New Approach to Construct Planar Triangulations with Minimum Degree 5

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Abstract

1 Introduction

We restrict ourselves to undirected, connected and planar graphs G which have no loops or multiple edges. Given a simple graph $G = (X, E)$ with node set X and edge set E. A graph G is said to be *embeddable* on a surface S if it can be drawn on S that its edges intersect only at their end vertices; and *planar* if it can be embedded on a plane. If we consider a planar graph with no loops or faces, bounded by two edges, it may be possible to add a new edge to the given representation of G, such that these properties are preserved. When no such adjunction can be made, we call G maximal planar. A planar graph is called triangulated when all faces have three corners. We have in [19], that a planar graph is maximal if, and only if, it is triangulated. In [17], a graph G is called an $MPG5$ graph if G is a Maximal Planar Graph with minimum degree five; we denote by $MPG5_n$ the set of the $MPG5$ graphs with n vertices.

The coloring problem of maximal planar graphs, will be *easier* if there is a vertex x with $dg(x) < 5$. If $dg(x) < 4$ then x can be colored by one of the four colors, and if $dg(x)$=4 then by using the Kempe chains [14], x can be also colored by one of the four colors. $MPG5$ graphs are the most difficult planar graphs to four color, so an application of this work, could be for example, in order to experiment and to compare, some planar graphs coloring algorithms.

Let $G = (X, E)$ be a planar graph, with vertex set $X = \{x_1, \ldots, x_n\}$ and let $3 \leq r \leq n$ be an integer. G is an r-rooted triangulation, denoted by $G_{n,r}$ if it can be embedded in the plane such that the outer face has labels $x_1, \ldots, x_r$ in clockwise order, and all interior faces are triangles.

The efficient generation of unrooted triangulations has received some attention in the literature. This appears to be harder than generating all rooted triangulations, and isomorphism testing is required by current algorithms. In [2], Avis has proposed an efficient algorithm which uses the *Reverse Search* [3] procedure to generate all 2 and 3-connected r-rooted triangulations without repetitions. Thus, an idea to generate all maximal planar graphs with minimum degree five, could be to use his program [1] by computing the 3-rooted triangulations; and with a filter to extract the $MPG5$ graphs. One drawback of this solution is its very expensive computational time. For example, for n=20, there exists 73 $MPG5$ non-isomorphic graphs. For n, r, let $g(n, r)$ be the number of non-isomorphic 3-connected r-rooted triangulations, W. Tutte, [23] found a closed formula for $g(n, r)$. For $n \geq 5$,

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$$g(n,3) = \frac{2}{(n-2)!}(3n-6)(3n-5)\dots(4n-11).$$

By Tutte's formula, we have approximately 10^{13} 3-rooted triangulations for n=20. There is some algorithms to generate the $MPG5$ graphs without enumeration control. The expansion method has been developed by Marble and Matula [9], and despite the fact that it has undergone no formal analysis (it is not even known whether every $MPG5$ graph can be generated by this approach), it is useful because it tends to generate graphs that contain vertices of high degree. In [16, 17] Morgenstern and Shapiro have introduced the reduction method, which has the virtue of generating all $MPG5$ graphs with finite probability, though the graphs it generates have a more uniform degree distribution. Both methods begin by constructing a circuit C, and then randomly triangulate the *outside* of C. The reduction method proceeds by randomly triangulating C again, but this time the edges are added to the *inside* of C and precautions are taken not to add edges that already exist on the *outside*. The final step is to reduce the graph by recursively removing, first, the vertices of degree three, and after by contracting vertices of degree four with a neighbor; we repeat these two stages until the current graph is $MPG5$. The expansion method triangulates the *inside* of C in a different manner. A single step of this method does the following:

1. Creates a new vertex, v, in the inside face of C.
2. Makes v adjacent to a randomly chosen circuit vertex u_0.
3. Sequentially adds the edges connecting v and the vertices $u_1, u_2, \dots, u_h$, where $u_0, u_1, u_2, \dots, u_h$ describe a path on C in clockwise direction from u_0. This addition phase terminates when u_h has degree four or smaller after the edge (v, u_h) has been added. Vertices $u_1, u_2, \dots, u_{h-1}$ are forced to the outside of C, and v is added to C.

In [21] Stamm-Wilbrandt has developed an algorithm to generate any maximal planar graphs starting by $K4$; we can use the final step of the reduction method to transform a non-$MPG5$ into an $MPG5$.

Some drawbacks of these solutions are :

- These algorithms:
  - do not generate the $MPG5$ graphs for a fixed order n.
  - do not control the degrees distribution.
- The expansion method is not guaranteed to terminate unless restriction.

In [15], Chiu Ming Kong have proposed some modifications of Avis's work in order to enumerate all rooted triangulation with minimum degree four.

More recently, Gunnar Brinkmann and Brendan McKay [6] have proposed best results and fastest tools, called `plantri`, about construction of planar triangulations with minimum degree 5. These results is based on Barnette [4], Butler [8] and Batagelj's [5] works. `Plantri`, is so fast than for example it can construct all $MPG5$ with 25 vertices, 25381 graphs, less than one second on a basic Personal Computer (PIII-700MHz). Their algorithm is based on three basic operations like *flip* or *diagonal* transformation.

2 Approach

Actually best methods to construct all $MPG5$ are inductive: by starting with an initial graph, like icosahedron, these methods applies three basic operations like *flip* from 14 to n vertices. The best algorithm to enumerate all $MPG5$ is actually describing in [6] and implemented in `plantri`. If your aim is defined to enumerate all $MPG5$ with n vertices or to build one of them, then we need to construct some or all $MPG5$ between initial graph and target. By example if your aim is to construct all $MPG5$ with 50 vertices, you must to construct all 14, all 15, all 16, all 17, ..., all 49 and finally all 50 $MPG5$ graphs.

Here, we will describe a different approach: we don't need to construct $MPG5$ between 14 and n, but only those with n vertices. Own approach is based on one basic transformation and two dedicated algorithms.

This paper will be structured in several parts. First we will describe basic definitions, part two and three, will present the two dedicated algorithms.

3 Basic definitions

We are using frequently a degree partitioning of X, so by commodity we have defining the following:

Definition 1 *Let $G = (X, E)$ be an $MPG5$ graph. We denote*

$$X_{sup6} = \{x \in X / dg(x) \geq 6\},$$
$$X_{inf6} = \{x \in X / dg(x) = 5\}.$$

Now, we are defining two basics and well knows operations.

