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Inaccessibility in constructive set theory and type theory

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Abstract

This paper is the first in a series whose objective is to study notions of large sets in the context of formal theories of constructivity. The two theories considered are Aczel's constructive set theory (**CZF**) and Martin-Löf's intuitionistic theory of types.

This paper treats Mahlo's π-numbers which give rise classically to the enumerations of inaccessibles of all transfinite orders. We extend the axioms of **CZF** and show that the resulting theory, when augmented by the tertium non-datur, is equivalent to **ZF** plus the assertion that there are inaccessibles of all transfinite orders. Finally, the theorems of that extension of **CZF** are interpreted in an extension of Martin-Löf's intuitionistic theory of types by a universe.

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1. Prefatory and historical remarks

The paper is organized as follows: After recalling Mahlo's π-numbers and relating the history of universes in Martin-Löf type theory in Section 1, we study notions of inaccessibility in the context of Aczel's constructive set theory **CZF**. Section 2 also introduces the intuitionistic set theory $\mathbf{CZF}_\pi$, an extension of **CZF**, which asserts the existence of inaccessible sets of all transfinite orders, thereby capturing Mahlo's π-hierarchy.

In a series of papers [2–4], Aczel gave constructive interpretations of **CZF** and various extensions of Martin-Löf's intuitionistic type theory. In a similar vein we

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vindicate the constructiveness of $\mathbf{CZF}_\pi$ by interpreting it in type theory. However, none of the Martin-Löf type theories with universes that have been proposed up till now suffices for this task. Therefore we introduce a new system, **MLQ**, of type theory in Section 3. **MLQ** is well in keeping with the spirit of Martin-Löf type theory as an open ended system. While previous extensions were mainly the result of adding new set constructors to existing formalizations, **MLQ** internalizes this process by allowing for the generation of sets (of codes) of set constructors. The new type of **MLQ** that brings about the interpretation of $\mathbf{CZF}_\pi$ is a universe **M** which is simultaneously generated with a set **Q** of (codes for) set constructors under which it is going to be closed.

Section 4 is devoted to interpreting $\mathbf{CZF}_\pi$ in **MLQ**.

1.1. Mahlo's π-numbers

In a paper from 1911 Mahlo [11] investigated two hierarchies of regular cardinals. In view of its early appearance this work is astounding for its refinement and its audacity in venturing into the higher infinite. Mahlo called the cardinals considered in the first hierarchy *π_α-numbers*. In modern terminology they are spelled out as follows:

κ is 0-*weakly inaccessible*	iff κ is regular;
κ is $(\alpha+1)$-*weakly inaccessible*	iff κ is a regular limit of α-weakly inaccessibles
κ is λ-*weakly inaccessible*	iff κ is α-weakly inaccessible for every $\alpha<\lambda$

for limit ordinals λ. This hierarchy could be extended through diagonalization, by taking next the cardinals κ such that κ is κ-weakly inaccessible and after that choosing regular limits of the previous kind, etc.

Mahlo also discerned a second hierarchy which is generated by a principle superior to taking regular fixed-points. Its starting point is the class of ρ_0-numbers which later came to be called *weakly Mahlo cardinals*. Weakly Mahlo cardinals are larger than any of those that can be obtained by the above processes from below. Remarkably, Gaifman [8] showed that in a mathematically precise sense a weakly Mahlo cardinal is the least upper bound of diagonalizing the regular fixed-point operation from below.

1.2. A brief history of universes

Martin-Löf, in 1975 [12] and in his 1984 monograph [13] on an intuitionistic theory of types, gave a framework for a theory of constructive types or sets. Aside from the set formation principles of *generalised Cartesian product* and *infinite disjoint union*, he adds two set constructions of considerable proof-theoretic strength. The first takes a family of sets indexed by a set and constructs the set of trees all of whose branchings are members of that family, the so-called *well-ordering type* or **W**-type. The idea of the second is to define a *universe* as the least set closed under certain specified set forming operations. In [12, 13] Martin-Löf only considers an infinite tower of universes $\mathbf{U}_0 \in \mathbf{U}_1 \in \cdots \in \mathbf{U}_n \in \ldots$ all of which are closed under the same ensemble of set forming operations. The next natural step was to implement a *universe operator* into type theory which takes a family of sets and constructs a universe above it. Such a universe operator was formalized by Palmgren while working on a domain-theoretic interpretation of the logical framework with an infinite sequence of universes (cf. [17]). Aiming at extensions of type theory with more powerful axioms, Martin-Löf then suggested finding axioms for a universe $\mathbb{V}$ which itself is closed under the universe operator. The type-theoretic formalization of the pertinent rules appeared first in Griffor–Palmgren [9] and are, in their final form, due to Palmgren [16], where the universe was referred to as a *superuniverse* for intuitionistic type theory.

Let $\mathbf{ML}_n$ denote the system with n universes but *without* the $\mathbf{W}$-type. The first indication of the proof-theoretic strength of type theory with universes came with P. Aczel's proof (cf. [1]) that the proof-theoretic ordinal of $\mathbf{ML}_1$ is $\varphi_{\varepsilon_0}0$. Feferman [6] then proved *Hancock's Conjecture* about the strength of the theories $\mathbf{ML}_n$ for general n. As a result (independently proved by Aczel), $\mathbf{ML}_{<\omega} := \bigcup_{n<\omega}\mathbf{ML}_n$ has the strength of ramified analysis or the ordinal Γ_0. Later Griffor–Rathjen [10] and Setzer [22] independently showed that type theory with a single universe closed under $\mathbf{W}$-types exceeds the proof-theoretic strength of $\mathbf{KPi}$ (the classical theory of one recursively inaccessible ordinal).

If $\mathbb{V}$ denotes the superuniverse noted above then the formation rules for $\mathbb{V}$ are

$$\frac{}{\mathbb{V}\ Set} \qquad \frac{a \in \mathbb{V}}{\mathbb{S}(a)\ Set}.$$

The $\mathbb{V}$-introduction rules stating the closure of $\mathbb{V}$ under universe formation have the form

$$\frac{a\in\mathbb{V} \quad b\in\mathbb{V}\,[x\in\mathbb{S}(a)]}{\mathbf{u}(a,(x)b)\in\mathbb{V}} \qquad \frac{a\in\mathbb{V} \quad b\in\mathbb{V}\,[x\in\mathbb{S}(a)]}{\mathbb{S}(\mathbf{u}(a,(x)b))=\mathbf{U}(\mathbb{S}(a),(x)\mathbb{S}(b))}$$

$$\frac{a\in\mathbb{V} \quad b\in\mathbb{V}\,[x\in\mathbb{S}(a)] \quad c\in\mathbb{S}(\mathbf{u}(a,(x)b))}{\mathbf{t}(a,(x)b,c)\in\mathbb{V}}$$

$$\frac{a\in\mathbb{V} \quad b\in\mathbb{V}\,[x\in\mathbb{S}(a)] \quad c\in\mathbb{S}(\mathbf{u}(a,(x)b))}{\mathbb{S}(\mathbf{t}(a,(x)b,c))=\mathbf{T}(\mathbb{S}(a),(x)\mathbb{S}(b),c)}.$$

The set $\mathbf{U}(\mathbb{S}(a),(x)\mathbb{S}(b))$ is described separately by a *module* of rules. Given an arbitrary family of sets $(A,(x)B)$ these rules state that $\mathbf{U}(A,(x)B)$ and $(v)\mathbf{T}(A,(x)B,v)$ give the Tarski formulation of a universe reflecting product, disjoint union and $\mathbf{W}$-type construction while containing the set A and all of the sets $B(x)$ for x in A.

We may then take the family $(\mathbb{V},\mathbb{S})$ as a basis for a new universe and repeating this process obtain universes $\mathbb{V}_n$. This process itself can be summarized as a module of rules parameterized by an arbitrary family of sets giving a new operator, a *superuniverse operator*. One may then form a *super superuniverse* closed under the *superuniverse operator*. Each step gives rise to successively stronger operators. For each such one can use $\mathbf{W}$-types and their elimination rules to obtain the αth superuniverse of order n, where α is an element in a $\mathbf{W}$-type. Let $\mathbb{V}_{n,\alpha}$ denote this universe.

Already here there is, at least informally, an analogy with Mahlo's π-numbers. The essence of the fundamental step in giving a *new universe operator* is collecting a set constructors defined thus far and asserting the existence of a universe closed under each constructor in this collection. The difficulty in extending the $\mathbb{V}_{n,\alpha}$'s mentioned above is internalizing the process of going from $\mathbb{V}_{n,\alpha}$ to the operator that is used to define $\mathbb{V}_{n+1,\alpha}$.

