Differentiation formulae

① $\frac{d}{dx}(uv) = u'v + uv'$

② $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

$= \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

③ $\frac{d}{dx}(x^n) = nx^{n-1}$

④ $\frac{d}{dx}(\sin x) = \cos x$

⑤ $\frac{d}{dx}(\cos x) = -\sin x$

⑥ $\frac{d}{dx}(\tan x) = \sec^2 x$

⑦ $\frac{d}{dx}(\cot x) = -\mathrm{Cosec}^2 x$

⑧ $\frac{d}{dx}(\sec x) = \sec x \tan x$

⑨ $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{Cosec}\, x \cot x$

⑩ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$= \frac{-d}{dx}(\cos^{-1}x)$

⑪ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$= \frac{-d}{dx}(\cot^{-1}x)$

⑫ $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{-d}{dx}(\mathrm{Cosec}^{-1}x)$

⑬ $\frac{d}{dx}(e^x) = e^x$

⑭ $\frac{d}{dx}(a^x) = a^x \log a$

⑮ $\frac{d}{dx}(\log x) = \frac{1}{x}$

⑯ $\frac{d}{dx}(|x|) = \frac{|x|}{x}$

⑰ $\frac{d}{dx}(\sinh x) = \cosh x$

⑱ $\frac{d}{dx}(\cosh x) = \sinh x$

⑲ $\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x$

⑳ $\frac{d}{dx}(\coth x) = -\mathrm{Cosech}^2 x$

(21) $\frac{d}{dx}(\mathrm{sech}\, x) = -\mathrm{sech}\, x \cdot \tanh x$

(22) $\frac{d}{dx}(\mathrm{cosech}\, x) = -\mathrm{cosech}\, x \coth x$

(23) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

(24) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

(25) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

$= \frac{d}{dx}(\coth^{-1}x)$

* Leibnitz rule

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + u'''v_4 - \cdots\cdots$

where $v_1 = \int v \cdot dx$, $v_2 = \iint v \cdot dx\,dx$, $v_3 = \iiint v\,dx\,dx\,dx$ ---

$u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$, $u''' = \frac{d^3u}{dx^3}$

Integration formulae

(1) $\int 1\,dx = x + c$

(2) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(3) $\int \frac{1}{x} dx = \log x + c$

(4) $\int a^x dx = \frac{a^x}{\log_e^a} + c$

(5) $\int \sin x\,dx = -\cos x + c$

(6) $\int \cos x\,dx = \sin x + c$

(7) $\int \sec^2 x\,dx = \tan x + c$

(8) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(9) $\int \sec x \tan x\,dx = \sec x + c$

(10) $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + c$

(11) $\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1} x = -\cos^{-1} x$

(12) $\int \frac{1}{1+x^2} dx = \tan^{-1} x = -\cot^{-1} x + c$

(13) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x = -\operatorname{cosec}^{-1} x$

(14) $\int \sinh x\,dx = \cosh x + c$

(15) $\int \cosh x\,dx = \sinh x + c$

(16) $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

(17) $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

(18) $\int \operatorname{sech} x \cdot \tanh x\,dx = -\operatorname{sech} x + c$

(19) $\int \operatorname{cosech} x \cdot \coth x\,dx = -\operatorname{cosech} x + c$

(20) $\int \frac{1}{\sqrt{1+x^2}} = \sinh^{-1} x = \log\left[x + \sqrt{x^2+1}\right]$

(21) $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x = \log\left[x + \sqrt{x^2-1}\right]$

(22) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

(24) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

(25) $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$

(26) $\int \tan x\,dx = \log|\sec x| + c$

(27) $\int \cot x\,dx = \log|\sin x| + c$

(28) $\int \sec x \cdot dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

$= \log|\sec x + \tan x| + c$

(29) $\int \operatorname{cosec} x\,dx = \log\left|\tan \frac{x}{2}\right| + c$

$= \log|\operatorname{cosec} x - \cot x| + c$

(30) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

(31) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c.$

(32) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$

(33) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

(34) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c$

(35) $\int \sqrt{x^2-a^2}\,dx = -\frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{x^2-a^2} + c$

(36) $\int \sqrt{x^2+a^2}\,dx = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{x^2+a^2}.$

(37) $\int uv \cdot dx = u\int v \cdot dx - \int\left(u' \cdot \int v\,dx\right)dx.$

(38) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx =$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n\text{-even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & \text{if } n\text{-odd} \end{cases}$$

Trigonometric formulae

① $\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta - \tan^2\theta = 1$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$

② $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

③ $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

$\boxed{\sin^2\theta = \dfrac{1-\cos 2\theta}{2}}$

$\boxed{\cos^2\theta = \dfrac{1+\cos 2\theta}{2}}$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

④ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

⑤ $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

$-[\cos(A+B) + \cos(A-B)] = 2\sin A\sin B$.

⑥ $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

⑦ $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

⑧ $\cosh^2 x - \sinh^2 x = 1$

$\operatorname{sech}^2 x + \tanh^2 x = 1$

$\coth^2 x - \operatorname{cosech}^2 x = 1$

⑨ $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

$\cosh^{-1} x = \log(x + \sqrt{x^2-1})$

$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

⑩ $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$(e^{i\theta})^n = e^{ni\theta} = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

⑪ $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

⑫ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

(1) $\int e^{ax}\sin bx \cdot dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

(2) $\int e^{ax}\cos bx \cdot dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\cos bx - b\sin bx\right]$

(3) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\cdot dx, & \text{if } f(x) \text{ - is even function} \\ 0, & \text{if } f(x) \text{ is odd function} \end{cases}$

even function: if $f(-x) = f(x)$

Ex: x^2, $x^4 + x^2$, $\cos x$, - - - etc.

odd function: if $f(-x) = -f(x)$

Ex: x, x^3+x, $\sin x$ - - - - etc.

(4) $\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

(5) $\int_{0}^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \begin{cases} \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2} \cdots \dfrac{1}{m+2}\,\dfrac{m-1}{m} \cdots \dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } \boxed{m,n\text{-even}} \\ \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2} \cdots \dfrac{1}{m+2}\,\dfrac{m-1}{m} \cdots \dfrac{2}{3} & \text{if } \left(\begin{matrix} n\text{-even} \\ m\text{-odd} \end{matrix}\right) \text{ (or) } \left(\begin{matrix} n\text{-odd} \\ m\text{-even} \end{matrix}\right) \end{cases}$

Trigonometric ratio values

θ (trigonometric angle)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0	∞
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞	-1
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\dfrac{2}{\sqrt{3}}$	1	∞

UNIT - I

1st order Differential Equations:

Definition:

* Differential Equation:

An equation involving derivatives of one or more ~~in~~dependent variables with respect to one or more independent variables is called a differential equation.

Ex: ① $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x$

* Types of D.E:

① Ordinary Differential Equations

② Partial Differential Equations

① Ordinary differential equation

→ A differential equation is said to be ordinary if the derivatives of one dependent variable with respect to one independent variable.

Ex: $\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 5y = \cos x$.

2) Partial Differential Equations:

→ A differential equation is said to be partial if the derivatives of one dependent variable w.r.t two or more independent variables.

Ex:- ① $\frac{\partial^2 y}{\partial x^2} = \frac{1}{e^2} \frac{\partial^2 y}{\partial t^2}$ → one dimensional. wave equation.

② $\frac{\partial^2 y}{\partial x^2} = \frac{1}{e^2} \frac{\partial y}{\partial t}$ → one dimensional. heat eq'n

③ $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0$ → Two-dimensional Laplace eq'n.

* Order of Differential equation:

The highest derivative in given differential equation is called order of differential equation

* Degree of Differential equation:

The highest power of highest derivative in given differential equation after made it free from radicals and fractions.

Ex:

(1) Differential Equation	Order	Degree
(1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x.$	2	1
(2) $x = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{dy}{dx}}$ $\left(x\frac{dy}{dx}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$	1	6
(3) $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{d^{\cancel{2}}y}{dx}\right)^2}$ $\left(y - x\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2.$ $\cancel{y^2+x^2}$	1	2.
(4) $x = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1/2}}{\frac{d^2y}{dx^2}}$ $x\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1/2}$ $x^2\left(\frac{d^2y}{dx^2}\right)^2 = \frac{1}{1+\left(\frac{dy}{dx}\right)^2}$	2	2

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1) Solution of differential equation:

Any relation between dependent and independent variables, not containing their derivatives, which satisfies the given differential equation is called solution of differential equation

Ex:- $y = A\cos x + B\sin x$ is solution of

$$\frac{d^2y}{dx^2} + y = 0$$

2) General solution:

A solution containing no. of independent arbitrary constants is equal to order of differential equation then it is called a general solution.

Ex:- $y = c_1 e^{x} + c_2 e^{2x}$ is general solution of

$$y'' - 3y' + 2y = 0$$

$$\boxed{\text{Order} = \text{no. of arbitrary constants}}$$

3) Particular solution:-

A solution obtained from general solution of differential eq'n by giving particular values to arbitrary constants is called a particular solution.

Ex:- ~~C_1~~ If $C_1 = 2$, $C_2 = 3$

then, $y = 2e^{x} + 3e^{2x}$ then particular

sol'n is $y'' - 3y' + 2y = 0$

4) Singular solution:

A solution which can't be obtained from any general solution of differential eq'n by any choice of arbitrary constants is called singular solution.

Ex: $y = (x+c)^2$ is general solution of

$$(y')^2 - 4y = 0.$$

Diff. wrt 'x'

$$y' = 2(x+c)$$

$$\frac{y'}{2} = x+c$$

$$y = \left(\frac{y'}{2}\right)^2$$

$$4y = (y')^2$$

$$\boxed{(y')^2 - 4y = 0}$$

But $\boxed{y=0}$ is also a sol'n which is not obtained by general solution. so, it is a singular solution.

* Exact Differential Eq'n :-

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a first order, first degree differential eq'n where $M(x,y)$ ~~and~~, $N(x,y)$ are real valued functions, it is called exact differential equation. if there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \;,\; \frac{\partial f}{\partial y} = N$$

Ex:- $2xy\,dx + x^2 dy = 0$

$\exists :\; f(x,y) = x^2 y$

$$\frac{\partial f}{\partial x} = 2xy = M \;,\; \frac{\partial f}{\partial y} = x^2 = N$$

$\therefore\; 2xy\,dx + x^2 dy = 0 = d(x^2 y)$

$\therefore\; x^2 y = c$ is solution.

Necessary and sufficient condition of EDE :-

The differential eq'n $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact differential equation if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

* General solution:

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

(y - constant) (terms independent of 'x' in N)

(or)

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

(terms independent of 'y' in M) (x - constant)

Q) Solve the differential equations

① $(2x-y+1)\,dx + (2y-x-1)\,dy = 0$

② $(y^2-2xy)\,dx = (x^2-2xy)\,dy$

③ $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

④ $2xy\,dy - (x^2-y^2+1)\,d\cancel{y}x = 0$

① $(2x-y+1)\,dx + (2y-x-1)\,dy = 0 \longrightarrow ①$

Compare ① $M\,dx + N\,dy = 0$

$M = 2x-y+1$, $N = 2y-x-1$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ It is Exact D.E

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* $\int M(x,y)\,dx + \int N(x,y)\,dy$

$\int (2x-y+1)\,dx + \int (2y-1)\,dy = c$

y constant

$\frac{\cancel{2}x^2}{\cancel{2}} - yx + x + \frac{\cancel{2}y^2}{\cancel{2}} - y = c$

$x^2 - yx + x + y^2 - y = c$

② $(y^2-2xy)\,dx = (x^2-2xy)\,dy \rightarrow \cancel{(1)}$

$(y^2-2xy)\,dx - (x^2-2xy)\,dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = y^2-2xy$, $N = 2xy-x^2$

$\frac{\partial M}{\partial y} = 2y - 2x$, $\frac{\partial N}{\partial x} = 2y - 2x$

$\Longrightarrow$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

It is Exact differential equation.

$\int M(x,y)\,dx + \int N(x,y)\,dy = c$

$\int (y^2-2xy)\,dx + \int 0\,dy = c$

$\cancel{y^2 x - \frac{2x^2 y}{2} + 0 = c}$

$\boxed{\Longrightarrow}$

$xy^2 - x^2 y = c$

$\boxed{xy(y-x) = c}$

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(3) $(1+e^{x/y})dx + e^{x/y}(1-\frac{x}{y})dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = 1+e^{x/y}$, $N = e^{x/y}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = -e^{x/y}\frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{-xe^{x/y}}{y^2}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + e^{x/y}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right)$$

$$= -\cancel{\frac{e^{x/y}}{y}} + \cancel{\frac{e^{x/y}}{y}} - \frac{xe^{x/y}}{y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{xe^{x/y}}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is Exact differential equation

$$\int M(x,y)dx + \int N(x,y)dy = c$$

$$\int(1+e^{x/y})dx + \cancel{\int e^{x/y}(1-\frac{x}{y})dy = c} \quad \int 0\,dy = c$$

$$\cancel{x + e^{x/y}\left(\frac{-y^2}{x}\right) = c} \qquad x + \frac{e^{x/y}}{\frac{\partial}{\partial x}\left(\frac{x}{y}\right)} = c$$

$$\cancel{\boxed{\frac{x^2 - y^2 e^{x/y}}{x} = c}} \qquad x + \frac{e^{x/y}}{1/y} = c$$

$$\boxed{x + ye^{x/y} = c}$$

4) $(x^2-y^2+1)\,dx - 2xy\,dy = 0 \longrightarrow ①$

Compare ① with $M\,dx + N\,dy = 0$

$$M = x^2-y^2+1, \quad N = 2xy$$

$$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

General sol :

$$\cancel{\int M dx} \int M(x,y)\,dx + \int N(x,y)\,dy = 0$$

$$\int (x^2-y^2+1)\,dx + \int -2xy\,dy = 0$$

$$\frac{x^3}{3} - y^2x + x - 2x\left[\frac{y^2}{2}\right] = 0.$$

$$\frac{x^3}{3} - y^2x + x - xy^2 = 0$$

$$\int (x^2-y^2+1)\,dx + \int 0\,dy = 0$$

$$\boxed{\frac{x^3}{3} - xy^2 + x = 0}$$

⑤ Solve $(xe^{xy}+2y)\frac{dy}{dx} + ye^{xy} = 0$

⑥ Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

⑦ solve $x^3\sec^2 y\frac{dy}{dx} + 3x^2\tan y = \cos x$

⑧ $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx + (x+\log|x| - x\sin y)\,dy = 0$

⑤ $(xe^{xy}+2y)\frac{dy}{dx} + ye^{xy} = 0$

$(xe^{xy}+2y)dy + ye^{xy}dx = 0$.

$ye^{xy}dx + (xe^{xy}+2y)dy = 0 \longrightarrow ①$

Compare ① with $Mdx + Ndy = 0$.

$M = ye^{xy}$, $N = xe^{xy}+2y$

$\frac{\partial M}{\partial y} = ye^{xy}(x) + e^{xy}$

$= xye^{xy} + e^{xy}$

$= e^{xy}(xy+1)$

$\frac{\partial N}{\partial x} = xe^{xy}(y) + e^{xy} + 0$

$= xye^{xy} + e^{xy}$

$= e^{xy}(xy+1)$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

It is Exact differerenential equation.

General solution

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

y-constant

$$\int e^{xy}(1+xy)\,dx + \int 2y\,dy = c$$

$$\int e^{xy}\,dx + \int xy\,e^{xy}\,dx + \frac{2y^2}{2} = c$$

$$\frac{e^{xy}}{y} + y\int x\,e^{xy}\,dx + y^2 = c$$

$$\frac{e^{xy}}{y} + y\left[x\int e^{xy}dx - \int\left(\frac{d}{dx}(x)\int e^{xy}dx\right)dx\right] + y^2$$

$$\int y e^{xy} + \int 2y\,dy = c$$

$$y\frac{e^{xy}}{y} + \frac{2y^2}{2} = c$$

$$\boxed{e^{xy} + y^2 = c}$$

(6) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$(\sin x + x\cos y + x)\,dy + (y\cos x + \sin y + y)\,dx = 0$

$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$ — (1)

Compare (1) with $M\,dx + N\,dy = 0$

$M = y\cos x + \sin y + y \quad , \quad N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad , \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is exact D.E

General solution:

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

y-constant

$$\int (y\cos x + \sin y + y)\,dx + \int \cancel{\sin x}\cdot 0\,dy = c$$

$$\boxed{y\sin x + x\sin y + xy = c}$$

(7) $x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x$

$$x^3 \sec^2 y\,dy + 3x^2 \tan y\,dx = \cos x\,dx$$

$$(3x^2 \tan y - \cos x)dx + x^3 \sec^2 y\,dy = 0 \rightarrow (1)$$

Compare (1) with $Mdx + Ndy = 0$

$$M = 3x^2 \tan y - \cos x, \quad N = x^3 \sec^2 y$$

$$\frac{\partial M}{\partial y} = 3x^2 \sec^2 y + 0 \cancel{+\sin x}$$

$$= 3x^2 \sec^2 y \cancel{+\sin x}$$

$$\frac{\partial N}{\partial x} = x^3 (0) + 3x^2 \sec^2 y.$$

$$= 3x^2 \sec^2 y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is exact D.E

equation and it is homogenous differential equation then we apply.

$$\boxed{I.F = \frac{1}{Mx+Ny}}$$

① Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

② Solve $y^2dx + (x^2 - xy - y^2)dy = 0$

③ Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

A ① $x^2y\,dx - (x^3+y^3)\,dy = 0 \longrightarrow (i)$

Compare ① with $Mdx + Ndy = 0$

$M = x^2y, \quad N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = -3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

It is not exact D.E

$\therefore$ ① is homogenous D.E

$I.F = \frac{1}{Mx+Ny}$

$= \frac{1}{x^3y - x^3y - y^4}$

$= \frac{-1}{y^4}$

Multiply ① with $I.F. = \frac{-1}{y^4}$

$$-\frac{x^2y}{y^4}dx + \left(\frac{x^3+y^3}{y^4}\right)dy = 0$$

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy = 0 \longrightarrow ②$$

Compare ② with $M_1dx + N_1dy = 0$

$$M_1 = \frac{-x^2}{y^3}, \quad N_1 = \frac{x^3}{y^4}+\frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4}, \quad \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$$

$\therefore$ ② is exact D.E

General solution:

$$\int M_1 dx + \int N_1 dy = c$$

(y-const) (terms independent of x)

$$\int \frac{-x^2}{y^3}dx + \int \frac{1}{y}dy = c$$

$$\frac{-x^3}{3y^3} + \log y = c$$

2) $y^2dx + (x^2-xy-y^2)dy = 0 \longrightarrow ①$

Compare ① with $Mdx + Ndy = 0$

$$M = y^2, \quad N = x^2-xy-y^2$$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x-y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is not exact D.E

$\therefore$ (1) is homogenous D.E

$$I.F = \frac{1}{Mx+Ny}$$

$$I.F = \frac{1}{xy^2 + x^2y - xy^2 - y^3}$$

$$= \frac{1}{y(x^2-y^2)}$$

Multiply (1) with $I.F = \frac{1}{y(x^2-y^2)}$

$$\frac{y^2}{y(x^2-y^2)}dx + \frac{(x^2-xy-y^2)}{y(x^2-y^2)}dy = 0$$

$$\frac{y}{x^2-y^2}dx + \left(\frac{x^2-y^2}{y(x^2-y^2)} - \frac{xy}{y(x^2-y^2)}\right)dy = 0$$

$$\frac{y}{x^2-y^2}dx + \left(\frac{1}{y} - \frac{x}{(x^2-y^2)}\right)dy = 0 \rightarrow (2)$$

