LECTURE NOTES

MATH 103A — SPRING 2022

COMPLEX ANALYSIS

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adapted from

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(MATH 103A — WINTER 2021)

Last Updated: Monday 9th May, 2022

Contents

1. Lecture 1 (3/29)

What is Complex Analysis? The main object of study is a holomorphic function $f : G \to \mathbf{C}$, where $G \subseteq \mathbf{C}$. Namely, a function for which the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite on an open set; that is, a complex-differentiable function on an open set. As a set, $\mathbf{C} = \mathbf{R}^2$, so one can naively expect the theory to be similar to that of real analysis, in this case the behaviour of differentiable functions. Interestingly, the requirement of holomorphicity can yield results that have no counterpart in the real case.

A prime example of this is *Louiville's Theorem. Every bounded holomorphic function is constant.*

Discussion 1.1. We begin with first addressing the existence and nature of $\mathbf{C}$ itself. Let $\mathbf{R}$ denote the (field of) real numbers. One immediately deduces that the equation

$$x^2 + 1 = 0 \tag{$*$}$$

has no solution in the real numbers. The (field of) complex numbers $\mathbf{C}$ stems from our desire to find a set containing $\mathbf{R}$ that extends the algebraic operations of addition and multiplication of real numbers and which contains not only solutions to the polynomial equation above but solutions to all polynomial equations.

Surprisingly enough, the construction amounts to defining a symbol i that is a solution to $(*)$ and then considering all expressions of the form

$$x + iy, \quad x, y \in \mathbf{R}$$

Part I. Preliminaries

Construction of the (field of) Complex Numbers

Definition 1.2 (The set of Complex Numbers)**.** A complex number z is simply an order pair $z := (x, y)$ of real numbers. Thus, the set of all complex numbers is given by

$$\mathbf{C} := \mathbf{R}^2 = \{(x, y) \ : \ x, y \in \mathbf{R}\}$$

If $z = (x, y)$ is a complex number, then we call

$$\operatorname{Re} z := x \quad \text{and} \quad \operatorname{Im} z := y$$

the real and imaginary parts of z respectively.

Two complex numbers z_1 and z_2 are equal if and only if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$.

If $\operatorname{Re} z = 0$ and $\operatorname{Im} z \neq 0$, we say that z is purely imaginary. The set of purely imaginary complex numbers corresponds to the y-axis and is called the imaginary axis in $\mathbf{C}$.

Definition 1.3 (Binary Operations on **C**). Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be complex numbers. Then their *sum* is

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and their *product* is

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Proposition 1.4. *There exists a subset of* $\mathbf{C}$ *that is algebraically indistinguishable from* $\mathbf{R}$.

Proof. Consider the set (the x-axis)

$$\mathbf{R} \times \{0\} = \{(x, 0) \ : \ x \in \mathbf{R}\} \subseteq \mathbf{C}.$$

There is a bijection

$$\phi : \mathbf{R} \to \mathbf{R} \times \{0\}, \ x \mapsto (x, 0).$$

Moreover,

$$\phi(x) + \phi(y) = (x, 0) + (y, 0) = (x + y, 0) = \phi(x + y)$$
$$\phi(x) \cdot \phi(y) = (x, 0) \cdot (y, 0) = (xy - 0 \cdot 0, x \cdot 0 + y \cdot 0) = (xy, 0) = \phi(xy) \qquad \square$$

According to the proposition, the operations of addition and multiplication on complex numbers we have defined extend the operations of addition and multiplication of real numbers. We therefore call the x-axis, the real axis.

Discussion 1.5. We identify each complex number $(x, 0)$ with the corresponding real number x; more than that, abusing notation, we write

$$1 = (1, 0) \quad \text{and} \quad (x, 0) = x(1, 0) = x$$

Now, define the imaginary unit $i := (0, 1)$. Then

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0^2 - 1^2, 1 \cdot 0 + 1 \cdot 0) = (-1, 0) = -1.$$

Moreover, for any $z = (x, y) \in \mathbf{C}$ we see that

$$\begin{aligned} z &= (x, y) \\ &= (x, 0) + y(0, 1) = x + iy = \operatorname{Re} z + i \operatorname{Im} z \end{aligned}$$

Hence, with our new notation

$$\mathbf{C} = \{x + iy \ : \ x, y \in \mathbf{R}, \ i^2 = -1\}$$

and

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Although we have expanded the real numbers and we will see that the complex numbers have several new and familiar properties. We do end up losing one property of the real numbers when working with complex numbers: total ordering (that extends the one on $\mathbf{R}$ or is compatible with multiplication). In the world of complex numbers, it no longer makes sense to ask if $z_1 > z_2$ (see Problem 1.9).

In practice, the product of complex numbers can be computed by multiplying the expressions as if they were polynomials in the variable i, and using $i^2 = -1$. The fact that this works is left as Problem 1.3.

Example 1.6. Compute $(1+i)(1-3i)$.

Answer. We note

$$\begin{aligned}(1+i)(1-3i) &= (1-3i) + i(1-3i)\\ &= (1-3i) + (i - 3i^2)\\ &= (1-3i) + (i+3) = 4 - 2i\end{aligned}$$

□

Proposition 1.7 (Algebraic Properties of $(\mathbf{C}, +, \cdot)$)**.**

(1) Additive Identity. *For every $z \in \mathbf{C}$*

$$z + 0 = z = 0 + z$$

(2) Associativity of Addition. *For every triple $z_1, z_2, z_3 \in \mathbf{C}$*

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(3) Commutativity of Addition. *For every pair $z_1, z_2 \in \mathbf{C}$*

$$z_1 + z_2 = z_2 + z_1$$

(4) Additive Inverses. *For every $z \in \mathbf{C}$, there exists a complex number, denoted $-z$, such that*

$$z + (-z) = 0 = (-z) + z$$

In fact, $-z := (-1)z$, which is described in Problem 1.2.

(5) Multiplicative Identity. *For every $z \in \mathbf{C}$*

$$z \cdot 1 = z = 1 \cdot z$$

(6) Associativity of Multiplication. *For every triple $z_1, z_2, z_3 \in \mathbf{C}$*

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

(7) Commutativity of Multiplication. *For every pair $z_1, z_2 \in \mathbf{C}$*

$$z_1 \cdot z_2 = z_2 \cdot z_1$$

(8) Multiplicative Inverses. *For every $z \in \mathbf{C}^* := \mathbf{C} \setminus \{0\}$, there exists a complex number, denoted z^{-1} or $1/z$, such that*

$$z \cdot z^{-1} = 1 = z^{-1} \cdot z$$

In fact, if $z = x + iy$, then $z^{-1} = \dfrac{1}{z} := \dfrac{x}{x^2+y^2} - i\,\dfrac{y}{x^2+y^2}$.

(9) Distributive Law. *For every triple* $z_1, z_2, z_3 \in \mathbf{C}$

$$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$

Proof. (1) - (7) and (9) are left as Problem 1.4. One proves these directly by showing that the left hand side matches the right hand side.

(8) We note that

$$\begin{aligned}
z \cdot \frac{1}{z} &= (x + iy)\left(\frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}\right) \\
&= (x + iy)\left(\frac{x}{x^2+y^2} + i\,\frac{(-y)}{x^2+y^2}\right) \\
&= \left(x \cdot \frac{x}{x^2+y^2} - y \cdot \frac{(-y)}{x^2+y^2}\right) + i\left(x \cdot \frac{(-y)}{x^2+y^2} + y \cdot \frac{x}{x^2+y^2}\right) \\
&= \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + i\left(\frac{-yx + xy}{x^2+y^2}\right) \\
&= \frac{x^2+y^2}{x^2+y^2} + i \cdot 0 \\
&= 1
\end{aligned}$$

Of course, we should comment that $z = (x, y) \neq (0, 0)$ if and only if $x^2 + y^2 \neq 0$ (one proves this by stating and proving the contrapositive). $\square$

Remark 1.8. In the language of algebra,

- (1) – (4) tells us that $(\mathbf{C}, +)$ is an abelian group.
- (5) – (8) tells us that $(\mathbf{C}^*, \cdot)$ is an abelian group.
- (1) – (9) tells us that $(\mathbf{C}, +, \cdot)$ is a field.

Definition 1.9. Consider $z_1, z_2 \in \mathbf{C}$. We define *subtraction* and *division* as follows, respectively:

$$z_1 - z_2 := z_1 + (-z_2)$$

$$\frac{z_1}{z_2} := z_1 \cdot z_2^{-1} = z_1 \cdot \left(\frac{1}{z_2}\right), \quad z_2 \neq 0$$

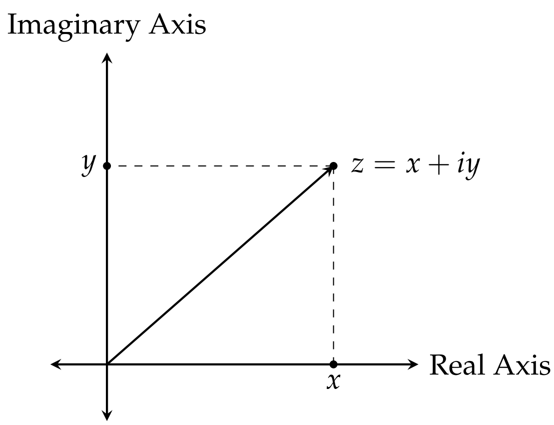
Writing down z_1/z_2 as $x + iy$ is not easy to remember, one obtains it by a method akin to "rationalising the denominator", in this case we could call it "realifiying the denominator"

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

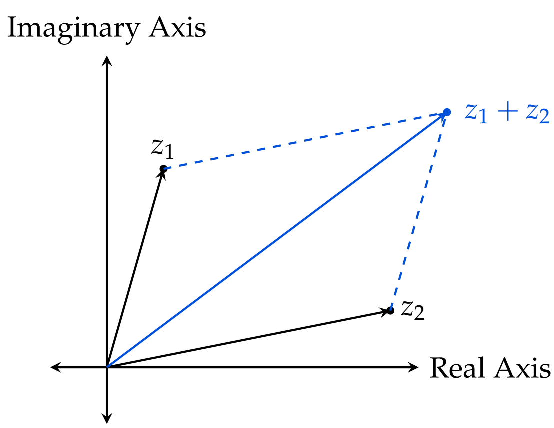
This method will be clarified soon when we talk about conjugates and absolute value.

Geometric Properties of Complex Numbers

As a set, we have $\mathbf{C} = \mathbf{R}^2$, so it's natural to visualise complex numbers as points in the complex plane (also called the Argand plane).



Geometrically, addition of complex numbers is just the addition of the corresponding vectors in the euclidean plane. We will soon see a geometric interpretation of multiplication.



Definition 1.10 (Modulus)**.** The modulus (or absolute value) of a complex number $z = x + iy$, denoted $|z|$, is the length of the vector (x, y), or equivalently its distance from the origin; namely

$$|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{x^2 + y^2} = \|(x, y)\|$$

Notice that this extends the usual absolute value of real numbers, as the modulus of a real number is its absolute value.

We can then immediately derive a useful inequality,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geqslant (\operatorname{Re} z)^2,\ (\operatorname{Im} z)^2,$$

giving us

$$\operatorname{Re} z \leqslant |\operatorname{Re} z| \leqslant |z| \quad \text{and} \quad \operatorname{Im} z \leqslant |\operatorname{Im} z| \leqslant |z|\,.$$

Definition 1.11 (Distance)**.** The distance between two complex numbers z_1 and z_2 is

$$|z_1 - z_2| = \|(x_1, y_1) - (x_2, y_2)\| = \|(x_1 - x_2, y_1 - y_2)\|$$

That is, it's the euclidean distance between the vectors representing these complex numbers.

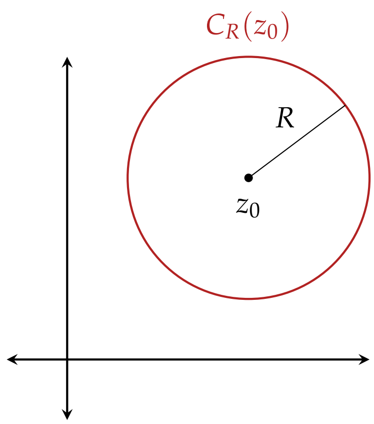
Discussion 1.12. The absolute value can be used to define various important subsets of $\mathbf{C}$.

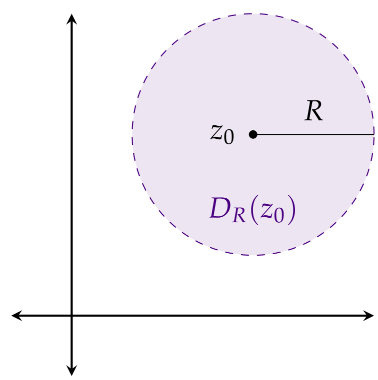
(1)
- The *circle of radius* $R > 0$ *centered at* z_0 is the set

$$C_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| = R\}$$

- The *open disk (or ball) of radius* $R > 0$ *centered at* z_0 is the set

$$D_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| < R\}$$



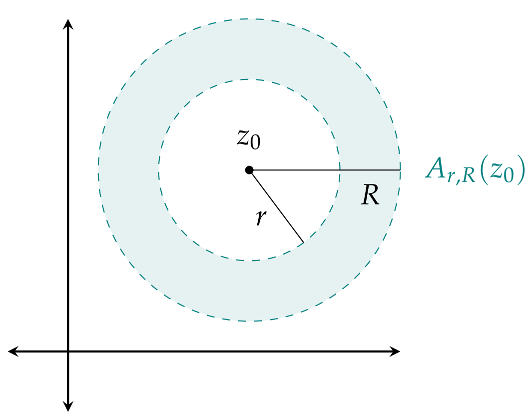


- The *closed disk (or ball) of radius* $R > 0$ *centered at* z_0 is the set

$$\overline{D}_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| \leqslant R\} = D_R(z_0) \cup C_R(z_0).$$

(2) The *(open) annulus of inner radius* $r > 0$ *and outer radius* $R > 0$ *centered at* z_0 is the set

$$A_{r,R}(z_0) = \{z \in \mathbf{C} \,:\, r < |z - z_0| < R\}$$



1.1. Problems

Problem 1.1. Consider the set of matrices

$$X := \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \ : \ x, y \in \mathbf{R} \right\}.$$

One can check (and you should if you're unconvinced) straightforwardly that X is closed under matrix addition and matrix multiplication; that is, if $A, B \in X$, then $A + B,\ AB \in X$.

(a) Let $\mathbf{C}$ denote the set of complex numbers. Show that the map $\phi : X \to \mathbf{C}$ defined by

$$\phi : X \to \mathbf{C}, \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x + iy$$

is a bijection.

(b) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the identity matrix. Consider $A, B \in X$, show that ϕ has the following properties.

  (i) $\phi(A + B) = \phi(A) + \phi(B)$

  (ii) $\phi(AB) = \phi(A)\phi(B)$

  (iii) $\phi(I) = 1$

(c) Find a matrix J satisfying $J^2 = -I$ and show that $\phi(J) = i$.

Remark 1.13. This indicates that one could very well define $\mathbf{C}$ to be X. The algebraic operations on $\mathbf{C}$ then seem less artificial, since product and sum of complex numbers correspond to the corresponding operations of matrices. Even taking the inverse and modulus is captured by X as taking inverse and the determinant of matrices. The copy of $\mathbf{R}$ corresponds to the set of diagonal matrices in X. One obtains X by considering the linear operator of multiplying by $x + iy$ on the $\mathbf{R}$-vector space $\mathbf{C}$ with basis 1 and i.

Problem 1.2. Using the definition of complex multiplication prove that

$$(a, 0) \cdot (x, y) = (ax, ay).$$

That is, $a(x + iy) = ax + iay$.

Problem 1.3. Consider complex numbers $z_1 = (x_1, y_1) = x_1(1, 0) + y_1(0, 1)$ and $z_2 = (x_2, y_2) = x_2(1, 0) + y_2(0, 1)$. Using the identity $(0, 1)^2 = (-1, 0)$. Prove that

$$(x_1(1,0) + y_1(0,1)) \cdot (x_2(1,0) + y_2(0,1)) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1),$$

where the former is computed distributively.

Problem 1.4. Prove properties (1) - (7) and (9) listed in Proposition 1.7.

Problem 1.5. Prove that if $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

Problem 1.6. Show that

(a) $\operatorname{Re} iz = -\operatorname{Im} z$;

(b) $\operatorname{Im} iz = \operatorname{Re} z$

Problem 1.7.

(a) Verify that $z = 1 \pm i$ satisfies the equation

$$z^2 - 2z + 2 = 0.$$

(b) Solve the equation

$$z^2 + z + 1 = 0$$

for $z = x + iy$ by solving a pair of simultaneous equations in x and y.

Problem 1.8. Let $p(z) = az^2 + bz + c$ be a polynomial with complex coefficients ($a \neq 0$).

(a) By completing the square, show that the solution to $p(z) = 0$ is

$$z = \frac{-b \pm \Delta^{1/2}}{2a},$$

where $\Delta := b^2 - 4ac$ is called the discriminant.

Remark. There's a subtlety with taking roots that we will address later in class.

(b) Consider the polynomial $p(z) = iz^2 - 1$

(i) Compute Δ.

(ii) For the Δ obtained in (b), compute $\Delta^{1/2}$ by solving a pair of simultaneous equations in x and y obtained by considering the equation

$$x^2 - y^2 + 2ixy = (x + iy)^2 = \Delta.$$

(iii) Finally, write down the roots of $p(z)$ in the form $u + iv$.

Problem 1.9. Suppose $\mathbf{C}$ had total ordering that extends the ordering on $\mathbf{R}$, arrive at a contradiction by comparing i and 0.

Problem 1.10. Locate the numbers $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$ in the complex plane when

(a) $z_1 = 2i$, $z_2 = \dfrac{2}{3} - i$.

(b) $z_1 = (-3, 1)$, $z_2 = (1, 4)$.

(c) $z_1 = (-\sqrt{3}, 1)$, $z_2 = (\sqrt{3}, 0)$.

(d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$.

Problem 1.11. Verify that $\sqrt{2}\,|z| \geqslant |\operatorname{Re} z| + |\operatorname{Im} z|$.

Problem 1.12. Let $z_0 \neq z_1 \in \mathbf{C}$ and let $\lambda > 0$.

(a) Show that if $\lambda \neq 1$, then the set of points

$$|z - z_0| = \lambda\,|z - z_1| \tag{$\bigstar$}$$

is a circle of radius $R = \dfrac{\lambda}{|1-\lambda^2|}\,|z_0 - z_1|$ centered at $w = \dfrac{z_0 - \lambda^2 z_1}{1 - \lambda^2}$.

(b) Show that every circle in the complex plane can be written in the form of ($\bigstar$) for some $\lambda > 0$, $\lambda \neq 1$ and $z_0 \neq z_1 \in \mathbf{C}$.

(c) If $\lambda = 1$, show that ($\bigstar$) defines a line. In fact, argue that the resulting line is perpendicular to and bisects the line segment joining z_0 and z_1, by producing the equation of this line as a subset of $\mathbf{R}^2$.

(d) Characterise points on the real (resp. imaginary) axis using (c). That is, find $z_0 \neq z_1 \in \mathbf{C}$ such that the points on the real (resp. imaginary) axis satisfy ($\bigstar$) for $\lambda = 1$.

(e) Consider the map

$$M(z) = \frac{z-3}{1-2z}.$$

For which values of $c \in \mathbf{R}$ is the image of the circle $|z - 1| = c$ under M a line? What is the equation of the line when considered as a subset of the plane $\mathbf{R}^2$?

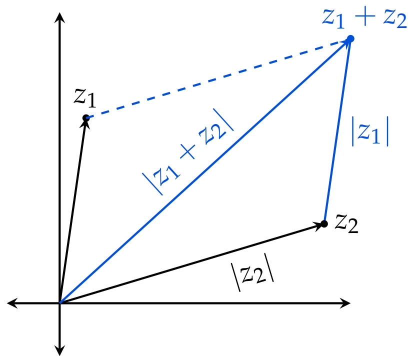
2. Lecture 2 (3/31)

Proposition 2.1 (Triangle Inequalities)**.** *For all $z_1, z_2 \in \mathbf{C}$, the following inequalities hold.*

(1) $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

(2) $|z_1 \pm z_2| \geqslant ||z_1| - |z_2||$. *We sometimes refer to this inequality as the* reverse triangle inequality.

Proof.



(1) A standard fact about triangles.

(2) We first assume that $|z_1| \geqslant |z_2|$. Then, $||z_1| - |z_2|| = |z_1| - |z_2|$. Now, note that

$$\begin{aligned} |z_1| - |z_2| &= |z_1 \pm z_2 \mp z_2| - |z_2| \\ &\leqslant |z_1 \pm z_2| + |\mp z_2| - |z_2|\,, \text{ triangle inequality} \\ &= |z_1 \pm z_2| + |z_2| - |z_2| \\ &= |z_1 \pm z_2| \end{aligned}$$

If we instead assumer $|z_2| \geqslant |z_1|$, then we do the same computation with the roles of z_1 and z_2 switched. ☐

Proposition 2.2 (Modulus is Multiplicative)**.** *For all $z, w \in \mathbf{C}$ and positive integers n,*

(1) $|zw| = |z|\,|w|$.

(2) $|z^n| = |z|^n$.

Proof.

(1) Left as Problem 2.1. One proves these directly by showing that the left hand side matches the right hand side.

(2) The proof of this is by induction. $n = 1$ is a tautology, and $n = 2$ is (1) in the case $w = z$. Assume the statement is true for $n = k$, that is $\left|z^k\right| = |z|^k$. Then, for $n = k + 1$

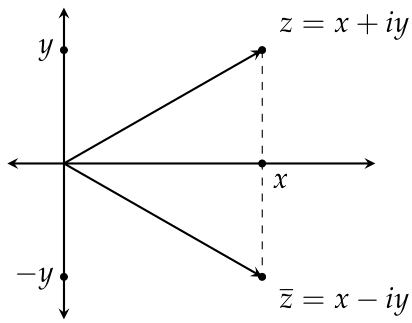
$$\begin{aligned} |z^{k+1}| = |z^k \cdot z| &= |z^k|\,|z|\,, \quad \text{using (1)} \\ &= |z|^k\,|z|\,, \quad \text{using the induction hypothesis} \\ &= |z|^{k+1} \end{aligned}$$

Therefore we have the result by the principle of mathematical induction. $\square$

Definition 2.3 (Complex Conjugation)**.** Given a complex number $z = x + iy$, its (complex) conjugate, denoted $\overline{z}$, is

$$\overline{z} := x - iy$$

Geometrically, $\overline{z}$ is the reflection of z about the real axis.



Proposition 2.4 (Properties of Conjugation)**.** *For all pairs $z, w \in \mathbf{C}$, we have*

(1) $\overline{\overline{z}} = z$

(2) $|\overline{z}| = |z|$

(3) $\overline{z + w} = \overline{z} + \overline{w}$

(4) $\overline{zw} = \overline{z}\,\overline{w}$

(5) $z\overline{z} = |z|^2$

(6) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(7) $z \in \mathbf{R}$ *if and only if* $z = \overline{z}$

Proof. (1) – (3) is clear geometrically. (4), (6) and (7) are left as Problem 2.2, (7) can be proved using (6) and can also be deduced geometrically. One proves these directly by showing that the left hand side matches the right hand side.

