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ELEMENTARY EMBEDDINGS AND INFINITARY COMBINATORICS

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§0. Introduction. One of the standard ways of postulating large cardinal axioms is to consider elementary embeddings, j, from the universe, V, into some transitive submodel, M. See Reinhardt–Solovay [7] for more details. If j is not the identity, and κ is the first ordinal moved by j, then κ is a measurable cardinal. Conversely, Scott [8] showed that whenever κ is measurable, there is such a j and M. If we had assumed, in addition, that $\mathscr{P}(2^\kappa) \subset M$, then κ would be the κth measurable cardinal; in general, the wider we assume M to be, the larger κ must be.

A natural question, posed by Reinhardt, is whether one can actually assume that $M = V$. The answer is no, however, as we shall show in §1 by a combinatorial argument. In §3, we introduce a combinatorial partition property, P. We show, using the existence of a super-compact cardinal, that there are cardinals possessing property P. By §5, this fact could not have been proved by using merely the existence of a measurable cardinal.

We emphasize that when we say that a map $k: M_1 \to M_2$ is elementary, we mean that M_1 and M_2 are considered to be first-order structures with the $\in$-relation understood; i.e., for every first-order formula $\varphi(v_1 \cdots v_m)$ in a symbol for $\in$, and every $x_1 \cdots x_m \in M_1$,

$$\varphi^{(M_1)}(x_1 \cdots x_m) \leftrightarrow \varphi^{(M_2)}(k(x_1) \cdots k(x_m)),$$

where $\varphi^{(M_i)}$ is the relativization of φ to M_i. It is intended that our results be formalized within the second-order Morse–Kelley set theory (as in the appendix to Kelley [4]), so that statements involving the satisfaction predicate for class models can be expressed.

§1. Embeddings from V into V. Our main tool here is the following:

THEOREM (ERDÖS–HAJNAL [1]). *Let λ be any infinite ordinal. There is a function $F: {}^\omega\lambda \to \lambda$ such that whenever $A \subseteq \lambda$ and $\overline{\overline{A}} = \overline{\overline{\lambda}}$, $F''({}^\omega A) = \lambda$.*

Here, as in Gödel [3], $F''X = \text{range}\,(F \restriction X)$.

The proof of the theorem for an arbitrary λ involves a rather ingenious construction, but in the one special case which we shall need, namely, when λ is a strong limit cardinal of cofinality ω, the proof is trivial. In this case, $\lambda^\omega = 2^\lambda$, so, if $\langle\langle A_\alpha, \xi_\alpha\rangle: \alpha < 2^\lambda\rangle$ enumerates $\{X \subseteq \lambda: \overline{\overline{X}} = \lambda\} \times \lambda$, we may simply pick, by induction on α, $s_\alpha \in {}^\omega A_\alpha \sim \{s_\beta: \beta < \alpha\}$, and set $F(s_\alpha) = \xi_\alpha$.

Now, as in §0, assume that j is a nontrivial elementary embedding from V into some transitive class M, with κ the first ordinal moved by j. Let $j^1 = j, j^{n+1} = j \circ j^n$.

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Let $\lambda = \sup\{j^n(\kappa): n < \omega\}$. Note that $j(\lambda) = \lambda$, since $j(\{j^n(\kappa): n < \omega\}) = \{j^n(\kappa): 1 \leq n < \omega\}$.

THEOREM. $\mathscr{P}(\lambda) \not\subset M$.

PROOF. Let $F: {}^\omega\lambda \to \lambda$ be as in the statement of the Erdös–Hajnal theorem. Let $A = \{j(\xi): \xi < \lambda\}$. If $A \in M$, then, by elementarity of j, there is an $s \in {}^\omega A \cap M$ such that $(j(F))(s) = \kappa$. But $s = j(t)$, where $j(t(n)) = s(n)$ $(n < \omega)$, so

$$\kappa = (j(F))(j(t)) = j(F(t)) \in \text{range}(j),$$

which is impossible.