Nonetheless, Mahlo's π-numbers are objects defined within the classical theory of sets. An important component of our understanding of these objects in a constructive setting is an extension by new axioms of Aczel's theory of constructive sets **CZF**. These axioms are formulated much as they are classically, while the underlying logic is intuitionistic. The approach we take is to interpret the theorems of **CZF** as propositions in the theory of types, while showing that augmenting this version of **CZF** by the law of the excluded middle results in the classical theory of Mahlo's π-numbers.

2. Large sets in constructive set theory

Analogues or constructivized versions of large cardinals have emerged in generalized recursion theory (in the shape of recursively large ordinals) and in ordinal representation systems used in proof theory (cf. [21]). In this section we embark on another approach to constructivizing large cardinals which might also contribute to a better understanding of the analogies alluded to above. To give an example, let us look at a largeness notion derived from regular cardinals:

Definition 2.1. A is inhabited if $\exists x\, x \in A$. An inhabited set A is *regular* if A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\, \exists y\, (\langle x, y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, (\langle x, y\rangle \in R) \;\wedge\; \forall y \in b\, \exists x \in a\, (\langle x, y\rangle \in R).^3$$

In the context of **ZFC** we have that V_κ is regular iff κ is a regular cardinal. The analogy between admissible sets and regular sets is drawn by restricting the class of relations (or functions) to the A-recursive ones. In contradistinction to the latter approach we suggest a study of regularity such that the only changes being made take place in the surrounding environment.[4] The particular environment will be Aczel's constructive set theory, **CZF**. The latter theory is due to Aczel (cf. [2–4]) and extends Myhill's constructive set theory **CST** (cf. [14]) which grew out of endeavours to discover a (simple) formalism that relates to Bishop's constructive mathematics as **ZFC** relates to classical Cantorian mathematics. The novel ideas were to replace the Powerset by the (classically equivalent) Exponentiation Axiom and to discard full Comprehension

[3] In particular, if $R : a \to A$ is a function, then the image of R is an element of A.

[4] Feferman [7] is in a similar vein, but undertakes a different approach.

while retaining full Collection. Aczel extended **CST** to **CZF** and corroborated the constructiveness of the latter theory by interpreting it in Martin-Löf's intuitionistic type theory (cf. [13]).

2.1. The system CZF

In this section we will summarize the language and axioms for Aczel's constructive set theory or **CZF**. The language of **CZF** is the first order language of ZF whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Infinity: $\exists x \forall u[u \in x \leftrightarrow (\emptyset \in x \vee \exists v \in x(u = v \cup \{v\}))]$.

Set Induction: $\forall x[\forall y \in x \phi(y) \rightarrow \phi(x)] \rightarrow \forall x \phi(x)$

Restricted Separation: $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$

for all *restricted* formulae ϕ. A set-theoretic formula is *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x \in y$ and $\exists x \in y$ only.

Strong Collection: For all formulae ϕ,

$$\forall a\, [\forall x \in a \exists y \phi(x,y) \rightarrow \exists b\, [\forall x \in a\, \exists y \in b\, \phi(x,y) \wedge \forall y \in b\, \exists x \in a\, \phi(x,y)]]$$

Subset Collection: For all formulae ψ,

$$\begin{aligned}&\forall a \forall b \exists c \forall u\ [\forall x \in a\, \exists y \in b\ \psi(x,y,u)\\ &\quad \rightarrow \exists d \in c\, [\forall x \in a\, \exists y \in d\, \psi(x,y,u) \wedge \forall y \in d\, \exists x \in a\, \psi(x,y,u)]]\end{aligned}$$

for all formulae ϕ.

The mathematically important axiom of *Dependent Choices* (**DC**) could be included among the axioms of **CZF** without changing any essential properties of **CZF**, including its interpretation in type theory.

The Subset Collection schema easily qualifies as the most intricate axiom of **CZF**. To explain this axiom in different terms, we introduce the notion of *fullness* (cf. [2]).

Definition 2.2. For sets A, B let ${}^{A}B$ be the class of all functions with domain A and with range contained in B. Let $\mathbf{mv}({}^{A}B)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u \in A\, \exists v \in B\, \langle u, v\rangle \in R$. A set C is said to be *full in* $\mathbf{mv}({}^{A}B)$ if $C \subseteq \mathbf{mv}({}^{A}B)$ and

$$\forall R \in \mathbf{mv}({}^{A}B)\, \exists S \in C\, S \subseteq R.$$

The expression $\mathbf{mv}({}^{A}B)$ should be read as the collection of *multi-valued functions* from the set A to the set B.

Additional axioms we shall consider are the following.

Exponentiation: $\forall x \forall y \exists z\, z = {}^{x}y$.

Fullness: $\forall x \forall y \exists z$ "z full in $\mathbf{mv}({}^{x}y)$".

The next result is an equivalent rendering of [2], Definition 2.2. We include a proof for the reader's convenience.

Proposition 2.3 (Aczel). *Let* **CZF**$^-$ *be* **CZF** *without* Subset Collection.
(i) **CZF**$^-$ $\vdash$ Subset Collection $\leftrightarrow$ Fullness.
(ii) **CZF** $\vdash$ Exponentiation.

Proof. (i): For "$\rightarrow$" let $\phi(x,y,u)$ be the formula $y \in u \wedge \exists z \in B\,(y = \langle x,z\rangle)$. Using the relevant instance of Subset Collection and noticing that for all $R \in \mathbf{mv}({}^{A}B)$ we have $\forall x \in A\,\exists y \in A \times B\,\phi(x,y,R)$, there exists a set C such that $\forall R \in \mathbf{mv}({}^{A}B)\,\exists S \in C\,S \subseteq R$.

"$\leftarrow$": Let C be full in $\mathbf{mv}({}^{A}B)$. Assume $\forall x \in A \exists y \in B \phi(x,y,u)$. Define $\psi(x,w,u) := \exists y \in B\,[w = \langle x,y\rangle \wedge \phi(x,y,u)]$. Then $\forall x \in A \exists w \psi(x,w,u)$. Thus, by Strong Collection, there exists $v \subseteq A \times B$ such that

$$\forall x \in A\,\exists y \in B\,[\langle x,y\rangle \in v \wedge \phi(x,y,u)] \wedge \forall x \in A\,\forall y \in B\,[\langle x,y\rangle \in v \rightarrow \phi(x,y,u)].$$

As C is full, we find $w \in C$ with $w \subseteq v$. Consequently, $\forall x \in A \exists y \in \mathbf{ran}(w) \phi(x,y,u)$ and $\forall y \in \mathbf{ran}(w) \exists x \in A\,\phi(x,y,u)$, where $\mathbf{ran}(w) := \{v\colon \exists z\,\langle z,v\rangle \in w\}$.

Whence $D := \{\mathbf{ran}(w) : w \in C\}$ witnesses the truth of the instance of Subset Collection pertaining to ϕ.

(ii) Let C be full in $\mathbf{mv}({}^{A}B)$. If now $f \in {}^{A}B$, then $\exists R \in C\,R \subseteq f$. But then $R = f$. Therefore ${}^{A}B = \{f \in C\colon\ f \text{ is a function}\}$. $\square$

Let **TND** be the principle of excluded third, i.e. the schema consisting of all formulae of the form $A \vee \neg A$. The first central fact to be noted about **CZF** is the following.

Proposition 2.4. **CZF** + **TND** = **ZF**.

Proof. Note that classically Collection implies Separation. Powerset follows classically from Exponentiation. $\square$

To stay in the world of **CZF** one has to keep away from any principles that imply **TND**. Moreover, it is fair to say that **CZF** is such an interesting theory owing to the non-derivability of Powerset and Separation. Therefore one ought to avoid any principles which imply Powerset or Separation.

In what follows we shall investigate largeness notions corresponding to inaccessibility.

2.2. Inaccessibility

Let **Reg**(A) be the statement that A is a regular set (cf. Definition 2.1). The next axiom states that the universe is a union of regular sets.

Regular Extension Axiom (REA). $\forall x \exists y[x \subseteq y \wedge \mathbf{Reg}(y)]$.

Definition 2.5. A set I is *set-inaccessible* if $\mathbf{Reg}(I)$, $\forall x \in I\, \exists y \in I\; [x \subseteq y \wedge \mathbf{Reg}(y)]$, and I is a model of **CZF**, i.e. the structure $\langle I, \in \restriction (I \times I)\rangle$ is a model of **CZF**.[5]

Let $\mathbf{inac}(I)$ denote the assertion that I is set-inaccessible.

Since Restricted Separation is an infinite schema of axioms one might wonder how the notion of set-inaccessibility is actually formalized in **CZF**. One way to do this is by formalizing satisfaction for restricted formulae in **CZF**. A probably simpler way consists in noticing that the schema can be replaced by a finite number of special cases (cf. [14], Appendix A).