3.1 About flip

Definition 2 *Let G be an $MPG5$ graph and $e = (x', x'')$ be an edge. Since G is maximal, e bounds two triangles described by x'', x_1, x' and x', x_k, x''. If $x', x'' \in X_{sup6}$, then we are denoting by $[x', x'']$ this edge. Let $[x', x'']$ be a such edge, we have calling by $D[x', x'']$ the graph G after the flip of (x', x''); i.e. a simple diagonal transformation. See figure 3.1.*

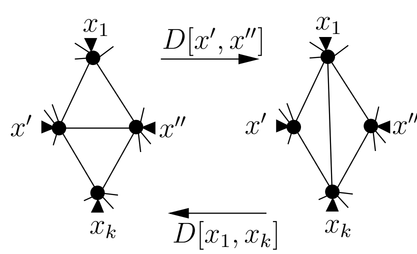

Figure 3.1. D Transformation.

3.2 About explosion

Definition 3 *Let $G = (X, E)$ be an $MPG5$ graph and $x_1, x, x_k \subset X$ be a simple path of length two. We denote $[x; x_1, x_k]$ this path when:*

- *$x \in X_{sup6}$.*
- *$N(x)$ can be described in clockwise order by $\{x_1, \ldots, x_k, \ldots, x_q\}$ where $k \in [4, q-2]$ and $q > 5$.*

$T[x; x_1, x_k]$ is the graph G after the explosion of x in two new adjacent vertices x', x'' such that in clockwise order $N(x') = \{x_1, x_2, \ldots, x_{k-1}, x_k, x''\}$ and $N(x'') = \{x_1, x', x_k, x_{k+1}, \ldots, x_q\}$. See figure 3.2.

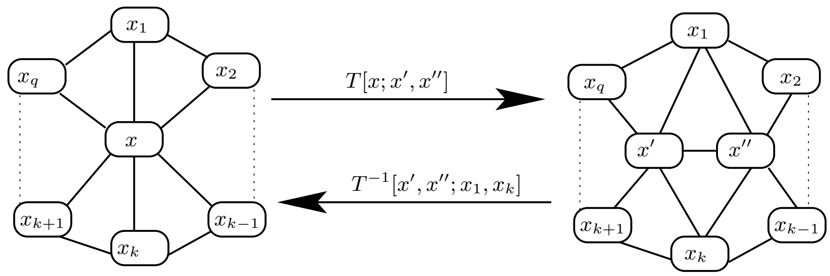

Figure 3.2. T and T^{-1} Transformations.

By using the previous notation, we can observe that, in $T[x; x_1, x_k]$, $\{x_1, x_k\} \subseteq X_{sup6}$.

Definition 4 *Let G be an $MPG5$ graph and $e = (x', x'')$ be an edge. Since G is maximal, e bounds two triangles described by x'', x_1, x' and x', x_k, x''. If $x_1, x_k \in X_{sup6}$, we call by $[x', x''; x_1, x_k]$ this polygon also denoted $\diamondsuit$. Let $[x', x''; x_1, x_k]$ be a such polygon $\diamondsuit$, we denote by $T^{-1}[x', x''; x_1, x_k]$ the graph G after contraction of x', x'' in a single vertex x. See figure 3.2.*

3.3 Partitioning induced by operations

Here, we are defining induced class of graphs by operations D and T^{-1}. In particular we looking for $MPG5$ such that :

- D cannot be apply: these graphs are called Π. Other graphs are denoted $\underline{\Pi}$.
- T^{-1} cannot be apply: these graphs are called Σ. Other graphs are denoted $\underline{\Sigma}$.

Now, by using these last lines, it must be possibly to give some precisions about own approach. In order to construct all $MPG5$ with n vertices, we will propose to built all Π by using two dedicated algorithms, and all $\underline{\Pi}$ graphs by using flip operation.

Before starting some proofs, we have to describing some basic definitions about induced graph partitioning.

3.3.1 About flip

Definition 5 *Let $G = (X, E)$ be an $MPG5$ graph. G is called Π_1, if $\forall x \in X_{sup6}, \{N(x) \cup N^2(x)\} \subseteq X_{inf6}$.*

Definition 6 *Let $G = (X, E)$ be an $MPG5$ graph not Π_1. G is called Π_2, if $\forall x \in X_{sup6}, \{N(x) \subseteq X_{inf6}$.*

We give two observations about this partitioning:

1. $\Pi = \Pi_1 \cup \Pi_2$.
2. Let G be an $MPG5$ graph $\underline{\Pi}$, there exists an edge $[x; y]$ to apply D.

3.3.2 About explosion

Definition 7 *Let G be an $MPG5$ such that set of polygon $\diamondsuit$ is empty; such graph are called Σ.*

4 Π graphs constructions

In this section we will describe two algorithms : first subsection about Π_1 construction and next subsection about Π_2.

4.1 Π_1 graphs constructions

There exists many ways to build Π_1 graphs. For example one approach to generate them could be in three steps as figure 4.1.a. Let x be a vertex in X_{sup6} with $dg(x) = q$. We can start by drawing first $N(x)$, since G must be maximal, we have to draw a wheel with at least six vertices of degree three in the outer face. Figure 4.1.a we have choose $q = 7$.

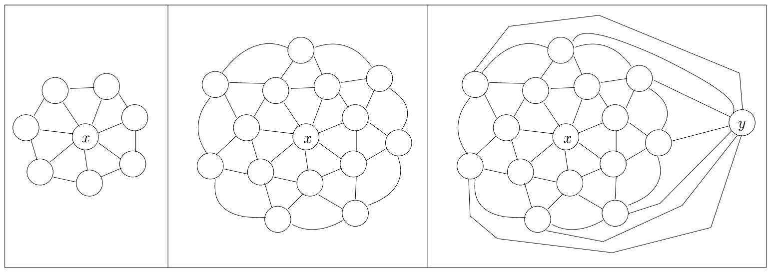

Figure 4.1.a. Π_1 graph construction (first proposition).

Since $N(x) \subset X_{inf6}$, between $N(x)$ and $N^2(x)$ we must add exactly two edges for each vertex in $N(x)$. G is maximal and planar, which implies that $|N(x)| = |N^2(x)| = q$. See second drawing figure 4.1a.

Since $N^2(x) \subset X_{inf6}$, between $N^2(x)$ and $N^3(x)$ we must add exactly one edge for each vertex in $N^2(x)$. G is maximal and planar, and so $N^3(x)$ is limited to a single vertex (called y) with a degree q, and $N^w(x) = \emptyset$ for $w > 3$. See last picture on figure 4.1.a. By this construction, we can add corollary 9.

Algorithm 8 and figure 4.1.b use a different way to generate an Π_1 graph with $n = (2 \times q) + 2$ vertices.

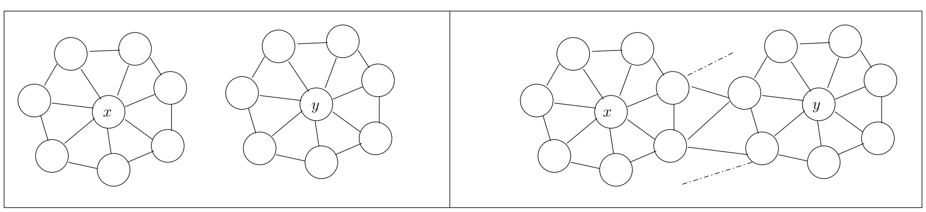

Figure 4.1.b. Π_1 graph construction (second proposition).