Compare (2) with $M_1dx + N_1dy = 0$

$$M = \frac{y}{x^2-y^2}, \quad N = \frac{1}{y} - \frac{x}{(x^2-y^2)}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2-y^2)(1) - y(-2y)}{(x^2-y^2)^2} = \frac{x^2-y^2+2y^2}{(x^2-y^2)^2}$$

$$= \frac{x^2+y^2}{(x^2-y^2)(x^2+y^2)}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2-y^2}$$

$$\frac{\partial N}{\partial x} = \frac{-((x^2-y^2)1 - x(2x))}{(x^2-y^2)^2}$$

$$= \frac{-(x^2-y^2-2x^2)}{(x^2-y^2)^2}$$

$$= \frac{\cancel{x^2+y^2}}{\cancel{(x^2+y^2)}(x^2-y^2)}$$

$$= \frac{1}{x^2-y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

Eq (2) is exact D.E

General solution:-

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

$$\int \frac{y}{x^2-y^2}\,dx + \int\left(\frac{1}{y} \cancel{- \frac{x}{(x^2-y^2)}}\right)dy = c.$$

$$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$\frac{\cancel{y}}{2\cancel{y}}\log\left(\frac{x-y}{x+y}\right) + \log y = \log$$

$$\frac{1}{2}\log\left(\frac{x-y}{x+y}\right) + \log y = \log c$$

$$\log\left(\frac{x-y}{x+y}\right)^{1/2} y = \log c$$

$$\left(\frac{x-y}{x+y}\right)^{1/2} y = c.$$

(3) $(3xy^2-y^3)\,dx - (2x^2y - xy^2)\,dy = 0 \rightarrow (1)$

Compare (1) with $M\,dx + N\,dy = 0$

$M = 3xy^2 - y^3, \quad N = xy^2 - 2x^2y$

$\frac{\partial M}{\partial y} = 6xy - 3y^2, \quad \frac{\partial N}{\partial x} = y^2 - 4xy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

It is not exact D.E

$\therefore$ (1) is homogenous D.E

~~multip~~ $I.F = \frac{1}{Mx + Ny}$

$$= \frac{1}{3x^2y^2 - \cancel{xy^3} + \cancel{xy^3} - 2x^2y^2}$$

$$= \frac{1}{x^2y^2 - xy^2 + xy^3}$$

$$= \frac{1}{xy(xy - y + y^2)}$$

multiply (1) with $I.F = \frac{1}{xy(xy-y+y^2)}$

$$\frac{3xy^2 - y^3}{xy(xy-y+y^2)}\,dx - \frac{(2x^2y - xy^2)}{xy(xy-y+y^2)}\,dy = 0$$

$$\frac{\cancel{xy}(3y - y^2)}{\cancel{xy}(xy - y + y^2)}\,dx - \frac{\cancel{xy}(2x - y)}{\cancel{xy}(xy - y + y^2)}\,dy = 0$$

$$I.F = \frac{1}{3x^2y^2 - 2x^2y^2}$$

$xy - 2x^2y$

$$= \frac{1}{x^2y^2 \cancel{(3y-2)}}$$

multiply ① with $IF = \frac{1}{x^2y^2\cancel{(3y-2)}}$

$$\left(\frac{3xy^2 - y^3}{x^2y^2\cancel{(3y-2)}}\right)dx - \left(\frac{2x^2y - xy^2}{x^2y^2\cancel{(3y-2)}}\right)dy = 0$$

$$\left(\frac{x\cancel{y}(3y - y^2)}{x^{\cancel{2}}y^{\cancel{2}}\cancel{(3y-2)}}\right)dx + \left(\frac{x\cancel{y}(1 - 2x)}{x^{\cancel{2}}y^{\cancel{2}}}\right)dy = 0$$

$$\left(\frac{3y}{x\cancel{y}} - \frac{y^2}{x^2y^2}\right)dx + \left(\frac{1}{xy} - \frac{2x}{xy}\right)dy = 0$$

$$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(\frac{1}{xy} - \frac{2}{y}\right)dy = 0$$

Compare ② with $M_1 dx + N_1 dy = 0$

$$M_1 = \frac{3}{x} - \frac{y}{x^2}, \quad N_1 = \frac{1}{x} - \frac{2}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{x^2}, \quad \frac{\partial N_1}{\partial x} = \cancel{\frac{-1}{xy}} = \frac{-1}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

It is exact D.E

General sol'n:

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

$$\int \left(\frac{3}{x} - \frac{y}{x^2}\right) dx + \int -\frac{2}{y}\,dy = c$$

$$3\log x - \cancel{y\log x}\left(\frac{-y}{x}\right) - 2\log y\,\cancel{dy} = \log c$$

$$\cancel{3\log x - \left(\log \frac{x^y}{y}\right) = \log c} \qquad 3\log x + \frac{y}{x} - 2\log y = c$$

$$\log x^3 + \log \frac{x^y}{y} = \log c \qquad \boxed{\log(x^3/y^2) + \frac{y}{x} = c}$$

$$\log\left(\frac{x^{3+y}}{y}\right) = \log c$$

$$\boxed{\frac{x^{3+y}}{y} = c}$$

III) Integrating factor $= \frac{1}{Mx - Ny}$

If $Mdx + Ndy = 0$ is not exact D.E and it is of the form

$$y f(xy)\,dx + x\,g(xy)\,dy = 0$$

then we apply $I.F = \frac{1}{Mx - Ny}$

(1) Solve $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0.$

(2) solve $y(xy\sin xy + \cos xy)\,dx + x(xy\sin xy - \cos xy)dy = 0$

(3) solve $y(1+xy)\,dx + x(1-xy)\,dy = 0$

(4) $(y - xy^2)dx - (x + x^2y)\,dy = 0$

(1) $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$$M = x^2y^3 + 2y \quad , \quad N = 2x - 2x^3y^2$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2 \quad , \quad \frac{\partial N}{\partial x} = 2 - 6x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(1) is not exact D.E

Eq (1) is in the form $y f(xy)dx + x g(xy)dy = 0$

$$I.F = \frac{1}{Mx - Ny}$$

$$I.F = \frac{1}{(x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3)}$$

$$I.F = \frac{1}{3x^3y^3}$$

multiply eq ① with $IF = \frac{1}{3x^3y^3}$

$$\left(\frac{x^2y^3+2y}{3x^3y^3}\right)dx + \left(\frac{2x-2x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{\cancel{x^2}\cancel{y^3}}{3x^{\cancel{3}}\cancel{y^3}} + \frac{2\cancel{y}}{3x^3y^{\cancel{3}2}}\right)dx + \left(\frac{2\cancel{x}}{3x^{\cancel{3}2}y^3} - \frac{2\cancel{x^3}\cancel{y^2}}{3\cancel{x^3}y^{\cancel{3}}}\right)dy = 0$$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0 \longrightarrow ②$$

Compare ② with $M_1dx + N_1dy = 0$

$$M_1 = \frac{1}{3x} + \frac{2}{3x^3y^2} \quad , \quad N_1 = \frac{2}{3x^2y^3} - \frac{2}{3y}$$

$$\frac{\partial M_1}{\partial y} = \frac{-4}{3x^3y^3} \quad , \quad \frac{\partial N_1}{\partial x} = \frac{-4}{3x^3y^3} + 0$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is exact D.E $\qquad \frac{y^{-2+1}}{-2+1} \qquad \frac{y^{-1}}{-1}$

General solution:

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

(y - constant)

$$\int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx \quad \int \frac{-2}{3y}\,dy = c$$

$$\cancel{\frac{-1}{3}\log x + \frac{2}{3x^3}\left(\frac{-1}{y}\right)}$$

$$\frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\left(\frac{-1}{\cancel{2}x^2}\right) - \frac{2}{3}\log y = \log c$$

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$$\log x^{1/3} - \frac{1}{3x^2y^2} - \log y^{2/3} = \log c$$

$$\log \frac{x^{1/3}}{y^{2/3}} - \frac{1}{3x^2y^2} = c$$

$$\frac{1}{3}\left[\log x - \frac{1}{x^2y^2} - 2\log y\right] = \log c.$$

$$\log x - \frac{1}{x^2y^2} - 2\log y = 3\log c$$

$$\log x - \log y^2 - \frac{1}{x^2y^2} = \log c^3$$

$$\log\left(\frac{x}{y^2}\right) - \log c^3 = \cancel{\log c^3}\frac{1}{x^2y^2}$$

$$\therefore \boxed{\log\left(\frac{x}{y^2c^3}\right) = \frac{1}{x^2y^2}}$$

(2) $y(xy\sin xy + \cos xy)dx + x(xy\sin xy - \cos xy)dy = 0 \rightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = xy^2\sin xy + y\cos xy$, $N = x^2y\sin xy - x\cos xy$

$$\frac{\partial M}{\partial y} = xy^2\cos xy(x) + x(2y)\sin xy + y(-\sin xy)x + \cos xy.$$

$$= x^2y^2\cos xy + 2xy\sin xy - xy\sin xy + \cos xy$$

$$= x^2y^2\cos xy + xy\sin xy + \cos xy.$$

$$\frac{\partial N}{\partial x} = x^2y\cos xy(y) + 2x\sin xy - (x(-\sin xy)y + \cos xy)$$

$$= x^2y^2\cos xy + 2x\sin xy + xy\sin xy - \cos xy$$

$$= x^2y^2\cos xy + 3xy\sin xy - \cos xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(1) is not exact D.E

Eq (1) is in the form $y f(xy)\,dx + x g(xy)\,dy = 0$

$$I.F = \frac{1}{Mx - Ny}$$

$$= \frac{1}{(xy^2 \sin xy + y \cos xy)x - (x^2 y \sin xy - x \cos xy)y}$$

$$= \frac{1}{x^2y^2 \cancel{\sin xy} + xy\cos xy - x^2y^2 \cancel{\sin xy} + xy\cos xy}$$

$$= \frac{1}{2xy.\cos xy}.$$

multiply (1) with $\frac{1}{2xy \cos xy}$

$$\left(\frac{\cancel{x}y^2 \sin xy}{2\cancel{x}\cancel{y} \cos xy} + \frac{\cancel{y} \cos\cancel{xy}}{2x\cancel{y}\cos\cancel{xy}}\right)dx + \left(\frac{x^2 y \sin xy}{2x\cancel{y} \cos xy} - \frac{\cancel{x}\cos\cancel{xy}}{2\cancel{x}y \cos\cancel{xy}}\right)dy$$

$$\left(\frac{y \tan xy}{2 \cancel{\tan xy}} + \frac{1}{2x.}\right)dx + \left(\frac{\cancel{x \sin x}\, x \tan xy}{2 \cancel{\tan xy}} - \frac{1}{2y}\right)dy = 0 \rightarrow (2)$$

Compare (1) with $M_1 dx + N_1 dy = 0$

$$M_1 = \frac{y \tan xy}{2 \cancel{\tan xy}} + \frac{1}{2x}, \quad N_1 = \frac{x \tan xy}{2 \cancel{\tan xy}} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{2\tan xy\,(\cancel{1}) - 1(\sec^2 xy)x}{4\tan^2 xy.} = \frac{1}{2}\left[1.\tan xy + xy\sec^2 xy\right.$$

$$= \frac{-x\sec^2 xy}{4\tan^2 xy} + 2\tan xy.$$

$$= \frac{-x - x\tan^2 xy}{4\tan^2 xy} + 2\tan xy$$

$$\cancel{= \frac{-x\sec^2}{}}$$

$$= -\frac{x\cot^2 xy}{4} - \frac{x}{4} + 2\tan xy$$

$$= -\frac{x}{4}(1+\cot^2 xy) + 2\tan xy$$

$$\frac{\partial N_1}{\partial x} = \frac{2\tan xy(1) - x\sec^2 xy}{4\tan^2 xy} = -\frac{1}{2}[\tan xy + xy\sec^2 xy)$$

$$= -\frac{x}{4}(1+\cot^2 xy) + 2\tan xy.$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

(2) is exact D.E

General sol'n:

$dy=0$

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = 0$$

$$\int\left(\frac{y\tan xy}{2} + \frac{1}{2x}\right)dx + \int \cancel{x\cot x}\,\frac{-1}{2y}\,dy = C$$

$$\frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)dx - \frac{1}{2}\int\frac{1}{y}\,dy = \log c$$

$$\cancel{\frac{y}{2}\log|\sin xy| + \log x} - \frac{1}{2}\log y = \log c$$

$$\frac{y}{2}\log|\frac{\sec xy}{y}| + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c$$

$$\log|\sec xy| + \log\left(\frac{x}{y}\right) = 2\log c$$

$$\log\frac{(\sec xy)x}{y} = \log c^2$$

$$\boxed{x\sec xy = yc^2}$$

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③ $y(1+xy)\,dx + x(1-xy)\,dy = 0 \longrightarrow ①$

Compare ① with $M\,dx + N\,dy = 0$

$$M = y + xy^2 \quad , \quad N = (x - x^2 y)$$

$$\frac{\partial M}{\partial y} = 1 + 2xy \quad , \quad \frac{\partial N}{\partial x} = 1 - 2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is not exact D.E

Eq ① in the form of $y f(xy)\,dx + x g(xy)\,dy = 0$

$$IF = \frac{1}{Mx - Ny}$$

$$IF = \frac{1}{yx + x^2y^2 - xy + x^2y^2}$$

$$= \frac{1}{2x^2y^2}$$

multiply ① with $\frac{1}{2x^2y^2}$

$$\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx + \left(\frac{x}{2x^2y^2} - \frac{x^2y}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \longrightarrow ②$$

Compare ② with $M_1\,dx + N_1\,dy = 0$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} \quad , \quad N_1 = \frac{1}{2xy^2} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2} \quad , \quad \frac{\partial N_1}{\partial x} = \frac{-1}{2x^2y^2}$$

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$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

(2) is exact D.E

General soln:

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

$$\cancel{\int y + xy^2\,dx}$$

$$\int \frac{1}{2x^2y} + \frac{1}{2x}\,dx + \int \frac{-1}{2y}\,dy = c$$

$$\cancel{\frac{1}{2x^2}\log}\ \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c$$

$$\frac{1}{2}\left[\log x - \log y - \frac{1}{xy}\right] = \log c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = \log c^2$$

$$\boxed{\log\left(\frac{x}{yc^2}\right) = \frac{1}{xy}}$$

$-x-x^2y$

$-x(1+xy)$

$-x$

$x(-1-xy)$

(4) $(y-xy^2)\,dx - (x+x^2y)\,dy = 0$

$y(1-xy)\,dx + x(-1-xy)\,dy = 0 \rightarrow (1)$

Compare (1) with $M\,dx + N\,dy = 0$

$M = y - xy^2$, $N = -x - x^2y$

$\frac{\partial M}{\partial y} = 1 - 2xy$, $\frac{\partial N}{\partial x} = -1 - 2xy.$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is not exact D.E

Eq ① is of the form $y f(xy)dx + x g(xy) dy = 0$

Then $I.F = \frac{1}{Mx - Ny}$

$$I.F = \frac{1}{yx - x^2y^2 + xy + x^2y^2}$$

$$I.F = \frac{1}{2xy}$$

multiply ① with $\frac{1}{2xy}$.

$$\left(\frac{y}{2xy} - \frac{xy^2}{2xy}\right) dx + \left(\frac{-x}{2xy} - \frac{x^2y}{2xy}\right) dy = 0$$

$$\left(\frac{1}{2x} - \frac{y}{2}\right) dx + \left(\frac{-1}{2y} - \frac{x}{2}\right) dy = 0 \rightarrow ②$$

Compare ② with $M_1 dx + N_1 dy = 0$

$$M_1 = \frac{1}{2x} - \frac{y}{2} \quad , \quad N_1 = \frac{-1}{2y} - \frac{x}{2}$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{2} \quad , \quad \frac{\partial N_1}{\partial x} = \frac{-1}{2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is Exact D.E

General solution:

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

$$\int \left(\frac{1}{2x} - \frac{y}{2}\right) dx + \int \frac{-1}{2y}\,dy = c$$

$$\frac{1}{2}\left[\log x - xy - \log y\right] = \log c$$

$$\log\left(\frac{x}{y}\right) - \log c^2 = xy.$$

$$\boxed{\log\left(\frac{x}{yc^2}\right) = xy} \quad (\text{or}) \quad \frac{x}{yc^2} = e^{xy}$$

$$\boxed{x = yc^2e^{xy}}$$

Integrating factor :- $e^{\int f(x)dx}$

If $M(x,y)dx + N(x,y)dy = 0$ is non-exact differential equation and there exists a single variable function $f(x)$ such that

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x) \text{ then}$$

we apply $I.F = e^{\int f(x)dx}$

$$\boxed{\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)}$$

we apply $e^{\int g(y)dy}$

(1) solve $2xy\,dy - (x^2+y^2+1)dx = 0$

(2) solve $y(2x^2 - xy + 1)dx + (x-y)dy = 0$

(3) solve $x\sin x \frac{dy}{dx} + y(x\cos x - \sin x) = 2$

(4) Solve $\left(xy^2 - \frac{1}{e^{x^3}}\right)dx - x^2y\,dy = 0$

(5) $(x^4e^x - 2mxy^2)dx + 2mx^2y\,dy = 0$

(1) $-(x^2+y^2+1)dx + (2xy)dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = -x^2 - y^2 - 1$, $N = 2xy$

$\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 2y$

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$\therefore$ ① is not exact D.E.

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2xy}\left[-2y-2y\right]=\frac{-4y}{2xy}=\frac{-2}{x}=f(x)$$

$$\therefore I.F = e^{\int f(x)\,dx} = e^{\int \frac{-2}{x}dx} = e^{-2\log x}$$

$$I.F = e^{\log x^{-2}} \qquad (\because e^{\log a} = a)$$

$$IF = \frac{1}{x^2}$$

Multiply ① with IF

$$\frac{-(x^2+y^2+1)}{x^2}dx + \frac{2xy}{x^2}dy = 0$$

$$\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx + \frac{2y}{x}dy = 0 \longrightarrow ②$$

Compare ② with $M_1dx + N_1dy = 0$

$$M_1 = -1-\frac{y^2}{x^2}-\frac{1}{x^2}, \quad N_1 = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{-2y}{x^2}, \quad \frac{\partial N_1}{\partial x} = \frac{-2y}{x^2}$$

Eq ② is exact D.E

General solution:-

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

(y-constant) (Terms independent of x in N_1)

$$\int\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx + \int 0\,dy = c$$

$$-x - y^2\left(\frac{-1}{x}\right) - \left(\frac{-1}{x}\right) = c$$

$$-\frac{x}{1} + \frac{y^2}{x} + \frac{1}{x} = c$$

$$\boxed{y^2 - x^2 + 1 = cx}$$

② $y(2x^2 - xy + 1)dx + (x-y)dy = 0 \rightarrow ①$

Compare ① with $Mdx + Ndy = 0$

$$M = 2x^2y - xy^2 + y \quad , \quad N = x - y$$

$$\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$2x(x$

$\therefore$ ① is not exact D.E

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x-y}\left[2x^2 - 2xy + \cancel{1} - \cancel{1}\right] = \frac{2x(\cancel{x-y})}{\cancel{x-y}}$$

$$= 2x = f(x)$$

$$\therefore I.F = e^{\int f(x)\,dx} = e^{\int 2x\,dx} = e^{\cancel{2}x^2/\cancel{2}}$$

$$= e^{x^2}$$

$$\therefore I.F = e^{x^2}$$

multiply ① with I.F

$$\left(\frac{y(2x^2 - xy + 1)}{e^{x^2}}\right)dx + \left(\frac{x-y}{e^{x^2}}\right)dy = 0$$

$$\cancel{\frac{2x^2y}{e^{x^2}} - \frac{xy^2}{e^{x^2}} + \frac{y}{e^{x^2}} + \frac{x}{e^{x^2}}} - \frac{y}{e^{x^2}} = 0$$

$$\frac{2x^2y}{e^{x^2}} - \frac{xy^2}{e}$$

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$e^{x^2}(2x^2y - xy^2 + y)\,dx + e^{x^2}(x-y)\,dy = 0$ ---- (2)

Compare (2) with $M_1 dx + N_1 dy = 0$

$M_1 = e^{x^2}(2x^2y - xy^2 + y)$, $N_1 = e^{x^2}(x-y)$

$\frac{\partial M_1}{\partial y} = e^{x^2}(2x^2 - 2xy + 1)$, $\frac{\partial N_1}{\partial x} = e^{x^2} + e^{x^2}(2x)(x-y)$

$= e^{x^2}[1+2x^2 - 2xy]$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq (2) is exact D.E

General solution:

$\int M(x,y)\,dx + \int N_1(x,y)\,dy = c$

(y-const) (terms independent of x in N)

(or)

$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$

(Terms independent of 'y' in M) (x-const)

