10.4 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_n y=0 \qquad \ldots (1)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is known as homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since degree of dependent variable y and all its differential coefficients is one.

Equation (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

$$f(D)y = 0$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$.

Here $D \equiv \frac{d}{dx}$ is known as differential operator.

The operator D obeys the laws of algebra.

10.4.1 General Solution of Homogeneous Linear Differential Equation

The homogeneous equation

$$f(D)y = 0 \qquad \ldots (2)$$

can be solved by replacing D by m in $f(D)$ and solving the auxiliary equation (A.E.)

$$f(m) = 0 \qquad \ldots (3)$$

The general solution of Eq. (2) depends upon the nature of the roots of auxiliary Eq. (3).

If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the A.E., following cases arise:

Case I: Real and distinct roots: If roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct, then the solution of Eq. (1) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II: Real and repeated roots: If two roots m_1, m_2 are real and equal and remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and remaining $(n-r)$ roots $m_{r+1}, m_{r+2}, \ldots, m_n$ are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})e^{m_1 x} + c_{r+1}e^{m_{r+1}x} + \ldots + c_n e^{m_n x}$$

Case III: Imaginary roots: If two roots m_1, m_2 are imaginary say, $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$ (conjugate pair) and remaining (n – 2) roots m_3, m_4, …, m_n are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pair of imaginary roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta,\ m_3 = m_4 = \alpha - i\beta$ and remaining (n – 4) roots m_5, m_6, …, m_n are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark:

(i) In all the above cases, c_1, c_2, …, c_n are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1: Solve $2D^2y + Dy - 6y = 0$.

Solution: The equation can be written as

$$(2D^2 + D - 6)y = 0$$

Auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = -2,\ \frac{3}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 2: Solve $(D^3 + D^2 - 2D)y = 0$.

Solution: Auxiliary equation is

$$m^3 + m^2 - 2m = 0$$

$$m(m^2 + m - 2) = 0$$

$$m(m - 1)(m + 2) = 0$$

$$m = 0,\ 1, -2$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-2x}$$

$$y = c_1 + c_2 e^{x} + c_3 e^{-2x}$$

Example 3: Solve $2D^2y - 2Dy - y = 0$.

Solution: The equation can be written as

$$(2D^2 - 2D - 1)y = 0$$

Auxiliary equation is

$$2m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

$$m = \frac{1+\sqrt{3}}{2}, \ \frac{1-\sqrt{3}}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{\left(\frac{1+\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{1-\sqrt{3}}{2}\right)x}$$

Example 4: Solve $D^2y + 6Dy + 9y = 0$.

Solution: The equation can be written as

$$(D^2 + 6D + 9)y = 0$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3, \ -3$$

The roots are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Example 5: Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$

$$m(m^3 - 6m^2 + 12m - 8) = 0$$

$$m(m-2)(m^2 - 4m + 4) = 0$$

$$m(m-2)(m-2)^2 = 0$$

$$m = 0, \ 2, \ 2, \ 2$$

The root $m = 2$ is repeated three times.
Hence, solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

$$y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

Example 6: Solve $(D^4 - 6D^3 + 13D^2 - 12D + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 13m^2 - 12m + 4 = 0$$

$$(m-1)^2(m-2)^2 = 0$$

$$m = 1,\ 1,\ 2,\ 2$$

The roots $m = 1$ and $m = 2$ are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{2x}$$

Example 7: Solve $(D^4 + 4D^2)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4m^2 = 0$$

$$m^2(m^2 + 4) = 0$$

$$m = 0,\ 0 \text{ and } m^2 = -4,\ m = \pm 2i$$

The root $m = 0$ is real and repeated twice and two roots are imaginary with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = (c_1 + c_2 x)e^{0x} + c_1 \cos 2x + c_2 \sin 2x$$

$$= c_1 + c_2 x + c_1 \cos 2x + c_2 \sin 2x$$

Example 8: Solve $(D^4 + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4 = 0$$

$$m^4 + 4 + 4m^2 - 4m^2 = 0$$

$$(m^2 + 2)^2 - (2m)^2 = 0$$

$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$

$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

$$m = -1 \pm i \text{ and } m = 1 \pm i$$

The roots are imaginary with $\alpha_1 = -1,\ \beta_1 = 1$ and $\alpha_2 = 1,\ \beta_2 = 1$.
Hence, solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$$

Example 9: Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution: Auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m^2 - 4m + 4) = 0$$
$$(m-1)(m-2)^2 = 0$$
$$m = 1,\ 2,\ 2$$

The roots are real and distinct, but second root $m = 2$ is repeated twice.
Hence, solution is

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

Example 10: Solve $(D^4 + 8D^2 + 16)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$
$$(m^2 + 4)^2 = 0$$
$$m = \pm 2i,\ \pm 2i$$

The pair of roots is imaginary and repeated twice with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$
$$= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Example 11: Solve $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.