Algorithm 8

TITLE: `Generation of an` Π_1 `graph with` $n \geq 14$ `vertices.(First Algorithm)`
/ First Step */*
Let x (resp. y) be a vertex;
Build a wheel in clockwise order with $q = \frac{n-2}{2}$ vertices
with x (y) as center such that $N(x)$ (resp. $N(y)$) is defined by
$\{x_1, x_2, \ldots, x_q\}$ ($\{y_1, y_2, \ldots, y_q\}$);
$i = 1$;
/ Second Step */*
While $i < q$ **do**
`Add` *(x_i, y_i) and (x_i, y_{i+1});*
$i = i + 1$;
End while
`Add` *(x_q, y_q) and (x_q, y_1);*

Corollary 9 *Let G be an $MPG5$ graph with n vertices. If $\exists x \in X_{sup6}$ with $\{N(x) \cup N^2(x)\} \subseteq X_{inf6}$ then G is an Π_1 graph.*

We can include the following observations:

- Let G be an Π_1 graph with n vertices. n is even, and if we denote X_{sup6} by $\{x, y\}$ then $\Delta(G) = \frac{n-2}{2} = dg(x) = dg(y)$.

- Smallest Π_1 graph G with :

  1. $X_{sup6} = \emptyset$. We have icosahedron.
  2. $X_{sup6} \neq \emptyset$. The current graph is unique, with 14 vertices and $\Delta(G) = 6$. $MPG5_{14}$ contains only this graph.

- Let $G = (X, E)$ be an $MPG5$ graph with $n \geq 14$ vertices (i.e. excepted for the icosahedron)

$$\nexists x \in X, \left\{ \bigcup_{i=1}^{k} N^i(x) \right\} \subseteq X_{inf6} \text{ ,with } k \geq 3$$

- All Π_1 graphs with n vertices are isomorphic.

Here we will known how to construct Π_1 graph.

4.2 Π_2 graphs constructions

4.2.1 Approach

In order to build G, a Π_2 graph, we propose three stages. Except some exceptions the sketch is something like that:

1. Build K, a connected 3-regular planar graph (see figure 4.2.1.a).
2. Build Q, by replacing all vertices in (1) by at choice 3 vertices (denoted $\triangle^3$) or six (denoted $\triangle^6$), see figure 4.2.1.b.
3. Build G. In all non triangular face in (2) fixe a vertex v and insert all edges between v and vertices describing the face around it (see figure 4.2.1.c).

These three stages are proposed in the three figure 4.2.1.a, 4.2.1.b, 4.2.1.c.

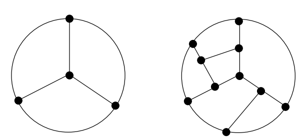

Figure 4.2.1.a. Stage 1: Two K graphs examples.

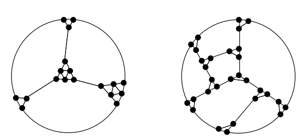

Figure 4.2.1.b. Stage 2: Examples of corresponding Q graphs.

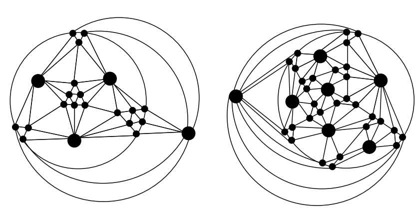

Figure 4.2.1.c. Stage 3: Examples of corresponding Π_2 graphs.

4.2.2 Basic definitions

As we will see there exists Π_2 graphs with some vertices in X_{inf6} with their first neighborhood in X_{inf6}. For this reason we'll use the following definition.

Definition 10 *Let $G = (X, E)$ be an $MPG5$ graph. Then we denote by X' the following vertex set:*

$$X' = \{x \in X/N(x) \subset X_{inf6}\}.$$

Π_2 graphs have some particular topology, in order to fixe it, we give another definition about some vertices in X_{inf6}. As we will see, most of these vertices are used in two kind of polygons denoted $\triangle^3$, $\triangle^6$.

Definition 11 *$\triangle^3$ is an empty triangle, i.e. 3 corners labelled by vertices a, b, and c. See figure 4.2.2.*

Definition 12 *$\triangle^6$ is a hexagon completely triangular, with six vertices on the external face labelled in clockwise order by a, c_1, b, a_1, c, b_1 and zero vertices on the internal face, for which, locally in $\triangle^6$, $dg(a) = dg(b) = dg(c) = 2$ and $dg(a_1) = dg(b_1) = dg(c_1) = 4$. Vertices a, b, c are called the corners. See figure 4.2.2.*

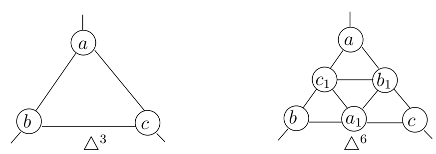

Figure 4.2.2: Polygons $\triangle^3$, $\triangle^6$.

4.2.3 Π_2 properties

The main result of this section designed by theorem 15, is a vertex cover of $X_{inf6} - X'$ by a polygon set defined by $\triangle^3$ and $\triangle^6$. Before proof of theorem 15, we are giving some properties of Π_2 graphs.

Lemma 13 *Let $G = (X, E)$ be a Π_2 graph. Then $X_{sup6} \subseteq X'$.*

Proof By definition of Π_2 graph $\forall x \in X_{sup6}, N(x) \subset X_{inf6}$. ∎

Lemma 14 *Let $G = (X, E)$ be a Π_2 graph. Then $\forall x \in X'$, $\exists y \in N^2(x) \cap X_{sup6}$.*

Proof

- Let $x \in X_{sup6}$. Since G is Π_2, $N(x) \subset X_{inf6}$ and since not Π_1, $\exists y \in N^2(x) \cap X_{sup6}$.
- Let $x \in X_{inf6}$. Suppose x with $\{N(x) \cup N^2(x)\} \subset X_{inf6}$, then G is the smallest $MPG5$ graph, i.e. 12 vertices of degree five, i.e. icosahedron. We have a contradiction since G is Π_2, i.e. not Π_1.

Thus in all case in Π_2 graph, $\forall x \in X'$, $\exists y \in N^2(x) \cap X_{sup6}$.

∎

Theorem 15 *Let $G = (X, E)$ be a Π_2 graph. There exists a vertex cover of $\{X_{inf6} - X'\}$ by a polygon set of $\triangle^3$ and $\triangle^6$.*

In order to prove the theorem 15, we are going to describe the neighborhood for every vertex in X'. We'll describe two cases (every one gives a lemma): $x \in \{X' \cap X_{sup6}\}$ (lemma 16) and $x \in \{X' \cap X_{inf6}\}$ (lemma 17).

