Homework 3 For Numeric II

1) Proof:

Since f is continous and by the definition of K_i, we know

$$\lim_{h\to 0} K_i(t,y,h) = f(t,y) \quad , \forall (t,y) \in [a,b] \times \mathbb{R}^n$$

Then, for $t = t_K$, $y = y(t_K)$ it means

$$\lim_{h\to 0} |f(t_K, y(t_K)) - \Phi(t_K, y(t_K), h)| = 0 \text{ since } f(t,y) = \Phi(t,y,h)$$

$$\Rightarrow \lim_{h\to 0} \left| f(t_K, y(t_K)) - \sum_{i=1}^{s} b_i K_i(t_K, y(t_K), h) \right|$$

$$\Rightarrow \lim_{h\to 0} \left| f(t_K, y(t_K)) - \sum_{i=1}^{s} b_i f(t_K, y(t_K)) \right|$$

$$\Rightarrow \left| f(t_K, y(t_K)) \left(1 - \sum_{i=1}^{s} b_i\right) \right| = 0$$

Thus, $\sum_{i=1}^{s} b_i = 1$. □

2) $y_{K+1} = y_K + \frac{1}{6} h (K_1 + 2K_2 + 2K_3 + K_4)$

where, $K_1 = f(t_K, y_K)$

$$K_2 = f\left(t_K + \frac{1}{2}h, y_K + \frac{1}{2}hK_1\right)$$

$$K_3 = f\left(t_K + \frac{1}{2}h, y_K + \frac{1}{2}hK_2\right)$$

$$K_4 = f(t_K + h, y_K + hK_3)$$

a) The Butcher tableau is

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

b) We know $y'(t) = f(t,y) = \lambda y$. Then

$$K_1 = f(t_K, Y_K) = \lambda Y_K$$

$$K_2 = f\left(t_K + \frac{1}{2}h, Y_K + \frac{1}{2}hK_1\right) = \lambda\left(Y_K + \frac{1}{2}h(\lambda Y_K)\right) = \lambda Y_K\left(1 + \frac{h\lambda}{2}\right)$$

$$K_3 = f\left(t_K + \frac{1}{2}h, Y_K + \frac{1}{2}hK_2\right) = \lambda\left(Y_K + \frac{1}{2}h\lambda Y_K\left(1 + \frac{h\lambda}{2}\right)\right)$$

$$= \lambda Y_K\left(1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{4}\right)$$

$$K_4 = f(t_K + h, Y_K + hK_3) = \lambda\left(Y_K + h\lambda Y_K\left(1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{4}\right)\right)$$

$$= \lambda Y_K\left(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{4}\right).$$

The Runge Kutta form is

$$Y_{K+1} = Y_K + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= Y_K + \frac{h}{6}\lambda Y_K\left[1 + 2\left(1 + \frac{h\lambda}{2}\right) + 2\left(1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{4}\right) + \left(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{4}\right)\right].$$

$$\Rightarrow \frac{Y_{K+1}}{Y_K} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4.$$

Thus, $Y_{K+1} = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right)Y_K$.

$$y(t_{K+1}) = y(t_K + h) = y(t_K) + hy'(t_K) + \frac{h}{2}y''(t_K) + \frac{h^3}{6}y'''(t_K) + \frac{h^4}{24}(t_K) + O(h^5)$$

Taylor expansion for $t = t_K$

The basic difference is that the classical Runge-Kutta method doesn't include $O(h^5)$ term, but Taylor expansion includes $O(h^5)$ terms.