$\int 0\,dx + \int e^{x^2}(x-y)\,dy = c$

$\int e^{x^2}(x)\,dy - \int e^{x^2} y\,dy = c$

$xye^{x^2} - \frac{y^2}{2}e^{x^2} = c.$

$2xye^{x^2} - y^2e^{x^2} = 2c$

$ye^{x^2}(2x - y) = 2c$

(3) $x\sin x \cdot \frac{dy}{dx} + y(x\cos x - \sin x) = 2$

$x\sin x\, dy + y(x\cos x - \sin x)dx = 2dx$

$(xy\cos x - y\sin x - 2)\,dx + x\sin x\, dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = xy\cos x - y\sin x - 2$, $N = x\sin x$

$\frac{\partial M}{\partial y} = x\cos x - \sin x$, $\frac{\partial N}{\partial x} = x\cos x + \sin x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is not exact D.E

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x\sin x}\left[x\cancel{\cos x} - \sin x - x\cancel{\cos x} - \sin x\right]$

$= \cancel{\frac{\cos ec x}{x}} \quad \cancel{= f(x)} = \frac{-2\sin x}{x\sin x} = \frac{-2}{x}.$

$I.F = e^{\int \cancel{\frac{-2\cos ec x}{x}}}$

$= \cancel{\cos ec x \int \frac{1}{x} dx} - \int\left(-\cos ec x \cot x \int \frac{1}{x} dx\right) dx$

$= \cancel{\cos ec x \log x}$

$= e^{-2\log x}$

$= e^{\log x^{-2}}$

$= \frac{1}{x^2}$

multiply ① with $M_1 dx + N_1 dy = 0$ IF $= \frac{1}{x^2}$

$$\left(\frac{xy\cos x - y\sin x - 2}{x^2}\right)dx + \left(\frac{x\sin x}{x^2}\right)dy = 0$$

$$\left(\frac{y\cos x}{x} - \frac{y\sin x}{x^2} - \frac{2}{x^2}\right)dx + \left(\frac{\sin x}{x}\right)dy = 0 \rightarrow ②$$

Compare ② with $M_1 dx + N_1 dy = 0$

$$M_1 = \frac{y\cos x}{x} - \frac{y\sin x}{x^2} - \frac{2}{x^2}, \quad N_1 = \frac{\sin x}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\cancel{x}\cos x}{x^{\cancel{2}}} - \frac{x^{\cancel{2}}\sin x}{x^{\cancel{4}2}}, \quad \frac{\partial N_1}{\partial x} = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{\cos x}{x} - \frac{\sin x}{x^2} \qquad\qquad = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is exact D.E

General soln :-

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

(terms independent of 'y' in M_1) ($\cancel{x}$ const)

$$\int \frac{-2}{x^2}\,dx + \int \frac{\sin x}{x}\,dy = c$$

$$-2\left(\frac{-1}{x}\right) + \frac{y\sin x}{x} = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

$$\boxed{2 + y\sin x = xc}$$

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(4) $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0 \longrightarrow (1)$

Compare (1) with $Mdx + Ndy = 0$

$M = xy^2 - e^{1/x^3}, \quad N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = -2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Eq (1) is not exact D.E

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{-x^2y}\left[2xy + 2xy\right] = \frac{4xy}{-x^2y} = \frac{-4}{x}$

$= f(x)$

$I.F = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} \quad e^{-4\log x} = e^{\log x^{-4}}$

$IF = \frac{1}{x^4}$

multiply (1) with $IF = \frac{1}{x^4}$

$\frac{1}{x^4}\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - \frac{x^2y}{x^{4}}dy = 0$

$\left(\frac{xy^2}{x^{4}} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$

$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0 \longrightarrow (2)$

Compare (2) with $M_1dx + N_1dy = 0$

$M_1 = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}, \quad N_1 = \frac{-y}{x^2}$

$\frac{\partial M_1}{\partial y} = \frac{2y}{x^3}, \quad \frac{\partial N_1}{\partial x} = \frac{-(-2y)}{x^3} = \frac{2y}{x^2}$

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$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is exact D.E

General solution:

$$\int M(x,y)\,dx + \int N(x,y)\,dy = c$$

(y-constant) (Terms independent of 'x' in N_1)

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = c$$

$\frac{1}{x^3} = t$

$\frac{-3}{x^4}\,dx = dt$

$\frac{1}{x^4}\,dx = \frac{-dt}{3}$

$$\frac{-y^2}{2x^2} + \frac{1}{3}\int \frac{-3e^{1/x^3}}{x^4}\,dx = c$$

$\frac{-y^2}{2x^2} + \int \frac{e^t\,dt}{-3} = c$

$\frac{-y^2}{2x^3} + \frac{e^t}{3} = c$

$$\frac{-y^2}{2x^2} + \frac{e^{1/x^3}}{3} = c$$

$$\boxed{\frac{e^{1/x^3}}{3} - \frac{y^2}{2x^2} = c}$$

$\left(\because \int e^{f(x)} f'(x) = f(x) + c\right)$

$\left(\because \int e^{f(x)} f'(x) = f(x) + c\right)$

⑤ $(x^4 e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0 \rightarrow ①$

Compare ① with $M\,dx + N\,dy = 0$

$M = x^4e^x - 2mxy^2$, $N = 2mx^2y$

$\frac{\partial M}{\partial y} = -4mxy$, $\frac{\partial N}{\partial x} = 4mxy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Eq ① is not exact D.E

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2mx^2y}\left[-4mxy-4mxy\right]$$

$$=\frac{-8mxy}{2mx^2y}$$

$$= \frac{-4}{x} = f(x)$$

$$I.F = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}}$$

$$I.F = \frac{1}{x^4}$$

Multiply ① with $I.F = \frac{1}{x^4}$

$$\left(\frac{x^4 e^x}{x^4}-\frac{2mxy^2}{x^{4}}\right)dx + \frac{2mx^2y}{x^{4}}dy = 0$$

$$\left(e^x - \frac{2my^2}{x^3}\right)dx + \frac{2my}{x^2}dy = 0 \longrightarrow ②$$

Compare ~~multiply~~ ② with $M_1dx + N_1dy = 0$

$\frac{2m(xy)}{x^2}$

$$M_1 = e^x - \frac{2my^2}{x^3}, \quad N_1 = \frac{2my}{x^2}$$

$2my\left(\frac{2}{x}\right.$

$$\frac{\partial M_1}{\partial y} = \frac{-4my}{x^3}, \quad \frac{\partial N_1}{\partial x} = \frac{-4my}{x^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$-2my^2\left(\frac{1}{x}\right)$

Eq ① is exact D.E

General solution

$$\int M_1(x,y)dx + \int N_1(x,y)dy = c$$

(y-const) (Terms independent of 'x' in N)

$$\int\left(e^x - \frac{2my^2}{x^3}\right)dx + \int 0 \, dy = c$$

$$e^x + \frac{2my^2}{x^2} = c$$

$$\boxed{x^n e^x + m y^2 = x^2 c}$$

V) Integrating factor

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact D.E and there exists a single variable function such that $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$

then we apply $IF = e^{\int g(y)\,dy}$

solve

① $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

② $y(2xy + e^x)\,dx - e^x\,dy = 0$

③ $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

① $(xy^3 + y)\,dx + (2x^2y^2 + 2x + 2y^4)\,dy = 0 \longrightarrow ①$

Compare ① with $M\,dx + N\,dy = 0$

$M = xy^3 + y \quad , \quad N = 2x^2y^2 + 2x + 2y^4$

$\frac{\partial M}{\partial y} = 3xy^2 + 1 \quad , \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is not exact D.E

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^3+y}\left[4xy^2+2 - 3xy^2 - 1\right]$$

$$= \frac{1}{y(xy^2+1)}\left[xy^2+1\right]$$

$$= \frac{1}{y} = g(y)$$

$$I.F = e^{\int g(y)dy} = e^{\int \frac{1}{y} dy} = e^{\log y}$$

$$= y$$

Multiply ① with $IF = y$.

$$\left(\frac{xy^3+y}{y}\right)dx + \left(\frac{2x^2y^2+2x+2y^4}{y}\right)dy = 0.$$

$$\left(\frac{xy^3}{y} + \frac{y}{y}\right)dx + \left(\frac{2x^2y^2}{y} + \frac{2x}{y} + \frac{2y^4}{y}\right)dy = 0$$

$$(xy^2+1)\,dx + \left(2x^2y + \frac{2x}{y} + 2y^3\right)dy = 0 \rightarrow ②$$

Compare ② with $M_1dx + N_1dy = 0$

$$M_1 = xy^2+1 \quad , \quad N_1 = 2x^2y + \frac{2x}{y} + 2y^3$$

$$\frac{\partial M_1}{\partial y} = 2xy \quad , \quad \frac{\partial N_1}{\partial x} = 4xy + \frac{2}{y}$$

$$(xy^4+y^2)\,dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0 \rightarrow ②$$

Compare ② with $M_1dx + N_1dy = 0$.

$$M_1 = xy^4 + y^2 \quad , \quad N_1 = 2x^2y^3 + 2xy + 2y^5$$

$$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y \quad , \quad \frac{\partial N_1}{\partial x} = 4xy^3 + 2y$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is exact D.E

General solution

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

(y - const) (Terms independent of x in N)

$$\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$$

$$\frac{x^2}{2}y^4 + xy^2 + \frac{\cancel{2}y^6}{\cancel{6}_3} = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$$

$$\frac{3x^2y^4 + 6xy^2 + 2y^6}{6} = c$$

$$\boxed{3x^2y^4 + 6xy^2 + 2y^6 = 6c}$$

② $y(2xy + e^x)\,dx - e^x\,dy = 0 \rightarrow ①$

Compare ① with $M\,dx + N\,dy = 0$

$M = 2xy^2 + ye^x$, $N = -e^x$.

$\frac{\partial M}{\partial y} = 4xy + e^x$, $\frac{\partial N}{\partial x} = -e^x$.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq. ① is not exact D.E

$$\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=\frac{1}{2xy^2+ye^x}\left[-e^x-4xy-e^x\right]$$

$$=\frac{1}{y\left[2xy+e^x\right]}\left[-2e^x-4xy\right]$$

$$=\frac{-2\left[e^x+2xy\right]}{y\left[2xy+e^x\right]}$$

$$=\frac{-2}{y}=g(y)$$

$$I.F=e^{\int g(y)\,dy}=e^{\int -\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\frac{1}{y^2}$$

multiply ① with $\frac{1}{y^2}$

$$\left(\frac{2xy^2+ye^x}{y^2}\right)dx-\left(\frac{e^x}{y^2}\right)dy=0$$

$$\left(\frac{2x\cancel{y^2}}{\cancel{y^2}}+\frac{ye^x}{y^2}\right)dx-\frac{e^x}{y^2}dy=0$$

$$\left(2x+\frac{e^x}{y}\right)dx-\frac{e^x}{y^2}dy=0 \rightarrow ②$$

compare ② with $M_1dx+N_1dy=0$

$$M_1=2x+\frac{e^x}{y}, \quad N_1=-\frac{e^x}{y^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{-e^x}{y^2}, \quad \frac{\partial N_1}{\partial x}=\frac{-e^x}{y^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

Eq ② is exact DE

General solution :-

$$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$$

(y-constant) (terms independent of x in c)

$$\int \left(2x + \frac{e^x}{y}\right) dx + \int 0\,dy = c$$

$$\frac{\cancel{2}x^2}{\cancel{2}} + \frac{e^x}{y} = c$$

$$x^2 + \frac{e^x}{y} = c$$

$$\boxed{yx^2 + e^x = yc}$$

(4) $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)\,dy = 0$

(3) $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$ —(1)

Compare (1) with $M dx + N dy = 0$

$M = xy^2 - x^2$, $N = 3x^2y^2 + x^2y - 2x^3 + y^2$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is not exact D.E

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^2 - x^2}\left[6xy^2 + \cancel{2xy} - 6x^2 - \cancel{2xy}\right]$$

$$= \frac{6xy^2 - 6x^2}{xy^2 - x^2}$$

$\frac{x(y^2-x)}{x(y^2-x)}$

$$= \frac{6[\cancel{xy^2 - x^2}]}{\cancel{xy^2 - x^2}} = 6$$

$IF = e^{\int g(y)\,dy} = e^{\int 6\,dy} = e^{6y}$

multiply (1) with e^{6y}

$$\left(\frac{xy^2 - x^2}{e^{6y}}\right)dx + \left(\frac{3x^2y^2 + x^2y - 2x^3 + y^2}{e^{6y}}\right)dy = 0$$

$$(xy^2e^{6y} - x^2e^{6y})\,dx + (3x^2y^2e^{6y} + x^2ye^{6y} - 2x^3e^{6y} + y^2e^{6y})\,dy = 0 \quad \text{-(2)}$$

Compare (2) with $M_1dx + N_1dy = 0$

$M_1 = e^{6y}(xy^2 - x^2)$, $N_1 = e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)$

$$\frac{\partial M_1}{\partial y} = e^{6y}(2xy) + e^{6y}(6)(xy^2 - x^2)$$

$$= 2xye^{6y} + 6e^{6y}xy^2 - 6e^{6y}x^2$$

$$= e^{6y}(6xy^2 + 2xy - 6x^2)$$

$$\frac{\partial N_1}{\partial x} = e^{6y}(6xy^2 + 2xy - 6x^2)$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq (2) is exact D.E

General sol'n

$\int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$

(y-constant) $\qquad \int e^{6y}y^2\,dy$

$\cancel{\int e^{6y}(xy^2 x^2}$ $\int e^{6y}(xy^2 - x^2)\,dx + \int e^{6y}y^2\,dy = c$

$$e^{6y}y^2\frac{x^2}{2} - e^{6y}\frac{x^3}{3} = c$$

$$e^{6y}\left[\frac{x^2y^2}{2} - \frac{x^3}{3}\right] = c$$

$$e^{6y}\left[\frac{3x^2y^2-2x^3}{6}\right]=c$$

$$\boxed{x^2e^{6y}\left[3y^2-2x\right]=6c}$$

$$e^{6y}\,y^2\frac{x^2}{2}-e^{6y}\frac{x^3}{3}+\int e^{6y}\,y^2\,dy=c$$

$$\frac{e^{6y}\,y^2x^2}{2}-\frac{e^{6y}\,x^3}{3}+\left[y^2\frac{e^{6y}}{6}-\frac{1}{3}\int \cancel{2}y\,\frac{e^{6y}}{\cancel{6}\,3}\,dy\right]=c$$

$$\frac{e^{6y}\,y^2x^2}{2}-\frac{e^{6y}\,x^3}{3}+\frac{y^2}{6}e^{6y}-\frac{1}{3}\left[\int y\,e^{6y}\,dy\right]=c$$

$$\frac{e^{6y}\,y^2x^2}{2}-\frac{e^{6y}\,x^3}{3}+\frac{y^2e^{6y}}{6}-\frac{1}{3}\left[y\,\frac{e^{6y}}{6}-\frac{1}{6}\int e^{6y}\right]=c$$

$$\frac{e^{6y}\,y^2x^3}{2}-\frac{e^{6y}\,x^3}{3}+\frac{e^{6y}y^2}{6}-\frac{y\,e^{6y}}{18}+\frac{1}{18}\,\frac{e^{6y}}{6}=c$$

$$e^{6y}\left[\frac{x^2y^2}{2}-\frac{x^3}{3}+\frac{y^2}{6}-\frac{y}{18}+\frac{1}{108}\right]=c$$

$$e^{6y}\left[\frac{54x^2y^2-36x^3+18y^2-6y+1}{108}\right]=c$$

$$e^{6y}\left[54x^2y^2-36x^3+18y^2-6y+1\right]=108c$$

V) Integrating factors by inspection :

(1) $d(xy) = x\,dy + y\,dx$

(2) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(3) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(4) $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$

(5) $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

(6) $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{x^2+y^2}$

(7) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

(8) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

(9) $d\left[\log(xy)\right] = \frac{x\,dy + y\,dx}{xy}$

(10) $d\left[\frac{e^x}{y}\right] = \frac{y\cdot e^x dx - e^x dy}{y^2}$

(11) $d\left(\frac{y^2}{x^2}\right) = \frac{x^2\cdot 2y\,dy - y^2\cdot 2x\,dx}{x^4}$

(12) $d\left[\log(x^2+y^2)\right] = \frac{2x\,dx + 2y\,dy}{x^2+y^2}$

solve.

① $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

② $y(2x^2y + e^x)dx = (e^x + y^3)dy$

③ $(1+xy)x\,dy + (1-yx)y\,dx = 0$

④ $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

① $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

$d\left(\dfrac{x^2+y^2}{2}\right) = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

Integration on both sides

$\int \cancel{x}\, d\left(\dfrac{x^2+y^2}{2}\right) = \int d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

$\dfrac{x^2+y^2}{2} = \tan^{-1}\left(\dfrac{y}{x}\right) + c$

② $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$

$\cancel{2x^2y^2d + ye^x}$

$2x^2y^2\,dx + ye^x\,dx = e^x\,dy + y^3\,dy$

$ye^x\,dx - e^x\,dy = y^3\,dy - 2x^2y^2\,dx$

$ye^x\,dx - e^x\,dy = y^2\left[y\,dy - 2x^2\,dx\right]$

$\dfrac{ye^x\,dx - e^x\,dy}{y^2} = y\,dy - 2x^2\,dx$

$d\left[\frac{e^x}{y}\right] + 2x^2 dx - y dy = 0$

$2x^2 dx + d\left[\frac{e^x}{y}\right] - y dy = 0$

Integrate on both sides

$\int 2x^3 dx + \int d\left[\frac{e^x}{y}\right] - \int y dy = c$

$\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c$

$\boxed{\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c.}$

(3) $(1+xy)x\,dy + (1-yx)y\,dx = 0$

$(x + x^2y)dy + (y - y^2x)dx = 0$

$x\,dy + x^2y\,dy + y\,dx - y^2x\,dx = 0$

$x\,dy + y\,dx + x^2y\,dy - y^2x\,dx = 0$

Divide with x^2y^2

$\frac{x\,dy + y\,dx}{x^2y^2} + \frac{x^2y\,dy}{x^2y^2} - \frac{y^2x\,dx}{x^2y^2} = 0$

$\frac{1}{x^2y^2}[x\,dy + y\,dx] + \frac{y\,dy}{yx} - \frac{x\,dx}{x^2} = 0$

$\frac{1}{x^2y^2}[x\,dy + y\,dx] + \frac{x\,dy - y\,dx}{xy} = 0.$

$\frac{(x\,dy + y\,dx)}{x^2y^2} + \frac{(x\,dy - y\,dx)}{xy} = 0.$

$$\frac{d(xy)}{(xy)^2} + d\left[\log\left(\frac{y}{x}\right)\right] = 0$$

Integration on both sides

$$\int \frac{d(xy)}{(xy)^2} + \int d\left[\log\left(\frac{y}{x}\right)\right] = c$$

$$xy = t \qquad \frac{t^{-2+1}}{-2+1}$$

$$d(xy) = dt \qquad = \frac{-1}{t}$$

$$\int \frac{dt}{t^2} + \log\left(\frac{y}{x}\right) = c$$

$$\frac{-1}{t} + \log\left(\frac{y}{x}\right) = c$$

$$\frac{-1}{xy} + \log\left(\frac{y}{x}\right) \cancel{-\log} = c.$$

(4) $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

Dividing with y^2 on both sides

$$\frac{y\,dx - x\,dy}{y^2} + \frac{3x^2y^2e^{x^3}}{y^2}dx = 0$$

$$d\left(\frac{x}{y}\right) + e^{x^3}\,3x^2dx = 0$$

Integrating on both sides

$$\int d\left(\frac{x}{y}\right) + \int e^{x^3}\,3x^2 = c$$

$$x^3 = t$$

$$3x^2dx = dt$$

$\frac{x}{y} + \int e^t dt = c$

$\frac{x}{y} + e^t = c$

$$\boxed{\frac{x}{y} + e^{x^3} = c}$$

(5) $y(2xy + e^x)dx = e^x dy$

(6) $ydx + x(x^2y - 1)dy = 0$

(7) $y(x^3e^{xy} - y)dx + x(y + x^3e^{xy})dy = 0$

(5) $2xy^2dx + ye^x dx - e^x dy = 0$

$\cancel{2xy^2dx + d\left[\frac{e^x}{y}\right] = 0}$

~~Divide with y^2~~

$\cancel{2x\,dx + \frac{d\left(\frac{e^x}{y}\right)}{y^2} = 0}$

$\frac{1}{y^2} = t$

$\frac{-1}{y} dy = dt$

$2xy^2dx + ye^xdx - e^xdy = 0$

Divide with y^2 on both sides

$$2x\,dx + \frac{ye^xdx - e^xdy}{y^2} = 0$$

$$2xdx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating on both sides

$$\int 2xdx + \int d\left(\frac{e^x}{y}\right) = c$$

$$\frac{2x^2}{2} + \frac{e^x}{y} = c$$

$$x^2 + \frac{e^x}{y} = c$$

⑤ $y\,dx + x(x^2y - 1)dy = 0$

$y\,dx + x^3y\,dy - x\,dy = 0$

$y\,dx - x\,dy + x^3y\,dy = 0$

~~Dividing with $\frac{y}{x^3}$~~ multiply with $\frac{y}{x^3}$

$$\frac{y^2dx - xy\,dy}{x^3} + y^2dy = 0$$

$$\frac{1}{2}\left(\frac{y^2 2x\,dx - x^2 2y\,dy}{x^4}\right) + y^2dy = 0$$

$$\int \frac{1}{2}\, d\left(\frac{-y^2}{x^2}\right) + \int y^2dy = c$$

$$\frac{-y^2}{2x^2} + \frac{y^3}{3} = c$$

Solve

④ $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)dy = 0$ ①

Compare ① with $M\,dx + N\,dy = 0$

$M = \frac{y}{x}\sec y - \tan y$, $N = \sec y \log x - x$

$$\frac{\partial M}{\partial y} = \frac{y}{x}\sec y\tan y + \frac{1}{x}\sec y - \sec^2 y$$

$$\frac{\partial N}{\partial x} = \sec y\frac{1}{x} - 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