(5) Let $z = x + iy$, then

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + iyx - i^2y^2 \\ &= x^2 + y^2 + i(yx - xy) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

$\square$

Discussion 2.5. Proposition 2.4 (5) gives us a nice formula for z^{-1} for $z \in \mathbf{C}^*$. For such a z, we have $z\overline{z} = |z|^2$, which gives us

$$z^{-1} = z^{-1} \cdot \frac{z\overline{z}}{|z|^2} = \frac{\overline{z}}{|z|^2}$$

This tells us that z^{-1} is just a scaled $\overline{z}$, which means, geometrically speaking, z^{-1} lies on the line passing through the origin and $\overline{z}$.

Recall that every non-zero point $(x, y) \in \mathbf{R}^2$ can be re-written in polar coordinates (r, θ) as

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This suggests the following definition.

Definition 2.6 (Polar Form)**.** If (r, θ) are polar coordinates for a non-zero (x, y), then the polar form of a non-zero complex number $z = x + iy$ is

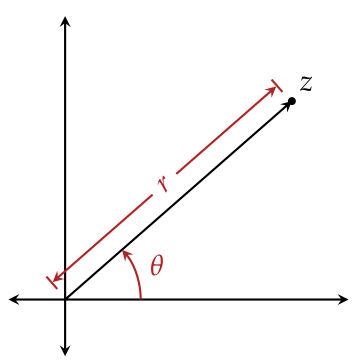
$$z = r(\cos\theta + i\sin\theta)$$

We sometimes abbreviate $\cos\theta + i\sin\theta$ as $\operatorname{cis}\theta$, so $z = r\operatorname{cis}\theta$.

Evidently, (r, θ) are related to (x, y) by the equations

$$|z| = r \quad \text{and} \quad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}},\ \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}},\ \text{so}\ \tan\theta = \frac{y}{x}$$

We have to be careful and take into account which quadrant (x, y) belongs to, if we think of θ with respect to its formulation using tan.



Since sin and cos are periodic functions, θ is not unique (you can replace θ with $\theta + 2\pi$). Each possible value of θ is called an argument of z, and the set of all such θ is denoted as $\arg z$. That is,

$$\arg z = \{\arctan(y/x) + 2k\pi \ :\ k \in \mathbf{Z}\}$$

The polar form, specifically θ is unique, as soon as we specify bounds on θ. The unique argument in the interval $(-\pi, \pi]$ is called the principal argument denoted $\operatorname{Arg} z$.

Notice that we can then write

$$\arg z = \{\operatorname{Arg} z + 2k\pi \ :\ k \in \mathbf{Z}\}$$

Definition 2.7 (Euler's Formula)**.** $e^{i\theta} := \operatorname{cis}\theta = \cos\theta + i\sin\theta$. Therefore $\left|e^{i\theta}\right| = 1$.

Remark on Definition 2.7. This is for now a stopgap, defining $e^{i\theta}$ in this way. In a few weeks, we'll see that this is truly an equality of holomorphic functions. Euler deduced this by looking at the Taylor series expansion of these functions. We haven't built or discussed enough machinery to give this reasoning a solid foundation yet.

Using Euler's formula, one can write the polar form of a non-zero complex number, even more succinctly in its exponential form

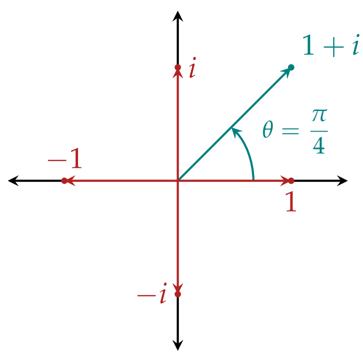
$$z = re^{i\theta}$$

Example 2.8.

(1) Exponential form of $1+i$,

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2} \quad \text{and} \quad \operatorname{Arg} z = \arctan(1) = \frac{\pi}{4}$$

So, $1+i = \sqrt{2}e^{i\pi/4}$.



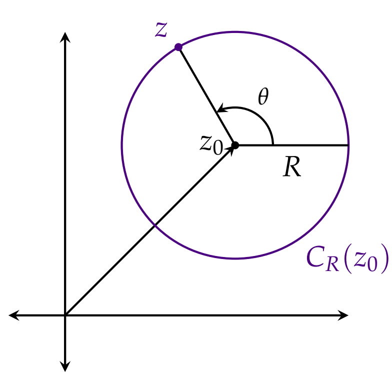
(2) Note that

$$1 = e^{i0} = e^{i2n\pi} \text{ for any } n \in \mathbf{Z}, \qquad i = e^{i\pi/2}, \qquad -1 = e^{i\pi} = e^{i(2n+1)\pi} \text{ for any } n \in \mathbf{Z}$$

One could write $-i = e^{i3\pi/2}$ but $3\pi/2 \neq \operatorname{Arg}(-i)$; instead we should write $-i = e^{-i\pi/2}$.

(3) The circle $C_R(z_0)$ has a nice parametrisation

$$C_R(z_0) = \{z = z_0 + Re^{i\theta} \,:\, 0 \leqslant \theta < 2\pi\}$$



Proposition 2.9 (Properties of Exponential Form)**.** *Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be non-zero complex numbers. Then*

(1) $zw = rs\, e^{i(\theta+\phi)}$

(2) $z^{-1} = (1/r)e^{-i\theta}$

(3) $z^n = r^n e^{in\theta}$*, for any* $n \in \mathbf{Z}$

(4) $\bar{z} = re^{-i\theta}$

(5) $z/w = (r/s)e^{i(\theta-\phi)}$

Proof.

(1) Note that

$$\begin{aligned} zw = (re^{i\theta})(se^{i\phi}) &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\ &= rs\, e^{i(\theta+\phi)} \end{aligned}$$

(2) It suffices to show that $(re^{i\theta})((1/r)e^{-i\theta}) = 1$, for which we use (1).

(3) We first prove this result for $n \geqslant 0$, the result is clear for $n = 0$ and $n = 1$. Assume the result is true for $n = k$, that is $z^k = r^k e^{ik\theta}$. Then, for $n = k+1$

$$\begin{aligned} z^{k+1} &= z^k z \\ &= (r^k e^{ik\theta})(re^{i\theta}) \text{ using the induction hypothesis} \\ &= r^{k+1}e^{ik\theta+\theta} \text{ by (1)} \\ &= r^{k+1}e^{i(k+1)\theta} \end{aligned}$$

Therefore we have the result by the principle of mathematical induction.

Suppose $n < 0$ instead, then write $n = -m$ for a positive $m > 0$. Now, we can apply the first case to $z^n := (z^{-1})^m$ to get our result.

(4) Using $z\bar{z} = |z|^2 = r^2$, we get that $\bar{z} = r^2 z^{-1}$, and the result follows from (2).

(5) Recall $z/w = zw^{-1}$, and the result follows from (2) and (1). □

Example 2.10. Let's use this to compute $(1+i)^{2021}$, then

$$\begin{aligned} (1+i)^{2021} &= (\sqrt{2}e^{i\pi/4})^{2021} \\ &= (\sqrt{2}e^{i\pi/4})^{2020}(\sqrt{2}e^{i\pi/4}) \\ &= (\sqrt{2})^{2020}(e^{i2020\pi/4})(1+i) \\ &= 2^{1010}(e^{i505\pi})(1+i) \\ &= -2^{1010}(1+i) \end{aligned}$$

Example 2.11 (in-class)**.** Compute $(1+i\sqrt{3})^{101}$.

Answer. We first note that $|1+i\sqrt{3}| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$, and since $1+i\sqrt{3}$ lies in the first quadrant of the complex plane

$$\operatorname{Arg} z = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

Therefore

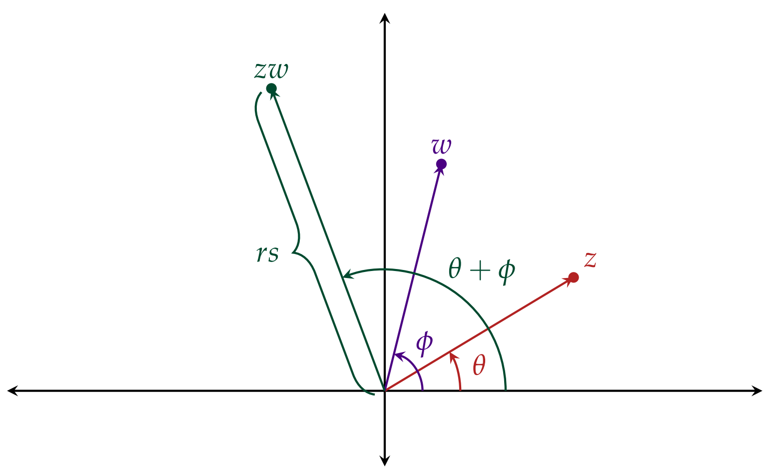
$$1+i\sqrt{3} = 2e^{i\pi/3}$$

and so

$$\begin{aligned}(1+i\sqrt{3})^{101} &= (2e^{i\pi/3})^{101} \\ &= (2e^{i\pi/3})^{99}(2e^{i\pi/3})^2 \\ &= 2^{99}e^{i33\pi}(1+i\sqrt{3})^2 \\ &= -2^{99}(1-3+2i\sqrt{3}), \quad \text{since 33 is odd} \\ &= -2^{99}(-2+2i\sqrt{3}) \\ &= 2^{100}(1-i\sqrt{3})\end{aligned}$$

□

Discussion 2.12. Proposition 2.9 (1) gives us a nice geometric interpretation of complex multiplication. If $z = re^{i\theta}$ and $w = se^{i\phi}$, then $zw = rs\, e^{i(\theta+\phi)}$. This can be interpreted as saying that zw is obtained from w by scaling w by $|z| = r$ and rotating w by an angle of $\operatorname{Arg} z$ (or vice versa).



A few more interesting consequences

(1) The *unit circle*

$$S^1 = \{z \in \mathbf{C} \,:\, |z| = 1\} = \{e^{i\theta} \,:\, \theta \in \mathbf{R}\}$$

is closed under multiplication. It's in fact an abelian group, usually denoted $U(1)$.

(2) *De Moivre's Theorem*. From Proposition 2.9 (4) applied to $z = e^{i\theta}$ we get

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

2.1. Problems

Problem 2.1. Prove Proposition 2.2 (1).

Problem 2.2. Prove the properties, other than (5), listed in Proposition 2.4.

Problem 2.3. Prove that z is either real or pure imaginary if and only if $z^2 = \bar{z}^2$.

Problem 2.4. Prove that $|z| = 1$ if and only if $\bar{z} = \dfrac{1}{z}$.

Problem 2.5. Follow the steps below to give an algebraic derivation of the triangle inequality (Proposition 2.1 (a))

(a) Show that
$$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2.$$

(b) Argue why
$$z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\operatorname{Re}(z_1\bar{z}_2) \leqslant 2|z_1||z_2|.$$

(c) Use (a) and (b) to obtain $|z_1 + z_2|^2 \leqslant (|z_1| + |z_2|)^2$. Finally note how the triangle inequality follows from this.

Problem 2.6. Let $z, w \in \mathbf{C}$.

(a) Prove the formula
$$|z + w|^2 = |z|^2 + 2\operatorname{Re} z\bar{w} + |w|^2$$

(b) Use (a) to deduce the *parallelogram law*
$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$$
Give a geometric interpretation of this formula.

Problem 2.7. Suppose p is a polynomial with *real coefficients*. Prove that

(a) $\overline{p(z)} = p(\bar{z})$.

(b) $p(z) = 0$ if and only if $p(\bar{z}) = 0$.

Problem 2.8. Find the principal argument $\operatorname{Arg} z$ when

(a) $-i(3 + 3i)^{-1}$.

(b) $(1 - i\sqrt{3})^6$.

3. Lecture 3 (4/05)

Proposition 3.1 (Arguments of Products). *Let z, w be non-zero complex numbers, then*

(1) $\arg(zw) = \arg z + \arg w$

(2) $\arg w^{-1} = -\arg w$

Note that this is *not* saying $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$, this is actually not true, we're claiming an equality of sets. (1) and (2) together give us $\arg(z/w) = \arg z - \arg w$.

Proof.

(1) Consider $\theta \in \arg z$ and $\phi \in \arg w$, so $z = re^{i\theta}$ and $w = se^{i\phi}$. By Proposition 2.9 (1), we have $zw = rs\, e^{i(\theta+\phi)}$ and therefore $\theta + \phi \in \arg(zw)$. Hence $\arg z + \arg w \subseteq \arg(z + w)$.

Consider $\psi \in \arg(z + w)$, and some $\theta \in \arg z$ then we claim that $\psi - \theta \in \arg w$. We have $rs\, e^{i\psi} = zw = re^{i\theta} w$, then by Proposition 2.9 (5), we get $w = sr^{i(\psi-\theta)}$. Hence $\psi - \theta \in \arg w$, and since $\psi = \theta + (\psi - \theta) \in \arg z + \arg w$, we have $\arg(z + w) \subseteq \arg z + \arg w$.

Therefore $\arg(zw) = \arg z + \arg w$.

(2) Consider $\theta \in \arg z$, so $z = re^{i\theta}$. By Proposition 2.9 (2), we have $z^{-1} = (1/r)e^{i(-\theta)}$ and therefore $-\theta \in \arg w^{-1}$. Hence $-\arg w \subseteq \arg w^{-1}$.

Note that $w = (w^{-1})^{-1}$, applying the above result to w^{-1} gets us $-\arg w^{-1} \subseteq \arg(w^{-1})^{-1} = \arg w$ and so $\arg w^{-1} \subseteq -\arg w$.

Therefore $\arg w^{-1} = -\arg w$. □

Remark 3.2. For a complex number, $\arg z$ is a set of all possible θ's such that we can write $z = |z|\, e^{i\theta}$, as you know. Therefore, we will abuse notation by sometimes calling any $\theta \in \arg z$ as an argument of z, and sometimes also writing $z = |z|\, e^{i \arg z}$. That is, we are not, or are careless about, distinguishing the set $\arg z$ and its element when we can be agnostic about the choice of θ; for example, the polar form of a complex number. It will be clear when we choose to care about out choice, it will be evident because we'll be then forcing θ to lie in an interval of length 2π; for example, the principal argument $-\pi < \operatorname{Arg} z \leqslant \pi$.

Example 3.3.

(1) The principal argument of $z = (\sqrt{3} - i)^6$. We first note that $\operatorname{Arg}(\sqrt{3} - i) = -\pi/6$. By Proposition 3.1 (1), applied inductively, we have

$$\arg(\sqrt{3} - i)^6 = \underbrace{\arg(\sqrt{3} - i) + \cdots + \arg(\sqrt{3} - i)}_{6 \text{ times}} = \{-\pi + 2k\pi \;:\; k \in \mathbf{Z}\}$$

Then $\operatorname{Arg}(\sqrt{3} - i)^6$ is the element in the set above in the interval $(-\pi, \pi]$ which is π.

(2) As mentioned previously, we can't just replace arg with Arg in the statement of Proposition 3.1 (1). Here's a simple example: let $z = w = -1$, then $\operatorname{Arg} z = \operatorname{Arg} w = \pi$ and $\operatorname{Arg} zw = \operatorname{Arg} 1 = 0$ but $0 \neq 2\pi = \operatorname{Arg} z + \operatorname{Arg} w$.

(3) Note that $\arg z + \arg z \neq 2 \arg z$.

Roots of Complex Numbers

Lemma 3.4. *Two non-zero complex numbers z, w are equal if and only if $|z| = |w|$ and $\arg z = \arg w$.*

Proof. If $|z| = |w|$ and $\arg z = \arg w$, then clearly $z = w$.

Suppose $z = w$, then we immediately get $|z| = |w|$. Consider $\theta \in \arg z$ and $\phi \in \arg w$, then we get $e^{i\theta} = e^{i\phi}$ which is equivalent to saying $\cos(\theta - \phi) + i\sin(\theta - \phi) = e^{i(\theta-\phi)} = 1$. This gives us

$$\sin(\theta - \phi) = 0.$$

The solution to this is $\theta - \phi = 2k\pi$ for some $k \in \mathbf{Z}$. This gives us $\arg z = \arg w$. $\square$

Definition 3.5 (Roots)**.** Let α be a non-zero complex number. An *n^{th} root of α* is a solution to the polynomial equation $z^n - \alpha = 0$.

The set of all n^{th} roots of α is denoted by $\alpha^{1/n}$, we reserve the symbol $\sqrt[n]{\cdot}$ for the unique positive n^{th} root of a positive real number.

Proposition 3.6 (Distinct Roots)**.** *There are precisely n distinct n^{th} roots of α, namely*

$$\beta_k = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, \dots, n-1$$

Proof. Let $z = re^{i\theta}$ and $\alpha = |\alpha|\, e^{i\operatorname{Arg}\alpha}$, we solve

$$r^n e^{in\theta} = z^n = \alpha = |\alpha|\, e^{i\operatorname{Arg}\alpha}.$$

By Lemma 3.4, this equality is true if and only if $r^n = |\alpha|$ and $n\theta = \operatorname{Arg}\alpha + 2k\pi$ for some $k \in \mathbf{Z}$. Therefore

$$z = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbf{Z}$$

We obtain distinct n complex numbers for $k = 0, \dots, n-1$ since they have distinct arguments, and they necessarily give us the n distinct n^{th} roots of α. $\square$

Discussion 3.7. With the notation of Proposition 3.6, the n^{th} principal root of α is

$$\beta_0 = \sqrt[n]{|\alpha|}\, e^{i\frac{\operatorname{Arg}\alpha}{n}}$$

If we introduce the notation $\zeta_n = e^{\frac{2\pi i}{n}}$, then

$$\zeta_n^k = e^{\frac{2k\pi i}{n}}$$

According to the proposition, the complex numbers

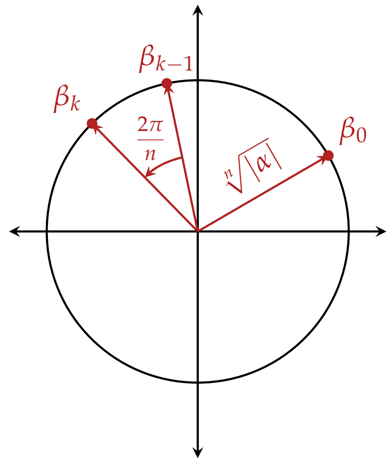
$$1,\ \zeta_n,\ \zeta_n^2, \dots, \zeta_n^{n-1}$$

are the distinct solutions to $z^n - 1 = 0$, the n^{th} roots of unity, making ζ_n the *principal n^{th} root of unity*.

Then we can write the roots of α in terms of the principal root and the roots of unity

$$\beta_k = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n} + \frac{2k\pi}{n}\right)} = \sqrt[n]{|\alpha|}\, e^{i\frac{\operatorname{Arg}\alpha}{n}} e^{\frac{2k\pi i}{n}} = \beta_0 \zeta_n^k$$

That is, β_k's all lie on the circle of radius $\sqrt[n]{|\alpha|}$ centered at the origin, and all of them are obtained by rotating β_0 by an angle of $2k\pi/n$. That is, they all lie on the vertices of an inscribed regular n-gon.

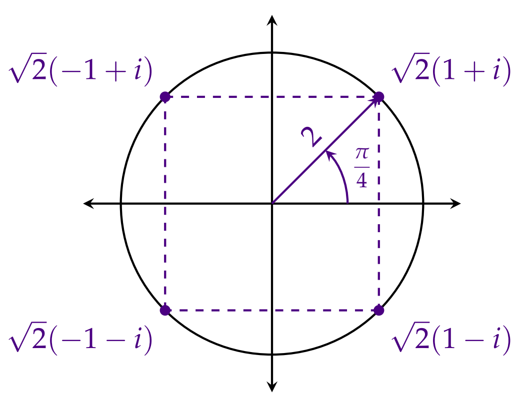


Example 3.8.

(1) We compute explicitly the 4^{th} roots of $\alpha = -16$. As a negative real number, $\operatorname{Arg}(-16) = \pi$, so

$$\begin{aligned}
\beta_k &= \sqrt[4]{16} e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)} = 2\, e^{i\frac{\pi}{4}} e^{\frac{ki\pi}{2}} \\
&= 2\, e^{i\frac{\pi}{4}} \left(e^{\frac{i\pi}{2}}\right)^k \\
&= 2\, \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = 2\, \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) i^k = \sqrt{2}(1+i)i^k
\end{aligned}$$

Therefore



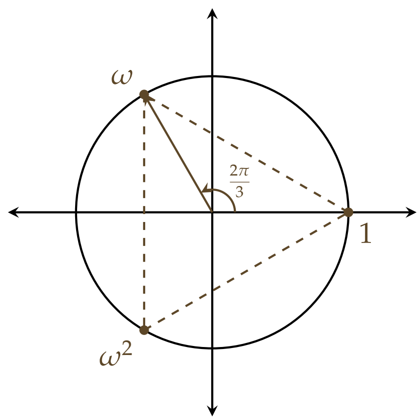
$$\beta_0 = \sqrt{2}(1+i), \quad \beta_1 = \sqrt{2}(-1+i), \quad \beta_2 = \sqrt{2}(-1-i), \quad \beta_3 = \sqrt{2}(1-i)$$

(2) In the course of the previous example, we have computed the 4^{th} roots of unity, since they are

$$e^{\frac{2ki\pi}{4}} = e^{\frac{ki\pi}{2}}, \quad k = 0, 1, 2, 3$$

as $\operatorname{Arg} 1 = 0$. Letting $\zeta_4 = e^{i\pi/2} = i$, the 4^{th} roots of unity are $\zeta_4^0, \zeta_4^1, \zeta_4^2, \zeta_4^3$, which are nothing but $\pm 1, \pm i$.

Example 3.9 (in-class)**.** Compute the 3^{rd} roots of unity, also called the cube roots of unity where we denote $\omega = \zeta_3$, explicitly.



Answer. Let the principal root be $\omega = \zeta_3$, then the cube roots of unity are

$$1,\ \omega,\ \omega^2$$

where we have

$$\omega = e^{\frac{2\pi i}{3}} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega^2 = e^{\frac{4\pi i}{3}} = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

□

Basic Topology of $\mathbf{C}$

Our purpose now is to define the kind of subsets of $\mathbf{C}$ that are suitable for doing complex analysis, namely *non-empty open connected sets*.

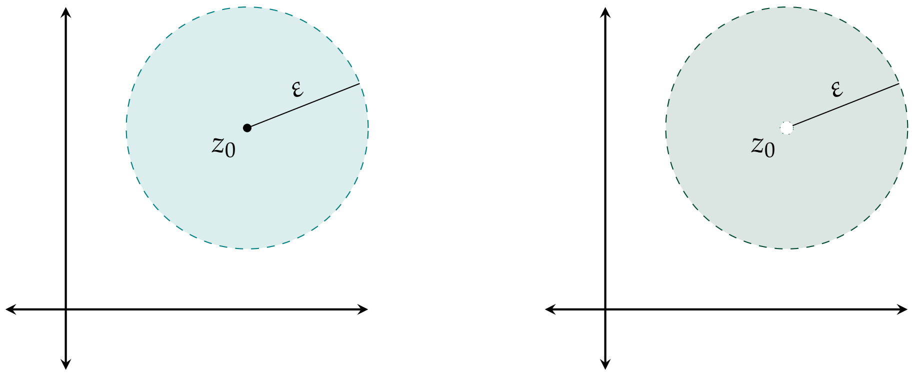
Definition 3.10 (Open Disks or Neighbourhoods)**.** Let $\varepsilon > 0$. Recall the open disk (of radius ε centered at z_0) is the set

$$D_\varepsilon(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| < \varepsilon\}.$$

We also refer to such an open disk as an ε-neighbourhood or simply a neighbourhood.

