ME2 Computing- Tutorial 7: Transient analysis of ODEs: Heat Transfer

Learning outcomes:

- Being familiar with the finite difference scheme
- Being familiar with the heat conduction problem
- Being able to assess the stability of the numerical method

Before you start

In your H drive create a folder *H:\ME2CPT\Tutorial7* and work within it.

Introduction

In Tutorial 5 we have solved numerically ODEs with time marching solutions, after providing an initial condition. In Tutorial 6 we have solved steady state (no time varying) ODEs, with solutions restrained at the boundaries.

In this Tutorial, we join both the initial value and the boundary value problems, and will be solving PDEs with time marching solutions and restrains at the boundaries.

A very representative example of this case is the heat conduction within a body, which is described, in one-dimension, by the PDE:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where T is the temperature of the body, α is the thermal diffusivity of the material (in m^2/s) and t and x are the time and spatial independent variables, respectively.

To solve numerically the PDE we use the backward finite difference scheme for the temporal discretisation and the central difference scheme for the spatial discretisation:

$\frac{dT}{dt} = \frac{T^p - T^{p-1}}{\Delta t}$	$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$

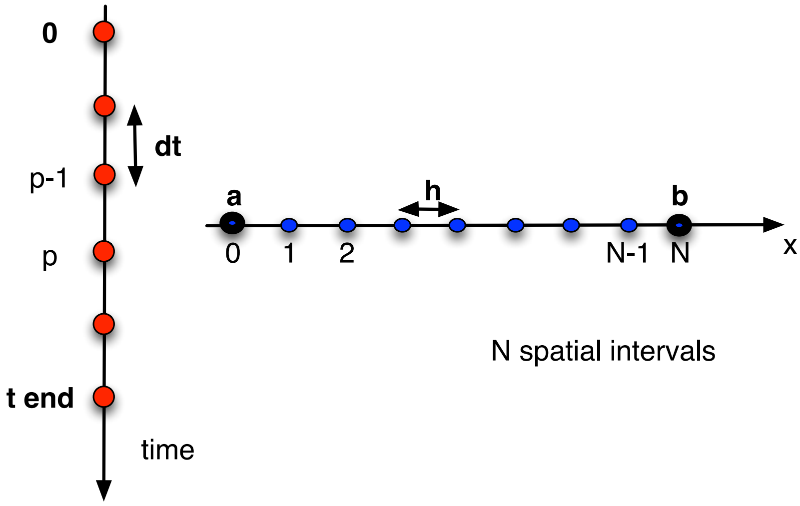
Therefore, the ODE is *discretised* as:

$$\frac{T_i^p - T_i^{p-1}}{\Delta t} = \alpha \frac{T_{i+1}^{p-1} - 2T_i^{p-1} + T_{i-1}^{p-1}}{\Delta x^2}$$

The superscript p refers to time, whilst the subscript i to space. So T_i^p indicates the Temperature at time step p and at node i of the domain. I.e., the temperature of spatial location $x = i \cdot \Delta x$ at time $t = p \cdot \Delta t$.

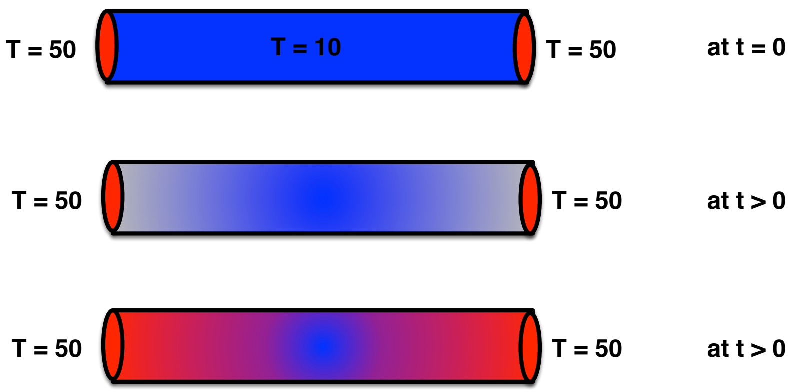
If we know the initial temperature of the body, at every node, we can compute the temperature at later times as:

$$T_i^p = \frac{\alpha\Delta t}{\Delta x^2}\left(T_{i+1}^{p-1} + T_{i-1}^{p-1}\right) + \left(1 - \frac{2\alpha\Delta t}{\Delta x^2}\right)T_i^{p-1} \quad i = 1 \ldots N-1$$	**Equation 1**



Task A: Heat conduction in a bar

A steel bar ($\alpha = 1.172 \times 10^{-5} m^2/s$) of length $0.5m$ is initially at a uniform temperature of $T = 10°C$. The two extremes are suddenly brought to a temperature of $T = 50°C$ and kept at this temperature. Determine the temperature distribution within the bar after one hour.



1. Write a script to solve numerically the heat conduction equation.
   Input data for the problem are: the endpoints of the domain *a* and *b*, the values of the solution at these points, $T(a)$ and $T(b)$, the initial uniform temperature T_0 for all spatial points, the time and spatial steps Δt and Δx, respectively, and the desired computational time span t_{end}.
   Output data for the problem are: the grid points x, the grid time t and the solution $T(x,t)$ at every grid points for anytime between 0 and t_{end}.

