Euler-Cauchy Equations

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Any ODE of the form

$$x^2y'' + axy' + by = 0 \quad \cdots (1)$$

is called a Euler-Cauchy eqⁿ. where a & b are constants

To solve eqⁿ (1), put

$$y = x^m, \; y' = mx^{m-1}, \; y'' = m(m-1)x^{m-2}$$

to eqⁿ (1)

$$x^2 * m(m-1)x^{m-2} + ax \cdot mx^{m-1} + bx^m = 0$$

$$\Rightarrow (m^2 + (a-1)m + b)x^m = 0$$

For a non trivial solⁿ of (1)

$$m^2 + (a-1)m + b = 0 \quad \cdots (2)$$

eqⁿ (2) is called the auxiliary eqⁿ

solving (2)

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$$

<u>case-I</u> (Real distinct roots m_1 & m_2)

$$y_1 = x^{m_1}, \; y_2 = x^{m_2}$$

are the two linearly independent (LI) solⁿˢ of (1). The general solⁿ is

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

<u>case-II</u> A real double root $m_1 = m_2 = m$ say

$m_1 = \frac{1}{2}(1-a)$ occurs if and only if $b = \frac{1}{4}(a-1)^2$

Then the solⁿ is $y_1 = x^{(1-a)/2}$

The ode (1) for $b = \frac{1}{4}(1-a)^2$ become

$$x^2y'' + axy' + \frac{1}{4}(1-a)^2 y = 0$$

$$\Rightarrow y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0 \quad (3)$$

$$\Rightarrow y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \frac{a}{x}$, $q(x) = \frac{(1-a)^2}{4x^2}$

The LI solⁿ

$$y_2 = uy_1, \; y_2' = uy_1' + u'y_1$$

$$y_2'' = uy_1'' + u'y_1' + u''y_1 + u'y_1'$$

using y_2'', y_2' and y_2 in (3)

$$(uy_1'' + u'y_1' + u''y_1 + u'y_1') + p(x)(uy_1' + u'y_1) + q(x)(uy_1) = 0$$

$$\Rightarrow \underbrace{u(y_1'' + py_1' + qy_1)}_{0} + 2u'y_1' + pu'y_1 + u''y_1 = 0$$

$$\Rightarrow 2u'y_1' + pu'y_1 + u''y_1 = 0$$

?

$$u = \log x, \quad y_2 = \log x \, y_1$$

$$y = (C_1 + C_2 \log x)x^m; \quad m = \frac{1}{2}(1-a)$$

<u>case-III</u> <u>complex conjugate roots</u>

$$\alpha \pm i\beta$$

$$y = x^\alpha [C_1 \cos(\beta \log x) + C_2 \sin(\beta \log x)]$$

<u>EX: 1</u> $x^2y'' + 1.5xy' - 0.5y = 0$

The auxiliary eqⁿ is

$$m^2 + (a-1)m + b = 0$$

$$\Rightarrow m^2 + (1.5-1)m + (-0.5) = 0$$

$$\Rightarrow m^2 + 0.5m - 0.5 = 0$$

$$\Rightarrow m = 0.5 \text{ and } -1$$

$$y_1 = x^{0.5}, \; y_2 = x^{-1}$$

The general solⁿ is

$$y = C_1 x^{0.5} + C_2 x^{-1}$$

$$= C_1\sqrt{x} + \frac{C_2}{x}$$

<u>Ex: 2</u> $x^2y'' - 5xy' + 9y = 0$

The auxiliary eqⁿ is

$$m^2 + (a-1)m + b = 0$$

$$m^2 + (-5-1)m + 9 = 0$$

$$\Rightarrow m^2 - 6m + 9 = 0$$

$$\Rightarrow m = 3, 3$$

The general solⁿ is

$$y = (C_1 + C_2 \log x)x^m$$

$$= (C_1 + C_2 \log x)x^3$$

<u>Ex: 3</u> solve $x^2y'' + 0.6xy' + 16.04y = 0$

<u>Solⁿ:</u> The auxiliary eqⁿ is

$$m^2 + (0.6-1)m + 16.04 = 0$$

$$\Rightarrow m^2 - 0.4m + 16.04 = 0$$

$$\Rightarrow m_1 = 0.2 + 4i, \; m_2 = 0.2 - 4i$$

The general solⁿ is

$$y = x^{0.2}[C_1 \cos(4\log x) + C_2 \sin(4\log x)]$$

<u>Ex: 4</u> $5x^2y'' + 23xy' + 16.2y = 0$

<u>Ex: 5.</u> $xy'' + 2y' = 0$

<u>Ex: 6</u> $(x^2D^2 - xD + 5I)y = 0$

EX: 7 $x^2y'' + xy' + 9y = 0$, $y(1) = 0$, $y'(1) = 2.5$