Derivation 01: Variational Principle and Field Equations

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1 The Total Action

We begin with the action functional proposed in the manuscript:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_m + \mathcal{L}_{constr} + \mathcal{L}_{FI} \right] \tag{1}$$

where the Fisher Information term is defined as:

$$S_{FI} = A \int d^4x \sqrt{-g} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi \tag{2}$$

with $\chi \equiv \ln n$.

2 Variation with respect to the Metric

To find the contribution of the Fisher Information sector to the Einstein Field Equations, we vary the action S_{FI} with respect to the metric $g^{\mu\nu}$. We use the standard identity for the variation of the determinant:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{3}$$

The variation is:

$$\delta S_{FI} = A \int d^4x \left[(\delta \sqrt{-g}) g^{\alpha\beta} \nabla_\alpha \chi \nabla_\beta \chi + \sqrt{-g} (\delta g^{\alpha\beta}) \nabla_\alpha \chi \nabla_\beta \chi \right] \tag{4}$$

$$= A \int d^4x \sqrt{-g} \left[-\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} (\nabla \chi)^2 + \delta g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi \right] \tag{5}$$

$$= \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[A \left(\nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \chi \nabla^\alpha \chi \right) \right] \tag{6}$$

3 The Stress-Energy Tensor

By definition, $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$. Applying this to our result above (noting the sign convention):

$$T_{\mu\nu}^{FI} = -2 \frac{\delta \mathcal{L}_{FI}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_{FI} \tag{7}$$

However, using the direct variation term derived in (5):

$$T_{\mu\nu}^{FI} = 2A \left(\nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \chi \nabla^\alpha \chi \right) \tag{8}$$

This matches Eq (4) in the main manuscript. The trace of this tensor is:

$$T^{FI} = g^{\mu\nu} T_{\mu\nu}^{FI} = 2A((\nabla \chi)^2 - 2(\nabla \chi)^2) = -2A(\nabla \chi)^2 \tag{9}$$

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4 Einstein Field Equations

Combining this with the standard Einstein-Hilbert variation, the full field equations are:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu}^{(m)} + T_{\mu\nu}^{FI}\right) \tag{10}$$