A deleted (or punctured) open disk (or neighbourhood) is a set of the form

$$D_\varepsilon(z_0) \setminus \{z_0\} = \{z \in \mathbf{C} \,:\, 0 < |z - z_0| < \varepsilon\}.$$



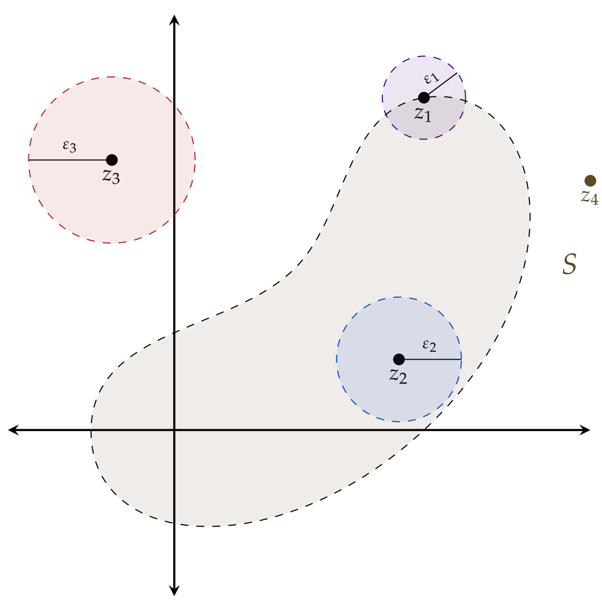
Points belonging to the same ε-neighbourhood are considered "close" to each other, in the sense that they are within a distance of 2ε from each other.

Definition 3.11 (Various kinds of Points)**.** Consider a $S \subseteq \mathbf{C}$.

- A point $z \in S$ is an interior point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \subseteq S$.
- A point $z \notin S$ is an exterior point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \cap S = \varnothing$.
- A point $z \in \mathbf{C}$ is a boundary point of S if it's neither an interior nor an exterior point of S. Equivalently, if every neighbourhood of z contains both a point in S and not in S.
- A point $z \in \mathbf{C}$ is a accumulation (or cluster) point of S if for every $\varepsilon > 0$ we have

$$D_\varepsilon(z) \setminus \{z\} \cap S \neq \varnothing.$$

- A point $z \in S$ is an isolated point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \setminus \{z\} \cap S = \varnothing$. Isolated points are examples of boundary point



Here z_1 is a boundary point, z_2 an interior point, z_3 an exterior point, and z_4 is an isolated point (and a boundary point).

Remark 3.12. The idea is that if we don't move too far from an interior point of S then we remain in S; a similar idea holds for an exterior point. But at a boundary point we can make an arbitrarily small move and get to a point inside S, and we can also make an arbitrarily small move and get to a point outside S. An accumulation point is one where it has other points from S within any arbitrarily small distance, i.e. points "accumulate" near it; an isolated point is the exact opposite.

3.1. Problems

Problem 3.1. Prove that

$$\arg z + \arg w = \{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \ : \ k \in \mathbf{Z}\}$$

Combining this with Proposition 3.1 we get that $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2k\pi$ for some $k \in \mathbf{Z}$ such that $-\pi < \operatorname{Arg} z + \operatorname{Arg} w + 2k\pi \leqslant \pi$. That is, to find $\operatorname{Arg} zw$, just add $\operatorname{Arg} z$ and $\operatorname{Arg} w$ and then add or subtract a suitable multiple of 2π to get it between $-\pi$ and π.

Problem 3.2. Prove that for any complex number z, we have $\operatorname{Arg} \overline{z} = \operatorname{Arg} z^{-1} = -\operatorname{Arg} z$.

Problem 3.3.

(a) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

(b) Show that if $\operatorname{Re} z > 0$, then $\operatorname{Arg}(-z) = -\pi + \operatorname{Arg} z$ if $\operatorname{Im} z > 0$ or $\operatorname{Arg}(-z) = \pi + \operatorname{Arg} z$ if $\operatorname{Im} z < 0$.

(c) Using (a) and (b), find an expression for $\operatorname{Arg} zw$ for any non-zero complex numbers z and w, in terms of $\operatorname{Arg} z$, $\operatorname{Arg} w$ and specific multiples of π.

Problem 3.4. Compute the 6^{th} roots of unity, explicitly. Show that the principal 6^{th} root of unity is $\zeta_6 = -\omega$, where ω is as in Example 3.9.

Problem 3.5.

(a) Let $z \in \mathbf{C}$. Using the principle of mathematical induction, show that the following formula holds for all integers $n \geqslant 1$

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) Use (a) to derive *Lagrange's Trigonometric Identity*.

$$1 + \cos\theta + \cos^2\theta + \cdots + \cos^n\theta = \frac{2\sin((2n+1)\theta/2)}{2\sin(\theta/2)}, \quad 0 < \theta < 2\pi.$$

(c) If $\zeta_1, \ldots, \zeta_n$ are the *distinct* n^{th} roots of unity, show that, using (a), $\sum_{i=1}^{n} \zeta_i = 0$.

(d) We compute the following sum of real numbers

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} \tag{†}$$

(i) Let $w = e^{\frac{\pi i}{7}}$. What is $\operatorname{Re} w$ and w^7? Furthermore, rewrite (†) as

$$\operatorname{Re}(w^{a_1} + w^{a_2} + w^{a_3}), \quad \text{for some } 0 \leqslant a_i < 7.$$

(ii) Replacing z by $-z$ in (a), find a formula for

$$\frac{z^7 + 1}{z + 1}.$$

Use this to deduce an identity involving w and its powers.

(iii) Using the identity you found in (iii), conclude that

$$w^{a_1} + w^{a_2} + w^{a_3} = \frac{1}{1 - w}$$

where the a_i's are the numbers you found in (ii).

(iv) Finally compute (†).

4. Lecture 4 (4/07)

Definition 4.1 (Open and Closed Sets)**.** Consider a $S \subseteq \mathbf{C}$.

- The interior of S is the set of all interior points of S, denoted S°.
- S is said to be open if $S = S^\circ$.
- The boundary of S is the set of all boundary points of S, denoted ∂S.
- S is said to be closed if $\partial S \subseteq S$. Equivalently, if its complement is open.
- The closure of S is the set $S \cup \partial S$, denoted $\overline{S}$.

Example 4.2.

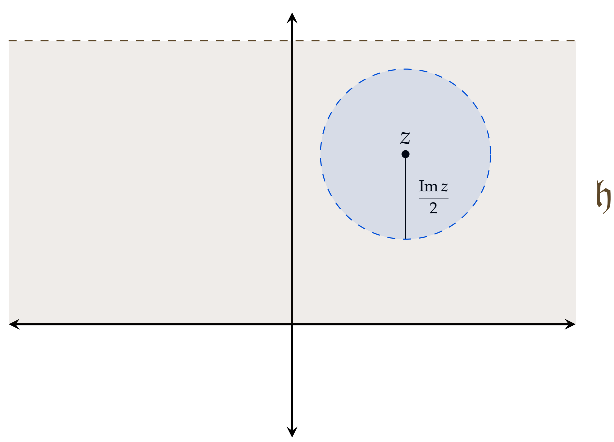
(1) The open disks $D_R(z_0)$ are truly open sets, and the closed disks $\overline{D}_R(z_0)$ are truly closed sets. The closure of the open disk $D_R(z_0)$ is $\overline{D}_R(z_0)$. The boundary of $D_R(z_0)$ is the circle $C_R(z_0)$.

(2) Consider the upper half-plane

$$\mathfrak{h} = \{z \in \mathbf{C} \ : \ \operatorname{Im} z > 0\},$$

then we have $\mathfrak{h}^\circ = \mathfrak{h}$. Since by definition $\mathfrak{h}^\circ \subseteq \mathfrak{h}$, it's enough to prove $\mathfrak{h} \subseteq \mathfrak{h}^\circ$. Consider any $z \in \mathfrak{h}$, then $\operatorname{Im} z > 0$. Let $\varepsilon = (\operatorname{Im} z)/2$, we claim that

$$D_\varepsilon(z) \subseteq \mathfrak{h}$$



Let $w \in D_\varepsilon(z)$, then

$$|w - z| < \varepsilon = \frac{\operatorname{Im} z}{2}$$

The end of Discussion 1.10 tells us

$$\begin{aligned} \frac{\operatorname{Im} z}{2} > |w - z| \geqslant |\operatorname{Im}(w - z)| \\ = |\operatorname{Im} w - \operatorname{Im} z| \end{aligned}$$

The later is simply the absolute value of a real number, which gives

$$-\frac{\operatorname{Im} z}{2} < \operatorname{Im} w - \operatorname{Im} z < \frac{\operatorname{Im} z}{2}$$

Adding $\operatorname{Im} z$ throughout the inequality, we get from the inequality on the left hand side

$$\operatorname{Im} w > \frac{\operatorname{Im} z}{2} > 0.$$

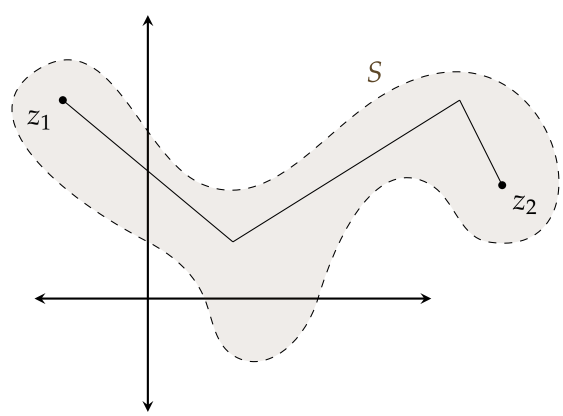
Therefore $w \in \mathfrak{h}$, and hence $D_\varepsilon(z) \subseteq \mathfrak{h}$. Thus $\mathfrak{h}^\circ = \mathfrak{h}$.

The points exterior to $\mathfrak{h}$ are points z such that $\operatorname{Im} z < 0$. That is, the exterior of the upper half-plane is the (open) lower half-plane. The boundary of $\mathfrak{h}$ consists of precisely points z whose $\operatorname{Im} z = 0$. That is, $\partial\mathfrak{h} = \mathbf{R}$.

The closure of $\mathfrak{h}$ is $\overline{\mathfrak{h}} = \{z \in \mathbf{C} \, : \, \operatorname{Im} z \geqslant 0\}$. While $\mathfrak{h} \cup \{0\}$ is neither open nor closed.

Definition 4.3 (Bounded Sets)**.** A set $S \subseteq \mathbf{C}$ is bounded if $S \subseteq D_M(0)$ for some $M > 0$. That is, there exists an $M > 0$ such that $|z| \leqslant M$ for every $z \in S$.

Definition 4.4 (Connected Sets)**.** A set $S \subseteq \mathbf{C}$ is said to be connected if each pair of points z_1 and z_2 in S can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. Otherwise, we say it is disconnected.



Definition 4.5 (Domain)**.** $S \subseteq \mathbf{C}$ is called a domain if it's a non-empty open and connected set.

A region is a domain together with some or all of its boundary points.

Remark 4.6. Domains and regions are sets we will find most suitable for stating elegant results about certain functions in a complex variable.

Example 4.7. $\mathfrak{h}$ is a domain since it's non-empty, open and any two points in $\mathfrak{h}$ can be connected by a straight line. It's an unbounded set. An example of a region is $\mathfrak{h} \cup \{0\}$.

Part II. Holomorphic Functions

Complex Functions

Definition 4.8. A *function* $f : G \to \mathbf{C}$ is a rule that assigns to each $z \in G$ a unique number $f(z) \in \mathbf{C}$. The set G is called the *domain (of definition)*. If $S \subseteq G$, then

$$f(S) := \{f(z) \ : \ z \in S\}$$

is called the *image of S under f*.

The set $f(G)$ is called the *image (or range) of f*. Points in $f(G)$ are called *values of f*.

Given a function f, we define its conjugate $\bar{f}$ by the rule $\bar{f}(z) := \overline{f(z)}$.

Discussion 4.9. If $f : G \to \mathbf{C}$ is a function, then the value $f(x+iy) = u + iv$ depends on a pair $(x, y) \in \mathbf{R}^2$. Collecting all values, we decompose f into its real and imaginary parts

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y); \quad \operatorname{Re} f = u \text{ and } \operatorname{Im} f = v,$$

where $u, v : \mathbf{R}^2 \to \mathbf{R}$ are real-valued functions in two real variables.

In practice, as the examples below tell us, this means replace your $z = x + iy$ and do the required operations to the output $f(x+iy)$. The resulting complex number will be, as a complex number, of the form $u + iv$. The real part is u, which you will obtain in terms of x and y, and the imaginary part is v, which you will also obtain in terms of x and y.

Example 4.10 (Some Complex Functions)**.**

(1) $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$. So,

$$u(x,y) = x^2 - y^2 \quad \text{and} \quad v(x,y) = 2xy.$$

(2) $f(z) = \bar{z} = x - iy$. So,

$$u(x,y) = x \quad \text{and} \quad v(x,y) = -y.$$

(3) (in-class) $f(z) = z\bar{z} = |z|^2 = x^2 + y^2$. So,

$$u(x,y) = x^2 + y^2 \quad \text{and} \quad v(x,y) = 0.$$

Such a function is *real-valued*.

(4) *Polynomials of degree n* are functions of the form

$$p(z) = a_0 + a_1 z + \cdots a_n z^n,$$

where $a_i \in \mathbf{C}$ and $a_n \neq 0$.

A polynomial of degree 0 is simply a non-zero complex number, sometimes also referred to as a *constant polynomial*.

(5) *Rational functions (or polynomials)* are functions of the form

$$\frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials. The domain of definition is wherever $q(z) \neq 0$. For example,

$$f : \mathbf{C}^* \to \mathbf{C},\ z \mapsto \frac{1}{z}$$

(6) If we express z in its polar form, then a function f, when we restrict its domain of definition within $\mathbf{C}^*$, can be written as

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

For example,

$$f : \mathbf{C}^* \to \mathbf{C},\ z = re^{i\theta} \mapsto \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{\cos\theta}{r} - i\,\frac{\sin\theta}{r}.$$

Here $u(r,\theta) = \dfrac{\cos\theta}{r}$ and $v(r,\theta) = -\dfrac{\sin\theta}{r}$.

(7) (in-class) Let's consider the function $f(z) = \bar{z}^2$, in polar form we have

$$\begin{aligned} f(re^{i\theta}) &= (\overline{re^{i\theta}})^2 \\ &= (re^{-i\theta})^2, \quad \text{by Proposition 2.9 (4)} \\ &= r^2e^{-i2\theta}, \quad \text{by Proposition 2.9 (3)} \\ &= r^2(\cos(-2\theta) + i\sin(-2\theta)) \\ &= r^2\cos(2\theta) - i\sin(2\theta)) \end{aligned}$$

Therefore, here $u(r,\theta) = r^2\cos(2\theta)$ and $v(r,\theta) = -r^2\sin(2\theta)$.

(8) Consider $f(z) = z^{1/n}$, where n is a non-zero integer. For no $n \neq 1$ is this a function! We have seen previously that $z^{1/n}$ has n-distinct values. Such a "function" is called multi-valued.

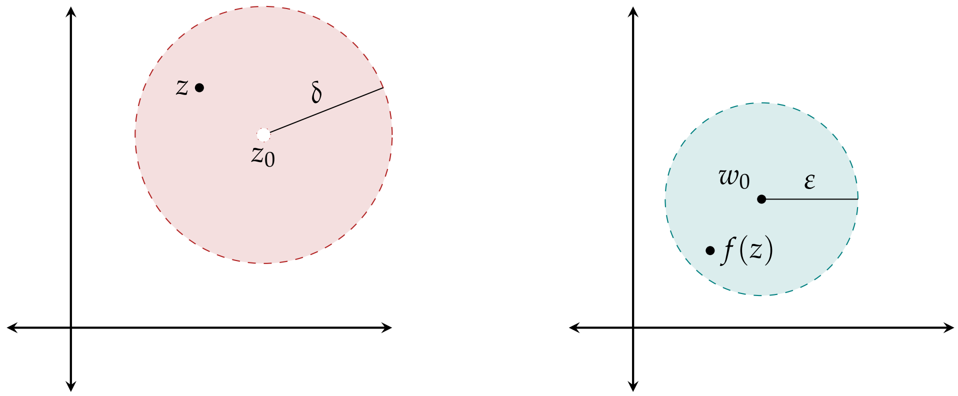
We can make this into a (single-valued) function by assigning a single value of $z^{1/n}$ to each z; taking the *principal n^{th} root of z*, for instance. More on such functions soon.

Limits of Functions

Definition 4.11 (Limit of a Function)**.** Consider a function $f : G \to \mathbf{C}$, and an accumulation point z_0 of G. We say that limit of f, as z approaches z_0, is $w_0 \in \mathbf{C}$ if *for all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\textit{if } 0 < |z - z_0| < \delta, \quad \textit{then } |f(z) - w_0| < \varepsilon$$

Equivalently, if $z \in D_\delta(z_0) \setminus \{z_0\}$, then $f(z) \in D_\varepsilon(w_0)$.



In this case we write $\lim_{z \to z_0} f(z) = w_0$ or $f(z) \to w_0$, as $z \to z_0$.

Intuitively, the limit of f at z_0 is w_0 if

"f is arbitrarily close to w_0 eventually, that is sufficiently, near z_0".

How close? Within an error of ε. How near, eventually? Within a distance of δ.

4.1. Problems

Problem 4.1.

(a) Recall that a set is open if every point of the set is an interior point. Prove that a set $U \subseteq \mathbf{C}$ is open if and only if it does not contain any of its boundary points; that is, $\partial U \cap U = \varnothing$. Then deduce that the complement of a closed set is open.

(b) Prove that an open disk $D_\varepsilon(z_0) = \{z \in \mathbf{C} : |z - z_0| < \varepsilon\}$ is a domain; that is, a non-empty open and connected subset of $\mathbf{C}$.

Problem 4.2. Sketch the sets defined by the following constraints and determine whether they are open, closed, or neither; bounded; connected. What are their boundaries?

(a) $|z + 3| < 2$.

(b) $|\mathrm{Im}(z)| < 1$.

(c) $0 < |z - 1| < 2$.

(d) $|z - 1| + |z + 1| = 2$.

(e) $|z - 1| + |z + 1| < 3$.

(f) $|z| \geqslant \mathrm{Re}(z) + 1$.

Problem 4.3. Let G be the set of points $z \in \mathbf{C}$ satisfying either z is real and $-2 < z < -1$, or $|z| < 1$, or $z = 1$ or $z = 2$.

(a) Sketch the set G, being careful to indicate exactly the points that are in G.

(b) Determine the interior points of G.

(c) Determine the boundary points of G.

(d) Determine the isolated points of G.

(e) G can be written in three diferent ways as the union of two disjoint nonempty disconnected subsets. Describe them.

Problem 4.4. For each of the functions below, describe the domain of definition that is understood.

(a) $f(z) = \dfrac{1}{1+z^2}$

(b) $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$

(c) $f(z) = \dfrac{z}{z+\overline{z}}$

(d) $f(z) = \dfrac{1}{1-|z|^2}$

Problem 4.5.

(a) Write the function $f(z) = z^3 + z + \overline{z} + 1$ in the form

$$f(z) = u(x,y) + i\,v(x,y).$$

(b) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use Proposition 2.4 (6) to write $f(z)$ in terms of z, and simplify the result.

(c) Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form $f(z) = u(r,\theta) + i\,v(r,\theta)$.

Problem 4.6. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Show that

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to z_0} |f(z) - w_0| = 0.$$

Thereby deduce that

$$\lim_{z\to z_0} \bar{f}(z) = \overline{w}_0 \quad \text{if and only if} \quad \lim_{z\to z_0} f(z) = w_0.$$

Problem 4.7. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Show that

$$\text{if } \lim_{z\to z_0} f(z) = w_0, \quad \text{then } \lim_{z\to z_0} |f(z)| = |w_0|\,.$$

Hint. Use the reverse triangle inequality.

Problem 4.8. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Writing $h = z - z_0$, show that

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{h\to 0} f(z+h) = w_0.$$

5. Lecture 5 (4/12)

Example 5.1. Let's show that $\lim_{z \to i} z^2 = -1$ using the definition.

Proof. Let $\varepsilon > 0$ be arbitrary. Note that $\left|z^2 - (-1)\right| = |z - i|\,|z + i|$. We make an initial estimate, suppose $0 < |z - i| < 1$, then

$$\begin{aligned} |z + i| &= |z - i + 2i| \\ &\leqslant |z - i| + |2i| \\ &< 1 + 2 \\ &= 3 \end{aligned}$$

Now, if we choose $\delta = \min\left\{\frac{\varepsilon}{3}, 1\right\}$, then if $0 < |z - i| < \delta$ we get

$$0 < |z - i| < 1 \text{ and } \frac{\varepsilon}{3}$$

So,

$$\begin{aligned} \left|z^2 - (-1)\right| &= |z - i|\,|z + i| \\ &< 3\,|z - i|, \quad \text{since } |z - i| < 1 \\ &< 3 \cdot \frac{\varepsilon}{3}, \quad \text{since } |z - i| < \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Therefore $\lim_{z \to i} z^2 = -1$. □

Theorem 5.2. *If f has a limit at z_0, then it is unique.*

Proof. Assume

$$\lim_{z \to z_0} f(z) = \alpha \quad \text{and} \quad \lim_{z \to z_0} f(z) = \beta$$

Consider an arbitrary $\varepsilon > 0$, then we can find a $\delta_1 > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - \alpha| < \frac{\varepsilon}{2}$$

and $\delta_2 > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - \beta| < \frac{\varepsilon}{2}$$

Define $\delta := \min\{\delta_1, \delta_2\} \leqslant \delta_1,\, \delta_2$, then if $0 < |z - z_0| < \delta$ we have

$$\begin{aligned} |\alpha - \beta| &= |f(z) - f(z) + \alpha - \beta| \\ &\leqslant |\alpha - f(z)| + |f(z) - \beta| \\ &= |f(z) - \alpha| + |f(z) - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We have proven that $|\alpha - \beta| < \varepsilon$ for any $\varepsilon > 0$. Now, suppose $\alpha \neq \beta$, then for $\varepsilon = |\alpha - \beta| > 0$ we get $|\alpha - \beta| < |\alpha - \beta|$, which is preposterous. Hence $\alpha = \beta$, and thus the limit is unique. □

Remark 5.3. The reason we require that z_0 be an accumulation point of the domain of f is just that we need to be sure that there are points z of the domain that are arbitrarily close to z_0. That is, there are indeed points satisfying $0 < |z - z_0| < \delta$.

Our definition (i.e., the part that says $0 < |z - z_0|$) does not require z_0 to be in the domain of f, and if z_0 is in the domain of f, the definition explicitly ignores the value of $f(z_0)$.

Uniqueness of limits can be used to show that a limit does not exist.

Example 5.4. The function $f(z) = \dfrac{\overline{z}}{z}$ has no limit at 0.

Discussion of Example 5.4. Let $z = x + iy$, then

$$f(z) = \frac{x - iy}{x + iy}$$

Along the real axis, $\operatorname{Im} z = 0$, and so $z = x$, giving us $f(z) = \dfrac{x}{x} = 1$.

Along the imaginary axis, $\operatorname{Re} z = 0$, and so $z = y$, giving us $f(z) = \dfrac{-y}{y} = -1$.

