* Note that for $|x| > 1$, $\lim_{n\to\infty} |R_n(x)| = \infty$

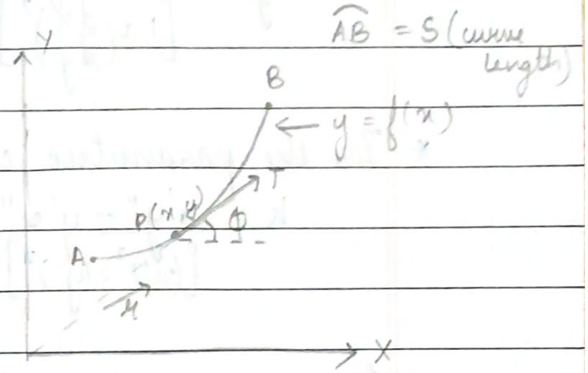
* $1 + x = y$ or $x = y - 1$

$$\log y = (y-1) - \frac{1}{2}(y-1)^2 + \dots + \frac{(-1)^{n-1}}{n}(y-1)^n + \dots \quad ; 0 < y \leq 2$$

<u>Curvature</u>

The unit tangent vector at $P(x, y)$:

$$T = \frac{d\vec{r}}{ds} \quad ; \quad |T| = 1$$



$\frac{dy}{dx} = \tan\phi$, is the slope of the curve at P

$\phi \leftarrow$ angle made by the tangent line w.r.t. x-axis.

$$\phi = \tan^{-1}\left(\frac{dy}{dx}\right)$$

The vector representation of the curve $y = f(x)$ is $\vec{r} = x\hat{i} + y\hat{j}$.

The rate at which the tangent line T turns as it moves along the curve by measuring the change in ϕ, the direction angle or slope angle that T makes with $\hat{i}$ (x-axis)

At each point, the absolute value of $\frac{d\phi}{ds}$, measured in radians per unit of length along the curve, is called curvature of the curve denoted as kappa, k.

$$k = \left|\frac{d\phi}{ds}\right|$$

$$\frac{d\phi}{ds} = \frac{d\phi}{dx}\frac{dx}{ds}$$

$$\phi = \tan^{-1}\left(\frac{dy}{dx}\right)$$

$$\frac{d\phi}{dx} = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \times \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$$k = \left|\frac{d\phi}{ds}\right| = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$ for the curve $y = f(x)$

* For the curve equation $x = g(y)$,

$$k = \frac{\left|d^2x/dy^2\right|}{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}$$

* For the parametric curve $x = f(t)$, $y = g(t)$,

$$k = \frac{x'y'' - y'x''}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

In space, there is ~~not~~ no natural choice of angle ϕ

* The generalised vector curvature formula is:

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$ where $\vec{v} \leftarrow$ velocity vector of the curve
$\vec{a} \leftarrow$ acceleration vector of the curve.

Examples

1. The curvature of a straight line.

On a straight line, ϕ has a constant value.
So, $\frac{d\phi}{d}$

2. Curvature of a circle of radius 'a'.

The parametric equation of a circle is:

$x(t) = a\cos t$, $y(t) = a\sin t$

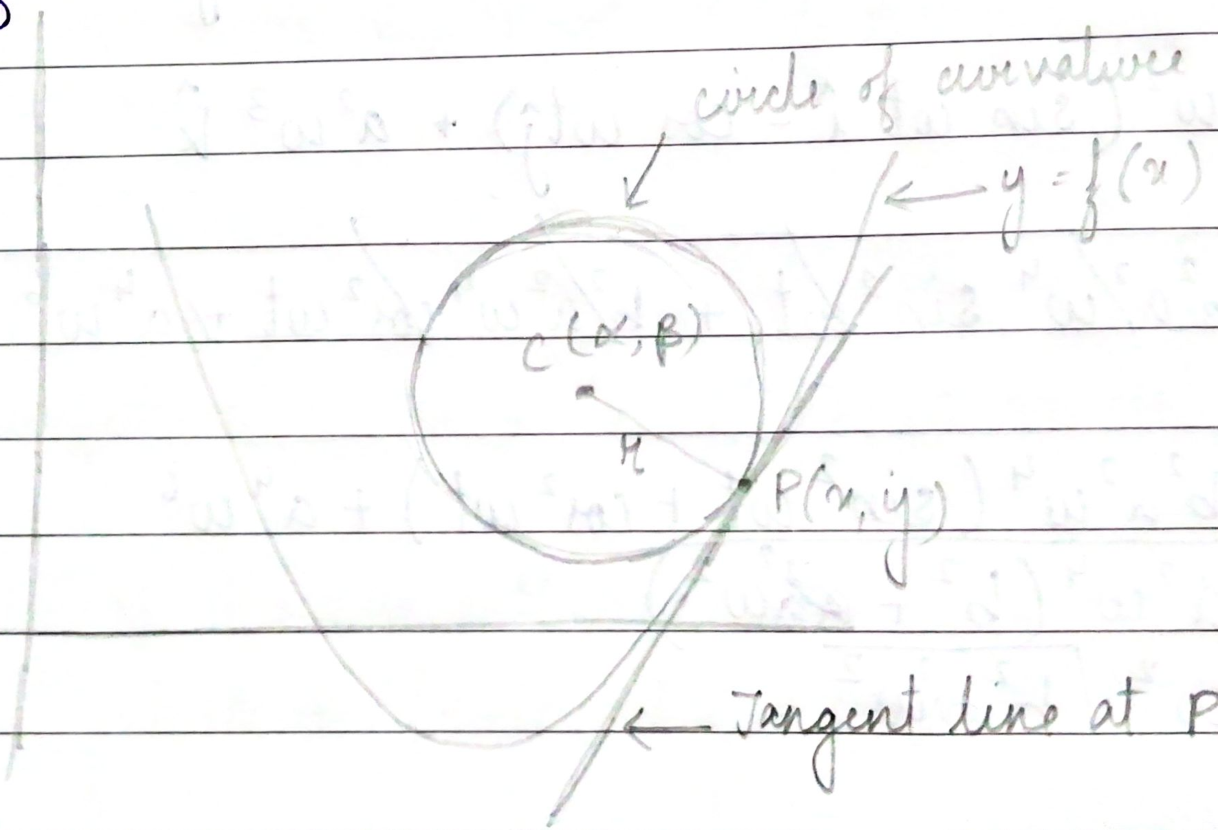
$x' = -a\sin t$, $y' = a\cos t$

$x'' = -a\cos t$, $y'' = -a\sin t$

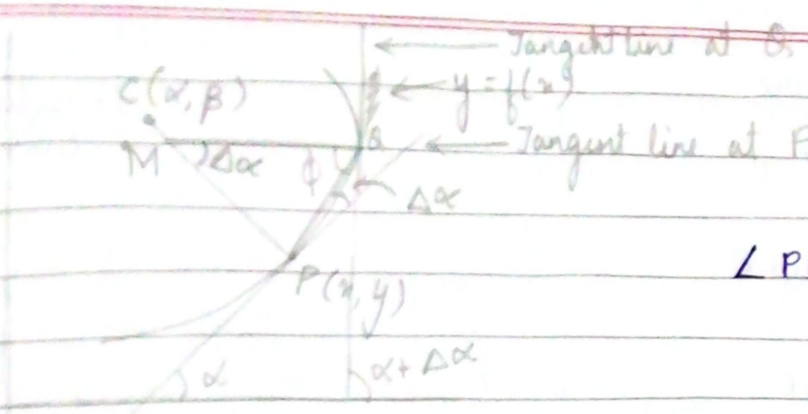
$\kappa =$

$k =$

Radius of curvature (ρ)



r = radius of curvature ; C = centre of curvature.