Solution: Auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$
$$m^2 + 1 = 0,\ m^2 + m + 1 = 0$$
$$m = \pm i, m = \frac{-1 \pm i\sqrt{3}}{2}$$

Both pair of roots are imaginary and first pair is repeated thrice with $\alpha = 0,\ \beta = 1$ and second pair is repeated twice with $\alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$$

$$+ e^{-\frac{x}{2}}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right]$$

Example 12: Solve $(D^3 - 2D^2 - 5D + 6)y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

Solution: Auxiliary equation is

$$m^3 - 2m^2 - 5m + 6 = 0$$
$$(m-1)(m^2 - m - 6) = 0$$
$$(m-1)(m+2)(m-3) = 0$$
$$m = 1,\ -2,\ 3$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x} \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = c_1 e^x - 2c_2 e^{-2x} + 3c_3 e^{3x} \qquad \text{... (2)}$$

Differentiating Eq. (2),

$$y'' = c_1 e^x + 4c_2 e^{-2x} + 9c_3 e^{3x} \qquad \text{... (3)}$$

Putting $x = 0$ in Eqs. (1), (2) and (3),

$$y(0) = c_1 + c_2 + c_3$$
$$0 = c_1 + c_2 + c_3$$
$$c_1 + c_2 + c_3 = 0 \qquad \text{... (4)}$$
$$y'(0) = c_1 - 2c_2 + 3c_3$$
$$0 = c_1 - 2c_2 + 3c_3$$
$$c_1 - 2c_2 + 3c_3 = 0 \qquad \text{... (5)}$$
$$y''(0) = c_1 + 4c_2 + 9c_3$$
$$1 = c_1 + 4c_2 + 9c_3$$
$$c_1 + 4c_2 + 9c_3 = 1 \qquad \text{... (6)}$$

Solving Eqs. (4), (5) and (6),

$$c_1 = -\frac{1}{6},\ c_2 = \frac{1}{15},\ c_3 = \frac{1}{10}$$
$$y = -\frac{1}{6}e^x + \frac{1}{15}e^{-2x} + \frac{1}{10}e^{3x}$$

Example 13: Solve $(D^3 + \pi^2 D)y = 0$, $y(0) = 0$, $y(1) = 0$, $y'(0) + y'(1) = 0$.

Solution: Auxiliary equation is

$$m^3 + \pi^2 m = 0, \qquad m(m^2 + \pi^2) = 0$$
$$m = 0, \qquad m = \pm\, i\pi$$

First root is real and second pair of roots is imaginary with $\alpha = 0,\ \beta = \pi$.
Hence, solution is

$$y = c_1 + c_2 \cos \pi x + c_3 \sin \pi x \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = 0 - c_2 \cdot \pi \sin \pi x + c_3 \cdot \pi \cos \pi x \quad \text{... (2)}$$

Putting $x = 0$ in Eqs. (1) and (2) and using given initial conditions,

$$y(0) = 0, \quad c_1 + c_2 = 0 \quad \text{... (3)}$$

$$y(1) = 0, \quad c_1 - c_2 = 0 \quad \text{... (4)}$$

$$y'(0) + y'(1) = 0$$

$$\pi c_3 - \pi c_3 = 0$$

Solving Eqs. (3) and (4),

$c_1 = 0$, $c_2 = 0$ and c_3 cannot be determined.

Hence, solution is

$y = c_3 \sin \pi x$, where c_3 is arbitrary constant.

Exercise 10.8

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{-2x} + c_2 e^{x}\right]$

2. $(4D^2 + 8D - 5y) = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$

3. $(D^2 - 4D - 12)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{6x} + c_2 e^{-2x}\right]$

4. $(D^2 + 2D - 8)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-4x}\right]$

5. $(D^2 + 4D + 1)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$

6. $(4D^2 + 4D + 1)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-\frac{x}{2}}\right]$

7. $(D^2 + 2\pi D + \pi^2)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-\pi x}\right]$

8. $(9D^2 - 12D + 4)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{2x}{3}}\right]$

9. $(25D^2 - 20D + 4)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{2x}{5}}\right]$

10. $(9D^2 - 30D + 25)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{5x}{3}}\right]$

11. $(D^2 - 6D + 25)y = 0.$

$\left[\textbf{Ans.: } y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)\right]$

