MATHS ASSIGNMENT - 01

1. Let $y = x^{1/3}$

where $x = 1000$ and $\Delta x = 5$

$$\frac{dy}{dx} = \frac{1}{3} x^{-2/3}$$

Also,

$$\Delta y = \frac{dy}{dx} \Delta x = \frac{1}{3x^{2/3}} \times 5$$

$$\Delta y = \frac{1 \times 5}{3\ (1000)^{2/3}} = \frac{1}{3} \times \frac{5}{100} = \frac{1}{60}$$

Also,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\frac{1}{60} = f(1000 + 5) - f(1000)$$

$$\frac{1}{60} + 10 = f(1005)$$

$$f(1005) = 10.0167$$

2. $(999)^{1/3}$

Let $y = x^{1/3}$

where $x = 1000$ and $\Delta x = -1$

$$\Delta y = \frac{dy}{dx} \Delta x = \frac{1}{3x^{2/3}} \times 5 =$$

$$= \frac{1}{3} \times \frac{1}{100} \times -1 = -\frac{1}{300}$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\Delta y = f(999) - f(1000)$$

$$\frac{-1}{300} = f(999) - 10$$

$$f(999) = 10 - \frac{1}{300} = 9.996$$

3. $(1.001)^3 + 2(1.001)^{4/3} + 5$

Let

$y = x^3$ where $x = 1$ and $\Delta x = 0.001$

$$\frac{dy}{dx} = 3x^2 \Rightarrow \Delta y = 3x^2 \Delta x$$

$$\Delta y = 3(1)^2 (0.001) = 0.003$$

$$\Delta y = f(x+\Delta x) - f(x)$$

$$0.003 = f(1+0.001) - f(1)$$

$$0.003 = f(1.001) - 1$$

$$(1.001)^3 = 1.003$$

Let

$y = x^{4/3}$ where $x = 1$ and $\Delta x = 0.001$

$$\frac{dy}{dx} = \frac{1}{3} x^{1/3} \Rightarrow \Delta y = \frac{4}{3} x^{1/3} \Delta x$$

$$\Delta y = \frac{1}{3} (1)^{1/3} (0.001) = \frac{0.004}{3}$$

$$(1.001)^{4/3} = \frac{0.004}{3} + 1 = 1.001$$

$$\therefore (1.001)^3 + 2(1.001)^{4/3} + 5 = 1.003 + 2.002 + 5$$

$$= 8.005$$

4. $\sin 60° \, 10'$

$1 \text{ min} = 0.016°$

$10 \text{ min} = 0.16°$

$\sin 60° \, 10' = \sin (60.16)°$ —— (i)

Let $y = \sin\theta$

$y' = \cos\theta \Rightarrow \Delta y = (\cos\theta) \Delta\theta$

In (i),

$\theta = 60°$

$\Delta\theta = 0.16°$

$\Delta y = (\cos 60°)(0.16) = 0.08$

$\Delta y = f(\theta + \Delta\theta) - f(\theta)$

$$0.08 = f(60.16) \cancel{8} - \frac{\sqrt{3}}{2}$$

$$\sin(60.16) = 0.08 + \frac{\sqrt{3}}{2} = \cancel{0.084} \ 0.97$$

5. $\tan 45° 5' 30''$

$1 \text{ min} = 0.016°$

$5 \text{ mins} = 0.08°$

$1 \text{ sec} = 0.0003°$

$30 \text{ secs} = 0.009°$

$$\tan(45 + 0.08 + 0.009)°$$

$$\tan(45.089)°$$

Let ~~to~~ $y = \tan\theta$ where $\theta = 45°$, $\Delta\theta = 0.089$

$$\frac{dy}{d\theta} = \sec^2\theta \Rightarrow \Delta y = \sec^2\theta \, \Delta\theta$$