Recall that (in **ZFC**) a cardinal κ is *strongly inaccessible* if κ is regular and a *strong limit*, i.e. $\forall \rho < \kappa\, (2^\rho < \kappa)$, where 2^ρ means cardinal exponentiation.

The aim of the above definition of inaccessibility is to capture as much as possible of the classical notion of a strongly inaccessible cardinal. In particular, when arguing in **ZFC**, set-inaccessibility of V_κ should imply that κ is a strongly inaccessible cardinal. To this end we included the following property in the original definition of set-inaccessibility:

$$(\mathbf{A}) \quad \forall A, B \in I\, \exists C \in I \text{ “}C \textit{ is full in } \mathbf{mv}({}^A B)\text{”}.$$

However, Peter Aczel has pointed out to us that $(\mathbf{A})$ is redundant. The point is that the regularity condition can be used to get $(\mathbf{A})$ from validity of Fullness in I.

Lemma 2.6 (Aczel) (**CZF**). *If I is set-inaccessible, then for all $A, B \in I$ there exists $C \in I$ such that C is full in $\mathbf{mv}({}^A B)$.*

Proof. Let I be a regular model of **CZF**. We first show:

$$\forall A \in I \text{ “}I \cap \mathbf{mv}({}^A I) \textit{ is full in } \mathbf{mv}({}^A I)\text{”}; \tag{1}$$

$$\forall A, B \in I\, \exists C \in I\;\; I \models \text{“}C \textit{ is full in } \mathbf{mv}({}^A B)\text{”}. \tag{2}$$

To prove (1), let $A \in I$ and $R \in \mathbf{mv}({}^A I)$. Then R is a subset of $A \times I$ such that for all $x \in A$ there is $y \in I$ such that xRy. Let R' be the set of all $(x, (x, y))$ such that xRy. Then $R' \in \mathbf{mv}({}^A I)$ also, as I is closed under Pairing. Hence, as I is regular, there is $S \in I$ such that $\forall x \in A\, \exists z \in S\; xR'z \wedge \forall z \in S\, \exists x \in A\; xR'z$. Hence $S \in (I \cap \mathbf{mv}({}^A I))$ and S is a subset of R. So (1) is proved. (2) is just stating that $I \models$ “*Fullness*”, which follows from Proposition 2.3 since I is a model of **CZF**.

To prove $(\mathbf{A})$, let $A, B \in I$ and choose $C \in I$ as in (2). It follows that $C \subseteq \mathbf{mv}({}^A B)$ and

$$\forall R' \in I\, [R' \in \mathbf{mv}({}^A B) \rightarrow \exists R_0 \in C\, (R_0 \subseteq R')].$$

[5] Several axioms are redundant here, e.g. $\omega \in I$ and $\mathbf{Reg}(I)$ imply $I \models$ *Pairing*; also $\mathbf{Reg}(I)$ entails $I \models$ *Subset Collection*.

So to complete the proof of (**A**) it suffices, given $R \in \mathbf{mv}({}^A B)$ to find a subset R' of R such that $R' \in (I \cap \mathbf{mv}({}^A B))$, as then we can get $R_0 \in C$, as above, a subset of R' and hence of R.

But, as B is a subset of I, $R \in \mathbf{mv}({}^A I)$ so that, by (1), there is a subset R' of R such that $R' \in (I \cap \mathbf{mv}({}^A I))$. It follows that $R' \in (I \cap \mathbf{mv}({}^A B))$ and we are done. □

Corollary 2.7. (i) (**ZF**) *If Z is set-inaccessible, then there exists a weakly inaccessible cardinal κ such that $Z = V_\kappa$.*

(ii) (**ZFC**) *If Z is set-inaccessible, then there exists a strongly inaccessible cardinal κ such that $Z = V_\kappa$.*

Proof. (i): If $x \in Z$, then ${}^x 2 \in Z$ by Lemma 2.6, thus the power set of x is in Z as $\{y : y \subseteq x\} = \{\{v \in x : f(v) = 0\} : f \in {}^x 2\}$. Consequently, for all $\alpha \in Z$, $(V_\alpha)^Z = V_\alpha$. Therefore $Z = V_\kappa$, where κ is the least ordinal not in Z. It is readily shown that κ is weakly inaccessible.

(ii): It remains to show that κ is a strong limit. Let $\rho < \kappa$. Using **AC** one finds an ordinal λ together with a bijection $G : {}^\rho 2 \to \lambda$. Set $D := \{f \in {}^\rho 2 \colon G(f) < \kappa\}$. As $D \subseteq {}^\rho 2$ and Z is closed under taking power sets, it follows $D \in Z$. If $\kappa \leqslant \lambda$, then $F := G \restriction D$ would provide a counterexample to the regularity of Z. Thus $\lambda < \kappa$. □

Corollary 2.8. *The following theories prove the same formulae*:

(i) $\mathbf{CZF} + \exists I \mathbf{inac}(I) + \mathbf{TND}$,

(ii) $\mathbf{ZF} + \exists I \mathbf{inac}(I)$.

They are equiconsistent with **ZFC** $+ \exists\kappa$ *"κ inaccessible cardinal".*

Let $\mathbf{TC}(a)$ denote the transitive closure of a. By [10], Definition 2.2, there is a definable class function **TC** such that

$$\mathbf{CZF} \vdash \forall a \left[\mathbf{TC}(a) = a \cup \bigcup\{\mathbf{TC}(x) : x \in a\}\right].$$

Definition 2.9. Let A, B be classes. A is said to be *unbounded in B* if

$$\forall x \in B\, \exists y \in A\, (x \in y \wedge y \in B).$$

Let a and Z be sets. Z is said to be *a-set-inaccessible* if Z is set-inaccessible and there exists a family $(X_b)_{b \in \mathbf{TC}(a)}$ of sets such that for all $b \in \mathbf{TC}(a)$ the following hold:

- X_b is unbounded in Z.
- X_b consists of set-inaccessible sets.
- $\forall y \in X_b\, \forall c \in \mathbf{TC}(b)$ "X_c is unbounded in y".

The function F with domain $\mathbf{TC}(a)$ satisfying $F(b) = X_b$ will be called a *witnessing function for the a-set-inaccessibility of Z*.

Let **PINUM** be the statement $\forall x \forall a \exists Z\, (x \in Z \wedge Z$ *is a-set-inaccessible*$)$. Set

$$\mathbf{CZF}_\pi := \mathbf{CZF} + \mathbf{PINUM}.$$

Corollary 2.10 (CZF). *If Z is a-set-inaccessible and $b \in \mathbf{TC}(a)$, then Z is b-set-inaccessible.*

Lemma 2.11 (CZF). *If Z is set-inaccessible, then Z is a-set-inaccessible iff for all $c \in a$ the c-set-inaccessibles are unbounded in Z.*

Proof. One direction is obvious. So suppose that for all $c \in a$ the c-set-inaccessibles are unbounded in Z. By Corollary 2.10 and the definition of $\mathbf{TC}(a)$, this implies that for all $b \in \mathbf{TC}(a)$ the b-set-inaccessibles are unbounded in Z; thus

$$\forall b \in \mathbf{TC}(a) \forall x \in Z \exists u \in Z (x \in u \wedge u \text{ is } b\text{-set-inaccessible}).$$

Using Strong Collection, there is a set S such that S consists of triples $\langle b, u, x\rangle$, where $b \in \mathbf{TC}(a)$, $x \in u \in Z$ and u is b-set-inaccessible, and for each $b \in \mathbf{TC}(a)$ and $x \in Z$ there is a triple $\langle b, u, x\rangle \in S$. Put

$$S_b = \{u : \exists x \in Z\, \langle b, u, x\rangle \in S\}.$$

Again by Strong Collection there exists a set $\mathscr{F}$ of functions such that for $b \in \mathbf{TC}(a)$ and any $u \in S_b$ there is a function $f \in \mathscr{F}$ witnessing the b-set-inaccessibility of u, and, conversely, any $f \in \mathscr{F}$ is a witnessing function for some $u \in S_b$ for some $b \in \mathbf{TC}(a)$. Now define a function F with domain $\mathbf{TC}(a)$ via

$$F(b) = S_b \cup \bigcup \{f(b) : f \in \mathscr{F}; b \in \mathbf{dom}(f)\}.$$

As S_b is unbounded in Z, so is $F(b)$. Let $y \in F(b)$ and suppose $c \in \mathbf{TC}(b)$. If $y \in S_b$, then there is an $f \in \mathscr{F}$ witnessing the b-set-inaccessibility of y, thus $f(c)$ is unbounded in y and a fortiori $F(c)$ is unbounded in y.