Taking the limit along these axes gives us different values of the limit, 1 and -1. Hence, by the uniqueness of limits, the limit doesn't exist. □

Theorems on Limits

Theorem 5.5 (Limit in terms of Real and Imaginary parts of a Function)**.** *Suppose that*

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$$

Then

$$\lim_{x+iy \to x_0 + iy_0} f(x + iy) = u_0 + iv_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Proof. ($\Rightarrow$) Consider an arbitrary $\varepsilon > 0$, then there exists a $\delta > 0$ such that if

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta$$

$$\text{then} \quad |f(x + iy) - (u_0 + iv_0)| = |(u(x, y) + i\,v(x, y)) - (u_0 + iv_0)| < \varepsilon$$

We first note that, by definition

$$\|(x, y) - (x_0, y_0)\| = |(x + iy) - (x_0 + iy_0)|$$

and the end of Discussion 1.10 tells us that

$$|u(x,y) - u_0| \leqslant |(u(x,y) + i\,v(x,y)) - (u_0 + iv_0)| < \varepsilon$$
$$|v(x,y) - v_0| \leqslant |(u(x,y) + i\,v(x,y)) - (u_0 + iv_0)| < \varepsilon$$

That is, we have that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta, \quad \text{then } |u(x,y) - u_0| < \varepsilon \text{ and } |v(x,y) - v_0| < \varepsilon$$

Therefore,

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

($\Leftarrow$) Consider an arbitrary $\varepsilon > 0$, then there exists a $\delta_1 > 0$ such that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta_1, \quad \text{then } |u(x,y) - u_0| < \frac{\varepsilon}{2}$$

and there exists a $\delta_2 > 0$ such that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta_2, \quad \text{then } |v(x,y) - v_0| < \frac{\varepsilon}{2}$$

Define $\delta := \min\{\delta_1, \delta_2\} \leqslant \delta_1,\ \delta_2$. Now, if

$$0 < |(x+iy) - (x_0+iy_0)| = \|(x,y) - (x_0,y_0)\| < \delta$$

then

$$\begin{aligned}
|f(x+iy) - (u_0+iv_0)| &= |(u(x,y) + i\,v(x,y)) - (u_0+iv_0)| \\
&= |(u(x,y) - u_0) + i(v(x,y) - v_0)| \\
&\leqslant |(u(x,y) - u_0)| + |i(v(x,y) - v_0)|\,, \text{ by triangle identity} \\
&= |(u(x,y) - u_0)| + |i|\,|(v(x,y) - v_0)| \\
&= |(u(x,y) - u_0)| + |(v(x,y) - v_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore,

$$\lim_{x+iy\to x_0+iy_0} f(x+iy) = u_0 + iv_0$$

□

Theorem 5.6 (Limit Laws). *Suppose*

$$\lim_{z\to z_0} f(z) = \alpha \quad \textit{and} \quad \lim_{z\to z_0} g(z) = \beta$$

Then

(1) $\lim_{z \to z_0}(f(z) + g(z)) = \alpha + \beta$

(2) $\lim_{z \to z_0}(f(z)\, g(z)) = \alpha\beta$

(3) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\alpha}{\beta}$*, provided* $\beta \neq 0$*.*

Proof. The proof follows from Theorem 5.5 and limit laws from Calculus. □

Example 5.7. Let $p(z)$ be a polynomial, then

$$\lim_{z \to z_0} p(z) = p(z_0)$$

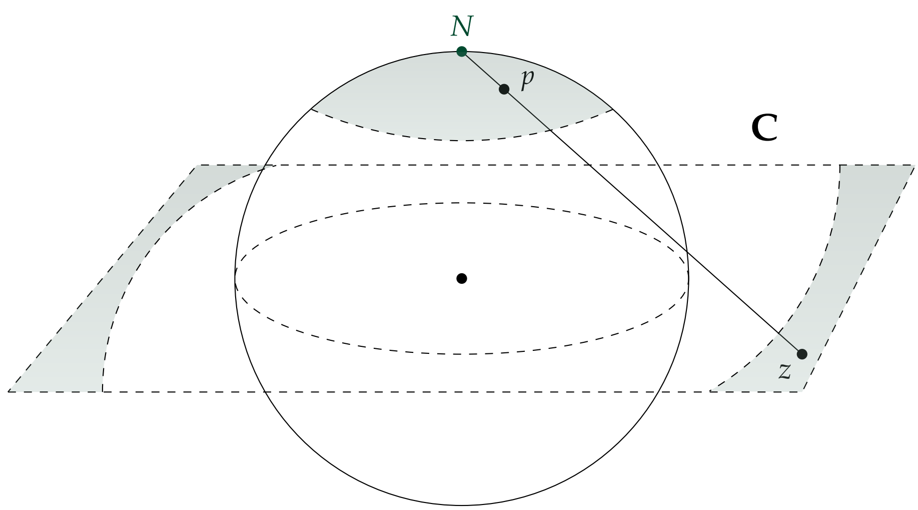
Write $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then by Theorem 5.6 we have

$$\begin{aligned}
\lim_{z \to z_0} p(z) &= \lim_{z \to z_0} (a_0 + a_1 z + \cdots + a_n z^n) \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 z + \cdots + \lim_{z \to z_0} a_n z^n, \text{ by Theorem 5.6 (1)} \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 \cdot \lim_{z \to z_0} z + \cdots + \lim_{z \to z_0} a_n \cdot \lim_{z \to z_0} z^n, \text{ by Theorem 5.6 (2)} \\
&= a_0 + a_1 z_0 + \cdots + a_n z_0^n, \text{ by Theorem 5.6 (2) and } \lim_{z \to z_0} z = z_0 \\
&= p(z_0)
\end{aligned}$$

□

Definition 5.8 (Extended Complex Plane or the Riemann Sphere)**.** The Extended Complex Plane is the set $\mathbf{C}$ together with a symbol ∞ called the *point at infinity*, denoted $\widehat{\mathbf{C}}$ or $\mathbf{C}_\infty$.

There is a bijection between the extended complex plane and the unit sphere given by the *stereographic projection*, and therefore the extended complex plane is also called the Riemann Sphere.



The point N (the north pole) corresponds to ∞, and any point p on the sphere corresponds uniquely to a point $z \in \mathbf{C}$ which is the unique point of intersection of the complex plane with the line passing through N and p.

Definition 5.9 (Neighbourhood of Infinity)**.** Let $\varepsilon > 0$, the set

$$\left\{ z \in \mathbf{C} \;:\; |z| > \frac{1}{\varepsilon} \right\}$$

is called a *neighbourhood of* ∞. Geometrically, a neighbourhood at infinity is the exterior of a circle centered at the origin, which corresponds to a neighbourhood of N on the unit sphere.

Discussion 5.10. We can now easily give meaning to limits

$$\lim_{z \to z_0} f(z) = w_0$$

where z_0 and w_0 are allowed to be ∞. We replace the appropriate neighbourhood in Definition 4.11 with neighbourhoods of ∞.

Theorem 5.11 (Limits involving Infinity)**.**

(1) $\displaystyle\lim_{z \to z_0} \frac{1}{f(z)} = 0$ *if and only if* $\displaystyle\lim_{z \to z_0} f(z) = \infty$.

(2) $\displaystyle\lim_{z \to \infty} f(z) = \lim_{z \to 0} f\left(\frac{1}{z}\right)$*, provided the limit exist.*

Combining (1) and (2), we get

$$\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0 \quad \text{if and only if} \quad \lim_{z \to \infty} f(z) = \infty.$$

Bottom line, we can simplify limits involving ∞ *to limits involving* 0.

Proof. The proofs are based on the simple observation that

$$\frac{1}{a} < b \quad \text{if and only if} \quad \frac{1}{b} < a$$

for non-zero real numbers a and b.

(1) Now $\displaystyle\lim_{z \to z_0} \frac{1}{f(z)} = 0$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \; 0 < |z - z_0| < \delta, \quad \text{then } \; \frac{1}{|f(z)|} = \left| \frac{1}{f(z)} - 0 \right| < \varepsilon$$

if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \; 0 < |z - z_0| < \delta, \quad \text{then } \; |f(z)| > \frac{1}{\varepsilon}$$

if and only if $\displaystyle\lim_{z \to z_0} f(z) = \infty$.

(2) $\lim_{z\to\infty} f(z) = \alpha$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad |z| > \frac{1}{\delta}, \quad \text{then} \quad |f(z) - \alpha| < \varepsilon$$

if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \;\; 0 < \left|\frac{1}{z}\right| < \delta, \quad \text{then} \quad |f(z) - \alpha| < \varepsilon$$

if and only if, by replacing z with $1/z$, $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \alpha$.

$\square$

Example 5.12. We want to show $\lim_{z\to\infty} \frac{2z^4+1}{z^3+1} = \infty$. This is equivalent to showing

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} = 0, \quad \text{for } f(z) = \frac{2z^4+1}{z^3+1}$$

Note that,

$$\begin{aligned}
\lim_{z\to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} &= \lim_{z\to 0} \frac{\frac{1+z^3}{z^3}}{\frac{2+z^4}{z^4}} \\
&= \lim_{z\to 0} z \cdot \frac{1+z^3}{2+z^4} \\
&= 0 \cdot \frac{1}{2} \\
&= 0
\end{aligned}$$

Therefore $\lim_{z\to\infty} \frac{2z^4+1}{z^3+1} = \infty$.

Example 5.13 (in-class)**.** Show $\lim_{z\to\infty} \frac{2+z^5}{z^2+3} = \infty$.

Answer. This is equivalent to showing

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{(1/z)^2+3}{2+(1/z)^5} = 0, \quad \text{for } f(z) = \frac{2+z^5}{z^2+3}$$

Note that,

$$\begin{aligned}
\lim_{z\to 0} \frac{(1/z)^2+3}{2+(1/z)^5} &= \lim_{z\to 0} \frac{\frac{1+3z^2}{z^2}}{\frac{2z^5+1}{z^5}} \\
&= \lim_{z\to 0} z^3 \cdot \frac{1+3z^2}{2z^5+1} \\
&= 0^3 \cdot \frac{1}{1} \\
&= 0
\end{aligned}$$

Therefore $\lim_{z\to\infty} \dfrac{2+z^5}{z^2+3} = \infty$. $\square$

5.1. Problems

Problem 5.1. Compute the following limits and prove your claim by using only the ε-δ definition.

(a) $\lim_{z\to i} \overline{z}$

(b) $\lim_{z\to 1+i} z^2$

(c) $\lim_{z\to 1} z^3$

(d) $\lim_{z\to 1-i} \overline{z}^2 - 1$

(e) $\lim_{z\to 1} z - \overline{z}$

(f) $\lim_{z\to i} \overline{z} + z$

Problem 5.2. Evaluate the following limits or explain why they don't exist.

(a) $\lim_{z\to i} \dfrac{iz^3-1}{z+i}$

(b) $\lim_{z\to 1-i} (x + i(2x+y))$

Problem 5.3. Define

$$f(z) = \frac{x^2y}{x^4+y^2} \quad \text{where } z = x+iy \neq 0.$$

Show that the limits of f at 0 along all straight lines through the origin exist and are equal, but $\lim_{z\to 0} f(z)$ does not exist.

Hint: Consider the limit along the parabola $y = x^2$.

Problem 5.4. Let $M(z) = \dfrac{z-3}{1-2z}$. Prove that

$$\lim_{z\to\infty} M(z) = -\frac{1}{2} \quad \text{and} \quad \lim_{z\to 1/2} M(z) = \infty$$

Problem 5.5. Let

$$M(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

Prove that

(a) $\lim_{z\to\infty} M(z) = \infty$ if $c = 0$.

(b) $\lim_{z\to\infty} M(z) = \dfrac{a}{c}$ and $\lim_{z\to -d/c} M(z) = \infty$, if $c \neq 0$.

6. Lecture 6 (4/14)

Continuous Functions

Definition 6.1 (Continuous Functions)**.** A function $f : G \to \mathbf{C}$ is *continuous at* $z_0 \in G$ if either z_0 is an isolated point or

$$\lim_{z \to z_0} f(z) = f(z_0) = f\left(\lim_{z \to z_0} z\right)$$

That is, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } |f(z) - f(z_0)| < \varepsilon.$$

A function is continuous if it is continuous at every point in its domain.

By the limit laws (Theorem 5.6), sum, product and quotient of continuous functions are continuous (whenever and wherever defined).

Theorem 6.2 (Composition of Continuous Functions)**.** *Suppose we have two functions $f : G_1 \to \mathbf{C}$ and $g : G_2 \to \mathbf{C}$ such that $f(G_1) \subseteq G_2$. If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0. That is,*

$$\lim_{z \to z_0} g(f(z)) = g(f(z_0)) = g\left(\lim_{z \to z_0} f(z)\right) = g\left(f\left(\lim_{z \to z_0} z\right)\right)$$

Therefore, if f and g are continuous, so is $g \circ f$.

Proof. By continuity of g at $f(z_0)$, for an arbitrary $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\text{if } 0 < |w - f(z_0)| < \delta_1, \quad \text{then } |g(w) - g(f(z_0))| < \varepsilon.$$

Now, by continuity of f at z_0, for $\delta_1 > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } |f(z) - f(z_0)| < \delta_1.$$

With these two statements, we have that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } |g(f(z)) - g(f(z_0))| < \varepsilon.$$

Therefore $g \circ f$ is continuous at z_0. □

Theorem 6.3. *Suppose $f : G \to \mathbf{C}$ is continuous at z_0 and $f(z_0) \neq 0$, then there exists a $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(z_0)$. That is, $|f(z)| > 0$ for all $z \in D_\delta(z_0)$.*

Proof. Since f is continuous and non-zero at z_0, for $\varepsilon = \dfrac{|f(z_0)|}{2} > 0$ there exists a $\delta > 0$ such that

$$\text{if } z \in D_\delta(z_0), \quad \text{then } |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

For such a z, the reverse triangle inequality gives us

$$||f(z)| - |f(z_0)|| \leqslant |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}; \quad \text{so,} \quad -\frac{|f(z_0)|}{2} < |f(z)| - |f(z_0)| < \frac{|f(z_0)|}{2}$$

since the former is the absolute value of real numbers. Therefore, adding $|f(z_0)|$ to this inequality gives us

$$|f(z)| > \frac{|f(z_0)|}{2} > 0$$

as needed. □

Theorem 6.4 (Continuity in terms of Real and Imaginary parts of a Function)**.** *Suppose that*

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y).$$

Then f is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

Proof. This is directly follows from Theorem 5.5. □

Definition 6.5 (Compact Sets)**.** A subset of $\mathbf{C}$ is said to be compact if it is closed and bounded.

Definition 6.6 (Bounded Functions)**.** A function $f : G \to \mathbf{C}$ is said to be a bounded function if the image $f(G)$ is bounded. Equivalently, if there exists $M > 0$ such that $|f(z)| \leqslant M$ for every $z \in G$.

Theorem 6.7 (Extreme Value Theorem)**.** *Suppose $K \subseteq \mathbf{C}$ is compact, and $f : K \to \mathbf{C}$ is continuous. Then f is bounded, that is there exists an $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in K$, and there exists a $z_0 \in K$ such that $|f(z_0)| = M$.*

Proof. Since $f = u + iv$ is continuous, so are $u, v : \mathbf{R}^2 \to \mathbf{R}$ by Theorem 6.4. Hence, so is

$$|f(z)| = |f(x + iy)| = \sqrt{u(x, y)^2 + v(x, y)^2}$$

as it's obtained as a sum, product and composition of continuous functions. This result then follows from standard Calculus, since $|f|$ is a real-valued function. □

Complex-Differentiable Functions

Definition 6.8 (Derivative)**.** Consider a function $f : G \to \mathbf{C}$, the derivative *of f at $z_0 \in G$* is the limit

$$\frac{d}{dz}(f(z_0)) = f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the limit exists, we say f is *differentiable at z_0*.

A function is differentiable if it is differentiable at every point in its domain.

Letting $h = \Delta_{z_0} z = z - z_0$, the limit can also be written as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Example 6.9. Consider $f(z) = z^2$, then

$$\begin{aligned} f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} \\ &= \lim_{h \to 0} \frac{2zh + h^2}{h} \\ &= \lim_{h \to 0} 2z + h \\ &= 2z \end{aligned}$$

Example 6.10. Where is $f(z) = |z|^2$ differentiable?

Consider $z \in \mathbf{C}$ and an arbitrary $h \in \mathbf{C}$, then we compute

$$\begin{aligned} f(z+h) - f(z) &= |z+h|^2 - |z|^2 \\ &= (z+h)\overline{(z+h)} - z\overline{z} \\ &= z\overline{z} + z\overline{h} + \overline{z}h + h\overline{h} - z\overline{z} \\ &= z\overline{h} + \overline{z}h + h\overline{h} \end{aligned}$$

Then

$$\frac{f(z+h) - f(z)}{h} = \frac{z\overline{h} + \overline{z}h + h\overline{h}}{h} = z\,\frac{\overline{h}}{h} + \overline{z} + \overline{h}$$

Along the real axis, $h = \overline{h}$, we have

$$\frac{f(z+h) - f(z)}{h} = z + \overline{z} + h;$$

therefore, as $h \to 0$, the limit is $z + \overline{z}$. Along the imaginary axis, $h = -\overline{h}$, we have

$$\frac{f(z+h) - f(z)}{h} = -z + \overline{z} - h;$$

therefore, as $h \to 0$, the limit is $-z + \overline{z}$.

Since limits are unique, if $f'(z)$ exists, then $z + \overline{z} = -z + \overline{z}$, which gives us $z = 0$. That is, if $f'(z)$ exists, it only exists for $z = 0$. So, does $f'(0)$ exist?

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h\overline{h}}{h} = \lim_{h \to 0} \overline{h} = 0$$

Proposition 6.11 (Differentiable Functions are Continuous)**.** *If f is differentiable at z_0, then f is continuous at z_0.*

Proof. Suppose f is differentiable at z_0, then

$$\lim_{z\to z_0} f(z) - f(z_0) = \left(\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}\right)\left(\lim_{z\to z_0} z - z_0\right) = f'(z_0)\cdot 0 = 0$$

Therefore $\lim_{z\to z_0} f(z) = f(z_0)$, and hence f is continuous at z_0. □

Theorem 6.12 (Differentiation Laws)**.** *Suppose f and g are differentiable at z. Then,*

(1) $(c)' = 0$, *for every* $c \in \mathbf{C}$.

(2) $(c\cdot f)'(z) = c\cdot f'(z)$, *for every* $c \in \mathbf{C}$. (Constant Rule)

(3) $(z^n)' = nz^{n-1}$, *for every* $n \in \mathbf{Z}$ *(assume* $z \neq 0$ *for* $n < 0$*).* (Power Rule)

(4) $(f+g)'(z) = f'(z) + g'(z)$. (Sum Rule)

(5) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$. (Product Rule)

(6) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$, *provided* $g(z) \neq 0$ (Quotient Rule)

Proof. (1) and (4) are proved directly using the limit definition, (2) can be proved directly or using (1) and (5), while (3) can be proven inductively using (5) for positive n and (6) for negative n.

(5) We first compute

$$\begin{aligned} f(z+h)g(z+h) - f(z)g(z) &= f(z+h)g(z+h) - f(z)g(z) + f(z+h)g(z) - f(z+h)g(z) \\ &= f(z+h)(g(z+h) - g(z)) + g(z)(f(z+h) - f(z)) \end{aligned}$$

So,

$$\begin{aligned} (fg)'(z) &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h\to 0} \frac{f(z+h)(g(z+h)-g(z))}{h} + \lim_{h\to 0} \frac{g(z)(f(z+h)-f(z))}{h} \\ &= \lim_{h\to 0} f(z+h)\cdot \lim_{h\to 0} \frac{g(z+h)-g(z)}{h} + g(z)\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \\ &= f(z)g'(z) + g(z)f'(z) \end{aligned}$$

(6) We first compute

$$\begin{aligned} \frac{1}{g(z+h)} - \frac{1}{g(z)} &= \frac{g(z)-g(z+h)}{g(z)g(z+h)} \\ &= -\frac{g(z+h)-g(z)}{g(z)g(z+h)} \end{aligned}$$

So,

$$\begin{aligned}
\left(\frac{1}{g}\right)'(z) &= \lim_{h\to 0}\frac{\dfrac{1}{g(z+h)}-\dfrac{1}{g(z)}}{h} \\
&= \lim_{h\to 0} -\frac{g(z+h)-g(z)}{g(z)g(z+h)}\cdot\frac{1}{h} \\
&= -\lim_{h\to 0}\frac{g(z+h)-g(z)}{h}\cdot\lim_{h\to 0}\frac{1}{g(z)g(z+h)} \\
&= -\frac{g'(z)}{g(z)^2}
\end{aligned}$$

(6) then follows from the computation above and using (5) on $\dfrac{f(z)}{g(z)} = f(z)\cdot\dfrac{1}{g(z)}$. □

Proposition 6.13 (Chain Rule)**.** *Suppose we have two functions $f: G_1 \to \mathbf{C}$ and $g: G_2 \to \mathbf{C}$ such that $f(G_1) \subseteq G_2$. If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then $g\circ f$ is differentiable at z_0 and*

$$(g\circ f)'(z_0) = g'(f(z_0))\cdot f'(z_0)$$

Proof. Let's start by defining an auxiliary function on G_2

$$\phi(w) = \begin{cases} \dfrac{g(w)-g(f(z_0))}{w-f(z_0)} - g'(f(z_0)) & w \neq f(z_0) \\ 0 & w = f(z_0) \end{cases}$$

Since g is differentiable at $f(z_0)$, then $\lim_{w\to f(z_0)}\phi(w) = 0 = \phi(f(z_0))$ and therefore ϕ is continuous at $f(z_0)$. Furthermore, since f is differentiable at z_0, it is continuous at z_0. So $\lim_{z\to z_0}\phi(f(z)) = \phi(f(z_0)) = 0$ by Theorem 6.2.

Rewriting the above expression, we get the following expression which is valid on all of G_2.

$$g(w) - g(f(z_0)) = (w - f(z_0))(\phi(w) + g'(f(z_0)))$$

Now, for $w = f(z) \in f(G_1)$, we have

$$\begin{aligned}
\frac{g(f(z))-g(f(z_0))}{z-z_0} &= \frac{(f(z)-f(z_0))(\phi(f(z))+g'(f(z_0)))}{z-z_0} \\
&= (\phi(f(z))+g'(f(z_0)))\cdot\frac{f(z)-f(z_0)}{z-z_0}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(g\circ f)'(z_0) &= \lim_{z\to z_0}\frac{g(f(z))-g(f(z_0))}{z-z_0} \\
&= \lim_{z\to z_0}(\phi(f(z))+g'(f(z_0)))\cdot\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \\
&= g'(f(z_0))\cdot f'(z_0), \text{ since } \lim_{z\to z_0}\phi(f(z)) = 0
\end{aligned}$$

□

6.1. Problems

Problem 6.1. Example 5.7 tells us that polynomials are continuous.

(a) Prove that the complex conjugation function $\sigma(z) := \overline{z}$ is continuous.

(b) Prove that a polynomial in $\overline{z}$ is continuous. That is, prove that a polynomial given as

$$p(\overline{z}) = a_n\overline{z}^n + \cdots + a_1\overline{z} + a_0, \quad a_i \in \mathbf{C},\ a_n \neq 0$$

is continuous.

(c) Prove that the following functions are continuous by writing them as a sum or product of polynomials $p(z)$ and $q(\overline{z})$

(i) $R(z) := \operatorname{Re} z$

(ii) $I(z) := \operatorname{Im} z$

(iii) $N(z) := |z|^2$

Problem 6.2. Show that the function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = \begin{cases} \dfrac{\overline{z}}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is continuous on $\mathbf{C}^*$.

