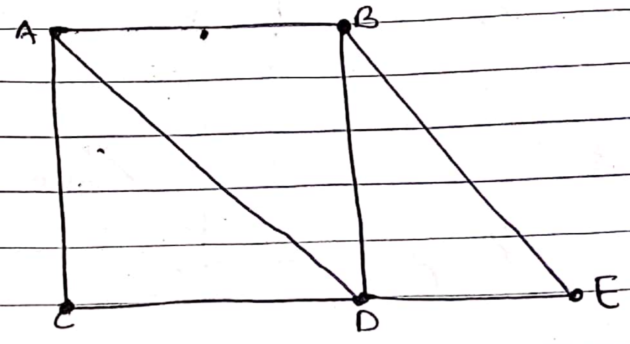
* Vertex Deleted Sub Graphs:

let, $G=(V,E)$ be a graph and S be non-empty subset of V. The induced sub graph denoted by $G-S$ is a sub graph, obtained by deleting vertices in S. In this method, to form a sub graph:

1. Remove all the vertices from $V(G)$ which are in S.
2. Remove all the edges which are incident on vertices in S.

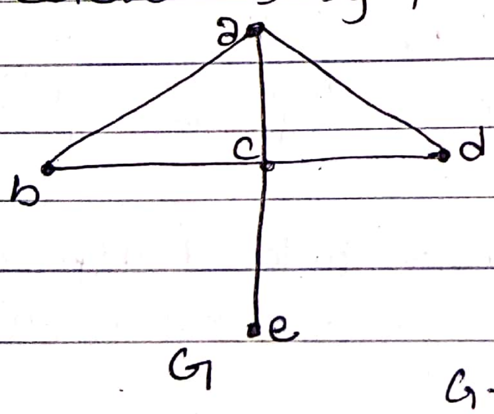
$V(G) = \{A, B, C, D, E\}$ Then $G-S$ can be obtained as let $V(S)=\{A\}$



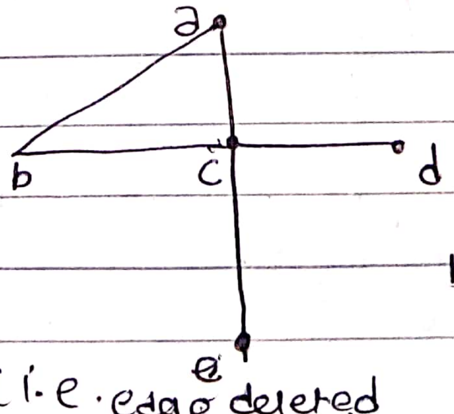


* Edges Deleted subgraphs:

If a subset F of E is deleted from Graph G then $G-F$ denoted the sub graph of G with Vertex V and edge $E-F$, then $G-F$ is called an edge deleted subgraph.

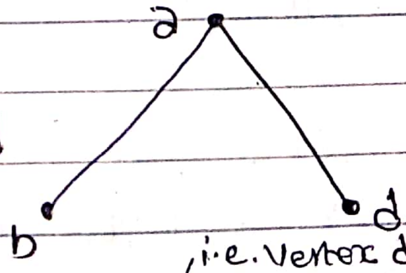


G



G-F (i.e. edge deleted sub graph)

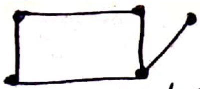
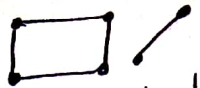
$F = \{a, d\}$



i.e. Vertex deleted Sub graph

$G-S$

$S = \{c\}$ subgraph

Note:  connected Graph  Disconnected graph

Union & Intersection of Graph:

- Union: Given two subgraphs G_1 and G_2, ~~theis~~ their union will be a graph such that.

$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

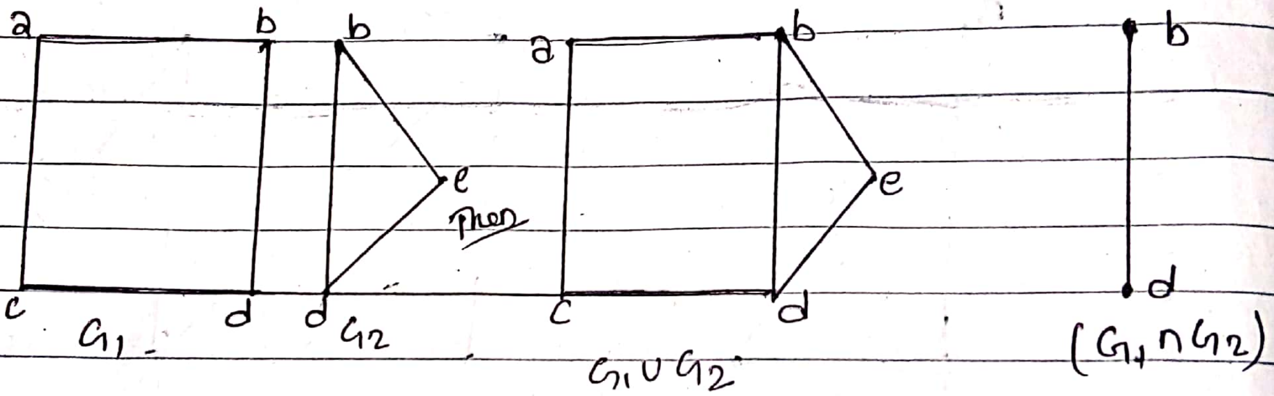
& $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

- Intersection:-

~~$V = ($~~ $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ ~~$\& E(G_1 \cap G_2) = L$~~

& $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$

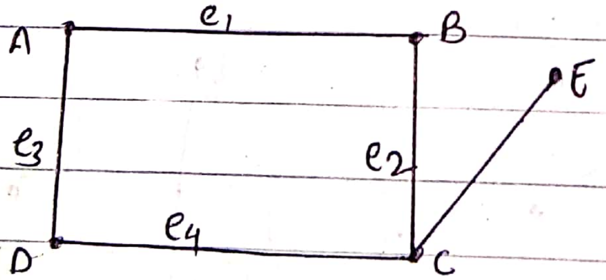
let,



Graph connectivity:

A graph G is said to be connected if there is a path between each possible pair of its vertices Otherwise G is disconnected.