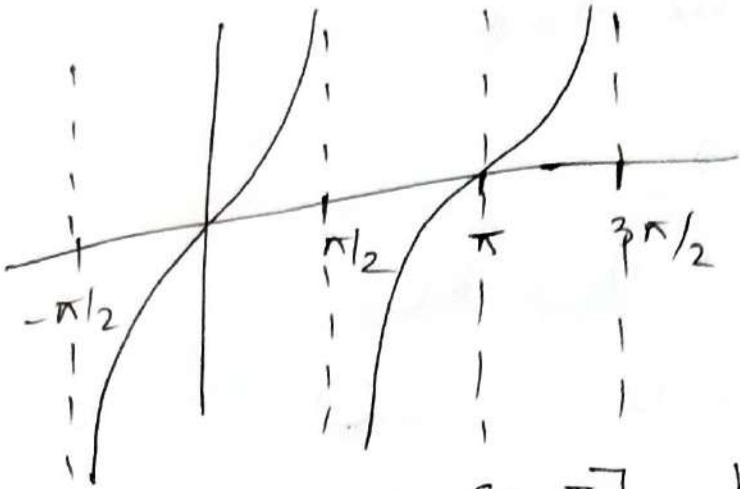
$$\Delta y = (\sec^2 45°)(0.089)$$

$$= (2)(0.089) = 0.178$$

$$\Delta y = \tan(45.089)° - \tan 45°$$

$$\tan(45.089)° = 1 + 0.178 = 1.178$$

6. (i)



In the interval of $[0, \pi]$, $\tan x$ is not continuous

(ii) $f(x) = \lfloor x \rfloor$ where $x \in [-1/2, 3/2]$

$f(x)$ is not continuous in the given interval.

(iii) $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$

$f'(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -1 & 1 \le x \le 2 \end{cases}$

LHD = RHD at $x = 1$

Hence, Rolle's theorem is not applicable here.

7. $f(x) = x^3 + bx^2 + cx$

$f(1) = f(2)$

$1 + b + c = 8 + 4b + 2c$

$3b + c + 7 = 0$ ——— (I)

$f'(4/3) = 0$

$3(4/3)^2 + 2b(4/3) + c = 0$

$\frac{16}{3} + \frac{8b}{3} + c = 0$

$16 + 8b + 3c = 0$ ——— (II)

Solving (I) and (II), we get

$\therefore b = -5$

$\therefore c = 8$

8. According to LMVT,

$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{10-4}{b-a} = \frac{6}{b-a}$

$g'(c) = \frac{g(b) - g(a)}{b-a} = \frac{3-1}{b-a} = \frac{2}{b-a}$

From above 2 eq^n. we observe

$f'(c) = 3g'(c)$

9. Given,

$$e^x \sin x = 1$$

Let a and b be 2 real roots of the above eq^n.

So, $e^a \sin a = 1$ and $e^b \sin b = 1$

Now, consider a function

$$f(x) = e^{-x} - \sin x$$

$f(x)$ is continuous and differentiable for $x \in (-\infty, \infty)$.

Also, $f(a) = e^{-a} - \sin a = e^{-a}(1 - e^a \sin a) = 0$

$f(b) = e^{-b} - \sin b = e^{-b}(1 - e^b \sin b) = 0$

Hence, Rolle's theorem is applicable to $f(x)$ in $x \in [a, b]$.

Now, $f'(x) = -e^{-x} - \cos x$

According to Rolle's theorem, there exists c such that $f'(c) = 0$ where $c \in [a, b]$

$$-e^{-c} - \cos c = 0 \Rightarrow -e^{-c} = \cos c$$

$-\frac{1}{e^c} = \cos c \Rightarrow e^c \cos c = -1 \Rightarrow e^c \cos c + 1 = 0$

c is a root of equation such that

~~$c \in [a, b]$~~. $c \in (a, b)$

10. $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

$f''(x) = \dfrac{f'(b) - f'(a)}{b - a}$

$g'(x) = \dfrac{g(b) - g(a)}{b - a}$

$g''(c) = \dfrac{g'(b) - g'(a)}{b - a}$

Using LMVT:

11. $a - 1 < a - \theta < a$

$f'(a - \theta) = \dfrac{f(a) - f(a-1)}{a - (a-1)} = \dfrac{f(a) - f(a-1)}{1}$

$a < a + \theta < a + 1$

$f'(a + \theta) = \dfrac{f(a+1) - f(a)}{a + 1 - a} = f(a+1) - f(a)$

$$f'(a+0) - f'(a-0) = f(a+1) - f(a) - f(a) + f(a-1)$$

$$f'(a+0) - f'(a-0) = f(a-1) - 2f(a) + f(a+1)$$

12 i) e^x is continuous and differentiable.