Problem 6.3. Consider the function

$$f : \mathbf{C}^* \to \mathbf{C},\ z \mapsto \frac{1}{z}.$$

Apply the definition of the derivative to give a direct proof that $f'(z) = -\dfrac{1}{z^2}$.

Problem 6.4. Find the derivative of the function

$$M(z) := \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

When is $M'(z) = 0$?

Problem 6.5. Using Example 5.4 as an inspiration, show that $f'(z)$ does not exist for any z for the functions

(a) $f(z) = \operatorname{Re} z$

(b) $f(z) = \operatorname{Im} z$

Problem 6.6. Show that the function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

is not differentiable at 0.

Problem 6.7.

(a) Show that a polynomial of degree n, $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, is differentiable everywhere, with

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

(b) Furthermore, show that for $p(z)$, as given in (a), we have

$$a_i = \frac{p^{(i)}(0)}{i!}$$

for $i = 0, \ldots, n$. Where $p^{(0)}(z) = p(z)$ and $p^{(i)}(z)$, for $i > 0$, is the i^{th} derivative of $p(z)$.

Problem 6.8. Let G be a domain and $f : G \to \mathbf{C}$ a function that is differentiable at every point in G. Consider the domain

$$G^* = \{z \in \mathbf{C} \,:\, \overline{z} \in G\}$$

and the function

$$f^* : G^* \to \mathbf{C},\ z \mapsto \overline{f(\overline{z})}$$

Show that f^* is differentiable at every point in G^*.

Problem 6.9. For each function, determine all points at which the derivative exists. When the derivative exists, find its value. Use Example 6.10 as an inspiration.

(a) $f(z) = z + i\overline{z}$

(b) $g(z) = (z + i\overline{z})^2$

(b) $h(z) = z \operatorname{Im} z$

Problem 6.10. By definition, a function $f : G \to \mathbf{C}$ is differentiable at $z_0 \in G$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Unpacking the limit definition, we see that f is differentiable at z_0 if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

By appealing only to the definition, we show that $\sigma : \mathbf{C} \to \mathbf{C}$ defined by $\sigma(z) = \bar{z}$ is not differentiable anywhere by completing the following steps.

(i) Let $z_0 \in \mathbf{C}$ and assume that $f'(z_0)$ exists. Choose $\delta > 0$ according to the definition using $\varepsilon = 1/2$ and write down the resulting statement.

(ii) Consider $z = z_0 + \delta/2$ and conclude from (a) that $|1 - f'(z_0)| < \varepsilon$.

(iii) Consider $z = z_0 + i\delta/2$ and conclude from (a) that $|1 + f'(z_0)| < \varepsilon$.

(iv) Using the triangle inequality together with (ii) and (iii), obtain a contradiction.

7. Lecture 7 (4/19)

Cauchy-Riemann Equations

Theorem 7.1 (Cauchy-Riemann Equations)**.** *Suppose that*

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

is differentiable at $z_0 = x_0 + iy_0$. Then

(a) *the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy the* Cauchy-Riemann Equations

$$\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned} \tag{CR}$$

(b) $f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$.

Proof. Since f is differentiable at z_0, we have, where we let $h = s + it$

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \frac{f((x_0+s) + i(y_0+t)) - f(x_0+iy_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0+s, y_0+t) - u(x_0,y_0)}{h} + i \cdot \lim_{h\to 0} \frac{v(x_0+s,y_0+t) - v(x_0,y_0)}{h} \end{aligned}$$

As we know by now, we must get the same result if we restrict h to be on the real axis and if we restrict it to be on the imaginary axis. In the former case, $t = 0$, giving us

$$\begin{aligned} f'(z_0) &= \lim_{s\to 0} \frac{u(x_0+s,y_0) - u(x_0,y_0)}{s} + i \cdot \lim_{s\to 0} \frac{v(x_0+s,y_0) - v(x_0,y_0)}{s} \\ &= u_x(x_0,y_0) + i\,v_x(x_0,y_0) \end{aligned}$$

In the latter case, $s = 0$, giving us

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{u(x_0,y_0+t) - u(x_0,y_0)}{it} + i \cdot \lim_{t\to 0} \frac{v(x_0,y_0+t) - v(x_0,y_0)}{it} \\ &= \frac{1}{i} \cdot \lim_{t\to 0} \frac{u(x_0,y_0+t) - u(x_0,y_0)}{t} + \lim_{t\to 0} \frac{v(x_0,y_0+t) - v(x_0,y_0)}{t} \\ &= -i\,u_y(x_0,y_0) + v_y(x_0,y_0) \end{aligned}$$

Therefore

$$u_x(x_0,y_0) + i\,v_x(x_0,y_0) = f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0),$$

and hence $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$. □

The Cauchy-Riemann equations (CR) are a *necessary* condition for f' to exist. We can use them to locate possible points where the derivative does not exist but not necessarily conclude where and if the derivative exists.

Example 7.2.

(1) Consider $f(z) = |z|^2 = x^2 + y^2$, so $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. The partial derivatives at (x,y) are

$$u_x = 2x \qquad\qquad v_x = 0$$

$$u_y = 2y \qquad\qquad v_y = 0$$

Therefore, the Cauchy-Riemann equations (CR) are only satisfied at $(x,y) = (0,0)$. Hence f is not differentiable at any $z \neq 0$. Again, note that this does not say anything about the existence of $f'(0)$.

(2) Consider $f(z) = \overline{z} = x - iy$, so $u(x,y) = x$ and $v(x,y) = -y$. The partial derivatives at (x,y) are

$$u_x = 1 \qquad\qquad v_x = 0$$

$$u_y = 0 \qquad\qquad v_y = -1$$

Note that $u_x \neq v_y$ for all (x,y) and therefore the Cauchy-Riemann equations (CR) are satisfied for no (x,y). Hence f is nowhere complex-differentiable.

(3) (in-class) Consider $f(z) = (z + i\overline{z})^2$, let's simplify f to identify its real and imaginary parts $u(x,y)$ and $v(x,y)$.

$$\begin{aligned} f(z) = f(x+iy) &= ((x+iy) + i(x-iy))^2 \\ &= ((x+iy) + (y+ix))^2 \\ &= ((x+y) + i(x+y))^2 \\ &= (x+y)^2(1+i)^2 \\ &= (x+y)^2(1^2 + i^2 + 2i) \\ &= 2i(x+y)^2 \end{aligned}$$

Therefore $u(x,y) = 0$ and $v(x,y) = 2(x+y)^2$. The partial derivatives at (x,y) are

$$u_x = 0 \qquad\qquad v_x = 4(x+y)$$

$$u_y = 0 \qquad\qquad v_y = 4(x+y)$$

Therefore, the Cauchy-Riemann equations (CR) are satisfied if and only if $4(x+y) = 0$, if and only if $y = -x$. Hence f is not differentiable any $z \in \mathbf{C}$ such that $\operatorname{Im} z \neq -\operatorname{Re} z$.

As commented, the Cauchy-Riemann equations (CR) are not a *sufficient* condition for the existence of the derivative as the example below shows. Problem 7.1 gives another example.

Example 7.3. Consider

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} = \dfrac{\overline{z}^3}{|z|^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Then,

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We show that u and v satisfy the Cauchy-Riemann equations (CR) at $(0,0)$.

$$u_x(0,0) = \lim_{s\to 0} \frac{u(s,0)-u(0,0)}{s} = \lim_{s\to 0} \frac{\frac{s^3}{s^2} - 0}{s} = 1$$

$$u_y(0,0) = \lim_{t\to 0} \frac{u(0,t)-u(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$v_x(0,0) = \lim_{s\to 0} \frac{v(s,0)-v(0,0)}{s} = \lim_{s\to 0} \frac{0-0}{s} = 0$$

$$v_y(0,0) = \lim_{t\to 0} \frac{v(0,t)-v(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{t^2} - 0}{t} = 1$$

Therefore $u_x(0,0) = 1 = v_y(0,0)$ and $u_y(0,0) = 0 = -v_x(0,0)$, and hence the Cauchy-Riemann equations (CR) are satisfied. But $f'(0)$ does not exist, as seen in Problem 6.6.

Imposing certain existence and continuity conditions on the first order partial derivatives of u and v, the Cauchy-Riemann equations (CR) can be upgraded to a sufficient condition for differentiability.

Theorem 7.4 (Sufficient Conditions for Differentiability)**.** *Consider a function*

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

and a z_0 in the domain of f, such that

(a) the first order partial derivatives of u and v exist and are continuous in an open disk centered at z_0; and

(a) the Cauchy-Riemann equations (CR) are satisfied at (x_0, y_0).

Then $f'(z_0)$ exists and is given by $u_x(x_0,y_0) + i\,v_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$.

Proof. We skip the proof. You can find a proof in [1, Section 22, Page 66]. □

Example 7.5. Let's revisit examples from Example 7.2 and 7.3.

(1) Consider $f(z) = |z|^2 = x^2+y^2$, we noted that $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. We have seen that the only point where $f(z)$ can be differentiable is $z = 0$. The partial derivatives in a neighbourhood of $(0,0)$ are

$$u_x = 2x \qquad\qquad v_x = 0$$

$$u_y = 2y \qquad\qquad v_y = 0$$

which clearly exist and are continuous. We have also seen that the Cauchy-Riemann equations (CR) are satisfied at $(0,0)$, trivially. Therefore $f'(0)$ exists and

$$f'(0) = u_x(0,0) + i\,v_x(0,0) = 0.$$

(2) Consider $f(z) = (z + i\bar{z})^2$, we noted that $u(x,y) = 0$ and $v(x,y) = 2(x+y)^2$. We have seen that the only point where $f(z)$ can be differentiable are $z = x + iy \in \mathbf{C}$ such that $y = \operatorname{Im} z = -\operatorname{Re} z = -x$. That is, at points of the form $(x, -x)$. The partial derivatives in a neighbourhood of $(x, -x)$ are

$$u_x = 0 \qquad\qquad v_x = 4(x+y)$$
$$u_y = 0 \qquad\qquad v_y = 4(x+y)$$

which clearly exist and are continuous. Note the Cauchy-Riemann equations (CR) are satisfied at $(x, -x)$ trivially, since

$$u_x(x,-x) = u_y(x,-x) = v_x(x,-x) = v_y(x,-x) = 0.$$

Therefore $f'(z)$ exists, for $z = x - ix$, and

$$f'(z) = u_x(x,-x) + i\,v_x(x,-x) = 0.$$

(3) The reason Example 7.3 doesn't contradict Theorem 7.4 is because, u_x, in particular, is not continuous at $(0,0)$. Note that we have

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$, we compute $u_x(x,y)$ using the quotient rule, while we have already computed $u_x(0,0) = 1$ in Example 7.3, giving us

$$u_x(x,y) = \begin{cases} \dfrac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Suppose $u_x(x,y)$ is continuous at $(0,0)$, then we have

$$\lim_{(x,y)\to(0,0)} u_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2} = u_x(0,0) = 1$$

Restricting the limit along the y-axis, where $x = 0$, we get

$$1 = \lim_{(0,y)\to(0,0)} \frac{-3y^4}{(y^2)^2} = \lim_{y\to 0} \frac{-3y^4}{y^4} = -3,$$

a contradiction. Hence, $u_x(x,y)$ is not continuous at $(0,0)$.

Example 7.6 (Complex Exponential)**.** Define, for any $z = x + iy \in \mathbf{C}$

$$\exp(z) = e^z := e^x e^{iy} = e^x(\cos y + i\,\sin y)$$

the *complex exponential function*. Note that e^x is the usual real exponential and e^{iy} is given by Euler's formula (Definition 2.7). Here,

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y$$

We then see that

$$u_x = e^x \cos y = v_y,$$
$$v_x = -e^x \sin y = -u_y;$$

so exp satisfies the Cauchy-Riemann equations (CR) everywhere. Furthermore, u_x, u_y, v_x and v_y are everywhere defined and continuous. Hence exp is everywhere complex-differentiable, an *entire* function. Furthermore $\exp(z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = \exp(z)$.

Discussion 7.7 (Polar Cauchy-Riemann Equations)**.** Recall that if the domain of a function f is contained in $\mathbf{C}^*$ or restricted to within $\mathbf{C}^*$, one can express in polar coordinates at $z = re^{i\theta}$ as

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

Then, the Cauchy-Riemann equations (CR) at a point (r_0, θ_0) can be expressed in polar coordinates, Polar Cauchy-Riemann Equations (see Problem 7.3)

$$\begin{aligned} ru_r &= v_\theta \\ u_\theta &= -rv_r \end{aligned} \tag{Polar CR}$$

and a differentiable function at $z_0 = r_0 e^{i\theta_0}$ is then expressed as

$$f'(z_0) = f'(r_0 e^{i\theta_0}) = e^{-i\theta_0}(u_r(r_0,\theta_0) + i\,v_r(r_0,\theta_0)).$$

Example 7.8. Consider the function

$$f(z) = f(re^{i\theta}) = \sqrt{r}\,e^{i\frac{\theta}{2}},$$

where $r > 0$ and $-\pi < \theta < \pi$. This is the function that outputs the principal square root of z. We compute $f'(z)$ at $z = re^{i\theta}$ using the polar form of Theorem 7.4. We first note that

$$f(z) = \underbrace{\sqrt{r}\cos\left(\frac{\theta}{2}\right)}_{u(r,\theta)} + i\underbrace{\sqrt{r}\sin\left(\frac{\theta}{2}\right)}_{v(r,\theta)}$$

Now, we compute

$$ru_r = r\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) = v_\theta$$
$$u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) = -r\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right) = -rv_r$$

Clearly the first order partial derivatives exist everywhere and the Polar Cauchy-Riemann equations (Polar CR) are also satisfied everywhere. Hence $f'(z)$ exists and

$$\begin{aligned}
f'(z) &= e^{-i\theta}(u_r(r,\theta) + i\, v_r(r,\theta)) \\
&= e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\,\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right) \\
&= \frac{e^{-i\theta}}{2\sqrt{r}}\left(\cos\left(\frac{\theta}{2}\right) + i\,\sin\left(\frac{\theta}{2}\right)\right) \\
&= \frac{1}{2\sqrt{r}} \cdot e^{-i\theta} \cdot e^{i\frac{\theta}{2}} \\
&= \frac{1}{2\sqrt{r}e^{i\frac{\theta}{2}}} \\
&= \frac{1}{2f(z)}
\end{aligned}$$

7.1. Problems

Problem 7.1. Define

$$f(z) = \begin{cases} 0 & \text{if } \operatorname{Re}(z) \cdot \operatorname{Im}(z) = 0, \\ 1 & \text{if } \operatorname{Re}(z) \cdot \operatorname{Im}(z) \neq 0 \end{cases}.$$

Show that f satisfies the Cauchy–Riemann equation at $z = 0$, yet f is not diferentiable at $z = 0$.

Problem 7.2. Show that when $f(z) = x^3 + i(1-y)^3$, it makes sense to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

Problem 7.3. Show that $f'(z)$ does not exist at any point if

(a) $f(z) = z - \overline{z}$

(b) $f(z) = 2x + ixy^2$

Problem 7.4. Show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$

(b) $f(z) = e^{-x}e^{-iy}$

Problem 7.5. Let $f : G \to \mathbf{C}$ be a function, such that $G \subseteq \mathbf{C}^*$, then we can write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y) \quad \text{or} \quad f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

Using the fact that $x = r\cos\theta$ and $y = r\sin\theta$ and the chain rule from calculus, write u_r and u_θ in terms of u_x and u_y. Assuming f is differentiable, rewrite the CR-equations and $f'(z)$ in terms of u_r and u_θ.

Problem 7.6. Prove that the function

$$f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$$

is differentiable when $r > 0$ and $0 < \theta < 2\pi$, and find $f'(z)$ in terms of $f(z)$.

8. Lecture 8 (4/21)

Holomorphic Functions

Definition 8.1 (Holomorphic Functions)**.** A function f is *holomorphic on an open set* U if $f'(z)$ exists for every $z \in U$.

We say f is *holomorphic at a point* z_0 if it holomorphic on some open disk $D_\varepsilon(z_0)$ for an $\varepsilon > 0$. We say f is holomorphic if it is holomorphic at every point in its domain.

A function that is holomorphic on all of $\mathbf{C}$ is said to be entire.

Example 8.2.

(1) $f(z) = \dfrac{1}{z}$ is holomorphic on any open set not containing 0, in particular on $\mathbf{C}^*$.

(2) $f(z) = |z|^2$ is nowhere holomorphic since we have already seen that f is only complex-differentiable at $z = 0$ and at no other point.

(3) Polynomials are entire.

(4) $f(z) = \bar{z}$ is nowhere holomorphic, since it's nowhere differentiable.

Discussion 8.3. Let G be a domain (open and connected subset of $\mathbf{C}$). We know several necessary and sufficient conditions for $f = u + iv$ to be holomorphic on G.

(Necessary)
(1) f is continuous on G.
(2) Cauchy-Riemann equations (CR) are satisfied on G.

(Sufficient)
(1) First order partial derivatives of u and v exist and continuous on G, and the Cauchy-Riemann equations (CR) are satisfied on G.
(2) Differentiation Laws. If f and g are holomorphic on G, then so are $f + g$, fg and f/g (if $g \neq 0$ on G).
(3) Composition of holomorphic functions is holomorphic.

Theorem 8.4 (Sufficient Condition for Constantness)**.** *Suppose G is a domain and $f'(z) = 0$ for all $z \in G$. Then $f(z)$ is constant on G.*

Proof. Write $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$, so we have

$$0 = f'(z) = u_x + iv_x = v_y - i\,u_y$$

Therefore $u_x = u_y = 0$ and $v_x = v_y = 0$. We consider points $p, q \in G$ such that there's a line segment L in G connecting them. Let $\vec{w} = (a, b)$ be a unit vector parallel to L, then the directional derivative of u along L is

$$(\operatorname{grad} u) \cdot \vec{w} = au_x + bu_y = 0.$$

So, u is constant along L. Since G is a domain, any two points can be connected by a polygon line. Applying the above argument along constituent line segments, we see that u has the same value along the endpoints of any polygon line. This shows that u is constant on G, say $u(x,y) = c$. A similar argument works for v, giving us $v(x,y) = d$. Hence

$$f(z) = c + id,$$

that is, f is constant. □

Theorem 8.4 has many interesting consequences.

Proposition 8.5. *Suppose f and $\bar{f}$ are holomorphic on a domain G. Then f is constant on G.*

Proof. We write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

$$\bar{f}(z) = \overline{f(x+iy)} = u(x,y) - i\,v(x,y)$$

Since f and $\bar{f}$ are holomorphic, they satisfy the Cauchy-Riemann equations (CR)

$$\text{for } f\text{:} \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\text{for } \bar{f}\text{:} \quad \begin{cases} u_x = (-v)_y = -v_y \\ u_y = -(-v)_x = v_x \end{cases}$$

This gives us $v_y = -v_y$ and $v_x = -v_x$, and therefore $u_x = v_x = 0$. Hence $f'(z) = u_x + i\,v_x = 0$, giving us that f is constant by Theorem 8.4. □

Corollary 8.6. *Suppose f is holomorphic on a domain G and always real-valued. Then f is constant on G.*

Proof. Since f is always real-valued, we have $f = \bar{f}$. Therefore $\bar{f}$ is holomorphic on G as well, and hence f is constant by Proposition 8.5. □

Corollary 8.7. *Suppose f is holomorphic on a domain G and $|f|$ is constant on it. Then f is also constant on G.*

Proof. By assumption $|f(z)| = c$, for all $z \in G$, for some $c \in \mathbf{C}$. This gives us

$$f(z)\overline{f(z)} = |f(z)|^2 = c^2 \tag{$*$}$$

Suppose $c = 0$, then $|f(z)| = 0$ and therefore $f(z) = 0$. Suppose $c \neq 0$, then necessarily $f(z) \neq 0$ for every $z \in G$ by $(*)$. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)},$$

and thus $\bar{f}$ is holomorphic. Therefore both f and $\bar{f}$ are holomorphic and hence f is constant by Proposition 8.5. □

Example 8.8. We apply Corollary 8.7 to $f(z) = \dfrac{\overline{z}}{z}$ to conclude that it's not holomorphic.

We first note that, for any $z \in \mathbf{C}$,

$$|f(z)| = \left|\frac{\overline{z}}{z}\right| = \frac{|\overline{z}|}{|z|} = 1;$$

that is, $|f|$ is constant. Suppose f was holomorphic on $\mathbf{C}$ (this argument can be specialised to any domain G), then f would be a holomorphic function such that $|f|$ is constant. Therefore, by Corollary 8.7, f is constant on $\mathbf{C}$. That's a contradiction, since f is non-constant, as $f(1) = 1$ and $f(i) = -1$.

Example 8.9 (in-class)**.** Is the function $f(z) = \operatorname{Re} z$ holomorphic?

Answer. Note that $f(z) = \operatorname{Re} z$ is a real-valued function, for any $z \in \mathbf{C}$. Suppose f was holomorphic on $\mathbf{C}$ (this argument can be specialised to any domain G), then f would be a holomorphic function such that f is always real-valued. Therefore, by Corollary 8.6, f is constant on $\mathbf{C}$. That's a contradiction, since f is non-constant, as $f(1) = 1$ and $f(i) = 0$. $\square$

We now discuss a large class of holomorphic functions, which are complex versions of functions you may have seen in your Calculus classes

The Exponential Function

Definition 8.10 (The Exponential Function)**.** The (complex) exponential function e^z (or $\exp(z)$) is defined on all of $\mathbf{C}$ as follows

$$e^z := e^{\operatorname{Re} z} e^{i \operatorname{Im} z} = e^{\operatorname{Re} z}(\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z)).$$

That is, writing $z = x + iy$, we have

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Since $x \in \mathbf{R}$, e^x is the usual real exponential function, while e^{iy} is given by Euler's formula.

Furthermore, the definitions give us $\overline{e^z} = e^{\overline{z}}$.

Note that when $z = x \in \mathbf{R}$, we have $e^z = e^x$, since then $\operatorname{Im} z = 0$.

Proposition 8.11 (Properties of the Exponential)**.** *Consider $z, w \in \mathbf{C}$.*

(1) $|e^z| = e^{\operatorname{Re} z}$ *and* $\arg e^z = \{\operatorname{Im} z + 2k\pi \ : \ k \in \mathbf{Z}\}$.

(2) $e^{z+w} = e^z e^w$.

(3) $e^{z-w} = \dfrac{e^z}{e^w}$.

(4) e^z *is entire, and* $(e^z)' = e^z$.

(5) *e^z is periodic: $e^{z+2k\pi i} = e^z$ for all $k \in \mathbf{Z}$.*

Proof.

(1) Write $z = x + iy$, then $|e^z| = |e^x| \, |\cos x + i \sin x| = |e^x|$. Which tells us

$$\arg e^z = \{y + 2k\pi \ : \ k \in \mathbf{Z}\} .$$

(2) Write $z = x + iy$ and $w = u + iv$, then

$$\begin{aligned} e^{z+w} &= e^{(x+u)+i(y+v)} \\ &= e^{x+u} e^{i(y+v)} \\ &= e^x e^u e^{iy} e^{iv} \\ &= e^x e^{iy} e^u e^{iv} \\ &= e^z e^w \end{aligned}$$

(3) From (2) we get $e^{z-w} e^w = e^z$.

(4) This was seen in Example 7.6.

(5) From (2) we have $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$.

