ME2 Computing- Session 2: Numerical Integration

Learning outcomes:

- Being able to compute numerically the integration of a proper integral
- Being able to compute numerical integration for a set of points not positioned equidistantly
- Being able to compute numerically integration of multi-dimensional functions

Before you start

In your H drive create a folder *H:\ME2MCP\Session2* and work within it.

Task A: Trapezium rule for functions with equidistant nodes

1. Write a Python function, *trapzeqd*, receiving a set of points *x* and *y*, and outputting the numerical integral of *y* within the interval specified by *x*. Assume that the nodes *x* are equidistant.
   Test the Python function by integrating:

$$I = \int_0^b \frac{1}{\sqrt{x^{17.10} + 2023}} dx$$

   in the interval *x = [0 : b=2]* with 5 nodes and then with 11 nodes.

2. Increase the interval of integration with *b = 10, 100, 1000, 10000*, and recompute the integral with same number of nodes (5).
   Plot the values of *I* vs *b*.

3. Repeat the numerical integration for the intervals in Part 2, but retaining the same interval *h = 0.5*, i.e. by increasing progressively the number of nodes.
   Replot the values of *I* vs *b*.

Task B: Numerical integration of diverging improper integrals

4. Recompute the numerical integrations as in Task A2 and A3, but with the integrand function:

$$I = \int_0^b \frac{1}{\sqrt{x^{1.10} + 2023}}$$

Task C: Trapezium rule for functions with non-equidistant nodes

1. Write another Python function, *trapz*, receiving a set of points *x* and *y*, and outputting the numerical integral of *y* within the interval specified by *x*. The values in x might not be distanced at same intervals.

Task D: The river Thames basin in London

The file *Thames.txt* contains $N = 72$ spatial coordinates (x_i, y_i) of the north and south banks of the river Thames (units in meters), within the Central London region (between Chiswick and Woolwich).
The nodal points are organised in the file within $N = 72$ lines, as follows:

x North bank	y North bank	x South bank	y South bank
xn_o	yn_o	xs_o	ys_o
xn_1	yn_1	xs_1	ys_1
xn_{N-1}	yn_{N-1}	xs_{N-1}	ys_{N-1}

1. Read in the data from the files and plot the two banks of the river together, to visualise the shape of the basin. (To plot with aspect ratio 1:1 for the two axes use *pl.axis('equal')*, after plotting.

2. Compute the surface occupied by the basin in Km^2.

Task E: Multiple integrals (with given analytical function): volume of the dome of the Royal Albert Hall

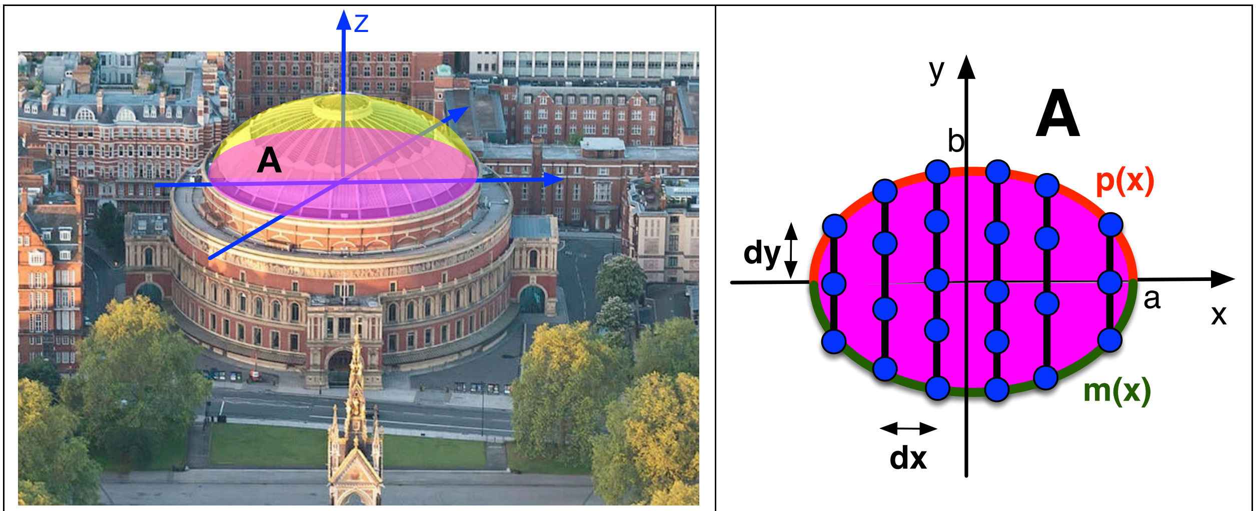
A two-dimensional integral has the form of:

$$I = \iint_A z(x,y)dA = \int_a^b dx \int_{m(x)}^{p(x)} z(x,y)dy = \int_a^b G(x)dx$$

where A is the domain of integration.
The two-dimensional integral can be computed numerically, by applying the trapezium method twice. Firstly, the integral

$$G(x) = \int_{m(x)}^{p(x)} z(x,y)dy$$

is computed for all values of x. Then, the total integral is obtained as: $I = \int_a^b G(x)dx$.



(a) Dome volume

(b) Domain of integration (base of the dome)

1. The dome of the Royal Albert Hall is described by the ellipsoid function:

$$z(x,y) = \sqrt{h^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$$

   with height $h = 25m$.
   The base of the dome A (domain of integration) is described by an ellipse, A: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with major and minor axes $a = 67m$ and $b = 56m$, respectively.
   Determine the numerical value of the dome volume, by discretising the domain A with a mesh of equidistant intervals both along the x and y axes, i.e., $dx = dy = 0.5m$.

2. Compare the numerical value against the analytical value. Repeat the calculation for a mesh with elements size: $dx = dy = 0.05m$

3. Plot the function $z(x,y)$.

Task F: Multiple integrals (with given nodes): volume of an aerofoil

The file *Aerofoil.txt* contains the nodal coordinates of an aerofoil. The aerofoil is defined by two surfaces $Z_t(x, y)$ and $Z_b(x, y)$, each defining the top surface and the bottom surface of the aerofoil, respectively. The domain has been discretised with a mesh of dimension ($N_x = 100$, $N_y = 15$).
Nodes in the file are organised within $N_x \cdot N_y = 1500$ lines, as follows:

x_o	y_o	$Z_t(x_o, y_o)$	$Z_b(x_o, y_o)$
x_1	y_o	$Z_t(x_1, y_o)$	$Z_b(x_1, y_o)$
x_{Nx-1}	y_o	$Z_t(x_{Nx-1}, y_o)$	$Z_b(x_{Nx-1}, y_o)$
x_o	y_1	$Z_t(x_0, y_1)$	$Z_b(x_0, y_1)$
x_1	y_1	$Z_t(x_1, y_1)$	$Z_b(x_1, y_1)$
x_{Nx-1}	y_1	$Z_t(x_{Nx-1}, y_1)$	$Z_b(x_{Nx-1}, y_1)$
$etc.$			

1. Compute the volume of the aerofoil.

2. Plot the aerofoil.