Homework 1 For Numeric II

1) $ty'(t)+3y(t)=t^2$, $t_0=0$, $y_0=0$.

Homogenous equation is $ty'(t)+3y(t)=0$.

$ty'(t)+3y(t)=0 \Rightarrow t\frac{dy}{dt}+3y=0$. Then we have,

$$t\frac{dy}{dt}+3y=0 \Rightarrow t\frac{dy}{dt}=-3y \Rightarrow \frac{dy}{-3y}=\frac{dt}{t}.$$

Integrate both sides, then

$$\int\frac{dy}{-3y}=\int\frac{dt}{t} \Rightarrow \frac{-1}{3}\ln|y|=\ln|t|+c_1 \Rightarrow \ln|y|=-3\ln|t|+C.$$

$$\Rightarrow y=Ct^{-3},\quad C\in\mathbb{R}.$$

As a result, $y_h(t)=Ct^{-3}$.

By the initial condition, since $t=0$, t^{-3} is undefined.

So, C would be equal to 0.

Hence $y_h(t)=0$.

We need to make a guess for particular solution.

Assume $y_p(t)=at^2$, then taking the derivative, we get

$y_p(t)=at^2 \Rightarrow y_p'(t)=2at$

Substituting into the given equation:

$$t(2at)+3(at^2)=t^2 \Rightarrow 5at^2=t^2 \Rightarrow a=\frac{1}{5}.$$

It means $y_p(t)=\frac{1}{5}t^2$.

Since general solution is the sum of homogenous and particular solution, we obtain

$$y(t)=y_h(t)+y_p(t)=0+\frac{1}{5}t^2=\frac{1}{5}t^2.$$

2) $y'(t) = \sqrt{|y(t)|}$, $y(0) = 0$

Let us try to guess.

Suppose $y(t) = 0$. Then substitute into the equation.

It implies the following:

$y(t) = 0 \Rightarrow y'(t) = 0 \Rightarrow y'(t) = \sqrt{|y(t)|} \Rightarrow 0 = 0.$

It holds.

So, $y(t) = 0$ is a solution.

Let $y(t) = \frac{t^2}{4}$. If we take the derivative, we have,

$y(t) = \frac{t^2}{4} \Rightarrow y'(t) = \frac{t}{2}.$

When we apply this into the given equation

It shows $y'(t) = \sqrt{|y(t)|} \Rightarrow \frac{t}{2} = \sqrt{\frac{t^2}{4}} \Rightarrow \frac{t}{2} = \frac{t}{2}.$

It also holds.

Hence, we have two solutions.

Thus, $y'(t) = \sqrt{|y(t)|}$ is not unique.

3) $y'(t) = A(t)y(t)$, $y(0) = y_0$, $A(t) \equiv A := \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Let $y'(t) = A(t)y(t)$.

We know $f(t,y) = y'(t)$. We can consider $f(t,y) = A(t)y(t)$

Assume y_1, y_2 as vectors in $\mathbb{R}^2$. Then

$$f(t,y_1) - f(t,y_2) = Ay_1 - Ay_2 = A(y_1 - y_2) = Az$$

where, $z = y_1 - y_2$.

So, $\langle y_1 - y_2, f(t,y_1) - f(t,y_2) \rangle = \langle z, Az \rangle$.

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, then

$$\langle z, Az \rangle = z^T A z = (z_1 \ z_2) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= (\alpha z_1 - \beta z_2 \quad \beta z_1 + \alpha z_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= (\alpha z_1 - \beta z_2) z_1 + (\beta z_1 + \alpha z_2) z_2$$

$$= \alpha z_1^2 - \beta z_1 z_2 + \beta z_1 z_2 + \alpha z_2^2$$

$$= \alpha (z_1^2 + z_2^2) = \alpha \|z\|^2$$

Thus, $\langle y_1 - y_2, f(t,y_1) - f(t,y_2) \rangle = \alpha \|y_1 - y_2\|^2$.

Hence, monotonocity holds with $\alpha = \phi$.

f is Lipschitz continous if $\exists L > 0$ s.t $\|f(t,y_1) - f(t,y_2)\| \leq L\|y_1 - y_2\|$ $\forall y_1, y_2$ i.e. $\|Az\| \leq L\|z\|$, $\forall z$.

$$\|Az\| = \sqrt{(\alpha z_1 + \beta z_2)^2 + (-\beta z_1 + \alpha z_2)^2}$$

$$= \sqrt{\alpha^2 z_1^2 + 2\alpha\beta z_1 z_2 + \beta^2 z_2^2 + \beta^2 z_1^2 - 2\beta\alpha z_1 z_2 + \alpha^2 z_2^2}$$

$$= \sqrt{(\alpha^2 + \beta^2)\|z\|^2} = \sqrt{(\alpha^2 + \beta^2)}\,\|z\|.$$

Thus, $\|Az\| \leq L\|z\|$ holds where $L = \sqrt{\alpha^2 + \beta^2}$.

Therefore, $f(t,y) = A(t)y(t)$ is Lipschitz continous with $L = \sqrt{\alpha^2 + \beta^2}$.

4) Programming Exercise

For a) and b)

Solution with $h=0.2$:

t: 0.00, y: 1.000000

t: 0.20, y: 1.000000

t: 0.40, y: 0.972222

t: 0.60, y: 0.935374

t: 0.80, y: 0.896577

t: 1.00, y: 0.858686

Error at $t=1$

$h=0.2$: Error = 0.012112

Solution with $h=0.1$:

t: 0.00, y: 1.000000

t: 0.10, y: 1.000000

t: 0.20, y: 0.991736

t: 0.30, y: 0.978535

t: 0.40, y: 0.962435

t: 0.50, y: 0.944710

t: 0.60, y: 0.926174

t: 0.70, y: 0.907350

t: 0.80, y: 0.888579

t: 0.90, y: 0.870078

t: 1.00, y: 0.851985

Error at $t=1$:

$h=0.1$: Error = 0.005411

For c)

If the step size increases, then the error increases.