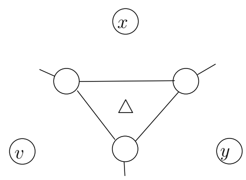

Figure 4.2.3.a. Around vertices X_{sup6} in Π_2 graph.

Lemma 16 *Consider $G = (X, E)$ be a Π_2 graph, and $x, y \in X_{sup6}$ with $y \in N^2(x)$. There exists a vertex $v \in X'$ in the common second neighborhood of x and y such that a polygon $\triangle$ separates x, y, v like figure 4.2.3.a.*

Proof

First since G is Π_2, $\forall x \in X_{sup6}, \exists y \in \{X_{sup6} \cap N^2(x)\}$ (see lemma 14) and $\{N(x) \cup N(y)\} \subset X_{inf6}$, i.e. there is an edge (z, t) which bounds two triangles (x, z, t), (z, t, y) as figure 4.2.3.b. We consider a plane orientation defined by the clockwise order. Suppose $N(x)$ and $N(y)$ are described by:

$$N(x) = \{x_1, \dots, x_k, t, z, x_{k+3}, \dots, x_{dg(x)}\}$$
$$N(y) = \{y_1, \dots, y_{k+2}, z, t, y_{k+5}, \dots, y_{dg(y)}\}$$

Since G is a Π_2 graph, $x_{k+3}, y_{k+2} \in X_{inf6}$. So there exists a vertex s such that $N(s) = \{\dots, x_{k+4}, x_{k+3}, y_{k+2}, y_{k+1}, \dots\}$ in order to enclose x_{k+3}, y_{k+2}. We have two choices: either s is in X_{sup6} or in X_{inf6}.

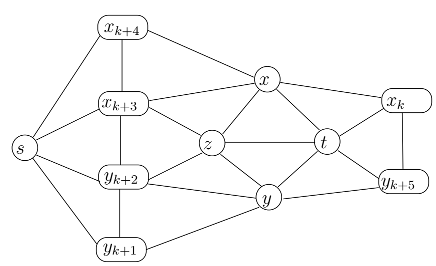

Figure 4.2.3.b. Neighborhood of $\triangle^3$.

- Suppose $s \in X_{sup6}$, see Figure 4.2.3.b. G is Π_2, so $N(s) \subset X_{inf6}$, i.e. $s \in X'$. Of course, s belongs to the common second neighborhood of x and y; thus we can use s as v. We have a triangle $\triangle^3$ describing by 3 corners z, x_{k+3}, y_{k+2}.

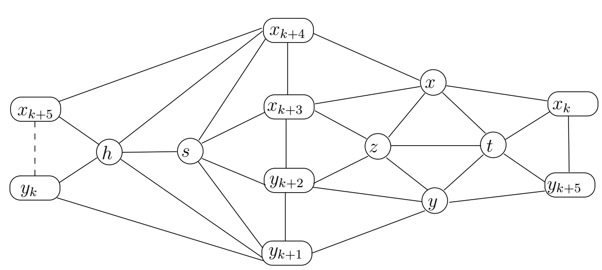

Figure 4.2.3.c. Neighborhood of $\triangle^6$.

- Suppose $s \in X_{inf6}$, see Figure 4.2.3.c. To enclose s, there exists a vertex h such that $N(h) = \{\dots, x_{k+4}, s, y_{k+1}, \dots\}$. Since $x_{k+4} \in N(x)$, we have $(h, x_{k+5}) \in E$, and then $N(x_{k+4}) = \{x_{k+5}, x, x_{k+3}, s, h\}$. Now, we use the same remark about y_{k+1}, i.e. we have $N(y_{k+1}) = \{h, s, y_{k+2}, y, y_k\}$, and so to enclose the vertex y_{k+1} the edge $(h, y_k) \in E$.

  Thus we have a polygon $\triangle^6$ describing by 3 corners z, x_{k+4}, y_{k+1}. Now, we have to talk about h; we have two cases:

– Suppose $h \in X_{sup6}$, then since G is Π_2, $h \in X'$. Thus we can use h as v. In this case the three vertices x, y and v are separating by a polygon $\triangle^6$.

– Suppose $h \in X_{inf6}$, see figure 4.2.3.d. By hypothesis $N(x) \subset X_{inf6}$ and in order to enclose h, there exists a vertex w, for which :

  * $w, x_{k+5} \in N(x)$,
  * $N(y) \subset X_{inf6}$,
  * $w, y_k \in N(y)$.

  Therefore $N(h) \subset X_{inf6}$, $h \in X'$ and h belongs to the common second neighborhood of x and y; we can use h as v. The vertices w, x_{k+5} and y_k are the three corners of a polygon $\triangle^6$. We can observe that $v \in X_{inf6}$ with $N(v) \in X_{inf6}$; and in this case around v we have exactly one $\triangle^3$ and one $\triangle^6$ like figure 4.2.3.d..

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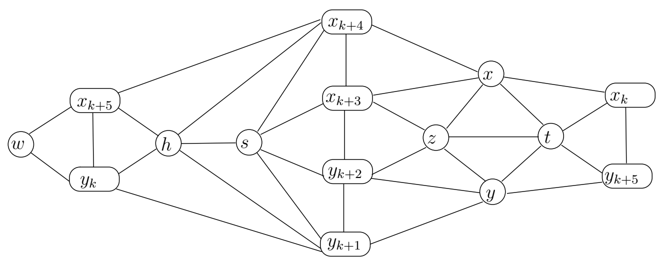

Figure 4.2.3.d. Neighborhood of $\triangle^6$: $h \in X_{inf6}$.

Lemma 17 *Let G be a Π_2 graph and x be a vertex in $X' \cap X_{inf6}$. Then $N^2(x)$ contains exactly two vertices y, z in X_{sup6} and there is exactly one $\triangle^6$ and one $\triangle^3$ around x.*

Proof

Let x be a vertex in $X' \cap X_{inf6}$. We consider value of $|N^2(x) \cap X_{sup6}|$.

- Suppose $\{N^2(x) \cap X_{sup6}\} = \emptyset$. Then $\{N(x) \cup N^2(x)\} \subset X_{inf6}$ and so by corollary 9, G is an Π_1 graph. There is a contradiction since a Π_2 graph can not be Π_1.

- Suppose $\{N^2(x) \cap X_{sup6}\} = \{y\}$. Let $N^2(x) = \{\dots, y_1, y, y_2 \dots\}$ the second neighborhood of x in clockwise order. In order to enclose y_1 and y_2 in X_{inf6} we must add a double edge.

