Fourier Series

Periodic Signal ⟶ Sum of Harmonics (sine & cosine)

Why?

Decomposing a signal (periodic) into a sum of sine and cosine harmonics can be useful bcz the sinusoidal waves are easier to work with. Their properties are already known.

Moreover, you can convert a given function from time-domain to frequency-domain (i.e., different sinusoidal harmonics are actually frequency components)

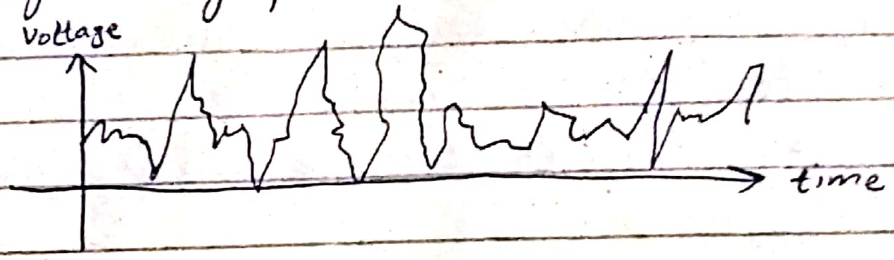
Where is it used?

Fourier Series & Transform is used in Signal Processing (especialey in audio signals processing), Image Processing, Communication, Circuit analysis, Control Systems, etc.

Understanding Fourier Series using a simple Problem

We have a sound wave that is being sent to the microphone. The microphone converts the air pressure to certain voltage signals.

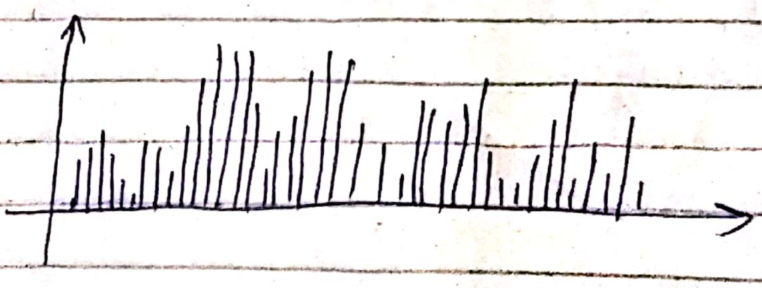
Plotting these voltage signals w.r.t time, we get a graph like



Now, we need to store this sound wave in the computer memory.

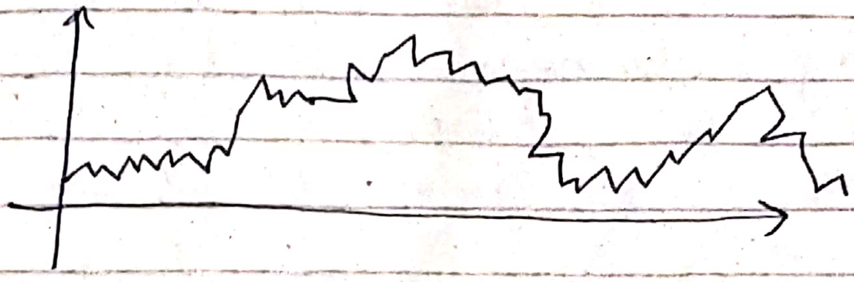
How can we store it?

One of the first solutions that comes to mind is to sample the signal at regular intervals and then store that samples in the memory.



But how can we reconstruct that original waveform from these samples?

One of the first methods that comes to mind is to just interpolate the samples to form a graph again.



But interpolation can't solve the problem because it doesn't match the original wave and the difference between the original wave and the interpolated wave remains non-trivial no matter how much you increase the sample size.

What are we doing wrong here?

We're trying to reconstruct signal using interpolation but that simply adds up more to our problem instead of solving it.

Interpolation introduces sharp-corners to our signal which then result in high-frequency components that is "noise" added to the signal

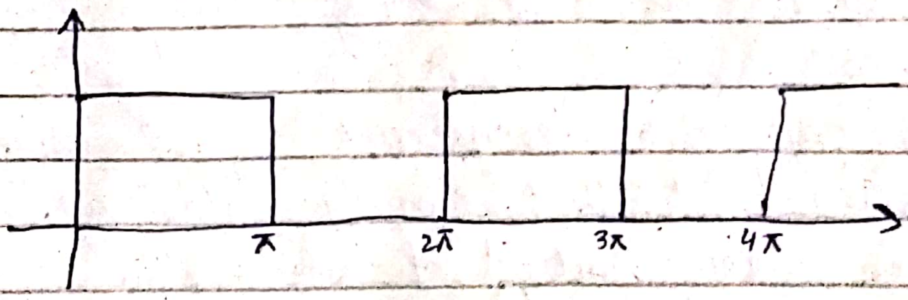
What else can we do?

