LECTURE NOTES

MATH 103A — SPRING 2022

COMPLEX ANALYSIS

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adapted from

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(all errors introduced are my own)

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Contents

1. Lecture 1 (3/29)

What is Complex Analysis? The main object of study is a holomorphic function $f : G \to \mathbf{C}$, where $G \subseteq \mathbf{C}$. Namely, a function for which the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite on an open set; that is, a complex-differentiable function on an open set. As a set, $\mathbf{C} = \mathbf{R}^2$, so one can naively expect the theory to be similar to that of real analysis, in this case the behaviour of differentiable functions. Interestingly, the requirement of holomorphicity can yield results that have no counterpart in the real case.

A prime example of this is *Louiville's Theorem. Every bounded holomorphic function is constant.*

Discussion 1.1. We begin with first addressing the existence and nature of $\mathbf{C}$ itself. Let $\mathbf{R}$ denote the (field of) real numbers. One immediately deduces that the equation

$$x^2 + 1 = 0 \tag{$*$}$$

has no solution in the real numbers. The (field of) complex numbers $\mathbf{C}$ stems from our desire to find a set containing $\mathbf{R}$ that extends the algebraic operations of addition and multiplication of real numbers and which contains not only solutions to the polynomial equation above but solutions to all polynomial equations.

Surprisingly enough, the construction amounts to defining a symbol i that is a solution to $(*)$ and then considering all expressions of the form

$$x + iy, \quad x, y \in \mathbf{R}$$

Part I. Preliminaries

Construction of the (field of) Complex Numbers

Definition 1.2 (The set of Complex Numbers)**.** A complex number z is simply an order pair $z := (x, y)$ of real numbers. Thus, the set of all complex numbers is given by

$$\mathbf{C} := \mathbf{R}^2 = \{(x, y) \ : \ x, y \in \mathbf{R}\}$$

If $z = (x, y)$ is a complex number, then we call

$$\operatorname{Re} z := x \quad \text{and} \quad \operatorname{Im} z := y$$

the real and imaginary parts of z respectively.

Two complex numbers z_1 and z_2 are equal if and only if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$.

If $\operatorname{Re} z = 0$ and $\operatorname{Im} z \neq 0$, we say that z is purely imaginary. The set of purely imaginary complex numbers corresponds to the y-axis and is called the imaginary axis in $\mathbf{C}$.

Definition 1.3 (Binary Operations on **C**). Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be complex numbers. Then their *sum* is

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and their *product* is

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Proposition 1.4. *There exists a subset of* **C** *that is algebraically indistinguishable from* **R**.

Proof. Consider the set (the x-axis)

$$\mathbf{R} \times \{0\} = \{(x, 0) \ : \ x \in \mathbf{R}\} \subseteq \mathbf{C}.$$

There is a bijection

$$\phi : \mathbf{R} \to \mathbf{R} \times \{0\}\,,\ x \mapsto (x, 0).$$

Moreover,

$$\begin{aligned} \phi(x) + \phi(y) &= (x, 0) + (y, 0) = (x + y, 0) = \phi(x + y) \\ \phi(x) \cdot \phi(y) &= (x, 0) \cdot (y, 0) = (xy - 0 \cdot 0, x \cdot 0 + y \cdot 0) = (xy, 0) = \phi(xy) \end{aligned}$$

□

According to the proposition, the operations of addition and multiplication on complex numbers we have defined extend the operations of addition and multiplication of real numbers. We therefore call the x-axis, the real axis.

Discussion 1.5. We identify each complex number $(x, 0)$ with the corresponding real number x; more than that, abusing notation, we write

$$1 = (1, 0) \quad \text{and} \quad (x, 0) = x(1, 0) = x$$

Now, define the imaginary unit $i := (0, 1)$. Then

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0^2 - 1^2, 1 \cdot 0 + 1 \cdot 0) = (-1, 0) = -1.$$

Moreover, for any $z = (x, y) \in \mathbf{C}$ we see that

$$\begin{aligned} z &= (x, y) \\ &= (x, 0) + y(0, 1) = x + iy = \operatorname{Re} z + i \operatorname{Im} z \end{aligned}$$

Hence, with our new notation

$$\mathbf{C} = \left\{x + iy \ : \ x, y \in \mathbf{R},\ i^2 = -1\right\}$$

and

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \\ z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Although we have expanded the real numbers and we will see that the complex numbers have several new and familiar properties. We do end up losing one property of the real numbers when working with complex numbers: total ordering (that extends the one on **R** or is compatible with multiplication). In the world of complex numbers, it no longer makes sense to ask if $z_1 > z_2$ (see Problem 1.9).

In practice, the product of complex numbers can be computed by multiplying the expressions as if they were polynomials in the variable i, and using $i^2 = -1$. The fact that this works is left as Problem 1.3.

Example 1.6. Compute $(1+i)(1-3i)$.

Answer. We note

$$\begin{aligned}(1+i)(1-3i) &= (1-3i) + i(1-3i) \\ &= (1-3i) + (i - 3i^2) \\ &= (1-3i) + (i+3) = 4 - 2i\end{aligned}$$

□

Proposition 1.7 (Algebraic Properties of $(\mathbf{C}, +, \cdot\,)$)**.**

(1) Additive Identity. *For every $z \in \mathbf{C}$*

$$z + 0 = z = 0 + z$$

(2) Associativity of Addition. *For every triple $z_1, z_2, z_3 \in \mathbf{C}$*

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(3) Commutativity of Addition. *For every pair $z_1, z_2 \in \mathbf{C}$*

$$z_1 + z_2 = z_2 + z_1$$

(4) Additive Inverses. *For every $z \in \mathbf{C}$, there exists a complex number, denoted $-z$, such that*

$$z + (-z) = 0 = (-z) + z$$

In fact, $-z := (-1)z$, which is described in Problem 1.2.

(5) Multiplicative Identity. *For every $z \in \mathbf{C}$*

$$z \cdot 1 = z = 1 \cdot z$$

(6) Associativity of Multiplication. *For every triple $z_1, z_2, z_3 \in \mathbf{C}$*

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

(7) Commutativity of Multiplication. *For every pair $z_1, z_2 \in \mathbf{C}$*

$$z_1 \cdot z_2 = z_2 \cdot z_1$$

(8) Multiplicative Inverses. *For every $z \in \mathbf{C}^* := \mathbf{C} \setminus \{0\}$, there exists a complex number, denoted z^{-1} or $1/z$, such that*

$$z \cdot z^{-1} = 1 = z^{-1} \cdot z$$

In fact, if $z = x + iy$, then $z^{-1} = \dfrac{1}{z} := \dfrac{x}{x^2+y^2} - i\,\dfrac{y}{x^2+y^2}$.

(9) Distributive Law. *For every triple* $z_1, z_2, z_3 \in \mathbf{C}$

$$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$

Proof. (1) - (7) and (9) are left as Problem 1.4. One proves these directly by showing that the left hand side matches the right hand side.

(8) We note that

$$\begin{aligned}
z \cdot \frac{1}{z} &= (x+iy)\left(\frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}\right) \\
&= (x+iy)\left(\frac{x}{x^2+y^2} + i\,\frac{(-y)}{x^2+y^2}\right) \\
&= \left(x \cdot \frac{x}{x^2+y^2} - y \cdot \frac{(-y)}{x^2+y^2}\right) + i\left(x \cdot \frac{(-y)}{x^2+y^2} + y \cdot \frac{x}{x^2+y^2}\right) \\
&= \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right) + i\left(\frac{-yx+xy}{x^2+y^2}\right) \\
&= \frac{x^2+y^2}{x^2+y^2} + i \cdot 0 \\
&= 1
\end{aligned}$$

Of course, we should comment that $z = (x, y) \neq (0, 0)$ if and only if $x^2 + y^2 \neq 0$ (one proves this by stating and proving the contrapositive).

□

Remark 1.8. In the language of algebra,

- (1) – (4) tells us that $(\mathbf{C}, +)$ is an abelian group.
- (5) – (8) tells us that $(\mathbf{C}^*, \cdot)$ is an abelian group.
- (1) – (9) tells us that $(\mathbf{C}, +, \cdot)$ is a field.

Definition 1.9. Consider $z_1, z_2 \in \mathbf{C}$. We define *subtraction* and *division* as follows, respectively:

$$z_1 - z_2 := z_1 + (-z_2)$$

$$\frac{z_1}{z_2} := z_1 \cdot z_2^{-1} = z_1 \cdot \left(\frac{1}{z_2}\right), \quad z_2 \neq 0$$

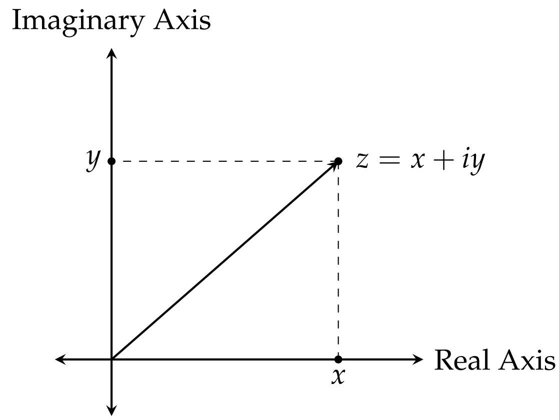
Writing down z_1/z_2 as $x + iy$ is not easy to remember, one obtains it by a method akin to "rationalising the denominator", in this case we could call it "realifiying the denominator"

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

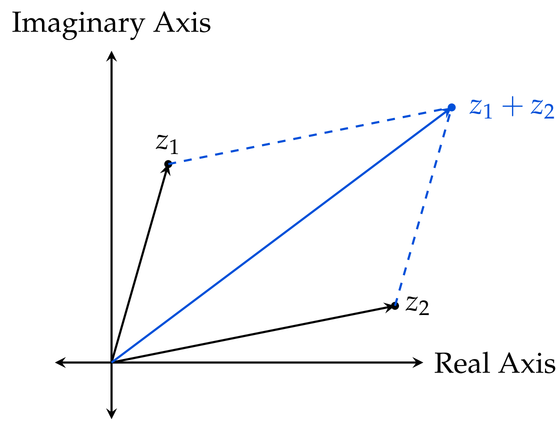
This method will be clarified soon when we talk about conjugates and absolute value.

Geometric Properties of Complex Numbers

As a set, we have $\mathbf{C} = \mathbf{R}^2$, so it's natural to visualise complex numbers as points in the complex plane (also called the Argand plane).



Geometrically, addition of complex numbers is just the addition of the corresponding vectors in the euclidean plane. We will soon see a geometric interpretation of multiplication.



Definition 1.10 (Modulus)**.** The modulus (or absolute value) of a complex number $z = x + iy$, denoted $|z|$, is the length of the vector (x, y), or equivalently its distance from the origin; namely

$$|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{x^2 + y^2} = \|(x, y)\|$$

Notice that this extends the usual absolute value of real numbers, as the modulus of a real number is its absolute value.

We can then immediately derive a useful inequality,

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geqslant (\operatorname{Re} z)^2,\ (\operatorname{Im} z)^2,$$

giving us

$$\operatorname{Re} z \leqslant |\operatorname{Re} z| \leqslant |z| \quad \text{and} \quad \operatorname{Im} z \leqslant |\operatorname{Im} z| \leqslant |z|\,.$$

Definition 1.11 (Distance)**.** The distance between two complex numbers z_1 and z_2 is

$$|z_1 - z_2| = \|(x_1, y_1) - (x_2, y_2)\| = \|(x_1 - x_2, y_1 - y_2)\|$$

That is, it's the euclidean distance between the vectors representing these complex numbers.

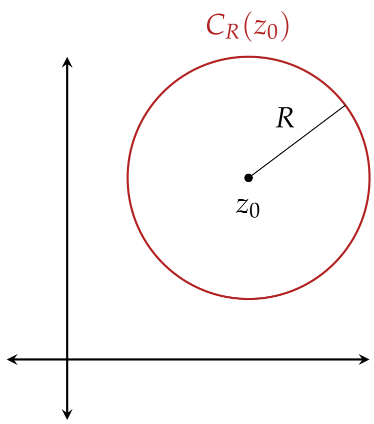
Discussion 1.12. The absolute value can be used to define various important subsets of $\mathbf{C}$.