2. Compute the temperature distribution with $\Delta t = 1s$ and $\Delta x = 0.01m$.
   Plot $T(x, t_{end})$.

3. Repeat the calculation with $T(a) = 50$ and $T(b) = 70$.

Task B: Heat conduction in a bar with a heat source

Consider the same bar as in Task A, with an initial uniform temperature of $T = 10°C$. The temperature of the middle point is suddenly increased to $T = 100°C$ through a source and kept constant at this value. The bar is immersed in a large pool of water with constant temperature $T_w = 5°C$ and is subject to convective heat exchange with the water, i.e. $k\left.\frac{dT}{dx}\right|_a = h(T_a - T_w)$ and $k\left.\frac{dT}{dx}\right|_b = -h(T_b - T_w)$. The thermal conductivity for steel is $k = 40\,\frac{W}{mK}$, and the heat transfer coefficient $h = 500\,\frac{W}{m^2K}$.

1. Amend the script of Task A, to incorporate the mixed boundary conditions.
   A few hints:
   - Set the temperature in the middle point of the grid to be $T = 100°C$, irrespectively of the rest, to simulate the source.
   - For the mixed boundary conditions, you need to write down the form of the numerical solution for the first and last points, $T_0^{p+1} = \cdots$, $T_N^{p+1} = \cdots$.

2. Compute the temperature distribution with $\Delta t = 1s$ and $\Delta x = 0.01m$.
   Plot the spatial distribution of the temperature within the bar at $t = 0$, $t = 500s$ and $t_{end} = 1200s$.

Task C: Stability of the finite difference numerical method

Solving numerically a time marching PDE with boundary conditions can be easily unstable and careful attention is needed.

Once the time step Δt becomes too large the last term in Equation 1, $\left(1 - \frac{2\alpha\Delta t}{\Delta x^2}\right)$, becomes negative. The same is true if Δx is reduced. When this term is negative the numerical computation is unstable: errors are introduced and amplified at every successive time step. Therefore, in order for the solution to converge it must be:

$\left(1 - \dfrac{2\alpha\Delta t}{\Delta x^2}\right) > 0 \qquad i.e.$	$\dfrac{\alpha\Delta t}{\Delta x^2} < \dfrac{1}{2}$

This is a stability criterion, also known as the Courant condition of stability, and pose a restrain between Δx and Δt. For a chosen Δx there must be a maximum Δt to ensure numerical stability.

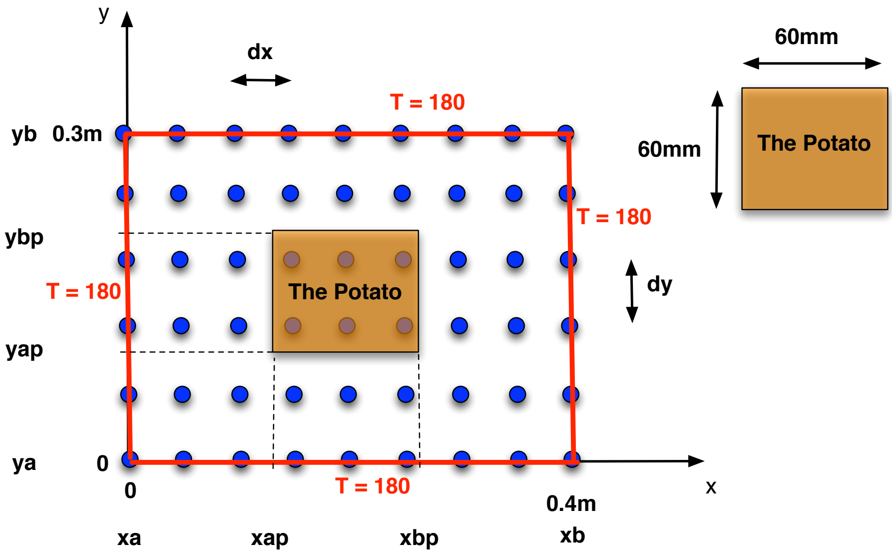
Use the script for Task A, with $T(a) = 50$ and $T(b) = 50$.

1. Compute $T(x, t_{end})$ with $\Delta t = 1s$ and $\Delta x = 0.01m$, $\Delta x = 0.05m$ and $\Delta x = 0.001m$. Calculate the Courant condition for all the three cases and plot $T(x, t_{end})$.

2. Repeat the calculation with $\Delta x = 0.001m$ and a reduced $\Delta t = 0.04s$.

Task D: PDE with multiple spatial dimensions: baking a freaking potato in the oven

A (two dimensional) traditional oven can be represented with the discretisation grid:



The heat equation within the oven is described by the Partial Differential Equation:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

The oven, inside, is initially at room temperature $T_0 = 25°C$. A frozen potato, with an initial temperature $T_0 = -15°C$, is positioned centrally in the middle of the oven. The walls of the oven are heat at a constant temperature of $T_0 = 180°C$.

1. Write a script to solve numerically the heat conduction equation.
   The thermal diffusivity of air in the oven is $\alpha = 1.9 \times 10^{-5} m^2/s$. The thermal diffusivity of the potato is $\alpha = 1.3 \times 10^{-7} m^2/s$.

2. Compute $T(x, y, t)$ with $\Delta x = \Delta y = 0.01m$, $\Delta t = 1s$.

3. Plot the 2-D temperature profile at various time steps, to observe how the potato is baking.
   Enjoy the baked potato with a filling of your choice (I like cheese).