In nature, all the signals have different frequency components which are also called harmonics.

In 1807, a french physicist noticed this interesting phenomena while solving a heat-flow problem. He saw that we can show any function as a sum of different sine and cosine harmonics.

So, we can decompose our sound wave as a superposition of different sine and cosine waves at different frequencies

Let's put the sound wave aside for a moment and work with a simpler wave. Say a square wave



Let's assume that time period of this function is 2π just for the sake of simplicity in calculations.

According to Fourier, we can represent this function as sum of harmonics of sine and cosine

i.e., $\sum_{n=0}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$

where a_n & b_n are amplitudes of the n^{th} harmonic.

ω is the angular frequency $\left(\frac{2\pi}{T}\right)$

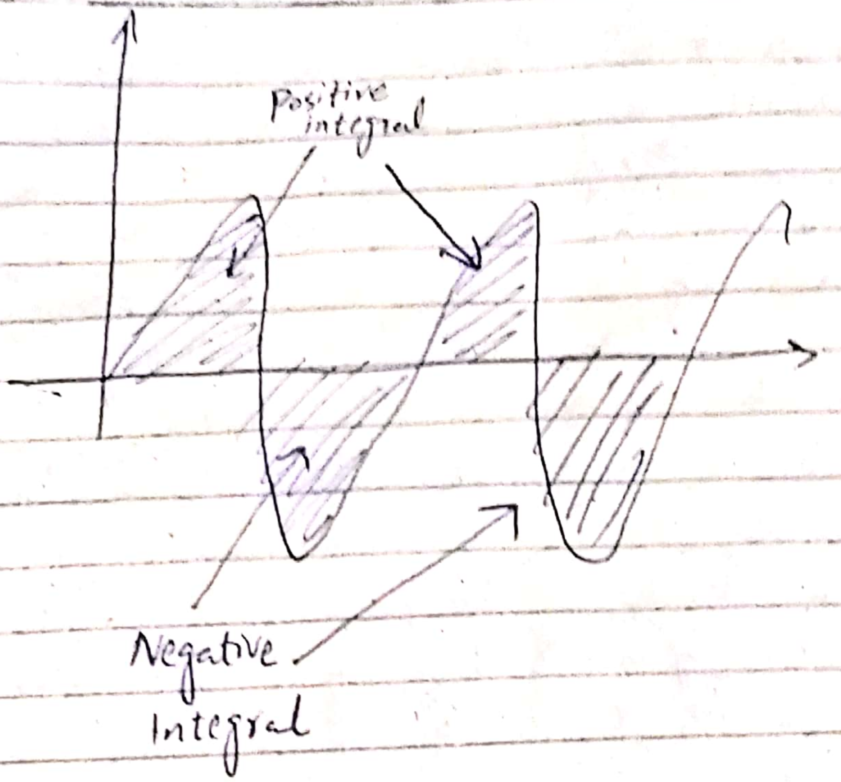
t is the time-domain if we're working with time. Usually we work with x instead of t.

Now, all we need to do is to find the values of amplitudes for each of these harmonics (a_n and b_n)

First of all, we notice that when $n=0$, that means time period is infinity for the harmonics. This implies that $n=0$ will deal with the steady-state value of the function.

Let's elaborate this further.

We know that the integral of sine waves and cosine waves over their complete time period is zero.