Applying LMVT for $(0, x)$ on e^x

$$f'(c) = \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x}$$

$$e^c = \frac{e^x - 1}{x}$$

$$e^c > 1 \Rightarrow \frac{e^x - 1}{x} > 1$$

$$\boxed{e^x > 1 + x}$$

(ii) $\ln(1+x)$ is continuous and differentiable

Applying LMVT for $(0, x)$ on $\ln(1+x)$

$$f'(c) = \frac{\ln(1+x) - \ln(1)}{x - 0} = \frac{\ln(1+x)}{1+x}$$

$$\frac{1}{1+c} = \frac{\ln(1+x)}{x}$$

$$1 < \frac{1}{\sqrt{1-x^2}} < \infty$$

$$\frac{1}{\sqrt{1-x^2}} > 1$$

$$\frac{\sin^{-1}x}{x} > 1 \Rightarrow \sin^{-1}x > x \quad \text{—— (1)}$$

Applying LMVT for $(x, 1)$

$$\frac{1}{\sqrt{1-x^2}} = \frac{\sin^{-1}x - \pi/2}{x-1}$$

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13. Applying LMVT on $f(x)$:

$x \in (-a, 0)$

$$f'(c) = \frac{f(0) - f(-a)}{0-(-a)} = \frac{f(0)+a}{a}$$

$$-1 \le f'(c) \le 1 \Rightarrow -1 \le \frac{f(0)+a}{a} \le 1$$

$$-a \le f(0) + a \le a$$

$$-2a \le f(0) \le 0 \quad \text{—— (i)}$$

for $x \in [0, a]$

$$f'(c) = \frac{f(a) - f(0)}{a} = \frac{a - f(0)}{a}$$

$$-1 \leq \frac{a - f(0)}{a} \leq 1 \Rightarrow -a \leq a - f(0) \leq a$$

$$-2a \leq -f(0) \leq 0$$

$$0 \leq f(0) \leq 2a \quad \text{—— (ii)}$$

Combining (i) & (ii), we get

$$f(0) = 0$$

1b (iii) $f(x) = \sin^{-1} x$

Applying the MVT to $f(x)$ on the interval $[0, x]$ gives a number c with $0 < c < x$ such that

$$\frac{\sin^{-1} x - \sin^{-1} 0}{x - 0} = \frac{1}{\sqrt{1 - c^2}}$$

$$\sin^{-1} x = \frac{x}{\sqrt{1 - c^2}}$$

Since $x > c$ and $x > 0$ i.e. $0 < c < x$

we have

$\cancel{\frac{x}{1+x^2}}$

$$x < \frac{x}{\sqrt{1-c^2}} < \frac{x}{\sqrt{1-x^2}}$$

$$x < \sin^{-1}x < \frac{x}{\sqrt{1-x^2}}$$

14. Given $F(x)$ and $G(x)$ satisfy the hypothesis of LMVT.

Let $g(x) = \frac{1}{G(x)}$

$$F'(c) = \frac{F(b) - F(a)}{b-a} \quad \text{where } c \in (a,b) \quad \text{——— (1)}$$

$$F(b) - F(a) = (b-a)\, F'(c)$$

$$g'(x) = \frac{-1}{G^2(x)} \cdot G'(x)$$

$$g'(c) = \frac{g(b) - g(a)}{b-a}$$

$$\frac{-1}{G^2(c)} \cdot G'(c) = \frac{\frac{1}{G(b)} - \frac{1}{G(a)}}{b-a}$$

$$= \frac{G(a) - G(b)}{(b-a)\, G(b)\, G(a)}$$

$$G(b) - G(a) = (b-a)\, G'(c) \left[\frac{G(a) \cdot G(b)}{G^2(c)} \right]$$

——— (2)

Eq^n (1) $\div$ Eq^n (2)

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} \left[\frac{G^2(c)}{G(a)\,G(b)} \right]$$

where $c \in (a,b)$.