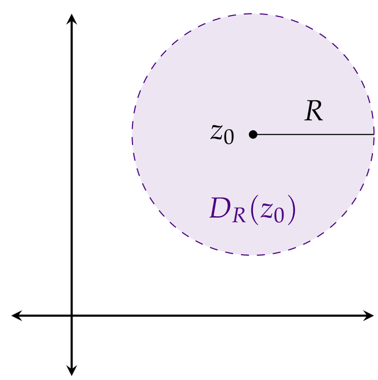
(1) • The *circle of radius $R > 0$ centered at z_0* is the set

$$C_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| = R\}$$

• The *open disk (or ball) of radius $R > 0$ centered at z_0* is the set

$$D_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| < R\}$$



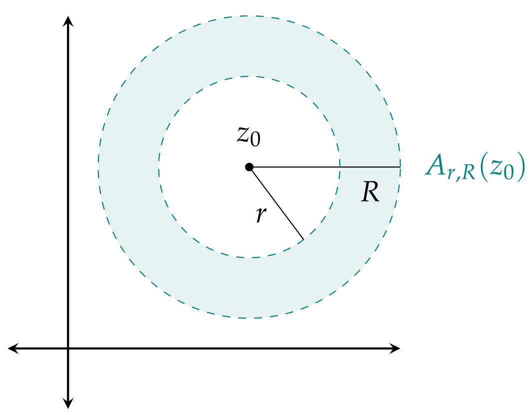


• The *closed disk (or ball) of radius $R > 0$ centered at z_0* is the set

$$\overline{D}_R(z_0) = \{z \in \mathbf{C} \,:\, |z - z_0| \leqslant R\} = D_R(z_0) \cup C_R(z_0).$$

(2) The *(open) annulus of inner radius $r > 0$ and outer radius $R > 0$ centered at z_0* is the set

$$A_{r,R}(z_0) = \{z \in \mathbf{C} \,:\, r < |z - z_0| < R\}$$



1.1. Problems

Problem 1.1. Consider the set of matrices

$$X := \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \; : \; x, y \in \mathbf{R} \right\}.$$

One can check (and you should if you're unconvinced) straightforwardly that X is closed under matrix addition and matrix multiplication; that is, if $A, B \in X$, then $A + B,\ AB \in X$.

(a) Let $\mathbf{C}$ denote the set of complex numbers. Show that the map $\phi : X \to \mathbf{C}$ defined by

$$\phi : X \to \mathbf{C}, \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x + iy$$

is a bijection.

(b) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the identity matrix. Consider $A, B \in X$, show that ϕ has the following properties.

  (i) $\phi(A + B) = \phi(A) + \phi(B)$

  (ii) $\phi(AB) = \phi(A)\phi(B)$

  (iii) $\phi(I) = 1$

(c) Find a matrix J satisfying $J^2 = -I$ and show that $\phi(J) = i$.

Remark 1.13. This indicates that one could very well define $\mathbf{C}$ to be X. The algebraic operations on $\mathbf{C}$ then seem less artificial, since product and sum of complex numbers correspond to the corresponding operations of matrices. Even taking the inverse and modulus is captured by X as taking inverse and the determinant of matrices. The copy of $\mathbf{R}$ corresponds to the set of diagonal matrices in X. One obtains X by considering the linear operator of multiplying by $x + iy$ on the $\mathbf{R}$-vector space $\mathbf{C}$ with basis 1 and i.

Problem 1.2. Using the definition of complex multiplication prove that

$$(a, 0) \cdot (x, y) = (ax, ay).$$

That is, $a(x + iy) = ax + iay$.

Problem 1.3. Consider complex numbers $z_1 = (x_1, y_1) = x_1(1, 0) + y_1(0, 1)$ and $z_2 = (x_2, y_2) = x_2(1, 0) + y_2(0, 1)$. Using the identity $(0, 1)^2 = (-1, 0)$. Prove that

$$(x_1(1, 0) + y_1(0, 1)) \cdot (x_2(1, 0) + y_2(0, 1)) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1),$$

where the former is computed distributively.

Problem 1.4. Prove properties (1) - (7) and (9) listed in Proposition 1.7.

Problem 1.5. Prove that if $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

Problem 1.6. Show that

(a) $\operatorname{Re} iz = -\operatorname{Im} z$;

(b) $\operatorname{Im} iz = \operatorname{Re} z$

Problem 1.7.

(a) Verify that $z = 1 \pm i$ satisfies the equation

$$z^2 - 2z + 2 = 0.$$

(b) Solve the equation

$$z^2 + z + 1 = 0$$

for $z = x + iy$ by solving a pair of simultaneous equations in x and y.

Problem 1.8. Let $p(z) = az^2 + bz + c$ be a polynomial with complex coefficients ($a \neq 0$).

(a) By completing the square, show that the solution to $p(z) = 0$ is

$$z = \frac{-b \pm \Delta^{1/2}}{2a},$$

where $\Delta := b^2 - 4ac$ is called the discriminant.

Remark. There's a subtlety with taking roots that we will address later in class.

(b) Consider the polynomial $p(z) = iz^2 - 1$

(i) Compute Δ.

(ii) For the Δ obtained in (b), compute $\Delta^{1/2}$ by solving a pair of simultaneous equations in x and y obtained by considering the equation

$$x^2 - y^2 + 2ixy = (x + iy)^2 = \Delta.$$

(iii) Finally, write down the roots of $p(z)$ in the form $u + iv$.

Problem 1.9. Suppose **C** had total ordering that extends the ordering on **R**, arrive at a contradiction by comparing i and 0.

Problem 1.10. Locate the numbers $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$ in the complex plane when

(a) $z_1 = 2i$, $z_2 = \frac{2}{3} - i$.

(b) $z_1 = (-3, 1)$, $z_2 = (1, 4)$.

(c) $z_1 = (-\sqrt{3}, 1)$, $z_2 = (\sqrt{3}, 0)$.

(d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$.

Problem 1.11. Verify that $\sqrt{2}\,|z| \geqslant |\operatorname{Re} z| + |\operatorname{Im} z|$.

Problem 1.12. Let $z_0 \neq z_1 \in \mathbf{C}$ and let $\lambda > 0$.

(a) Show that if $\lambda \neq 1$, then the set of points

$$|z - z_0| = \lambda\,|z - z_1| \tag{$\bigstar$}$$

is a circle of radius $R = \frac{\lambda}{|1-\lambda^2|}\,|z_0 - z_1|$ centered at $w = \frac{z_0 - \lambda^2 z_1}{1 - \lambda^2}$.

(b) Show that every circle in the complex plane can be written in the form of ($\bigstar$) for some $\lambda > 0$, $\lambda \neq 1$ and $z_0 \neq z_1 \in \mathbf{C}$.

(c) If $\lambda = 1$, show that ($\bigstar$) defines a line. In fact, argue that the resulting line is perpendicular to and bisects the line segment joining z_0 and z_1, by producing the equation of this line as a subset of $\mathbf{R}^2$.

(d) Characterise points on the real (resp. imaginary) axis using (c). That is, find $z_0 \neq z_1 \in \mathbf{C}$ such that the points on the real (resp. imaginary) axis satisfy ($\bigstar$) for $\lambda = 1$.

(e) Consider the map

$$M(z) = \frac{z-3}{1-2z}.$$

For which values of $c \in \mathbf{R}$ is the image of the circle $|z - 1| = c$ under M a line? What is the equation of the line when considered as a subset of the plane $\mathbf{R}^2$?

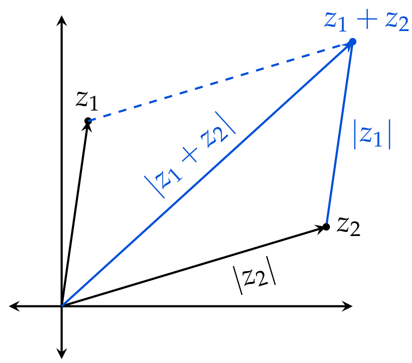
2. Lecture 2 (3/31)

Proposition 2.1 (Triangle Inequalities)**.** *For all $z_1, z_2 \in \mathbf{C}$, the following inequalities hold.*

(1) $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

(2) $|z_1 \pm z_2| \geqslant ||z_1| - |z_2||$. *We sometimes refer to this inequality as the* reverse triangle inequality*.*

Proof.



(1) A standard fact about triangles.

(2) We first assume that $|z_1| \geqslant |z_2|$. Then, $||z_1| - |z_2|| = |z_1| - |z_2|$. Now, note that

$$\begin{aligned} |z_1| - |z_2| &= |z_1 \pm z_2 \mp z_2| - |z_2| \\ &\leqslant |z_1 \pm z_2| + |\mp z_2| - |z_2|, \text{ triangle inequality} \\ &= |z_1 \pm z_2| + |z_2| - |z_2| \\ &= |z_1 \pm z_2| \end{aligned}$$

If we instead assumer $|z_2| \geqslant |z_1|$, then we do the same computation with the roles of z_1 and z_2 switched.

□

Proposition 2.2 (Modulus is Multiplicative)**.** *For all $z, w \in \mathbf{C}$ and positive integers n,*

(1) $|zw| = |z|\,|w|$.

(2) $|z^n| = |z|^n$.

Proof.

(1) Left as Problem 2.1. One proves these directly by showing that the left hand side matches the right hand side.

(2) The proof of this is by induction. $n = 1$ is a tautology, and $n = 2$ is (1) in the case $w = z$. Assume the statement is true for $n = k$, that is $\left|z^k\right| = |z|^k$. Then, for $n = k + 1$

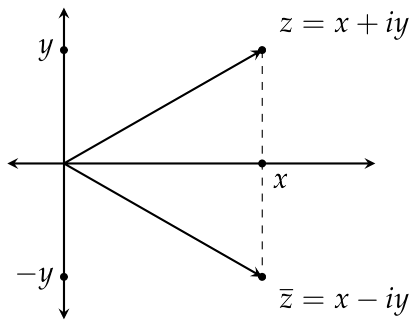
$$\begin{aligned} |z^{k+1}| = |z^k \cdot z| &= |z^k|\,|z|, \quad \text{using (1)} \\ &= |z|^k\,|z|, \quad \text{using the induction hypothesis} \\ &= |z|^{k+1} \end{aligned}$$

Therefore we have the result by the principle of mathematical induction. $\square$

Definition 2.3 (Complex Conjugation)**.** Given a complex number $z = x + iy$, its (complex) conjugate, denoted $\overline{z}$, is

$$\overline{z} := x - iy$$

Geometrically, $\overline{z}$ is the reflection of z about the real axis.



Proposition 2.4 (Properties of Conjugation)**.** *For all pairs $z, w \in \mathbf{C}$, we have*

(1) $\overline{\overline{z}} = z$

(2) $|\overline{z}| = |z|$

(3) $\overline{z + w} = \overline{z} + \overline{w}$

(4) $\overline{zw} = \overline{z}\,\overline{w}$

(5) $z\overline{z} = |z|^2$

(6) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(7) $z \in \mathbf{R}$ *if and only if* $z = \overline{z}$

Proof. (1) – (3) is clear geometrically. (4), (6) and (7) are left as Problem 2.2, (7) can be proved using (6) and can also be deduced geometrically. One proves these directly by showing that the left hand side matches the right hand side.

(5) Let $z = x + iy$, then

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - ixy + iyx - i^2y^2 \\ &= x^2 + y^2 + i(yx - xy) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

$\square$

Discussion 2.5. Proposition 2.4 (5) gives us a nice formula for z^{-1} for $z \in \mathbf{C}^*$. For such a z, we have $z\overline{z} = |z|^2$, which gives us

$$z^{-1} = z^{-1} \cdot \frac{z\overline{z}}{|z|^2} = \frac{\overline{z}}{|z|^2}$$

This tells us that z^{-1} is just a scaled $\overline{z}$, which means, geometrically speaking, z^{-1} lies on the line passing through the origin and $\overline{z}$.

Recall that every non-zero point $(x, y) \in \mathbf{R}^2$ can be re-written in polar coordinates (r, θ) as

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

This suggests the following definition.

Definition 2.6 (Polar Form)**.** If (r, θ) are polar coordinates for a non-zero (x, y), then the polar form of a non-zero complex number $z = x + iy$ is

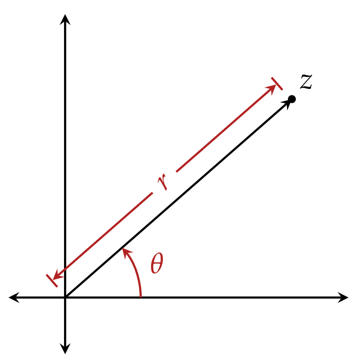
$$z = r(\cos\theta + i\sin\theta)$$

We sometimes abbreviate $\cos\theta + i\sin\theta$ as $\operatorname{cis}\theta$, so $z = r\operatorname{cis}\theta$.

Evidently, (r, θ) are related to (x, y) by the equations

$$|z| = r \quad \text{and} \quad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}},\ \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}},\ \text{so}\ \tan\theta = \frac{y}{x}$$

We have to be careful and take into account which quadrant (x, y) belongs to, if we think of θ with respect to its formulation using tan.



Since sin and cos are periodic functions, θ is not unique (you can replace θ with $\theta + 2\pi$). Each possible value of θ is called an argument of z, and the set of all such θ is denoted as $\arg z$. That is,

$$\arg z = \{\arctan(y/x) + 2k\pi \ : \ k \in \mathbf{Z}\}$$

The polar form, specifically θ is unique, as soon as we specify bounds on θ. The unique argument in the interval $(-\pi, \pi]$ is called the principal argument denoted $\operatorname{Arg} z$.

Notice that we can then write

$$\arg z = \{\operatorname{Arg} z + 2k\pi \ : \ k \in \mathbf{Z}\}$$

Definition 2.7 (Euler's Formula). $e^{i\theta} := \operatorname{cis}\theta = \cos\theta + i\sin\theta$. Therefore $\left|e^{i\theta}\right| = 1$.

