Homework 2 For Numeric II

1) $t_K = a + Kh$, $K = \overline{0,N}$, $h = \frac{b-a}{N}$

$y'(t) = f(t, y(t))$, $y(a) = y_0$

$$Y_{K+1} = Y_K + h(1-\delta) f(t_K, Y_K) + h\delta f\left(t_K + \frac{h}{2\delta}, Y_K + \frac{h}{2\delta} f(t_K, Y_K)\right), \delta \in \mathbb{R}_+$$

a) Let us define the truncation error

$$\tau_K(h,\delta) = \frac{1}{h}\Big[y(t_{K+1}) - y(t_K) - h\delta f(t_K, y(t_K)) - h\delta f\left(t_K + \frac{h}{2\delta}, y(t_K) + \frac{h}{2\delta} f(t_K, y(t_K))\right)\Big]$$

We know that

$$Y_{K+1} = Y_K + h\delta f(t_K, Y_K) + h\delta f\left(t_K + \frac{h}{2\delta}, Y_K + \frac{h}{2\delta} f(t_K, Y_K)\right)$$

Applying the Taylor expansion to $Y_{K+1} = y(t_{K+1}) = y(t_K + h)$, we have

$$y(t_{K+1}) = y(t_K) + hy'(t_K) + \frac{h^2}{2} y''(t_K) + \frac{h^3}{6} y'''(t_K) + O(h^4).$$

We also apply Taylor expansion for function f.

$$f\left(t_K + \frac{h}{2\delta}, y(t_K) + \frac{h}{2\delta} f(t_K, y(t_K))\right) = f(t_K, y(t_K)) + \frac{h}{2\delta}\frac{\partial f}{\partial t}(t_K, y(t_K)) + \frac{h}{2\delta}\frac{\partial f}{\partial y}(t_K, y(t_K)) f(t_K, y(t_K)) + \frac{h^2}{8\delta^2} y'''(t_K) + O(h^3)$$

By expansion of numerical solution, we obtain

$$Y_{K+1} - Y_K = h\delta f(t_K, y(t_K)) + h\delta\left[f(t_K, y(t_K)) + \frac{h}{2\delta} y'' + \frac{h^2}{8\delta^2} y'''\right]$$

$$= 2h\delta f(t_K, y(t_K)) + \frac{h^2}{2} y''(t_K) + \frac{h^3}{8\delta} y'''(t_K) + O(h^4)$$

Substract the numerical solution from the real solution, then

$$y(t_{k+1}) - y(t_k) - (y_{k+1} - y_k) = h(1-2\delta) f(t_k, y(t_k)) + h^3\left(\frac{1}{6} - \frac{1}{8\delta}\right) y'''(t_k) + O(h^4)$$

For consistency, $1-2\delta = 0 \Rightarrow \delta = \frac{1}{2}$.

The result which we obtain would be divided by h in order to find the truncation error. Then, it implies:

$$\tau_k(h,\delta) = \frac{h^3\left(\frac{1}{6} - \frac{1}{8\delta}\right) y'''(t_k) + O(h^4)}{h}$$

$$\tau_k(h,\delta) = h^2\left(\frac{1}{6} - \frac{1}{8\delta}\right) y'''(t_k) + O(h^3)$$, but when $\frac{\partial f}{\partial y} \neq 0$. Then

$$\Rightarrow \tau_k(h,\delta) = \frac{h^2}{8\delta}\left[\left(\frac{4}{3}\delta - 1\right) y'''(t_k) + y''(t_k)\frac{\partial f}{\partial y}(t_k, y(t_k))\right] + O(h^3).$$

□

b) For $\alpha = 1$. IVP is $y'(t) = f(t, y(t)) = -y$, $\frac{\partial f}{\partial y} = -1$.

$$(y'(t))' = (f(t, y(t)))'$$

$$y'' = f_t + f_y f \underset{\uparrow\, f_t = 0}{=} f_y f = (-1)(-y) = y.$$

$$y''' = (y'')' = (y)' = y' = -y. \quad \text{(By chain rule)}$$

Substituting into the equation

$$\tau_k(h,\delta) = \frac{h^2}{8\delta}\left[\left(\frac{4}{3}\delta - 1\right)(-y) + y(-1)\right] + O(h^3)$$

$$= \frac{h^2}{8\delta}\left[-\left(\frac{4}{3}\delta y\right)\right] + O(h^3) = -\frac{h^2}{6} y + O(h^3)$$

$$\frac{\tau_k(h,\delta)}{h} = -\frac{h}{6} y + O(h^2)$$

For $\alpha \geq 2$. $y' = -y^\alpha$, $\frac{\partial f}{\partial y} = -\alpha y^{\alpha-1}$.

$$y' = -y^\alpha$$

$$y'' = f_y f = (-\alpha y^{\alpha-1})(-y^\alpha) = \alpha y^{2\alpha-1}$$

$$y''' = (y'')' = (\alpha y^{2\alpha-1})' = \alpha(2\alpha-1)y^{2\alpha-2}\, y' = \alpha(2\alpha-1)y^{2\alpha-2}(-y^\alpha)$$

$$= -\alpha(2\alpha-1)y^{3\alpha-2}$$

$$y'' f_y = (\alpha y^{2\alpha-1})(-\alpha y^{\alpha-1}) = -\alpha^2 y^{3\alpha-2}$$