15. $\lim_{x \to 1} \frac{x-1}{x^n - 1} = \lim_{x \to 1} \frac{x-1}{x^n - 1^n}$

$= \dfrac{1}{\lim_{x \to 1} \frac{x^n - 1^n}{x-1}} = \dfrac{1}{n}$

16. $\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} \Rightarrow 0/0$ form

$$\lim_{x \to 0} \frac{e^x + 2\sin x - e^{-x}}{x \cos x + \sin x}$$

Series expansion: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots$

$\sin x = x - \frac{x^3}{3!} \dots$

$$\cos x = 1 - \frac{x^2}{2} \cdots$$

$$\lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} - 2\left(1 - \frac{x^2}{2}\right) + \left(1 - x + \frac{x^2}{2}\right)}{x^2}$$

$$= 2$$

17. $$\lim_{x\to\pi/2} \frac{\ln(\sin x)}{(\pi - 2x)^2} = \lim_{h\to 0} \frac{\ln(\sin(\pi/2 + h))}{\left(\pi - 2\left(\frac{\pi}{2} + h\right)\right)^2}$$

$$\{x = \pi/2 + h\}$$

$$\lim_{h\to 0} \frac{\ln(\cos(h))}{4h^2} = \lim_{h\to 0} \frac{-\sin h}{\cos h\,(8h)}$$

$$\lim_{h\to 0} \frac{-\tanh}{8h} = \lim_{h\to 0} \frac{(-\sec^2 h)}{8} = \frac{-1}{8}$$

18. $$\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{\frac{\sin^2 x}{x^2 \cdot x^2} - \frac{1}{x^2}}{\frac{\sin^2 x}{x^2}}$$

$$\lim_{x\to 0} \frac{\frac{1}{x^2} - \frac{1}{x^2}}{1} = 0$$

19. $\lim\limits_{x\to 1} (1-x)\tan\left(\pi/_2 - x\right)$

$\lim\limits_{x\to 1} (1-x)\cot x = 0$

19. $\lim\limits_{x\to 1} (1-x)\tan\left(\pi/_2\, x\right)$

$\lim\limits_{h\to 0} \left(1-(1+h)\right)\tan\left(\pi/_2\,(1+h)\right)$

$\lim\limits_{h\to 0} (-h)\cdot\left(-\cot \pi/_2 h\right)$

$$\lim_{h\to 0} \cancel{h}\ \frac{\cos(\pi/2\, h)}{\dfrac{\sin(\pi/_2 h)}{h(\pi/2)} \times \pi/2} = \frac{2}{\pi}$$

20. $\cancel{\lim\limits_{h\to 0}}\ \lim\limits_{x\to 2}\left[\dfrac{x-1}{x-2} - \dfrac{1}{\ln(x-1)}\right]$

$\lim\limits_{h\to 0}\left[\dfrac{2+h-1}{2+h-2} - \dfrac{1}{\ln(2+h-1)}\right]$

$\lim\limits_{h\to 0}\left[\dfrac{h+1}{h} - \dfrac{1}{\ln(1+h)}\right]$

$\lim\limits_{h\to 0}\left[\dfrac{h+1}{h}\right] - \lim\limits_{h\to 0}\dfrac{1}{\ln(1+h)}$

$= 1/_2$

21. $\lim_{x \to 1} x^{1/x-1}$

$e^{\lim_{x \to 1} (x-1)/x-1} = e$

22. $\lim_{x \to \pi/2} (\sin x)^{\tan x}$

$\lim_{x \to \pi/2} (\sin -1) \frac{\sin x}{\cos x} = e^{\lim_{x \to \pi/2} \frac{2\sin x \cos x - \cos x}{-\sin x}}$

$e^{\cancel{E}\ 0}$

$= e^{0} = 1$

23. $\lim_{x \to 0} \frac{e^{b(x)} - 1}{b(x)} = \lim_{x \to 0} \frac{e^{b(x)} \cdot \cancel{b'(x)}}{\cancel{b'(x)}} = e^{0} = 1$

24. $\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \frac{2}{\infty} = 0$

25. $\lim_{x \to \infty} \left[1 + b(x)\right]^{\frac{1}{b(x)}}$

$= e^{\lim_{x \to \infty} (\cancel{b(x)} \times \frac{1}{\cancel{b(x)}})} = e$

26. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e^{\lim_{x \to \infty} \frac{1}{\cancel{x}} \times \cancel{x}} = e$

27. $\lim_{x \to \infty} \sqrt{\dfrac{x + \sin x}{x - \cos^2 x}}$

$\lim_{y \to 0} \sqrt{\dfrac{\frac{1}{y} + \sin\left(\frac{1}{y}\right)}{\frac{1}{y} - \cos^2\left(\frac{1}{y}\right)}}$ $\qquad y = \frac{1}{x}$

$\lim_{y \to 0} \sqrt{\dfrac{1 + y\sin\left(\frac{1}{y}\right)}{1 - y\cos^2\left(\frac{1}{y}\right)}} = 1$