In the proof, we could have first assumed, by way of obtaining a contradiction, that $\mathscr{P}(\lambda) \subset M$. Then, by induction on n, each $j^n(\kappa)$ is measurable, so λ is a strong limit cardinal of cofinality ω. Thus, we really needed only this special case of the Erdös–Hajnal theorem.

Of course, we have immediately,

COROLLARY. *If j is an elementary embedding from V into V, j is the identity.*

Remark. At the suggestion of the referee, we point out that the Axiom of Choice (AC) is used in an essential way in the proof of the Erdös–Hajnal theorem. For example, one may ask whether the Axiom of Determinateness (AD) plus the Axiom of Dependent Choices (DC) implies that the Erdös–Hajnal theorem fails for some λ. Furthermore, it is unknown whether the Corollary can be proved in ZF, or, for that matter, whether it can be refuted in ZF + AD + DC.

§2. **Iterating elementary embeddings.** We continue to assume that j, M, κ are as in §1. In this section, we show that each j^n is an elementary embedding of V into some transitive submodel. There is really nothing essentially new here; it is merely a matter of checking that Gaifman's construction [2] for the case that j is definable from a set goes through in general.

DEFINITION. If K is any class and k is a function on V,

$$k(K) = \bigcup\{k(K \cap R(\alpha)): \alpha \in \text{ORD}\}.$$

Note in particular that $j(V) = M$. By an application of the reflection theorem,

LEMMA. *If $K_1, \cdots, K_m$ are any classes, j is an elementary embedding from $(V; \in, K_1, \cdots, K_m)$ into $(M; \in, j(K_1), \cdots, j(K_m))$.*

THEOREM. *Each j^n is an elementary embedding from V into some transitive class M_n. If ${}^\alpha M \subset M$, then ${}^\alpha M_n \subset M_n$.*

PROOF. We proceed by induction on n. Assuming the theorem for n, let $M_{n+1} = j(M_n)$. By the lemma applied to $(V; j^n, M_n)$, M_{n+1} is transitive and $j(j^n): M \to M_{n+1}$ is elementary, so that $j(j^n) \circ j: V \to M_{n+1}$ is elementary. But $j(j^n) \circ j = j^{n+1}$, since

$$j(j^n) \supseteq \{\langle j(x), j^{n+1}(x)\rangle: x \in V\}.$$

If ${}^\alpha M \subset M$ and we assume inductively that ${}^\alpha M_n \subset M_n$, then by applying the lemma to $(V; M_n)$ we have that ${}^{j(\alpha)}M_{n+1} \cap M \subset M_{n+1}$, so ${}^\alpha M_{n+1} = {}^\alpha M_{n+1} \cap M \subset M_{n+1}$.

The M_n of the theorem are actually determined uniquely, since $M_n = j^n(V)$.

We could, following Gaifman, define M_ξ for $\xi \in \text{ORD}$, and elementary embeddings $J_{\xi\eta}: M_\xi \to M_\eta$ for $\xi \leq \eta$. Then j^n would be j_{0n}. Gaifman's proof that all the M_ξ are wellfounded goes over without change here.

§3. Super-compact cardinals.

DEFINITION. κ is *super-compact* iff for all α, there is a transitive M and an elementary embedding $j: V \to M$ such that:

(i) ${}^{\alpha}M \subset M$;

(ii) $j(\xi) = \xi$ for all $\xi < \kappa$;

(iii) $j(\kappa) > \alpha$.

This is the usual definition which, in [7], is shown to be equivalent to the existence of certain types of ultrafilters on $S_\kappa(\alpha)$ for arbitrarily large α. As a technical point, we remark that

THEOREM. *In the above definition, clause* (iii) *may be weakened to* $j(\kappa) > \kappa$.

PROOF. Suppose M, j satisfy (i) and (ii) for a particular α. By (i) and §1, $j^n(\kappa) > \alpha$ for some n. By §2, $j^n(V)$, j^n satisfy (i)–(iii) for α.

Now consider the following combinatorial property:

DEFINITION. $P(\lambda)$ is the assertion: Whenever $F: {}^{\omega}\lambda \to \lambda$, there is an infinite $A \subseteq \lambda$ such that $\bar{\bar{A}} \notin F''({}^{\omega}A)$.