□

8.1. Problems

Problem 8.1. Let $f = u + iv$ be a complex-valued function defined on an open set $G \subseteq \mathbf{C}$. Suppose that the first-order partial derivatives of $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ exist and are continuous on G.

(a) Recall that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}$$

Treat $f = f(x, y)$ as a function in two real-variables, and *formally* apply the chain rule in Calculus to obtain the expressions

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

(b) Define $\dfrac{\partial f}{\partial x} := \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$, and similarly for $\dfrac{\partial f}{\partial y}$.

Prove that f is holomorphic on G if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$.

(c) (i) If f is holomorphic on G, prove that $f'(z) = \dfrac{\partial f}{\partial z}$.

(ii) The *Jacobian* of $(x, y) \mapsto (u(x, y), v(x, y))$ is the determinant of the matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

If f is holomorphic on G, prove that the Jacobian equals $|f'(z)|^2 \geqslant 0$.

Problem 8.2. Suppose f is entire and can be written as

$$f(z) = u(x) + i\,v(y),$$

that is, the real part of f depends only on $x = \mathrm{Re}(z)$ and the imaginary part of f depends only on $y = \mathrm{Im}(z)$.

Prove that $f(z) = az + b$ for some $a \in \mathbf{R}$ and $b \in \mathbf{C}$.

Problem 8.3. Suppose f is entire, with real and imaginary parts u and v satisfying

$$u(x, y)\,v(x, y) = 3$$

for all $z = x + iy$. Show that f is constant.

Problem 8.4. Prove that, if $G \subseteq \mathbf{C}$ is a domain and $f : G \to \mathbf{C}$ is a complex-valued function with $f''(z)$ defined and equal to 0 for all $z \in G$, then $f(z) = az + b$ for some $a, b \in \mathbf{C}$.

Problem 8.5. Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$

(b) $\exp\left(\dfrac{2+\pi}{4}\right) = \sqrt{\dfrac{e}{2}}(1+i)$

(c) $\exp(z + \pi i) = -\exp z$.

Problem 8.6. Prove that

(a) $f(z) = \exp \overline{z}$ is nowhere holomorphic.

(b) $f(z) = \exp z^2$ is entire. What is its derivative?

Problem 8.7. Show that

(a) $\left|\exp(2z + i) + exp(iz^2)\right| \leqslant e^{2x} + e^{-2xy}$.

(b) $\left|\exp(z^2)\right| \leqslant \exp(|z|^2)$.

(c) $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

Problem 8.8. Find all values of z such that

(a) $\exp z = -2$

(b) $\exp z = 1 + i\sqrt{3}$

(c) $\exp(2z - 1) = 1$.

Problem 8.9. Find all solutions to the equation $e^{2z} - 2ie^{z} = 1$.

Problem 8.10. Let $G \subseteq \mathbf{C}^*$ be an open set and let f be a function that is continuous on G with the property

$$e^{f(z)} = z, \quad z \in G.$$

Show that f is holomorphic on G.

Remark 8.12. This shows that a *continuously* defined logarithm on an open set is immediately holomorphic.

9. Lecture 9 (4/26)

The Logarithmic Function

Discussion 9.1. The complex logarithmic function arises, just the like the usual real logarithmic function, from trying to solve the following equation for w

$$e^w = z \quad (z \neq 0)$$

Write $z = re^{i\theta}$ and $w = u + iv$, then

$$e^u e^{iv} = e^w = z = re^{i\theta}.$$

So, $e^u = r$, giving us $u = \ln r = \ln|z|$, and $v = \theta + 2k\pi$ for some $k \in \mathbf{Z}$, that is the possible values of v are exactly $\arg z = \operatorname{Arg} z + 2k\pi,\ k \in \mathbf{Z}$.

Therefore,

$$\begin{aligned} w &= \ln|z| + i\arg(z) \\ &= \ln|z| + i\operatorname{Arg}(z) + 2k\pi i,\ k \in \mathbf{Z} \end{aligned}$$

Essentially, w is not unique, as v is not unique. This is to be expected, since e^z is not injective as it is periodic.

Multiple functions satisfy the equation we considered, which we package into a *multi-valued function* using $\arg z$.

Definition 9.2 (The Logarithmic Function)**.** We define the logarithmic function $\log z$ for any $z \neq 0$, following the discussion above, as

$$\log z := \ln|z| + i\arg(z)$$

Note that $\log z$ is not really a function but a *multi-valued function*, as $\arg z$ is not single-valued.

The principal logarithm, denoted $\operatorname{Log} z$, is defined by taking the principal argument of z

$$\operatorname{Log} z := \ln|z| + i\operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z \leqslant \pi$$

The principal branch of log is a single-valued function.

Proposition 9.3 (Properties of the Logarithm)**.** *Consider* $z \in \mathbf{C}$.

(1) $e^{\log z} = z$.

(2) $\log e^z = z + 2k\pi i,\ k \in \mathbf{Z}$.

(3) $\log z = \operatorname{Log} z + 2k\pi i,\ k \in \mathbf{Z}$.

(4) If $z = x \in \mathbf{R}_{>0}$, *then* $\operatorname{Log} z = \ln x$.

Proof.

(1) Note that

$$\begin{aligned} e^{\log z} &= e^{\ln|z|+i\arg z} \\ &= e^{\ln|z|}e^{i(\operatorname{Arg} z+2k\pi)},\ k\in\mathbf{Z} \\ &= e^{\ln|z|}e^{i\operatorname{Arg} z}e^{2k\pi i},\ k\in\mathbf{Z} \\ &= |z|\, e^{i\operatorname{Arg} z} \\ &= z \end{aligned}$$

(2) Note that

$$\begin{aligned} \log e^z &= \ln|e^z| + i\arg(e^z) \\ &= \ln e^{\operatorname{Re} z} + i(\operatorname{Im} z + 2k\pi),\ k\in\mathbf{Z} \\ &= \operatorname{Re} z + i\operatorname{Im} z + 2k\pi i,\ k\in\mathbf{Z} \\ &= z + 2k\pi i,\ k\in\mathbf{Z} \end{aligned}$$

(3) Note that

$$\begin{aligned} \log z &= \log e^{\operatorname{Log} z}, \text{ by (1)} \\ &= \operatorname{Log} z + 2k\pi i,\ k\in\mathbf{Z}, \text{ by (2)} \end{aligned}$$

(4) Note that if $z = x \in \mathbf{R}_{>0}$, then $\operatorname{Arg} z = 0$, therefore

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z = \ln x.$$

□

Example 9.4.

(1) $$\begin{aligned} \log(1+i\sqrt{3}) &= \ln\left|1+i\sqrt{3}\right| + i\arg(1+i\sqrt{3}) \\ &= \ln 2 + i\left(\frac{\pi}{3}+2k\pi\right),\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log}(1+i\sqrt{3}) = \ln 2 + \frac{\pi i}{3}$$

(2) $$\begin{aligned} \log 1 &= \ln|1| + i\arg 1 \\ &= 0 + i\,(0+2k\pi)\,,\ k\in\mathbf{Z} \\ &= 2k\pi i,\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log} 1 = 0$$

(3) $$\begin{aligned} \log -1 &= \ln|-1| + i\arg 1 \\ &= \ln 1 + i\,(\pi+2k\pi)\,,\ k\in\mathbf{Z} \\ &= (2k+1)\pi i,\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log} -1 = \pi i$$

(4) Familiar properties of logarithms that you know may not hold.

(a) $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$

$$\begin{aligned}
\operatorname{Log}(-1+i)^2 = \operatorname{Log}(-2i) &= \ln|-2i| + i\operatorname{Arg}(-2i) \\
&= \ln 2 + i\left(-\frac{\pi}{2}\right) \\
&= \ln 2 - \frac{\pi i}{2}
\end{aligned}$$

$$\begin{aligned}
2\operatorname{Log}(-1+i) &= 2\ln|-1+i| + 2i\arg(-1+i) \\
&= 2\ln\sqrt{2} + 2i\left(\frac{3\pi}{4}\right) \\
&= \ln 2 + \frac{3\pi i}{2}
\end{aligned}$$

(b) $\log i^2 \neq 2\log i$

$$\log i^2 = \log -1 = (2k+1)\pi i,\ k \in \mathbf{Z}$$

$$\begin{aligned}
2\log i &= 2\ln|i| + 2i\arg i \\
&= 0 + 2i\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbf{Z} \\
&= (4k+1)\pi i,\ k \in \mathbf{Z}
\end{aligned}$$

Proposition 9.5. *For all $z,\ w \in \mathbf{C}^*$*

(1) $\log zw = \log z + \log w$ *(2)* $\log w^{-1} = -\log w$

One treats this as an equality of sets. (1) and (2) also gives you $\log z/w = \log z - \log w$.

Proof.

(1) We have

$$\begin{aligned}
\log z + \log w &= \ln|z| + i\arg z + \ln|w| + i\arg w \\
&= \ln|z|\,|w| + i(\arg z + \arg w) \\
&= \ln|zw| + i\arg zw, \text{ by Proposition 3.1 (1)} \\
&= \log zw
\end{aligned}$$

(2) We have

$$\begin{aligned}
\log w^{-1} &= \ln|w^{-1}| + i\arg w^{-1} \\
&= \ln|w|^{-1} + i(-\arg w), \text{ by Proposition 3.1 (2)} \\
&= -(\ln|w| + i\arg w) \\
&= -\log w
\end{aligned}$$

This statement does not hold if we replace $\log z$ with $\operatorname{Log} z$. □

Definition 9.6 (Branch of a Multi-Valued Functions)**.** A branch of a multi-valued function f is a single-valued function F such that

- F is holomorphic on some domain G; and
- F assigned to each $z \in G$ precisely one value $F(z)$ of $f(z)$.

A portion of a line or curve in the complex plane is called a branch cut for f if a branch f is defined on its complement. A point belonging to *every* branch cut of f is a branch point.

Proposition 9.7 (Branches of log)**.** *Let $\alpha \in \mathbf{R}$. The function*

$$L_\alpha(z) = L_\alpha(re^{i\theta}) = \ln r + i\theta, \quad \alpha < \theta < \alpha + 2\pi$$

is a branch of $f(z) = \log z$. Note that $\operatorname{Re} L_\alpha = u(r,\theta) = \ln r$ and $\operatorname{Im} L_\alpha = v(r,\theta) = \theta$.

Proof. We first remark that if we were to define L_α also on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, as one notes that $\lim_{\theta \to \alpha^-} \theta = \alpha$ but $\lim_{\theta \to \alpha^+} \theta \neq \alpha$ as the points close to the ray to the right have arguments near $\alpha + 2\pi$.

It is clear that $L_\alpha(z)$ is single-valued and, for each z, $L_\alpha(z)$ is a value of $\log z$. We need to show L_α is holomorphic. Note that $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ have continuous partial derivatives on the domain of definition

$$u_r = \frac{1}{r} \qquad\qquad v_r = 0$$

$$u_\theta = 0 \qquad\qquad v_\theta = 1$$

Clearly, the Polar Cauchy Riemann equations (Polar CR) are satisfied, and therefore L_α is holomorphic. In fact,

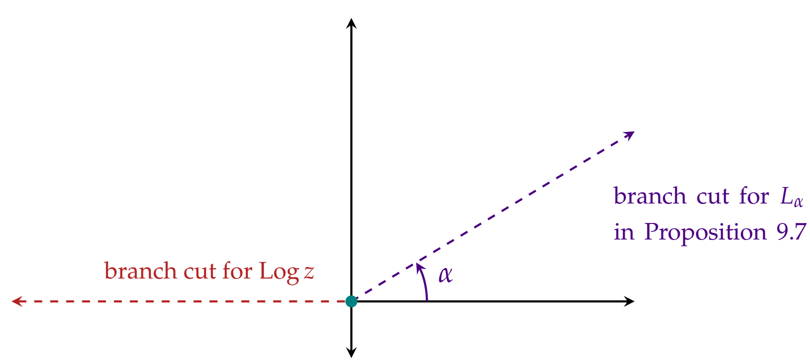
$$L'_\alpha(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{z}$$

In particular, $\operatorname{Log} z$ for those z such that $-\pi < \operatorname{Arg} z < \pi$ is a branch of $\log z$, called the principal branch of the logarithm and

$$(\operatorname{Log} z)' = \frac{1}{z}$$

□

Remark 9.8. The branch cut for $\log z$ in Proposition 9.7 is the ray $r > 0$, $\theta = \alpha$



The branch cut for $\operatorname{Log} z$ is the ray $r > 0$, $\theta = \pi$, i.e., the negative real axis. The origin is a branch point of $\log z$.

Example 9.9 (Integer Powers and Roots)**.** The logarithmic function can be used to compute integer powers and roots (as previously seen and defined).

(1) $z^n = e^{n \log z}$

(2) $z^{1/n} = e^{\frac{\log z}{n}}$

Proof. We note that

$$\begin{aligned} e^{n \log z} &= e^{n(\ln|z| + i \arg z)} \\ &= e^{n \ln|z|} \cdot e^{in \arg z} \\ &= |z|^n \cdot (e^{i \arg z})^n \\ &= (|z|\, e^{i \arg z})^n \\ &= z^n \end{aligned} \qquad \begin{aligned} e^{\frac{\log z}{n}} &= e^{\frac{1}{n}(\ln|z| + i \arg z)} \\ &= e^{\frac{1}{n} \ln|z|} \cdot e^{i\left(\frac{\arg z}{n}\right)} \\ &= e^{\frac{1}{n} \ln|z|} \cdot e^{i\left(\frac{\operatorname{Arg} z + 2k\pi}{n}\right)} \\ &= \sqrt[n]{|z|} \cdot e^{i\left(\frac{\operatorname{Arg} z + 2k\pi}{n}\right)} \\ &= z^{1/n} \end{aligned}$$

Recall that z^n is single-valued, but $z^{1/n}$ is multi-valued, as complex numbers have n distinct n^{th} roots (Proposition 3.6). In fact, using the the principal logarithm, the complex number

$$e^{\frac{\operatorname{Log} z}{n}}$$

gives the principal n^{th} root of z. ◻

Power and Exponential Functions

Definition 9.10 (Power Function)**.** The power function z^α for a fixed $c \in \mathbf{C}$ is the *multi-valued* function

$$z^c := e^{c \log z}, \quad z \neq 0$$

Proposition 9.11 (Branches of z^c)**.** *A branch of z^c is determined by specifying a branch of* $\log z$

$$\log z = \ln|z| + i \arg z, \quad z \neq 0,\ \alpha < \arg z < \alpha + 2\pi$$

Moreover,

$$(z^c)' = cz^{c-1},$$

whenever $z \neq 0$, $\alpha < \arg z < \alpha + 2\pi$.

Proof. We only need to verify that z^c is holomorphic, once a branch of $\log z$ has been specified. Since $z^c = e^{c \log z}$ is a composition of two holomorphic functions, z^c itself is holomorphic. Moreover, by the chain rule

$$\begin{aligned} (z^c)' = (e^{c \log z})' &= e^{c \log z}(c \log z)' \\ &= e^{c \log z} \cdot \frac{c}{z} \\ &= c \cdot \frac{e^{c \log z}}{e^{\log z}} = c \cdot e^{(c-1)\log z} = cz^{c-1} \end{aligned}$$

◻

Discussion 9.12. The principal branch of z^c is defined by specifying the principal branch $\operatorname{Log} z$ of $\log z$. The principal branch of z^c reduces to the usual power function when $z = x \in \mathbf{R}$.

9.1. Problems

Problem 9.1. Find the all possible values of

(a) $\log(-5)$

(b) $\log(-2+2i)$

(c) $\log(\sqrt{2}+i\sqrt{6})$

(d) $\log(-ei)$

(e) $\log(1+i)$

(f) $\log(-\sqrt{3}+i)$

Problem 9.2. Compute

(a) $\operatorname{Log}(6-6i)$

(b) $\operatorname{Log}(-e^2)$

(c) $\operatorname{Log}(-12+5i)$

(d) $\operatorname{Log}((1+i\sqrt{3})^5)$

(e) $\operatorname{Log}(3-4i)$

(f) $\operatorname{Log}((1+i)^4)$

Problem 9.3.

(a) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

(b) Show that for any two non-zero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i,$$

where $N \in \{0, \pm 1\}$.

Problem 9.4. Example 9.4 (4) tells us that it's not necessarily true that $\log z^n = n \log z$, for $n \in \mathbf{Z}_{>0}$. Writing $z = re^{i \operatorname{Arg} z}$, show that, where $n \in \mathbf{Z}_{>0}$

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i \left(\frac{\operatorname{Arg} z + 2(pn+k)\pi}{n} \right), \quad k = 0, \ldots, n-1.$$

Now, after writing

$$\frac{1}{n} \log z = \frac{1}{n} \ln r + i \left(\frac{\operatorname{Arg} z + 2qz}{n} \right), \quad q \in \mathbf{Z},$$

show that we have equality of sets

$$\log(z^{1/n}) = \frac{1}{n} \log z$$

Problem 9.5. Find a domain in which the given function f is holomorphic; then find the derivative f'.

(a) $f(z) = 3z^2 - e^{2iz} + i \operatorname{Log} z$

(b) $f(z) = (z+1) \operatorname{Log} z$

(c) $f(z) = \dfrac{\operatorname{Log}(2z - i)}{z^2 + 1}$

(d) $f(z) = \operatorname{Log}(z^2 + 1)$

10. Lecture 10 (4/28)

Definition 10.1 (Exponential Function with Base c)**.** The exponential function with base c, where $c \in \mathbf{C}^*$, is the *multi-valued* function

$$c^z := e^{z \log c}$$

Discussion 10.2. Once a branch of $\log z$ has been specified, c^z is an entire function. In that case, using chain rule we have

$$\begin{aligned}(c^z)' = (e^{z\log c})' &= e^{z \log c}(z \log c)' \\ &= e^{c \log z} \cdot \log c \\ &= c^z \log c\end{aligned}$$

What happens if we take $c = e$? Specifying the principal branch $\operatorname{Log} z$ we see

$$e^z = e^{z \operatorname{Log} e} = e^{z(\ln e + i \operatorname{Arg} e)} = e^{z(1+0)} = e^z$$

Example 10.3.

(1) We compute

$$\begin{aligned} i^i &= e^{i \log i} \\ &= e^{i(\ln|i| + i \arg i)} \\ &= e^{i(\ln 1 + i(\frac{\pi}{2} + 2k\pi))},\ k \in \mathbf{Z} \\ &= e^{i^2(\frac{\pi}{2}+2k\pi)},\ k \in \mathbf{Z} \\ &= e^{-\frac{\pi}{2}} e^{-2k\pi},\ k \in \mathbf{Z}\end{aligned}$$

(2) We compute

$$\begin{aligned} (-1)^{\frac{1}{\pi}} &= e^{\frac{1}{\pi}\log -1} \\ &= e^{\frac{1}{\pi}(\ln|-1| + i \arg -1)} \\ &= e^{\frac{1}{\pi}(\ln 1 + i(\pi + 2k\pi))},\ k \in \mathbf{Z} \\ &= e^{\frac{1}{\pi}(\pi i(2k+1))},\ k \in \mathbf{Z} \\ &= e^{i(2k+1)},\ k \in \mathbf{Z}\end{aligned}$$

Trigonometric Functions

Discussion 10.4. Recall that for any $z \in \mathbf{C}$,

$$\operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}$$

Therefore, for $x \in \mathbf{R}$,

$$\begin{aligned}\cos x &= \operatorname{Re}(e^{ix}) \\ &= \frac{e^{ix} + \overline{e^{ix}}}{2} \\ &= \frac{e^{ix} + e^{-ix}}{2}\end{aligned} \qquad \begin{aligned}\sin x &= \operatorname{Im}(e^{ix}) \\ &= \frac{e^{ix} - \overline{e^{ix}}}{2i} \\ &= \frac{e^{ix} - e^{-ix}}{2i}\end{aligned}$$

This suggests a way to extend the domain of definition of sine and cosine functions to all of $\mathbf{C}$.

Definition 10.5 (Sine and Cosine)**.** The (complex) sine and cosine functions are defined as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

respectively. Moreover, this gives us $e^{iz} = \cos z + i \sin z$. And our calculations above tell us that these functions reduce to the usual sine and cosine for $z = x \in \mathbf{R}$.

Proposition 10.6 (Holomorphicity of sin and cos)**.**

(1) $\sin z$ *and* $\cos z$ *are entire.*

(2) $(\sin z)' = \cos z$ *and* $(\cos z)' = -\sin z$*.*

Proof.

(1) Since $\sin z$ and $\cos z$ are linear combinations of entire functions, they themselves are entire functions.

(2) We note that

$$\begin{aligned} (\sin z)' &= \frac{(e^{iz})' - (e^{-iz})'}{2i} \\ &= \frac{ie^{iz} - (-i)e^{-iz}}{2i} \\ &= \frac{ie^{iz} + ie^{-iz}}{2i} \\ &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \cos z \end{aligned} \qquad \begin{aligned} (\cos z)' &= \frac{(e^{iz})' + (e^{-iz})'}{2} \\ &= \frac{ie^{iz} - ie^{-iz}}{2} \\ &= i \cdot \frac{e^{iz} - e^{-iz}}{2} \\ &= -\frac{e^{iz} - e^{-iz}}{2i} \\ &= -\sin z \end{aligned}$$

□

Discussion 10.7 (Trigonometric Identities)**.** Various familiar identities hold, here are a few.

(1) $\sin(-z) = -\sin z$

(2) $\cos(-z) = \cos z$

(3) $\sin(z + w) = \sin z \cos w + \cos z \sin w$

(4) $\cos(z + w) = \cos z \cos w - \sin z \sin w$

(5) $\sin(z + 2\pi) = \sin z$

(6) $\cos(z + 2\pi) = \cos z$

(7) $\sin(\pi/2 - z) = \cos z$

(8) $\sin^2 z + \cos^2 z = 1$

To define other trigonometric functions, we need to understand the zeros of $\sin z$ and $\cos z$.

Theorem 10.8 (Zeros of Sine and Cosine)**.** *The zeros of* $\sin z$ *and* $\cos z$ *are precisely the zeros of sine and cosine functions in a real variable:*

$$\sin z = 0 \quad \textit{if and only if} \quad z = k\pi,\ k \in \mathbf{Z}$$

$$\cos z = 0 \quad \textit{if and only if} \quad z = k\pi + \frac{\pi}{2},\ k \in \mathbf{Z}$$

Proof. We immediately note that

$$\sin z = \sin k\pi = 0 \quad \text{and} \quad \cos z = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$$

since the inputs are real numbers and sine and cosine reduce to the usual real sine and cosine for real inputs.

Conversely, suppose

$$\frac{e^{iz} - e^{-iz}}{2i} = \sin z = 0,$$

this gives us $e^{iz} = e^{-iz}$, and therefore $e^{2iz} = 1$. Applying log gives us

$$2iz + 2m\pi i = 2n\pi i, \quad \text{for } m, n \in \mathbf{Z}$$

by Proposition 9.3 (2) and Example 9.4 (2). Giving us $z = (n-m)\pi = k\pi$ for any $k \in \mathbf{Z}$.