28. $\lim_{x \to \infty} \left(\dfrac{x+4}{x+2}\right)^{x+3}$

$\lim_{h \to 0} \left(\dfrac{1+4h}{1+2h}\right)^{\frac{1+3h}{h}} = e^{\lim_{h \to 0} \left(\frac{1+3h}{h}\right)\left(\frac{1+4h-1-2h}{1+2h}\right)}$

$= e^{\lim_{h \to 0} \left(\frac{1+3h}{1+2h}\right) 2} = e^3$

29. $f(x) = \ln(2+x) - \dfrac{2x}{2+x}$

$f'(x) = \dfrac{1}{2+x} - \left[\dfrac{(2+x)2 - 2x(1)}{(2+x)^2}\right]$

$= \dfrac{1}{2+x} - \dfrac{4}{(2+x)^2}$

$= \dfrac{2+x-4}{(2+x)^2} = \dfrac{x-2}{(2+x)^2}$

Critical point : $x = 2$

decreasing in $(-\infty, 2)$

increasing in $[2, \infty)$

30. $f(x) = x|x|$

~~$f(x)$ is increasing from $[0, \infty)$ and decreasing from $(-\infty, 0)$~~

30. $f(x) = x|x|$

$f(x)$ is always increasing from $(-\infty, \infty)$.

31. $f(x) = \tan^{-1}x + x$

$$f'(x) = \frac{1}{1+x^2} + 1$$

$$f'(x) > 0$$

The function is always increasing.

32. $f(x) = \sin x + |\sin x|$

$$f(x) = \begin{cases} 2\sin x & 0 < x \leq \pi \\ 0 & \pi < x \leq 2\pi \end{cases}$$

$$f'(x) = \begin{cases} 2\cos x & 0 < x \leq \pi \\ 0 & \pi < x \leq 2\pi \end{cases}$$

$f(x)$ is increasing from $[0, \pi/2]$ and decreasing from $[\pi/2, \pi]$

33. $f(x) = \ln(\sin x)$

$f'(x) = \cot x$

$f'(x) > 0$ for $x \in (0, \pi/2]$

$f'(x) < 0$ for $x \in (\pi/2, \pi)$

34. $f(x) = \dfrac{\ln x}{x}$

$$f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$\cancel{1 - \ln x = 0} \Rightarrow \cancel{\ln x = 1}$

$\cancel{x = 0}$

$$\cancel{f''(x) = \frac{2\ln x - 3}{x^3}}$$

$f'(x) > 0 \Rightarrow 1 - \ln x > 0$

$1 > \ln x \Rightarrow \ln_e x < 1$

$x < e^1$

$f(x)$ is increasing from $(-\infty, e)$ and decreasing from (e, ∞)

35. $f(x) = \sin x(1+\cos x) = \sin x + \frac{\sin 2x}{2}$

$f'(x) = \cos x + \frac{2\cos 2x}{2} = \cos x + \cos 2x$

$= \cos x + 2\cos^2 x - 1 = (2\cos x - 1)(\cos x + 1)$

Increasing :

$f'(x) > 0 \Rightarrow$

$2\cos x > 1$ and $\cos x > -1$

$\cos x > 1/2$ and $\cos x > -1$

$x \in (0, \pi/3)$ and $x \in (0, \pi/2)$

So, $x \in (0, \pi/3)$

Decreasing :