Remark on Definition 2.7. This is for now a stopgap, defining $e^{i\theta}$ in this way. In a few weeks, we'll see that this is truly an equality of holomorphic functions. Euler deduced this by looking at the Taylor series expansion of these functions. We haven't built or discussed enough machinery to give this reasoning a solid foundation yet.

Using Euler's formula, one can write the polar form of a non-zero complex number, even more succinctly in its exponential form

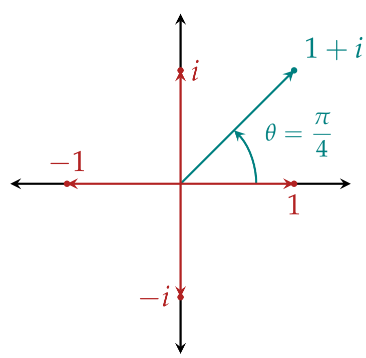
$$z = re^{i\theta}$$

Example 2.8.

(1) Exponential form of $1+i$,

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2} \quad \text{and} \quad \operatorname{Arg} z = \arctan(1) = \frac{\pi}{4}$$

So, $1+i = \sqrt{2}e^{i\pi/4}$.



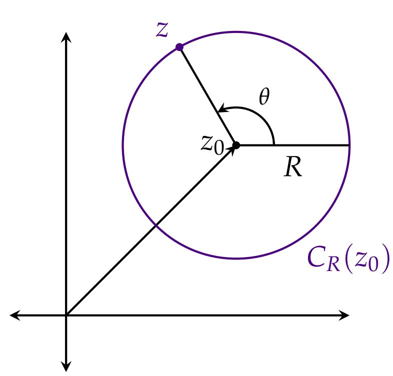
(2) Note that

$$1 = e^{i0} = e^{i2n\pi} \text{ for any } n \in \mathbf{Z}, \qquad i = e^{i\pi/2}, \qquad -1 = e^{i\pi} = e^{i(2n+1)\pi} \text{ for any } n \in \mathbf{Z}$$

One could write $-i = e^{i3\pi/2}$ but $3\pi/2 \neq \operatorname{Arg}(-i)$; instead we should write $-i = e^{-i\pi/2}$.

(3) The circle $C_R(z_0)$ has a nice parametrisation

$$C_R(z_0) = \{z = z_0 + Re^{i\theta} \;:\; 0 \leqslant \theta < 2\pi\}$$



Proposition 2.9 (Properties of Exponential Form). *Let $z = re^{i\theta}$ and $w = se^{i\phi}$ be non-zero complex numbers. Then*

(1) $zw = rs\, e^{i(\theta+\phi)}$

(2) $z^{-1} = (1/r)e^{-i\theta}$

(3) $z^n = r^n e^{in\theta}$, *for any* $n \in \mathbf{Z}$

(4) $\bar{z} = re^{-i\theta}$

(5) $z/w = (r/s)e^{i(\theta-\phi)}$

Proof.

(1) Note that

$$\begin{aligned} zw = (re^{i\theta})(se^{i\phi}) &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\ &= rs\, e^{i(\theta+\phi)} \end{aligned}$$

(2) It suffices to show that $(re^{i\theta})((1/r)e^{-i\theta}) = 1$, for which we use (1).

(3) We first prove this result for $n \geqslant 0$, the result is clear for $n = 0$ and $n = 1$. Assume the result is true for $n = k$, that is $z^k = r^k e^{ik\theta}$. Then, for $n = k+1$

$$\begin{aligned} z^{k+1} &= z^k z \\ &= (r^k e^{ik\theta})(re^{i\theta}) \text{ using the induction hypothesis} \\ &= r^{k+1} e^{ik\theta+\theta} \text{ by (1)} \\ &= r^{k+1} e^{i(k+1)\theta} \end{aligned}$$

Therefore we have the result by the principle of mathematical induction.

Suppose $n < 0$ instead, then write $n = -m$ for a positive $m > 0$. Now, we can apply the first case to $z^n := (z^{-1})^m$ to get our result.

(4) Using $z\bar{z} = |z|^2 = r^2$, we get that $\bar{z} = r^2 z^{-1}$, and the result follows from (2).

(5) Recall $z/w = zw^{-1}$, and the result follows from (2) and (1). □

Example 2.10. Let's use this to compute $(1+i)^{2021}$, then

$$\begin{aligned} (1+i)^{2021} &= (\sqrt{2}e^{i\pi/4})^{2021} \\ &= (\sqrt{2}e^{i\pi/4})^{2020}(\sqrt{2}e^{i\pi/4}) \\ &= (\sqrt{2})^{2020}(e^{i2020\pi/4})(1+i) \\ &= 2^{1010}(e^{i505\pi})(1+i) \\ &= -2^{1010}(1+i) \end{aligned}$$

Example 2.11 (in-class)**.** Compute $(1+i\sqrt{3})^{101}$.

Answer. We first note that $|1+i\sqrt{3}| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$, and since $1+i\sqrt{3}$ lies in the first quadrant of the complex plane

$$\operatorname{Arg} z = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

Therefore

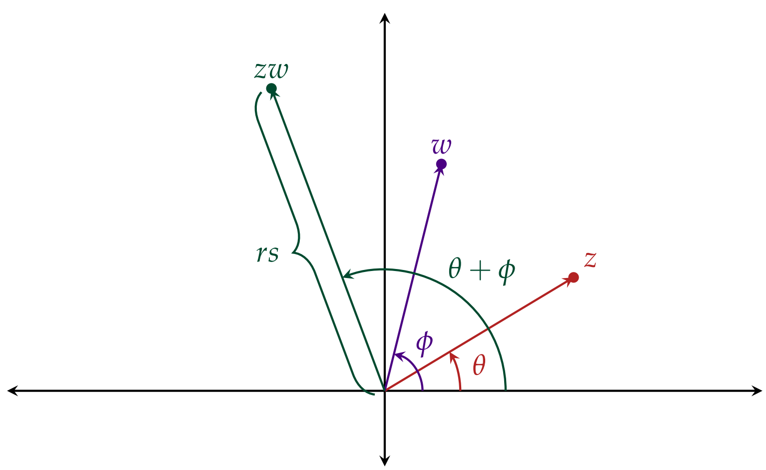
$$1+i\sqrt{3} = 2e^{i\pi/3}$$

and so

$$\begin{aligned}(1+i\sqrt{3})^{101} &= (2e^{i\pi/3})^{101}\\ &= (2e^{i\pi/3})^{99}(2e^{i\pi/3})^2\\ &= 2^{99}e^{i33\pi}(1+i\sqrt{3})^2\\ &= -2^{99}(1-3+2i\sqrt{3}), \quad \text{since 33 is odd}\\ &= -2^{99}(-2+2i\sqrt{3})\\ &= 2^{100}(1-i\sqrt{3})\end{aligned}$$

□

Discussion 2.12. Proposition 2.9 (1) gives us a nice geometric interpretation of complex multiplication. If $z = re^{i\theta}$ and $w = se^{i\phi}$, then $zw = rs\, e^{i(\theta+\phi)}$. This can be interpreted as saying that zw is obtained from w by scaling w by $|z| = r$ and rotating w by an angle of $\operatorname{Arg} z$ (or vice versa).



A few more interesting consequences

(1) The *unit circle*

$$S^1 = \{z \in \mathbf{C} \ : \ |z| = 1\} = \{e^{i\theta} \ : \ \theta \in \mathbf{R}\}$$

is closed under multiplication. It's in fact an abelian group, usually denoted $U(1)$.

(2) *De Moivre's Theorem*. From Proposition 2.9 (4) applied to $z = e^{i\theta}$ we get

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

2.1. Problems

Problem 2.1. Prove Proposition 2.2 (1).

Problem 2.2. Prove the properties, other than (5), listed in Proposition 2.4.

Problem 2.3. Prove that z is either real or pure imaginary if and only if $z^2 = \overline{z}^2$.

Problem 2.4. Prove that $|z| = 1$ if and only if $\overline{z} = \dfrac{1}{z}$.

Problem 2.5. Follow the steps below to give an algebraic derivation of the triangle inequality (Proposition 2.1 (a))

(a) Show that
$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = z_1\overline{z}_1 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + z_2\overline{z}_2.$$

(b) Argue why
$$z_1\overline{z}_2 + \overline{z_1\overline{z}_2} = 2\,\mathrm{Re}(z_1\overline{z}_2) \leqslant 2\,|z_1|\,|z_2|\,.$$

(c) Use (a) and (b) to obtain $|z_1 + z_2|^2 \leqslant (|z_1| + |z_2|)^2$. Finally note how the triangle inequality follows from this.

Problem 2.6. Let $z, w \in \mathbf{C}$.

(a) Prove the formula
$$|z + w|^2 = |z|^2 + 2\,\mathrm{Re}\, z\overline{w} + |w|^2$$

(b) Use (a) to deduce the *parallelogram law*
$$|z + w|^2 + |z - w|^2 = 2\,|z|^2 + 2\,|w|^2$$

Give a geometric interpretation of this formula.

Problem 2.7. Suppose p is a polynomial with *real coefficients*. Prove that

(a) $\overline{p(z)} = p(\overline{z})$.

(b) $p(z) = 0$ if and only if $p(\overline{z}) = 0$.

Problem 2.8. Find the principal argument $\mathrm{Arg}\, z$ when

(a) $-i(3 + 3i)^{-1}$.

(b) $(1 - i\sqrt{3})^6$.

3. Lecture 3 (4/05)

Proposition 3.1 (Arguments of Products). *Let z, w be non-zero complex numbers, then*

(1) $\arg(zw) = \arg z + \arg w$

(2) $\arg w^{-1} = -\arg w$

Note that this is *not* saying $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$, this is actually not true, we're claiming an equality of sets. (1) and (2) together give us $\arg(z/w) = \arg z - \arg w$.

Proof.

(1) Consider $\theta \in \arg z$ and $\phi \in \arg w$, so $z = re^{i\theta}$ and $w = se^{i\phi}$. By Proposition 2.9 (1), we have $zw = rs\, e^{i(\theta+\phi)}$ and therefore $\theta + \phi \in \arg(zw)$. Hence $\arg z + \arg w \subseteq \arg(z + w)$.

Consider $\psi \in \arg(z + w)$, and some $\theta \in \arg z$ then we claim that $\psi - \theta \in \arg w$. We have $rs\, e^{i\psi} = zw = re^{i\theta} w$, then by Proposition 2.9 (5), we get $w = sr^{i(\psi-\theta)}$. Hence $\psi - \theta \in \arg w$, and since $\psi = \theta + (\psi - \theta) \in \arg z + \arg w$, we have $\arg(z + w) \subseteq \arg z + \arg w$.

Therefore $\arg(zw) = \arg z + \arg w$.

(2) Consider $\theta \in \arg z$, so $z = re^{i\theta}$. By Proposition 2.9 (2), we have $z^{-1} = (1/r)e^{i(-\theta)}$ and therefore $-\theta \in \arg w^{-1}$. Hence $-\arg w \subseteq \arg w^{-1}$.

Note that $w = (w^{-1})^{-1}$, applying the above result to w^{-1} gets us $-\arg w^{-1} \subseteq \arg(w^{-1})^{-1} = \arg w$ and so $\arg w^{-1} \subseteq -\arg w$.

Therefore $\arg w^{-1} = -\arg w$. □

Remark 3.2. For a complex number, $\arg z$ is a set of all possible θ's such that we can write $z = |z|\, e^{i\theta}$, as you know. Therefore, we will abuse notation by sometimes calling any $\theta \in \arg z$ as an argument of z, and sometimes also writing $z = |z|\, e^{i \arg z}$. That is, we are not, or are careless about, distinguishing the set $\arg z$ and its element when we can be agnostic about the choice of θ; for example, the polar form of a complex number. It will be clear when we choose to care about out choice, it will be evident because we'll be then forcing θ to lie in an interval of length 2π; for example, the principal argument $-\pi < \operatorname{Arg} z \leqslant \pi$.

Example 3.3.

(1) The principal argument of $z = (\sqrt{3} - i)^6$. We first note that $\operatorname{Arg}(\sqrt{3} - i) = -\pi/6$. By Proposition 3.1 (1), applied inductively, we have

$$\arg(\sqrt{3} - i)^6 = \underbrace{\arg(\sqrt{3} - i) + \cdots + \arg(\sqrt{3} - i)}_{\text{6 times}} = \{-\pi + 2k\pi \ : \ k \in \mathbf{Z}\}$$

Then $\operatorname{Arg}(\sqrt{3} - i)^6$ is the element in the set above in the interval $(-\pi, \pi]$ which is π.

(2) As mentioned previously, we can't just replace arg with Arg in the statement of Proposition 3.1 (1). Here's a simple example: let $z = w = -1$, then $\operatorname{Arg} z = \operatorname{Arg} w = \pi$ and $\operatorname{Arg} zw = \operatorname{Arg} 1 = 0$ but $0 \neq 2\pi = \operatorname{Arg} z + \operatorname{Arg} w$.

(3) Note that $\arg z + \arg z \neq 2 \arg z$.