Of course, $\lambda' \geq \lambda \wedge P(\lambda) \to P(\lambda')$.

THEOREM. *If κ is super-compact then* $\exists\lambda P(\lambda)$.

PROOF. We show $P(\kappa^+)$. Let $F: {}^{\omega}(\kappa^+) \to \kappa^+$. Let M, j satisfy the definition of super-compactness for $\alpha = \kappa^+$. Let $A = \{j(\xi): \xi < \kappa^+\}$. Then $A \in M$ and, as in §1, $\kappa \notin (j(F))''({}^{\omega}A)$, so

$$M \models \exists X \subseteq j(\kappa^+)[\bar{\bar{X}} \notin (j(F))''({}^{\omega}X)].$$

By elementarity of j,

$$\exists X \subseteq \kappa^+[\bar{\bar{X}} \notin F''({}^{\omega}X)].$$

Of course, by super-compactness, $\exists\lambda P(\lambda) \to \exists\lambda < \kappa P(\lambda)$.

The Erdös–Hajnal theorem is a refutation of a stronger version of P in which we require that $\bar{\bar{A}} = \bar{\bar{\lambda}}$. Many similar partition properties involving infinite sequences are refutable by similar arguments, although we are unable to refute P itself. However, by the results of §5, $\exists\lambda P(\lambda)$ is not provable in ZFC, even under the assumption of the existence of several measurable cardinals.

§4. Remarks on partition properties. Examining the proof of $P(\kappa^+)$ for κ super-compact yields a seemingly stronger property, in which $F''({}^{\omega}A)$ is replaced by the closure of A under F. However, by part (ii) of the following lemma, these properties are always equivalent.

LEMMA. *Suppose B is any class and, for $\alpha < 2^{\omega}$, $F_\alpha: {}^{\omega}B \to B$. Then there are functions $G_1, G_2: {}^{\omega}B \to B$ such that whenever $A \subseteq B$ and $\bar{\bar{A}} \geq 2$,*

(i) $G_1''({}^{\omega}A) = \bigcup\{F_\alpha''({}^{\omega}A): \alpha < 2^{\omega}\}$;

(ii) $G_2''({}^{\omega}A)$ *is the closure of A under the F_α; i.e.,*

$$G_2''({}^{\omega}A) = \bigcap\{X \subseteq B: A \subseteq X \wedge \forall s \in {}^{\omega}X \forall\alpha < 2^{\omega}[F_\alpha(s) \in X]\}.$$

PROOF. For (i), let $f_1: {}^{\omega}B \to 2^{\omega}$, $f_2: {}^{\omega}B \to {}^{\omega}B$ be such that

$$\forall x \subseteq B \forall\alpha < 2^{\omega} \forall t \in {}^{\omega}B[[2 \leq \bar{\bar{x}} \leq \omega \wedge \text{range}(t) \subseteq x] \to \exists s \in {}^{\omega}B[\text{range}(s) = x \wedge f_1(s) = \alpha \wedge f_2(s) = t]].$$

Now, let $G(s) = F_{f_1(s)}(f_2(s))$.

For (ii), let $I_n(s) = s(n)$ for $s \in {}^{\omega}B$, $n < \omega$. We may assume that the I_n are among the F_α since this will not change closures. Let H_β ($\beta < 2^\omega$) enumerate the collection of all functions obtainable from the F_α by composition—i.e. the least collection of functions, $\mathscr{H}$, containing the F_α, such that whenever $H_\mu \in \mathscr{H}$ ($\mu \leq \omega$),

$$\{\langle s, H_\omega(\langle H_n(s): n < \omega\rangle)\rangle : s \in {}^\omega B\} \in \mathscr{H}.$$

Then the closure of any A under the H_β is the same as the closure under the F_α. Furthermore, this closure is just $\bigcup\{H_\beta{}''({}^\omega A): \beta < 2^\omega\}$, since any $H_\gamma(\langle H_{\beta_n}(s_n): n < \omega\rangle)$ is equal to

$$H_\gamma(\langle H_{\beta_n}(\langle I_{2^n \cdot 3^m}(t): m < \omega\rangle): n < \omega\rangle),$$

where $t(2^n \cdot 3^m) = s_n(m)(n, m < \omega)$. (ii) now follows by applying (i) to the H_β.