$\angle PMQ = \Delta\alpha$

In ΔPQM, $\angle PQM = \phi$

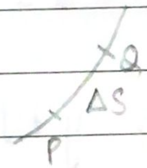
In ΔPQM,

$$\frac{\overline{PM}}{\sin\phi} = \frac{\overline{PQ}}{\sin\Delta\alpha} \qquad [\because \overline{PQ} = \text{chord length}]$$

$$\Rightarrow PM = \overline{PQ}\left(\frac{\sin\phi}{\sin\Delta\alpha}\right)$$

$$= \frac{\overline{PQ}}{\widehat{PQ}} \cdot \frac{\widehat{PQ}}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\sin\Delta\alpha} \cdot \sin\phi \qquad [\because \widehat{PQ} = \text{arc length}]$$

$$= \frac{\overline{PQ}}{\widehat{PQ}} \cdot \frac{\Delta S}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\sin\Delta\alpha} \cdot \sin\phi \qquad [\widehat{PQ} \simeq \Delta S]$$



As $Q \to P$, $\phi \to \pi/2$, $\Delta\alpha \to 0$

$$\lim_{Q\to P} \frac{\overline{PQ}}{\widehat{PQ}} = 1, \quad \lim_{\Delta\alpha\to 0} \frac{\Delta S}{\Delta\alpha} = \frac{ds}{d\alpha}, \quad \lim_{\Delta\alpha\to 0} \frac{\Delta\alpha}{\sin\Delta\alpha} = 1$$

When $Q \to P$, Q and P become a point.
So $\overline{PQ} = \widehat{PQ}$

$$\lim_{Q\to P} PM = \lim_{Q\to P}\left(\frac{\overline{PQ}}{\widehat{PQ}} \cdot \frac{\Delta S}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\sin\Delta\alpha} \cdot \sin\phi\right)$$

$$\Rightarrow PC = 1 \times \frac{ds}{d\alpha} \times 1 \times 1$$

$$\Rightarrow \rho = \frac{ds}{d\alpha} = \frac{1}{(d\alpha/ds)} = \frac{1}{K}$$

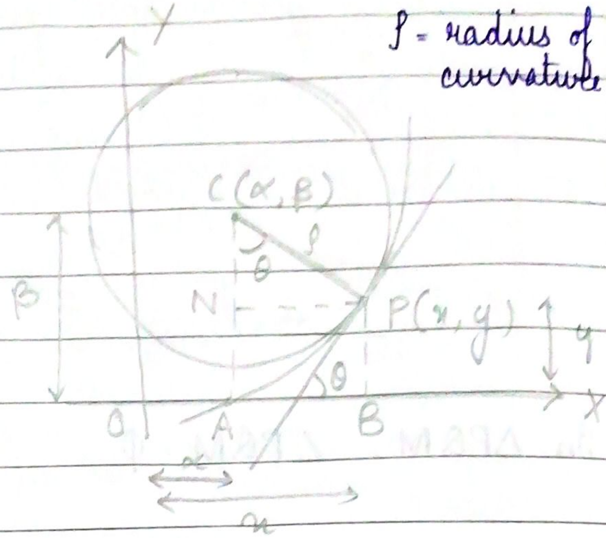
$$\boxed{\rho = \frac{1}{k}} \longrightarrow \text{Radius of curvature}$$

k = curvature

Centre of curvature

ρ = radius of curvature

$\alpha = OA = OB - AB$
$= x - PN$
$= x - \rho \sin\theta$



$\beta = AC = AN + NC$
$= y + \rho\cos\theta$

We know that,

$y' = \frac{dy}{dx} = \tan\theta$, $\sin\theta = \frac{y'}{\sqrt{1+(y')^2}}$, $\cos\theta = \frac{1}{\sqrt{1+(y')^2}}$

$\alpha = x - \rho\sin\theta$

$= x - \frac{[1+(y')^2]^{3/2}}{y''} \times \frac{y'}{\sqrt{1+(y')^2}}$

$= x - \frac{y'}{y''}[1+(y')^2]$

$\beta = y + \rho\cos\theta$

$= y + \frac{[1+(y')^2]^{3/2}}{y''} \times \frac{1}{\sqrt{1+(y')^2}}$

$= y + \frac{1}{y''}[1+(y')^2]$

Circle of curvature is : $(x-\alpha)^2 + (y-\beta)^2 = \rho^2$

Examples

1. Find the circle of curvature of the curve $y = e^x$ at $(0, 1)$.

~~k~~ $y = e^x$

$y' = e^x$, $y'|_{x=0} = \cancel{e}\,1$

$y'' = e^x$, $y''|_{x=0} = 1$

$$k = \frac{y''}{[1+(y')^2]^{3/2}} = \frac{1}{(1+1^2)^{3/2}} = \frac{1}{2\sqrt{2}}$$