Roots of Complex Numbers

Lemma 3.4. *Two non-zero complex numbers z, w are equal if and only if $|z| = |w|$ and $\arg z = \arg w$.*

Proof. If $|z| = |w|$ and $\arg z = \arg w$, then clearly $z = w$.

Suppose $z = w$, then we immediately get $|z| = |w|$. Consider $\theta \in \arg z$ and $\phi \in \arg w$, then we get $e^{i\theta} = e^{i\phi}$ which is equivalent to saying $\cos(\theta - \phi) + i\sin(\theta - \phi) = e^{i(\theta-\phi)} = 1$. This gives us

$$\sin(\theta - \phi) = 0.$$

The solution to this is $\theta - \phi = 2k\pi$ for some $k \in \mathbf{Z}$. This gives us $\arg z = \arg w$. □

Definition 3.5 (Roots)**.** Let α be a non-zero complex number. An n^{th} *root of* α is a solution to the polynomial equation $z^n - \alpha = 0$.

The set of all n^{th} roots of α is denoted by $\alpha^{1/n}$, we reserve the symbol $\sqrt[n]{\cdot}$ for the unique positive n^{th} root of a positive real number.

Proposition 3.6 (Distinct Roots)**.** *There are precisely n distinct n^{th} roots of α, namely*

$$\beta_k = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, \ldots, n-1$$

Proof. Let $z = re^{i\theta}$ and $\alpha = |\alpha|\, e^{i \operatorname{Arg}\alpha}$, we solve

$$r^n e^{in\theta} = z^n = \alpha = |\alpha|\, e^{i\operatorname{Arg}\alpha}.$$

By Lemma 3.4, this equality is true if and only if $r^n = |\alpha|$ and $n\theta = \operatorname{Arg}\alpha + 2k\pi$ for some $k \in \mathbf{Z}$. Therefore

$$z = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbf{Z}$$

We obtain distinct n complex numbers for $k = 0, \ldots, n-1$ since they have distinct arguments, and they necessarily give us the n distinct n^{th} roots of α. □

Discussion 3.7. With the notation of Proposition 3.6, the n^{th} principal root of α is

$$\beta_0 = \sqrt[n]{|\alpha|}\, e^{i\frac{\operatorname{Arg}\alpha}{n}}$$

If we introduce the notation $\zeta_n = e^{\frac{2\pi i}{n}}$, then

$$\zeta_n^k = e^{\frac{2k\pi i}{n}}$$

According to the proposition, the complex numbers

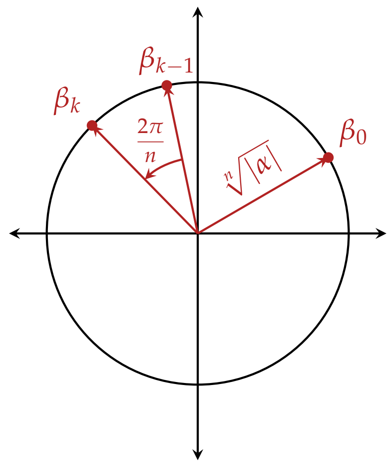
$$1,\ \zeta_n,\ \zeta_n^2, \ldots, \zeta_n^{n-1}$$

are the distinct solutions to $z^n - 1 = 0$, the n^{th} roots of unity, making ζ_n the *principal n^{th} root of unity*.

Then we can write the roots of α in terms of the principal root and the roots of unity

$$\beta_k = \sqrt[n]{|\alpha|}\, e^{i\left(\frac{\operatorname{Arg}\alpha}{n}+\frac{2k\pi}{n}\right)} = \sqrt[n]{|\alpha|}\, e^{i\frac{\operatorname{Arg}\alpha}{n}} e^{\frac{2k\pi i}{n}} = \beta_0 \zeta_n^k$$

That is, β_k's all lie on the circle of radius $\sqrt[n]{|\alpha|}$ centered at the origin, and all of them are obtained by rotating β_0 by an angle of $2k\pi/n$. That is, they all lie on the vertices of an inscribed regular n-gon.

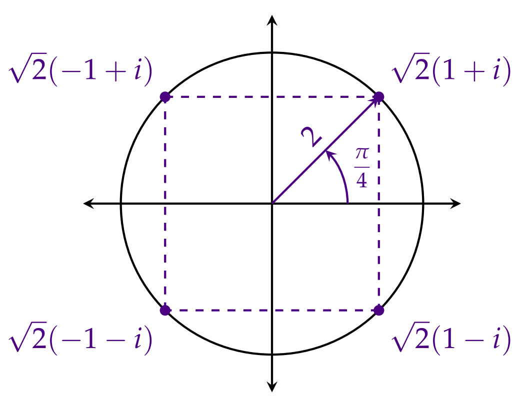


Example 3.8.

(1) We compute explicitly the 4^{th} roots of $\alpha = -16$. As a negative real number, $\operatorname{Arg}(-16) = \pi$, so

$$\begin{aligned}
\beta_k &= \sqrt[4]{16} e^{i\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)} = 2\, e^{i\frac{\pi}{4}} e^{\frac{ki\pi}{2}} \\
&= 2\, e^{i\frac{\pi}{4}} \left(e^{\frac{i\pi}{2}}\right)^k \\
&= 2\, \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = 2\, \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) i^k = \sqrt{2}(1+i) i^k
\end{aligned}$$

Therefore



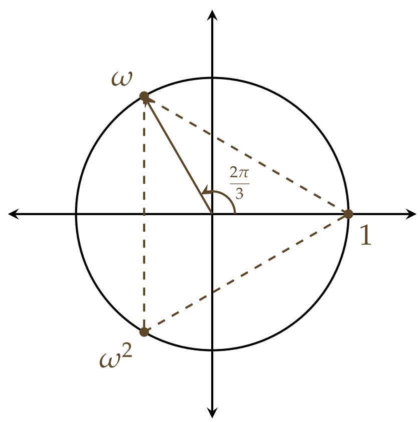
$$\beta_0 = \sqrt{2}(1+i), \quad \beta_1 = \sqrt{2}(-1+i), \quad \beta_2 = \sqrt{2}(-1-i), \quad \beta_3 = \sqrt{2}(1-i)$$

(2) In the course of the previous example, we have computed the 4^{th} roots of unity, since they are

$$e^{\frac{2ki\pi}{4}} = e^{\frac{ki\pi}{2}}, \quad k = 0, 1, 2, 3$$

as $\operatorname{Arg} 1 = 0$. Letting $\zeta_4 = e^{i\pi/2} = i$, the 4^{th} roots of unity are $\zeta_4^0, \zeta_4^1, \zeta_4^2, \zeta_4^3$, which are nothing but $\pm 1, \pm i$.

Example 3.9 (in-class)**.** Compute the 3^{rd} roots of unity, also called the cube roots of unity where we denote $\omega = \zeta_3$, explicitly.



Answer. Let the principal root be $\omega = \zeta_3$, then the cube roots of unity are

$$1,\ \omega,\ \omega^2$$

where we have

$$\omega = e^{\frac{2\pi i}{3}} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega^2 = e^{\frac{4\pi i}{3}} = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

□

Basic Topology of $\mathbf{C}$

Our purpose now is to define the kind of subsets of $\mathbf{C}$ that are suitable for doing complex analysis, namely *non-empty open connected sets*.

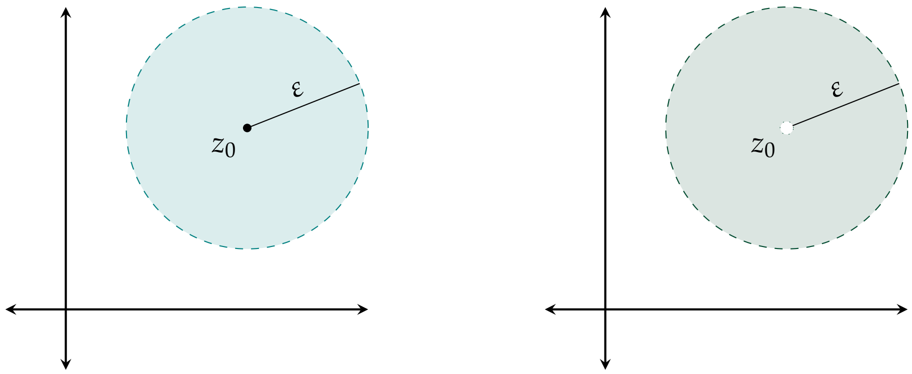
Definition 3.10 (Open Disks or Neighbourhoods)**.** Let $\varepsilon > 0$. Recall the open disk (of radius ε centered at z_0) is the set

$$D_\varepsilon(z_0) = \{z \in \mathbf{C} \ : \ |z - z_0| < \varepsilon\}.$$

We also refer to such an open disk as an ε-neighbourhood or simply a neighbourhood.

A deleted (or punctured) open disk (or neighbourhood) is a set of the form

$$D_\varepsilon(z_0) \setminus \{z_0\} = \{z \in \mathbf{C} \ : \ 0 < |z - z_0| < \varepsilon\}.$$



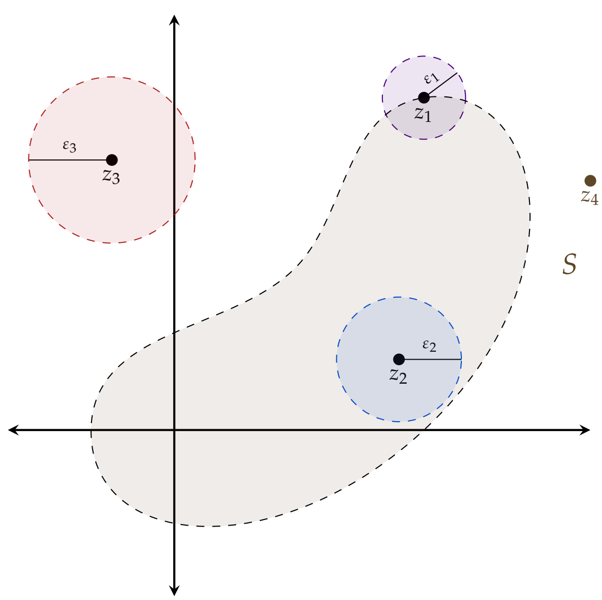
Points belonging to the same ε-neighbourhood are considered "close" to each other, in the sense that they are within a distance of 2ε from each other.

Definition 3.11 (Various kinds of Points)**.** Consider a $S \subseteq \mathbf{C}$.

- A point $z \in S$ is an interior point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \subseteq S$.
- A point $z \notin S$ is an exterior point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \cap S = \varnothing$.
- A point $z \in \mathbf{C}$ is a boundary point of S if it's neither an interior nor an exterior point of S. Equivalently, if every neighbourhood of z contains both a point in S and not in S.
- A point $z \in \mathbf{C}$ is a accumulation (or cluster) point of S if for every $\varepsilon > 0$ we have

$$D_\varepsilon(z) \setminus \{z\} \cap S \neq \varnothing.$$

- A point $z \in S$ is an isolated point of S if there exists an $\varepsilon > 0$ such that $D_\varepsilon(z) \setminus \{z\} \cap S = \varnothing$. Isolated points are examples of boundary point



Here z_1 is a boundary point, z_2 an interior point, z_3 an exterior point, and z_4 is an isolated point (and a boundary point).

Remark 3.12. The idea is that if we don't move too far from an interior point of S then we remain in S; a similar idea holds for an exterior point. But at a boundary point we can make an arbitrarily small move and get to a point inside S, and we can also make an arbitrarily small move and get to a point outside S. An accumulation point is one where it has other points from S within any arbitrarily small distance, i.e. points "accumulate" near it; an isolated point is the exact opposite.

3.1. Problems

Problem 3.1. Prove that

$$\arg z + \arg w = \{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \ : \ k \in \mathbf{Z}\}$$

Combining this with Proposition 3.1 we get that $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2k\pi$ for some $k \in \mathbf{Z}$ such that $-\pi < \operatorname{Arg} z + \operatorname{Arg} w + 2k\pi \leqslant \pi$. That is, to find $\operatorname{Arg} zw$, just add $\operatorname{Arg} z$ and $\operatorname{Arg} w$ and then add or subtract a suitable multiple of 2π to get it between $-\pi$ and π.

Problem 3.2. Prove that for any complex number z, we have $\operatorname{Arg} \overline{z} = \operatorname{Arg} z^{-1} = -\operatorname{Arg} z$.

Problem 3.3.

(a) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

(b) Show that if $\operatorname{Re} z > 0$, then $\operatorname{Arg}(-z) = -\pi + \operatorname{Arg} z$ if $\operatorname{Im} z > 0$ or $\operatorname{Arg}(-z) = \pi + \operatorname{Arg} z$ if $\operatorname{Im} z < 0$.

(c) Using (a) and (b), find an expression for $\operatorname{Arg} zw$ for any non-zero complex numbers z and w, in terms of $\operatorname{Arg} z$, $\operatorname{Arg} w$ and specific multiples of π.

Problem 3.4. Compute the 6^{th} roots of unity, explicitly. Show that the principal 6^{th} root of unity is $\zeta_6 = -\omega$, where ω is as in Example 3.9.

Problem 3.5.