$f'(x) < 0$

$2\cos x - 1 < 0$ and $\cos x + 1 > 0$

$\cos x < 1/2$ and $\cos x > -1$

$x \in (\pi/3, \pi/2)$ and $x \in (0, \pi/2)$

So, $x \in (\pi/3, \pi/2)$

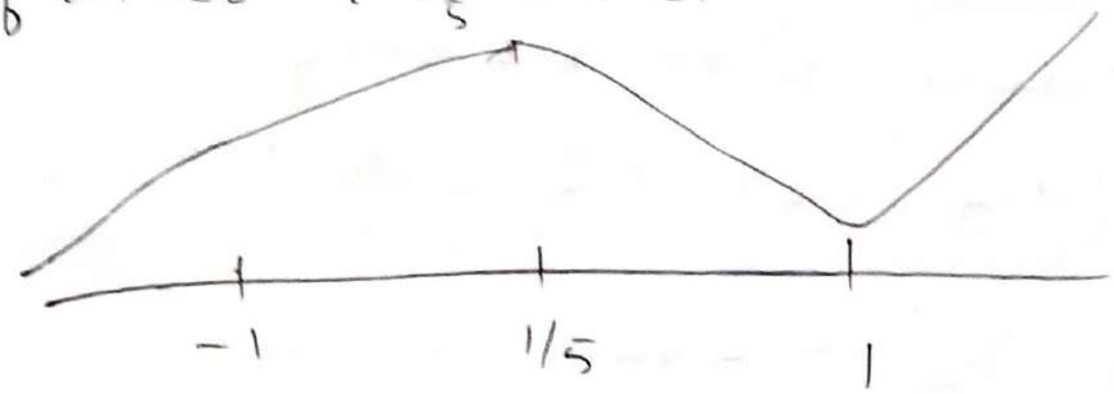
38. $f(x) = (x-1)^2 (x+1)^3$

$f'(x) = (x+1)^2 (x-1)(5x-1)$

critical points : $x = -1$, $x = 1/5$, $x = 1$

$f'(x) > 0$: $x < -1$ or $-1 < x < \frac{1}{5}$ or $x > 1$

$f'(x) < 0$: $\frac{1}{5} < x < 1$



Maximum at $x = 1/5$

Maximum value at $f(1/5) = \frac{3456}{3125}$

Minimum at $x = 1$ and the value is 0

Extreme values $\left(0, \frac{3456}{3125}\right)$

39. $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f'(x) = 0$

$\tan x = 1$

$x = \pi/4$ ~~$2n\pi$~~ $, 5\pi/4$

Maxima is at $x = \pi/4$

Minima is at $x = 5\pi/4$

$f(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \sqrt{2}$

$f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

40. $f(x) = x^{1/x}$

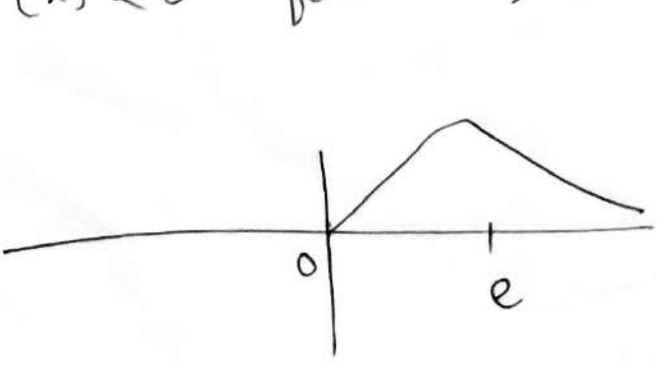
$f'(x) = x^{1/x^{-2}}(1 - \ln x)$

$f'(x) = 0$

$x = 0, e$ are the critical points

$f'(x) > 0$ for $0 < x < e$

$f'(x) < 0$ for $x > e$



45. $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{b^2 + 3b + 2} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Continuity at $x = 1$,

$$LHL = RHL$$

$$-1 + \frac{(b^3 - b^2 + b - 1)}{b^2 + 3b + 2} = -1$$

$$b^3 - b^2 + b - 1 = 0$$

$$b^2(b-1) + (b-1) = 0$$

$$(b^2+1)(b-1) = 0$$

$$b = 1$$

$$f(x) = \begin{cases} -x^3 + 1/6 & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$$

$$f'(x) = \begin{cases} -3x^2 & 0 \le x < 1 \\ 2 & 1 \le x \le 3 \end{cases}$$

for $x \in [0, 1)$, $f'(x) < 0$ i.e decreasing

$x \in [1, 3]$ $f(x)$ is increasing

$f_{min}(1) = 2(1) - 3 = -1$

16. Taylor's polynomial

$f(x) = \sqrt{x} \quad n = 3, \; a = 1, \; 1 \le x \le 1.5$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = \frac{1}{2} \times \left(\frac{-1}{2}\right) x^{-3/2}$

$= \frac{-1}{4} x^{-3/2}$

$f'''(x) = \frac{-1}{4} \times \left(\frac{-3}{2}\right) \times x^{-5/2}$

$f(1) = 1$

$f'(1) = 1/2$

$f''(1) = -1/4$

$f'''(1) = 3/8$

$f(x-a+a) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$

for $(n=3)$

$\Rightarrow 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

error : $R_3 = \frac{(x-a)^4}{4!} \times f^{IV}(z)$

$$R_3 = \left| \frac{(x-1)^4}{4!} \left(\frac{-15}{16}\right) \left(z^{7/2}\right) \right|$$

$$R_3 = \left| \frac{(1.5-1)^4}{4!} \left(\frac{-15}{16}\right) \right| = \left| \frac{(-1.5-1)^4}{4!} \times \left(\frac{-15}{16}\right) \right|$$