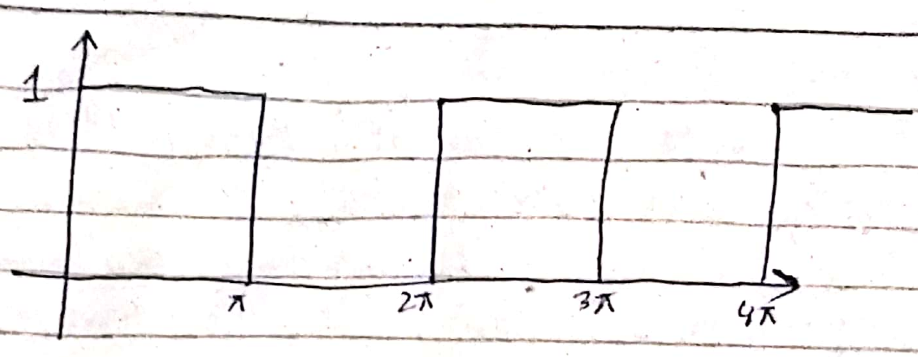


Positive and negative integrals cancel each other out. So, the integral is zero. Or we can say that the average value of function over its whole time-period is zero.

Since the average value of sine and cosine waves over their whole period is zero, the average value of all the harmonics from $n=1$ to $n=\infty$ will be zero too.

But the average value of the given function isn't necessarily zero.

In our case, we're working with the square wave.



It is clear that unlike sinusoidal waves, this is clearly not symmetric about x-axis.

And the average value can be calculated as

$$\frac{1}{T}\int_{0}^{\cancel{2}\pi} 1 \cdot dt$$

$$= \frac{1}{2\pi} \cdot t \Big|_{0}^{\pi}$$

$$= \frac{1}{2\pi}(\pi - 0) = \boxed{\frac{1}{2}}$$

So, 1/2 is the steady-state component for this periodic function.

Hence 1/2 will also be the steady-state component for our required series.

In the fouries series, we can get the Steady-state component by putting frequency $= 0 \Rightarrow n = 0$

when $n=0$, we have the harmonic as

$$a_0 \cos(0) + b_0 \sin(0)$$

$$= a_0 + 0 = a_0 \quad \text{--- (i)}$$

Hence, our steady-state component (aka offset) is equal to $a_0 = 1/2$

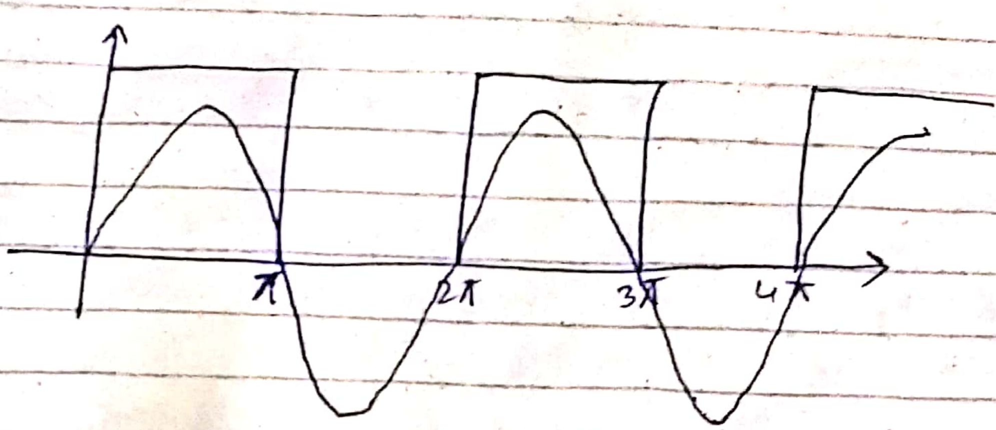
from eq. (i) we can see that the 0^{th} harmonic will always be equal to a_0. And we've already said that the steady-state value of function would be equal to the 0^{th} harmonic of series.

Hence, a_0 = Steady-State value of given function

$$\Rightarrow a_0 = \frac{1}{T} \int^{T} f(t) \cdot dt$$

We've calculated the 0^{th} harmonic. Now, how can we calculate the other harmonics?

Let's visualize the harmonics alongwith the function.



Above is the first harmonic ~~drawn~~ of sine function drawn alongside the original function.

Now how do we estimate the amplitude for this harmonic (b_1) ?

Let's think about this intuitively, After adding all of the harmonics, we want to get the function value. So, the amplitude of each harmonic directly depends on how much they overlap with the original function.

How can we find the overlap b/w function and harmonic?

One of the ways to find the overlap is to multiply the harmonic with function

i.e.,

$$f(x) \cdot \sin(\omega x) \qquad (\text{i.e., } k = 1)$$

In this case, whenever both the function and the harmonic are in the same direction, we'll get a positive value and when they're in the ~~negative di~~ opposite directions, we'll get a negative value.