Actually, the statement of the Erdös–Hajnal theorem in [1] is with the closure of A under F, rather than with $F''({}^\omega A)$, and was intended as a proof of the existence of Jónsson algebras in every cardinality if one admits infinitary functions. Of course, by our lemma, the two forms of the theorem are equivalent. Also, a trivial modification of the proof in [1] yields directly the form quoted here.

§5. **Further discussion of property P.** It follows from the Erdös–Hajnal theorem and the lemma of §4 that $P(\lambda)$ implies $\lambda > (2^\omega)^+$. We do not know whether it is possible that $P((2^\omega)^{++})$. However, P is a large cardinal property in the sense that if $\exists\lambda P(\lambda)$, then there are inner models with several measurable cardinals. More precisely,

THEOREM. *Suppose $\exists\lambda P(\lambda)$. Then for all ordinals $\theta < \omega_1$, there is a transitive proper class, N, such that*

$$N \models [\text{ZFC} + \textit{there are } \theta \textit{ measurable cardinals}].$$

[5, §10] and [6] present two somewhat different methods for deriving the same conclusion (but for all θ) from the assumption of the existence of a strongly compact cardinal. Many parts of the present proof can be transcribed directly from corresponding parts of one or the other of these earlier proofs. We shall concentrate here on showing how the parts fit together, referring the reader to [5] and [6] for those details which are not essentially new. The general outline will follow [6]; in particular, Definitions 1–4 will be taken verbatim from there.

The proof of the theorem occupies the rest of this section, and we assume $\exists\lambda P(\lambda)$ from now on. We first derive, from the combinatorial property P, a statement involving elementary embeddings. This type of argument is due to Rowbottom and Silver.

LEMMA 1. *There at least $(2^\omega)^+$ different ordinals, δ, such that there exist M, i with the following properties:*

(i) $\delta^+ \subset M$.

(ii) *M is a transitive model of* ZFC.

(iii) *i is an elementary embedding from M into V.*

(iv) $i(\delta) > \delta$.

(v) *$i \restriction \delta$ is the identity.*

(vi) ${}^\omega M \subset M$.

PROOF. We first show that there is at least one such δ. Let $F_n: {}^\omega V \to V$ be Skolem functions for $\langle V; \in\rangle$. More precisely, we assume that whenever $\varphi(v_0 \cdots v_{k-1})$

is a first order formula of set theory of the form $\exists v_k \psi(v_0 \cdots v_{k-1} v_k)$, then for some n and all $s \in {}^\omega V$,

$$\varphi(s(0) \cdots s(k-1)) \rightarrow \psi(s(0) \cdots s(k-1) F_n(s)).$$

Also, let $F_\omega: {}^\omega V \rightarrow V$ be the identity (i.e. $F_\omega(s) = s$ for $s \in {}^\omega V$). By (ii) of the lemma of §4, let $G: {}^\omega V \rightarrow V$ be such that $G''({}^\omega A)$ is the closure of A under the $F_\alpha(\alpha \leq \omega)$ whenever $\bar{\bar{A}} \geq 2$. Assume $P(\lambda)$, and define $\tilde{G}: {}^\omega\lambda \rightarrow \lambda$ so that $\tilde{G}(s) = G(s)$ if $G(s) \in \lambda$; $\tilde{G}(s) = 0$ otherwise. Let A be an infinite subset of λ such that $\bar{\bar{A}} \nleq \tilde{G}''({}^\omega A)$. Let $C = G''({}^\omega A)$, and let δ be the first ordinal not in C. Then $\langle C; \in\rangle \prec \langle V; \in\rangle$, ${}^\omega C \subset C$, and card $(C \cap \mathrm{ORD}) \geq \delta^+$. Thus, if M is the transitive model isomorphic to C and i is the isomorphism from M onto C, it follows that δ, M, i satisfy (i)–(vi).