① is not exact D.E

$\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] = \frac{y}{x}$

$$= \frac{\left[\frac{1}{x}\cancel{s}ecy - 1 - \frac{y}{x}\sec y\tan y - \frac{1}{\cancel{x}}\sec y + \sec^2 y\right]}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{\sec^2 y - 1 - \frac{y}{x}\sec y\tan y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{\tan^2 y - \frac{y}{x}\sec y\tan y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\tan y\left(\frac{y}{x}\sec y - \tan y\right)}{\frac{y}{x}\sec y - \tan y}$$

$= -\tan y$

$= g(y)$

$$I.F = e^{\int g(y)dy} = e^{\int -\tan y dy} = e^{-\log|\sec y|}$$

$$= \frac{1}{\sec y} = \cos y.$$

Multiply ① with $IF = \frac{1}{\sec y}$

$$\left(\frac{y\sec y - \tan y}{x\sec y}\right)dx + \left(\frac{\sec y\log x - x}{\sec y}\right)dy = 0$$

$$\left(\frac{y}{x} - \sin y\right)dx + (\log x - x\cos y)dy = 0$$

$$M_1 = \frac{y}{x} - \sin y, \quad N_1 = \log x - x\cos y$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x} - \cos y \quad , \quad \frac{\partial N_1}{\partial x} = \frac{1}{x} - \cos y$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

General solution:

$\int M_1(x,y)dx + \int N_1(x,y)dy = c$

(y-const) (Terms independent of x in N)

$$\int\left(\frac{y}{x} - \sin y\right) dy + \int 0\, dy = c$$

$$\boxed{y \log x - x\sin y = c}$$

(7) $y(x^3e^{xy} - y)dx + x(y + x^3e^{xy})dy = 0$

sol) $x^3ye^{xy}dx - y^2dx + xydy + x^4e^{xy}dy = 0$

$x^3ye^{xy}dx + x^4e^{xy}dy - y^2dx + xydy = 0$

Dividing with x^3.

$$ye^{xy}dx + xe^{xy}dy - \left(\frac{y^2dx - xydy}{x^3}\right) = 0$$

$$e^{xy}(ydx + xdy) - \frac{1}{2}\left(\frac{y^2\cdot(2x)dx - x^2(2y)dy}{x^4}\right) = 0$$

$$e^{xy}\, d(xy) - \frac{1}{2} d\left(\frac{-y^2}{x^2}\right) = 0$$

Integrating on both sides

$$\int e^{xy} d(xy) - \int \frac{1}{2} d\left(\frac{-y^2}{x^2}\right) = C$$

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Put $xy = t$

$$d(xy) = dt$$

$$\int e^t\,dt - \frac{1}{2}\left(\frac{-y^2}{x^2}\right) = c$$

$$e^t + \frac{y^2}{2x^2} = c$$

$$e^{xy} + \frac{y^2}{2x^2} = c$$

* Linear differential equations :

(1) The differential equation is of the form

$\frac{dy}{dx} + P(x)y = Q(x)$ is called linear differential equation in 'y'.

$$I.F = e^{\int P(x)\,dx}$$

Gen. sol'n: $y(IF) = \int Q(x)\cdot(IF)\,dx + c$

(2) The differential equation is of the form $\frac{dx}{dy} + P(y)x = Q(y)$ is called linear differential equation in 'x'.

$$I.F = \cancel{e^{\int P(y)\,d}}\; e^{\int P(y)\,dy}$$

General solution:

$$x(I.F) = \int Q(y)\,(I.F)\,dy + c$$

Solve:

① $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

② $\frac{dy}{dx} + (\tan x)y = \sin 2x$, by $y(0)=0$

③ $x\log x \frac{dy}{dx} + y = 2\log x$

④ $y\log y\, dx + (x - \log y)dy = 0$

⑤ $(1+y^2)dx = (\tan^{-1}y - x)dy$

⑥ $dr + (2r\cos\theta + \sin 2\theta)d\theta = 0$

1 sol'n: $(x+1)\frac{dy}{dx} - y = e^{3x}(1+x)^2$

divide with $(1+x)$ on both sides

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(1+x) \longrightarrow ①$$

It is linear D.E, it is in the form of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$\because P = \frac{-1}{x+1}$, $Q(x) = e^{3x}(1+x)$

$$I.F = \cancel{e^{\sin x}}\, e^{\int P(x)dx} = e^{\int \frac{-1}{x+1}dx} = e^{-\log|x+1|}$$

$$= e^{\log(x+1)^{-1}}$$

$$I.F. = \frac{1}{1+x}$$

General solution : $y(I.F) = \int Q(x)\,(I.F)\,dx + c$

$$\frac{y}{x+1} = \int e^{3x}(1+x)\cdot\frac{1}{(x+1)}\,dx + c$$

$$\frac{y}{x+1} = \int e^{3x}\,dx + c$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$$

(2) $\frac{dy}{dx} + (\tan x)y = \sin 2x,\ y(0) = 0 \longrightarrow (1)$

It is linear in 'y'

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$\therefore P = \tan x,\ Q = \sin 2x$

$$I.F = e^{\int P(x)dx} = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x$$

$$IF = \sec x$$

General sol'n :

$$y(I.F) = \int Q\,(I.F)\,dx + c$$

$$y\sec x = \int \sin 2x\cdot\sec x\,dx + c$$

$$\frac{y}{\cos x} = \int 2\sin x\cos x\cdot\frac{1}{\cos x}\,dx + c$$

$$\frac{y}{\cos x} = 2\int \sin x\,dx + c$$

$$= 2(-\cos x) + c$$

$$\frac{y}{\cos x} = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \longrightarrow (2)$$

Given,

$$y(0)=0$$

$y=0$ and when $x=0$

from (2)

$$0 = -2\cos^2 0 + c\cos 0$$

$$0 = -2+c$$

$$\boxed{c=2}$$

$\therefore$ General soln : $y = -2\cos^2 x + 2\cos x$

(3) soln $x\log x \cdot \frac{dy}{dx} + y = 2\log x \longrightarrow (1)$

Divide with $x\log x$ on both sides

$$\frac{\cancel{x\log x}}{\cancel{x\log x}} \cdot \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2\cancel{\log x}}{x\cancel{\log x}}$$

$$\frac{dy}{dx} + y\frac{1}{x\log x} = \frac{2}{x}$$

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

It is linear in 'y'

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$$\because P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x\log x} dx}$$

Put $\log x = t$

$$\frac{1}{x} dx = dt$$

$IF = e^{\int \frac{1}{t} dt} = e^{\log t} = t = \log x$

$\boxed{IF = \log x}$

General sol'n:

$y(IF) = \int Q\,(I.F)\,dx + c$

$y \log x = \int \frac{2}{x} \log x \, dx + c$

Let, $\log x = t$.

$\Rightarrow \frac{1}{x} dx = dt$

$y \log x = 2\int t\,dt + c$

$= \frac{2t^2}{2} + c$

$y \log x = (\log x)^2 + c$

$$\boxed{y = \log x + \frac{c}{\log x}}$$

4) $y \log y \, dx + (x - \log y)\, dy = 0$

$y \log y \, dx = -(x - \log y)\, dx$

$\frac{dx}{dy} = \frac{-(x - \log y)}{y \cdot \log y}$

$\frac{dx}{dy} = \frac{\log y - x}{y \log y}$

$\frac{dx}{dy} = \frac{1}{y} - \frac{x}{y \log y}$

$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$

It is linear in 'x'

$$\frac{dx}{dy} + P(y)x = Q(y)$$

$$P = \frac{1}{y\log y}, \quad Q = \frac{1}{y}$$

$$I.F = e^{\int P(y)dy}$$

$$= e^{\int \frac{1}{y\log y}} = \log y$$

General sol'n :-

$$x(I.F) = \int Q(y)\,(I.F)\,dy + c$$

$$x\log y = \int \frac{1}{y}\log y\,dy + c$$

Put $\log y = t$

$$\frac{1}{y}dy = dt$$

$$x\log y = \int t\,dt + c$$

$$x\log y = \frac{t^2}{2} + c$$

$$x\log y = \frac{(\log y)^2}{2} + c$$

$$x\log y = \log y + c$$

$$\boxed{x = 1 + \frac{c}{\log y}}$$

(5) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

It is linear in 'x'

$$\frac{dx}{dy} + p(y)x = Q(y)$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int p(y)dy}$$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1}y}$$

General solution

$$x(IF) = \int Q(y)(I.F)\,dy + c$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy + c$$

$$\tan^{-1}y = t$$

$$\frac{1}{1+y^2}\,dy = dt$$

$$x e^{\tan^{-1}y} = \int t\, e^t + c$$

$$= te^t - \int (1\, e^t)\,dt + c$$

$$= te^t - e^t + c$$

$$= e^t(t-1) + c$$

$$x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\boxed{x = \tan^{-1}y - 1 + c e^{\tan^{-1}y}}$$

(6) $dr + (2r\cot\theta + \sin 2\theta)\, d\theta = 0$

$$dr = (-2r\cot\theta - \sin 2\theta)\, d\theta$$

$$\frac{dr}{d\theta} = -2r\cot\theta - \sin 2\theta$$

$$\frac{dr}{d\theta} + (2\cot\theta)\, r = -\sin 2\theta$$

It is linear in 'r'

$$\frac{dr}{d\theta} + P(\theta)\, r = Q(\theta)$$

$$P = 2\cot\theta, \quad Q = -\sin 2\theta$$

$$IF = e^{\int P(\theta)\, d\theta} = e^{\int 2\cot\theta\, d\theta} = \cancel{e^{2\sin\theta}}$$
$$= e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

General sol'n:-

$$r(IF) = \int Q(\theta)(IF)\, d\theta + c$$

$$\cancel{r\, e^{2\sin\theta} = \int -2\sin 2\theta\, (e^{2\sin\theta})\, d\theta} + c$$

$$r\sin^2\theta = \int \sin^2\theta\, (-\sin 2\theta)\, d\theta + c$$

$$= -2\int \sin^2\theta\, (\sin\theta\cos\theta)\, d\theta + c$$

$$= -2\int \sin^3\theta \cos\theta\, d\theta + c$$

$$\left(\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right)$$

$$= -2\left(\frac{\sin^{3+1}\theta}{3+1}\right) + c$$

$$r\sin^2\theta = -\cancel{2}\left(\frac{\sin^4\theta}{\cancel{4}_{2}}\right) + c$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

$$\boxed{r = -\frac{\sin^2\theta}{2} + c\,\mathrm{cosec}^2\theta}$$

solve

(7) $\frac{dy}{dx} + (y-1)\cos x = e^{-\sin x}.\cos^2 x$

(8) $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$

(9) $\sin 2x \frac{dy}{dx} - y = \tan x$

(10) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

(7) $\frac{dy}{dx} + y\cos x - \cos x = e^{-\sin x}\cos^2 x$

$\frac{dy}{dx} + y\cos x = e^{-\sin x}.\cos^2 x + \cos x$

$P = \cos x$, $Q = e^{-\sin x}\cos^2 x + \cos x$.

It is linear in 'y'

$\frac{dy}{dx} + P(x)y = Q(x)$

$P = \cos x$, $Q = e^{-\sin x}\cos^2 x + \cos x$

$IF = e^{\int P(x)dx} = e^{\int \cos x\, dx} = e^{\sin x}$

General sol'n :

$y(I.F) = \int Q(I.F)\,dx + c$

$y e^{\sin x} = \int e^{-\sin x}\cos^2 x\, e^{\sin x}\,dx + c$

$$y e^{\sin x} = \int \cos^2 x \, dx + c$$

$$= \int \frac{1+\cos 2x}{2} dx + c$$

$$= \frac{1}{2} x + \frac{1}{2} \frac{\sin 2x}{2} + c$$

$$\boxed{y e^{\sin x} = \frac{1}{2} x + \frac{\sin 2x}{4} + c}$$

$$y e^{\sin x} = \int (e^{-\sin x} \cos^2 x + (\cos x) e^{\sin x}) \, dx$$

$$y e^{\sin x} = \int \cos^2 x \, dx + \int \cos x \, e^{\sin x} \, dx$$

$$= \int \frac{1+\cos 2x}{2} dx + \quad \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array}$$

$$= \frac{1}{2} x + \frac{\sin 2x}{4} + \int e^t \, dt$$

$$= \frac{1}{2} x + \frac{\sin 2x}{4} + e^t$$

$$y e^{\sin x} = \frac{1}{2} x + \frac{\sin 2x}{4} + e^{\sin x} + c$$

(8) $$(1+y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$$

$$(x - e^{\tan^{-1} y}) \frac{dy}{dx} = -(1+y^2)$$

$$(x - e^{\tan^{-1} y}) = -(1+y^2) \frac{dx}{dy}$$

$$(1+y^2) \frac{dx}{dy} + x = e^{\tan^{-1} y}$$

Divide with $1+y^2$ on b.s

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right) x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

It is linear in 'x'

$$\frac{dx}{dy} + P(y) x = Q(y)$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{e^{\tan^{-1} y}}{1+y^2}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Gen soln:

$$x(IF) = \int Q(IF)\,dy + c$$

$$x(e^{\tan^{-1}y}) = \int \frac{e^{\tan^{-1}y}}{1+y^2}\,e^{\tan^{-1}y}\,dy + c$$

let $\tan^{-1}y = t$

$$\frac{1}{1+y^2}\,dy = dt$$

$$x(e^{\tan^{-1}y}) = \int e^{2t}\,dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$x e^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + c.$$

$$\boxed{x = \frac{e^{\tan^{-1}y}}{2} + c e^{-\tan^{-1}y}}$$

(9) $\sin 2x \dfrac{dy}{dx} - y = \tan x$

Divide with '$\sin 2x$'

$$\frac{dy}{dx} + \left(\frac{-1}{\sin 2x}\right) y = \frac{\tan x}{\sin 2x}$$

It is linear D.E in 'y'

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P = \frac{-1}{\sin 2x} \quad , \quad Q = \frac{\tan x}{2\sin x\cos x} = \frac{\frac{\cancel{\sin} x}{\cos x}}{2\cancel{\sin} x\cos x}$$

$$= -\operatorname{cosec} 2x \qquad\qquad = \frac{\cancel{x}}{\cancel{2\sec^2} x}$$

$$= \frac{1}{2\cos^2 x}$$

$$= \frac{\sec^2 x}{2}$$

$$IF = e^{\int P dx} = e^{-\int \operatorname{cosec} 2x \, dx}$$

$$= e^{-\frac{1}{2}\log|\operatorname{cosec} 2x - \cot 2x|} = e^{\log|\operatorname{cosec} 2x - \cot 2x|^{-1/2}}$$

$$IF = \frac{1}{\sqrt{\operatorname{cosec} 2x - \cot 2x}} = \sqrt{\frac{\sin 2x}{1 - \cos 2x}} = \sqrt{\frac{\cancel{2}\sin x \cos x}{\cancel{2}\sin^2 x}}$$

$$= \sqrt{\cot x}$$

Gen sol'n

$$y(IF) = \int Q(I \cdot F)\, dx + c$$

$$y\sqrt{\cot x} = \int \sec^2 x \left(\sqrt{\cot x}\right) dx + c$$

$$= \frac{1}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx + c \qquad \left(\because \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}\right)$$

$$= \frac{1}{\cancel{2}}\, \cancel{2}\sqrt{\tan x} + c$$

(or) $\tan x = t$

$\sec^2 x \, dx = dt$

$\int \frac{1}{\sqrt{t}}\, dt$

$\frac{1}{2}(2\sqrt{t}) + c$

$\sqrt{t} + c$

$$y\sqrt{\cot x} = \sqrt{\tan x} + c$$

$$\frac{y}{\sqrt{\tan x}} = \sqrt{\tan x} + c$$

$$y = \tan x + c\sqrt{\tan x}$$

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10) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \longrightarrow (1)$$

It is linear in 'y'

It is of the form.

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$$P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$IF = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx}$$

$$IF = e^{2\sqrt{x}}$$

Gen. sol'n

$$y(I.F) = \int Q(IF)\, dx + c$$

$$ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}}\, dx + c$$

$$= \int \frac{e^{0}}{\sqrt{x}}\, dx + c$$

$$ye^{2\sqrt{x}} = 2\sqrt{x} + c$$

Divide $e^{2\sqrt{x}}$

$$\boxed{y = 2\sqrt{x}\cdot e^{-2\sqrt{x}} + c\cdot e^{-2\sqrt{x}}}$$

* Bernoullis Differential Equation:

The D.E is of the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n \longrightarrow (1)$$

It is called Bernoullis D.E in 'y'

General Solution:

Divide eq (1) with y^{-n} on both sides

$$\frac{1}{y^n}\frac{dy}{dx} + P(x) \cdot y^{1-n} = Q(x) \longrightarrow (2)$$

(8) let $y^{1-n} = u$

$$(1-n)\, y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{du}{dx}$$

(2) becomes

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x) \cdot u = Q(x)$$

multiply with $(1-n)$ on both sides

$$\frac{du}{dx} + (1-n)\, P(x) \cdot u + (1-n)\, Q(x)$$

It is linear in 'u'

Solve

(1) $x\dfrac{dy}{dx} + y = x^3 \cdot y^6$

(2) $\dfrac{dy}{dx} + y\cos x = y^3 \sin 2x$

(3) 3. $\frac{dy}{dx} - y\cos x = y^2(\sin 2x - \cos x)$

(4) $\cos x\, dy = y(\sin x - y^3)\, dx$

(1) $x\frac{dy}{dx} + y = x^3 y^6$

Divide with 'x' in both sides.

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \longrightarrow (1)$$

It is Bernoulli's DE in 'y'

Divide with 'y^6' on both sides

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^2 \longrightarrow (2)$$

Let $\boxed{\frac{1}{y^5} = u}$

differentiate wrt 'x'

$$-5y^{-6}\frac{dy}{dx} = \frac{du}{dx}$$

$$\boxed{\frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{du}{dx}}$$

Substitute above terms in (2)

(2) $\Rightarrow$

$$\frac{-1}{5}\frac{du}{dx} + \frac{1}{x}\cdot u = x^2$$

multiply -5 on both sides

$$\frac{du}{dx} - \frac{5}{x}u = -5x^2 \longrightarrow (3)$$

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It is of the form

$$\frac{du}{dx} + P(x) \cdot u = Q(x)$$

$$P = \frac{-5}{x}, \quad Q = -5x^2$$

$$I.F = e^{\int P dx} = e^{\int \frac{-5}{x} dx} = e^{-5 \log x}$$

$$I.F = e^{\log x^{-5}} = \frac{1}{x^5}$$

General sol'n of ③

$$u(I.F) = \int Q(x)(I.F)\, dx + c$$

$$\frac{1}{y^5} \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\, dx + c$$

$$= -5 \int \frac{1}{x^3}\, dx + c$$

$$= -5 \left(\frac{x^{-3+1}}{-3+1} \right) + c$$

$$= -5 \left(\frac{1}{-2x^2} \right) + c$$

$$\boxed{\frac{1}{(xy)^5} = \frac{5}{2x^2} + c}$$

② $\frac{dy}{dx} + y \cos x = y^3 \sin 2x$

Divide with y^3 on both sides.