(a) Let $z \in \mathbf{C}$. Using the principle of mathematical induction, show that the following formula holds for all integers $n \geqslant 1$

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) Use (a) to derive *Lagrange's Trigonometric Identity*.

$$1 + \cos\theta + \cos^2\theta + \cdots + \cos^n\theta = \frac{2\sin((2n+1)\theta/2)}{2\sin(\theta/2)}, \quad 0 < \theta < 2\pi.$$

(c) If $\zeta_1, \ldots, \zeta_n$ are the *distinct* n^{th} roots of unity, show that, using (a), $\sum_{i=1}^{n} \zeta_i = 0$.

(d) We compute the following sum of real numbers

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} \tag{†}$$

(i) Let $w = e^{\frac{\pi i}{7}}$. What is $\operatorname{Re} w$ and w^7? Furthermore, rewrite (†) as

$$\operatorname{Re}(w^{a_1} + w^{a_2} + w^{a_3}), \quad \text{for some } 0 \leqslant a_i < 7.$$

(ii) Replacing z by $-z$ in (a), find a formula for

$$\frac{z^7+1}{z+1}.$$

Use this to deduce an identity involving w and its powers.

(iii) Using the identity you found in (iii), conclude that

$$w^{a_1} + w^{a_2} + w^{a_3} = \frac{1}{1-w}$$

where the a_i's are the numbers you found in (ii).

(iv) Finally compute (†).

4. Lecture 4 (4/07)

Definition 4.1 (Open and Closed Sets)**.** Consider a $S \subseteq \mathbf{C}$.

- The interior of S is the set of all interior points of S, denoted S°.
- S is said to be open if $S = S^\circ$.
- The boundary of S is the set of all boundary points of S, denoted ∂S.
- S is said to be closed if $\partial S \subseteq S$. Equivalently, if its complement is open.
- The closure of S is the set $S \cup \partial S$, denoted $\overline{S}$.

Example 4.2.

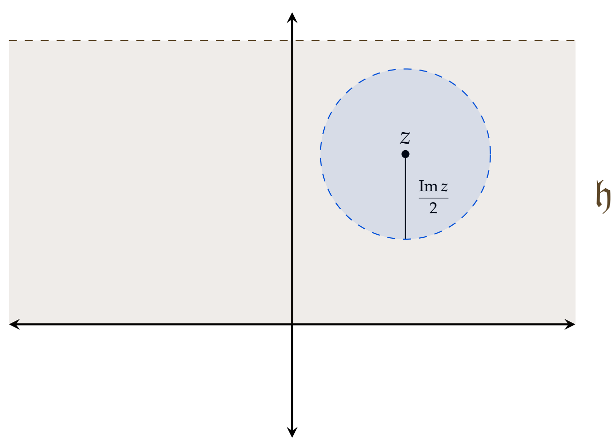
(1) The open disks $D_R(z_0)$ are truly open sets, and the closed disks $\overline{D}_R(z_0)$ are truly closed sets. The closure of the open disk $D_R(z_0)$ is $\overline{D}_R(z_0)$. The boundary of $D_R(z_0)$ is the circle $C_R(z_0)$.

(2) Consider the upper half-plane

$$\mathfrak{h} = \{z \in \mathbf{C} \;:\; \operatorname{Im} z > 0\}\,,$$

then we have $\mathfrak{h}^\circ = \mathfrak{h}$. Since by definition $\mathfrak{h}^\circ \subseteq \mathfrak{h}$, it's enough to prove $\mathfrak{h} \subseteq \mathfrak{h}^\circ$. Consider any $z \in \mathfrak{h}$, then $\operatorname{Im} z > 0$. Let $\varepsilon = (\operatorname{Im} z)/2$, we claim that

$$D_\varepsilon(z) \subseteq \mathfrak{h}$$



Let $w \in D_\varepsilon(z)$, then

$$|w - z| < \varepsilon = \frac{\operatorname{Im} z}{2}$$

The end of Discussion 1.10 tells us

$$\begin{aligned} \frac{\operatorname{Im} z}{2} > |w - z| &\geqslant |\operatorname{Im}(w - z)| \\ &= |\operatorname{Im} w - \operatorname{Im} z| \end{aligned}$$

The later is simply the absolute value of a real number, which gives

$$-\frac{\operatorname{Im} z}{2} < \operatorname{Im} w - \operatorname{Im} z < \frac{\operatorname{Im} z}{2}$$

Adding $\operatorname{Im} z$ throughout the inequality, we get from the inequality on the left hand side

$$\operatorname{Im} w > \frac{\operatorname{Im} z}{2} > 0.$$

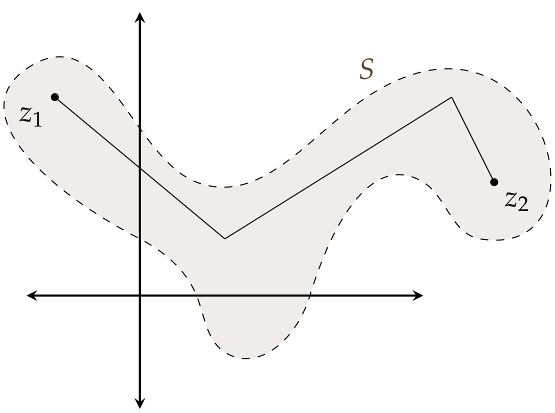
Therefore $w \in \mathfrak{h}$, and hence $D_\varepsilon(z) \subseteq \mathfrak{h}$. Thus $\mathfrak{h}^\circ = \mathfrak{h}$.

The points exterior to $\mathfrak{h}$ are points z such that $\operatorname{Im} z < 0$. That is, the exterior of the upper half-plane is the (open) lower half-plane. The boundary of $\mathfrak{h}$ consists of precisely points z whose $\operatorname{Im} z = 0$. That is, $\partial\mathfrak{h} = \mathbf{R}$.

The closure of $\mathfrak{h}$ is $\overline{\mathfrak{h}} = \{z \in \mathbf{C} \ : \ \operatorname{Im} z \geqslant 0\}$. While $\mathfrak{h} \cup \{0\}$ is neither open nor closed.

Definition 4.3 (Bounded Sets)**.** A set $S \subseteq \mathbf{C}$ is bounded if $S \subseteq D_M(0)$ for some $M > 0$. That is, there exists an $M > 0$ such that $|z| \leqslant M$ for every $z \in S$.

Definition 4.4 (Connected Sets)**.** A set $S \subseteq \mathbf{C}$ is said to be connected if each pair of points z_1 and z_2 in S can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. Otherwise, we say it is disconnected.



Definition 4.5 (Domain)**.** $S \subseteq \mathbf{C}$ is called a domain if it's a non-empty open and connected set.

A region is a domain together with some or all of its boundary points.

Remark 4.6. Domains and regions are sets we will find most suitable for stating elegant results about certain functions in a complex variable.

Example 4.7. $\mathfrak{h}$ is a domain since it's non-empty, open and any two points in $\mathfrak{h}$ can be connected by a straight line. It's an unbounded set. An example of a region is $\mathfrak{h} \cup \{0\}$.

Part II. Holomorphic Functions

Complex Functions

Definition 4.8. A *function* $f : G \to \mathbf{C}$ is a rule that assigns to each $z \in G$ a unique number $f(z) \in \mathbf{C}$. The set G is called the *domain (of definition)*. If $S \subseteq G$, then

$$f(S) := \{f(z) \ : \ z \in S\}$$

is called the *image of S under f*.

The set $f(G)$ is called the *image (or range) of f*. Points in $f(G)$ are called *values of f*.

Given a function f, we define its conjugate $\bar{f}$ by the rule $\bar{f}(z) := \overline{f(z)}$.

Discussion 4.9. If $f : G \to \mathbf{C}$ is a function, then the value $f(x+iy) = u + iv$ depends on a pair $(x, y) \in \mathbf{R}^2$. Collecting all values, we decompose f into its real and imaginary parts

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y); \quad \operatorname{Re} f = u \ \text{ and } \ \operatorname{Im} f = v,$$

where $u, v : \mathbf{R}^2 \to \mathbf{R}$ are real-valued functions in two real variables.

In practice, as the examples below tell us, this means replace your $z = x + iy$ and do the required operations to the output $f(x+iy)$. The resulting complex number will be, as a complex number, of the form $u + iv$. The real part is u, which you will obtain in terms of x and y, and the imaginary part is v, which you will also obtain in terms of x and y.

Example 4.10 (Some Complex Functions)**.**

(1) $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$. So,

$$u(x,y) = x^2 - y^2 \quad \text{and} \quad v(x,y) = 2xy.$$

(2) $f(z) = \bar{z} = x - iy$. So,

$$u(x,y) = x \quad \text{and} \quad v(x,y) = -y.$$

(3) (in-class) $f(z) = z\bar{z} = |z|^2 = x^2 + y^2$. So,

$$u(x,y) = x^2 + y^2 \quad \text{and} \quad v(x,y) = 0.$$

Such a function is *real-valued*.

(4) *Polynomials of degree n* are functions of the form

$$p(z) = a_0 + a_1 z + \cdots a_n z^n,$$

where $a_i \in \mathbf{C}$ and $a_n \neq 0$.

A polynomial of degree 0 is simply a non-zero complex number, sometimes also referred to as a *constant polynomial*.

(5) *Rational functions (or polynomials)* are functions of the form

$$\frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials. The domain of definition is wherever $q(z) \neq 0$. For example,

$$f : \mathbf{C}^* \to \mathbf{C},\ z \mapsto \frac{1}{z}$$

(6) If we express z in its polar form, then a function f, when we restrict its domain of definition within $\mathbf{C}^*$, can be written as

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

For example,

$$f : \mathbf{C}^* \to \mathbf{C},\ z = re^{i\theta} \mapsto \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{\cos\theta}{r} - i\,\frac{\sin\theta}{r}.$$

Here $u(r,\theta) = \dfrac{\cos\theta}{r}$ and $v(r,\theta) = -\dfrac{\sin\theta}{r}$.

(7) (in-class) Let's consider the function $f(z) = \bar{z}^2$, in polar form we have

$$\begin{aligned} f(re^{i\theta}) &= (\overline{re^{i\theta}})^2 \\ &= (re^{-i\theta})^2, \quad \text{by Proposition 2.9 (4)} \\ &= r^2e^{-i2\theta}, \quad \text{by Proposition 2.9 (3)} \\ &= r^2(\cos(-2\theta) + i\sin(-2\theta)) \\ &= r^2\cos(2\theta) - i\sin(2\theta)) \end{aligned}$$

Therefore, here $u(r,\theta) = r^2\cos(2\theta)$ and $v(r,\theta) = -r^2\sin(2\theta)$.

(8) Consider $f(z) = z^{1/n}$, where n is a non-zero integer. For no $n \neq 1$ is this a function! We have seen previously that $z^{1/n}$ has n-distinct values. Such a "function" is called multi-valued.

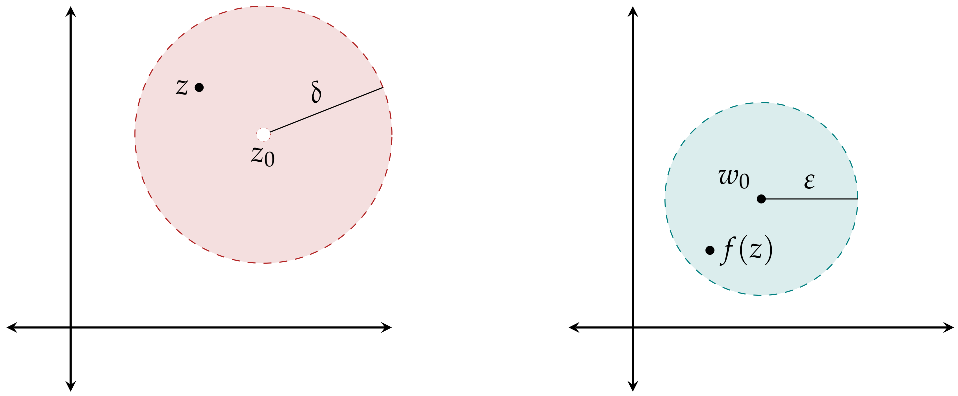
We can make this into a (single-valued) function by assigning a single value of $z^{1/n}$ to each z; taking the *principal n^{th} root of z*, for instance. More on such functions soon.

Limits of Functions

Definition 4.11 (Limit of a Function)**.** Consider a function $f : G \to \mathbf{C}$, and an accumulation point z_0 of G. We say that limit of f, as z approaches z_0, is $w_0 \in \mathbf{C}$ if *for all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\textit{if } 0 < |z - z_0| < \delta, \quad \textit{then } |f(z) - w_0| < \varepsilon$$

Equivalently, if $z \in D_\delta(z_0) \setminus \{z_0\}$, then $f(z) \in D_\varepsilon(w_0)$.



In this case we write $\lim_{z \to z_0} f(z) = w_0$ or $f(z) \to w_0$, as $z \to z_0$.

Intuitively, the limit of f at z_0 is w_0 if

"f is arbitrarily close to w_0 eventually, that is sufficiently, near z_0".