(error max, z min
z = 1)

$$R_3 = \left(\frac{1}{2}\right)^4 \times \frac{1}{4!} \times \frac{15}{16}$$

$$= \frac{15}{6144} = 0.002441$$

47. $f(x) = e^{-x^2}$, $n = 3$, $a = 0$, $-1 \leq x \leq 1$

$f'(x) = e^{-x^2}(-2x)$, $f(0) = 1$

$f''(x) = -2x e^{-x^2}$ $\quad f'(0) = 0$

$= -2\left(e^{-x^2} + x e^{-x^2}(-2x)\right)$

$= -2 e^{-x^2} (2x)(-4x^2 + 6)$

$f'''(x) = e^{-x^2}(2x)(-4x^2+6)$ $\qquad f''(0) = -2$

$f'''(0) = 0$

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$$f(x) = f(a) + \frac{f'(a)x}{1!} + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f^{iv}(a)x^4}{4!}$$

for $n = 3$

$$f(a) + \frac{f'(a)x}{1!} + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!}$$

Substituting the values,

$$e^{-x^2} = 1 - x^2$$

error : $R_3 = \frac{x^4 f^{iv}(z)}{4!}$

$$f^{iv}(x) = 12e^{-x^2} + 12xe^{-x^2}(-2x) - 8x^3e^{-x^2}(-2x) - 8e^{-x^2}(3x^2)$$

$$f^{iv}(c) = 12e^{-c^2} - 48c^2e^{-c^2} + 16c^4e^{-c^2}$$

$$R_3 = \frac{1 \times 12}{4!} = \frac{12}{24} = 0.5$$

48.

$f(x) = x\sin x$ $\qquad f(0) = 0$

$f'(x) = \sin x + x\cos x$ $\qquad f'(0) = 0$

$f''(x) = \cos x + \cos x - x\sin x$ $\qquad f''(0) = 2$

$= 2\cos x - x\sin x$ $\qquad f'''(0) = 0$

$f'''(x) = -2\sin x - \sin x - x\cos x$ $\qquad f^{iv}(0) = -4$

$= -3\sin x - x\cos x$

$f^{iv}(x) = -4\cos x + x\sin x$

$f^{v}(x) = 4\sin x + \sin x + x\cos x = 5\sin x + x\cos x$

$$x\sin x = f(a) + \frac{f'(a)x}{1!} + \frac{f''(a)\cdot x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f^{iv}(a)x^4}{4!}$$

$$x\sin x = 0 + 0 + \frac{2x^2}{2!} + 0 - \frac{4x^4}{4!}$$

$$x\sin x = x^2 - \frac{x^4}{6}$$

error $= R_4 = \left| \frac{x^5}{5!} \times f^{v}(z) \right|$

$n = 4$

$$= \left| \frac{x^5}{5!} (5\sin z + z\cos z \right|$$

$$= \left| \frac{1}{2 \times 3 \times 6} \right| \qquad |\sin z + z \cos z|$$

19. $f(x) = x^2 e^{-x}$

$f'(x) = x^2 e^{-x}(-1) + e^{-x}(2x)$

$f^{iv}(x) = -7 e^{-x} x + 12 e^{-x}$

$$x^2 e^{-x} = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a)$$

$$x^2 e^{-x} = e^{-x} + (x-1) e^{-1} + \frac{(x-1)^2 (-e^{-1})}{2!} + \frac{(x-1)^3 (-e)^{-1}}{3!} + \frac{(x-1)^4 \, 5 e^{-1}}{4!}$$

$$x^2 e^{-x} = e^{-1} \left[1 + (x-1) - \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 + \frac{5}{4!}(x-1)^4 \right]$$

Error $f^{v}(x) = -12e^{-x} + 7xe^{-x} - 7e^{-x}$

$= -19e^{-x} + 7xe^{-x} = \frac{7x-19}{e^{x}}$

$n = 4$

$\text{Error} = \frac{(x-1)^5}{5!} \times \left| \frac{7c-19}{e^{c}} \right|$

$= \frac{(1.5-1)^5}{5!} \left| \frac{7\times 0.5 - 19}{e^{0.5}} \right| = 0.004e^{-0.5}$

