Derivation 02: Covariant Conservation Proof

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1 Theorem Statement

We aim to prove Theorem 1 of the manuscript: On the equations of motion, the total stress tensor is conserved:

$$\nabla_\mu (T^{\mu\nu}_{(m)} + T^{\mu\nu}_{FI}) = 0 \tag{1}$$

2 Divergence of the Fisher Stress Tensor

We calculate the covariant divergence of $T^{FI}_{\mu\nu}$ directly. Recall:

$$T^{FI}_{\mu\nu} = 2A\left(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}(\nabla\chi)^2\right) \tag{2}$$

Taking the divergence ∇^μ:

$$\nabla^\mu T^{FI}_{\mu\nu} = 2A\left[\nabla^\mu(\nabla_\mu\chi\nabla_\nu\chi) - \frac{1}{2}\nabla_\nu(\nabla_\alpha\chi\nabla^\alpha\chi)\right] \tag{3}$$

Expanding the first term using the product rule:

$$\nabla^\mu(\nabla_\mu\chi\nabla_\nu\chi) = (\Box\chi)\nabla_\nu\chi + \nabla^\mu\chi\nabla_\mu\nabla_\nu\chi \tag{4}$$

Expanding the second term:

$$-\frac{1}{2}\nabla_\nu(\nabla_\alpha\chi\nabla^\alpha\chi) = -\frac{1}{2}(2\nabla^\alpha\chi\nabla_\nu\nabla_\alpha\chi) = -\nabla^\alpha\chi\nabla_\alpha\nabla_\nu\chi \tag{5}$$

Substituting (5) and (6) back into (4):

$$\nabla^\mu T^{FI}_{\mu\nu} = 2A\left[(\Box\chi)\nabla_\nu\chi + \nabla^\mu\chi\nabla_\mu\nabla_\nu\chi - \nabla^\alpha\chi\nabla_\alpha\nabla_\nu\chi\right] \tag{6}$$

The last two terms cancel identically (since dummy indices μ and α are equivalent). Thus, we arrive at the result in Appendix A (Eq A1):

$$\nabla^\mu T^{FI}_{\mu\nu} = 2A(\Box\chi)\nabla_\nu\chi \tag{7}$$

3 Equation of Motion Cancellation

The equation of motion for χ (derived from the variation w.r.t χ in the full action) yields:

$$\Box\chi = \mathcal{S}_{matter} \tag{8}$$

where $\mathcal{S}_{matter}$ arises from the coupling to the baryon density n. Due to diffeomorphism invariance of the total action S, the Noether identity guarantees that the variation of the matter sector exactly balances the variation of the Fisher sector. Specifically, the divergence of the matter stress tensor satisfies:

$$\nabla^\mu T_{\mu\nu}^{(m)} = -2A(\Box\chi)\nabla_\nu\chi \tag{9}$$

Summing (8) and (10):

$$\nabla^\mu T_{\mu\nu}^{tot} = 2A(\Box\chi)\nabla_\nu\chi - 2A(\Box\chi)\nabla_\nu\chi = 0 \tag{10}$$

Q.E.D.