Suppose $\cos z = 0$. By Discussion 10.7 (1) and (7), we have

$$\sin\left(z - \frac{\pi}{2}\right) = -\cos z = 0$$

Hence, $z - \frac{\pi}{2} = k\pi,\ k \in \mathbf{Z}$. □

Definition 10.9 (Other Trigonometric Functions)**.** The (complex) tangent, cotangent, secant and cosecant functions are defined in terms of sine and cosine.

$$\tan z := \frac{\sin z}{\cos z}, \quad z \neq k\pi + \frac{\pi}{2} \qquad \sec z := \frac{1}{\cos z}, \quad z \neq k\pi + \frac{\pi}{2}$$

$$\cot z := \frac{\cos z}{\sin z}, \quad z \neq k\pi \qquad \csc z := \frac{1}{\sin z}, \quad z \neq k\pi$$

These functions are entire in their stated domains of definition since $\sin z$ and $\cos z$ are. They also all reduce to the usual real trigonometric functions when z is real, since $\sin z$ and $\cos z$ do. The derivatives are exactly as expected.

Part III. Integration

We now want to develop a theory of integration of complex-valued functions in a single complex variable. Integrals will be defined over suitable curves (contours) in the complex plane. This theory of integration is a surprisingly powerful tool in the study of holomorphic functions.

Using this theory, we will obtain powerful characterisations of holomorphic functions. Roughly speaking we will prove the following: let G be a domain and $f : G \to \mathbf{C}$ a function. The following are equivalent.

(1) f is holomorphic on G.

(2) For all $n \in \mathbf{Z}_{>0}$, $f^{(n)}$ exists and is holomorphic on G.

(3) In each *simply connected* subdomain D of G, there exists a holomorphic function $F : D \to \mathbf{C}$ such that $F' = f|_D$.

(4) f is continuous on G and
$$\int_C f(z)\,dz = 0$$
for every *contour* C lying in a *simply connected* subdomain.

(5) If C is a *simple closed contour* in G and z_0 is interior to C, then
$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz.$$

Additionally, as an application of the theory, we will prove

- *Liouville's theorem*. Every bounded holomorphic function is constant.
- *Fundamental Theorem of Algebra*. Every polynomial of degree $n \geqslant 1$ has atleast one complex root.

Derivatives of Functions of a Real-variable

To define an integral of a complex-valued functions in a single complex variable, we need to understand how to differentiate a complex-valued function in a single real variable
$$\gamma : [a, b] \to \mathbf{C},$$
where $[a, b] \subseteq \mathbf{R}$.

Definition 10.10. For $\gamma : [a, b] \to \mathbf{C}$, writing $\gamma(t) = u(t) + i\,v(t)$, where $u,\ v : [a, b] \to \mathbf{R}$, we define the *derivative* of γ to be
$$\gamma'(t) = u'(t) + iv'(t),$$
provided that $u'(t)$ and $v'(t)$ exist. In this case, we say γ is differentiable.

Proposition 10.11. *Suppose $\gamma_1(t) = u_1(t) + iv_1(t)$ and $\gamma_2(t) = u_2(t) + iv_2(t)$ are differentiable, then*

(1) $(\gamma_1 + \gamma_2)'(t) = \gamma_1'(t) + \gamma_2'(t)$

(2) $(\gamma_1\gamma_2)'(t) = \gamma_1'(t)\gamma_2(t) + \gamma_1(t)\gamma_2'(t)$

Proof.

(1)
$$\begin{aligned}(\gamma_1 + \gamma_2)' &= ((u_1 + u_2) + i(v_1 + v_2))' \\ &= (u_1 + u_2)' + i(v_1 + v_2)' \\ &= (u_1' + u_2') + i(v_1' + v_2') \\ &= (u_1' + iv_1') + (u_2' + iv_2') \\ &= \gamma_1' + \gamma_2'\end{aligned}$$

(2) $$\begin{aligned}(\gamma_1\gamma_2)' &= ((u_1+iv_1)(u_2+iv_2))' \\ &= ((u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1))' \\ &= (u_1u_2 - v_1v_2)' + i(u_1v_2 + u_2v_1)' \\ &= (u_1u_2)' - (v_1v_2)' + i(u_1v_2)' + i(u_2v_1)' \\ &= (u_1'u_2 + u_1u_2') - (v_1'v_2 + v_1v_2') + i(u_1'v_2 + u_1v_2') + i(u_2'v_1 + u_2v_1') \\ &= (u_1'u_2 - v_1'v_2) + i(u_1'v_2 + u_2v_1') + (u_1u_2' - v_1v_2') + i(u_1v_2' + u_2'v_1) \\ &= (u_1' + iv_1')(u_2 + iv_2) + (u_1 + iv_1)(u_2' + iv_2') \\ &= \gamma_1'\gamma_2 + \gamma_1\gamma_2'\end{aligned}$$

Hence, $(\gamma_1\gamma_2)' = \gamma_1'\gamma_2 + \gamma_1\gamma_2'$. □

Example 10.12. We will often encounter the function $\gamma : [a,b] \to \mathbf{C}$, where

$$\gamma(t) = e^{z_0 t}, \quad z_0 \in \mathbf{C}$$

Let's compute $\gamma'(t)$, for which we first need to express it as $u(t) + iv(t)$. Let $z_0 = x_0 + iy_0$,

$$\begin{aligned}\gamma(t) = e^{z_0 t} &= e^{(x_0+iy_0)t} \\ &= e^{x_0 t + iy_0 t} \\ &= e^{x_0 t}e^{iy_0 t} = e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t))\end{aligned}$$

Therefore, $u(t) = e^{x_0 t}\cos(y_0 t)$ and $v(t) = e^{x_0 t}\sin(y_0 t)$. We note,

$$\begin{aligned}u'(t) &= (e^{x_0 t})'(\cos(y_0 t)) + (e^{x_0 t})(\cos(y_0 t))' & v'(t) &= (e^{x_0 t})'(\sin(y_0 t)) + (e^{x_0 t})(\sin(y_0 t))' \\ &= x_0 e^{x_0 t}\cos(y_0 t) - y_0 e^{x_0 t}\sin(y_0 t) & &= x_0 e^{x_0 t}\sin(y_0 t) + y_0 e^{x_0 t}\cos(y_0 t)\end{aligned}$$

Hence,

$$\begin{aligned}\gamma'(t) = u'(t) + iv'(t) &= x_0 e^{x_0 t}\cos(y_0 t) - y_0 e^{x_0 t}\sin(y_0 t) + ix_0 e^{x_0 t}\sin(y_0 t) + iy_0 e^{x_0 t}\cos(y_0 t) \\ &= x_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) + iy_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) \\ &= (x_0 e^{x_0 t} + iy_0 e^{x_0 t})(\cos(y_0 t) + i\sin(y_0 t)) \\ &= (x_0 + iy_0)e^{x_0 t}e^{iy_0 t} \\ &= z_0 e^{z_0 t}\end{aligned}$$

To summarise, for $\gamma(t) = e^{z_0 t}$, we have $\gamma'(t) = z_0 e^{z_0 t}$.

10.1. Problems

Problem 10.1. Find the all possible values of

(a) $(-1)^{3i}$

(b) $3^{2i/\pi}$

(c) $(1+i)^{1-i}$

(d) $(1+i\sqrt{3})^{i}$

(e) $(-i)^{i}$

(f) $(ei)^{\sqrt{2}}$

(g) $(-1)^{1/\pi}$

(h) $i^{i/\pi}$

Problem 10.2. Compute the principal value of the given complex powers.

(a) $(-1)^{3i}$

(b) $3^{2i/\pi}$

(c) 2^{4i}

(d) $(1+i\sqrt{3})^{3i}$

(e) $i^{i/\pi}$

(f) $(1+i)^{2-i}$

(g) $\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i}$

(h) $(1-i)^{4i}$

Problem 10.3.

(a) Verify that $(z^{\alpha})^{n} = z^{n\alpha}$ for $z \neq 0$ and $n \in \mathbf{Z}$.

(b) Find a counterexample to the statement: $(z^{\alpha})^{\beta} = z^{\alpha\beta}$, where $z \neq 0$ and $\alpha, \beta \in \mathbf{C}$.

Problem 10.4. Let z^{α} represent the principal value of the complex power. Find the derivative of the given function at the given point.

(a) $z^{3/2}; \quad z = 1+i$

(b) $z^{1+i}; \quad z = 1+i\sqrt{3}$

(c) $z^{2i}; \quad z = i$

(d) $z^{\sqrt{2}}; \quad z = -i$

Problem 10.5. Let $z \in \mathbf{C}$.

(a) Prove that $|1^{z}|$ is single-valued if and only if $\operatorname{Im} z = 0$.

(b) Find a necessary and sufficient condition for $|i^{iz}|$ to be single-valued.

(c) Find a counterexample to the statement: 1^{z} is single-valued if and only if $\operatorname{Im} z = 0$.

Problem 10.6. Express the value of the given trigonometric function in the form $x+iy$.

(a) $\sin(4i)$

(b) $\cos(-3i)$

(c) $\cos(2-4i)$

(d) $\sin\left(\frac{\pi}{4}+i\right)$

(e) $\tan(2i)$

(f) $\cot(\pi+2i)$

(g) $\sec\left(\frac{\pi}{2} - i\right)$

(h) $\csc(1+i)$

Problem 10.7. Find all complex values z satisfying the given equation.

(a) $\sin z = i$

(b) $\cos z = 4$

(c) $\sin z = \cos z$

(d) $\cos z = i \sin z$

Problem 10.8. Prove the properties stated in Discussion 10.7.

Problem 10.9.

(a) Prove that $\overline{\cos z} = \cos \overline{z}$.

(b) What is $\operatorname{Re} \cos z$ and $\operatorname{Im} \cos z$?

(c) Using the identity $e^{iz} = \cos z + i \sin z$, prove $\overline{\sin z} = \sin \overline{z}$ and find $\operatorname{Re} \sin z$ and $\operatorname{Im} \sin z$.

11. Lecture 11 (5/03)

Integral of $\gamma : [a, b] \to \mathbf{C}$

Definition 11.1 (Definite Integral of γ)**.** Consider a function $\gamma : [a, b] \to \mathbf{C}$ with

$$\gamma(t) = u(t) + iv(t),$$

where u, $v : [a, b] \to \mathbf{R}$. The definite integral of γ is defined as

$$\int_a^b \gamma(t)\, dt := \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

provided the integrals of u and v exist.

Improper integrals can be defined in a similar manner.

Example 11.2. We illustrate this definition by integrating $\gamma(t) = e^{it}$ on $[0, \pi]$.

$$\begin{aligned}\int_0^\pi e^{it}\, dt &= \int_0^\pi \cos t\, dt + \int_0^\pi \sin t\, dt \\ &= \Big[\sin t\Big]_0^\pi + i\Big[-\cos t\Big]_0^\pi \\ &= (\sin\pi - \sin 0) + i(-\cos\pi + \cos 0) = 2i\end{aligned}$$

Definition 11.3 (Piecewise Continuity)**.** A function $u : [a, b] \to \mathbf{R}$ is piecewise continuous on $[a, b]$ if it is continuous on $[a, b]$ except at a finite number of points, where despite its discontinuity on those points, both one sided limits exist.

We call $\gamma(t) = u(t) + iv(t)$ *piecewise continuous* if both u and v are.

Remark 11.4. The existence of the integrals

$$\int_a^b u(t)\, dt \quad \text{and} \quad \int_a^b v(t)\, dt$$

is guaranteed when γ is piecewise continuous.

Proposition 11.5 (Properties of the Integral of γ)**.** *Suppose γ and γ_1 are piecewise continuous on $[a, b]$, then*

(1) $\displaystyle\int_a^b z_0\gamma(t)\, dt = z_0 \int_a^b \gamma(t)\, dt$*, for any* $z_0 \in \mathbf{C}$.

(2) $\displaystyle\int_a^b \gamma(t) + \gamma_1(t)\, dt = \int_a^b \gamma(t)\, dt + \int_a^b \gamma_1(t)\, dt.$

(3) $\displaystyle\int_a^b \gamma(t)\, dt = \int_a^c \gamma(t)\, dt + \int_c^b \gamma(t)\, dt$, *for any* $c \in [a, b]$.

(4) $\displaystyle\int_b^a \gamma(t)\, dt = -\int_a^b \gamma(t)\, dt.$

Proof. These properties follow from the properties of regular real integrals applied to the real and imaginary part of γ and γ_1. □

Proposition 11.6 (Extension of Fundamental Theorem of Calculus)**.** *Suppose that* $\gamma(t) = u(t) + iv(t)$ *is continuous on* $[a, b]$ *and* $\Gamma(t) = U(t) + iV(t)$ *is differentiable such that* $\Gamma'(t) = \gamma(t)$ *on* $[a, b]$. *Then*

$$\int_a^b \gamma(t)\, dt = \Gamma(b) - \Gamma(a)$$

Proof. By assumption $\Gamma' = \gamma$, therefore $U'(t) = u(t)$ and $V'(t) = v(t)$, therefore

$$\begin{aligned}
\int_a^b \gamma(t)\, dt &= \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt \\
&= U(b) - U(a) + i(V(b) - V(a)), \text{ by the Fundamental Theorem of Calculus} \\
&= U(b) + iV(b) - (U(a) + iV(a)) \\
&= \Gamma(b) - \Gamma(a)
\end{aligned}$$

□

Example 11.7. We use this proposition to integrate e^{it} on $[0, \pi]$. For this, we first note that

$$\left(\frac{e^{it}}{i}\right)' = \frac{1}{i}\left(e^{it}\right)' = \frac{i}{i} e^{it} = e^{it}.$$

Therefore,

$$\begin{aligned}
\int_0^\pi e^{it}\, dt = \left[\frac{e^{it}}{i}\right]_0^\pi &= \left[-ie^{it}\right]_0^\pi \\
&= -ie^{i\pi} + ie^{i\cdot 0} \\
&= i + i = 2i
\end{aligned}$$

Contours

So far, we have only defined the integral of a complex-valued function in a single real variable over an interval. Integrals of complex-valued functions in a single complex variable are defined over suitable curves in the complex plane called *contours*.

Definition 11.8 (Arcs)**.**

(1) An arc, or curve, is a collection of points

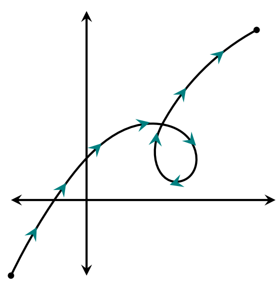
$$C = \{z(t) \ : \ t \in [a, b]\},$$

where $z(t) = x(t) + iy(t)$ and $x, y : [a, b] \to \mathbf{R}$ are continuous functions. The function $z(t)$ is called a parametrization of C.

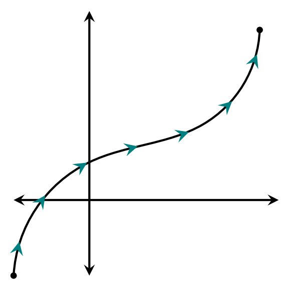
(2) An arc (or curve) C is called simple or a Jordan arc if it does not cross itself, which is equivalent to saying the function $z(t)$ is injective; that is, if $z(t_1) = z(t_2)$ then $t_1 = t_2$.

(3) If C is simple except for the fact that $z(a) = z(b)$, then C is called a simple closed curve or a Jordan curve.

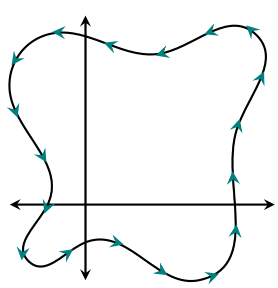
(4) A simple closed curve is positively oriented if it is transversed counter-clockwise as t increases from a to b. It is called negatively oriented if it is transversed clockwise.



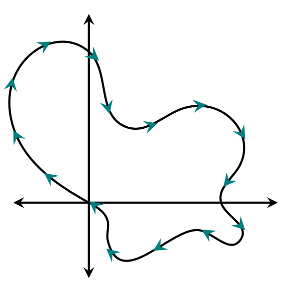
a not simple arc



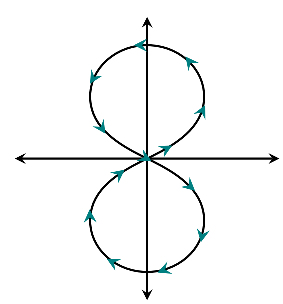
a simple arc



a simple closed curve with positive orientation



a simple closed curve with negative orientation

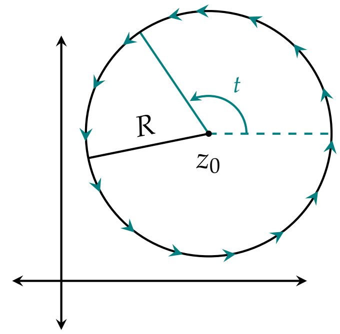


a not simple closed non-orientable curve

Example 11.9. The most frequently encountered arcs and curves are line segments and circles.

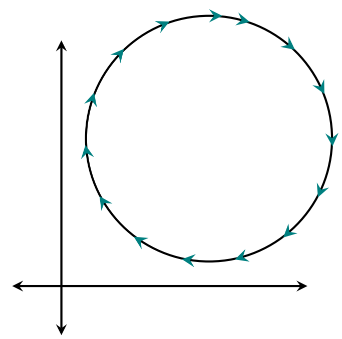
(1) The circle of radius R centered at z_0 with positive orientation has as a parametrisation

$$z(t) = z_0 + Re^{it}, \quad t \in [0, 2\pi]$$



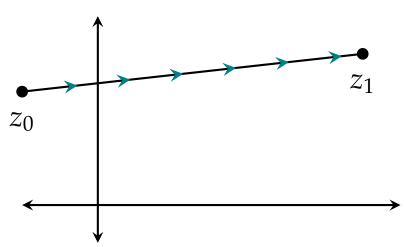
(2) The circle of radius R centered at z_0 with negative orientation has as a parametrisation

$$z(t) = z_0 + Re^{-it}, \quad t \in [0, 2\pi]$$



(3) The line segment from z_0 to z_1 in $\mathbf{C}$ has as a parametrisation

$$z(t) = z_0 + (z_1 - z_0)t = (1-t)z_0 + tz_1, \quad t \in [0, 1]$$



Definition 11.10 (Reparametrisation of an arc)**.** Suppose an arc C is parametrised by $z : [a, b] \to \mathbf{C}$. A map

$$w : [c, d] \to \mathbf{C}$$

is called an orientation-preserving reparametrisation of C if there exists a surjective function

$$\phi : [c, d] \to [a, b]$$

with continuous derivative such that $\phi(c) = a$ (preserves initial point), $\phi(d) = b$ (preserves final point), $\phi'(s) > 0$ and $w(s) = z(\phi(s))$ (w and z trace out the same arc C).

Example 11.11. Note that $z(t) = e^{it}$ for $t \in [0, 2\pi]$ is a parametrisation of the unit circle. Now, consider

$$w : [0, \pi] \to \mathbf{C},\ s \mapsto e^{2is},$$

this is, in fact, an orientation-preserving reparametrisation of the unit circle. To conclude this, we produce the following surjective map

$$\phi : [0, \pi] \to [0, 2\pi],\ s \mapsto 2s,$$

we note that $\phi(0) = 0$ and $\phi(\pi) = 2\pi$, furthermore $\phi'(s) = 2 > 0$ which is clearly continuous. Lastly, $z(\phi(s)) = z(2s) = e^{2is} = w(s)$.

Remark 11.12. Suppose an arc C is parametrised by $z : [a, b] \to \mathbf{C}$, a map $w : [c, d] \to \mathbf{C}$ is called an orientation-reversing reparametrisation of C if there exists a surjective function

$$\psi : [c, d] \to [a, b]$$

with continuous derivative such that $\psi(c) = b$ and $\psi(d) = b$ (swaps initial and final points), $\psi'(s) < 0$ and $w(s) = z(\psi(s))$ (w and z trace out the same arc C).

Consider the unit circle, which has parametrisation $z(t) = e^{it}$, $t \in [0, 2\pi]$. Then $w(t) = e^{-it}$ for $0 \leqslant t \leqslant 2\pi$ is an orientation-reversing parametrisation. To see this, we consider the surjective function

$$\psi : [0, 2\pi] \to [0, 2\pi],\ s \mapsto 2\pi - s;$$

we note that $\psi(0) = 2\pi$ and $\psi(2\pi) = 0$, furthermore $\psi'(s) = -1 < 0$ and

$$z(\psi(s)) = z(2\pi - s) = e^{2\pi i - is} = e^{-is} = w(s),$$

since $e^{2\pi i} = 1$.

Definition 11.13 (Arc length and Smooth arcs)**.**

(1) If C is parametrised by $z(t) = x(t) + iy(t)$ and $x'(t)$, $y'(t)$ exist and are continuous on $[a, b]$, then C is called a differentiable arc.

(2) The arc length of such a differentiable arc C is

$$L(C) = \int_a^b |z'(t)|\ dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

(3) A differentiable curve parametrised by $z(t)$ is called smooth if $z'(t) \neq 0$ on $[a, b]$.

11.1. Problems

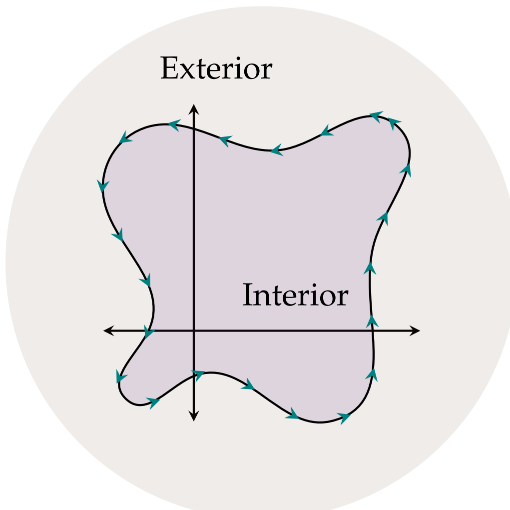
To be added

12. Lecture 12 (5/05)

Definition 12.1 (Contours)**.** A contour is an arc consisting of a finite number of smooth arcs joined end to end.

A simple closed contour is a contour that does not cross itself except that the initial and final points are the same.

Discussion 12.2 (Jordan Curve Theorem)**.** A deep theorem known as the *Jordan Curve theorem* tells us that every simple closed contour C is the boundary of two distinct domains called the interior of C, which is bounded, and the exterior of C, which is unbounded.



The theorem is geometrically evident but the proof is not easy. We will assume its truth so that we can refer to the interior of a simple closed contour.

Contour Integration

Definition 12.3 (Contour Integral)**.** Suppose $f : G \to \mathbf{C}$ is a complex function and C is a contour lying in G. If $z(t)$, $t \in [a, b]$, is a parametrisation of C and $f(z(t))$ is piecewise continuous, then the contour integral of f over C is

$$\int_C f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt$$

Remark 12.4. Since C is a contour, $z'(t)$ is piecewise continuous and therefore the above integral exists.