① walk:



$A\, e_1\, B\, e_2\, C\, e_4\, D\, e_3\, A$

A walk in a graph G is finite ordered ~~step~~ set

$W = \{V_0, e_1, V_1, e_2, V_2, e_3, V_3, \ldots, V_{k-1}, e_k, V_k\}$

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whose element are alternately vertices & edges such that $1 \leq i \leq k$, the edge e_i has ends v_{i-1} and v_i.

This walk is a $v_0 - v_k$ walk or walk from v_0 to v_k. The number of edges appearing in a sequence of path is called its length. If the length of walk is zero i.e. the walk has no edges, it contains only a single vertex and is called trivial walk. A graph is closed if its starts and ends in same point (vertex) otherwise, the walk is open.

② Trail:

A walk $w(u,v)$ in which all the edges are distinct. ~~A walk $w(u,v)$ in wt~~ is called a Trial.

③ path: A walk in which all the edges and vertices are distinct is called a path.

④ Circuit: A closed trail which contains atleast 3 edges is called a Circuit.

⑤ cycle: A ~~co~~ circuit which doesnot repeat vertices except the intial and the final vertex is called a cycle. Thus, a cycle is a non intersecting circuit and must have length 3 or more. A cycle of length 'k' is called k-cycle.

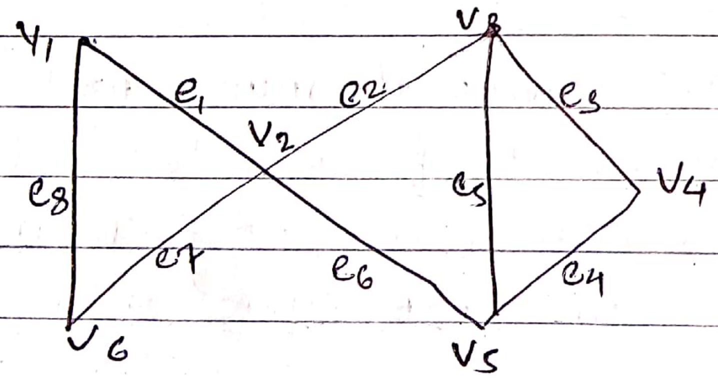
It should be noted that every

cycle is a circuit but the converse is not always true. A circuit may have repeated vertices other than the end vertices but in a cycle. the repeated vertices are ~~first~~ only the first & last.

The ~~definita~~ definition are summerized in the following table

Term	Repeated Edge	Repeated vertices
walk	~~No~~ yes	~~No(open)~~ & yes (closed & termial vertex)
Trial	No	Yes
path	No	No
circuit	No	Yes
cycle	No	Yes (first & last only)

Example:



(a) walk from v_1 to v_4 : $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4$ (open walk

walk from V_1 to V_1 = $V_1 e_1 V_2 e_6 V_5 e_5 V_3 e_2 V_2 e_7 V_6$
$e_8 V_1$ (closed walk)

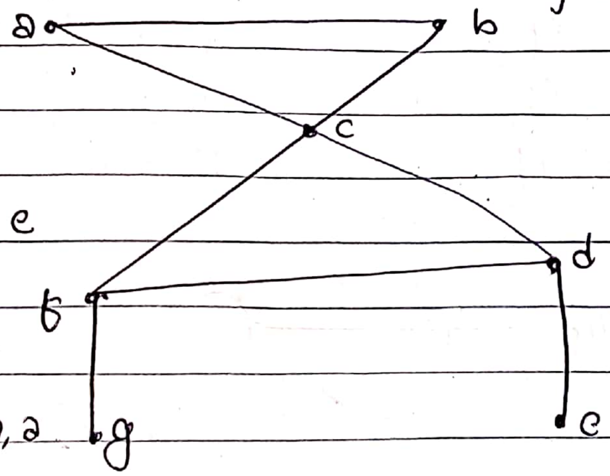
(b) Trail: walk from V_1 to V_5 : $V_1 e_1 V_2 e_6 V_5$

(c) path: $V_3 e_5 V_5 e_6 V_2 e_7 V_6$

(d) Circuit: $V_1 e_1 V_2 e_6 V_5 e_5 V_3 e_2 V_2 e_7 V_6 e_8 V_1$

(e) cycle: $V_1 e_1 V_2 e_7 V_6 e_8 V_1$

2007 spring * Explain the terms path, walk & Circuit for a given graph find cut vertices & cut edges.



path:
a, ac, c, cd, d, de, e

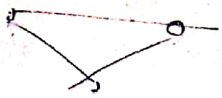
walk: a, ac, c, cd, d,
df, f, fc, c, cb, b, ba, a

circuit: " (same)

cut vertex: c, f, d
cut edges: fg, de

* Representation of Graph:

Graph can be represented in many ways. One of the representation method of graph is matrix representation, based on adjacency matrix and other is based on, of incidence of vertices and edges



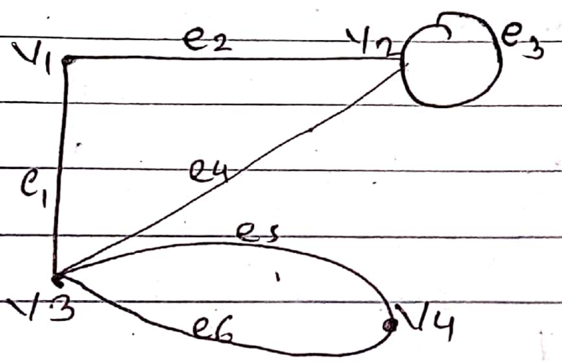
called incidence matrix. The other representation of graph is adjacency list based on the list of adjacent vertices.