12. $(D^2 + 6D + 11)y = 0.$

$\left[\textbf{Ans.: } y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0.$

$\left[\textbf{Ans.: } y = e^{ax}(c_1 \cos bx + c_2 \sin bx)\right]$

14. $(D^3 - 9D)y = 0.$

$\left[\textbf{Ans.: } y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$

15. $(D^3 - 3D^2 - D + 3)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$

16. $(D^3 - 6D^2 + 11D - 6)y = 0.$

$\left[\textbf{Ans.: } y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$

17. $(D^3 - 6D^2 + 12D - 8)y = 0.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x + c_3 x^2)e^{2x}\right]$

18. $(D^3 + D)y = 0.$

$\left[\textbf{Ans.: } y = c_1 + c_2 \cos x + c_3 \sin x\right]$

19. $(D^3+5D^2+8D+6)y=0.$

$\left[\textbf{Ans. :}\ y=c_1e^{-3x}+e^{-x}(c_2\cos x+c_3\sin x)\right]$

20. $(8D^4-6D^3-7D^2+6D-1)y=0.$

$\left[\textbf{Ans.:}\ y=c_1e^{\frac{x}{4}}+c_2e^{\frac{x}{2}}+c_3e^{x}+c_4e^{-x}\right]$

21. $(D^4-2D^3+D^2)y=0.$

$\left[\textbf{Ans. :}\ y=c_1+c_2x+(c_3+c_4x)e^x\right]$

22. $(D^4-3D^3+3D^2-D)y=0.$

$\left[\textbf{Ans. :}\ y=c_1+(c_2+c_3x+c_4x^2)e^x\right]$

23. $(D^4+8D^2-9)y=0.$

$\left[\textbf{Ans. :}\ y=c_1e^x+c_2e^{-x}+c_3\cos 3x+c_4\sin 3x\right]$

24. $(D^4+D^3+14D^2+16D-32)y=0.$

$\left[\textbf{Ans. :}\ y=c_1e^x+c_2e^{-2x}+c_3\cos 4x+c_4\sin 4x\right]$

25. $(D^4+2D^3-9D^2-10D+50)y=0.$

$\left[\textbf{Ans. :}\ y=e^{2x}(c_1\cos x+c_2\sin x)+e^{-3x}(c_3\cos x+c_4\sin x)\right]$

26. $(D^4+18D^3+81)y=0.$

$\left[\textbf{Ans. :}\ y=(c_1+c_2x)\cos 3x+(c_3+c_4x)\sin 3x\right]$

27. $(D^4-4D^3+14D^2-20D+25)y=0.$

$\left[\textbf{Ans. :}\ y=e^x[(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x]\right]$

28. $(D^2+D-2)y=0, y(0)=4, y'(0)=-5.$

$\left[\textbf{Ans. :}\ y=e^x+3e^{-2x}\right]$

29. $(4D^2+12D+9)y=0,$
$y(0)=-1,\ y'(0)=2.$

$\left[\textbf{Ans. :}\ y=\left(\frac{x}{2}-1\right)e^{-\frac{3x}{2}}\right]$

30. $(D^2-4D+5)y=0,$
$y(0)=2,\ y'(0)=-1.$

$\left[\textbf{Ans. :}\ y=e^{2x}(2\cos x-5\sin x)\right]$

31. $(9D^2-6D+1)y=0,$
$y(1)=e^{\frac{1}{3}},\ y(2)=1.$

$\left[\textbf{Ans. :}\ y=\left[\left(e^{-\frac{2}{3}}-1\right)x+\left(2-e^{-\frac{2}{3}}\right)\right]e^{\frac{x}{3}}\right]$

32. $(4D^3-4D^2-9D+9)y=0,$
$y(0)=1,\ y'(0)=0,\ y''(0)=0.$

$\left[\textbf{Ans. :}\ y=\frac{1}{5}\left(9e^x-5e^{\frac{3x}{2}}+e^{\frac{-3x}{2}}\right)\right]$

33. $(D^3+D^2-2)y=0,\ y(0)=2,$
$y'(0)=2,\ y''(0)=-3.$

$\left[\textbf{Ans. :}\ y=e^x+e^{-x}(\cos x+2\sin x)\right]$

34. $(D^4-3D^3)=0,\ y(0)=2,$
$y'(0)=5,\ y''(0)=15,\ y'''(0)=27.$

$\left[\textbf{Ans. :}\ y=1+2x+3x^2+e^{3x}\right]$

35. $(D^4-3D^3+2D^2)y=0,\ y(0)=2,$
$y'(0)=0,\ y''(0)=2, y'''(0)=2.$

$\left[\textbf{Ans. :}\ y=2(e^x-x)\right]$