ME1 Computing- Session 10: Sets of Linear Equations and Gauss Elimination

Learning outcomes:

- Being able to solve a set of linear equations numerically
- Being able to implements Gauss elimination
- Being able to express engineering problems in matrix-vector terms

Before you start

In your H drive create a folder *H:**ME1MCP*\\Session10 and work within it.

Introduction

A set of linear equations can be written numerically in the matrix-vector form as:

$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$	**Eq. 1**	$\begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & \mathbf{a_{14}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{24}} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} & \mathbf{a_{34}} \\ \mathbf{a_{41}} & \mathbf{a_{42}} & \mathbf{a_{43}} & \mathbf{a_{44}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \boldsymbol{b_3} \\ \boldsymbol{b_4} \end{bmatrix}$ $A \cdot x = b$

In Session 8 we learned how to invert a matrix and we applied Cramer's method to compute the numerical values of the solutions for a given set of linear equation. An alternative and faster approach is to apply Gauss elimination, as learned in the Maths class.

Task A: Gauss elimination

If the given set of equations can be manipulated into the form:

$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ \qquad d_{22}x_2 + d_{23}x_3 + d_{24}x_4 = d_2 \\ \qquad\qquad e_{33}x_3 + e_{34}x_4 = e_3 \\ \qquad\qquad\qquad f_{44}x_4 = f_4 \end{cases}$	$\begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & \mathbf{a_{14}} \\ \mathbf{0} & \mathbf{d_{22}} & \mathbf{d_{23}} & \mathbf{d_{24}} \\ \mathbf{0} & \mathbf{0} & \mathbf{e_{33}} & \mathbf{e_{34}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f_{44}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{d_2} \\ \boldsymbol{e_3} \\ \boldsymbol{f_4} \end{bmatrix}$

then it is straightforward to find the solutions, starting from x_4 and proceeding backwards by substitution to find x_3, x_2 and x_1.

This manipulation can be done in repeated steps. At each step *i*, the variable x_i is eliminated from the $(i+1)^{th}$ row downwards, (i.e. all the coefficients in column *i* are set to zero, starting from the row below *i*).
For example, at step 1, starting from Eq. 1, the variable x_1 is eliminated from the second row downwards, (i.e. all the coefficients in column 1 are set to zero starting from row 2):

row 1 is unchanged new row 2 = row 2 - $a_{21}/\boldsymbol{a_{11}}$ · row 1 new row 3 = row 3 - $a_{31}/\boldsymbol{a_{11}}$ · row 1 new row 4 = row 4 - $a_{41}/\boldsymbol{a_{11}}$ · row 1	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & d_{32} & d_{33} & d_{34} \\ 0 & d_{42} & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$	**Eq. 2**

The common coefficient, $\boldsymbol{a_{11}}$, is also knows as the **pivot**.

At step 2, starting from Eq. 2, the variable x_2 is eliminated from the third row downwards, (i.e. all the coefficients in column 2 are set to zero starting from row 3):

row 1 is unchanged new row 2 is unchanged new row 3 = new row 3 - $d_{32}/\boldsymbol{d_{22}}$ · new row 2 new row 4 = new row 4 - $d_{42}/\boldsymbol{d_{22}}$ · new row 2	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 0 & e_{43} & e_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_2 \\ e_3 \\ e_4 \end{bmatrix}$	**Eq. 3**

The pivot at this step is the coefficient $\boldsymbol{d_{22}}$.

These steps are repeated, until the matrix A is reduced into an upper triangular form.

If you find difficult to understand this logic, read the supplemental document on Gauss elimination (on BB), by paying particular attention to the colour code adopted in it.

1. Write a function, *GaussElimination*, that receives a set of *n* linear equations, in the form of a matrix *A* and a vector *b*, and outputs the vector solution *x*.

2. Test the function to solve the set of equations:

$$\begin{cases} 8x_1 - 2x_2 + x_3 + 3x_4 = 9 \\ x_1 - 5x_2 + 2x_3 + x_4 = -7 \\ -x_1 + 2x_2 + 7x_3 + 2x_4 = -1 \\ 2x_1 - x_2 + 3x_3 + 8x_4 = 5 \end{cases}$$

   Solution: [1.3232   1.5657   -0.6061   0.7172]

Task B: Further skills

Sometimes the set of equations does not have a solution, even though the numerical method may still provide a solution, which is obviously wrong. If we consider the matrix-vector form:

$$A \cdot x = b$$

the solution can be expressed as:

$$x = A^{-1} \cdot b$$

where A^{-1} is the inverse of matrix A. Therefore, there will exist a solution if the matrix *A* is invertible. This is equivalent of saying that the determinant of matrix *A* is non-zero.