Now assume that $y \in f(b)$ for some $f \in \mathscr{F}$. As $f \restriction \mathbf{TC}(b)$ witnesses the b-set-inaccessibility of y, $f(c)$ is unbounded in y, thus $F(c)$ is unbounded in y. □

The preceding lemma shows that the notion of being α-set-inaccessible is closely related to Mahlo's π_α-numbers. To state this precisely, we recall the notion of κ being *α-strongly inaccessible* (for ordinals α and cardinals κ) which is defined as α-weak inaccessibility except that κ is also required to be a strong limit, i.e. $\forall \rho < \kappa (2^\rho < \kappa)$.

Corollary 2.12 (ZF). *Let $Z = V_\kappa$ be set-inaccessible.*

(i) *If Z is α-set-inaccessible, then κ is α-weakly inaccessible.*

(ii) (**ZFC**) *κ is α-strongly inaccessible iff V_κ is α-set-inaccessible.*

3. A type-theoretic universe for π-numbers

The formalisation of universes for intuitionistic type theory we use in this section is that referred to as the *Tarski formulation* in Martin-Löf's monograph [13]. It involves the simultaneous definition of a universe $\mathbf{U}$ and of a mapping $\mathbf{T}$ which, given an element of **U**, produces the set *coded* by that element. The codes are built up using codes for ground sets as well as codes for the set constructors. The mapping **T** on codes for ground sets gives them as its values and on codes for constructors applied to other codes produces those constructors applied to corresponding sets. Thus **U** together with **T** give a family of sets. This family of sets can then be taken as the ground sets of a *second universe* in a similar fashion. The definition of the *nth universe* constructed in this way was given by Martin-Löf (cf. [12, 13]).

It is essential for the results here and forthcoming extensions of them that one consider the simultaneous generation of a universe of (codes for) sets and (codes for) functionals or quantifiers of finite (dependent) type over that universe. For this and other purposes Martin-Löf has formalized a finite dependent type structure referred to as the *logical framework.*

3.1. Types in the logical framework

The logical framework, **LF**, is a finite dependent type structure over a collection of ground types. The only ground types we will be concerned with here are *Set* and the type of elements, $\widehat{A}$, for sets A. The logical framework's elementary statements are the judgements that something is a type, that two given types are equal, that some object is of a given type, and that two objects the same objects of a given type. For each A such that we have made the judgement A *Set* in constructive set theory we have in the logical framework the judgement $A : Set$. The notation used for the four judgement forms of **LF** is: α *type*, $\alpha = \beta$ *type*, $a : \alpha$, and $a = b : \alpha$, which are read α *is a type, the types* α *and* β *are identical, a is of type* α, *a and b are identical objects of type* α, respectively. In the logical framework every set A, i.e. $A : Set$, gives rise to a type $\widehat{A}$ which is the type of A's elements, and that a is an element of A is expressed by $a : \widehat{A}$. The rules of the logical framework have a form similar to those for constructive type theory. We declare the *ground type Set* by

$$\frac{}{Set\ type} \qquad \frac{A : Set}{\widehat{A}\ type} \qquad \frac{A = B : Set}{\widehat{A} = \widehat{B}\ type}.$$

Other types are formed using the *type formation rule* for dependent families of types: [6]

$$\frac{\alpha\ type \qquad \beta\ type\ [x : \alpha]}{(x : \alpha)\beta\ type}$$

together with the rules for abstraction and application

$$\frac{b : \beta\ [x : \alpha]}{(x)b : (x : \alpha)\beta} \qquad \frac{c : (x : \alpha)\beta \qquad a : \alpha}{c(a) : \beta(a/x)}.$$

In addition to these rules there are rules corresponding to β- and η-conversion giving equalities at the level of objects and types. For more details on the logical framework we refer to the book [15], Part III or [18], chapter 8.

[6] The judgements within brackets are the discharged assumptions.

Instead of $f(a_1)(a_2)\dots(a_n)$ we shall often write $f(a_1, a_2, \dots, a_n)$. When β does not depend on x, we sometimes emphasize this by writing $(\alpha)\beta$ or $\alpha \to \beta$ instead of $(x : \alpha)\beta$. The difference between the notations A and $\widehat{A}$ will be neglected. We shall continue to use $a \in A$ (as in the theory of constructive sets) instead of $a : \widehat{A}$.

3.2. *A universe operator*

Let $\mathbf{K}$ be the type of functors Φ which take a set A together with family of sets $B : A \to Set$ over A and produce a set $\Phi(A, B)$. We shall refer to $\mathbf{K}$ as the type of *quantifiers*. Expressed in the language of the logical framework $\mathbf{K}$ is the type $(X : Set)((X)Set)Set$.

In this subsection we introduce a new universe operator $\mathbf{U}$ in analogy with the original universe operator mentioned at the outset of this section. Given five types of the shapes

$$\begin{array}{ll} A : Set, & B : (x \in A)Set, \\ C : Set, & F : (y \in C)\mathbf{K}, \\ & G : (y \in C)(X : Set)(Y : (X)Set)(F(y)(X, Y))Set, \end{array} \tag{3}$$

$\mathbf{U}$ and the decoding functional $\mathbf{T}$ produce a universe of sets $\mathbf{U}(C, F, G, A, B)$ whose decoding functional is $\mathbf{T}(C, F, G, A, B)$. In addition to being closed under the standard set formers $\mathbf{\Pi}, \mathbf{\Sigma}, \mathbf{W}, \dots$, the universe $\mathbf{U}(C, F, G, A, B)$ contains the sets $A, C, B(a)$, where $a \in A$, and is "closed under" the quantifiers $F(c)$, where $c \in C$. The need to express a strong form of closure under the quantifiers $F(c)$ $[c \in C]$ necessitates the introduction of the third parameter G. The reasons are the following: We are only interested in quantifiers which when fed a family of sets produce a universe, i.e. a set of codes for sets. One way to view such universe-valued quantifiers is to conceive of them as pairs Q, T_Q, where $Q : \mathbf{K}$ and $T_Q(X, Y)$ decodes the elements of $Q(X, Y)$ for any family of sets $Y : (X)Set$, $X : Set$, i.e. $T_Q(X, Y) : (Q(X, Y))Set$. The meaning of G in (3) is that, for every $c \in C$, $G(c)$ provides the set-valued decoding functional pertaining to the quantifier $F(c)$.

3.2.1. $\mathbf{U}(\mathscr{P})$*-formation*

The typing of $\mathbf{U}$ and $\mathbf{T}$ is spelled out in the introduction rules. Let $\mathfrak{A}$ abbreviate the standing assumptions of (3). We shall use the abbreviation $\mathscr{P} := C, F, G, A, B$.

$$\frac{\mathfrak{A}}{\mathbf{U}(\mathscr{P}) : Set} \qquad \frac{\mathfrak{A} \qquad z \in \mathbf{U}(\mathscr{P})}{\mathbf{T}(\mathscr{P}, z) : Set}.$$

3.2.2. $\mathbf{U}(\mathscr{P})$*-introduction*

We refrain from repeating the standard introduction rules for universes. That A, C, $B(a)$ are in $\mathbf{U}(\mathscr{P})$ (though only via codes) is expressed by

$$\frac{\mathfrak{A}}{\star(\mathscr{P}) \in \mathbf{U}(\mathscr{P})} \qquad \frac{\mathfrak{A}}{\mathbf{T}(\mathscr{P}, \star(\mathscr{P})) = C : Set}$$

$$\frac{\mathfrak{A}}{\diamond(\mathscr{P})\in\mathbf{U}(\mathscr{P})}\qquad\frac{\mathfrak{A}}{\mathbf{T}(\mathscr{P},\diamond(\mathscr{P}))=A:Set}$$

$$\frac{\mathfrak{A}\qquad a\in A}{\jmath(\mathscr{P},a)\in\mathbf{U}(\mathscr{P})}\qquad\frac{\mathfrak{A}\qquad a\in A}{\mathbf{T}(\mathscr{P},\jmath(\mathscr{P},a))=B(a):Set}.$$

Thus $\star(\mathscr{P})$ and $\diamond(A)$ are the respective codes for C and A in $\mathbf{U}(\mathscr{P})$. $\jmath(\mathscr{P},a)$ (for $a\in A$) provides a code for $B(a)$ in $\mathbf{U}(\mathscr{P})$.