① Adjacency matrix Representation:-

let, $G = (V, E)$ be a graph with n-vertices, $V_1, V_2, V_3 \ldots V_n$. The adjacency matrix of G with respect to given ordered list of vertices is $n \times n$ matrix denoted by $A_G = (a_{ij})_{n \times n}$ such that

$$a_{ij} = \begin{cases} 0 & \text{if there is no edges bet}^n \text{ the vertex } V_i \,\&\, V_j \\ 1 & \text{if " " 1 edge " " "} \\ K & \text{if " " K of edges " " "} \end{cases}$$

Imp Ex:-



$$A(G) = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 1 & 0 \\ V_2 & 1 & 1 & 1 & 0 \\ V_3 & 1 & 1 & 0 & 2 \\ V_4 & 0 & 0 & 2 & 0 \end{array}$$

② Incidence matrix representation:

let G be a graph with vertices $v_1, v_2 - - - v_n$ and edges $e_1, e_2, e_3, - - - e_n$. The incidence matrix I(G) of graph G is m×n matrix with $I(G) = (M_{ij})\ m \times n$ where,

$$M_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{if } e_j \text{ is not incident with } v_i \\ 2 & \text{if } e_j \text{ is make loop on } v_i \end{cases}$$

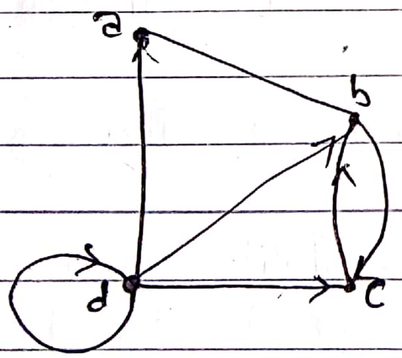
M_{ij} =

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	1	2	1	0	0
v_3	1	0	0	1	1	1
v_4	0	0	0	0	1	1

③ ~~Aja~~ Adjacency list representation (linked list):

The adjacent vertices are represented by linked list in this representation.

The linked list representation can be shown as:



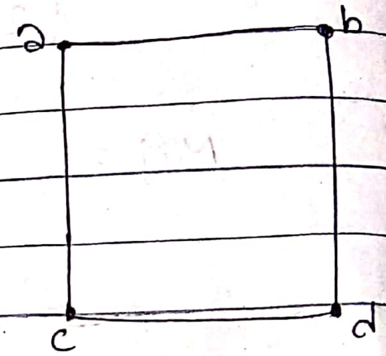
Vertex	Adjacent vertices
a	b
b	c
c	b
d	a, b, c

③ Adjacent list Representation:

One of the way of representing a graph without multiple edges is by listing its edges. This types of representation is suitable for the undirected graphs without multiple edges and directed graphs. This representation looks as below:

Edge list for simple graph	
vertex	Adjacent vertex
a	b, c
b	a, d
c	a, d
d	b, c

=>



* Isomorphism of Graphs:

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a ~~bieactive~~ bijective mapping ϕ between $V_1(G_1)$ and $V_2(G_2)$ such that $\{u, v\}$ is in E_1 if and only if $\{\phi(u), \phi(v)\}$ is in E_2. The function ϕ is called isormphism.

If two graph G_1 & G_2 are isomorphic then they must have:

(i) Same number of vertices i.e. $(|V_1| = |V_2|)$

(ii) Some number of edges i.e. $(|E_1| = |E_2|)$

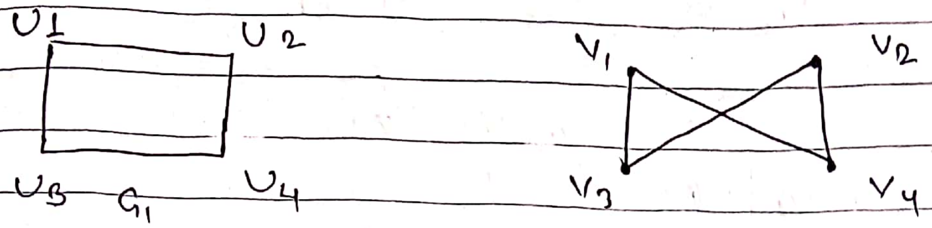
(iii) If (u, v) in E_1 then $\{\phi(u), \phi(v)\}$ in E_2.

(iv) If $\deg(u) = k$ in G_1 then $\deg(\phi(u)) = k$ in G_2

if not same order of degree write from highest to lowest degree.

(v) $A(G_1) = A(G_2)$ with respect to orders of vertices $v_1, v_2, v_3 \ldots v_n$ and $\phi(v_1), \phi(v_2), \phi(v_3) \ldots \phi(v_n)$.

Ex: Show that the graphs G_1 and G_2 are isomorphic.



Solution: The isomorphic ~~invariety~~ invariant of two graphs are:

(a) No. of vertices in G_1 $|V(G_1)| = 4$

" " G_2 $|V(G_2)| = 4$

(b) No. of edges in G_1 $|E(G_1)| = 4$

" " in G_2 $|E(G_2)| = 4$

In graph G_1 there are four vertices each of degree 2 (i.e. 2, 2, 2, 2) Similar is true in graph G_2 also.