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{\cos x}{y^2} = \sin 2x \longrightarrow ①$$

Let $y^{-2} = u$

$$-2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{du}{dx}$$

substitute above terms in (1)

$$\frac{-1}{2}\frac{du}{dx} + \cos x\, u = \sin 2x$$

multiply '-2' on both sides

$$\frac{du}{dx} - 2(\cos x)u = -2\sin 2x \longrightarrow (2)$$

It is of the form

$$\frac{du}{dx} + P(x)\,u = Q(x)$$

$$P = -2\cos x, \quad Q = -2\sin 2x$$

$$IF = e^{\int P dx} = e^{\int -2\cos x} = e^{-2\int \cos x} = e^{-2\sin x}.$$

$$\boxed{IF = e^{-2\sin x}}$$

General sol'n of (2)

$$u(IF) = \int Q(x)(IF)\, dx + c$$

$$\frac{1}{y^2} \times e^{-2\sin x} = \int -2\sin 2x\,(e^{-2\sin x})\, dx + c$$

$$= 2\int -2\sin x\cos x\,(e^{-2\sin x})\, dx + c$$

$$= -\int (-2\sin x)(-2\cos x)(e^{-2\sin x})\, dx + c$$

$$-2\sin x = t$$

$$-2\cos x\, dx = dt$$

$$= -\int t e^t\, dt + c$$

$$= -\left[te^t - \int 1 e^t \, dt\right] + c$$

$$= -\left[te^t - e^t\right] + c$$

$$= -\left[-2\sin x\,(e^{-2\sin x}) - e^{-2\sin x}\right] + c$$

$$= e^{-2\sin x}\left[2\sin x + 1\right] + c$$

$$\frac{1}{y^2} \times e^{-2\sin x} = e^{-2\sin x}\left[2\sin x + 1\right] + c$$

$$\frac{1}{y^2} = \left[2\sin x + 1\right] + c e^{-2\sin x}$$

(3) $3\dfrac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$

Divide with $(3y^4)$

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{\cos x}{3y^3} = \frac{1}{3}(\sin 2x - \cos x)$$

Let $\dfrac{1}{y^3} = u$

Diff. wrt (x)

$$-3y^{-4}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^4}\frac{dy}{dx} = \frac{-1}{3}\frac{du}{dx}$$

substitute above terms in (2)

$$\frac{-1}{3}\frac{du}{dx} - \frac{\cos x}{3} \cdot u = \frac{1}{3}(\sin 2x - \cos x)$$

multiply '-3'

$$\frac{du}{dx} + u\cos x = -(\sin 2x - \cos x)$$

It is linear in 'u'

It is of the form

$$\frac{du}{dx} + P(x)\cdot u = Q(x)$$

$$P = \cos x, \quad Q = (\cos x - \sin 2x)$$

$$IF = e^{\int P dx} = e^{\int \cos x\, dx} = e^{\sin x}$$

General sol'n of ③

$$u(I.F) = \int Q(IF)\, dx + C$$

$$\frac{1}{y^3} e^{\sin x} = \int (\cos x - \sin 2x) e^{\sin x} dx + c$$

$$= \int (\cos x - 2\sin x \cos x) e^{\sin x} dx + c$$

$$\frac{e^{\sin x}}{y^3} = \int (1 - 2\sin x) e^{\sin x} \cdot \cos x\, dx + c$$

let $\sin x = t$

$\cos x\, dx = dt$

$$\frac{e^{\sin x}}{y^3} = \int \underset{u}{(1-2t)} \underset{v}{e^t} dt + c$$

$$= (1-2t) e^t - \int (-2) e^t dt + c$$

$$= (1-2t) e^t + 2e^t + c$$

$$= (1 - 2t + 2) e^t + c$$

$$\frac{e^{\sin x}}{y^3} = (3-2t)e^t + c$$

$$\frac{e^{\sin x}}{y^3} = (3-2\sin x)e^{\sin x} + c$$

$$\boxed{\frac{1}{y^3} = (3-2\sin x) + ce^{-\sin x}}$$

(5) $xy(1+xy^2)\frac{dy}{dx} = 1$

$$xy(1+xy^2) = \frac{dx}{dy}$$

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3 \longrightarrow (1)$$

It is Bernoullis D.E in 'x'

It is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)\cdot x^n \longrightarrow (2)$$

Divide 'x^2' on both sides of eq (1)

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$$

let $\frac{-1}{x} = u.$

$$\frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$$

∴ sub. above terms in eq (2)

$$\frac{du}{dy} + y\cdot u = y^3 \longrightarrow (3)$$

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It is linear in 'y'

It is of the form

$$\frac{du}{dy} + P_1(y)\,u = Q_1(y)$$

$$P_1(y) = y, \quad Q_1(y) = y^3.$$

$$IF = e^{\int P_1 dy} = e^{\int y\,dy} = e^{y^2/2}.$$

Gen. sol'n of eq (3).

$$u\,(I.F) = \int Q_1(y)\,(I.F)\,dy + c$$

$$-\frac{1}{x}\,e^{y^2/2} = \int y^3\,e^{y^2/2}\,dy + c$$

Let
$$\frac{y^2}{2} = t$$
$$y^2 = 2t$$
$$2y\,dy = 2\,dt$$
$$\boxed{y\,dy = dt}$$

$$-\frac{1}{x}\,e^{y^2/2} = \int y^2\,e^{y^2/2}\,y\,dy + c$$

$$= \int (2t)\,e^t\,dt + c$$

$$= 2\int t\,e^t\,dt + c$$

$$= 2\left[t e^t - \int 1\cdot e^t\,dt\right] + c$$

$$= 2\left[t e^t - e^t\right] + c$$

$$= 2e^t\,[t-1] + c$$

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$$\frac{-1}{x} e^{y^2/2} = 2e^{y^2/2}\left[\frac{y^2}{2}-1\right] + c$$

$$\frac{-1}{x} = 2\left(\frac{y^2}{2}-1\right) + c\,e^{-y^2/2}$$

$$\frac{-1}{x} = \cancel{2}\left(\frac{y^2-1}{\cancel{2}}\right) + c\,e^{-y^2/2}$$

$$\boxed{\frac{-1}{x} = y^2-1 + ce^{-y^2/2}}$$

(4) $\cos x \, dy = y(\sin x - y)\, dx.$

$$\frac{dy}{dx} = \frac{y\sin x - y^2}{\cos x}$$

$$\frac{dy}{dx} = y\tan x - y^2\sec x.$$

~~$\frac{dy}{dx} = y[\tan x -$~~

$$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

Divide with $\frac{-1}{y^2}$

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{y^2}\tan x = \sec x$$

$$\frac{-1}{y^2}\frac{dy}{dx} + \frac{\tan x}{y} = \sec x \rightarrow (1)$$

Let $\frac{1}{y} = u$

Diff w.r.t 'x'

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + (\tan x)u = \sec x \rightarrow (2)$$

It is linear in 'u'

It is of the form

$$\frac{du}{dx} + P(x) \cdot u = Q(x)$$

$P = \tan x, \quad Q = \sec x$

$$IF = e^{\int P dx} = e^{\int \tan x \, dx} = e^{\log|\sec x|}$$

$$IF = \sec x$$

$\frac{\tan x}{\sec x}$

$\frac{\sin x}{\cos x} \cdot \cos x$

General sol'n of (2)

$$u(IF) = \int Q(IF)\, dx + c$$

$$\frac{1}{y}(\sec x) = \int \sec x (\sec x)\, dx + c$$

$$\frac{1}{y}\sec x = \int \sec^2 x \, dx + c$$

$$\frac{1}{y}\sec x = \tan x + c$$

Divide with $\sec x$

$$\boxed{\frac{1}{y} = \sin x + c\cos x}$$

6) $$\frac{dr}{d\theta} = r\tan\theta - \frac{r^2}{\cos\theta}$$

(6) $\frac{dr}{d\theta} = r\tan\theta - \frac{r^2}{\cos\theta}$

$$\frac{dr}{d\theta} - r\tan\theta = \frac{-r^2}{\cos\theta} \longrightarrow (1)$$

Dividing with $\frac{1}{r^2}$ on both sides

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = \frac{-1}{\cos\theta} \longrightarrow (2)$$

let $-\frac{1}{r} = u$

Differentiate wrt 'θ'

$$+\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}$$

substituting the terms in eq (2)

$$\frac{du}{d\theta} + u\tan\theta = \frac{-1}{\cos\theta} \longrightarrow (3)$$

The equation is linear in 'u'.

It is of the form

$$\frac{du}{d\theta} + (\tan\theta)u = \cancel{Q(\theta)}\ \frac{-1}{\cos\theta}$$

$$\frac{du}{d\theta} + P(\theta)(u) = Q(\theta)$$

$$P(\theta) = \tan\theta \quad , \quad Q(\theta) = \frac{-1}{\cos\theta}$$

$$I.F = e^{\int P(\theta)d\theta} = e^{\int \tan\theta d\theta} = e^{\log\sec\theta}$$

$$= \sec\theta$$

General sol'n of eq (3)

$$u(IF) = \int (I \cdot F)\, Q(\theta)\, d\theta + c$$

$$u \sec\theta = \int \sec\theta \left(\frac{-1}{\cos\theta}\right) d\theta + c$$

$$= \int -\sec\theta\, (\sec\theta)\, d\theta + c$$

$$= -\int \sec^2\theta\, d\theta + c$$

$$u\sec\theta = -\tan\theta + c$$

Divide with $\sec\theta$ on both sides

$$\boxed{u = -\sin\theta + c\cos\theta}$$

$$\boxed{\frac{-1}{r} = -\sin\theta + c\cos\theta} \quad \left(\because u = \frac{-1}{r}.\right)$$

$$\frac{1}{r} = \sin\theta - c\cos\theta$$

* Reducible to Linear Differential Equation:

The D.E $f'(y)\frac{dy}{dx} + P(x)\,f(y) = Q(x)$

which is reducible to linear D.E

let $f(y) = u$

Diff. wrt 'x'

$$f'(y)\cdot\frac{dy}{dx} = \frac{du}{dx}$$

From ①

$$\frac{du}{dx} + P(x)\cdot u = Q(x)$$

It is linear in 'u'.

Solve:

① $\frac{dy}{dx} - x\sin 2y = x^3\cos^2 y$

② $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$.

③ $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$

④ $e^y\left(\frac{dy}{dx} + 1\right) = e^x$

⑤ $\frac{dy}{dx} - \frac{1}{x} = \frac{e^y}{x^2}$

⑥ $2x\frac{dy}{dx} + y = \frac{2x^2}{y^3}$, $y(1) = 2$

⑦ $\frac{dy}{dx} + \tan x\tan y = \cos x\cdot\sec y$

(8) $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$

~~Divide~~

(1) $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Divide with $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x \cdot \frac{2\sin y\cancel{\cos y}}{\cos^{\cancel{2}} y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \longrightarrow (1)$$

let $\tan y = u$

$$\sec^2 y\frac{dy}{dx} = \frac{du}{dx}$$

Substitute above terms in (1)

$$\frac{du}{dx} + 2x(u) = x^3 \longrightarrow (2)$$

It is Linear in 'u'.

It is of the form

$$\frac{du}{dx} + P(x)\cdot u = Q(u)$$

$$P = 2x, \quad Q = x^3$$

$$IF = e^{\int P dx} = e^{\int 2x dx} = e^{\int \frac{\cancel{2}x^2}{\cancel{2}}} = e^{x^2}$$

Gen. sol'n of ②

$u(I.F) = \int Q\,(I.F)\,dx + c$

$(\tan y)\, e^{x^2} = \int x^3 e^{x^2} dx + c$

Let $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$(\tan y)\, e^{x^2} = \int x^2 \cdot e^{x^2} \cdot x\,dx + c$

$= \int t \cdot e^t \frac{dt}{2} + c$

$= \frac{1}{2}\left[te^t - \int 1 \cdot e^t\,dt \right] + c$

$= \frac{1}{2}\left[te^t - e^t \right] + c$

$= \frac{1}{2}(t-1)e^t + c$

$(\tan y)\, e^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c$

Divide with e^{x^2}

$\frac{(\tan y)\, e^{x^2}}{e^{x^2}} = \frac{1}{2}\frac{(x^2-1)e^{x^2}}{e^{x^2}} + \frac{c}{e^{x^2}}$

$\tan y = \frac{x^2 - 1}{2} + ce^{-x^2}$

$2\tan y = x^2 - 1 + 2ce^{-x^2}$

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(5) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Divide $\sec y$ on both sides

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y}\left(\frac{1}{1+x}\right) = (1+x)e^x \cancel{\sec y}$$

$$\cos y \frac{dy}{dx} - \sin y\left(\frac{1}{1+x}\right) = (1+x)e^x \rightarrow (1)$$

$$\sin y = u$$

$$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{du}{dx} - \left(\frac{1}{1+x}\right)u = (1+x)e^x \rightarrow (2)$$

It is linear in 'u'

It is of form

$$\frac{du}{dx} + \left(\frac{-1}{1+x}\right)u = (1+x)e^x$$

$$P(x) = \frac{-1}{1+x}, \quad Q(x) = (1+x)e^x$$

$$IF = e^{\int P dx} = e^{\int \frac{-1}{1+x} dx} = e^{-\log|1+x|}$$

$$= \frac{1}{1+x}.$$

Gen. sol'n of (2)

$$u(IF) = \int Q(IF) + c$$

$$\sin y \frac{1}{1+x} = \int \cancel{(1+x)}e^x \frac{1}{\cancel{1+x}} + c$$

$$\frac{\sin y}{1+x} = e^x + c$$

$$\boxed{\sin y = (1+x)e^x + c(1+x)}$$

(3) $$\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$$

mul ~~Divide~~ with $\frac{1}{z(\log z)^2}$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{\cancel{z}}{x}\frac{\log z}{z(\log z)^2} = \frac{1}{x}$$

$$\frac{(\log z)^{-2}}{z}\frac{dz}{dx} + \frac{1}{x}\frac{1}{\log z} = \frac{1}{x} \longrightarrow (1)$$

$$\therefore \frac{1}{\log z} = u$$

$$\frac{\log z(0) - \frac{1}{z}}{(\log z)^2}\frac{dz}{dx} = \frac{du}{dx}$$

$$\frac{(\log z)^{-2}}{z}\frac{dz}{dx} = \frac{-du}{dx}$$

$$\frac{-du}{dx} + \frac{1}{x}(u) = \frac{1}{x} \Rightarrow \frac{du}{dx} - \frac{1}{x}(u) = \frac{-1}{x}$$

It is linear in 'u'

$$P(x) = \frac{-1}{x}, \quad Q(x) = \frac{-1}{x} \longrightarrow (2)$$

Gen sol'n of (2)

$$u(IF) = \int Q(IF) + c$$

$$\frac{1}{\log z}\cdot\frac{1}{x} = \int \frac{1}{x}\times\frac{1}{x} + c$$

x^{-2+}

$$\frac{1}{x\log z} = \int \frac{1}{x^2} + c$$

$$\frac{1}{x\log z} = \frac{-1}{x} + c$$

multiply with 'x' on both sides

$$\boxed{\frac{1}{\log z} = -1 + cx}$$

(5) $e^y\left(\frac{dy}{dx}+1\right) = e^x$

$$IF = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{-1}{x}\,dx} = e^{-\log x} = \frac{1}{x}.$$

Gen sol'n

$$u\,(IF) = \int Q\,(IF) + c$$

$$u\left(\frac{1}{x}\right) = \int \frac{1}{x} \times \frac{-1}{x} + c$$

$$\boxed{\frac{1}{x(\log z)} = \frac{1}{x} + c}$$

(4) $e^y\left(\frac{dy}{dx}+1\right) = e^x. \rightarrow (1)$

$$e^y\frac{dy}{dx} + e^y = e^x.$$

$$e^y = u$$

$$e^y \frac{dy}{dx} - e^x = -e^y.$$

$$\frac{dy}{dx} - \frac{e^x}{e^y} = -1$$

$$e^y = u$$

$$e^y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + (1)u = e^x \longrightarrow (2)$$

It is linear in 'u'.

$P(x) = 1$, $Q(x) = e^x$

$$I.F = e^{\int P dx} = e^{\int 1 dx} = e^x.$$

Gen. solution of (2)

$$u(IF) = \int Q(IF) + c$$

$$e^y e^x = \int e^x e^x dx + c$$

$$e^x = t, \quad e^x dx = dt$$

$$e^y e^x = \int t\, dt + c$$

$$e^x e^y = \frac{t^2}{2} + c.$$

$$\boxed{\cancel{e^y = 1 + ce^{-x}}}$$

$$\boxed{e^x e^y = \frac{e^{2x}}{2} + c}$$

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(5) $\frac{dy}{dx} - \frac{1}{x} = \frac{e^y}{x^2} \longrightarrow (1)$

Divide with e^y

$$\frac{1}{e^y}\frac{dy}{dx} - \frac{1}{e^y}\frac{1}{x} = \frac{1}{x^2}$$

$$e^{-y}\frac{dy}{dx} - (e^{-y})\frac{1}{x} = \frac{1}{x^2}$$

$$e^{-y} = u$$

$$-e^{-y}\frac{dy}{dx} = \frac{du}{dx}$$

$$e^{-y}\frac{dy}{dx} = -\frac{du}{dx}$$

$$-\frac{du}{dx} - \frac{1}{(x)}u = \frac{1}{x^2} \longrightarrow (2)$$

Multiply with (-1) of (2)

$$\frac{du}{dx} + \frac{1}{x}u = \frac{-1}{x^2}$$

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{-1}{x^2}$$

$$IF = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Gen sol'n

$$u(I.F) = \int Q(IF) + c$$

$$e^{-y}(x) = \int \frac{-1}{x^2}(x) + c$$

$$xe^{-y} = -\log x + c$$

$$\boxed{x = -e^{y}\log x + ce^{y}}$$

(6) $2x\frac{dy}{dx} + y = \frac{2x^2}{y^3} \rightarrow (1),\ y(1) = 2$ $\frac{2x^2}{y^3} \times \frac{y^3}{x}$

~~Divide with~~ $2x^2$

$$\cancel{\frac{1}{x}\frac{dy}{dx} + \frac{y}{2x^2} = \frac{1}{y^3}}$$

multiply with $\frac{2y^3}{x}$

$$\frac{4xy^3}{x}\frac{dy}{dx} + y\left(\frac{2y^3}{x}\right) = 4x$$

$$4y^3\frac{dy}{dx} + y^4\left(\frac{2}{x}\right) = 4x$$

$$y^4 = u$$

$$4y^3\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + \left(\frac{2}{x}\right)u = \frac{2}{x}.$$

$$P(x) = \frac{2}{x},\quad Q(x) = 4x$$

$$IF = e^{\int P\,dx} = e^{\int\frac{2}{x}} = e^{2\log x} = e^{\log x^2} = x^2$$

Gen. Sol'n

$$u(IF) = \int Q(IF) + c$$

$$y^4x^2 = \int 4x^3 + c$$

$$y^4x^2 = x^4 + c$$

(1) $\frac{dy}{dx} + \tan x \tan y = \cos x \cdot \sec y$.

