Ex Sheet 4

1i) $Y_{k+1} = R(z) Y_k$ where $R(z) = 1 + z b^T (I - zA)^{-1} \mathbb{1}$

$$I - zA = -zA\left(\frac{-1}{z}A^{-1} + I\right) = -zA\left(I - \frac{1}{z}A^{-1}\right)$$

hence $(I - zA)^{-1} = \left[-zA\left(I - \frac{1}{z}A^{-1}\right)\right]^{-1} = -\frac{1}{z}\left(I - \frac{1}{z}A^{-1}\right)^{-1} A^{-1}$

we now expand $\left(I - \frac{1}{z}A^{-1}\right)^{-1}$.

for large z, our $\frac{1}{z}A^{-1}$ is small. Using Neumann series: $(I - X)^{-1} = I + X + X^2 + X^3 + \dots$ for $\|X\| < 1$

$$\Rightarrow \left(I - \frac{1}{z}A^{-1}\right)^{-1} = I + \frac{1}{z}A^{-1} + \left(\frac{1}{z}A^{-1}\right)^2 + \dots$$

$$\Rightarrow (I - zA)^{-1} = -\frac{1}{z}\left[I + \frac{1}{z}A^{-1} + \left(\frac{1}{z}A^{-1}\right)^2 + O\left(\frac{1}{z^3}\right)\right] A^{-1}$$

$$= -\frac{1}{z}A^{-1} - \frac{1}{z^2}(A^{-1})^2 - \frac{1}{z^3}(A^{-1})^3 + O\left(\frac{1}{z^4}\right)$$

$$\Rightarrow R(z) = 1 + z b^T \left(-\frac{1}{z}A^{-1} - \frac{1}{z^2}(A^{-1})^2 - \frac{1}{z^3}(A^{-1})^3 + O\left(\frac{1}{z^4}\right)\right) \mathbb{1}$$

$$= 1 + b^T\left(-A^{-1} - \frac{1}{z}(A^{-1})^2 - \frac{1}{z^2}(A^{-1})^3 + O\left(\frac{1}{z^3}\right)\right) \mathbb{1}$$

$$\Rightarrow R(\infty) = 1 + b^T(-A^{-1})\mathbb{1} = 1 - b^T A^{-1} \mathbb{1}$$

if $a_{si} = b_i \ \forall i \ (i = 1, \dots, s)$ | i.e. if each entry in the last row of A equals the corresponding weight in b

then $b^T = (b_1, b_2, \dots, b_s) = (a_{s1}, a_{s2}, \dots, a_{ss}) =$ row s of $A = (0, 0, \dots, 1) A$

let $e_s = (0,0,\ldots,1) \Rightarrow b^T = e_s A$

$\Rightarrow R(\infty) = 1 - e_s A A^{-1} \underline{1} = 1 - e_s \underline{1} = 1 - 1 = 0$

ii) if $a_{i1} = b_1 \ \forall i \ (i = 1, \ldots, s)$, $b_1 \neq 0$

ie. first column of A has all entries equal to b_1.

$$A(1,0,0,\ldots,0)^T = (b_1, b_1, \ldots, b_1)^T = b_1 \underline{1}$$

let $e_1 = (1,0,0,\ldots,0)^T \Rightarrow Ae_1 = b_1 \underline{1} \Rightarrow e_1 = b_1 A^{-1} \underline{1}$

$\Rightarrow A^{-1} \underline{1} = \frac{1}{b_1} e_1$

$\Rightarrow R(\infty) = 1 - b^T \left(\frac{1}{b_1} e_1\right) = 1 - \frac{1}{b_1} b^T e_1$ (since $b^T e_1 = b_1$)

$= 1 - \frac{b_1}{b_1} = 1 - 1 = 0$

2) $\Phi(t, Y, h) = b_1 f(t, Y) + b_2 f(t + c_2 h, Y + h a_{21} f(t, Y))$ $\qquad y'(t) = f(t, y(t))$

By taylor exp. we have:

$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$ $\qquad (y''(t) = f_t + f_y \cdot f)$

$= y(t) + h f(t, y(t)) + \frac{h^2}{2}(f_t + f_y \cdot f) + O(h^3)$

$\Rightarrow \tau(t,h) = \frac{y(t+h) - y(t)}{h} - \Phi(t, y(t), h)$

$= f + \frac{h}{2}(f_t + f_y \cdot f) + O(h^2) - \Phi(t, y(t), h)$

we apply Taylor exp. to second term in Φ:

$$f(t + c_2 h, Y + h a_{21} f(t,Y)) = f + c_2 h f_t + a_{21} h f_y \cdot f + O(h^2)$$

$\Rightarrow \Phi(t, y(t), h) = b_1 f + b_2 (f + c_2 h f_t + a_{21} h f_y \cdot f) + O(h^2)$

$= (b_1 + b_2) f + h(b_2 c_2 f_t + b_2 a_{21} f_y \cdot f) + O(h^2)$

$\Rightarrow \tau(t,h) = \left[f + \frac{h}{2}(f_t + f_y \cdot f) + O(h^2)\right] - \left[(b_1 + b_2) f + h(b_2 c_2 f_t + b_2 a_{21} f_y f) + O(h^2)\right]$

