Derivation 03: FRW Cosmology and Stiff Fluid Limit

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October 2025

1 FRW Ansatz

We assume a flat Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \tag{1}$$

We assume the scalar field χ is homogeneous, i.e., $\chi = \chi(t)$, so $\nabla_i \chi = 0$.

2 Energy Density and Pressure

Using the stress tensor derived in Derivation 01:

$$T_{\mu\nu}^{FI} = 2A \left(\nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} (\nabla\chi)^2 \right) \tag{2}$$

First, calculate the kinetic term $(\nabla\chi)^2$:

$$(\nabla\chi)^2 = g^{tt}(\dot{\chi})^2 = -\dot{\chi}^2 \tag{3}$$

Now, compute the energy density $\rho_{FI} = -T_t^t = -g^{tt} T_{tt}$:

$$T_{tt} = 2A \left(\dot{\chi}^2 - \frac{1}{2}(-1)(-\dot{\chi}^2) \right) = 2A \left(\dot{\chi}^2 - \frac{1}{2}\dot{\chi}^2 \right) = A\dot{\chi}^2 \tag{4}$$

$$\rho_{FI} = A\dot{\chi}^2 \tag{5}$$

Now, compute the pressure p_{FI} from the spatial component T_{xx}:

$$T_{xx} = 2A \left(0 - \frac{1}{2} g_{xx} (-\dot{\chi}^2) \right) = Aa^2\dot{\chi}^2 \tag{6}$$

$$p_{FI} = g^{xx} T_{xx} = \frac{1}{a^2}(Aa^2\dot{\chi}^2) = A\dot{\chi}^2 \tag{7}$$

3 Equation of State

From (5) and (7), we find the equation of state parameter w:

$$w_{FI} = \frac{p_{FI}}{\rho_{FI}} = \frac{A\dot{\chi}^2}{A\dot{\chi}^2} = 1 \tag{8}$$

This corresponds to a **Stiff Fluid**.

4 Cosmological Evolution

The continuity equation for a fluid in FRW is:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{9}$$

Substituting $p = \rho$:

$$\dot{\rho} + 6H\rho = 0 \implies \frac{\dot{\rho}}{\rho} = -6\frac{\dot{a}}{a} \tag{10}$$

Integrating both sides:

$$\ln \rho \propto -6 \ln a \implies \rho_{FI} \propto a^{-6} \tag{11}$$

This confirms the result in Eq (9) of the manuscript. The rapid decay (a^{-6}) compared to radiation (a^{-4}) and matter (a^{-3}) ensures that the Fisher Information density does not disrupt Big Bang Nucleosynthesis (BBN) constraints at late times.