Suppose now that δ_β $(\beta < 2^\omega)$ satisfy the conditions of Lemma 2. Define $F_{\omega+1+\beta}$: ${}^\omega V \rightarrow V$ so that $F_{\omega+1+\beta}(s) = \delta_\beta$ for all s. Go through the above proof, but now using all the F_α $(\alpha < 2^\omega)$. Then the C we obtain will contain all the δ_β, so the δ we obtain will be different from all the δ_β. Thus, there are at least $(2^\omega)^+$ such δ, proving Lemma 1.

We remark on some technical points. In carrying out the proof of Lemma 1 in Morse–Kelley set theory, the global form of the axiom of choice (needed to define the F_n) could be replaced by the local form by using Skolem functions just for some $R(\sigma)$, where $\sigma > \lambda$, $cf(\sigma) > \omega$, and $\langle R(\sigma), \in\rangle \prec \langle V, \in\rangle$. Actually, the whole proof of the theorem could be formalized within ZFC by considering elementary submodels of a suitable $R(\sigma)$ rather than of V.

We really need only ω_1 ordinals satisfying Lemma 1. Let δ_ζ $(\zeta < \omega_1)$ be an increasing sequence of such ordinals, with M_ζ and i_ζ the corresponding models and embeddings. Also, fix κ to be some regular cardinal greater than all the δ_ζ.

We shall now forget about property P and argue directly with the δ_ζ, M_ζ, i_ζ. The power of Lemma 1 lies in condition (vi). If we had omitted this condition, we could have dealt just with the Skolem functions, F_n, which are essentially finitary. The lemma would then follow from the existence of a Ramsey cardinal, which is clearly insufficient to imply the theorem.

DEFINITION 1. For any ordinal θ, a θ-set is a set $b \subset \mathrm{ORD}$ such that b has order type $\omega \cdot \theta$ and such that for each $\zeta \in b$, $\zeta > \sup(b \cap \zeta)$.

DEFINITION 2. If b is a θ-set, $\xi < \theta$, and $n < \omega$,

(a) $\gamma(b, \xi, n)$ is the $\omega \cdot \xi + n$th ordinal in b and $\lambda(b, \xi) = \sup(\{\gamma(b, \xi, m): m < \omega\})$.

(b) $\mathscr{F}(b, \xi) = \{x \subseteq \lambda(b, \xi): \exists m \forall k > m[\gamma(b, \xi, k) \in x]\}$; $\vec{\mathscr{F}}(b) = \langle \mathscr{F}(b, \eta): \eta < \theta\rangle$.

(c) $\Phi(b, \xi)$ is the statement that $\mathscr{F}(b, \xi) \cap L[\vec{\mathscr{F}}(b)]$ is, in $L[\vec{\mathscr{F}}(b)]$, a normal ultrafilter on $\lambda(b, \xi)$.

DEFINITION 3.

(a) $K_0 = \{\sigma: \sigma$ is a cardinal $\wedge\, cf(\sigma) > \kappa \wedge \forall\tau < \sigma[\tau^\kappa < \sigma]\}$.

(b) $K_{\alpha+1} = \{\sigma \in K_\alpha: \mathrm{card}\,(K_\alpha \cap \sigma) = \sigma\}$.

(c) $K_\beta = \bigcap_{\alpha<\beta} K_\alpha$ for limit ordinals β.

DEFINITION 4. If $a \subset \mathrm{ORD}$ and $X \subseteq L[a]$, then $H(a, X) = \{y \in L[a]: y$ is definable in $L[a]$ from elements of $a \cup X \cup \{a\}\}$.

Similarly to [6], we want to show that if $\theta < \omega_1$, b is a θ-set, and $b \subset K_{\omega\cdot\theta+1}$, then $\forall\xi < \theta\Phi(b, \xi)$. There, we obtained our results by iterating ultrapowers by an ultrafilter on κ. Now, κ need not even be measurable. Instead, we shall use the ultrafilters on the δ_ζ which arise from the embeddings i_ζ.