Since, both graphs agree so many isomorphic invariants so, it is reasonable to find and isomorphism ϕ.

let, $\phi : V(G_1) \to V(G_2)$ defined by

$\phi(u_1) = v_1,\ \phi(u_2) = v_4,\ \phi(u_3) = v_3,\ \phi(u_4) = v_2$

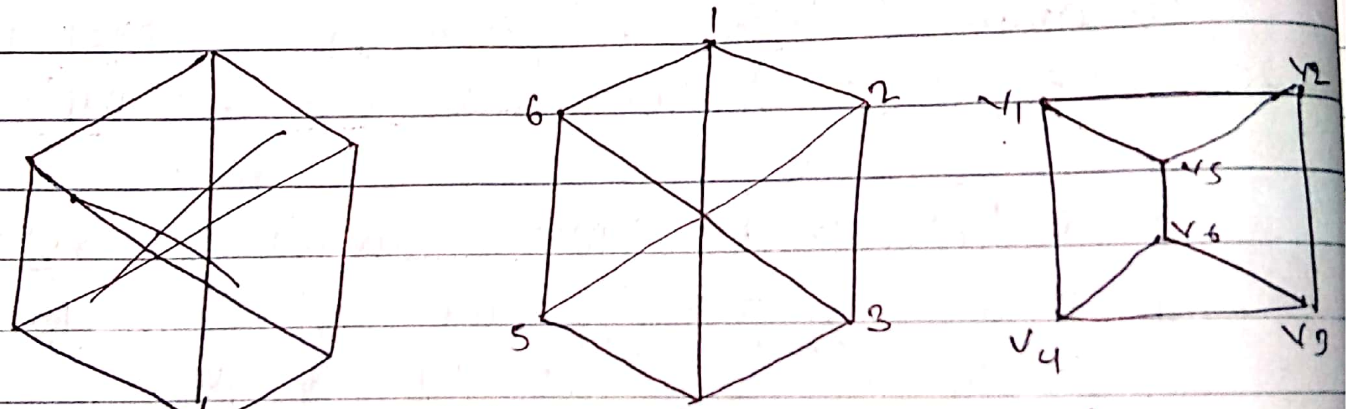
Now, adjacency matrix with respect to ordering vertices (u_1, u_2, u_3, u_4) and $(\phi(u_1)), \phi(u_2), \phi(u_3), \phi(u_4))$ are

$$A(G_1) = \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 1 & 0 \\ u_2 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_4 & 0 & 1 & 1 & 0 \end{array}$$

$$A(G_2) = \begin{array}{c|cccc} & v_1 & v_4 & v_3 & v_2 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 1 & 0 \end{array}$$

since: $A(G_1) = A(G_2)$ with respect to ordering of vertices so, ϕ is an isomorphism and graphs G_1 and G_2 are isomorphic.

* Show that G_1 and G_2 are not isomorphic.



G_2

G_1

solution:

The isomorphic invariant of two graph are:

$|V(G_1)| = |V(G_2)| = 6$

$|E(G_1)| = |E(G_2)| = 9$

Deg sequence of G_1: $(3, 5, 3, 3, 3)$

" " " G_2: $(3, 3, 3, 3, 3)$

since both graphs agree so many invariants so it is reasonable to find an isomorphic ϕ.

let,

$\phi : V(G_1) \to V(G_2)$ is defined by.

$\phi(1) = v_1$ $\quad$ $\phi(3) = v_3$ $\quad$ $\phi(5) = v_5$

$\phi(2) = v_2$ $\quad$ $\phi(4) = v_4$ $\quad$ $\phi(6) = v_6$

Now, Adjacency matrices with respect to ordering of vertices in ϕ are:

$$A(G_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

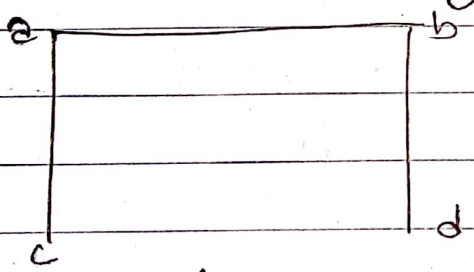
$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

since $A(G_1) \neq A(G_2)$ with respect to ordering of vertices of so ϕ is not isomorphic.

Self Complementary graph:

A Graph $G = (V, E)$ is said to be self complementary if G is isomorphic to its complement graph.

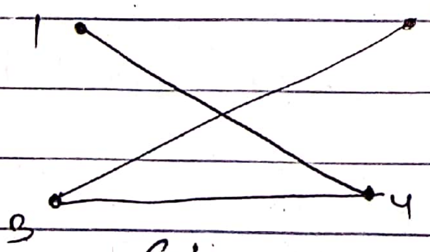
* show the following graph itself-complementary.



G_1

Soln:

As we know that the graph G is self complementary if it is isomorphic to its complement graph. Thus the complement of the given graph.



G'

Here,

$|V(G_1)| = 4 = |V(G')|$

$|E(G_1)| = 3 = |E(G')|$

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The degree of vertices in G_1 are in order: 2, 2, 1, 1
The degree of vertices in G' are in order : 2, 2, 1, 1

Since basic invariant of isomorphic are valid for G_1 and G'. ~~are in order~~ Thus we can find the mapping ϕ as follows:

$$\phi(a) = 3, \quad \phi(b) = 4$$
$$\phi(c) = 2, \quad \phi(d) = 1$$

$$A(G_1) = \begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 0 & 0 \end{array} \qquad A(G') = \begin{array}{c|cccc|} & 3 & 4 & 2 & 1 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}$$

Since,
$A(G_1) = A(G')$
i.e. G_1 and G' are isomorphic
Hence, The G is self complementary.