Divide with $\sec y$.

$$\frac{1}{\sec y}\frac{dy}{dx} + \frac{\tan x \tan y}{\sec y} = \cos x$$

$$\cos y \frac{dy}{dx} + \sin y \tan x = \cos x$$

$$\sin y = u$$

$$\cos y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + (\tan x) u = \cos x.$$

$$P(x) = \tan x, \quad Q(x) = \cos x$$

$$IF = e^{\int P(x)dx} = e^{\int \tan x\, dx} = e^{\log|\sec x|} = \sec x$$

Gen sol'n:-

$$u\,(IF) = \int Q\,(IF)\,dx + c$$

$$\sin y \sec x = \int \cos x \sec x\, dx + c$$

$$\boxed{\sin y \sec x = x + c}$$

8) $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$

$$1+\frac{dy}{dx} + x(x+y) = x^3(x+y)^3$$

let $x+y = z$

$$1+\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + z(x) = x^3(x+y)^3$$

~~$P(x) = x$, $Q(x) = x^3(x+y)^3$~~

~~$IF = e^{\int P(x)dx} = e^{\int x dx} = e^{x^2/2}$.~~

$$\frac{dz}{dx} + z(x) = x^3 z^3$$

$$\frac{1}{z^3}\frac{dz}{dx} + \frac{1}{z^2}(x) = x^3$$

$$\frac{1}{z^2} = u$$

$$\frac{-2}{z^3}\frac{dz}{dx} = \frac{du}{dx}$$

$$\frac{1}{z^3}\frac{dz}{dx} = \frac{-1}{2}\frac{du}{dx}$$

$$-\frac{1}{2}\frac{du}{dx} + x(u) = x^3$$

multiply with '-2'

$$\frac{du}{dx} + (-2x)u = -2x^3$$

$$P(x) = -2x, \quad Q(x) = -2x^3$$

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$IF = e^{\int P dx} = e^{\int -2x\, dx} = e^{-x^2}$

Gen. sol'n

$u(IF) = \int Q\,(IF)\, dx + c$

$\frac{1}{z^2} e^{-x^2} = \int -2x^3 e^{-x^2} dx + c$

$-x^2 = t$

$-2x\, dx = dt$

$\frac{1}{z^2} e^{-x^2} = -\int -x^2 (-2x) e^{-x^2} dx + c$

$\frac{1}{(x+y)^2} e^{-x^2} = -\int t\, e^t\, dt + c$

$\frac{1}{(x+y)^2} e^{-x^2} = -\left[te^t - e^t \right] + c$

$= e^t - te^t + c$

$= e^{-x^2}\left[1 + x^2\right] + c$

$\frac{1}{(x+y)^2} e^{-x^2} = e^{-x^2}(1+x^2) + c$

$$\boxed{\frac{1}{(x+y)^2} = (1+x^2) + c e^{x^2}}$$

* Applications of D.E :-

Newton's law of Cooling

- Statement:- The rate of change of temp. of a body is proportional to the difference of temperature of the body under that surrounding medium.

Let 'θ' - be temperature of body ~~any~~ at ~~'θ_0' be~~ any time 't'

θ_0 - be temperature of surrounding medium.

From Newton's Law

$$\boxed{\frac{d\theta}{dt} \propto (\theta-\theta_0)}$$

$$\frac{d\theta}{dt} = -k(\theta-\theta_0)$$

$$\frac{d\theta}{\theta-\theta_0} = -k\,dt$$

Integrating on both sides

$$\int \frac{d\theta}{\theta-\theta_0} = -k\int dt + \log c$$

$$\log(\theta-\theta_0) = -kt + \log c$$

$$\log\left(\frac{\theta-\theta_0}{c}\right) = -kt$$

$$\frac{\theta-\theta_0}{C} = e^{-kt}$$

$$\theta - \theta_0 = Ce^{-kt}$$

$$\boxed{\theta = \theta_0 + Ce^{-kt}} \longrightarrow (1)$$

which gives temperature of a body at a time 't'.

(1) The temperature of body drops from 100°c to 75°c in 10 minutes when the surrounding air is at 20°C. What will be it's temperature after half an hour. When will the temperature be 25°C.

2) If the temperature of air is 20°C and the temperature of body drops from 100°c to 80°c in 10 minutes. What will be it's temperature after 20 minutes. When will be its temperature 40°C.

3) A copper ball is heated to temperature 80°C, then at that time t=0, it is placed in water which is maintained at 30°C. If 't' = 3minutes, the temperature of ball

is reduced to 50°C. Find the time at which the temperature of the ball is 40°C

4) Suppose that an object is heated to 300°F and allowed to cool down in a room whose air temperature is 80°F. If after 10 minutes the temperature of object is 250°F, what will be its temperature after 20 minutes.

① Let 'θ' be temp of body at any time 't'

'θ_0' be temp of air

$\theta_0 = 20°C$

From Newton's law,

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

we get $\boxed{\theta = \theta_0 + ce^{-kt}} \longrightarrow (1)$

Given,

if $t = 0$ min, $\theta = 100°c$

if $t = 10$ min, $\theta = 35°c$

we have to find

if $t = 30$ min, $\theta = ?$

if $\theta = 25°c$, $t = ?$

if $t = 0$ min, $\theta = 100°c$

from (1),

$$100 = 20 + ce^{0}$$

$$\boxed{C = 80}$$

(1) becomes

$$\theta = 20 + 80 \cdot e^{-kt} \longrightarrow (2)$$

if $t = 10$ min's, $\theta = 75°c$

From (2),

$$75 = 20 + 80e^{-10k}$$

$$55 = 80e^{-10k}$$

$$\frac{55}{80} = (e^{-k})^{10}$$

$$\boxed{e^{-k} = \left(\frac{11}{16}\right)^{1/10}}$$

if $t = 30$ min, temp $\theta = ?$

from (2)

$$\theta = 20 + 80e^{-30k}$$

$$\theta = 20 + 80\left(\frac{11}{16}\right)^{\frac{30}{10}}$$

$$\theta = 20 + 80\left(\frac{11}{16}\right)^{3}$$

$$\theta = 45.99$$

$$\boxed{\theta \simeq 46°C}$$

Temp of body is 46°c at time t = 30 min.

if $\theta = 25°C$ then time $(t) = ?$

From ②,

$$25 = 20 + 80\,e^{-kt}$$

$$5 = 80\,e^{-kt}$$

$$\frac{5}{80} = \left(\frac{11}{16}\right)^{t/10}$$

$$\frac{1}{16} = \left(\frac{11}{16}\right)^{t/10}$$

Applying log on both side

$$\log\left(\frac{1}{16}\right) = \log\left(\frac{11}{16}\right)^{t/10}.$$

$$\log 1 - \log 16 = \frac{t}{10}\left[\log 11 - \log 16\right]$$

$$t = 10 \times \frac{\log 1 - \log 16}{\log 11 - \log 16}$$

$$t = 10 \times 7.486$$

$$t = 74.86 \text{ min}.$$

Temp of body is 25°C then $t = 74.86$ min.

② $$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

we get $\boxed{\theta = \theta_0 + c e^{-kt}} \rightarrow (1)$

if $t = 0$ min , $\theta = 100°c$

if $t = 10$ min , $\theta = 80°C$

we have to find

if $t = 20$ min , $\theta = ?$

if $\theta = 40°c$, $t = ?$

From (1)

$$100 = 20 + c e^0$$

$$\boxed{c = 80}$$

(1) becomes

$$\theta = 20 + 80 e^{-kt} \rightarrow (2)$$

If $t = 10$ min $\theta = 80°C$

From (2)

$$80 = 20 + 80 e^{-10k}$$

$$60 = 80 e^{-10k}$$

$$\frac{60}{80} = (e^{-k})^{10}$$

$$\boxed{e^{-k} = \left(\frac{3}{4}\right)^{\frac{1}{10}}}$$

If $t = 20$ mins , $\theta = ?$

$$\theta = 20 + 80 e^{-20k}$$

$$\theta = 20 + 80\,(e^{-k})^{20}$$

$$\theta = 20 + 80\left(\frac{3}{4}\right)^{20/10}$$

$$= 20 + 80\left(\frac{3}{4}\right)^{2}$$

$$= 20 + \cancel{80}^{5} \times \frac{9}{\cancel{16}}$$

$$= 20 + 45$$

$$= 65^\circ c$$

Temp of body is $65^\circ c$

if $\theta = 40^\circ c$, then

$$t = ?$$

From (2)

$$40 = 20 + 80e^{-kt}$$

$$20 = 80\left(\frac{3}{4}\right)^{t/10}$$

$$\log\left(\frac{20}{80}\right) = \frac{t}{10}\left[\log\left(\frac{3}{4}\right)\right]$$

$$\log 1 - \log 4 = \frac{t}{10}\left[\log 3 - \log 4\right]$$

$$t = 10 \times \frac{\log 1 - \log 4}{\log 3 - \log 4}$$

$$= 10 \times 4.818$$

$$= 48.188 \text{ min.}$$

3) From newton's law

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

we get $\boxed{\theta = \theta_0 + Ce^{-kt}}$ $\longrightarrow$ (1)

if $\theta_0 = 30°$, / if $t = 0$, $\theta = 80\,°c$

if $t = 0$, $\theta = 80°c$

we have to find

if $\theta = 40°c$, $t = ?$

From eq (1)

$$\theta = \theta_0 + Ce^{-kt} \qquad (\text{at } t = 0)$$

$$\cancel{\theta = 30} \cdot 80 = 30 + Ce^{0}$$

$$\boxed{C = 50}$$

$$\theta = 30 + 50e^{-kt} \longrightarrow (2)$$

At $t = 3\text{min}$, $\theta = 50°c$

$$50 = 30 + 50e^{-k(3)}$$

$$20 = 50(e^{-k})^3$$

$$\frac{20}{50} = (e^{-k})^3$$

$$\boxed{e^{-k} = \left(\frac{2}{5}\right)^{1/3}}$$

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when $\theta = 40°c$

$$40 = 30 + 50\, e^{-kt}$$

$$10 = 50\,(e^{-k})^{t}$$

$$\frac{1}{5} = \left(\left(\frac{2}{5}\right)^{1/3}\right)^{t}$$

$$\log\frac{1}{5} = \frac{t}{3}\left[\log\frac{2}{5}\right]$$

$$\log 1 - \log 5 = \frac{t}{3}\left[\log 2 - \log 5\right]$$

$$t = 3 \times \frac{\log 1 - \log 5}{\log 2 - \log 5}$$

$$\boxed{t = 5.269 \text{ min}}$$

(4) $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$$\boxed{\theta = \theta_0 + ce^{-kt}} \longrightarrow (i)$$

if $t = 0$, $\theta = 300°F$

if $t = 10$ min, $\theta = 250°F$

we have to find at $t = 20$ min, $\theta = ?$

If $t = 0$, $\theta = 300°F$

$$300 = 80 + ce^{0}$$

$$\boxed{c = 220° F}$$

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$$\theta = 80°F + 220°F\, e^{-kt} \longrightarrow (2)$$

At $t = 10$ min, $\theta = 250°F$

$$250 = 80 + 220\,(e^{-k})^{10}$$

$$\frac{170}{220} = (e^{-k})^{10}$$

$$e^{-k} = \left(\frac{17}{22}\right)^{1/10}$$

At $t = 20$ min, from eq (2)

$$\theta = 80 + 220\, e^{-kt}$$

$$\theta = 80 + 220\,(e^{-k})^{20}$$

$$\theta = 80 + 220\left(\frac{17}{22}\right)^{2}$$

$$\theta = 80 + 220 \times \frac{289}{484}$$

$$= 80 + 220 \times 0.597$$

$$\boxed{\theta = 211.363°F}$$

(5) A murder victim is discovered and an officer from forensic science laboratory is summoned to estimate at the time of death. The body is located in a room that is kept at a constant temperature of 68°F an officer arrived at

9:40 pm and measured the body temperature as 94.4°F at that time. Another measurement of body temperature at 11pm is 89.2°F. Find the estimated time of death?

Sol) Let 'θ' be temp. of body at anytime

Let, θ_0 = constant temp

$\theta_0 = 68°F$

From Newton's Law,

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\boxed{\theta = \theta_0 + ce^{-kt}} \longrightarrow (1)$$

Assume the time 9:40 pm as initial time

if $t = 0$ min's , $\theta = 94.4°F$

at 11pm , if $t = 80$ min's , $\theta = 89.2°F$

if $\theta = 98.6°F$, $t = ?$

($\because$ temp. of human body $= 98.6°F$)

if $t = 0$ min's , $\theta = 94.4°F$

From (1)

$94.4 = 68 + c \cdot e^0$

$\boxed{c = 26.4}$

Eq (1) becomes $\boxed{\theta = 68 + 26.4e^{-kt}}$

if $t = 80$ min , $\theta = 89.2°F$

$$① \to 89.2 = 68 + (26.4)e^{-80k}$$

$$21.2 = (26.4)e^{-80k}$$

$$\frac{21.2}{26.4} = (e^{-k})^{80}$$

$$0.8 = (e^{-k})^{80}$$

$$e^{-k} = (0.8)^{\frac{1}{80}}$$

if $\theta = 98.6°F$ the time $t = ?$

From ②

$$98.6 = 68 + (26.4)e^{-kt}$$

$$30.6 = (26.4)(e^{-k})^{t}$$

$$\frac{30.6}{26.4} = (0.8)^{t/80}$$

$$1.159 = (0.8)^{t/80}$$

Apply 'log$_e$' on both side

$$\log(1.159) = \frac{t}{80}\log(0.8)$$

$$t = 80 \times \frac{\log(1.159)}{\log(0.8)}$$

$$t = -52.9 \text{ min's}$$

$\therefore$ Exact time of death is approximately is 52.9 mins before the first measurement at 9:40pm

$\therefore$ The time of death approximately at 8:47pm.

* Natural growth (or) decay.

Let $x(t)$ be available amount of a substance at a time t. The rate of change of amount ~~chemically~~ 'x' of chemically changing substance is proportional to the available amount of substance at that time.

i.e. $\boxed{\frac{dx}{dt} \propto x}$

For growth

$\frac{dx}{dt} = kx$

$\frac{dx}{x} = kdt$

Integrate on b/s

$\int \frac{1}{x} dx = k\int dt + \log c$

$\log x = kt + \log c$

$\log_e\left(\frac{x}{c}\right) = kt$

$\frac{x}{c} = e^{kt}$

$\boxed{x = ce^{kt}}$

For decay

$\frac{dx}{dt} = -kx$

$\frac{dx}{x} = -kdt$

Integrate on b/s

$\int \frac{1}{x} dx = -\int kdt + \log c$

$\log x = -kt + \log c$

$\log_e\left(\frac{x}{c}\right) = -kt$

$\frac{x}{c} = e^{-kt}$

$\boxed{x = ce^{-kt}}$

ely

nt

If 'N' is amount of substance at any time 't'.

$N = ce^{kt}$ (Natural growth) | $N = c \cdot e^{-kt}$ (Natural decay)

* Natural growth:

Eg:- Bacteria, virus, human body cell, etc.

* Natural decay:

Eg:- Chemical reactions, metals, uranium, plutonium, carbon.

1) The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in 1 hour. What was the value of 'N' after 1½ hour.

2) A bacterial culture, growing exponentially increases from 200 to 500 grams in the period from 6 am to 7 am. How many grams will be present at noon.

3) If radioactive carbon-14 (C^{14}) has half life of 5750 years. What will remain of 1 gram after 3000 years.

4) A bacteria in a culture grows exponentially so that initial number has doubled in 3 hours. How many times the initial number will be present after 9 hours.

5) In a chemical reaction, a given substance being converted to another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)$th of the original amount has been transformed in 4 minutes, how much time will be required to transform one half.

6) If 30% of radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear.

7) Uranium disintegrates at a rate proportional to the amount present. If M_1 and M_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half-life of uranium.

1A) Let 'N' be amount of bacteria at any time 't'.

From law of natural growth,

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

we get $\boxed{N = ce^{kt}}$ → (1)

if $t = 0\,hr$, $N = 100$

if $t = 1\,hr$, $N = 332$

if $t = 1\frac{1}{2}\,hr = \frac{3}{2}\,hr$, $N = ?$

if $t = 0\,hr$, $N = 100$

from (1)

$$100 = ce^0$$

$$\boxed{c = 100}$$

(1) becomes

$$N = 100e^{kt} \rightarrow (2)$$

if $t = 1\,hr$, $N = 332$

(2) becomes,

$$332 = 100e^k$$

$$\boxed{e^k = 3.32}$$

if $t=\frac{3}{2}$ hr, $N=?$

From ②,

$$N = 100\cdot e^{3/2 k}$$

$$N = 100\,(e^{k})^{3/2}$$

$$N = 100\,(3.32)^{3/2}$$

~~$N = 604.9$~~ $N = 604.9 \cong 605$.

② Let 'N' be amount of bacteria at any time 't'.

From law of natural growth,

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

we get $\boxed{N = ce^{kt}} \longrightarrow ①$

if $t = 0$ hr, $N = 200$ grams

if $t = 3$ hr, $N = 500$ grams

if $t = 6$ hrs, $N = ?$

If $t=0$ hr, $N = 200$ grams

from ①

$$200 = ce^{0}$$

$$c = 200$$

① becomes

$$\boxed{N = 200e^{kt}} \rightarrow ②$$

if $t = 3hrs$, $N = 500grams \Rightarrow$ eq ② becomes

$$500 = 200e^{k(3)}$$

$$\frac{5}{2} = e^{3k}$$

$$\boxed{e^k = \left(\frac{5}{2}\right)^{1/3}}$$

if $t = 6hrs$, $N = ?$

From ②,

$$N = 200e^{6k}$$

$$N = 200\left(\frac{5}{2}\right)^2$$

$$N = \cancel{200}^{50} \times \frac{25}{\cancel{4}}$$

$$\boxed{N = 1250 \text{ grams}}$$

③ Let 'N' be the amount of radioactive carbon in any time 't'.

From law of natural decay.

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -kN$$

we get $\boxed{N = ce^{-kt}} \rightarrow ①$

if ~~t=575~~ t = 0 years, N = 1 gram

if t = 5750 years, N = 0.5 gram

if t = 3000 years, N = ?

if t = 0 years, N = 1 gram. then

~~eq ① becomes~~ from eq ①

$$1 = ce^0$$

$$\boxed{c = 1}$$

eq ① becomes

$$\boxed{N = e^{-kt}} \longrightarrow ②$$

if t = 5750 years, N = 0.5 grams

From eq ②

$$0.5 = e^{-5750k}$$

$$\boxed{e^{-k} = (0.5)^{1/5750}}$$

if t = 3000 years, N = ?

$$N = (e^{-k})^{3000}$$

$$N = (0.5)^{3000/5750}$$

$$\boxed{N = 0.696 \text{ grams}}$$

(4) Let 'N' be the amount of bacteria at any time 't'

From law of natural growth,

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

we get $\boxed{N = Ce^{+kt}}$

Given:-

If $t = 0\,hr$, $N = m$

If $t = 3\,hr's$, $N = 2m$.

If $t = 9\,hr's$, $N = ?$

If $t = 0\,hr$, $N = m$.

From (1)

$$m = c \cdot e^{0}$$

$$\boxed{c = m}$$

eq (1) becomes

$$\boxed{N = me^{kt}} \longrightarrow (2)$$

If $t = 3\,hrs$, $N = 2m$.

From (2)

$$2\cancel{m} = \cancel{m}e^{3k}$$

$$e^{3k} = 2$$

$$\boxed{e^{k} = (2)^{1/3}}$$

if $t = 9$ hrs, $N = ?$

From (2),

$$N = m \cdot e^{9k}$$

$$N = m(e^{k})^{9}$$

$$= m(2)^{\cancel{9}^{3}/\cancel{3}}$$

$$\boxed{N = 8m}$$

8 times of initial amount will be present after 9 hrs.

(6) let 'N' be the amount of bacteria at any time 't'

From law of natural decay.

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -kN$$

$$\boxed{N = ce^{-kt}} \longrightarrow (1)$$

if $t=0$, let $N = m$

if $t = 10$ days $N = \left(\frac{100-30}{100}\right)m = \frac{70}{100} = 0.7m.$

if $N = \left(\frac{100-90}{100}\right)m = 0.1m$, $t = ?$

$$\left(\because N = \cancel{m} \frac{100m - 30m}{100}\right)$$

At $t=0$, $N=m$

From ①

$$m = ce^{0}$$

$$\boxed{c = m}$$

eq ② becomes

$$\boxed{N = me^{-kt}} \rightarrow ③$$

At $t = 10$ days, $N = 0.7m$

From eq ③

$$0.7m = me^{-kt}$$

$$0.7 = (e^{-k})^{10}$$

$$\boxed{e^{-k} = (0.7)^{1/10}}$$

At $N = 0.1m$, $t = ?$

From eq ③

$$0.1m = me^{-kt}$$

$$0.1 = (0.7)^{t/10}$$

$$0.1 = [0.7]^{t/10}$$

Apply log on both sides

$$\log(0.1) = \frac{t}{10}\log[0.7]$$

$$t = 10 \times \frac{\log(0.1)}{\log(0.7)}$$

$\boxed{t = 64.556 \text{ days}}$

⑦ Let 'N' be available amount of substance at any time 't'.