- Suppose $\{N^2(x) \cap X_{sup6}\} = \{u, v\}$. Since G is Π_2, u and v are not adjacent. We have using the notation proposed in figure 4.2.3.f: in clockwise order $N(x) = \{x_1, x_2, x_3, x_4, x_5\}$, $N^2(x) = \{x_6, v, x_7, u, x_8\}$, and finally $N(v) = \{x_1, x_6, x_9 \dots, x_7, x_2\}$. Since x_9 and x_7 belong to the first neighborhood of u and v, they are in X_{inf6}. Therefore we have one polygon $\triangle^6$ and one polygon $\triangle^3$ around $N(x)$; describing by the three following corners : x_1, x_4, x_9 for $\triangle^6$, x_2, x_7, x_3 for $\triangle^3$. This result is equivalent to those proved in last part of lemma 16.

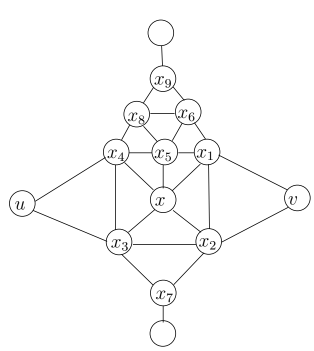

Figure 4.2.3.f. Neighborhood of a vertex x in $X' \cap X_{inf6}$.

- Suppose $|N^2(x) \cap X_{sup6}| \geq 3$. Then since $dg(x) = 5$, there are two adjacent vertices in $N^2(x) \cap X_{sup6}$. There is a contradiction since G is a Π_2 graph.

■

Lemma 16 are describing only one side, the *left* side of $[z, t; x, y]$. We can observe the same results on the *right*; i.e. there is a vertex v' in the common second neighborhood of x and y with $v' \in X'$ and a polygon p' with t as corner such that p' separate these three previous vertices x, y, v'. Clearly, there is $v = v'$ only in two cases:

- $|X_{sup6}| = 3$. In this case we have three vertices of degree six, two polygons $\triangle^6$ and the graph is unique and like figure 4.2.3.g.
- $|X_{sup6}| = 0$. In some sense, we have one polygon $\triangle^3$, one $\triangle^6$. We have icosahedron.

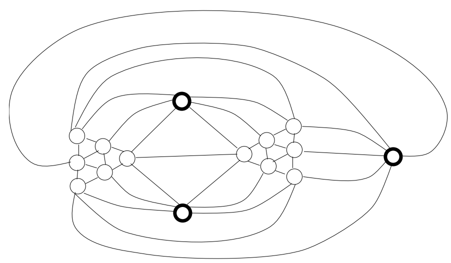

Figure 4.2.3.g. Π_2 graph with $|X_{sup6}| = 3$ and $\Delta = 6$.

Now, we can present a consequence of precedent lemmas :

Corollary 18 *Let G be a Π_2 graph, $|X'| \geq 3$.*

Lemma 14, 16, and 17, imply that there exists a vertex cover of $X_{inf6} - X'$ by a polygon set of $\triangle^3$ and $\triangle^6$. So we have proved theorem 15. ■

4.2.4 Π_2 graph construction algorithm

In order to construct Π_2 graphs we'll give algorithm 23. This algorithm is based on two basics stages, called γ and δ, by starting on connected 3-regular planar graphs with or without double edges. In order to prove this construction's property, we'll propose to consider his inverse algorithm defined by algorithm 20 and composed of inverse basics stages called α and β. We'll proof that any Π_2 graph can be reduced by algorithm 20 on a 3-regular planar graph with or without double edges.

So we'll want to prove the following:

- All Π_2 graphs can be construct by algorithm 23.
- All graphs building by algorithm 23 are Π_2 graphs.

Definition 19 *Let G be a Π_2 graph. We call P the polygon set of $\triangle^3$ and $\triangle^6$ in G.*

Algorithm 20

TITLE: `Inverse` Π_2 `graphs Construction.`

Let $G = (X, E)$ be a Π_2 graph with n vertices

/— BEGIN Operation α —*/*

    While $X' \neq \emptyset$ **do**

        Let x be a vertex in X'

        Remove x; $X' = X' - \{x\}$

    End while

/— END Operation α —*/*

Let $Q = (X_Q, E_Q)$ be this current graph.

Let P the polygon's set (ie set of $\triangle^3$ and $\triangle^6$) on Q.

/— BEGIN Operation β —*/*

    While $P \neq \emptyset$ **do**

        Let p_q be a polygon in P;

        Replace p_q by a single vertex v_k; $P = P$ -$\{p_q\}$

    End while

/— END Operation β —*/*

Let $K = (X_K, E_K)$ be this current graph

We'll give some properties of graph Q and K.

Lemma 21 *Let G be a Π_2 graph. In algorithm 20,*

- *Q is a subgraph induced by $X_{inf6} - X'$ in a Π_2 graph.*
- *There exists a vertex cover by $\triangle^3$ and $\triangle^6$ in Q.*

Proof Since $X_{sup6} \subseteq X'$ (lemma 13) and by definition of X', the graph Q is defined :

- if $X_{inf6}(G) \cap X' = \emptyset$, by the induced graph of $X_{inf6}(G)$.
- otherwise, by the induced graph of $X_{inf6}(G) - X'$.

Thus in all case Q is the subgraph induced by $X_{inf6}(G) - X'$ in G. And so by using corollary 15, there exists a vertex cover by $\triangle^3$ and $\triangle^6$ on Q.

■

Lemma 22 *Let G be a Π_2 graph. In algorithm 20, K is a connected 3-regular planar graph with or without double edges.*

Proof Q is such that there exists a vertex cover by $\triangle^3$ and $\triangle^6$. Each of these polygons contains exactly 3 corners. Since K has been constructed by replacing each polygon p_q by a vertex v_k and connectivity, planarity is not modified; K is connected 3-regular planar graphs. Double edge we'll be product in K when $X_{inf6}(G) \cap X' \neq \emptyset$; i.e. there exists in G vertices $v \in X'$ for which $dg_G(v) = 5$ (see lemma 17 and figure 4.2.4.a).

■

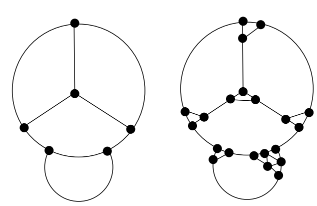

Figure 4.2.4.a. K and Q with an double edge.

Algorithm 23

TITLE: Π_2 `graph construction.`

Build $K = (X_K, E_K)$.

Generating a connected 3-regular planar graph with or without double edges.

Build $Q = (E_Q, V_Q)$.

/— BEGIN Operation γ —*/*

For each edge $e_k = (x_k, y_k)$ in K.

Place a polygon $\triangle^3$ or $\triangle^6$ on x_k and y_k;

If e_k is an double edge place at least one $\triangle^6$.

/— END Operation γ —*/*

Build $G = (X, E)$.

/— BEGIN Operation δ —*/*

For each non triangular face f_q in Q

Fixing a vertex v_q into f_q.

Triangulated f_q by addition of edges

between v_q and all vertices describing f_q.