How close? Within an error of ε. How near, eventually? Within a distance of δ.

4.1. Problems

Problem 4.1.

(a) Recall that a set is open if every point of the set is an interior point. Prove that a set $U \subseteq \mathbf{C}$ is open if and only if it does not contain any of its boundary points; that is, $\partial U \cap U = \varnothing$. Then deduce that the complement of a closed set is open.

(b) Prove that an open disk $D_\varepsilon(z_0) = \{z \in \mathbf{C} : |z - z_0| < \varepsilon\}$ is a domain; that is, a non-empty open and connected subset of $\mathbf{C}$.

Problem 4.2. Sketch the sets defined by the following constraints and determine whether they are open, closed, or neither; bounded; connected. What are their boundaries?

(a) $|z+3| < 2$.

(b) $|\mathrm{Im}(z)| < 1$.

(c) $0 < |z-1| < 2$.

(d) $|z-1| + |z+1| = 2$.

(e) $|z-1| + |z+1| < 3$.

(f) $|z| \geqslant \mathrm{Re}(z) + 1$.

Problem 4.3. Let G be the set of points $z \in \mathbf{C}$ satisfying either z is real and $-2 < z < -1$, or $|z| < 1$, or $z = 1$ or $z = 2$.

(a) Sketch the set G, being careful to indicate exactly the points that are in G.

(b) Determine the interior points of G.

(c) Determine the boundary points of G.

(d) Determine the isolated points of G.

(e) G can be written in three diferent ways as the union of two disjoint nonempty disconnected subsets. Describe them.

Problem 4.4. For each of the functions below, describe the domain of definition that is understood.

(a) $f(z) = \dfrac{1}{1+z^2}$

(b) $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$

(c) $f(z) = \dfrac{z}{z+\overline{z}}$

(d) $f(z) = \dfrac{1}{1-|z|^2}$

Problem 4.5.

(a) Write the function $f(z) = z^3 + z + \overline{z} + 1$ in the form

$$f(z) = u(x, y) + i\, v(x, y).$$

(b) Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use Proposition 2.4 (6) to write $f(z)$ in terms of z, and simplify the result.

(c) Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + i\, v(r, \theta)$.

Problem 4.6. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \to z_0} |f(z) - w_0| = 0.$$

Thereby deduce that

$$\lim_{z \to z_0} \bar{f}(z) = \overline{w}_0 \quad \text{if and only if} \quad \lim_{z \to z_0} f(z) = w_0.$$

Problem 4.7. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Show that

$$\text{if } \lim_{z \to z_0} f(z) = w_0, \quad \text{then } \lim_{z \to z_0} |f(z)| = |w_0|\,.$$

Hint. Use the reverse triangle inequality.

Problem 4.8. Let $f : G \to \mathbf{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Writing $h = z - z_0$, show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{h \to 0} f(z + h) = w_0.$$

5. Lecture 5 (4/12)

Example 5.1. Let's show that $\lim_{z\to i} z^2 = -1$ using the definition.

Proof. Let $\varepsilon > 0$ be arbitrary. Note that $\left|z^2 - (-1)\right| = |z-i|\,|z+i|$. We make an initial estimate, suppose $0 < |z-i| < 1$, then

$$\begin{aligned} |z+i| &= |z-i+2i| \\ &\leqslant |z-i| + |2i| \\ &< 1+2 \\ &= 3 \end{aligned}$$

Now, if we choose $\delta = \min\left\{\frac{\varepsilon}{3}, 1\right\}$, then if $0 < |z-i| < \delta$ we get

$$0 < |z-i| < 1 \text{ and } \frac{\varepsilon}{3}$$

So,

$$\begin{aligned} \left|z^2 - (-1)\right| &= |z-i|\,|z+i| \\ &< 3\,|z-i|\,, \quad \text{since } |z-i| < 1 \\ &< 3 \cdot \frac{\varepsilon}{3}, \quad \text{since } |z-i| < \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Therefore $\lim_{z\to i} z^2 = -1$. □

Theorem 5.2. *If f has a limit at z_0, then it is unique.*

Proof. Assume

$$\lim_{z\to z_0} f(z) = \alpha \quad \text{and} \quad \lim_{z\to z_0} f(z) = \beta$$

Consider an arbitrary $\varepsilon > 0$, then we can find a $\delta_1 > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - \alpha| < \frac{\varepsilon}{2}$$

and $\delta_2 > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - \beta| < \frac{\varepsilon}{2}$$

Define $\delta := \min\{\delta_1, \delta_2\} \leqslant \delta_1, \delta_2$, then if $0 < |z - z_0| < \delta$ we have

$$\begin{aligned} |\alpha - \beta| &= |f(z) - f(z) + \alpha - \beta| \\ &\leqslant |\alpha - f(z)| + |f(z) - \beta| \\ &= |f(z) - \alpha| + |f(z) - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We have proven that $|\alpha - \beta| < \varepsilon$ for any $\varepsilon > 0$. Now, suppose $\alpha \neq \beta$, then for $\varepsilon = |\alpha - \beta| > 0$ we get $|\alpha - \beta| < |\alpha - \beta|$, which is preposterous. Hence $\alpha = \beta$, and thus the limit is unique. $\square$

Remark 5.3. The reason we require that z_0 be an accumulation point of the domain of f is just that we need to be sure that there are points z of the domain that are arbitrarily close to z_0. That is, there are indeed points satisfying $0 < |z - z_0| < \delta$.

Our definition (i.e., the part that says $0 < |z - z_0|$) does not require z_0 to be in the domain of f, and if z_0 is in the domain of f, the definition explicitly ignores the value of $f(z_0)$.

Uniqueness of limits can be used to show that a limit does not exist.

Example 5.4. The function $f(z) = \dfrac{\overline{z}}{z}$ has no limit at 0.

Discussion of Example 5.4. Let $z = x + iy$, then

$$f(z) = \frac{x - iy}{x + iy}$$

Along the real axis, $\operatorname{Im} z = 0$, and so $z = x$, giving us $f(z) = \dfrac{x}{x} = 1$.

Along the imaginary axis, $\operatorname{Re} z = 0$, and so $z = y$, giving us $f(z) = \dfrac{-y}{y} = -1$.

Taking the limit along these axes gives us different values of the limit, 1 and -1. Hence, by the uniqueness of limits, the limit doesn't exist. $\square$

Theorems on Limits

Theorem 5.5 (Limit in terms of Real and Imaginary parts of a Function)**.** *Suppose that*

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$$

Then

$$\lim_{x+iy \to x_0+iy_0} f(x + iy) = u_0 + iv_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Proof. $(\Rightarrow)$ Consider an arbitrary $\varepsilon > 0$, then there exists a $\delta > 0$ such that if

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta$$

$$\text{then} \quad |f(x + iy) - (u_0 + iv_0)| = |(u(x, y) + i\,v(x, y)) - (u_0 + iv_0)| < \varepsilon$$

We first note that, by definition

$$\|(x, y) - (x_0, y_0)\| = |(x + iy) - (x_0 + iy_0)|$$

and the end of Discussion 1.10 tells us that

$$|u(x,y) - u_0| \leqslant |(u(x,y) + i\,v(x,y)) - (u_0 + iv_0)| < \varepsilon$$
$$|v(x,y) - v_0| \leqslant |(u(x,y) + i\,v(x,y)) - (u_0 + iv_0)| < \varepsilon$$

That is, we have that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta, \quad \text{then } |u(x,y) - u_0| < \varepsilon \text{ and } |v(x,y) - v_0| < \varepsilon$$

Therefore,

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

($\Leftarrow$) Consider an arbitrary $\varepsilon > 0$, then there exists a $\delta_1 > 0$ such that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta_1, \quad \text{then } |u(x,y) - u_0| < \frac{\varepsilon}{2}$$

and there exists a $\delta_2 > 0$ such that

$$\text{if } 0 < \|(x,y) - (x_0,y_0)\| < \delta_2, \quad \text{then } |v(x,y) - v_0| < \frac{\varepsilon}{2}$$

Define $\delta := \min\{\delta_1, \delta_2\} \leqslant \delta_1,\ \delta_2$. Now, if

$$0 < |(x+iy) - (x_0 + iy_0)| = \|(x,y) - (x_0,y_0)\| < \delta$$

then

$$\begin{aligned}
|f(x+iy) - (u_0 + iv_0)| &= |(u(x,y) + i\,v(x,y)) - (u_0 + iv_0)| \\
&= |(u(x,y) - u_0) + i(v(x,y) - v_0)| \\
&\leqslant |(u(x,y) - u_0)| + |i(v(x,y) - v_0)|\,, \text{ by triangle identity} \\
&= |(u(x,y) - u_0)| + |i|\,|(v(x,y) - v_0)| \\
&= |(u(x,y) - u_0)| + |(v(x,y) - v_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore,

$$\lim_{x+iy\to x_0+iy_0} f(x+iy) = u_0 + iv_0$$

□

Theorem 5.6 (Limit Laws). *Suppose*

$$\lim_{z\to z_0} f(z) = \alpha \quad \textit{and} \quad \lim_{z\to z_0} g(z) = \beta$$

Then

(1) $\lim_{z\to z_0}(f(z)+g(z)) = \alpha + \beta$

(2) $\lim_{z\to z_0}(f(z)\,g(z)) = \alpha\beta$

(3) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\alpha}{\beta}$*, provided* $\beta \neq 0$*.*

Proof. The proof follows from Theorem 5.5 and limit laws from Calculus. $\square$

Example 5.7. Let $p(z)$ be a polynomial, then

$$\lim_{z\to z_0} p(z) = p(z_0)$$

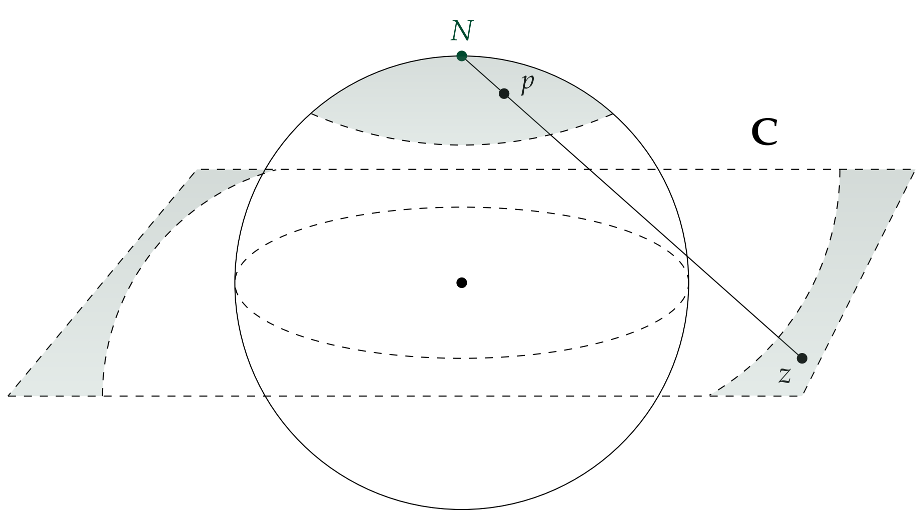
Write $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then by Theorem 5.6 we have

$$\begin{aligned}
\lim_{z\to z_0} p(z) &= \lim_{z\to z_0}(a_0 + a_1 z + \cdots + a_n z^n) \\
&= \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 z + \cdots + \lim_{z\to z_0} a_n z^n, \text{ by Theorem 5.6 (1)} \\
&= \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 \cdot \lim_{z\to z_0} z + \cdots + \lim_{z\to z_0} a_n \cdot \lim_{z\to z_0} z^n, \text{ by Theorem 5.6 (2)} \\
&= a_0 + a_1 z_0 + \cdots + a_n z_0^n, \text{ by Theorem 5.6 (2) and } \lim_{z\to z_0} z = z_0 \\
&= p(z_0)
\end{aligned}$$

$\square$

Definition 5.8 (Extended Complex Plane or the Riemann Sphere)**.** The Extended Complex Plane is the set $\mathbf{C}$ together with a symbol ∞ called the *point at infinity*, denoted $\widehat{\mathbf{C}}$ or $\mathbf{C}_\infty$.

There is a bijection between the extended complex plane and the unit sphere given by the *stereographic projection*, and therefore the extended complex plane is also called the Riemann Sphere.



The point N (the north pole) corresponds to ∞, and any point p on the sphere corresponds uniquely to a point $z \in \mathbf{C}$ which is the unique point of intersection of the complex plane with the line passing through N and p.

Definition 5.9 (Neighbourhood of Infinity)**.** Let $\varepsilon > 0$, the set

$$\left\{ z \in \mathbf{C} \;:\; |z| > \frac{1}{\varepsilon} \right\}$$

is called a *neighbourhood of* ∞. Geometrically, a neighbourhood at infinity is the exterior of a circle centered at the origin, which corresponds to a neighbourhood of N on the unit sphere.

Discussion 5.10. We can now easily give meaning to limits

$$\lim_{z \to z_0} f(z) = w_0$$

where z_0 and w_0 are allowed to be ∞. We replace the appropriate neighbourhood in Definition 4.11 with neighbourhoods of ∞.