$$\rho = \frac{1}{k} = 2\sqrt{2}$$

$$\alpha = x - \frac{y'}{y''}(1+y'^2) = 0 - \frac{1}{1}(1+1^2) = -2$$

$$\beta = y + \frac{1}{y''}(1+y'^2) = 1 + \frac{1}{1}(1+1^2) = 3$$

Centre of curvature is $(-2, 3)$

Circle of curvature is : $(x+2)^2 + (y-3)^2 = (2\sqrt{2})^2$

2. Find the circle of curvature for the curve $y = e^{2x}$ at $P(0,1)$.

$y = e^{2x}$

$y' = 2e^{2x}$, $y'|_{x=0} = 2$

$y'' = 4e^{2x}$, $y''|_{x=0} = 4$

$$k = \frac{y''}{[1+(y')^2]^{3/2}} = \frac{4}{(1+4)^{3/2}} = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}}$$

$$\rho = \frac{1}{k} = \frac{5\sqrt{5}}{4}$$

$$\alpha = x - \frac{y'}{y''}(1+y'^2) = 0 - \frac{2}{4}(1+4) = -\frac{5}{2}$$

$$\beta = y + \frac{1}{y''}(1+y'^2) = 1 + \frac{1}{4}(1+4) = 1 + \frac{5}{4} = \frac{9}{4}$$

Centre of curvature is $\left(-\frac{5}{2}, \frac{9}{4}\right)$

Circle of curvature is : $\left(x+\frac{5}{2}\right)^2 + \left(y-\frac{9}{4}\right)^2 = \left(\frac{5\sqrt{5}}{4}\right)^2$

3. Find the circle of curvature of the ~~curve~~ cycloid $x = a(t - \sin t)$ and $y = a(1 - \cos t)$ at $t = \pi$

$x' = \frac{dx}{dt} = a(1 - \cos t)$, $y' = \frac{dy}{dt} = a(0 + \sin t) = a \sin t$

$\frac{dy}{dx}$ $y' = \frac{dy}{dx} = \frac{\cancel{a} \sin t}{\cancel{a}(1 - \cos t)}$, $y'|_{t=\pi} = \frac{a \sin \pi}{\cancel{a}(1 - \cos \pi)} = 0$

$y'' = \frac{\cancel{2} \sin t/2 \cos t/2}{\cancel{2} \cos^2 t/2}$, $y''|_{t=\pi} = \tan \frac{\pi}{2} = \infty$

$= \tan \frac{t}{2}$

$k = \frac{y''}{[1 + (y')^2]^{3/2}} = 0$

$x'' = a(0 + \sin t) = a \sin t$, $y'' = -a \cos t$

$x'|_{t=\pi} = a(1+1) = 2a$, $y'|_{t=\pi} = 0$

$x''|_{t=\pi} = 0$, $y''|_{t=\pi} = -a$

$$\rho = \frac{1}{k} = \frac{[(x')^2 + (y')^2]^{3/2}}{|x'y'' - y'x''|}$$

$$= \frac{[(2a)^2 + 0]^{3/2}}{|-2a^2 - 0|}$$

$$= \frac{(2a)^3}{2a^2}$$

$$= \frac{8a^3}{2a^2}$$

$$= 4a$$

$x|_{t=\pi} = a(\pi - 0) = a\pi$, $y|_{t=\pi} = a(1 + 1) = 2a$

$$\alpha = x - \frac{y'}{y''}\left[1+(y')^2\right]^{3/2} = a\pi - 0 = a\pi$$

$$\beta = y + \frac{1}{y''}\left[1+(y')^2\right]^{3/2} = 2a + \frac{1}{-a}\left[1+0^2\right]^{3/2} = 2a - \frac{1}{a} \cancel{\frac{2a^2-1}{a}}$$

The circle of curvature is:

$$(x - a\pi)^2 + (y - 2a)^2 = (4a)^2$$