/— END Operation δ —*/*

Lemma 24 *All Π_2 graphs can be construct by algorithm 23.*

Proof We can give the operations's correspondence between algorithm 20 and 23:

- β is equivalent to inverse of γ. β remove all polygons $\triangle^3$ and $\triangle^6$ on Π_2 graph. γ add all polygons $\triangle^3$ and $\triangle^6$ in Π_2 graph.

- α is equivalent to inverse of δ. α remove X' vertices of Π_2 graph. δ fixe X' vertices. This last process in algorithm 23 finish the Π_2 graph construction.

■

We can see an example of this Π_2 graph composition and decomposition in figures 4.2.4.b, 4.2.4.c, 4.2.4.d.

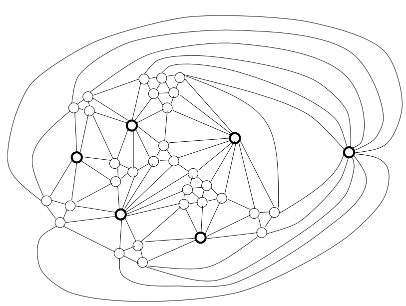

Figure 4.2.4.b
Example of a Π_2 graph.

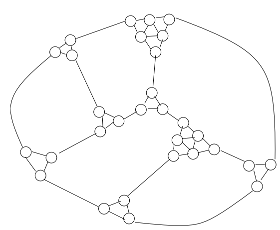

Figure 4.2.4.c
Example of Q.

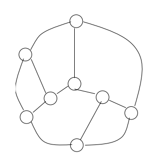

Figure 4.2.4.d
Example of K.

We have some observations about K. We consider G and x a vertex in X'. The minimum of polygons $\triangle^3$, $\triangle^6$ around x is two. This minimum is defined by two cases:

1. $dg(x) = 6$ and we have two $\triangle^6$ or three $\triangle^3$ around x.
2. $dg(x) = 5$, we have one $\triangle^3$, one $\triangle^6$.

At this stage, we can add the following main theorem about Π_2 graph construction:

Theorem 25 *Algorithm 23 building all Π_2 graphs and only those.*

Proof

1. All graphs G building by algorithm 23 are Π_2: all vertices in X_{sup6} are such that $N(x) \in X_{inf6}$ and $N^2(x) \cap X_{sup6}$ is not empty, i.e. G is not A. Thus G is Π_2.

2. All Π_2 graphs can be build by algorithm 23. We use precedent lemma 24.

■

In conclusion, by this section, we will know to construct Π graph by using algorithm 8 for Π_1 graph and algorithm 23 for Π_2. Now, in the next section we will describing a method using only diagonal operation to construct all $\underline{\Pi}$ graphs.

5 $\underline{\Pi}$ graphs constructions

In this section we have denoting D^k, k operations D applied iteratively on a graph such that all intermediary graphs are $MPG5$. Parallelization of D operation has been already treated for example by J. Galtier and al. in [10] and [11].

5.1 Approach

Own approach is based on main theorem 27 proving that all $\underline{\Pi}$ graphs with n vertices are equivalent under operation D. Before introducing this main result we propose to present relationship between graph Π, $\underline{\Pi}$ and operation T,T^{-1}.

Lemma 26 *Let G be Π with $n > 15$. By using two operations based on T and T^{-1} we can construct a $\underline{\Pi}$ with n vertices.*

Proof Let G be a Π with $n > 15$ and $x \in X_{sup6}$. We have to prove that there exists a triplet $[x; x_1, x_k]$ for which in $T[x; x_1, x_k]$ there exists a polygon $\diamondsuit$ called p such that $T^{-1}(p)$ is $\underline{\Pi}$. We have two cases since $\Pi = \Pi_1 \cup \Pi_2$. In the two cases the sketch is describing like figure 5.1.a.

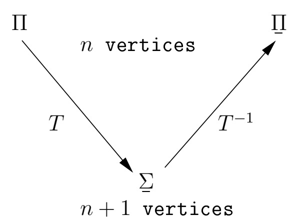

Figure 5.1.a. Lemma 26 proof sketch: Π to $\underline{\Pi}$ by T, T^{-1}.

- Let G a Π_1 graph. We can observe that $n > 15 \Longrightarrow \Delta \geq 7$. We consider $X_{sup6} = \{x, y\}$ and choose $k = 4$ for x_k. Thus $T[x; x_1, x_k]$ is such that $dg(x'') = 5$ and $dg(x') = dg(x) - 1$. Since $\Delta \geq 7$, $dg(x') \geq 6$. So $G' = T^{-1}[y_1, y_2; y, x_1]$ contains the edge $[x', x_k]$, where y_1, y_2 the two common neighbors of y and x_1. In consequence G' is $\underline{\Pi}$.

- Let G a Π_2 graph. We can see that a graph Π_2 with $n > 15 \Longrightarrow \mid X_{sup6} \mid \geq 4$. Let U the set of second neighborhood of x in X_{sup6}, denoted in clockwise order by :

  $$U = N^2(x) \cap X_{sup6} = \{u_1, u_2, \ldots, u_r\}, r \geq 3.$$

  We consider a triplet $[x; x_1, x_k]$ with $x_1 \in N(u_1)$ a corner and the graph $G' = T[x; x_1, x_k]$. This T operation have adding an edge $[x_1, u_1]$, and we would like to keep it.

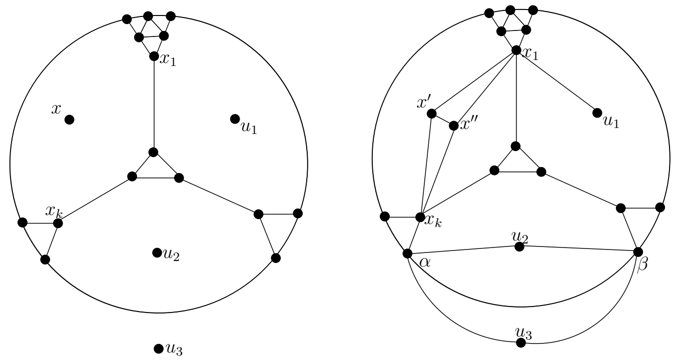

Figure 5.1.b. Lemma 26 proof sketch when G is a Π_2 graph.

In order to reduce number of vertices, we can realize $T^{-1}[\alpha,\beta;u_2,u_3]$ on G' giving G'', where α and β are the two common neighbors of u_2, u_3. This last operation give G'' a $\underline{\Pi}$ graph since we have conserved the edge $[x_1, u_1]$.

■

Now the main result:

Theorem 27 *Let G_1 and G_2 two $\underline{\Pi}$ graphs with n vertices. There exists k such that $G_1 = D^k(G_2)$.*

Proof For $n \in [12, 17]$ is manually provable. Suppose this theorem true for n vertices, thus we have to demonstrate for $n+1$.

We have to distinguish three cases 1) G_1, G_2 are $\underline{\Sigma}$. 2) G_1, G_2 are Σ. 3)Only one of these two graphs is $\underline{\Sigma}$.