It remains to assert closure under the quantifiers $F(e)$ for $e\in C$. This is done by introducing two new constants $\sharp$ and $\dagger$ with the following rules:

$$\frac{\mathfrak{A}\qquad e\in C\qquad a\in\mathbf{U}(\mathscr{P})\qquad b\in\mathbf{U}(\mathscr{P})\ [y\in\mathbf{T}(\mathscr{P},a)]}{\sharp(\mathscr{P},e,a,(y)b)\in\mathbf{U}(\mathscr{P})}$$

$$\frac{\mathfrak{A}\quad e\in C\quad a\in\mathbf{U}(\mathscr{P})\qquad b\in\mathbf{U}(\mathscr{P})\ [y\in\mathbf{T}(\mathscr{P},a)]\qquad c\in\mathbf{T}(\mathscr{P},\sharp(\mathscr{P},e,a,(y)b))}{\dagger(\mathscr{P},e,a,(y)b,c)\in\mathbf{U}(\mathscr{P})}$$

$$\frac{\mathfrak{A}\qquad e\in C\qquad a\in\mathbf{U}(\mathscr{P})\qquad b\in\mathbf{U}(\mathscr{P})\ [y\in\mathbf{T}(\mathscr{P},a)]}{\mathbf{T}(\mathscr{P},\sharp(\mathscr{P},e,a,(y)b))=F(e)(\mathbf{T}(\mathscr{P},a),(y)\mathbf{T}(\mathscr{P},b)):Set}$$

$$\frac{\mathfrak{A}\quad e\in C\quad a\in\mathbf{U}(\mathscr{P})\qquad b\in\mathbf{U}(\mathscr{P})\ [y\in\mathbf{T}(\mathscr{P},a)]\qquad c\in\mathbf{T}(\mathscr{P},\sharp(\mathscr{P},e,a,(y)b))}{\mathbf{T}(\mathscr{P},\dagger(\mathscr{P},e,a,(y)b,c))=G(e)(\mathbf{T}(\mathscr{P},a),(y)\mathbf{T}(\mathscr{P},b),c):Set}.$$

The impact of the last four rules is that (modulo coding) the universe created by the quantifier $F(e)$ applied to $(\mathbf{T}(\mathscr{P},a),(y)\mathbf{T}(\mathscr{P},b))$ is a subuniverse of $\mathbf{U}(\mathscr{P})$: $\sharp(\mathscr{P},e,a,(y)b)$ provides a code for $F(e)(\mathbf{T}(\mathscr{P},a),(y)\mathbf{T}(\mathscr{P},b))$ in $\mathbf{U}(\mathscr{P})$ while $(x)\dagger(\mathscr{P},e,a,(y)b,x)$ injects codes for elements of $F(e)(\mathbf{T}(\mathscr{P},a),(y)\mathbf{T}(\mathscr{P},b))$ into $\mathbf{U}(\mathscr{P})$.

3.3. *A type-theoretic universe with the strength of* $\mathbf{CZF}_\pi$

We shall introduce a type theory **MLQ**, an extension of Martin-Löf's 1984 type theory, with the help of which we show the constructiveness of $\mathbf{CZF}_\pi$. The universe operator **U** of the previous section falls short of achieving this goal. To obtain a universe of sets for type theory which allows for an interpretation of $\mathbf{CZF}_\pi$ we need to introduce a universe **M** (closed under **U**) jointly with a set **Q** of codes for constructors by simultaneous induction (as well as their decoding mappings in the Tarski formulation of universes). **M** will be the reflection of the type of sets (or set formation) while **Q** will be a reflection of the type of *quantifiers*, **K**. **S**, **F** and **G** will denote their respective decoding functions. The pertinent rules are

$$\frac{}{\mathbf{M}:Set}\qquad\frac{a\in\mathbf{M}}{\mathbf{S}(a)\ :Set}$$

$$\frac{}{\mathbf{Q}:Set}\qquad\frac{f\in\mathbf{Q}}{\mathbf{F}(f):\mathbf{K}}\qquad\frac{f\in\mathbf{Q}}{\mathbf{G}(f):(X:Set)(Y:(X)Set)(\mathbf{F}(f)(X,Y))Set}.$$

The introduction rules for **M** include the usual ones for ground sets and closure under the usual set constructors (including the formation of **W**-types) together with

$$\frac{f\in\mathbf{Q}\qquad a\in\mathbf{M}\qquad b\in\mathbf{M}\ \ [x\in\mathbf{S}(a)]}{\mathfrak{q}(f,a,(x)b)\in\mathbf{M}}$$

$$\frac{f \in \mathbf{Q} \qquad a \in \mathbf{M} \qquad b \in \mathbf{M}\,[x \in \mathbf{S}(a)]}{\mathbf{S}(\mathfrak{q}(f,a,(x)b)) = \mathbf{F}(f)(\mathbf{S}(a),(x)\mathbf{S}(b)) : Set}$$

$$\frac{f \in \mathbf{Q} \qquad a \in \mathbf{M} \qquad b \in \mathbf{M}\ \ [x \in \mathbf{S}(a)] \qquad d \in \mathbf{S}(\mathfrak{q}(f,a,(x)b))}{\boldsymbol{\ell}(f,a,(x)b,d) \in \mathbf{M}}$$

$$\frac{f \in \mathbf{Q} \qquad a \in \mathbf{M} \qquad b \in \mathbf{M}\,[x \in \mathbf{S}(a)] \qquad d \in \mathbf{S}(\mathfrak{q}(f,a,(x)b))}{\mathbf{S}(\boldsymbol{\ell}(f,a,(x)b,d)) = \mathbf{G}(f)(\mathbf{S}(a),(x)\mathbf{S}(b),d) : Set}.$$

Peeling off the coding lays bare the meaning of the last four rules: $\mathfrak{q}(f,a,(x)b))$ is a code for $\mathbf{F}(f)(\mathbf{S}(a),(x)\mathbf{S}(b))$ in $\mathbf{M}$ and $(y)\boldsymbol{\ell}(f,a,(x)b,y)$ injects codes for elements of the universe $\mathbf{F}(f)(\mathbf{S}(a),(x)\mathbf{S}(b))$ into $\mathbf{M}$.

The introduction rules for $\mathbf{Q}$ are

$$\frac{c \in \mathbf{M} \qquad f \in \mathbf{Q}\,[x \in \mathbf{S}(c)]}{\mathbf{u}(c,(x)f) \in \mathbf{Q}}$$

$$\frac{c \in \mathbf{M} \qquad f \in \mathbf{Q}\,[z \in \mathbf{S}(c)]}{\mathbf{F}(\mathbf{u}(c,(z)f)) = \mathbf{U}(\mathbf{S}(c),(z)\mathbf{F}(f),(z)\mathbf{G}(f)) : \mathbf{K}}$$

$$\frac{c \in \mathbf{M} \qquad f \in \mathbf{Q}\,[z \in \mathbf{S}(c)]}{\mathbf{G}(\mathbf{u}(c,(z)f)) = \mathbf{T}(\mathbf{S}(c),(z)\mathbf{F}(f),(z)\mathbf{G}(f)) : \mathbf{H}(\mathbf{u}(c,(z)f))},$$

where $\mathbf{H}(\mathbf{u}(c,(z)f)) = (X : Set)(Y : (X)Set)(\mathbf{F}(\mathbf{u}(c,(z)f))(X,Y))\,Set$.

Remark 3.1. It is worth noting that after completion of this paper we verified together with Peter Dybjer that the rules for **MLQ** fall under his schema of simultaneous inductive-recursive definition as delineated in his paper [5].

To demonstrate the potential of **MLQ**, we shall show how to build the superuniverses of [9, 16]. Let $\mathbf{N}_0$ and $\mathbf{N}_1$ be the empty set and the one element set, respectively. Let $\mathbf{n}_0$ and $\mathbf{n}_1$ be their respective codes in $\mathbf{M}$. Set $f := \mathbf{u}(\mathbf{n}_0,(y)\mathbf{R}_0((x)\mathbf{Q},y))$. Then $f \in \mathbf{Q}$. $\mathbf{F}(f)$ corresponds to Palmgren's universe operator, as for any family of sets $B : A \to Set$ over a set A, we have $\mathbf{F}(f)(A,B) = \mathbf{U}(\mathbf{N}_0,(y)\mathbf{F}(y),(y)\mathbf{G}(y),A,B)$; thus $\mathbf{F}(f)(A,B)$ is a universe above (A,B). Further, set $g := \mathbf{u}(\mathbf{n}_1,(z)f)$. Then $g \in \mathbf{Q}$. For any family of sets $D : C \to Set$ over a set C, we have $\mathbf{F}(g)(C,D) = \mathbf{U}(\mathbf{N}_1,(z)\mathbf{F}(f),(z)\mathbf{G}(f),C,D)$, which is a universe above (C,D) (with decoding function $\mathbf{G}(g)(C,D)$) closed under $\mathbf{F}(f)$; hence it is a superuniverse above (C,D).