From law of Natural decay.

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -kN$$

we get $\boxed{N = ce^{-kt}}$

Let 'M' be initial amount of substance

if $t = 0$ yr, $N = M$ grams.

if $t = T_1$, $N = M_1$

if $t = T_2$, $N = M_2$.

if $N = \frac{M}{2}$ grams, $t = ?$

if $t = 0$ yr, $N = M$ grams

From ①,

$$M = ce^0$$

$$\boxed{c = M}$$

① becomes $\boxed{N = Me^{-kt}} \longrightarrow$ ②

if $t = T_1$, $N = M_1$

$$M_1 = M e^{-kT_1}$$

$$\frac{M_1}{M} = e^{-kT_1}$$

if $t = T_2$, $N = M_2$

$$③ \rightarrow \quad M_2 = M e^{-kT_2}$$

$$\frac{M_2}{M} = e^{-kT_2} \longrightarrow ④$$

$$\frac{③}{④} \rightarrow \frac{\frac{M_1}{M}}{\frac{M_2}{M}} = \frac{e^{-kT_1}}{e^{-kT_2}}$$

$$\frac{M_1}{M_2} = e^{-kT_1 + kT_2}$$

$$\frac{M_1}{M_2} = e^{-k(T_1 - T_2)}$$

$$\boxed{e^{-k} = \left(\frac{M_1}{M_2}\right)^{1/T_1 - T_2}}$$

if $N = \frac{M}{2}$ grams , $t = ?$

From eq ②

$$\frac{M}{2} = M e^{-kt}$$

$$\frac{1}{2} = \left(\frac{M_1}{M_2}\right)^{\frac{t}{T_1 - T_2}}$$

Apply log on both sides

$$\log 1 - \log 2 = \frac{t}{T_1 - T_2}\left[\log M_1 - \log M_2\right]$$

$$t = T_1 - T_2 \left[\frac{\log 1 - \log 2}{\log m_1 - \log m_2} \right]$$

$$t = T_1 - T_2 \left[\frac{-\log 2}{\log m_1 - \log m_2} \right]$$

$$t = T_2 - T_1 \left[\frac{\log 2}{\log m_1 - \log m_2} \right]$$

$$\boxed{t = \frac{T_2 - T_1 (0.301)}{\log m_1 - \log m_2}}$$

⑤ let 'N' be the amount of chemical substance at any time 't'

From law of Natural decay.

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -KN$$

$$\boxed{N = ce^{-kt}}$$

Given:

if $t = 0$ min, $N = m$

if $t = 4$ min, $N = \left(1 - \frac{1}{5}\right)m = \frac{4}{5}m$

if $N = \frac{m}{2}$, $t = ?$

At $t = 0\,min$, $N = m$

From ①

$$m = ce^{0}$$

$$\boxed{c = m}$$

Eq ① ⇒ becomes $N = me^{-kt}$

At $t = 4\,min$, $N = \frac{4m}{5}$

From eq ②

$$\frac{4}{5}\cancel{m} = \cancel{m}e^{-k(4)}$$

$$\boxed{e^{-k} = \left(\frac{4}{5}\right)^{1/4}}$$

At $N = \frac{m}{2}$, $t = ?$

$$\frac{\cancel{m}}{2} = \cancel{m}e^{-kt}$$

$$\frac{1}{2} = \left(\frac{4}{5}\right)^{t/4}$$

$$\log\frac{1}{2} = \frac{t}{4}\left[\log\frac{4}{5}\right] \quad (\because \text{Apply log on b.s})$$

$$\log 1 - \log 2 = \frac{t}{4}\left[\log 4 - \log 5\right]$$

Radium

$$t = \frac{4(-\log 2)}{-\log 5}$$

$$\boxed{t = 1.722\ min}$$

$$t = 4 \times \frac{\log 1 - \log 2}{\log 4 - \log 5}$$

$$\boxed{t = 12.425\ min}$$

1) Radium decomposes at a rate proportional to the amount present if 5% of the original amount disappears in 50 years, how much will remain after 100 years.

Sol) Let 'N' be the amount of substance at any time 't'

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -KN$$

$$\boxed{N = ce^{-kt}} \longrightarrow (1)$$

If $t = 0$ years, $N = m$

If $t = 50$ yrs, $N = \left(\frac{100-5}{100}\right)m = \frac{95}{100}m = 0.95m$

If $t = 100$ yrs, $N = ?$

If $t = 0$ yrs, $N = m$

From (1) $\cancel{N} \quad m = ce^{0}$

$$\boxed{c = m}$$

Eq (1) becomes $\boxed{N = me^{-kt}} \longrightarrow (2)$

if $t = 50$ yrs, $N = 0.95m$

$$0.95\cancel{m} = \cancel{m}(e^{-k})^{50}$$

$$\boxed{(0.95)^{1/50} = e^{-K}}$$

if $t = 100$ yrs, $N = ?$

$$N = m(e^{-K})^{100}$$

$$N = m(0.95)^2$$

$$\boxed{N = 0.9025\, m}$$

$\therefore$ $\boxed{N = 90.25\ \%}$ available amount after 100 years.

*4.10

1) A bacterial culture, growing exponentially, increases from 100 to 400 grams in 10 hrs. How much was present after 3 hours.

* First order, higher degree ordinary DE :

If the degree of D.E of first order is more than or equal to 2, it is Convenient to denote $\boxed{\frac{dy}{dx} = P}$

The differential equation is of the form

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots a_{n-1} P + a_n = 0 \rightarrow (1)$$

where $\boxed{P = \frac{dy}{dx}}$

where $a_1, a_2, \ldots, a_n$ are the function is x, y (or) constants is called first order, higher(nth) degree ordinary differential equations

* To solve above D.E, there are 4 methods:

1) Equations solvable for p.
2) Equations solvable for y.
3) Equations solvable for x.
4) Clairaut's Differential Equation.

1) Differential Equations solvable for p:

Express eq ① as 'n' rational factors

$$[p - f_1(x,y)][p - f_2(x,y)] \cdots [p - f_n(x,y)] = 0$$

Equate each factor to '0'.

$$p - f_1(x,y) = 0, \quad p - f_2(x,y) = 0, \ldots \quad p - f_n(x,y) = 0$$

$$\frac{dy}{dx} = f_1(x,y), \quad \frac{dy}{dx} = f_2(x,y), \ldots, \frac{dy}{dx} = f_n(x,y)$$

Apply variable separable method, then we get

$$F_1(x, y, c_1) = 0, \; F_2(x, y, c_2) = 0, \ldots, F_n(x, y, c) = 0$$

General solution of ① is.

$$F_1(x,y,c), \; F_2(x,y,c), \; F_3(x,y,c) \ldots, F_n(x,y,c) =$$

Solve

(1) $x^2p^2 + 3xyp + 2y^2 = 0$

(2) $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

(3) $p^2 + 2py \cot x - y^2 = 0$

(4) $p^2 - 2p \cosh x + 1 = 0$

(5) $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

(1) $x^2p^2 + 3xyp + 2y^2 = 0$

$$x^2p^2 + 2xyp + xyp + 2y^2 = 0$$

$$xp[xp + 2y] + y[xp + 2y] = 0$$

$$(xp + 2y)(xp + y) = 0$$

Equate each term to zero

$xp + 2y = 0$	$xp + y = 0$

replace $\boxed{p = \frac{dy}{dx}}$

$x\frac{dy}{dx} + 2y = 0$	$x\frac{dy}{dx} + y = 0$
$x\frac{dy}{dx} = -2y$	$x\frac{dy}{dx} = -y$
$\frac{dy}{y} = \frac{-2dx}{x}$	$\frac{dy}{y} = \frac{-dx}{x}$

Integrating on both sides

$$\int \frac{dy}{y} = -2\int \frac{dx}{x}$$

$$\log y = -2\log x + \log c_1$$

$$\log y = \log x^{-2} c_1$$

$$y = \frac{c_1}{x^2}$$

$$\boxed{x^2 y = c_1}$$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\log y = -\log x + \log c_2$$

$$y = \frac{c_2}{x}$$

$$\boxed{xy = c_2}$$

$\therefore$ General sol'n.

$$\boxed{(x^2y - c)(xy - c) = 0}$$

(2) $xyp^2 + 3x^2p - 2y^2p - 6xy = 0$

$$xp[yp + 3x] - 2y[yp + 3x] = 0$$

$$(yp+3x)(xp-2y) = 0.$$

Equate each term to zero

$yp+3x=0$, $xp - 2y = 0$

replace $\boxed{p = \frac{dy}{dx}}$

$$y\frac{dy}{dx} + 3x = 0$$

$$y\frac{dy}{dx} = -3x$$

$$y\,dy = -3x\,dx$$

$$x\frac{dy}{dx} - 2y = 0$$

$$x\frac{dy}{dx} = 2y.$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

Integrate on both sides.

$$\int y\,dy = -\int 3x\,dx$$

$$\frac{y^2}{2} = \frac{-3x^2}{2} + c_1$$

$$y^2 = -3x^2 + 2c_1$$

$$3x^2 + y^2 - 2c_1 = 0$$

$$\int \frac{dy}{y} = 2\int \frac{dx}{x}$$

$$\log y = 2\log x + \log c_2$$

$$\log y = \log(c_2 x^2)$$

$$y = x^2 c_2$$

$$y - x^2 c_2 = 0$$

$\therefore$ General sol'n

$$(3x^2 + y^2 - 2c)(y - x^2 c) = 0$$

(3) $p^2 + 2py\cot x - y^2 = 0 \longrightarrow (1)$

It is a quadratic equation in p

roots are

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$p = -y\cot x \pm y\operatorname{cosec} x$$

$p = -y\cot x + y\operatorname{cosec} x$ | $p = -y\cot x - y\operatorname{cosec} x.$

replace $\boxed{p = \frac{dy}{dx}}$

$$\frac{dy}{dx} = y[-\cot x + \operatorname{cosec} x]$$

$$\frac{dy}{y} = [-\cot x + \operatorname{cosec} x]dx$$

Integrate on both sides

$$\int \frac{dy}{y} = \int [-\cot x + \operatorname{cosec} x]dx$$

$$\log y = \int -\cot x\, dx + \int \operatorname{cosec} x\, dx$$

$$\log y = -\log|\sin x| + \log|\tan\tfrac{x}{2}| + \log c_1$$

$$\log y = \log\left|\frac{c_1 \tan\frac{x}{2}}{\sin x}\right|$$

$$y = \frac{c_1 \tan\frac{x}{2}}{\sin x}$$

$$y = c_1 \frac{\cancel{\sin\frac{x}{2}}}{\cos x/2} \times \frac{1}{2\cancel{\sin\frac{x}{2}}\cos\frac{x}{2}}$$

$$y = \frac{c_1}{2\cos^2 x/2}$$

$$y = \frac{c_1}{(1+\cos x)}$$

$$\frac{dy}{dx} = y[-\cot x - \operatorname{cosec} x]$$

$$\frac{dy}{y} = [-\cot x - \operatorname{cosec} x]dx$$

$$\int \frac{dy}{y} = \int [-\cot x - \operatorname{cosec} x]dx$$

$$\log y = \int -\cot x\, dx - \int \operatorname{cosec} x\, dx$$

$$\log y = -\log|\sin x| - \log|\tan\tfrac{x}{2}| + \log c_2$$

$$\log y = \log\left|\frac{c_2}{\sin x \tan\frac{x}{2}}\right|$$

$$y = \frac{c_2}{\sin x \tan\frac{x}{2}}$$

$$y = \frac{c_2}{2\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}} \times \frac{\cancel{\cos x/2}}{\sin x/2}$$

$$y = \frac{c_2}{2\sin^2 x/2}$$

$$y = \cancel{\frac{c_2}{1+\sin}} \quad \frac{c_2}{1-\cos x}.$$

$$\left(\because 2\cos^2\frac{x}{2} = 1+\cos x,\ 2\sin^2\frac{x}{2} = 1-\cos x\right)$$

$$y(1+\cos x) = c_1$$

$$y(1+\cos x) - c_1 = 0$$

$$y(1-\cos x) = c_2.$$

$$y(1-\cos x) - c_2 = 0$$

General sol'n :-

$$\left(y(1+\cos x)-c_1\right)\left(y(1-\cos x)-c_2\right)=0$$

(4) $p^2 - 2p\cosh x + 1 = 0$

let $\cosh x = \dfrac{e^x+e^{-x}}{2}$

Replace 1 as $e^x - e^{-x}$

$$p^2 - 2p\left(\frac{e^x+e^{-x}}{2}\right)+\left(e^x-e^{-x}\right)=0$$

$$p^2 - p(e^x+e^{-x}) + (e^x-e^{-x})=0.$$

$\cancel{p\left[p-(e^x+e^{-x})\right]}$

$\cancel{p^2 - pe^x - pe^{-x} + e^x - e^{-x}} = 0$

$\cancel{p^2 - e^x(p-1) - e^{-x}(p+1)} = 0.$

$$p\left[p-e^x\right]-e^{-x}\left[p-e^{-x}\right]=0$$

$$(p-e^x)(p-e^{-x})=0$$

$p - e^x = 0$

$\dfrac{dy}{dx} - e^x = 0$

$dy = e^x\,dx$

$p - e^{-x} = 0$

$\dfrac{dy}{dx} - e^{-x} = 0$

$dy = e^{-x}\,dx$

Integrating on both sides

$\int dy = \int e^x\,dx$

$y = e^x + c_1$

$\int dy = \int e^{-x}\,dx$

$y = -e^{-x} + c_2.$

$y-e^{x}-c_1=0$ | $y+e^{-x}-c_2=0$

General solution is

$$(y-e^{x}-c)(y+e^{-x}-c)=0$$

(5) $p^3+2xp^2-y^2p^2-2xy^2p=0$

$$p^2[p+2x]-y^2p[p+2x]=0$$

$$[p+2x][p^2-y^2p]=0$$

$$(p+2x)p(p-y^2)=0$$

$p+2x=0$ | $p=0$ | $p-y^2=0$

replace $\boxed{p=\frac{dy}{dx}}$

$\frac{dy}{dx}+2x=0$ | $\frac{dy}{dx}=0$ | $\frac{dy}{dx}-y^2=0$

$dy=-2x\,dx$ | $dy=0$ | $dy=+y^2dx$

$\frac{dy}{y^2}=+dx$

Integrating on both sides

$\int dy=-2\int x\,dx$ | $\int dy=\int 0\,dx$ | $\int\frac{1}{y^2}dy=+\int dx$

$y=\frac{-2x^2}{2}+c$ | $\boxed{y=c}$ | $-\frac{1}{y}=+x+c$

$y+x^2-c=0$ | | $-\frac{1}{y}\mp x-c=0$

$1+xy+cy=0$

The general solution is

$$(y+x^2-c)(y-c)(xy+cy+1)=0$$

① Let 'N' be the amount of bacteria in any time 't'

From law of natural growth

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

$$\boxed{N = ce^{kt}} \longrightarrow ①$$

Given

if $t = 0$ hrs, $N = 100$ gms

if $t = 10$ hrs, $N = 400$ gms.

we have to find if $t = 3$ hrs, $N = ?$

if $t = 0$ hrs, $N = 100$ gms.

~~eq~~ From ①

$$100 = ce^0$$

$$\boxed{c = 100}$$

eq ① becomes $\boxed{N = 100e^{kt}} \longrightarrow ②$

if $t = 10$ hrs, $N = 400$ gms.

From ②

$$\cancel{400}^{4} = \cancel{100}(e^k)^{10}$$

$$\boxed{e^k = (4)^{1/10}}$$

if $t = 3$ hrs, $N = ?$

$$N = 100(e^{k})^3$$

$$N = 100(4)^{3/10}$$

$$\boxed{N = 151.571} \text{ gms.}$$

6) $4xp^2 = (3x-a)^2$

7) $xp^2 = (x-a)^2$

8) $p^2 - 7p + 12 = 0$

9) $p^2 - xy = y^2 - px$

6 A) $4xp^2 = (3x-a)^2$

$$p^2 = \frac{(3x-a)^2}{4x}$$

Apply square ~~on~~ root on b.s.

$$p = \pm \frac{(3x-a)}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \pm\left[\frac{3x}{2\sqrt{x}} - \frac{a}{2\sqrt{x}}\right]$$

$$dy = \pm\left[\frac{3}{2}x^{1/2} - \frac{a}{2}x^{-1/2}\right] dx.$$

Integrate on both sides

$$\int dy + c = \pm\left[\frac{3}{2}\int x^{1/2}\,dx - \frac{a}{2}\int x^{-1/2}\,dx\right]$$

$$y + c = \pm\left[\frac{8}{2}\frac{x^{3/2}}{3/2} - \frac{a}{2}\frac{x^{1/2}}{1/2}\right]$$

$$y + c = \pm\left[x^{3/2} - a x^{1/2}\right]$$

$$y + c = \pm\left[x - a\right] x^{1/2}$$

Square on both sides.

$$(y+c)^2 = (x-a)^2 x.$$

8) $p^2 - 7p + 12 = 0$

$$p^2 - 3p - 4p + 12 = 0$$

$$p(p-3) - 4(p-3) = 0$$

$$(p-3)(p-4) = 0$$

equate every term to zero

$p - 3 = 0$	$p - 4 = 0$

replace $p = \frac{dy}{dx}$

$\frac{dy}{dx} - 3 = 0$	$\frac{dy}{dx} - 4 = 0$
$dy = 3dx$	$dy = 4\,dx$

Integrate on both sides

$\int dy = 3\int dx + c_1$	$\int dy = 4\int dx + c_2$
$y = 3x + c_1$	$y = 4x + c_2$

General sol'n

$$(y - 3x - c)(y - 4x - c) = 0$$

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a) $p^2 - xy = y^2 - px$

$p^2 - xy - y^2 + px = 0$

$p^2 - y^2 - xy + px = 0$

$(p+y)(p-y) - x(y-p) = 0$

$(p-y)[p+y+x] = 0$

$p - y = 0$

$\frac{dy}{dx} - y = 0$

Integrate on both sides

$\int \frac{dy}{y} = \int dx$

$\log y = x + C_1$

$y = e^{x+C_1}$

$y = e^x \cdot e^{C_1}$

$y = C_2 \cdot e^x$ $\boxed{\because C_2 = e^{C_1}}$

$p + y + x = 0$

$\frac{dy}{dx} + y + x = 0$

$\frac{dy}{dx} + y = -x$

It is linear D.E in 'y'

$\frac{dy}{dx} + P(x) \cdot y = Q(x)$

$P = 1, \ Q = -x$

$I.F = e^{\int P dx} = e^{\int 1 dx} = e^x$

General solution

$y(I.F) = \int Q(I.F)\, dx + C$

$ye^x = \int \underset{u}{-x}\, \underset{v}{e^x}\, dx + c$

$ye^x = -[xe^x - \int 1 \cdot e^x dx] + c$

$ye^x = -[xe^x - e^x] + c$

$ye^x = -e^x(x-1) + c$

$y = -(x-1) + \frac{c}{e^x}$

$y + (x-1) - c_3 e^{-x}$.

General solution:

$[y - ce^x][y + (x-1) - ce^{-x}] = 0$

(7) $xp^2 = (x-a)^2$

$$p^2 = \frac{(x-a)^2}{x}$$

$$p = \pm \frac{x-a}{\sqrt{x}}$$

Now $p = \frac{dy}{dx}$

$$\frac{dy}{dx} = \pm \frac{x-a}{\sqrt{x}}$$

$$dy = \pm \left[\frac{x-a}{\sqrt{x}}\right] dx$$

$$dy = \pm \left[\sqrt{x} - \frac{a}{\sqrt{x}}\right] dx$$

Integrate on both sides.

$$\int dy + c = \pm \int \sqrt{x}\, dx - \int \frac{a}{\sqrt{x}} dx$$

$$y + c = \pm \left[\frac{x^{3/2}}{3/2} - a \cdot \frac{x^{1/2}}{1/2}\right]$$

$$y + c = \pm \left[\frac{2x^{3/2}}{3} - 2a x^{1/2}\right]$$

$$y + c = \pm \left[\frac{2}{3} \cdot x - 2a\right] \sqrt{x}$$

squaring on both sides

$$(y+c)^2 = \left(\frac{2}{3}x - 2a\right)^2 x$$

* Equations solvable for "y" :-

Express given differential equation in the form, $y = (x, p) \longrightarrow (1)$

eq(1) Diff. w.r.t 'x'.

$$\frac{dy}{dx} = p = f\left(x, p, \frac{dp}{dx}\right)$$

$$p = f\left(x, p, \frac{dp}{dx}\right) \longrightarrow (2)$$

It is differential eq'n in 'x' & 'p'.