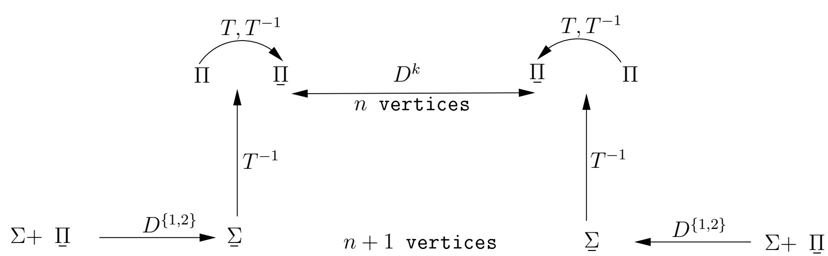

Figure 5.1.c. Theorem 27 proof sketch.

- Let G_1 and G_2 two $\underline{\Sigma}$ graphs; i.e. there exists at least one polygon $\diamondsuit$ in each graph. So let $G_1' = T^{-1}(G_1)$ and $G_2' = T^{-1}(G_2)$. Here we have two cases, these graphs G_1', G_2' are Π or not : in the first case we use lemma 26, in order to build a $\underline{\Pi}$ graph with n vertices. So without restriction we consider G_1' and G_2' two $\underline{\Pi}$ graphs, by hypothesis for n, there exists k' such that $G_1' = D^{k'}(G_2')$ and so for $n+1$ there exists k such that:

$$G_1 = D^k(G_2)$$

  We have proposing the sketch of this approach figure 5.1.c.

- Let G_1, G_2 two Σ graphs; i.e. we cannot applying T^{-1}, i.e. we cannot reducing vertices number. Now, own interest is based on Σ and $\underline{\Pi}$ graph: since induction is based on $\underline{\Pi}$ and case of $\underline{\Sigma}$ has been already treated.

  In order to proof this part we will propose to give two intermediary results about Σ and $\underline{\Pi}$ graphs. First result are giving a minimum number of vertices of X_{sup6} in the first neighborhood in each vertex in X_{sup6}.

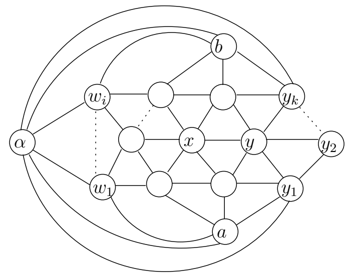

Figure 5.1.d. Lemma 28 proof.

Lemma 28 *Let G be a Σ and $\underline{\Pi}$ graph.* $\forall x \in X_{sup6}$, $\mid N(x) \cap X_{sup6} \mid \geq 2$.

Proof We will describe all cases:

- Suppose $\mid N(x) \cap X_{sup6} \mid = 0$. Since G is Σ, $\nexists \diamondsuit$, and so $N^2(x) \cap X_{sup6} = \emptyset$. Thus we have $N(x) \cup N^2(x) \subseteq X_{inf6}$; i.e. this graph must be Π_1. We have a contradiction since G is $\underline{\Pi}$.
- Suppose $\exists x \in X_{sup6}$ for which $\mid N(x) \cap X_{sup6} \mid = 1$. Since current graph is Σ and $\underline{\Pi}$, we have a configuration like figure 5.1.d : i.e. we have a vertex $\alpha \in X_{sup6}$ for which we have the polygon $[y_1, y_2; y, \alpha]$ in order to encapsulated y_1 in X_{inf6}.

■

In the following lemma, we are giving a result about the number of vertices in X_{inf6} between consecutive vertices in X_{sup6} in a first neighborhood of each vertex in X_{sup6}.

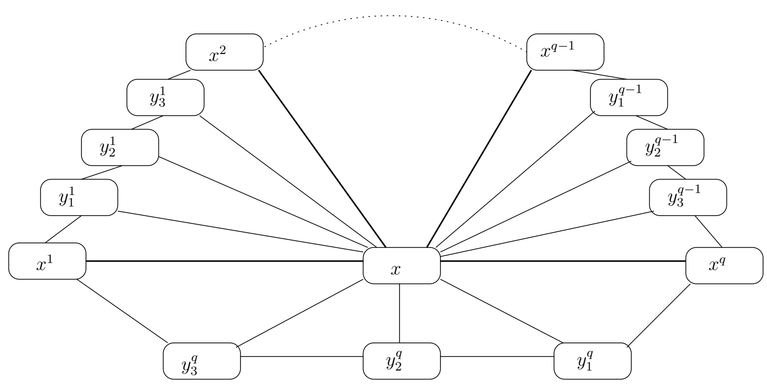

Figure 5.1.e. Σ graph design around vertex in X_{sup6}.

Lemma 29 *Let G be a Σ and $\underline{\Pi}$ graph. We are denoting in clockwise order first neighborhood of x by :*

$$N(x) = \{(x^1, y_1^1, y_2^1, \ldots, y_{k^1}^1), (x^2, y_1^2, y_2^2, \ldots, y_{k^2}^2), \ldots (x^q, y_1^q, y_2^q, \ldots, y_{k^q}^q)\}$$

where $q = \mid N(x) \cap X_{sup6} \mid$, $x^i \in X_{sup6}$, $y_j^i \in X_{inf6}$ with $i \in [1, q]$. We have $k^i \in \{0, 2, 3\}$.

Proof Basically k^i value 0,2,3 is possible, 1 give a polygon $\diamondsuit$. For $k^i > 3$, manually it is easy to see contradictions.

■

Here own aim is to prove that all Σ and $\underline{\Pi}$ graph can be transforming in $\underline{\Sigma}$ graph. This Σ graph will be $\underline{\Pi}$ or Π : without consequence since we only looking for to reduce vertices number. The sketch of this proof is describing figure 5.1.c. By using the two precedent lemmas 28, 29 we are going to work around x^1, x, x^{q-1}, x^q and more precisely about k^q. Suppose $q = 2$, then in the following $k^1 = k^{q-1}$.

Lemma 30 *Let G be a Σ and $\underline{\Pi}$ graph. By applying one or two diagonal operation we can transform G in a $\underline{\Sigma}$ graph.*

Proof We will use notation used in figure 5.1.e. We will describing all cases on k^q values; i.e. $\{0, 2, 3\}$.