4. An interpretation of $\mathbf{CZF}_\pi$ in MLQ

In this section we work informally in **MLQ** to facilitate the exposition. Similarly, as in the Russell-style formulation of universes in [13], we shall frequently neglect the difference between a code and the result of its decoding. Thus $\mathbf{T},\mathbf{S},\mathbf{F},\mathbf{G}$ as well as the constants $\star, \diamond, \jmath, \sharp, \dagger, \mathfrak{q}, \boldsymbol{\ell}$ are liable to disappear. The informal reasoning presented here can be rendered formally precise (in the sense of the formulation of **MLQ** in the previous section) by making explicit all the coding and decoding steps that we omit in this section.

Definition 4.1. Let $\mathbf{V}$ denote the type of iterative sets over $\mathbf{M}$, i.e. $\mathbf{V} = (\mathbf{W}x \in \mathbf{M})x$. $\mathbf{V}$ is inductively given by the rule

$$\frac{A \in \mathbf{M} \qquad b \in A \to \mathbf{V}}{\sup(A, b) \in \mathbf{V}}.$$

We shall also write $(\sup x \in A)b(x)$ for $\sup(A, (x)b(x))$. Let $\alpha, \beta, \gamma, \ldots$ range over elements of $\mathbf{V}$.

Associated with the rule inductively specifying the type $\mathbf{V}$ is the method of definition by *transfinite recursion on* $\mathbf{V}$ corresponding to the elimination rule for W-types. If d is a 3-place function such that $d(A, b, e) \in C(\sup(A, b))$ for all $A \in \mathbf{M}$, $b \in A \to \mathbf{V}$ and $e \in (\Pi x \in A)C(b(x))$, where C is a family of types over $\mathbf{V}$, then we have $h \in \Pi(\mathbf{V}, C)$ defined so that

$$h(\sup(A, b)) = d(A, b, (u)h(b(u)))$$

for $A \in \mathbf{M}$ and $b \in A \to \mathbf{V}$. An important property of $\alpha \in \mathbf{V}$ is that we can always recover its *branching-index* $\bar{\alpha} \in \mathbf{M}$ and the corresponding mapping $\tilde{\alpha} : \bar{\alpha} \to \mathbf{V}$.

Lemma 4.2. *There are one-place functions assigning* $\bar{\alpha} \in \mathbf{M}$ *and* $\tilde{\alpha} : \bar{\alpha} \to \mathbf{V}$ *to* $\alpha \in \mathbf{V}$ *such that if* $\alpha = \sup(A, b)$ *where* $A \in \mathbf{M}$ *and* $b \in A \to \mathbf{V}$ *then* $\bar{\alpha} = A$ *and* $\tilde{\alpha} = b$. *Moreover,* $\alpha = \sup(\bar{\alpha}, \tilde{\alpha})$ *for* $\alpha \in \mathbf{V}$.

Proof. [3], Theorem 2.1. □

If $B(x)$ is a proposition for $x \in \mathbf{V}$ then define

$$\begin{aligned} &\forall \beta \mathrel{\dot{\in}} \alpha\, B(\beta) := \forall x \in \bar{\alpha}\, B(\tilde{\alpha}(x)) \\ &\exists \beta \mathrel{\dot{\in}} \alpha\, B(\beta) := \exists x \in \bar{\alpha}\, B(\tilde{\alpha}(x)). \end{aligned} \tag{4}$$

Lemma 4.3. *If* F *is a species over* $\mathbf{V}$ *(i.e. a family of propositions over* $\mathbf{V}$*), then*

$$\forall \alpha [\forall \beta \mathrel{\dot{\in}} \alpha F(\beta) \to F(\alpha)] \to \forall \alpha F(\alpha).$$

Proof. This is an immediate application of $\mathbf{V}$-induction. □

Lemma 4.4. *There is a species on* $\mathbf{V} \times \mathbf{V}$ *assigning a small proposition* $(\alpha \doteq \beta)$ *to* $\alpha, \beta \in \mathbf{V}$ *such that*

$$(\alpha \doteq \beta) = [\forall \gamma \mathrel{\dot{\in}} \alpha\, \exists \delta \mathrel{\dot{\in}} \beta\, (\gamma \doteq \delta) \wedge \forall \delta \mathrel{\dot{\in}} \beta\, \exists \gamma \mathrel{\dot{\in}} \alpha\, (\gamma \doteq \delta)]. \tag{5}$$

Proof. See [3], 2.2. □

In [3, 4] Aczel has shown that **V** validates **CZF** + **REA** when elementhood is interpreted by $\dot{\in}$ and equality is interpreted as $\doteq$. To be precise, under the above interpretation each theorem ϕ of **CZF** + **REA** is translated into a proposition ϕ^* of **MLQ** such that $\mathbf{MLQ} \vdash t \in \phi^*$ for a suitable term t. The latter will be shortened into

$$\langle \mathbf{V}; \dot{\in}; \doteq \rangle \models \phi.$$

By the foregoing, for any $\alpha \in \mathbf{V}$ there exists $\beta \in \mathbf{V}$ such that

$$\langle \mathbf{V}; \dot{\in}; \doteq \rangle \models \text{“}\beta \textit{ is the transitive closure of } \alpha\text{”}.$$

Invoking the axiom of choice in type theory (cf. [13]), there is a function $\alpha \mapsto \alpha_{tc}$ from **V** to **V** such that the latter holds with $\beta := \alpha_{tc}$.

Definition 4.5. By transfinite recursion on **V** we define a mapping $\Phi : \mathbf{V} \to \mathbf{Q}$ via

$$\Phi(\alpha) := \mathbf{u}(\overline{\alpha_{tc}}, (x)\Phi(\widetilde{\alpha_{tc}}(x))).$$

Let $\alpha \in \mathbf{V}$, $A \in \mathbf{M}$ and $f \in A \to \mathbf{M}$. We put

$$\begin{aligned} &\mathbf{Q}^\alpha := \mathbf{F}(\Phi(\alpha)) \\ &\mathscr{M}^\alpha_{(A,f)} := \mathbf{Q}^\alpha(A, f) \\ &\mathscr{V}^\alpha_{(A,f)} := (\mathbf{W}x \in \mathscr{M}^\alpha_{(A,f)})x. \end{aligned}$$

By transfinite recursion on the **W**-type $\mathscr{V}^\alpha_{(A,f)}$ define $\mathbf{h}^\alpha_{(A,f)} : \mathscr{V}^\alpha_{(A,f)} \to \mathbf{V}$ via

$$\mathbf{h}^\alpha_{(A,f)}(\sup(a,g)) := \sup(a, (u)\mathbf{h}^\alpha_{(A,f)}(g(u))) \tag{6}$$

and put

$$\mathbf{V}^\alpha_{(A,f)} := \sup(\mathscr{V}^\alpha_{(A,f)}, \mathbf{h}^\alpha_{(A,f)}).$$

For $\beta \in \mathbf{V}$ define $\hat{\beta} : \bar{\beta} \to \mathbf{M}$ by

$$\hat{\beta}(u) := \overline{\tilde{\beta}(u)}$$

and set $\mathscr{M}^\alpha_\beta := \mathscr{M}^\alpha_{(\bar{\beta},\hat{\beta})}$, $\mathscr{V}^\alpha_\beta := \mathscr{V}^\alpha_{(\bar{\beta},\hat{\beta})}$, $\mathbf{h}^\alpha_\beta := \mathbf{h}^\alpha_{(\bar{\beta},\hat{\beta})}$ and $\mathbf{V}^\alpha_\beta := \sup(\mathscr{V}^\alpha_\beta, \mathbf{h}^\alpha_\beta)$.

Above we used a sloppy notation which ignores the difference between codes and sets. The definition of $\mathbf{h}^\alpha_{(A,f)}$ requires that $\mathscr{M}^\alpha_{(A,f)}$ is a subuniverse of **M**. This is the case only via the injection ℓ. Thus, on the right-hand side of (6), a ought to be $\ell(\Phi(\alpha), A, f, a)$. However, we think that identifications help the presentation and trust that the reader can always restore the official language of **MLQ**.

Remark 4.6. As in the case of **V**, for any $x \in \mathscr{V}^\alpha_{(A,f)}$ we can retrieve its branching-index $\bar{x}$ and the corresponding mapping $\tilde{x} : \bar{x} \to \mathscr{V}^\alpha_{(A,f)}$ such that $x = \sup(\bar{x}, \tilde{x})$.

Corresponding to $\dot{\in}$ and $\dot{=}$ we can define $\dot{\in}^{\alpha}_{(A,f)}$ and $\dot{=}^{\alpha}_{(A,f)}$, respectively, such that (by [3,4]) $\mathscr{V}^{\alpha}_{(A,f)}$ validates **CZF + REA** when equipped with these relations.