In order to vanish the part of h^2, we apply

$$\left(\frac{4}{3}\delta - 1\right) y''' + y'' f_y = 0$$

$$\Rightarrow \left(\frac{4}{3}\delta - 1\right)\left(-\alpha(2\alpha-1)y^{3\alpha-2}\right) + \left(-\alpha^2 y^{3\alpha-2}\right) = 0$$

$$-\left(\frac{4}{3}\delta - 1\right)\alpha(2\alpha-1) - \alpha^2 = 0$$

$$-\left(\frac{4}{3}\delta - 1\right)(2\alpha-1) - \alpha = 0 \Rightarrow -\left(\frac{4}{3}\delta - 1\right)(2\alpha-1) = \alpha$$

$$\Rightarrow \frac{4}{3}\delta - 1 = -\frac{\alpha}{2\alpha-1} \Rightarrow \frac{4}{3}\delta = 1 - \frac{\alpha}{2\alpha-1}$$

$$\Rightarrow \frac{4}{3}\delta = \frac{\alpha-1}{2\alpha-1}$$

$$\Rightarrow \delta = \frac{3(\alpha-1)}{4(2\alpha-1)}$$, since $\alpha \geq 2$, $\delta > 0$.

So, we find $\delta_0 \in \mathbb{R}^+$.

Thus, the truncation error would be the term of $O(h^3)$.

2) $y'(t) = f(t, y(t)), \quad a \leqslant t \leqslant b$

$y(a) = y_0$

$\|\Phi(t, y_1, h) - \Phi(t, y_2, h)\| \leqslant L\|y_1 - y_2\|, \quad y_1, y_2 \in \mathbb{R}^n$

$\langle y_1 - y_2, \Phi(t, y_1, h) - \Phi(t, y_2, h)\rangle \leqslant \theta \|y_1 - y_2\|^2$ for $\theta \in \mathbb{R}$.

Proof:

Let us define the global error $e_K = y(t_K) - Y_K$. Then, by the one-step method, we have

$$e_{K+1} = e_K + h_K [\Phi(t_K, y(t_K), h_K) - \Phi(t_K, Y_K, h_K)] + h_K \tau(t_K, h_K).$$

Take the inner product of e_{K+1}, then

$$\|e_{K+1}\|^2 = \|e_K\|^2 + 2h_K \langle e_K, \Phi(t_K, y(t_K), h_K - \Phi(t_K, Y_K, h_K)\rangle$$
$$+ h_K^2 \|\Phi(t_K, y(t_K), h_K) - \Phi(t_K, Y_K, h_K)\|^2$$
+ error terms from τ

By the given inequalities, it means

$$\langle e_K, \Phi(t_K, y(t_K), h_K) - \Phi(t_K, Y_K, h_K)\rangle \leqslant \theta \|e_K\|^2$$
$$\|\Phi(t_K, y(t_K), h_K) - \Phi(t_K, Y_K, h_K)\| \leqslant L\|e_K\|.$$

So,

$$\|e_{K+1}\|^2 \leqslant (1 + 2h_K\theta + h_K^2 L^2)\|e_K\|^2 + 2h_K \langle e_K, \tau(t_K, h_K)\rangle + h_K^2 \|\tau(t_K, h_K)\|^2$$
$$\leqslant (1 + 2h_K\theta + h_K^2 L^2)\|e_K\|^2 + h_K^2 (1 + L^2)\|\tau(t_K, h_K)\|^2.$$

Cauchy-Schwarz

If we take the square root, then

$$\|e_{K+1}\| \leqslant \sqrt{1 + 2h_K\theta + h_K^2 L^2}\,\|e_K\| + h_K \|\tau(t_K, h_K)\|.$$

Denote $\sqrt{1+2h_K\Theta+h_K^2L^2} \leq \underbrace{e^{\left(\frac{1}{2}(2h_K\Theta+h_K^2L^2)\right)}}_{=e^{\left(h_K\Theta+\frac{1}{2}h_K^2L^2\right)}}$

Hence, $\|e_{K+1}\| \leq e^{\left(h_K\Theta+\frac{1}{2}h_K^2L^2\right)}\|e_K\| + h_K\|\tau(t_K,h_K)\|$.

Now, let us apply the Gronwall, then

$$\|e_K\| \leq \sum_{J=0}^{K-1}\left(h_J\|\tau(t_J,h_J)\|\prod_{m=J+1}^{K-1} e^{\left(h_m\Theta+\frac{1}{2}h_m^2L^2\right)}\right)$$

$$= \sum_{J=0}^{K-1} h_J\|\tau(t_J,h_J)\| e^{\left(\sum_{m=J+1}^{K-1}\left(h_m\Theta+\frac{1}{2}h_m^2L^2\right)\right)}$$

In order to bound the sums, we use

$$\sum_{m=J+1}^{K-1} h_m \leq t_K-a \quad , \quad \sum_{m=J+1}^{K-1} h_m^2 \leq h\sum_{m=J+1}^{K-1} h_m \leq h(t_K-a)$$

Then,

$$\sum_{m=J+1}^{K-1}\left(h_m\Theta+\frac{1}{2}h_m^2L^2\right) \leq \Theta(t_K-a)+\frac{1}{2}L^2h(t_K-a).$$

Thus, $\|e_K\| \leq e^{\left(\Theta(t_K-a)+\frac{1}{2}L^2h(t_K-a)\right)}\sum_{J=0}^{K-1} h_J\|\tau(t_J,h_J)\|$.

Therefore,

$$\|y(t_K)-Y_K\| \leq e^{\left(\Theta(t_K-a)+\frac{1}{2}L^2h(t_K-a)\right)}\sum_{J=0}^{K-1} h_J\|\tau(t_J,h_J)\|$$

where, $K=1,\ldots,N$ and for $h=\max_K h_K$.

$\square$