DEFINITION 5. For $\zeta < \omega_1$,

$$\mathscr{U}_\zeta = \{x \in \mathscr{P}(\delta_\zeta) \cap M_\zeta : \delta_\zeta \in i_\zeta(x)\}.$$

LEMMA 2. *For $\zeta < \omega_1$,*

(i) *$\mathscr{U}_\zeta$ is an ultrafilter in the Boolean algebra $\mathscr{P}(\delta_\zeta) \cap M_\zeta$.*

(ii) *$\mathscr{U}_\zeta$ is closed under countable intersections.*

LEMMA 3. *If b is a countable set of ordinals and $\zeta < \omega_1$, then $\mathscr{P}(\delta_\zeta) \cap L[b] \subset M_\zeta$.*

PROOF. If $x \in \mathscr{P}(\delta_\zeta) \cap L[b]$, then, by a Löwenheim–Skolem and collapsing argument, $x \in L_\mu[b']$ for some $\mu < \delta_\zeta^+$ and b' a countable subset of μ. Hence, $x \in M_\zeta$ by (i) and (vi) of Lemma 1.

LEMMA 4. *Suppose b is a countable set of ordinals, $\zeta < \omega_1$, and $b \sim \delta_\zeta \subset K_0$. Then*

(i) *$\mathscr{U}_\zeta \cap L[b]$ is an $L[b]$-ultrafilter on δ_ζ (see [5, Definition 1.1]),*

(ii) *$\mathrm{Ult}_\nu(L[b], \mathscr{U}_\zeta \cap L[b])$ is well founded for all ν.*

PROOF. By Lemma 2 (ii), part (ii) will follow once we establish (i) (see [5, Theorem 3.6]). Let $\mathscr{V} = \mathscr{U}_\zeta \cap L[b]$. By Lemmas 2 (i) and 3, $\mathscr{V}$ satisfies all the conditions for being an $L[b]$-ultrafilter except possibly (v) of [5, Definition 1.1]: namely, that whenever $\langle x_\mu : \mu < \delta_\zeta\rangle \in M_\zeta$, $\{\mu : x_\mu \in \mathscr{V}\} \in M_\zeta$. This condition is necessary for the iterated ultrapowers to be defined, but in any case we may define $\mathrm{Ult}_1(L[b], \mathscr{V})$, which will be well founded by Lemma 2(ii). The assumption that $b \sim \delta_\zeta \subset K_0$ insures that b is fixed by the embedding $i_{01}^{\mathscr{V}} : L[b] \to \mathrm{Ult}_1(L[b], \mathscr{V})$. Hence, $\mathrm{Ult}_1(L[b], \mathscr{V}) = L[b]$, so $\mathscr{V}$ (which equals $\{x \in \mathscr{P}(\delta_\zeta) \cap L[b] : \delta_\zeta \in i_{01}^{\mathscr{V}}(x)\}$) is an $L[b]$-ultrafilter on δ_ζ (see [5, Lemma 4.6]).

LEMMA 5. *If b is a θ-set, $\zeta < \omega_1$, $\theta < \omega_1$, and $\xi < \theta$ is such that $\forall \eta < \xi[\lambda(b, \eta) < \delta_\zeta]$ and $\forall n \forall \eta[\xi \le \eta < \theta \to \gamma(b, \eta, n) \in K_0]$, then $\Phi(b, \xi)$.*

PROOF. Use iterated ultrapowers of $L[b]$ by $\mathscr{U}_\zeta \cap L[b]$. The details are exactly the same as in the proof of [5, Lemma 10.10], so we omit them.

Lemma 5 is analogous to Lemma 6 of [6]. The easier proof given there will not work, however, since the δ_ζ may not really be measurable.

We now state a more complicated version of [6, Lemma 7], which will enable us to collapse down a θ-set for an application of Lemma 5. The proof is almost identical to the one in [6]. If $x \subset \mathrm{ORD}$, we use ord (x) to denote the order type of x.

