B120051 4/2/21

1. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$1-x = t \Rightarrow x = 1-t \Rightarrow dx = -dt$

$\beta(m,n) = \int_1^0 (1-t)^{m-1}(t)^{n-1}\,dt$

$= \int_0^1 (1-x)^{m-1}(x)^{n-1}\,dx$

$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$

$= \beta(n,m)$

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2. $\beta(m,n) = \cancel{2\int_0^{\pi/2}}$

$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Let $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$\int_0^{\pi/2} (\sin\theta)^{2(m-1)} (\cancel{}\cos^2\theta)^{n-1}\, d\theta$

$2\sin\theta\cos\theta$

$= \int_0^{\pi/2} (2)(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta$

We know,

$\beta(m,n) = \beta(n,m)$

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$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2m-1}(\theta)\,d\theta$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2m-1}(\theta)\,d\theta$$

3. $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$x = \frac{u}{1+u}$$

$$\frac{dx}{du} = \frac{(1+u)(1) - u}{(1+u)^2} = \frac{1}{(1+u)^2}$$

$$dx = \frac{1}{(1+u)^2}\,du$$

$$\beta(m,n) = \int_0^1 \left(\frac{u}{1+u}\right)^{m-1}\left(\frac{1}{1+u}\right)^{n-1}\frac{du}{(1+u)^2}$$

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$$\beta(m,n) = \int \frac{u^{m-1}}{(1+u)^{m+n}} du$$

$$= \int \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

4. $$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

$$\Gamma m = \int_0^\infty x^{m-1} e^{-x} dx$$

$$x = u^2$$

$$2\int_0^\infty u^{2m-1} e^{-u^2} du$$

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

$$x = v^2$$

$$= 2\int_0^\infty v^{2n-1} e^{-v^2} dv$$

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$$\Gamma m \, \Gamma n = 4 \int_0^{\infty} \int_0^{\infty} u^{2m-1} v^{2n-1} e^{-(u^2+v^2)} \, du \, dv$$

Changing to polar coordinates.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

5. RHS: $\beta(m+1,n) + \beta(m,n+1)$

$$= \int_0^1 x^{m} (1-x)^{n-1} \, dx + \int_0^1 x^{m-1} (1-x)^{n} \, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} (x + 1 - x) \, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$$= \beta(m,n).$$