Theorem 5.11 (Limits involving Infinity)**.**

(1) $\displaystyle\lim_{z \to z_0} \frac{1}{f(z)} = 0$ *if and only if* $\displaystyle\lim_{z \to z_0} f(z) = \infty$.

(2) $\displaystyle\lim_{z \to \infty} f(z) = \lim_{z \to 0} f\left(\frac{1}{z}\right)$*, provided the limit exist.*

Combining (1) and (2), we get

$$\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0 \quad \text{if and only if} \quad \lim_{z \to \infty} f(z) = \infty.$$

Bottom line, we can simplify limits involving ∞ *to limits involving* 0*.*

Proof. The proofs are based on the simple observation that

$$\frac{1}{a} < b \quad \text{if and only if} \quad \frac{1}{b} < a$$

for non-zero real numbers a and b.

(1) Now $\displaystyle\lim_{z \to z_0} \frac{1}{f(z)} = 0$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \; 0 < |z - z_0| < \delta, \quad \text{then } \; \frac{1}{|f(z)|} = \left| \frac{1}{f(z)} - 0 \right| < \varepsilon$$

if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \; 0 < |z - z_0| < \delta, \quad \text{then } \; |f(z)| > \frac{1}{\varepsilon}$$

if and only if $\displaystyle\lim_{z \to z_0} f(z) = \infty$.

(2) $\lim_{z\to\infty} f(z) = \alpha$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \ |z| > \frac{1}{\delta}, \quad \text{then } \ |f(z) - \alpha| < \varepsilon$$

if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } \ 0 < \left|\frac{1}{z}\right| < \delta, \quad \text{then } \ |f(z) - \alpha| < \varepsilon$$

if and only if, by replacing z with $1/z$, $\lim_{z\to 0} f\left(\frac{1}{z}\right) = \alpha$.

$\square$

Example 5.12. We want to show $\lim_{z\to\infty} \frac{2z^4+1}{z^3+1} = \infty$. This is equivalent to showing

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} = 0, \quad \text{for } f(z) = \frac{2z^4+1}{z^3+1}$$

Note that,

$$\begin{aligned}
\lim_{z\to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} &= \lim_{z\to 0} \frac{\frac{1+z^3}{z^3}}{\frac{2+z^4}{z^4}} \\
&= \lim_{z\to 0} z \cdot \frac{1+z^3}{2+z^4} \\
&= 0 \cdot \frac{1}{2} \\
&= 0
\end{aligned}$$

Therefore $\lim_{z\to\infty} \frac{2z^4+1}{z^3+1} = \infty$.

Example 5.13 (in-class)**.** Show $\lim_{z\to\infty} \frac{2+z^5}{z^2+3} = \infty$.

Answer. This is equivalent to showing

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{(1/z)^2+3}{2+(1/z)^5} = 0, \quad \text{for } f(z) = \frac{2+z^5}{z^2+3}$$

Note that,

$$\begin{aligned}
\lim_{z\to 0} \frac{(1/z)^2+3}{2+(1/z)^5} &= \lim_{z\to 0} \frac{\frac{1+3z^2}{z^2}}{\frac{2z^5+1}{z^5}} \\
&= \lim_{z\to 0} z^3 \cdot \frac{1+3z^2}{2z^5+1} \\
&= 0^3 \cdot \frac{1}{1} \\
&= 0
\end{aligned}$$

Therefore $\lim_{z\to\infty} \frac{2+z^5}{z^2+3} = \infty$. □

5.1. Problems

Problem 5.1. Compute the following limits and prove your claim by using only the ε-δ definition.

(a) $\lim_{z\to i} \overline{z}$

(b) $\lim_{z\to 1+i} z^2$

(c) $\lim_{z\to 1} z^3$

(d) $\lim_{z\to 1-i} \overline{z}^2 - 1$

(e) $\lim_{z\to 1} z - \overline{z}$

(f) $\lim_{z\to i} \overline{z} + z$

Problem 5.2. Evaluate the following limits or explain why they don't exist.

(a) $\lim_{z\to i} \frac{iz^3 - 1}{z+i}$

(b) $\lim_{z\to 1-i} (x + i(2x+y))$

Problem 5.3. Define

$$f(z) = \frac{x^2 y}{x^4 + y^2} \quad \text{where } z = x + iy \neq 0.$$

Show that the limits of f at 0 along all straight lines through the origin exist and are equal, but $\lim_{z\to 0} f(z)$ does not exist.

Hint: Consider the limit along the parabola $y = x^2$.

Problem 5.4. Let $M(z) = \frac{z-3}{1-2z}$. Prove that

$$\lim_{z\to\infty} M(z) = -\frac{1}{2} \quad \text{and} \quad \lim_{z\to 1/2} M(z) = \infty$$

Problem 5.5. Let

$$M(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

Prove that

(a) $\lim_{z\to\infty} M(z) = \infty$ if $c = 0$.

(b) $\lim_{z\to\infty} M(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} M(z) = \infty$, if $c \neq 0$.

6. Lecture 6 (4/14)

Continuous Functions

Definition 6.1 (Continuous Functions)**.** A function $f : G \to \mathbf{C}$ is *continuous at* $z_0 \in G$ if either z_0 is an isolated point or

$$\lim_{z \to z_0} f(z) = f(z_0) = f\left(\lim_{z \to z_0} z\right)$$

That is, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - f(z_0)| < \varepsilon.$$

A function is continuous if it is continuous at every point in its domain.

By the limit laws (Theorem 5.6), sum, product and quotient of continuous functions are continuous (whenever and wherever defined).

Theorem 6.2 (Composition of Continuous Functions)**.** *Suppose we have two functions $f : G_1 \to \mathbf{C}$ and $g : G_2 \to \mathbf{C}$ such that $f(G_1) \subseteq G_2$. If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0. That is,*

$$\lim_{z \to z_0} g(f(z)) = g(f(z_0)) = g\left(\lim_{z \to z_0} f(z)\right) = g\left(f\left(\lim_{z \to z_0} z\right)\right)$$

Therefore, if f and g are continuous, so is $g \circ f$.

Proof. By continuity of g at $f(z_0)$, for an arbitrary $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\text{if } \ 0 < |w - f(z_0)| < \delta_1, \quad \text{then } \ |g(w) - g(f(z_0))| < \varepsilon.$$

Now, by continuity of f at z_0, for $\delta_1 > 0$, there exists a $\delta > 0$ such that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |f(z) - f(z_0)| < \delta_1.$$

With these two statements, we have that

$$\text{if } \ 0 < |z - z_0| < \delta, \quad \text{then } \ |g(f(z)) - g(f(z_0))| < \varepsilon.$$

Therefore $g \circ f$ is continuous at z_0. □

Theorem 6.3. *Suppose $f : G \to \mathbf{C}$ is continuous at z_0 and $f(z_0) \neq 0$, then there exists a $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(z_0)$. That is, $|f(z)| > 0$ for all $z \in D_\delta(z_0)$.*

Proof. Since f is continuous and non-zero at z_0, for $\varepsilon = \dfrac{|f(z_0)|}{2} > 0$ there exists a $\delta > 0$ such that

$$\text{if } \ z \in D_\delta(z_0), \quad \text{then } \ |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

For such a z, the reverse triangle inequality gives us

$$||f(z)| - |f(z_0)|| \leqslant |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}; \quad \text{so,} \quad -\frac{|f(z_0)|}{2} < |f(z)| - |f(z_0)| < \frac{|f(z_0)|}{2}$$

since the former is the absolute value of real numbers. Therefore, adding $|f(z_0)|$ to this inequality gives us

$$|f(z)| > \frac{|f(z_0)|}{2} > 0$$

as needed. □

Theorem 6.4 (Continuity in terms of Real and Imaginary parts of a Function)**.** *Suppose that*

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y).$$

Then f is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

Proof. This is directly follows from Theorem 5.5. □

Definition 6.5 (Compact Sets)**.** A subset of $\mathbf{C}$ is said to be compact if it is closed and bounded.

Definition 6.6 (Bounded Functions)**.** A function $f : G \to \mathbf{C}$ is said to be a bounded function if the image $f(G)$ is bounded. Equivalently, if there exists $M > 0$ such that $|f(z)| \leqslant M$ for every $z \in G$.

Theorem 6.7 (Extreme Value Theorem)**.** *Suppose $K \subseteq \mathbf{C}$ is compact, and $f : K \to \mathbf{C}$ is continuous. Then f is bounded, that is there exists an $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in K$, and there exists a $z_0 \in K$ such that $|f(z_0)| = M$.*

Proof. Since $f = u + iv$ is continuous, so are $u, v : \mathbf{R}^2 \to \mathbf{R}$ by Theorem 6.4. Hence, so is

$$|f(z)| = |f(x + iy)| = \sqrt{u(x, y)^2 + v(x, y)^2}$$

as it's obtained as a sum, product and composition of continuous functions. This result then follows from standard Calculus, since $|f|$ is a real-valued function. □

Complex-Differentiable Functions

Definition 6.8 (Derivative)**.** Consider a function $f : G \to \mathbf{C}$, the derivative *of f at $z_0 \in G$* is the limit

$$\frac{d}{dz}(f(z_0)) = f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the limit exists, we say f is *differentiable at z_0*.

A function is differentiable if it is differentiable at every point in its domain.

Letting $h = \Delta_{z_0} z = z - z_0$, the limit can also be written as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Example 6.9. Consider $f(z) = z^2$, then

$$\begin{aligned} f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} \\ &= \lim_{h \to 0} \frac{2zh + h^2}{h} \\ &= \lim_{h \to 0} 2z + h \\ &= 2z \end{aligned}$$

Example 6.10. Where is $f(z) = |z|^2$ differentiable?

Consider $z \in \mathbf{C}$ and an arbitrary $h \in \mathbf{C}$, then we compute

$$\begin{aligned} f(z+h) - f(z) &= |z+h|^2 - |z|^2 \\ &= (z+h)\overline{(z+h)} - z\overline{z} \\ &= z\overline{z} + z\overline{h} + \overline{z}h + h\overline{h} - z\overline{z} \\ &= z\overline{h} + \overline{z}h + h\overline{h} \end{aligned}$$

Then

$$\frac{f(z+h) - f(z)}{h} = \frac{z\overline{h} + \overline{z}h + h\overline{h}}{h} = z\frac{\overline{h}}{h} + \overline{z} + \overline{h}$$

Along the real axis, $h = \overline{h}$, we have

$$\frac{f(z+h) - f(z)}{h} = z + \overline{z} + h;$$

therefore, as $h \to 0$, the limit is $z + \overline{z}$. Along the imaginary axis, $h = -\overline{h}$, we have

$$\frac{f(z+h) - f(z)}{h} = -z + \overline{z} - h;$$

therefore, as $h \to 0$, the limit is $-z + \overline{z}$.

Since limits are unique, if $f'(z)$ exists, then $z + \overline{z} = -z + \overline{z}$, which gives us $z = 0$. That is, if $f'(z)$ exists, it only exists for $z = 0$. So, does $f'(0)$ exist?

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h\overline{h}}{h} = \lim_{h \to 0} \overline{h} = 0$$

Proposition 6.11 (Differentiable Functions are Continuous)**.** *If f is differentiable at z_0, then f is continuous at z_0.*

Proof. Suppose f is differentiable at z_0, then

$$\lim_{z\to z_0} f(z)-f(z_0) = \left(\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\right)\left(\lim_{z\to z_0} z-z_0\right) = f'(z_0)\cdot 0 = 0$$

Therefore $\lim_{z\to z_0} f(z) = f(z_0)$, and hence f is continuous at z_0. □

Theorem 6.12 (Differentiation Laws)**.** *Suppose f and g are differentiable at z. Then,*

(1) $(c)' = 0$, *for every* $c \in \mathbf{C}$.

(2) $(c\cdot f)'(z) = c\cdot f'(z)$, *for every* $c \in \mathbf{C}$. (Constant Rule)

(3) $(z^n)' = nz^{n-1}$, *for every* $n \in \mathbf{Z}$ *(assume* $z \neq 0$ *for* $n < 0$*).* (Power Rule)

(4) $(f+g)'(z) = f'(z)+g'(z)$. (Sum Rule)

(5) $(fg)'(z) = f'(z)g(z)+f(z)g'(z)$. (Product Rule)

(6) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}$, *provided* $g(z)\neq 0$ (Quotient Rule)

Proof. (1) and (4) are proved directly using the limit definition, (2) can be proved directly or using (1) and (5), while (3) can be proven inductively using (5) for positive n and (6) for negative n.