Exam question: show that the maximum no. of edges in a simple graph with n-vertices is $\frac{1}{2} n(n-1)$

Soln:
By handshaking theorem, we know that,

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

where $|E|$ is no. of edges in Graph G having n vertices.

Since, maximum degree of a vertex in a simple graph can be $(n-1)$ so that expanding the handshaking theorem we get.

$(n-1) + (n-1) + (n-1) \ldots + (n-1) = 2|E|_{max}$

or $n(n-1) = 2|E|_{max}$

$$\boxed{|E|_{max} = \frac{n(n-1)}{2}}$$

Hence, the maximum no. of edge in a simple graph with n-vertices is $\frac{1}{2} n(n-1)$

* How many vertices do the following graph have if contain.

(a) 16 edges and all the vertices of degree 2.

(b) 21 edges, 3 vertices of degree 4 and other vertices of degree 3.

Soln:

(a) let n = total number of vertices

we know that by handshaking theorem

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

i.e. $n \times 2 = 2 \times 16$

$n = 16$

(b) Soln:
Let n = total number of vertices

$$\sum_{i=1}^{n} deg(v_i) = 2|E|$$

$$3 \times 4 + (n-3)3 = 2 \times 21$$

$\therefore n =$

imp # Euler and Hamiltonian Graphs:

\# Eulerian Graph:

The development of the concept of all the Eulerian graph is due to the solution of the famous Konigsberg bridge problem by Swiss mathematician Leonard Euler in 1736.

Eulerian Trail:

A trail in a graph G is called Eulerian trail if it includes every edge of G and then G is called a Traversable Graph.

Eulerian circuit:

A circuit (closed trail) containing all the edges of the graph G is called an Eulerian circuit.

A graph containing an Eulerian circuit is called an Eulerian graph a simply Euler's graph.

* ~~Euler & Hand Shaking~~

* Euler & Hamiltonian Graph:

1) Initial and final vertex should be same.
2) Must include every edges.
3) Allowed repeated vertices

Eulerian trail:

1) Not same initial and final vertex.
2) Must include every edges.

Euler & Hamiltonian Graph:

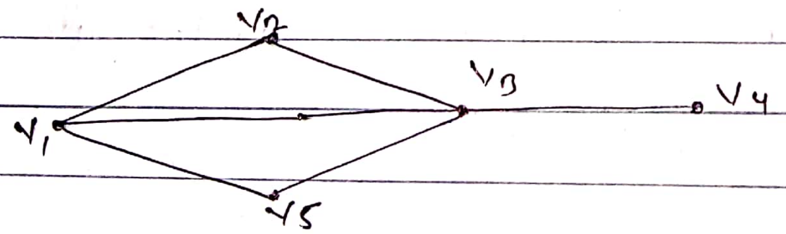
* Theorem 1:

A connected graph (multigraph) G is eulerian if and only if each vertices has even degree.

Theorem 2:

A connected graph G has an Eulerian trail if has exactly two odd degree vertices.

* Show that the graph in the figure contains as eulerian trail but no eulerian circuit.

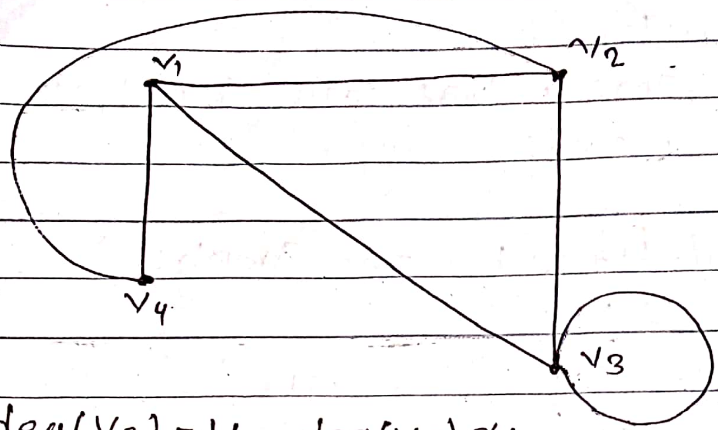


soln:

Here, G is connected and $deg(V_1) = 3$, $deg(V_2) = 2$, $deg(V_3) = 4$, $deg(V_4) = 1$, $deg(V_5) = 2$

Since the graph has exactly two odd vertices, it contains Eulerian trail but no. Eulerian circuit.

* Show that the graph has no Eulerian but has Eulerian trail.



Soln.
Given:
$deg(v_1) = 3$ $deg(v_2) = 4$ $deg(v_3) = 4$
$deg(v_4) = 3$
Eulerian Trail: $v_1 \to v_2 \to v_3 \to v_3 \to v_1 \to v_4$ or

According to Euler's theorem, A graph has Euler's trail it is contain exactly two add degree vertices. Thus, we can say that the graph has Euler's trail.
Also A graph has Eulers circuit, if it has all vertices of even degree. Thus, the given graph has no Euler's circuit.

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* Hamiltonian Graph:

Hamitonian Graph are named after sir william Hamilton and iris mathematician who introduce the problem of finding a circuit in which all the vertices of graph appear exactly once.

A cycle that contains every vertex of graph exactly one (except first & last) is called hamiltonian circuit.

A graph G is hamiltonian if it has hamiltonian cycle. A hamiltonian path is the simple path that contains all vertices of G.

* Hamiltonian cycle properties

↳ same initial and final vertex

↳ May not include all edges.

↳ Not allowed repeated vertices.