Find solution of (2), then we get,

$$F(x, p, c) = 0 \longrightarrow (3)$$

Eliminate 'p' from eqn's (1) & (3)

Then we get, General sol'n

$$\phi(x, y, c) = 0$$

Note: (i) If elimination of 'p' is not possible then express eqn's (1), (3) as

$$\left.\begin{aligned} x &= F_1(p, c) \\ y &= F_2(p, c) \end{aligned}\right\} \longrightarrow (4)$$

(ii) If the eq'n (4) is not possible then eqn's (1), (3) together gives the general solution.

(iii) If eq (2) can be expressed as product of two factors, ignore the term which do not contains $\frac{dp}{dx}$, Contine with the term of '$\frac{dp}{dx}$'.

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Solve:

(1) $y = 2px - xp^2$

(2) $y + px = x^4 p^2$

(3) $3x^4 p^2 - xp - y = 0$

(4) $y = 2px + p^n$

(5) $y = x + a\tan^{-1}p$.

(1) Sol'n: $y = 2px - xp^2$

Diff wrt 'x',

$$\frac{dy}{dx} = p = 2x \cdot \frac{dp}{dx} + 2p - p^2 - x \cdot 2p \cdot \frac{dp}{dx}$$

$$\frac{dp}{dx}(2x - 2xp) + p - p^2 = 0$$

$$2x \cdot \frac{dp}{dx}(1-p) + p(1-p) = 0$$

$$(1-p)\left(2x\frac{dp}{dx} + p\right) = 0$$

Ignore 1st term, continue with 2nd term

has $\frac{dp}{dx}$

$$2x\frac{dp}{dx} + p = 0$$

$$2x\frac{dp}{dx} = -p$$

$$\frac{dp}{p} = \frac{-dx}{2x}$$

$$\frac{dp}{p} + \frac{dx}{2x} = 0$$

Integrate on both sides

$$\int \frac{dp}{p} + \int \frac{dx}{2x} = \int 0$$

$$\log p + \frac{1}{2}\log x = \log c$$

$$2\log p + \log x = 2\log c$$

$$\log xp^2 = \log c^2$$

$$xp^2 = c^2$$

$$\boxed{p^2 = \frac{c^2}{x}}$$

$$\boxed{p = \frac{c}{\sqrt{x}}} \longrightarrow (2)$$

Sub eqn (2) in (1)

$$y = 2x\left(\frac{c}{\sqrt{x}}\right) - x \cdot \frac{c^2}{x}$$

$$\boxed{y = 2\sqrt{x}\,c - c^2}$$

(2) $y + px = x^4 p^2$

Diff. w.r.t 'x'

$$\frac{dy}{dx} + \frac{dp}{dx}x + p = x^4\, 2p\frac{dp}{dx} + 4x^3p^2$$

$$\frac{dy}{dx} = \cancel{x^4(2p)\frac{dp}{dx} + 4x^3p^2 - x\frac{dp}{dx} - p}$$

$$\frac{dy}{dx} = \cancel{x\frac{dp}{dx}(2x^3p - 1)} + \cancel{2p(2x^3p - 1)}$$

$$p = \quad 2$$

$$p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

$$2p + x\frac{dp}{dx} - 2px^3\left[2p + x\frac{dp}{dx}\right] = 0$$

$$\left[2p + x\frac{dp}{dx}\right](1 - 2px^3) = 0$$

Ignore 2nd term, continue with term containing $\frac{dp}{dx}$

$$2p + x\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{p} = -2\frac{dx}{x}$$

Integrating on both sides

$$\int \frac{dp}{p} = -2\int \frac{dx}{x}$$

$$\log p = -2\log x + \log c$$

$$\log p + \log x^2 = \log c$$

$$\cancel{\log}(px^2) = \cancel{\log} c$$

$$\boxed{px^2 = c}$$

Substitute in 1

$$y = -\frac{c}{x} + x^4\left(\frac{c^2}{x^4}\right)$$

$$\boxed{y = -\frac{c}{x} + c^2}$$

3) $3x^4p^2 - xp - y = 0$

$$y = 3x^4p^2 - xp \longrightarrow 1$$

Differentiate 1 wrt 'x'

$$\frac{dy}{dx} = 12x^3p^2 + 6x^4 p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$p = 12x^3p^2 + 6x^4 p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$\left(\because p = \frac{dy}{dx}\right)$$

$$2p - 12x^3p^2 + x\frac{dp}{dx} - 6x^4 p\frac{dp}{dx} = 0$$

$$2p(1-6x^3p) + x\frac{dp}{dx}(1-6x^3p) = 0$$

$$(1-6x^3p)\left[2p + x\frac{dp}{dx}\right] = 0$$

Ignore the 1st term, continue with 2nd term containing $\frac{dp}{dx}$,

$$\left[2p + x\frac{dp}{dx}\right] = 0$$

$$2p\,dx + x\,dp = 0$$

$$2p\,dx = -x\,dp$$

$$2\frac{dx}{x} = -\frac{dp}{p}$$

Integrating on both sides.

$$2\int\frac{dx}{x} = -\int\frac{dp}{p}$$

$$2\log x = -\log p + \log c$$

$$\log x^2 + \log p = \log c$$

$$\log px^2 = \log c$$

$$\boxed{px^2 = c}$$

Sub in eq (1)

$$y = 3x^4\left(\frac{c}{x^2}\right)^2 - x\left(\frac{c}{x^2}\right)$$

$$y = 3x^4 \frac{c^2}{x^4} - \frac{c}{x}$$

$$\boxed{y = 3c^2 - \frac{c}{x}}$$

(ii) $y = 2px + p^n \rightarrow (1)$

Differentiate (1) w.r.t 'x'

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \left(\because p = \frac{dy}{dx}\right)$$

$$p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} = 0$$

$$\cancel{p + (2x} \; p + \frac{dp}{dx}(2x + np^{n-1}) = 0$$

$$p\,dx + (2x + np^{n-1})\,dp = 0$$

$$p\,dx = -(2x + np^{n-1})\,dp$$

$$p\frac{dx}{dp} = -(2x + np^{n-1})$$

$$p\frac{dx}{dp} + 2x = -np^{n-1}$$

$$\frac{dx}{dp} + \frac{2}{p}x = -np^{n-2} \rightarrow (2)$$

If ② is linear in 'x',

$P(p) = \frac{2}{p}$, $Q = -np^{n-2}$

$I.F = e^{\int P dp} = e^{\int \frac{2}{p} dp} = e^{2\log p} = e^{\log p^2}$

$\boxed{I.F = p^2}$

General sol'n of ② is.

$x(IF) = \int (IF) Q(p) dp + c$

$x p^2 = \int p^2 (-np^{n-2}) dp + c$

$x p^2 = -n \int p^{\cancel{2}+n-\cancel{2}} dp + c$

$x p^2 = -n \int p^n dp + c$

$x p^2 = -n \left(\frac{p^{n+1}}{n+1}\right) + c$

Divide with p^2 on both sides

$x = \frac{-np^{n+1-2}}{n+1} + \frac{c}{p^2}$

$\boxed{x = \frac{-np^{n-1}}{n+1} + \frac{c}{p^2}} \rightarrow ③$

Substitute ③ in ①

$y = 2p\left(\frac{-np^{n-1}}{n+1} + \frac{c}{p^2}\right) + p^n$

$y = \frac{-2np^n}{n+1} + \frac{2cp}{p^2} + p^n$

$y = \frac{-2np^n}{n+1} + \frac{2c}{p} + p^n$

$$\cancel{y = \frac{2c}{p} - \frac{n-1}{n+1}p^n}$$

$$y = \frac{2c}{p} + p^n - \frac{2np^n}{n+1}$$

$$y = \frac{2c}{p} + p^n\left(\frac{n+1-2n}{n+1}\right)$$

$$\boxed{y = \frac{2c}{p} - p^n\left(\frac{n-1}{n+1}\right)}$$

(5) $y = x + a\tan^{-1}p \longrightarrow (1)$

Diff (1) wrt 'x'

$$\frac{dy}{dx} = 1 + a \times \frac{1}{1+p^2}\,\frac{dp}{dx}$$

$$\cancel{\frac{a}{1+p^2}\frac{dp}{dx}} = p = 1 + \frac{a}{1+p^2}\,\frac{dp}{dx}$$

$$\left(\because \frac{dy}{dx} = p\right)$$

$$p - 1 = \frac{a}{1+p^2}\,\frac{dp}{dx}$$

$$(p-1)dx = \left(\frac{a}{1+p^2}\right)dp$$

$$dx = \frac{a}{(p-1)(p^2+1)}\,dp$$

$$dx = \frac{a}{(\cancel{p^2}-1^2)}\,dp. \quad dx = \frac{a}{(p-1)(p^2+1)}\,dp$$

dx

Integrate on both sides.

$$\int dx = \int \frac{a}{(p-1)(p^2+1)}\, dp + \log c$$

partial ~~fact~~ fractions,

$$\frac{1}{(p-1)(p^2+1)} = \frac{A}{(p-1)} + \frac{Bp+C}{p^2+1} \rightarrow (2)$$

$$\frac{1}{(p-1)(p^2+1)} = \frac{A(p^2+1)+(Bp+C)(p-1)}{(p-1)(p^2+1)}$$

$$1 = A(p^2+1) + B(p^2-1) + C(p-1) \rightarrow (3)$$

if $p = 1$

$$1 = 2A$$

$$\boxed{A = \frac{1}{2}}$$

Compare p^2 coefficients on both sides.

$$0 = A + B$$

$$\frac{1}{2} + B = 0$$

$$\boxed{B = \frac{-1}{2}}$$

Compare constants on both side.

$$1 = A - C.$$

$$C = A - 1$$

$$C = \frac{1}{2} - 1$$

$$\boxed{C = \frac{-1}{2}}$$

sub. in eq (3)

$$\int dx = \int a\left[\frac{1}{2(p-1)} - \frac{1(p+1)}{2(p^2+1)}\right] dp + \cancel{\log} c$$

$$x = a\int \frac{1}{2(p-1)}dp - \int \frac{1}{2}\,\frac{(p+1)}{p^2+1}dp + \cancel{\log} c.$$

$$x = \frac{a}{2}\left[\int \frac{1}{p-1}\,dp - \frac{1}{2}\int \frac{2(p+1)}{p^2+1}\right] + \cancel{\log} c.$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}p\right] + c$$

$$x = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p\right] + c \rightarrow (4)$$

substitue x in (1)

$$y = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p\right] + a\tan^{-1}p + c$$

$$y = \frac{a}{2}\log\frac{p-1}{\sqrt{p^2+1}} - \frac{a}{2}\tan^{-1}p + a\tan^{-1}p + c$$

$$y = \frac{a}{2}\log\frac{p-1}{\sqrt{p^2+1}} + \frac{2a\tan^{-1}p - a\tan^{-1}p}{2} + c$$

$$\boxed{y = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1}p\right] + c} \rightarrow (5)$$

Eq (4) & Eq (5) together gives the solution of equation.

* Equations solvable for 'x' :-
express give eq'n in the form
$x = F(y, p) \longrightarrow ①$

Differentiate ① w.r.t 'y'

$\frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right)$ $\left[\because \frac{dx}{dy} = \frac{1}{p}\right]$

$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \longrightarrow ②$

Find General solution of ②

$\phi(y, p, c) = 0 \longrightarrow ③$

Try to eliminate 'p' from ① and ③, then, we get ~~general~~ general solution.

Note:-

① if elimination of 'p' is not possible from ① & ③ then express them as

$$\left.\begin{array}{l} x = F_1(p, c) \\ y = F_2(p, c) \end{array}\right\} \longrightarrow ④$$

where 'p' parameter.

② If eq ④ is not possible then eq'ns ① & ③ together gives general solution

③ If eq ② is product of two factors, ignore that term ~~tho~~ which do not

Contains $\frac{dp}{dy}$ and continue with another factor.

Solve

① $p^2 - xp + y = 0$

② $xp^2 - yp - y = 0$

③ $y = 2px + y^2p^3$

④ $y^2 \log y = xyp + p^2$

① $p^2 - xp + y = 0$

$xp = p^2 + y$

$x = \frac{p^2+y}{p}$

$x = p + \underset{u}{y} \cdot \underset{v}{\frac{1}{p}} \longrightarrow ①$

Diff. ① wrt 'y'

$\frac{dx}{dy} = \frac{dp}{dy} + 1 \cdot \frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy}$

$\cancel{\frac{1}{p}} = \cancel{\frac{1}{p}} + \left(1 - \frac{y}{p^2}\right) \cdot \frac{dp}{dy} \quad \left(\because \begin{array}{l} p = \frac{dy}{dx} \\ \frac{dx}{dy} = \frac{1}{p} \end{array}\right)$

$\left(1 - \frac{y}{p^2}\right)\frac{dp}{dy} = 0.$

Ignore 1st term & continue with term containing $\frac{dp}{dy}$

$$\frac{dp}{dy} = 0$$

$$dp = 0$$

Integrate on both sides

$$\int dp = 0 + c$$

$$\boxed{p = c}$$

substitute it in (1)

~~$c^2 - cp + y$~~ $\boxed{c^2 - cx + y = 0}$

(2) $xp^2 - yp - y = 0 \longrightarrow (1)$

$$xp^2 = yp + y$$

$$x = \frac{yp + y}{p^2}$$

$$x = \frac{y}{p} + \frac{y}{p^2} \longrightarrow (2)$$

Diff. (2) wrt 'y'

$$\frac{dx}{dy} = 1 \cdot \frac{1}{p} + y\left(\frac{-1}{p^2}\right)\frac{dp}{dy} + 1 \cdot \frac{1}{p^2} + y\left(\frac{-2}{p^3}\right)\frac{dp}{dy}.$$

$$\cancel{\frac{1}{p}} = \cancel{\frac{1}{p}} + \frac{1}{p^2} + \left(\frac{-y}{p^2} - \frac{2y}{p^3}\right)\frac{dp}{dy}$$

$$\frac{-y}{p^2}\left[1 + \frac{2}{p}\right]\frac{dp}{dy} + \frac{1}{p^2} = 0$$

$$\frac{1}{p^2}\left[-y\frac{dp}{dy}\left(1 + \frac{2}{p}\right) + 1\right] = 0$$

$$-y \cdot \frac{dp}{dy}\left(1+\frac{2}{p}\right)+1=0$$

multiply with $\frac{dy}{dp}$ on both sides.

$$-y\left(1+\frac{2}{p}\right)+\frac{dy}{dp}=0$$

$$\frac{dy}{dp}=y\left(1+\frac{2}{p}\right)$$

$$\frac{dy}{y}=\left(1+\frac{2}{p}\right)dp$$

Integrate on both sides

$$\int\frac{dy}{y}=\int\left(1+\frac{2}{p}\right)dp+\log c$$

$$\log y=p+2\log p+\log c$$

$$\log y=p\log_e e+\log p^2+\log c$$

$$\log y=\log e^p+\log p^2+\log c$$

$$\log y=\log(e^p p^2 c)$$

$$\boxed{y=e^p p^2 c}\longrightarrow (3)$$

Substitute 'y' in (1)

$$x=\frac{e^p p^2 c}{p}+\frac{e^p p^2 c}{p^2}$$

$$x=e^p pc+e^p c$$

$$\boxed{x=e^p c\,[p+1]}\longrightarrow (4)$$

Eq 3 & 4 together gives general sol'n.

(4) $y^2 \log y = xyp + p^2 \longrightarrow (1)$

$$xyp = y^2 \log y - p^2$$

$$x = \frac{y^2 \log y}{yp} - \frac{p^2}{yp}$$

$$x = \frac{y \log y}{p} - \frac{p}{y}$$

$$x = y \log y \cdot \frac{1}{p} - p \cdot \frac{1}{y} \longrightarrow (2)$$

diff ② wrt 'y'

$$\frac{dx}{dy} = \frac{d}{dy}(y\log y)\frac{1}{p} + (y\log y)\left(\frac{-1}{p^2}\right)\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} - p\left(\frac{-1}{y^2}\right)$$

$$\frac{1}{p} = \frac{1}{p}\left[\log y + y\frac{1}{y}\right] + \frac{dp}{dy}\left[\frac{-y\log y}{p^2} - \frac{1}{y}\right] + \frac{p}{y^2}$$

$$\cancel{\frac{1}{p}} = \frac{1}{p}\log y + \cancel{\frac{1}{p}} - \frac{dp}{dy}\left[\frac{y\log y}{p^2} + \frac{1}{y}\right] + \frac{p}{y^2}$$

$$\frac{1}{p}\log y - \frac{dp}{dy}\left[\frac{y\log y}{p^2} + \frac{1}{y}\right] + \frac{p}{y^2} = 0$$

$$\left(\frac{p}{y^2} + \frac{1}{p}\log y\right) - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) = 0$$

$$\left(\frac{p}{y^2} + \frac{1}{p}\log y\right) - \frac{y}{p}\frac{dp}{dy}\left(\frac{\log y}{p} + \frac{1}{y^2}\right) = 0$$

$$\left(\frac{p}{y^2} + \frac{1}{p}\log y\right)\left(1 - \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Ignore 1st, continue with 2nd term having $\frac{dp}{dy}$.

$$1 - \frac{y}{p}\frac{dp}{dy} = 0$$

$$1 = \frac{y}{p}\frac{dp}{dy}$$

$$\frac{dp}{p} = \frac{dy}{y}$$

Integrate on both sides

$$\log p = \log y + \log c$$

$$\log p = \log(yc)$$

$\boxed{p = cy}$

substitute it in (1)

$$y^2 \log y = xy(cy) + (cy)^2$$

$$y^2 \log y = y^2[xc + c^2]$$

$$\log y = xc + c^2.$$

Clairout's Equation:- An eqn of the form $y = px + f(p)$ —(1) is called Clairout's eqn. diff w.r.t x.

$$dy/dx = p(1) + x \cdot dp/dx + f'(p)\, dp/dx$$

$$p = p + dp/dx\,[x + f'(p)]$$

$$dp/dx\,[x + f'(p)] = 0$$

$$dp/dx = 0$$

$$dp = 0$$

Integrate

$$\boxed{p = c}$$

Eliminate p from eqn (1)

$\boxed{y = cx + f(c)}$ is clairout eqn.

The solution of clairaut equation is obtained by on replacing P by C.

Solve

1) $(y-Px)(P-1)=P$

$$(y-Px)=\frac{P}{(P-1)}$$

$$y=Px+\frac{P}{(P-1)} \quad -(1)$$

It is clairaut eqn

$\therefore$ general soln

replace P by C

$$y=cx+\frac{c}{c-1}$$ //.

2) $P=\log(Px-y)$

$$e^P=(Px-y)$$

$$y=Px-e^P \quad -(2)$$

It is clairauts eqn.

replace P by C

general soln

$$y=cx-e^c$$ //

5) $P = \tan(Px - y)$

$\tan^{-1}p = Px - y$

$y = Px - \tan^{-1}p$

It is clairouts eqn.

replace P by 'C'

$y = cx - \tan^{-1}c$ //.

4) $y = Px + P^2$

It is clairouts eqn

replace P by C

General solⁿ

$y = cx + c^2$ //.

6) $\sin px \cos y = \cos px \sin y + P$

$\sin px \cos y - \cos px \sin y = P$

$\boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$

$\sin(Px - y) = P$

$Px - y = \sin^{-1}p$

$y = Px - \sin^{-1} P$

It is clairauts eqn

replace P by C

$y = Cx - \sin^{-1} C$ //

2) $y^2 + x^2 (dy/dx)^2 - 2xy\, dy/dx = 4 (dx/dy)^2$

$y^2 + x^2 (P)^2 - 2xy P = 4\, 1/P^2$

$y^2 - 2xyP + P^2x^2 = 4/P^2$

$(Px - y)^2 = 4/P^2$

$Px - y = 2/P$

$y = Px - 2/P$

It is clairauts' eqn

replace P by C

$y = Cx - 2/C$ //.

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