Dr. Robert Gruhlke, André-Alexander Zepernick

Exercise Sheet 3

Deadline: Friday, November 7th, 2025 at 10 am

Exercise 1 (Consistency condition; 4 points)
Let $f \in C^1([a,b] \times \mathbb{R}^n)$. Show that the explicit Runge-Kutta scheme is consistent if the condition $\sum_{i=1}^{s} b_i = 1$ holds.

Exercise 2 (Fourth-order Runge-Kutta method; 2 + 3 = 5 points)
The so-called *classical fourth-order Runge-Kutta method* is given by

$$Y_{k+1} = y_n + \frac{1}{6}h(K_1 + 2K_2 + 2K_3 + K_4),$$

where

$$\begin{aligned} K_1 &= f(t_k, Y_k), \\ K_2 &= f\left(t_k + \frac{1}{2}h, Y_k + \frac{1}{2}hK_1\right), \\ K_3 &= f\left(t_n + \frac{1}{2}h, Y_k + \frac{1}{2}hK_2\right), \\ K_4 &= f(t_k + h, Y_k + hK_3). \end{aligned}$$

a) Write down the Butcher tableau for this method.

b) When the classical fourth-order Runge-Kutta method is applied to the differential equation $y' = \lambda y$, where λ is a real constant, show that

$$Y_{k+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right) Y_k.$$

Compare this with the Taylor series expansion of $y(t_{k+1}) = y(t_k + h)$ around the point $t = t_k$ (cf. Thm. 1.46).

Programming Exercise (Implementation of an explicit Runge-Kutta scheme; 7 points)
Implement

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runge_kutta_ex(f, y0, I, h, weights, A)
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to solve the autonomous IVP

$$y'(t) = f(y(t)), \quad t \in (a,b], \qquad y(a) = y_0, \qquad f : \mathbb{R}^d \to \mathbb{R}^d,$$

using an explicit Runge–Kutta (RK) method with Butcher tableau (A, b) and stepsize $h \in \{0.5, 0.25, 0.125, 0.0625\}$. Test your code using the derived Butcher-Tableau in

Exercise 2 and step size h. Use the scalar logistic equation

$$y'(t) = g\,y(t)\big(1 - y(t)/K\big), \qquad y(0) = y_0, \qquad y(t) = \frac{K}{1 + \big(\frac{K-y_0}{y_0}\big)e^{-gt}},$$

with $g = 2$, $K = 1$, $y_0 = 0.1$, $I = [0, 5]$. For each h, compute the maximal error

$$\max_k |y_{\text{num}}(t_k) - y_{\text{exact}}(t_k)|.$$

Rules for submission & passing criterion:
Exercises must be completed in **groups of three** and **submitted electronically via the Whiteboard system** under "**Assignments**." Late submissions will not be corrected. Please **state the names of all group members** on the submission. The programming language of this course is **Python**. The exercises will be discussed in the tutorial the following week. To pass the tutorial component of the module, you must achieve **at least 50% of the homework points**.