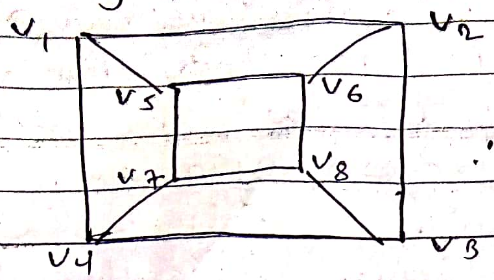
Hamiltonian circuit:

↳ same initial & final vertex

↳ include all edges

↳ Allowed repeated vertices.

* If the graph is Hamiltonian:



∴ Hamiltonian cycle

Solution:

The graph in the figure is Hamiltonian because it Contain a cycle covering all the Vertices:

$V_1 \to V_2 \to V_3 \to V_4 \to V_7 \to V_8 \to V_6 \to V_5 \to V_1$

Theorem1:

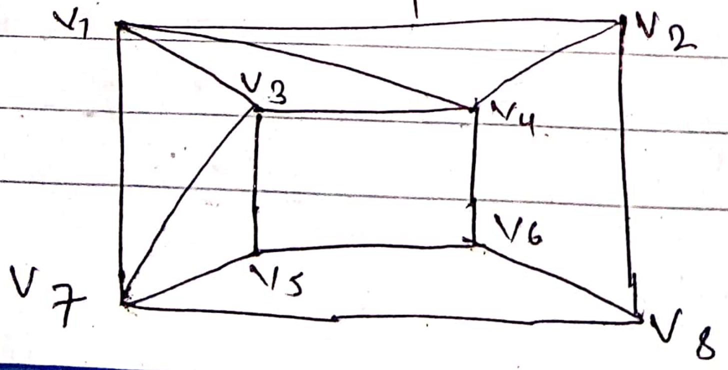
A simple Connected graph with $n \geq 2$ Vertices is Hamiltonian if ~~for any~~ degree of every vertex is at least $n/2$.

Theorem2: A connected graph with n vertices is Hamiltonian, if for any two non-adjacent vertices u and v,

$\deg(u) + \deg(v) \geq n$

This is also Known as Ore's theorem.

* Determine in Graph of fig: if there is an Eulerian circuit and/or Hamiltonian cycle.

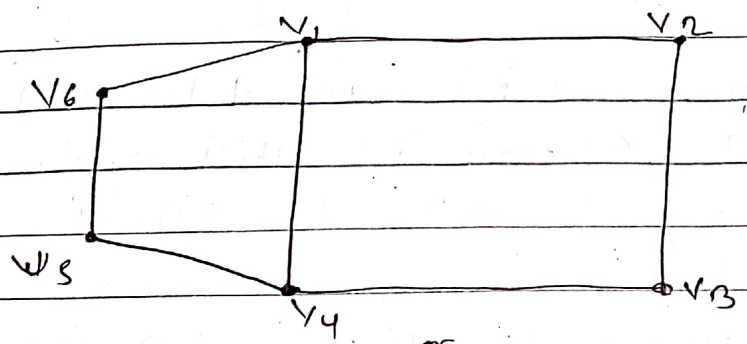


deg(v_1) = 4   deg(v_2) = 3   deg(v_3) = 4   deg(v_5) = 3

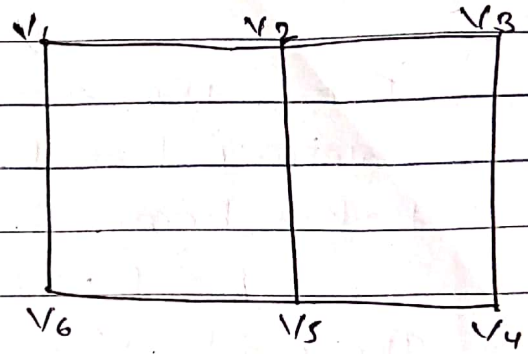
A graph is Eulerian graph if it contain all the vertices of even degree, Here the given graph has some odd degree vertices. so the graph is not Eulerian graph.

Here we can find the Hamiltonian cycle as.
$V_1 \to V_2 \to V_8 \to V_7 \to V_5 \to V_6 \to V_4 \to V_3 \to V_1$

Give an example of graph six vertices which is Hamiltonian but not Eulerian.



or,



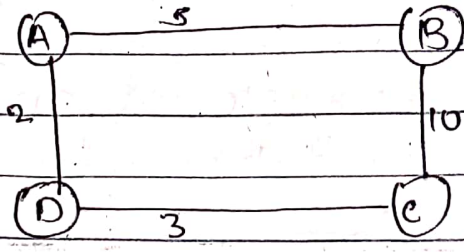
Travelling salesman problem (TSP - problem)

The problem was first formulated in 1930 and is one of the most intensively studied problem in graph theory and optimization.

The general TSP problem can be formulated as follows:

Given a list of cities represented by vertex or class and distance between each pair of cities, the problem is to find the shortest possible router path that visit each exactly one and return to the origin or starting point.

The name travelling salesman problem is given as it is analogues to the problem like a salesman one to visit each of H city exactly once for his sales and return to his house (starting city) at the end of journey.



From the above graph, the minimum distance a salesman has travel to cover all the cities and return to his home A starting from A. The path is

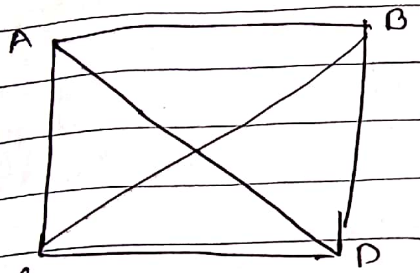
A → B → C → D → A

The most straight forward solution to TSP is to examine all possible Hamiltonian cycle and select the circuit with minimum length.