Lemma 4.7. (i) $\langle \mathbf{V}; \dot{\in}; \dot{=} \rangle \models$ "$\mathbf{V}^{\alpha}_{(A,f)}$ *is transitive*".
(ii) $\forall z, w \in \mathscr{V}^{\alpha}_{(A,f)}[z \,\dot{\in}^{\alpha}_{(A,f)}\, w \leftrightarrow \mathbf{h}^{\alpha}_{(A,f)}(z) \,\dot{\in}\, \mathbf{h}^{\alpha}_{(A,f)}(w)]$.
(iii) $\forall z, w \in \mathscr{V}^{\alpha}_{(A,f)}[z \,\dot{=}^{\alpha}_{(A,f)}\, w \leftrightarrow \mathbf{h}^{\alpha}_{(A,f)}(z) \,\dot{=}\, \mathbf{h}^{\alpha}_{(A,f)}(w)]$.

Proof. (i): Assume $\eta \,\dot{\in}\, \delta \,\dot{\in}\, \mathbf{V}^{\alpha}_{(A,f)}$. Then $\delta \dot{=} \mathbf{h}^{\alpha}_{(A,f)}(x)$ for some $x \in \mathscr{V}^{\alpha}_{(A,f)}$. Further, $x = \sup(a,p)$ for some $a \in \mathscr{M}^{\alpha}_{(A,f)}$ and $p : a \to \mathscr{V}^{\alpha}_{(A,f)}$. Then $\mathbf{h}^{\alpha}_{(A,f)}(x) = \sup(a, (u)\mathbf{h}^{\alpha}_{(A,f)}(p(u)))$; thus, $\eta \dot{=} \mathbf{h}^{\alpha}_{(A,f)}(p(u))$ for some $u \in a$. As $p(u) \in \mathscr{V}^{\alpha}_{(A,f)}$, it follows $\mathbf{h}^{\alpha}_{(A,f)}(p(u)) \,\dot{\in}\, \mathbf{V}^{\alpha}_{(A,f)}$. Hence $\eta \,\dot{\in}\, \mathbf{V}^{\alpha}_{(A,f)}$.

(ii) and (iii): We prove both claims simultaneously by induction on the ordering

$$(x', y') \lhd^{\alpha}_{(A,f)} (x,y) := (x' \,\dot{\in}^{\alpha}_{(A,f)}\, x \wedge y' \,\dot{=}^{\alpha}_{(A,f)}\, y) \vee (x' \,\dot{=}^{\alpha}_{(A,f)}\, x \wedge y' \,\dot{\in}^{\alpha}_{(A,f)}\, y).$$

We begin with (ii). Suppose $x \,\dot{\in}^{\alpha}_{(A,f)}\, y$. Then $x \,\dot{=}^{\alpha}_{(A,f)}\, \tilde{y}(u)$ for some $u \in \bar{y}$. By the induction hypothesis (i.h.) for (iii), we get $\mathbf{h}^{\alpha}_{(A,f)}(x) \dot{=} \mathbf{h}^{\alpha}_{(A,f)}(\tilde{y}(u))$. As $\mathbf{h}^{\alpha}_{(A,f)}(\tilde{y}(u)) \,\dot{\in}\, \mathbf{h}^{\alpha}_{(A,f)}(y)$, it follows $\mathbf{h}^{\alpha}_{(A,f)}(x) \,\dot{\in}\, \mathbf{h}^{\alpha}_{(A,f)}(y)$.

Conversely, assume $\mathbf{h}^{\alpha}_{(A,f)}(x) \,\dot{\in}\, \mathbf{h}^{\alpha}_{(A,f)}(y)$. Then $\mathbf{h}^{\alpha}_{(A,f)}(x) \dot{=} \mathbf{h}^{\alpha}_{(A,f)}(\tilde{y}(u))$ for some $u \in \bar{y}$. Using the i.h. for (iii), one gets $x \,\dot{=}^{\alpha}_{(A,f)}\, \tilde{y}(u)$, hence $x \,\dot{\in}^{\alpha}_{(A,f)}\, y$.

The proof of (iii) is similar. □

Corollary 4.8. $\langle \mathbf{V}; \dot{\in}; \dot{=} \rangle \models$ "$\mathbf{V}^{\alpha}_{(A,f)}$ *is set-inaccessible*".

Proof. Aczel's proofs in [3,4] show that $\mathscr{V}^{\alpha}_{(A,f)}$ is regular and $\langle \mathscr{V}^{\alpha}_{(A,f)}; \dot{\in}^{\alpha}_{(A,f)}; \dot{=}^{\alpha}_{(A,f)} \rangle$ is a model of **CZF + REA** in a strong sense. By Lemma 4.7, via the structural isomorphism $\mathbf{h}^{\alpha}_{(A,f)}$, this carries over to $\mathbf{V}^{\alpha}_{(A,f)}$. □

Lemma 4.9. *If* $B \in \mathscr{M}^{\alpha}_{(A,f)}$, $g \in B \to \mathscr{M}^{\alpha}_{(A,f)}$ *and* $u \in \overline{\alpha_{tc}}$, *then*

$$\mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \,\dot{\in}\, \mathbf{V}^{\alpha}_{(A,f)}.$$

Proof. As $\mathscr{M}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \in \mathscr{M}^{\alpha}_{(A,f)}$, we get $\mathscr{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \in \mathscr{M}^{\alpha}_{(A,f)}$. Put $\mathscr{V} := \mathscr{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. By recursion on the **W**-type $\mathscr{V}$ define

$$\ell : \mathscr{V} \to \mathscr{V}^{\alpha}_{(A,f)}$$

via $\ell(\sup(D,q)) = \sup(D, (v)\ell(q(v)))$, and put $\mathbb{V} := \sup(\mathscr{V}, \ell)$. Then $\mathbb{V} \in \mathscr{V}^{\alpha}_{(A,f)}$. Finally, put $\boldsymbol{\beta} := \mathbf{h}^{\alpha}_{(A,f)}(\mathbb{V})$.

Then $\boldsymbol{\beta} \,\dot{\in}\, \mathbf{V}^{\alpha}_{(A,f)}$. We claim that $\boldsymbol{\beta} \dot{=} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. First, we show that for all $a \in \mathscr{V}$,

$$\mathbf{h}^{\alpha}_{(A,f)}(\ell(a)) \dot{=} \mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(a) \tag{7}$$

by induction over the **W**-type $\mathcal{V}$. Now, if $a \in \mathcal{V}$, then $a = \sup(C, p)$ for some $C \in \mathcal{M}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$ and $p : C \to \mathcal{V}$. Inductively, we have for all $z \in C$,

$$\mathbf{h}^{\alpha}_{(A,f)}(\ell(p(z))) \doteq \mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(p(z)).$$

Hence

$$\mathbf{h}^{\alpha}_{(A,f)}(\ell(a)) = \sup(C, (z)\mathbf{h}^{\alpha}_{(A,f)}(\ell(p(z)))) \doteq \sup(C, (z)\mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(p(z))) = \mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(a).$$

Notice that

$$\boldsymbol{\beta} = \sup(\mathcal{V}, (z)\mathbf{h}^{\alpha}_{(A,f)}(\ell(z))).$$

Thus, if $\delta \dot\in \boldsymbol{\beta}$, then $\delta \doteq \mathbf{h}^{\alpha}_{(A,f)}(\ell(a))$ for some $a \in \mathcal{V}$. As $\mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(a) \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$, we get $\delta \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$ by (7).

Conversely, if $\delta \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$, then $\delta \doteq \mathbf{h}^{\widetilde{\alpha(u)}}_{(B,g)}(a)$ for some $a \in \mathcal{V}$. As $\mathbf{h}^{\alpha}_{(A,f)}(\ell(a)) \dot\in \mathbf{h}^{\alpha}_{(A,f)}(\mathbb{V})$, we get $\delta \dot\in \boldsymbol{\beta}$ by (7). □

Lemma 4.10. *If $\xi \dot\in \mathbf{V}^{\alpha}_{(A,f)}$, then for all $u \in \overline{\alpha_{tc}}$ there exist $B \in \mathcal{M}^{\alpha}_{(A,f)}$ and $g : B \to \mathcal{M}^{\alpha}_{(A,f)}$ such that $\xi \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$.*

Proof. Since $\mathbf{V}^{\alpha}_{(A,f)}$ is set-inaccessible, we may select $\beta \dot\in \mathbf{V}^{\alpha}_{(A,f)}$ such that $\xi \dot\in \beta$ and β is transitive. There exists $x \in \mathcal{V}^{\alpha}_{(A,f)}$ so that $\beta \doteq \mathbf{h}^{\alpha}_{(A,f)}(x)$. x is of the form $\sup(B, g_0)$ for some $B \in \mathcal{M}^{\alpha}_{(A,f)}$ and $g_0 : B \to \mathcal{V}^{\alpha}_{(A,f)}$.