1. $k^q = 0$.
   Since current graph is Σ, i.e. without polygon $\diamondsuit$: $(k^q = 0) \Longrightarrow (k^1, k^{q-1} > 0)$ and $(x^1, x^q) \in E$. We have to enumerate each cases:
   (a) $k^1 = 2, k^{q-1} > 0$. Consider $G^D = D(x^1, x)$, we have the new edge $(x^1, y^{q-1}_{k^{q-1}})$ and so $dg(y^{q-1}_{k^{q-1}}) = 6$, we have at least the polygon $[x, y^{q-1}_1; x^{q-1}, y^{q-1}_2]$. So, we can applying T^{-1} on G^D.
   (b) $k^1 > 0, k^{q-1} = 2$. Quasi similarly than precedent case.
   (c) $k^1 = k^{q-1} = 3$. Note : $dg(x) > 6$. Consider G^D the current graph after two operations D : $D(x, x^q)$ and now since $dg(y^{q-1}_3) > 6$, $D(x, y^{q-1}_3)$. Then we have at least the polygon $[x, y^{q-1}_1; x^{q-1}, y^{q-1}_2]$. So, we can applying T^{-1} on G^D.
2. $k^q = 2$.
   (a) $k^1 > 0, \forall k^{q-1}$. Consider $G^D = D(x^1, x)$, we have the new edge (y^1_1, y^q_2) and so $dg(y^1_1) = dg(y^q_2) = 6$, we have at least the polygon $[x, y^q_1; x^q, y^q_2]$. So, we can applying T^{-1} on G^D.
   (b) $k^1 = 0$. See case (1).
3. $k^q = 3$.
   (a) $k^1, k^{q-1} > 0$. Consider G^D the graph after two D operations : $D(x^1, x)$ and $D(x^q, x)$. We have two new edges (y^1_1, y^q_3), and $(y^{q-1}_{k^{q-1}}, y^q_1)$ with $k^{q-1} = \{2, 3\}$ and so $dg(y^q_1) = dg(y^q_3) = 6$, we have at least the polygon $[x, y^q_2; y^q_1, y^q_3]$. So, we can applying T^{-1} on G^D.
   (b) Others. See case (1).

After these enumeration of all possible cases, we can concluding that we can always transforming a Σ graph in a $\underline{\Sigma}$ by using one or two operations D. In other word, we can create a polygon $\diamondsuit$ by applying successively one or two D operations on a Σ graph.

Thus, here we have two cases, current graph after flip operation is Π or $\underline{\Pi}$. No matter since in the two cases we can reduce number of vertices, and so without restriction consider only first case i.e. G_1 and G_2 are $\underline{\Sigma}$.

■

- G_1 is $\underline{\Sigma}$, G_2 is Σ (resp. G_2 is $\underline{\Sigma}$, G_1 is Σ). We realize one or two D operations like a precedent case in order to transform G_2 (resp. G_1) in a $\underline{\Sigma}$ graph. Thus we could have the same configuration than the first, i.e. original graphs G_1 and G_2 are $\underline{\Sigma}$.

■

6 $MPG5$ construction

Approach to construct all $MPG5$ is describing by two parts: Π and $\underline{\Pi}$ constructions. In order to construct directly all $MPG5$ with n vertices and only those, we will describe in the following some propositions.

6.1 Π construction with n vertices

We have proposing two algorithms : 8 to construct Π_1 graphs and 23 for Π_2.

- Π_1 graph. We have two cases:
  - n is odd. There is no Π_1 with a number of vertices odd.
  - n is even. There exists only one Π_1 graph with n vertices even: use algorithm 8 to built it.
- Π_2 graph. We have using Euler formula applied on planar graph:

$$n = m - f + 2$$

  By this way we can see :

6.2 $\underline{\Pi}$ construction with n vertices

By using main theorem 27, we have just to built a $\underline{\Pi}$ with n vertices and after to apply D operations to construct all other $\underline{\Pi}$ with n vertices.

At the stage we consider two cases:

- $n > 16$ is odd. First we will construct a Π_1 graph with $n-1$ vertices by using algorithm 8. We will apply T such that $k = 4$ for x_k: in all case we have at least edges $[x', x_1]$, $[x', x_k]$ and so current graph is $\underline{\Pi}$ with n vertices.
- $n > 15$ is even. First we will construct a Π_1 graph with n vertices by using algorithm 8. We will apply two operations T and T^{-1} by using lemma 26 in order to give a $\underline{\Pi}$ graph with n vertices.

7 Graph coloring

Four color theorem (briefly $4CT$) have been proved two times in 1977 by K. Appel, W. Haken and J. Koch [12], [13] and in 1996 by N. Robertson, D. Sander, P. Seymour and R. Thomas [20]. Recently, update and precision about the last proof can be found in [22], [18]. Approach to prove $4CT$ are the same in the two success. Thus a manual proof of $4CT$ (i.e. without computer calculation) is an open problem.

Here we have giving manually a small and partial contribution to this problem. We will prove that all Π graph are four colorable.

7.1 Π graph coloring

Theorem 31 *All Π graphs are $four$-$colorable$.*

Proof We will prove four coloration for Π_1 and Π_2.

7.1.1 Π_1 graph coloring

Theorem 32 *All Π_1 graphs are $four$-$colorable$.*

Proof Let x, y the two vertices in X_{sup6}.

- $dg(x) = dg(y)$ is even. Color x by color 0 and y by 2. Bi-color $N(x)$ by color 1 and 2. Bi-color $N(y)$ by 0 and 3.

- $dg(x) = dg(y)$ is odd. Color x by color 0 and y by 2. Bi-color in clockwise order $N(x)$ by color 1 and 2. Use color 3 on only one neighbor of x. Bi-color $N(y)$ in clockwise order by 0 and 3, by starting by 0 such that this first color has a color 3 and 1 as neighbor in $N(x)$. Color the last vertex in $N(y)$ by 1.

- Other case is not possible.

■

7.1.2 Π_2 graph coloring

We recall Brooks theorem:

Theorem 33 *[7] Let $G = (X, E)$ be a connected simple graph with $\Delta(G) = d$. Then G is D-colorable if and only if $G \neq K_{d+1}$.*

Theorem 34 *Every Π_2 graph is $four$ colorable.*

Proof We present some observations about the coloring of G, Q, and K in the algorithm 23.

- The vertex set X' is one-colorable since X' is stable, i.e. there exists a k-coloring of G such that all the vertices of X' use the same color, for example we could color the vertex set X' by 3.

- Suppose $\{X_{inf6} - X'\}$ contains only triangles $\triangle^3$; then K is a simple 3-regular planar graph, and so $K \neq K_4$. By using the theorem of Brooks, K is 3 colorable.

Let G be a Π_2 graph without polygon $\triangle^6$. By the two previous observations, there exists a four coloring of G such that X' is one-colorable (we will suppose by the color 3).

Consider a polygon $\triangle^3$. We use the notation of figure 4.2.2. Suppose that we use the following coloring: $(a, 0)$, $(b, 1)$, $(c, 2)$. Now, in graph G, we replace the polygon $\triangle^3$ by a polygon $\triangle^6$. Without conflicts of coloring, we can color the polygon $\triangle^6$ as follows $(a, 0)$, $(b, 1)$, $(c, 2)$ and $(a_1, 0)$, $(b_1, 1)$, $(c_1, 2)$.

Continue this process, until the graph G will be without polygon $\triangle^3$ (step by step, each polygon $\triangle^3$ has been replaced with a polygon $\triangle^6$).

■

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