50. $f(x) = \frac{1}{1-x}$, $n=3$, $a=0$, $0 \le x \le 0.25$

$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$

$f'''(x) = \frac{6}{(1-x)^4}$

$\frac{1}{1-x} = f(a) + \frac{f'(a)x}{1!} + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!}$

$\frac{1}{1-x} = 1 + 1(x) + \frac{x^2}{2!} \times 2 + \frac{6 \times x^3}{3!}$

$\frac{1}{1-x} = 1 + x + x^2 + x^3$

Error:

$n = 3$

$$R_3 = \left| \frac{x^4}{4!} \times \frac{24}{(1-c)^5} \right|$$

$x_{max} = 0.25$

$1 - c = \min$

$c = 0.25$

$$R_3 = \frac{4}{243}$$

51. $f(x) = \sin 3x$

$f'(x) = \cos 3x\,(3)$

$f''(x) = -\sin 3x \times 3^2$

$f'''(x) = -\cos 3x \times 3^3$

$f^{iv}(x) = 3^4 \sin 3x$

$$f(x) = \sin 3x = 0 + \frac{3x}{1!} + 0 - \frac{(3x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(3x)^7}{7!} + \ldots$$

52. $f(x) = x^2 \ln x, \quad a = 1$

$f'(x) = x^2 \times \frac{1}{x} + \ln x \times 2x$

$= x(1 + 2\ln x)$

$f''(x) = 1 + 2x \times \frac{1}{x} + 2\ln x = 3 + 2\ln x$

$f^{(n)}(x) = \frac{[illegible]}{x}$

$f^{(n)}(x) = \frac{[illegible]}{x^2}$

$f^{(n)}(x) = [illegible]$

$[illegible] = (x-1) - \frac{1}{2!}(x-1)^2 + 2\sum_{m=3}^{\infty}\frac{(m-3)![illegible]}{m!}(x-1)^m$

[illegible] $f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{1+x^2}$

$f''(x) = \frac{-(2x)}{(1+x^2)^2}$

$f'''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$

$f^{iv}(x) = \frac{-24x(x^2-1)}{(x^2+1)^4}$

$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$+ \dots$

$$= 0 + \frac{x}{1!} + 0 + \frac{2(-1)x^3}{3!}$$

$$= x - \frac{x^3}{3}$$

So $n = 2$

So

$$|R_n(c)|_{max} = \frac{x^4}{4!} f^5(c) < 0.005$$

at $x = c$

$$c^4(c + c^3) < 0.005$$

$$c \approx 0.33$$

6! $\quad y = a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \dots$

$$y = a^x$$
$$y'(x) = a^x \ln a$$
$$y''(x) = a^x (\ln a)^2$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$= 1 + x \ln a + \frac{x^2}{2!} (\ln a)^2 + \frac{x^3}{3!} \ln a^3$$

3. $y = f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$ $\quad f(0) = 0$

$f'(0) = 1+1$
$= 2$

$f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}$ $\quad f''(x) = 0$

$f'''(x) = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3}$ $\quad f'''(x) = 4$

So,

$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots$

$= \frac{2x}{1!} + \frac{2x^3}{3} + \frac{2x^5}{5} \cdots$

$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots\right], -1<x<1$

4. $y = \left[\frac{x-1}{x+1}\right]$

$x = \left[\frac{1+y}{1-y}\right] \quad -1<y<1$

$\ln x = \ln(1+y) - \ln(1-y) = f(y)$

$f'(y) = \frac{1}{1+y} + \frac{1}{1-y}$

$f''(y) = \frac{[illegible]}{(1+y)^2} + \frac{[illegible]}{(1-y)^2}$

$f'''(y) = \frac{2}{(1+y)^3} - \frac{2}{(1-y)^3}$

So,

$f(y) = \ln x$

$= f(0) + \frac{f'(0)y}{1!} + \frac{f''(0)y^2}{2!} + \frac{f'''(0)y^3}{3!}$

$= 0 + 2\left[\frac{x-1}{x+1}\right] + 0 + \frac{4}{3!}\left[\frac{x-1}{x+1}\right]^3 + \ldots$

$\ln(x) = 2\left[\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \ldots\right]$