Put $g(z) := \overline{g_0(z)}$. Then $g : B \to \mathcal{M}^{\alpha}_{(A,f)}$. Thus $\mathcal{M}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \in \mathcal{M}^{\alpha}_{(A,f)}$, and consequently $\mathcal{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \in \mathcal{M}^{\alpha}_{(A,f)}$. We show

$$\delta \dot\in \beta \to \delta \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \tag{8}$$

by $\dot\in$-induction on δ. So assume $\delta \dot\in \beta$. Then $\delta \doteq \mathbf{h}^{\alpha}_{(A,f)}(g_0(z))$ for some $z \in B$. Let $C := \overline{\mathbf{h}^{\alpha}_{(A,f)}(g_0(z))}$. Then also $C = \overline{g_0(z)}$, so $C \in \mathcal{M}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. By the inductive assumption,

$$\mathbf{h}^{\alpha}_{(A,f)}(\widetilde{g_0(z)}(v)) \dot\in \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$$

for any $v \in C$. Thus for any $v \in C$ there exists $y \in \mathcal{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$ such that

$$\mathbf{h}^{\alpha}_{(A,f)}(\widetilde{g_0(z)}(v)) \doteq \mathbf{h}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}(y).$$

Using the axiom of choice in type theory, there is a function $q : C \to \mathcal{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$ such that for all $v \in C$,

$$\mathbf{h}^{\alpha}_{(A,f)}(\widetilde{g_0(z)}(v)) \doteq \mathbf{h}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}(q(v)). \tag{9}$$

Now, as $C \in \mathcal{M}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$, we have $\sup(C,q) \in \mathcal{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. Thus

$$\mathbf{h}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}(\sup(C,q)) = \sup(C,(v)\mathbf{h}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}(q(v))) \mathrel{\dot\in} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}.$$

Moreover, using (9),

$$\sup(C,(v)\mathbf{h}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}(q(v))) \doteq \sup(C,(v)\mathbf{h}^{\alpha}_{(A,f)}(\widetilde{g_0(z)}(v))) \doteq \mathbf{h}^{\alpha}_{(A,f)}(g_0(z)) \doteq \delta.$$

Whence, $\delta \mathrel{\dot\in} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$.

This concludes the proof of (8). As a consequence, $\beta \mathrel{\dot\subseteq} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. Thus $\xi \mathrel{\dot\in} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$. □

Lemma 4.11. *Let $A \in \mathbf{M}$, $f : A \to \mathbf{M}$ and $\alpha \in \mathbf{V}$. Then*

$$\langle \mathbf{V};\ \dot\in;\ \doteq \rangle \models \text{“}\mathbf{V}^{\alpha}_{(A,f)} \text{ is } \alpha\text{-set-inaccessible”}.$$

Proof. Proceed by $\dot\in$-induction on α. If $\alpha_0 \mathrel{\dot\in} \alpha$, then $\alpha_0 \mathrel{\dot\in} \alpha_{tc}$, and hence $\alpha_0 \doteq \widetilde{\alpha_{tc}}(u)$ for some $u \in \overline{\alpha_{tc}}$. If now $\xi \mathrel{\dot\in} \mathbf{V}^{\alpha}_{(A,f)}$, then by Lemma 4.10 together with Lemma 4.9 we have

$$\xi \mathrel{\dot\in} \mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)} \mathrel{\dot\in} \mathbf{V}^{\alpha}_{(A,f)}.$$

By the inductive assumption, $\mathbf{V}^{\widetilde{\alpha_{tc}(u)}}_{(B,g)}$ is α_0-set-inaccessible. Employing Corollary 4.8 and Lemma 2.11, it follows that $\mathbf{V}^{\alpha}_{(A,f)}$ is α-set-inaccessible. □

Lemma 4.12. *If β is transitive, then $\beta \mathrel{\dot\subseteq} \mathbf{V}^{\rho}_{\beta}$.*

Proof. We show

$$\delta \mathrel{\dot\in} \beta \to \delta \mathrel{\dot\in} \mathbf{V}^{\rho}_{\beta}$$

by $\dot\in$-induction on δ. $\delta \mathrel{\dot\in} \beta$ implies $\delta \doteq \tilde\beta(x)$ for some $x \in \bar\beta$. Set $\gamma := \tilde\beta(x)$. Note that $\bar\gamma \in \mathcal{M}^{\rho}_{\beta}$. By the inductive assumption, $\gamma \mathrel{\dot\subseteq} \mathbf{V}^{\rho}_{\beta}$. Hence, for any $u \in \bar\gamma$ there exists $v \in \mathcal{V}^{\rho}_{\beta}$ such that $\tilde\gamma(u) \doteq \mathbf{h}^{\rho}_{\beta}(v)$. Invoking the axiom of choice, there is a function $\ell : \bar\gamma \to \mathcal{V}^{\rho}_{\beta}$ such that $\mathbf{h}^{\rho}_{\beta}(\ell(u)) \doteq \tilde\gamma(u)$ for $u \in \bar\gamma$. Thence, $\sup(\bar\gamma,\ell) \in \mathcal{V}^{\rho}_{\beta}$, and further,

$$\mathbf{h}^{\rho}_{\beta}(\sup(\bar\gamma,\ell)) = \sup(\bar\gamma,(u)\mathbf{h}^{\rho}_{\beta}(\ell(u))) \doteq \sup(\bar\gamma,\tilde\gamma) = \gamma \mathrel{\dot\in} \mathbf{V}^{\rho}_{\beta}.$$

Hence, $\delta \mathrel{\dot\in} \mathbf{V}^{\rho}_{\beta}$. □

Corollary 4.13. $\langle \mathbf{V};\ \dot\in;\ \doteq \rangle \models \forall\beta,\alpha\,\exists\delta(\beta \mathrel{\dot\subseteq} \delta \wedge \delta$ *is α-set-inaccessible*).

Proof. This follows from Lemmas 4.11 and 4.12. □

Theorem 4.14. $\langle \mathbf{V};\ \dot\in;\ \doteq \rangle$ *is a model of* $\mathbf{CZF}_{\pi}$.

Proof. This ensues from Aczel's work [3,4] and Corollary 4.13. □

By the last theorem we also know that **MLQ** has at least the proof-theoretic strength of $\mathbf{CZF}_\pi$. In point of fact, **MLQ** is slightly stronger than $\mathbf{CZF}_\pi$. Using techniques from [10], the exact proof-theoretic strength of $\mathbf{CZF}_\pi$ can be expressed in terms of the ordinal representation system expounded in Rathjen [19], namely

$$|\mathbf{CZF}_\pi| = \psi_{\Omega_1}(\varepsilon_{\chi_{\mathrm{M}}(0)+1}),$$

where $|T|$ signifies the proof-theoretic ordinal of a theory T. Let τ be a notation for the least inaccessible above $\chi_{\mathbf{M}}(0)$. Regarding **MLQ** we can show

$$\psi_{\Omega_1}(\Omega_{\chi_{\mathrm{M}}(0)+\omega}) \leqslant |\mathbf{MLQ}| \leqslant \psi_{\Omega_1}(\varepsilon_{\tau+1}).$$

In particular, **MLQ** is proof-theoretically weaker than the classical theory **KPM** (introduced in Rathjen [20]) which formalizes a recursively Mahlo universe of sets.

5. Comments and extensions

The functional $\mathrm{U}(\mathscr{P})$ studied in the second section of this paper can be generalised to all *finite types* giving the closure conditions of higher type quantifiers. In a sequel we will study a universe of sets brought about by carrying out its definition while also reflecting (defining sets of codes for) quantifiers of all finite types in the sense of the logical framework. There we intend to show that intuitionistic type theory extended by the universe of sets so constructed has the proof-theoretic strength of the theory **KPM** introduced in [19], i.e., Kripke–Platek set theory extended by a recursive Mahlo rule. We have shown that an extension of **CZF** by an axiom schema asserting that the universe is Mahlo has the same strength as **KPM**. Thus, if the above delineated approach turns out to be successful, we would have shown that extension of type theory is a constructive analogue of Mahlo's least ρ-number $\rho_{0,0}$.

Another route of formalizing Mahloness in type theory has been taken by Setzer in [23]. The main difference to the approach presented here seems to be that [23] does not focuss on a constructive explication but rather postulates the Mahloness of the universe, thereby taking this notion as a point of departure.

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