Assignment 2.2

1 Reduction of Order

Reduce to first order and solve, showing each step in detail

1. $y'' + y' = 0$
2. $2xy'' = 3y'$
3. $yy'' = 3y'^2$
4. $xy'' + 2y' + xy = 0, y_1 = \frac{cosx}{x}$
5. $x^2y'' - 5xy' + 9y = 0, y_1 = x^3$

2 Applications of reducible ODEs

6. **Curve**. Find the curve through the origin in the xy-plane which satisfies $y'' = 2y$ and whose tangent at the origin has slope 1.
7. **Hanging cable.** It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called catenary (from Latin catena = the chain). Find and graph $y(x)$, assuming that $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.
8. **Motion.** If, in the motion of a small body on a straight line, the sum of velocity and acceleration equals a positive constant, how will the distance depend on the initial velocity and position?
9. **Motion.** In a straight-line motion, let the velocity be the reciprocal of the acceleration. Find the distance for arbitrary initial position and velocity.

3 General solution. Initial Value Problem (IVP)

(a) Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. (b) Solve the IVP. Graph or sketch the solution.

10. $4y'' + 25y = 0; y(0) = 3.0, y'(0) = -2.5,$
    $cos(2.5x), sin(2.5x)$
11. $y'' + 0.6y' + 0.09y = 0, y(0) = 2.2y'(0) = 0.14,$
    $e^{-0.3x}, xe^{-0.3x}$
12. $4x^2y'' - 3y = 0, y(1) = -3, y'(1) = 0,$
    $x^{\frac{3}{2}}, x^{\frac{-1}{2}}$

4 Existence and uniqueness of solutions

13.

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

Assume that the coefficients p and q of the ODE (1) are continuous on some open interval I, to which the subsequent statements refer.

(i) Solve $y'' - y = 0$
(a) by exponential functions,
(b) by hyperbolic functions.
How are the constants in the corresponding general solutions related?

(ii) Prove that the solutions of a basis cannot be 0 at the same point.

(iii) Prove that the solutions of a basis cannot have a maximum or minimum at the same point.

(iv) Sketch $y_1(x) = x^3$ if and $x \geq 0$ and 0 if $x \leq 0, y_2(x) = 0$ if $x \geq 0$ and x^3 if $x < 0$. Show linear independence on $-1 < x < 1$. What is their Wronskian? What Euler–Cauchy equation do satisfy?

5 ODE for given basis. Wronskian. IVP

(a) Find a second-order homogeneous linear ODE for which the given functions are solutions. (b) Show linear independence by the Wronskian. (c) Solve the initial value problem.

14. $cos5x, sin5x, y(0) = 3, y'(0) = -5$

15. $x^{m_1}, x_{m_2}, y(1) = -2, y'(1) = 2m_1 - 4m_2$

16. $e^{-2.5x}cos0.3x, e^{-2.5x}sin0.3x, y(0) = 3, y'(0) = -7.5$

17. x^2, x^2logx

18. $1, e^{-2x}, y(0) = 1, y'(0) = -1$

6 Structure of Solutions of Initial Value Problems.

19. Using the present method, find, graph, and discuss the solutions y of initial value problems of your own choice. Explore effects on solutions caused by changes of initial conditions. Graph separately, to see the separate effects. Find a problem in which (a) the part of y resulting from decreases to zero, (b) increases, (c) is not present in the answer y.

7 Euler-Cauchy Equations

20. $5x^2y'' + 23xy' + 16.2y = 0$

21. $xy'' + 2y' = 0$

22. $(x^2D^2 - xD + 5I)y = 0$

23. $x^2y'' + xy' + 9y = 0; y(1) = 0, y'(1) = 2.5$

24. $(x^2D^2 + xD + I)y = 0; y(1) = 1, y'(1) = 0$

8 Solving non-homogeneous ODE

Use method of undetermined coefficient and method of variation of parameter to solve the following ODEs.

25. $(D^2 + 2D + \frac{3}{4}I)y = 3e^x + \frac{9}{2}x$

26. $(D^2 + 2D + I)y = 2xsinx$

27. $8y'' - 6y' + y = 6coshx, y(0) = 0.2, y'(0) = 0.05$

28. $y'' + 4y' + 4y = e^{-2x}sin2x, y(0) = 1, y'(0) = -1.5$

29. $(x^2D^2 - 3xD + 3I)y = 3logx - 4, y(1) = 0, y'(1) = 1; y_p = logx$

9 Programming problems:

30. **Linear Independence.**
    Write a program for testing linear independence and dependence. Try it out on some of the problems in this and the next problem set and on examples of your own.

Answers:

1. $y = c_1 e^{-x} + c_2$
3. $y = (c_1 x + c_2)^{-1/2}$
5. $y_2 = x^3 lnx$
6. $y = c_1 e^{2x} + c_2$
8. $y(t) = c_1 e^{-t} + kt + c_2$
10. $y = 3cos2.5x - sin2.5x$
12. $y = -0.75x^{\frac{3}{2}} - 2.25x^{\frac{-1}{2}}$
14. $y'' + 25y = 0, W = 5, y = 3cos5x - sin5x$
16. $y'' + 5y + 6.34 = 0, W = 0.3e^{-5x}, 3e^{-2.5x}cos0.3x$
18. $y'' + 2y' = 0, W = -2e^{-2x}, y = 0.5(1 + e^{-2x})$
27. $e^{x/4} - 2e^{x/2} + \frac{1}{5}e^{-x} + e^{x}$
29. $y = logx$