But to find the amplitude of the harmonic, we need to take into account all the values of x throughout the time-period. So, we integrate this product over the whole time-period to get the total overlap between the function and harmonic.

i.e.,

$$\int^{T} f(t) \cdot \sin(\omega t) \cdot dt \qquad (\text{i.e., } k = 1)$$

Now we end up with another problem. This is a problem of scaling.

To understand this problem, let's take

$$f(t) = \sin(\cancel{4}t)$$

Now our function is just the first harmonic of sine.

So, logically, all other amplitudes for harmonics should be zero and only the amplitude b_1 should be 1.

But when we calculate the integral $\int^{T} f(t) \cdot \sin(\omega t)\, dt$

we get

$$\int_{0}^{2\pi} \sin(\omega t) \cdot \sin(\omega t)\, dt$$

$$= \int_{0}^{2\pi} \sin^2(\omega t)\, dt$$

Since $\omega = 1$ $(T = 2\pi)$, we get the above integral

$$= \int_{0}^{2\pi} \frac{1-\cos(2t)}{2}\, dt = \int_{0}^{2\pi} \frac{1}{2} dt - \int_{0}^{2\pi} \frac{\cos(2t)}{2}\, dt$$

$$= \frac{2\pi}{2} - \int_{0}^{2\pi} \frac{\cos 2t}{2}\, dt = \pi - \frac{1}{4} \sin(2t) \Big|_{0}^{2\pi}$$

$= \pi - 0 = \pi$

So, our integral is $\pi = \frac{T}{2}$

Solving the same problem for $\omega = K$, we find out that integral comes out to be $\pi/_{K}$

Since we know that $\omega = 2\pi/_{T}$,

$$\boxed{\text{integral becomes } \frac{T}{2}}$$

So, due to integration over the whole time-period, the amplitude has been scaled up by factor $\frac{T}{2}$.

To normalize this effect, we can simply multiply the integral by $\frac{2}{T}$

So, we get

$$b_1 = \frac{2}{T} \int^{T} f(t) \sin(\omega t)\, dt. \quad (\text{i.e., } k=1)$$

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So, we can write the expression for any value of k as

$$b_k = \frac{2}{T} \int^{T} f(t) \sin(k\omega t)\, dt$$

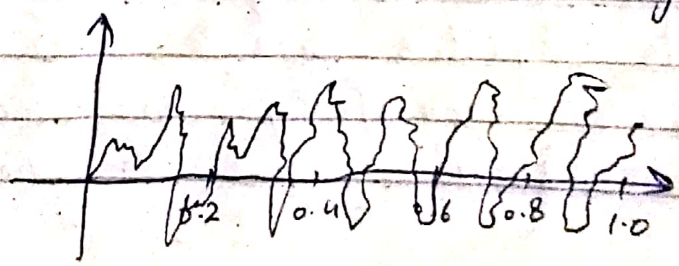
Using the same procedure, we can calculate the amplitudes for cosine harmonics.

$$a_k = \frac{2}{T} \int^{T} f(t) \cos(k\omega t)\, dt$$

So now we know what fourier ~~transform~~ series is and how we can calculate the fourier series for ~~ead~~ any given function.

Coming back to our sound wave, now we can treat the wave as a function with the period equal to the given sample of signal.

For example, if we have the audio signal for 1 second interval, it might look like



We can now sample this ~~time period~~ sound wave based on the nyquist criteria.

Typical audio drivers in our computers use the sampling rate of 44 kHZ which is slightly higher than twice the highest audible frequency (20kHZ)

Once we have the samples, we can use the fourier series formulae

$$a_n = \frac{2}{T} \int^{T} f(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int^{T} f(t) \sin(n\omega t)\, dt$$

The integral will be calculated in discrete intervals and this method is typically referred as Discrete Fourier Transform but ~~all~~ both the continuous & discrete transforms are using the fourier series.

Now, after dividing the given sound wave into its respective sine & cosine harmonics, we have successfully transformed the time-domain signal to the frequency domain signal.

This transformation helps us in processing the signal because we can do a lot of processing like noise reduction, bandwidth limiting, etc., when we're working in the frequency domain.

Moreover, when the signal is reconstructed by adding all the harmonics of sines and cosines, we get the exact same signal as before.