Proposition 12.5 (Integral is Parametrisation-independent)**.** *Suppose $z : [a, b] \to \mathbf{C}$ parametrises C and $w : [c, d] \to \mathbf{C}$ is an orientation-preserving reparametrisation of C, then*

$$\int_C f(z)\, dz = \int_C f(w)\, dw$$

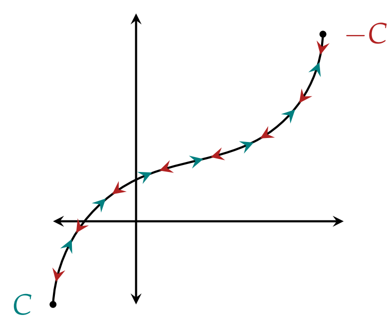
Proof. By definition of an orientation-preserving reparametrisation, there exists a surjective map $\phi : [c, d] \to [a, b]$ such that $\phi(c) = a$, $\phi(d) = b$, $\phi'(s) > 0$ and $w(s) = \phi(z(s))$. Then

$$\begin{aligned}
\int_C f(w)\, dw &= \int_c^d f(w(s))\, w'(s)\, ds \\
&= \int_c^d f(z(\phi(s)))\, \phi'(z(s))\, z'(s)\, ds, \text{ apply chain rule to } w(s) = \phi(z(s)) \\
&= \int_a^b f(z(t))\, z'(t)\, dt, \text{ set } t = \phi(s) \\
&= \int_C f(z)\, dz
\end{aligned}$$

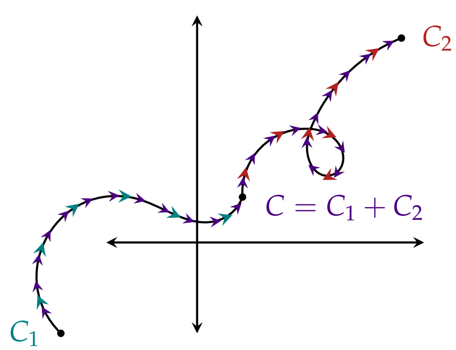
$\square$

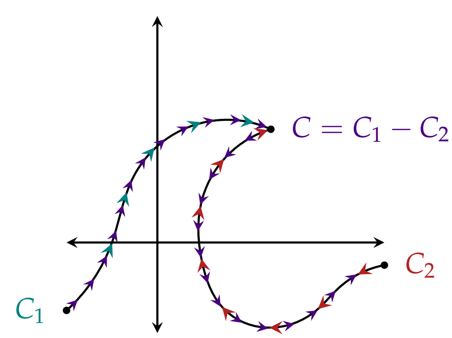
Discussion 12.6 (Notation for Contours)**.**

(1) Suppose C is a contour, then $-C$ denotes the same set of points as C but with opposite orientation. If $z : [a, b] \to \mathbf{C}$ is a parametrisation of C, then $w : [-b, -a] \to \mathbf{C}$ defined as $w(t) := z(-t)$ is a parametrisation of $-C$.



(2) If C_1 is a contour from z_1 to z_2 and C_2 is a contour from z_2 to z_3, then their sum $C = C_1 + C_2$ is the contour obtained by transversing C_1 and then C_2.





If C_1 and C_2 have the same final point, then we can consider the sum of C_1 and $-C_2$ and is written as $C_1 - C_2 := C_1 + (-C_2)$.

Proposition 12.7 (Properties of Contour Integral)**.** *Assume f, g are piecewise continuous on the contours we consider below.*

(1) $\displaystyle\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, *for any* $z_0 \in \mathbf{C}$.

(2) $\displaystyle\int_C f(z) + g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(3) $\displaystyle\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$.

(4) $\displaystyle\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ *if* $C = C_1 + C_2$.

Proof.

(1) Suppose C is parametrised by $z : [a, b] \to \mathbf{C}$

$$\begin{aligned}\int_C z_0 f(z)\,dz &= \int_C z_0 f(z(t))\,z'(t)\,dz \\ &= z_0 \int_a^b f(z(t))\,z'(t)\,dz, \text{ by Proposition 11.5 (1)} \\ &= z_0 \int_C f(z)\,dz\end{aligned}$$

(2) This will follow from Proposition 11.5 (2).

(3) Suppose C is parametrised by $z : [a, b] \to \mathbf{C}$, then, as we note before, a parametrisation of $-C$ is $w : [-b, -a] \to \mathbf{C}$ where $w(t) = z(-t)$. Then

$$\begin{aligned}\int_{-C} f(w)\,dw &= \int_{-b}^{-a} f(w(t))\,w'(t)\,dt \\ &= -\int_{-b}^{-a} f(z(-t))\,z'(-t)\,dt, \text{ apply chain rule to } w(t) = z(-t) \\ &= -\int_{-a}^{-b} f(z(-t))\,z'(-t)\,dt, \text{ by Proposition 11.5 (4)} \\ &= \int_a^b f(z(s))\,z'(s)\,ds, \text{ set } s = -t \\ &= \int_C f(z)\,dz\end{aligned}$$

(4) We leave this as an exercise (Problem **??**) for the motivated student.

□

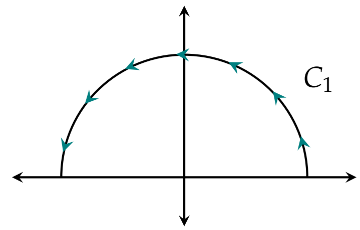
Example 12.8.

(1) Integrate $f(z) = \dfrac{1}{z}$ over the following contours:

- C_1: upper semicircle of the unit circle, from 1 to -1.
- C_2: lower semicircle of the unit circle, from 1 to -1.
- C_3: $C_1 - C_2$.

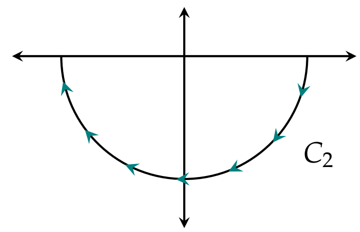
For C_1, parametrise C_1 as $z(t) = e^{it}$, $0 \leqslant t \leqslant \pi$. Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{it}}\, ie^{it}\, dt = i\int_0^{\pi} dt = \pi i$$



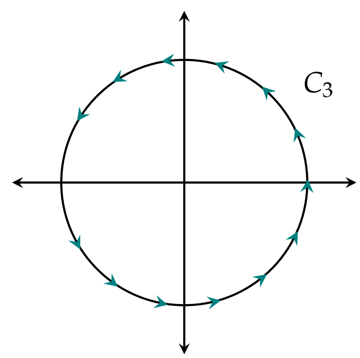
For C_1, parametrise C_1 as $z(t) = e^{-it}$, $0 \leqslant t \leqslant \pi$. Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{-it}}\, (-ie^{-it})\, dt = -i\int_0^{\pi} dt = -\pi i$$



For C_1, parametrise C_1 as $z(t) = e^{-it}$, $0 \leqslant t \leqslant \pi$. Then

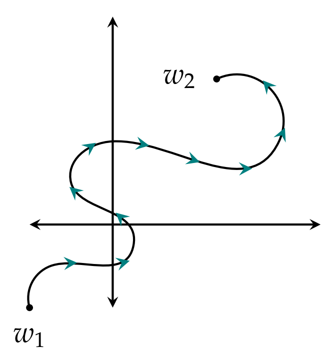
$$\begin{aligned}\int_{C_3} \frac{1}{z}\, dz = \int_{C_1 - C_2} \frac{1}{z}\, dz &= \int_{C_1} \frac{1}{z}\, dz + \int_{-C_2} \frac{1}{z}\, dz \\ &= \int_{C_1} \frac{1}{z}\, dz - \int_{C_2} \frac{1}{z}\, dz \\ &= \pi i - (-\pi i) \\ &= 2\pi i\end{aligned}$$



This example shows that the integral may depend on the path taken and not just on the endpoints. Also, the integral over a closed contour may be non-zero.

(2) Integrate $f(z) = z$ over *any* contour C connecting a point w_1 to a point w_2.

First, suppose C is a smooth arc joining w_1 and w_2 with parametrisation $z : [a, b] \to \mathbf{C}$.



Since,
$$\left(\frac{z(t)^2}{2}\right)' = \frac{z'(t)z(t) + z(t)z'(t)}{2} = z(t)z'(t).$$
Therefore,
$$\begin{aligned}\int_C f(z)\,dz = \int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt \\ &= \frac{z(b)^2}{2} - \frac{z(a)^2}{2}, \text{ by Proposition 11.6} \\ &= \frac{w_2^2 - w_1^2}{2}\end{aligned}$$

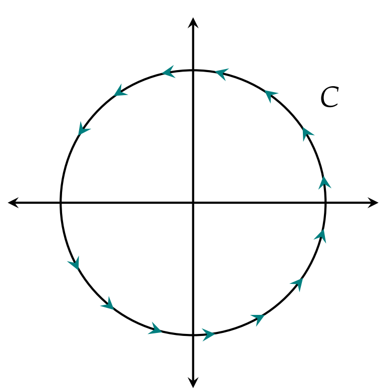
Now, if C is a contour, we can write $C = C_1 + \cdots + C_n$, where C_i is a smooth arc joining z_i to z_{i+1} with $z_1 = w_1$ and $z_{n+1} = w_2$. Then,

$$\begin{aligned}\int_C z\,dz &= \sum_{i=1}^{n} \int_{C_i} z\,dz, \text{ by Proposition 12.7 (4)} \\ &= \sum_{i=1}^{n} \frac{z_{i+1}^2 - z_i^2}{2} \\ &= \frac{z_{n+1}^2 - z_1^2}{2} \\ &= \frac{w_2^2 - w_1^2}{2}\end{aligned}$$

This example shows that some integrals do depend only on the end points and not the path taken. Also, for any contour C is closed, that is, when $w_2 = w_1$, we have shown hence that

$$\int_C z\,dz = 0.$$

(3) Integrate $f(z) = z^m \overline{z}^n$, for $m, n \in \mathbf{Z}$, over the unit circle C.



Parametrise C as $z(t) = e^{it}$, $0 \leqslant t \leqslant 2\pi$. Then,

$$
\begin{aligned}
\int_C f(z)\,dz &= \int_C z^m \overline{z}^n\,dz \\
&= \int_0^{2\pi} (e^{it})^m (\overline{e^{it}})^n\, ie^{it}\,dt \\
&= i\int_0^{2\pi} (e^{it})^m (e^{-it})^n\, e^{it}\,dt \\
&= i\int_0^{2\pi} e^{imt}e^{-int}e^{it}\,dt \\
&= i\int_0^{2\pi} e^{(m-n+1)it}\,dt
\end{aligned}
$$

Case I. $m = n-1$

$$
\begin{aligned}
\int_C f(z)\,dz &= i\int_0^{2\pi} e^{(m-n+1)it} \\
&= i\int_0^{2\pi} dt \\
&= 2\pi i
\end{aligned}
$$

Case II. $m \neq n-1$

$$
\begin{aligned}
\int_C f(z)\,dz &= i\int_0^{2\pi} e^{(m-n+1)it} \\
&= i\left[\frac{e^{(m-n+1)it}}{i(m-n+1)}\right]_0^{2\pi} \\
&= \frac{1}{m-n+1}\left(e^{2(m-n+1)\pi i} - e^0\right) \\
&= \frac{1}{m-n+1}\,(1-1) \\
&= 0
\end{aligned}
$$

12.1. Problems

To be added

13. Lecture 13 (5/10)

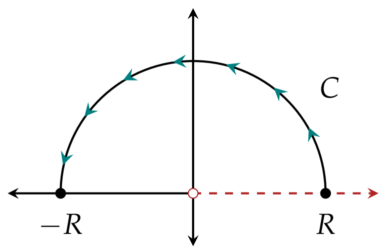
Some examples involving a branch of a multi-valued function.

Example 13.1.

(4) Integrate the branch of square root

$$f(z) = z^{1/2} = e^{(1/2)\log z}, \quad |z| > 0,\ 0 < \arg z < 2\pi$$

over the contour



$$C:\ z(t) = Re^{it}, \quad R > 0,\ 0 \leqslant t \leqslant \pi$$

Note that $f(z)$ is not defined at the initial point $z = R$ of the contour C as $\arg R = 0$, that is, $f(z(t))$ is not defined for $t = 0$. The integral

$$\int_C f(z)\, dz = \int_0^\pi f(z(t))\, z'(t)\, dt$$

nevertheless exists as the integrand $f(z(t))\, z'(t)$ is piecewise continuous on $[0, \pi]$. To see this, we note that for $0 < t \leqslant \pi$

$$\begin{aligned} f(z(t))\, z'(t) = e^{(1/2)\log Re^{it}} Rie^{it} &= iRe^{(\ln R + it)/2} e^{it} \\ &= iR(R^{1/2}e^{it/2})e^{it} \\ &= iR^{3/2}e^{3it/2} \\ &= iR^{3/2}\left(\cos\frac{3t}{2} + i\sin\frac{3t}{2}\right) = R^{3/2}\left(-\sin\frac{3t}{2} + i\cos\frac{3t}{2}\right) \end{aligned}$$

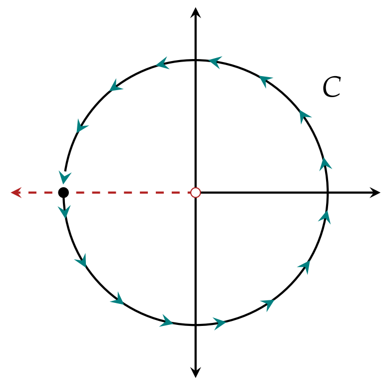
The right hand limits of the real and imaginary parts of $f(z(t))\, z'(t)$ at $t = 0$ exist, and equal 0 and $R^{3/2}$. Therefore, $f(z(t))\, z'(t)$ is continuous on $[0, \pi]$ with its value at $t = 0$ defined as $iR^{3/2}$. Hence,

$$\begin{aligned} \int_C f(z)\, dz = \int_0^\pi f(z(t))\, z'(t)\, dt = \int_0^\pi iR^{3/2}e^{3it/2}\, dt & \\ &= iR^{3/2}\int_0^\pi e^{3it/2}\, dt \\ &= iR^{3/2}\left[\frac{e^{3it/2}}{3i/2}\right]_0^\pi \\ &= \frac{2}{3}R^{3/2}\left(e^{3\pi i/2} - e^0\right) = \frac{2}{3}R^{3/2}(-i-1) = -\frac{2}{3}R^{3/2}(1+i) \end{aligned}$$

(5) Integrate the principal branch of

$$f(z) = z^{i-1} = e^{(i-1)\operatorname{Log} z}, \quad |z| > 0, \ -\pi < \operatorname{Arg} z < \pi$$

over the contour



$$C: \ z(t) = e^{it}, \quad R > 0, \ -\pi \leqslant t \leqslant \pi$$

Since the curve crosses the branc cut, we need to check if integrand $f(z(t))\, z'(t)$ is piecewise continuous on $[-\pi, \pi]$. To see this, we note that for $-\pi < t \leqslant \pi$

$$\begin{aligned} f(z(t))\, z'(t) = e^{(i-1)\operatorname{Log} e^{it}} ie^{it} &= ie^{(i-1)(\ln 1 + it)} e^{it} \\ &= ie^{(i-1)it} e^{it} \\ &= ie^{(i-1)it + it} = ie^{i^2 t} = ie^{-t} \end{aligned}$$

The right hand limits of the real and imaginary parts of $f(z(t))\, z'(t)$ at $t = \pi$ exist, and equal 0 and $e^{-\pi}$. Therefore, $f(z(t))\, z'(t)$ is continuous on $[-\pi, \pi]$ with its value at $t = -\pi$ defined as $ie^{-\pi}$. Hence,

$$\begin{aligned} \int_C f(z)\, dz = \int_{-\pi}^{\pi} f(z(t))\, z'(t)\, dt = \int_{-\pi}^{\pi} ie^{-t}\, dt \\ &= i \int_{-\pi}^{\pi} e^{-t}\, dt \\ &= i \Big[-e^{-t} \Big]_{-\pi}^{\pi} \\ &= i \left(-e^{-\pi} - (-e^{-(-\pi)}) \right) = i \left(e^{\pi} - e^{-\pi} \right) \end{aligned}$$

Estimating Contour Integrals

Lemma 13.2 (Triangle Inequality for Integrals). *Suppose $\gamma : [a, b] \to \mathbf{C}$ is piecewise continuous. Then*

$$\left| \int_a^b \gamma(t)\, dt \right| \leqslant \int_a^b |\gamma(t)|\, dt$$

Proof. Let's first assume
$$\int_a^b \gamma(t)\ dt = 0,$$
then the lemma holds as $|\gamma(t)| \geqslant 0$ for all $t \in [a, b]$ and so its integral is non-negative.

Otherwise, let
$$r_0 e^{it_0} = \int_a^b \gamma(t)\ dt \neq 0.$$
Then,
$$\begin{aligned}
\left|\int_a^b \gamma(t)\ dt\right| &= |r_0 e^{it}| \\
&= r_0 \\
&= \operatorname{Re} r_0 \\
&= \operatorname{Re}(r_0 e^{it_0} e^{-it_0}) \\
&= \operatorname{Re}\left(e^{-it_0} \int_a^b \gamma(t)\ dt\right) \\
&= \operatorname{Re}\left(\int_a^b e^{-it_0}\gamma(t)\ dt\right) \\
&= \int_a^b \operatorname{Re}(e^{-it_0}\gamma(t))dt \\
&= \int_a^b \operatorname{Re}(e^{-it_0}) \operatorname{Re}(\gamma(t))dt \\
&\leqslant \int_a^b |e^{-it_0}|\, |\gamma(t)|\, dt, \text{ using Discussion 1.10} \\
&= \int_a^b |\gamma(t)|\, dt
\end{aligned}$$
$\square$

Theorem 13.3 (Triangle Inequality for Integrals)**.** *Suppose that C is a contour of length L and f is piecewise continuous on C. Then*
$$\left|\int_C f(z)\ dz\right| \leqslant \max_{z\in C} |f(z)| \cdot L(C)$$

Proof. Suppose $z : [a, b] \to \mathbf{C}$ parametrises C. By assumption $f(z(t))$ is piecewise continuous on $[a, b]$. Hence,
$$\max_{z\in C} |f(z)| = \max_{t\in[a,b]} |f(z(t))|$$
is finite as $f(z(t))$ is continuous on a closed and bounded interval.

Thus,

$$\begin{aligned}
\left|\int_C f(z)\,dz\right| &= \left|\int_a^b f(z(t))\,z'(t)\,dz\right| \\
&\leqslant \int_a^b \left|f(z(t))\,z'(t)\right|\,dz, \text{ by Lemma 13.2} \\
&= \int_a^b |f(z(t))|\,\left|z'(t)\right|\,dz \\
&\leqslant \int_a^b \max_{t\in[a,b]} |f(z(t))|\,\left|z'(t)\right|\,dz \\
&= \max_{t\in[a,b]} |f(z(t))| \int_a^b \left|z'(t)\right|\,dz \\
&= \max_{t\in[a,b]} |f(z(t))| \cdot L(C) \\
&= \max_{z\in C} |f(z)| \cdot L(C)
\end{aligned}$$

□

Example 13.4.

(1) Finding an upper bound for

$$\int_C \frac{z^2+1}{z^3+2}\,dz,$$

where C is the semicircle $z(t) = 2e^{it},\ 0 \leqslant t \leqslant \pi$.

All we need to find is an $M > 0$ such that, for all $z \in C$

$$\left|\frac{z^2+1}{z^2+3}\right| \leqslant M, \qquad \text{because then} \quad \max_{z\in C}\left|\frac{z^2+1}{z^2+3}\right| \leqslant M$$

Suppose $z \in C$, then $|z| = C$, and therefore

$$|z^2+1| \leqslant |z|^2 + 1 = 5;$$

also,

$$|z^3+2| \geqslant |\,|z|^3 - 2| = |2^3 - 2| = 6.$$

Together, we get, for any $z \in C$

$$\left|\frac{z^2+1}{z^2+3}\right| \leqslant \frac{5}{6}.$$

Hence,

$$\left|\int_C \frac{z^2+1}{z^3+2}\,dz\right| \leqslant \max_{z\in C}\left|\frac{z^2+1}{z^2+3}\right| \cdot L(C) \leqslant \frac{5}{6}\cdot L(C) = \frac{5}{6}\cdot 2\pi = \frac{5\pi}{3}$$

(2) Show that

$$\lim_{R\to\infty}\int_{C_R}\frac{z^2+z}{z^4+2z^2+1}\,dz=0,$$

where C_R is the semicircle $z(t)=Re^{it}$, $0\leqslant t\leqslant 2\pi$. Note that $L(C)=2\pi R$.

Let $z\in C_R$, then $|z|=R$, and therefore

$$|z^2+z|\leqslant |z|^2+z=R^2+R;$$

also,

$$|z^4+2z^2+1|\geqslant|(z^2+1)|=|z^2+1|^2\geqslant||z|^2-1|^2=|R^2-1|^2=(R^2-1)^2.$$

Together, we get, for any $z\in C$ and $R>1$

$$\left|\frac{z^2+z}{z^4+2z^2+1}\right|\leqslant\frac{R^2+R}{(R^2-1)^2}.$$

Hence,

$$\left|\int_{C_R}\frac{z^2+z}{z^4+2z^2+1}\,dz\right|\leqslant\frac{R^2+R}{(R^2-1)^2}\cdot 2\pi R\to 0,\text{ as } R\to\infty$$

Therefore,

$$\lim_{R\to\infty}\int_{C_R}\frac{z^2+z}{z^4+2z^2+1}\,dz=0,$$

by the Sandwich theorem.

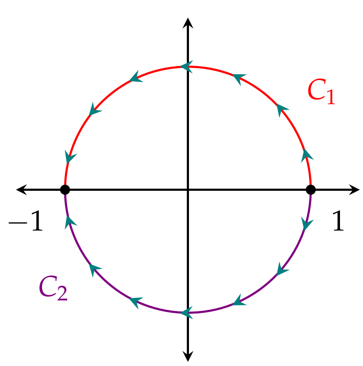
Antiderivatives & Fundamental Theorem of Contour Integrals

Discussion 13.5. Suppose C is a contour joining z_1 to z_2. In general, the value of the integral

$$\int_C f(z)\,dz$$

depends on C. For example, we have seen that

$$\int_{C_1}\frac{1}{z}\,dz=\pi i\quad\text{and}\quad\int_{C_2}\frac{1}{z}\,dz=-\pi i$$



But on the other hand we have also seen that

$$\int_C z\,dz = \frac{z_2^2 - z_1^2}{2}$$

for any contour C with initial point z_1 and end point z_2.

The difference between these functions turns out to be that $f(z) = z$ has an antiderivative on $\mathbf{C}$ while $g(z) = 1/z$ does not any domain containing C_1 and C_2.

Definition 13.6 (Antiderivative)**.** Suppose that f is a continuous function on a domain G. Any holomorphic function $F : G \to \mathbf{C}$ is called an antiderivative of f if $F'(z) = f(z)$ for every $z \in G$.

Definition 13.7 (Independence of Path)**.** Let $f : G \to \mathbf{C}$ be a continuous function on a domain G and fix $z_1, z_2 \in G$. If

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

for any pair of contours C_1 and C_2 joining z_1 to z_2, then the integral of f from z_1 to z_2 is *independent of path* and we denote the unique value by

$$\int_{z_1}^{z_2} f(z)\,dz.$$

So, for instance, we would write

$$\int_{z_1}^{z_1} z\,dz = \frac{z_2^2 - z_1^2}{2},$$

since we have already proved the integral of $f(z) = z$ from z_1 to z_2, for any $z_1, z_2 \in \mathbf{C}$, is independent of path.

Theorem 13.8 (Fundamental Theorem of Contour Integrals)**.** *Suppose f is continuous on a domain G. The following are equivalent.*

(1) f has an antiderivative $F : G \to \mathbf{C}$.

(2) For all $z_1, z_2 \in G$, the integral of f from z_1 to z_2 are independent of path.

(3) If C is any closed contour lying in G, then

$$\int_C f(z)\,dz = 0$$

If any of these conditions hold, then the unique value of the integral in (2) is given as

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where F is the antiderivative given in (1).

13.1. Problems

To be added

References

[1] Brown, James Ward and Churchill, Ruel V.. *Complex Variables and Applications*. McGraw-Hill, 2009.

[2] Beck, Matthias; Marchesi, Gerald; Pixton, Dennis and Sabalka, Lucas. *A First Course in Complex Analysis*. Version 1.54. Available online.