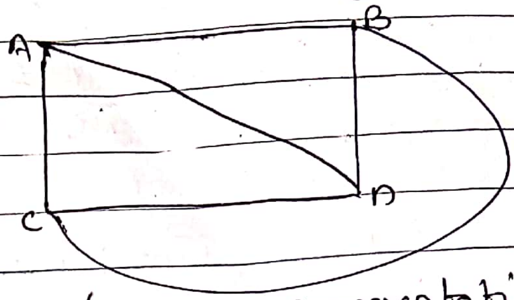
plannar graph:

A graph G=V is set to be plannar graph it can be drawn in a plane such that no intersection of edges exist at a point other than their common end point i.e. common vertex. A graph is plannar if it is drawn without crossing the edges, such drawing is called a planar representation of a graph.

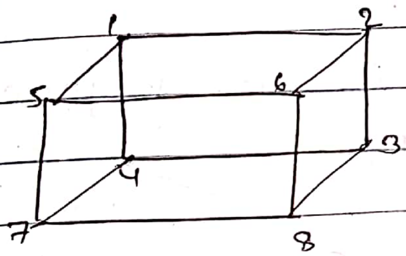
A graph may be plannar if it is usually drawn with crossing, Because it may be possible to draw it in a different way without crossing.



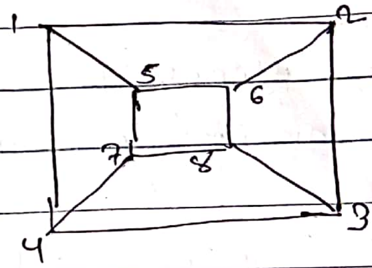
K_4: (non-plannar representation)



K_4: (plannar representation)



Q_3: (non-planner repr^n)



Q_3: (plannar representation)

Euler's Formula:

let G be a connected simple graph with V-vertices e-edges and r - no. of regions in plannar representation of G. Then

$$r = e - v + 2$$

(1) suppose that a connected planar graph has 26 vertices, each of degree 4. Into how many regions does a representation of this plannar split the plane?

Soln:

Given that,

no. of vertices $(V) = 20$

The degree of each vertex $= 4$

H.S.T we know that,

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

i.e. $20 \times 4 = 2|E|$

$\therefore |E| = 40$

again, From Euler's Formula,

$r = e - v + 2$

$= 40 - 20 + 2$

$= 22$

Hence, The plannar representation of given graph split the plane into 22 regions.

q. Suppose that a connected plannar graph has 30 edge. If a plannar representation of this graph divides the plane into 20 regions. How many vertices does this graph has:

Soln:- given.

Q. Suppose that a simple plannar has 6 vertices where 3 vertices are of degree 2 and 3 vertices are of degree 4. Into how many regions does a representation of this plannar graph split the plane.

Soln :-

no. of vertices $(V) = 6$

degree of each vertices $(2 \times 3) + (3 \times 4)$

$= 6 + 12 = 18$

By H.S.T, we get,

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

$$6 \times 6 = 2|E|$$

* ~~Example~~ The shortest path Algorithm * :

A weighted graph G in which each edge 'e' assign a non negative real number $w(e)$, called the weight of edge e.

Consider a weighted graph G. The ~~weight~~ length of a path in a weighted group is the sum of the weights of the edges of this path and the shortest path between the two vertices is the minimum length of the path. There are several different algorithm to find the shortest path between two vertices in a weighted graph.

Dijkstra's Algorithm

Step 1: label the initial vertex of the group with weight zero.

Step 2: calculate the weights of all vertices adjacent to the initial vertex corresponding to the weights of the edges incident on the initial ~~vertices~~ vertex.

Step 3: label these vertices with smallest possible values of their weight.

Step 4: calculates the weight of all those vertices which are adjacent to the vertices with the minimum weight determine in step 3.

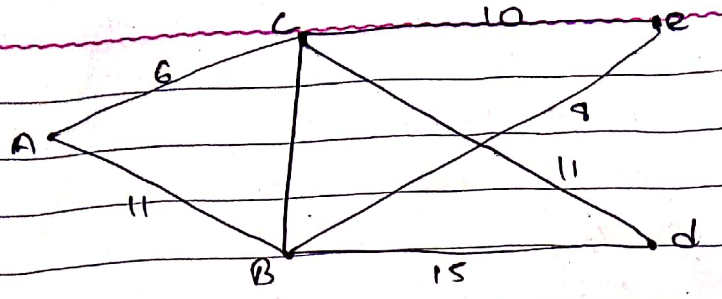
Step 5: label these vertices with minimum weight.

~~Step 6:~~

continue this process until all the vertices of weight graph is labeled.

Step 7: Trace the path ~~[illegible]~~ cumulative minimum weight from initial vertex to desire vertex.

* Apply Dijkstra's algorithm to find the shortest path from vertex A to each of the other vertices of the following weighted graph.



Soln:
Since we are applying dijkastra's shortesta path algorithm, assign the weight 0 to the initial vertex a and ∞ to all other vertices, And calculating the minimum weight to the adjacent vertices the Repeating the process we can find the shortest path from source vertex (a) to all other vertices as follows:

Vertices	a	b	c	d	e	f	g	h
Initially	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
a		8	$\boxed{2}$	5	∞	∞		
c		8		$\boxed{4}$	7	∞		
d		$\boxed{6}$			7	10		
b					$\boxed{7}$	10		
e						10		
g								

~~f~~

Vertices	a	b	c	d	e
initially	$\boxed{0}$	∞	∞	∞	∞
a		11a	$\boxed{6a}$	∞	∞
c		11c		17c	16c
b				17c	$\boxed{16c}$
e				$\boxed{17c}$	