LEMMA 6. *Let $a \subset K_\alpha$ be of order type α, where $\alpha < \omega_1$. Say $a = \{\sigma_\zeta : \zeta < \alpha\}$ in increasing enumeration. Then for each $\zeta < \alpha$,* $\mathrm{ord}\,(\sigma_\zeta \cap H(a, K_{\zeta+1} \sim \sigma_\zeta)) \le \delta_\zeta$.

Now, to prove the theorem, let b be a θ-set, $\theta < \omega_1$, and $b \subset K_{\omega\cdot\theta+1}$. Let $\xi < \theta$. Let $A = H(b, K_{\omega\cdot\theta} \sim \sup(\{\lambda(b, \eta) : \eta < \xi\}))$, and j the isomorphism from A onto the transitive collapse of A. For $\xi \le \eta < \theta$ and $n < \omega$, $j(\gamma(b, \eta, n)) = \gamma(b, \eta, n) \in K_0$, and for $\eta < \xi$ and $n < \omega$, $j(\gamma(b, \eta, n)) \le \delta_{\omega\cdot\eta+n}$ (by Lemma 6). By Lemma 5 (applied with some ζ between $\omega\cdot\xi$ and ω_1), $\Phi(j(b), \xi)$, and hence $\Phi(b, \xi)$. Thus, $\forall \xi < \theta \Phi(b, \xi)$.

§6. **Conclusion.** There are many other properties related to P that one may obtain by suitable large cardinal axioms. For example, let $Q(\kappa, \pi, \lambda)$ be the statement that κ, π, λ are cardinals, $\kappa < \pi < \lambda$, and whenever $F : {}^\omega(\lambda) \to \pi$, there is an $A \subset \lambda$ such that $\overline{\overline{A}} = \pi$ and card $(F''({}^\omega A)) \le \kappa$. Clearly $Q(\kappa, \pi, \lambda)$ implies $P(\lambda)$.

To obtain a triple satisfying Q, assume in the notation of §1, that $j: V \to M$ is such that ${}^{j^2(\kappa)}M \subset M$. Then $Q(\kappa, j(\kappa), j^2(\kappa))$.

Similarly to Lemma 1 in §5, we can, via Skolem functions, translate $Q(\kappa, \pi, \lambda)$ into a statement about models, obtaining a form of Chang's two-cardinal conjecture in the language $\mathscr{L}_{\omega_1\omega_1}$. Namely, whenever we have a two-cardinal model, $\langle B, U, \cdots \rangle$ for $\mathscr{L}_{\omega_1\omega_1}$ of type $\langle \lambda, \pi \rangle$, it has an elementary submodel (in the sense of $\mathscr{L}_{\omega_1\omega_1}$) of type $\langle \pi^\omega, \kappa \rangle$. Conceivably, these ideas may be used to prove theorems about $\mathscr{L}_{\omega_1\omega_1}$. We have been unable to discover any such theorems, however, that do not follow, as above, by trivially tracing through definitions.

REFERENCES

[1] P. ERDÖS and A. HAJNAL, *On a problem of B. Jónsson*, ***Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques***, vol. 14 (1966), pp. 19–23.

[2] H. GAIFMAN, *Pushing up the measurable cardinal*, Lecture notes 1967 Institute on Axiomatic Set Theory (University of California, Los Angeles, Calif., 1967), American Mathematical Society, Providence, R.I., 1967, pp. IV R 1–16. (Mimeographed).

[3] K. GÖDEL, *The consistency of the continuum hypothesis*, Princeton Univ. Press, Princeton, 1940.

[4] J. L. KELLEY, ***General topology***, D. Van Nostrand Co., Inc., Princeton, 1965.

[5] K. KUNEN, *Some applications of iterated ultrapowers in set theory*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 179–227.

[6] ———, *On the GCH at measurable cardinals*, ***Proceedings of 1969 Logic Summer School at Manchester***, pp. 107–110.

[7] W. REINHARDT and R. SOLOVAY, *Strong axioms of infinity and elementary embeddings*, to appear.

[8] D. SCOTT, *Measurable cardinals and constructible sets*, ***Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques***, vol. 9 (1961) pp. 521–524.

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