1. For this set of linear equations:

$$\begin{cases} 4x + 3y = 2 \\ 8x + 6y = 1 \end{cases}$$

Plot the two equations, each as a line *y* vs *x*, on the same graph.
Solve them numerically: what does it happen?

2. There are cases, when the solution exists, where numerical methods can provide completely inaccurate results. These cases are said to be **ill conditioned**.

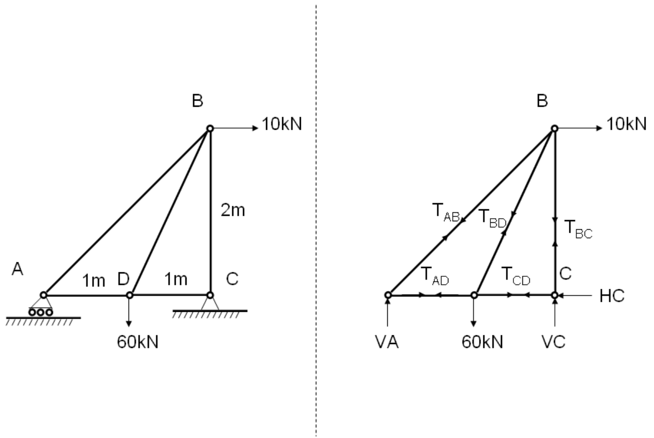
   Consider these two sets of linear equations:

   $$\begin{cases} 400x - 201y = 200 \\ -800x + 401y = -200 \end{cases} \qquad \begin{cases} \mathbf{401}\boldsymbol{x} - \mathbf{201}\boldsymbol{y} = \mathbf{200} \\ -\mathbf{800}\boldsymbol{x} + \mathbf{401}\boldsymbol{y} = -\mathbf{200} \end{cases}$$

   Solve the two sets of equations separately. How do the numerical solutions differ from each other?
   This is a case of an ill conditioned problem: with a modest change in one of the coefficients (often induced by a numerical error) one would expect only a small change in the solution. However, in ill conditioned cases the change is quite significant: the solution is very sensitive to the values of the coefficients.

Task C: Forces in a pin jointed frame

Consider the pin jointed frame:



To work out the reaction forces and tensions/compressions in the bars:

a) Resolve the whole structure horizontally:	HC = 10 kN
b) Resolve the whole structure vertically:	VA +VC = 60 kN
c) Taking moments about C:	VA *2 = 60 kN *1 – 10kN * 2
d) Vertical forces at joint A:	T_{AB} sin(45°) + VA = 0
e) Horizontal forces at joint A:	T_{AB} cos(45°) + T_{AD} = 0
f) Vertical forces at C:	VC + T_{CB} = 0
g) Horizontal forces at C:	HC + T_{CD} = 0
h) Vertical forces at D:	T_{DB} sin (63.43°) = 60 kN

1. Solve numerically the frame to find the forces and the tensions.

Task D: Solutions of a linear system

Write a function *LinSystems*, that receives a set of *n* linear equations, in the form of a matrix A and a vector p: $A \cdot x = p$, and prints out one of the possible results, as learned in Maths lecture (pag. 86):

<table>
<tr><td></td><td>Non-homogeneous system
$\mathbf{p} \neq \mathbf{0}$</td><td>Homogeneous system
$\mathbf{p} = \mathbf{0}$</td></tr>
<tr><td>$|\mathbf{A}| \neq 0$</td><td>The system has a unique solution, which is non-trivial,
$\mathbf{x} = \mathbf{A}^{-1}\mathbf{p}$</td><td>The system has unique solution, which is the trivial solution, $\mathbf{x} = \mathbf{0}$</td></tr>
<tr><td rowspan="2">$|\mathbf{A}| = 0$</td><td>The system has an infinite number of solutions</td><td rowspan="2">The system has an infinite number of solutions.</td></tr>
<tr><td>The system has no solutions, the system is inconsistent.</td></tr>
</table>

If the system has a solution, the function should also return the values of the array x (make use of the function *GaussElimination*, to determine the solution, and previous functions form Session 9 to compute the determinant).