Therefore,

The shortest path from

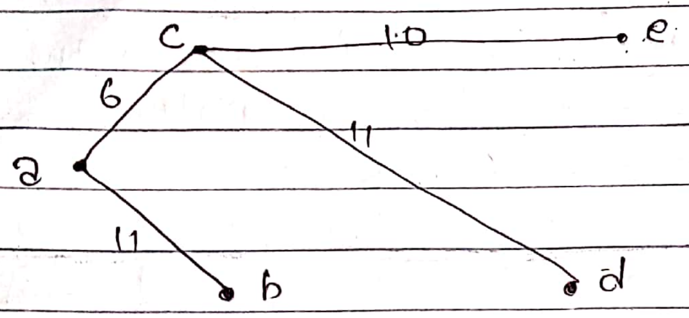
a to b is 11

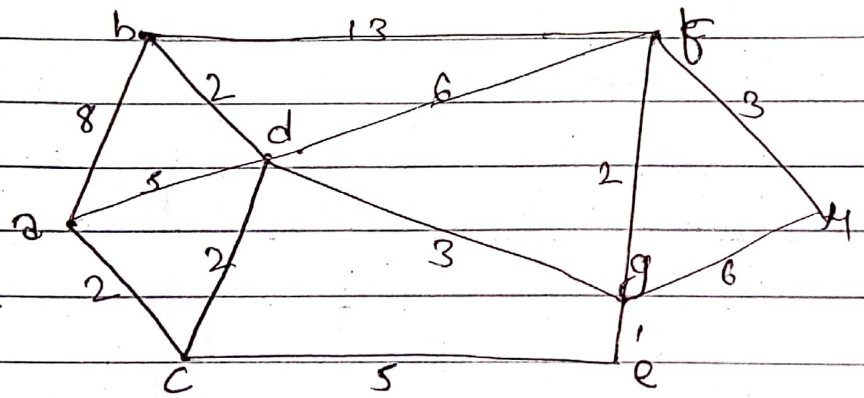
a to c is 6

a to d is 17

a to e is 16

The shortest path graph from a to all other vertices

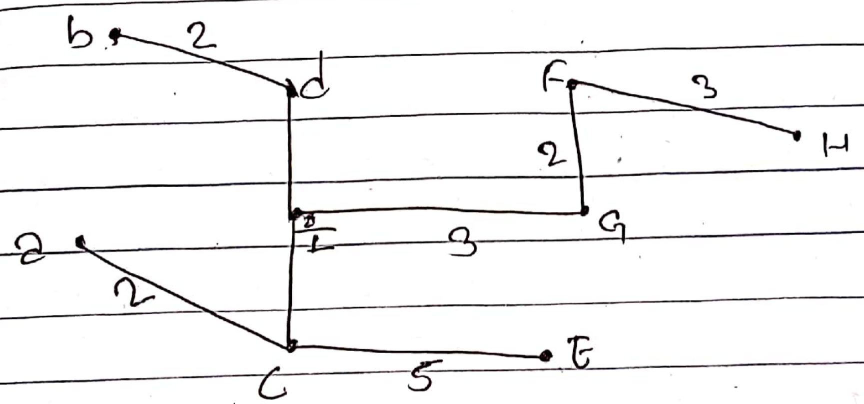




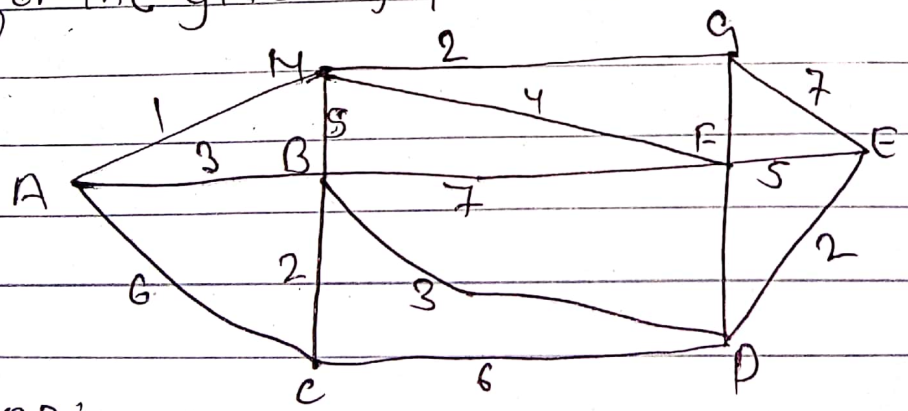
soln:

Since we are applying dijkastra's shortest path algorithm, assign the weight 0 to the initial calculating the minimum weight to the adjacent vertices the Repeating the process we can find the shortest path from source vertex (a) to all other vertices as follows:

Vertices	a	b	c	d	e	f	g	h
Initially	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
a		8	$\boxed{2}$	5	∞	∞	∞	∞
c		8		$\boxed{4}$	7	∞	∞	∞
d		$\boxed{6}$			7	10	7	∞
b					$\boxed{7}$	10	7	∞
e						10	$\boxed{7}$	∞
g						$\boxed{10}$		$\boxed{4}$
f								$\boxed{12}$



* Find the shortest path from vertex A to vertex E for the given graph.



soln:

Since we are applying dijkastra's shortest path algorithm assign the weight 0 to the initial

Vertex a and ∞ for all other vertex.

Vertical	A	B	C	D	E	F	G	H
Initially	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
A		3	6	∞	∞	∞	∞	$\boxed{1}$
H		3	6	∞	∞	5	$\boxed{3}$	
G		$\boxed{3}$	6	∞	10	4		
B			5	6	10	$\boxed{4}$		
F			$\boxed{5}$	6	10			
C				$\boxed{6}$	$\boxed{8}$			
D