$= [1 - (b_1 + b_2)] f + h\left[\frac{f_t}{2} + \frac{f_y \cdot f}{2} - b_2 c_2 f_t - b_2 a_{21} f_y \cdot f\right] + O(h^2)$

$$= (1-(b_1+b_2))f + h\left(\left(\tfrac{1}{2}-b_2c_2\right)f_t + \left(\tfrac{1}{2}-b_2a_{21}\right)f.f_y\right) + O(h^2)$$

for consistency order 2 we require $\tau(t,h) = O(h^2)$

$\Rightarrow b_1+b_2=1$ and $\frac{1}{2}-b_2c_2 = \frac{1}{2}-b_2a_{21} = 0$ $(c_2=a_{21})$

ie. $\sum_{i=1}^{2} b_i = 1$ and $\sum_{i=1}^{2} b_ic_i = \frac{1}{2}$ since $c_1=0$.

3) For order 3 we require:

$\sum_{i=1}^{5} b_i = 1$, $\sum_{i=1}^{5} b_ic_i = \frac{1}{2}$, $\sum_{i=1}^{5} b_ic_i^2 = \frac{1}{3}$, $\sum_{i=1}^{5} b_i a_{ij} c_j = \frac{1}{6}$

we know $c_1=0$, $c_2=\frac{1}{2}=c_3$, $c_4=1$, $c_5=a_{54}$ $(a_{51}=a_{52}=a_{53}=0)$

Our requirements give:

$b_1+b_2+b_3+b_4+b_5=1$ $-①$; $\frac{1}{2}b_2+\frac{1}{2}b_3+b_4+c_5b_5=\frac{1}{2}$ $-②$

$\frac{1}{4}b_2+\frac{1}{4}b_3+b_4+c_5^2b_5=\frac{1}{3}$ $-③$; $b_2a_{21}c_1 + b_3a_{32}c_2 + b_4a_{43}c_3 + b_5a_{54}c_4 = \frac{1}{6}$

$\Rightarrow \frac{1}{4}b_3 + \frac{1}{2}b_4 + a_{54}b_5 = \frac{1}{6}$ $-④$

Lets choose $a_{54}=\frac{1}{2}\ (=c_5)$

$\Rightarrow$ ② becomes $\frac{1}{2}b_2+\frac{1}{2}b_3+b_4+\frac{1}{2}b_5=\frac{1}{2}$

③ becomes $\frac{1}{4}b_2+\frac{1}{4}b_3+b_4+\frac{1}{4}b_5=\frac{1}{3}$

④ becomes $\frac{1}{4}b_3+\frac{1}{2}b_4+\frac{1}{2}b_5=\frac{1}{6}$

(② × 2): $b_2+b_3+2b_4+b_5=1$ $-⑤$

③ × 4 : $b_2+b_3+4b_4+b_5=\frac{4}{3}$ $-⑥$

⑥ − ⑤ : $2b_4=\frac{1}{3}$ $\Rightarrow b_4=\frac{1}{6}$

subbing $b_4=\frac{1}{6}$ into ⑤ gives: $b_2+b_3+b_5=\frac{2}{3}$ $-⑦$

subbing ⑦ into ① gives: $b_1+\frac{2}{3}+\frac{1}{6}=1$ $\Rightarrow b_1=\frac{1}{6}$

④ becomes $\frac{1}{4}b_3+\frac{1}{12}+\frac{1}{2}b_5=\frac{1}{6}$ $\Rightarrow b_3+2b_5=\frac{1}{3}$ $\Rightarrow b_3=\frac{1}{3}-2b_5$

subbing this equation of b_3 into ⑦ gives: $b_2+\frac{1}{3}-b_5=\frac{2}{3}$ $\Rightarrow b_2=\frac{1}{3}+b_5$

so we have: $b_1=\frac{1}{6}$, $b_2=\frac{1}{3}+b_5$, $b_3=\frac{1}{3}-2b_5$, $b_4=\frac{1}{6}$, $b_5=b_5$

choose $b_5 = \frac{1}{12} \Rightarrow b_1 = \frac{1}{6}, b_2 = \frac{5}{12}, b_3 = \frac{1}{6}, b_4 = \frac{1}{6}, b_5 = \frac{1}{12}$

We see with our system that.

$$\sum_{i=1}^{5} b_i = \frac{1}{6} + \frac{5}{12} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{12}{12} = 1$$

$$\sum_{i=1}^{5} b_i c_i = 0 + \frac{1}{12} + \frac{5}{24} + \frac{1}{12} + \frac{1}{6} + \frac{1}{24} = \frac{12}{24} = \frac{1}{2}$$

$$\sum_{i=1}^{5} b_i c_i^2 = 0 + \frac{5}{48} + \frac{1}{24} + \frac{1}{6} + \frac{1}{48} = \frac{16}{48} = \frac{1}{3}$$

$$\sum_{i=1}^{5} b_i a_{ij} c_j = 0 + 0 + b_3 a_{32} c_2 + b_4 a_{43} c_3 + b_5 a_{54} c_4$$
$$= \frac{1}{24} + \frac{1}{12} + \frac{1}{24} = \frac{4}{24} = \frac{1}{6} \quad \boxtimes$$

4. Observation: The code demonstrates the typical stiff behavior of the van der Pol oscillator and shows how numerical errors decrease with smaller step sizes, consistent with the 4-th order accuracy of the RK4 method.