(5) We first compute

$$\begin{aligned} f(z+h)g(z+h)-f(z)g(z) &= f(z+h)g(z+h)-f(z)g(z)+f(z+h)g(z)-f(z+h)g(z) \\ &= f(z+h)(g(z+h)-g(z))+g(z)(f(z+h)-f(z)) \end{aligned}$$

So,

$$\begin{aligned} (fg)'(z) &= \lim_{h\to 0}\frac{f(z+h)g(z+h)-f(z)g(z)}{h} \\ &= \lim_{h\to 0}\frac{f(z+h)(g(z+h)-g(z))}{h} + \lim_{h\to 0}\frac{g(z)(f(z+h)-f(z))}{h} \\ &= \lim_{h\to 0} f(z+h)\cdot\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} + g(z)\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \\ &= f(z)g'(z)+g(z)f'(z) \end{aligned}$$

(6) We first compute

$$\begin{aligned} \frac{1}{g(z+h)}-\frac{1}{g(z)} &= \frac{g(z)-g(z+h)}{g(z)g(z+h)} \\ &= -\frac{g(z+h)-g(z)}{g(z)g(z+h)} \end{aligned}$$

So,

$$\left(\frac{1}{g}\right)'(z) = \lim_{h\to 0} \frac{\dfrac{1}{g(z+h)} - \dfrac{1}{g(z)}}{h}$$

$$= \lim_{h\to 0} -\frac{g(z+h)-g(z)}{g(z)g(z+h)} \cdot \frac{1}{h}$$

$$= -\lim_{h\to 0} \frac{g(z+h)-g(z)}{h} \cdot \lim_{h\to 0} \frac{1}{g(z)g(z+h)}$$

$$= -\frac{g'(z)}{g(z)^2}$$

(6) then follows from the computation above and using (5) on $\dfrac{f(z)}{g(z)} = f(z) \cdot \dfrac{1}{g(z)}$. □

Proposition 6.13 (Chain Rule)**.** *Suppose we have two functions $f : G_1 \to \mathbf{C}$ and $g : G_2 \to \mathbf{C}$ such that $f(G_1) \subseteq G_2$. If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then $g \circ f$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$$

Proof. Let's start by defining an auxiliary function on G_2

$$\phi(w) = \begin{cases} \dfrac{g(w) - g(f(z_0))}{w - f(z_0)} - g'(f(z_0)) & w \neq f(z_0) \\ 0 & w = f(z_0) \end{cases}$$

Since g is differentiable at $f(z_0)$, then $\lim_{w\to f(z_0)} \phi(w) = 0 = \phi(f(z_0))$ and therefore ϕ is continuous at $f(z_0)$. Furthermore, since f is differentiable at z_0, it is continuous at z_0. So $\lim_{z\to z_0} \phi(f(z)) = \phi(f(z_0)) = 0$ by Theorem 6.2.

Rewriting the above expression, we get the following expression which is valid on all of G_2.

$$g(w) - g(f(z_0)) = (w - f(z_0))(\phi(w) + g'(f(z_0)))$$

Now, for $w = f(z) \in f(G_1)$, we have

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{(f(z) - f(z_0))(\phi(f(z)) + g'(f(z_0)))}{z - z_0}$$

$$= (\phi(f(z)) + g'(f(z_0))) \cdot \frac{f(z) - f(z_0)}{z - z_0}$$

Therefore,

$$(g \circ f)'(z_0) = \lim_{z\to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0}$$

$$= \lim_{z\to z_0} (\phi(f(z)) + g'(f(z_0))) \cdot \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= g'(f(z_0)) \cdot f'(z_0), \text{ since } \lim_{z\to z_0} \phi(f(z)) = 0$$

□

6.1. Problems

Problem 6.1. Example 5.7 tells us that polynomials are continuous.

(a) Prove that the complex conjugation function $\sigma(z) := \overline{z}$ is continuous.

(b) Prove that a polynomial in $\overline{z}$ is continuous. That is, prove that a polynomial given as

$$p(\overline{z}) = a_n\overline{z}^n + \cdots + a_1\overline{z} + a_0, \quad a_i \in \mathbf{C},\ a_n \neq 0$$

is continuous.

(c) Prove that the following functions are continuous by writing them as a sum or product of polynomials $p(z)$ and $q(\overline{z})$

(i) $R(z) := \operatorname{Re} z$

(ii) $I(z) := \operatorname{Im} z$

(iii) $N(z) := |z|^2$

Problem 6.2. Show that the function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = \begin{cases} \dfrac{\overline{z}}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is continuous on $\mathbf{C}^*$.

Problem 6.3. Consider the function

$$f : \mathbf{C}^* \to \mathbf{C},\ z \mapsto \frac{1}{z}.$$

Apply the definition of the derivative to give a direct proof that $f'(z) = -\dfrac{1}{z^2}$.

Problem 6.4. Find the derivative of the function

$$M(z) := \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

When is $M'(z) = 0$?

Problem 6.5. Using Example 5.4 as an inspiration, show that $f'(z)$ does not exist for any z for the functions

(a) $f(z) = \operatorname{Re} z$

(b) $f(z) = \operatorname{Im} z$

Problem 6.6. Show that the function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

is not differentiable at 0.

Problem 6.7.

(a) Show that a polynomial of degree n, $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, is differentiable everywhere, with

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

(b) Furthermore, show that for $p(z)$, as given in (a), we have

$$a_i = \frac{p^{(i)}(0)}{i!}$$

for $i = 0, \ldots, n$. Where $p^{(0)}(z) = p(z)$ and $p^{(i)}(z)$, for $i > 0$, is the i^{th} derivative of $p(z)$.

Problem 6.8. Let G be a domain and $f : G \to \mathbf{C}$ a function that is differentiable at every point in G. Consider the domain

$$G^* = \{z \in \mathbf{C} \,:\, \overline{z} \in G\}$$

and the function

$$f^* : G^* \to \mathbf{C},\ z \mapsto \overline{f(\overline{z})}$$

Show that f^* is differentiable at every point in G^*.

Problem 6.9. For each function, determine all points at which the derivative exists. When the derivative exists, find its value. Use Example 6.10 as an inspiration.

(a) $f(z) = z + i\overline{z}$

(b) $g(z) = (z + i\overline{z})^2$

(b) $h(z) = z \operatorname{Im} z$

Problem 6.10. By definition, a function $f : G \to \mathbf{C}$ is differentiable at $z_0 \in G$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Unpacking the limit definition, we see that f is differentiable at z_0 if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

By appealing only to the definition, we show that $\sigma : \mathbf{C} \to \mathbf{C}$ defined by $\sigma(z) = \bar{z}$ is not differentiable anywhere by completing the following steps.

(i) Let $z_0 \in \mathbf{C}$ and assume that $f'(z_0)$ exists. Choose $\delta > 0$ according to the definition using $\varepsilon = 1/2$ and write down the resulting statement.

(ii) Consider $z = z_0 + \delta/2$ and conclude from (a) that $|1 - f'(z_0)| < \varepsilon$.

(iii) Consider $z = z_0 + i\delta/2$ and conclude from (a) that $|1 + f'(z_0)| < \varepsilon$.

(iv) Using the triangle inequality together with (ii) and (iii), obtain a contradiction.

7. Lecture 7 (4/19)

Cauchy-Riemann Equations

Theorem 7.1 (Cauchy-Riemann Equations)**.** *Suppose that*

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

is differentiable at $z_0 = x_0 + iy_0$*. Then*

(a) the first order partial derivatives of u and v exist at (x_0, y_0) *and satisfy the* Cauchy-Riemann Equations

$$\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned} \tag{CR}$$

(b) $f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$.

Proof. Since f is differentiable at z_0, we have, where we let $h = s + it$

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \frac{f((x_0+s) + i(y_0+t)) - f(x_0+iy_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0+s, y_0+t) - u(x_0,y_0)}{h} + i\cdot \lim_{h\to 0} \frac{v(x_0+s,y_0+t) - v(x_0,y_0)}{h} \end{aligned}$$

As we know by now, we must get the same result if we restrict h to be on the real axis and if we restrict it to be on the imaginary axis. In the former case, $t = 0$, giving us

$$\begin{aligned} f'(z_0) &= \lim_{s\to 0} \frac{u(x_0+s,y_0) - u(x_0,y_0)}{s} + i\cdot \lim_{s\to 0} \frac{v(x_0+s,y_0) - v(x_0,y_0)}{s} \\ &= u_x(x_0,y_0) + i\,v_x(x_0,y_0) \end{aligned}$$

In the latter case, $s = 0$, giving us

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{u(x_0,y_0+t) - u(x_0,y_0)}{it} + i\cdot \lim_{t\to 0} \frac{v(x_0,y_0+t) - v(x_0,y_0)}{it} \\ &= \frac{1}{i}\cdot \lim_{t\to 0} \frac{u(x_0,y_0+t) - u(x_0,y_0)}{t} + \lim_{t\to 0} \frac{v(x_0,y_0+t) - v(x_0,y_0)}{t} \\ &= -i\,u_y(x_0,y_0) + v_y(x_0,y_0) \end{aligned}$$

Therefore

$$u_x(x_0,y_0) + i\,v_x(x_0,y_0) = f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0),$$

and hence $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$. $\square$

The Cauchy-Riemann equations (CR) are a *necessary* condition for f' to exist. We can use them to locate possible points where the derivative does not exist but not necessarily conclude where and if the derivative exists.

Example 7.2.

(1) Consider $f(z) = |z|^2 = x^2 + y^2$, so $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. The partial derivatives at (x, y) are

$$u_x = 2x \qquad v_x = 0$$
$$u_y = 2y \qquad v_y = 0$$

Therefore, the Cauchy-Riemann equations (CR) are only satisfied at $(x, y) = (0, 0)$. Hence f is not differentiable at any $z \neq 0$. Again, note that this does not say anything about the existence of $f'(0)$.

(2) Consider $f(z) = \bar{z} = x - iy$, so $u(x, y) = x$ and $v(x, y) = -y$. The partial derivatives at (x, y) are

$$u_x = 1 \qquad v_x = 0$$
$$u_y = 0 \qquad v_y = -1$$

Note that $u_x \neq v_y$ for all (x, y) and therefore the Cauchy-Riemann equations (CR) are satisfied for no (x, y). Hence f is nowhere complex-differentiable.

(3) (in-class) Consider $f(z) = (z + i\bar{z})^2$, let's simplify f to identify its real and imaginary parts $u(x, y)$ and $v(x, y)$.

$$\begin{aligned} f(z) = f(x+iy) &= ((x+iy) + i(x-iy))^2 \\ &= ((x+iy) + (y+ix))^2 \\ &= ((x+y) + i(x+y))^2 \\ &= (x+y)^2(1+i)^2 \\ &= (x+y)^2(1^2 + i^2 + 2i) \\ &= 2i(x+y)^2 \end{aligned}$$

Therefore $u(x, y) = 0$ and $v(x, y) = 2(x + y)^2$. The partial derivatives at (x, y) are

$$u_x = 0 \qquad v_x = 4(x+y)$$
$$u_y = 0 \qquad v_y = 4(x+y)$$

Therefore, the Cauchy-Riemann equations (CR) are satisfied if and only if $4(x + y) = 0$, if and only if $y = -x$. Hence f is not differentiable any $z \in \mathbf{C}$ such that $\operatorname{Im} z \neq -\operatorname{Re} z$.

As commented, the Cauchy-Riemann equations (CR) are not a *sufficient* condition for the existence of the derivative as the example below shows. Problem 7.1 gives another example.

Example 7.3. Consider

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} = \dfrac{\bar{z}^3}{|z|^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Then,

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We show that u and v satisfy the Cauchy-Riemann equations (CR) at $(0,0)$.

$$u_x(0,0) = \lim_{s\to 0} \frac{u(s,0)-u(0,0)}{s} = \lim_{s\to 0} \frac{\frac{s^3}{s^2} - 0}{s} = 1$$

$$u_y(0,0) = \lim_{t\to 0} \frac{u(0,t)-u(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$v_x(0,0) = \lim_{s\to 0} \frac{v(s,0)-v(0,0)}{s} = \lim_{s\to 0} \frac{0-0}{s} = 0$$

$$v_y(0,0) = \lim_{t\to 0} \frac{v(0,t)-v(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{t^2} - 0}{t} = 1$$

Therefore $u_x(0,0) = 1 = v_y(0,0)$ and $u_y(0,0) = 0 = -v_x(0,0)$, and hence the Cauchy-Riemann equations (CR) are satisfied. But $f'(0)$ does not exist, as seen in Problem 6.6.

Imposing certain existence and continuity conditions on the first order partial derivatives of u and v, the Cauchy-Riemann equations (CR) can be upgraded to a sufficient condition for differentiability.

Theorem 7.4 (Sufficient Conditions for Differentiability)**.** *Consider a function*

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

and a z_0 in the domain of f, such that

(a) the first order partial derivatives of u and v exist and are continuous in an open disk centered at z_0; and

(a) the Cauchy-Riemann equations (CR) are satisfied at (x_0, y_0).

Then $f'(z_0)$ exists and is given by $u_x(x_0,y_0) + i\,v_x(x_0,y_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$.

Proof. We skip the proof. You can find a proof in [1, Section 22, Page 66]. □

Example 7.5. Let's revisit examples from Example 7.2 and 7.3.

(1) Consider $f(z) = |z|^2 = x^2 + y^2$, we noted that $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. We have seen that the only point where $f(z)$ can be differentiable is $z = 0$. The partial derivatives in a neighbourhood of $(0,0)$ are

$$u_x = 2x \qquad\qquad v_x = 0$$

$$u_y = 2y \qquad\qquad v_y = 0$$

which clearly exist and are continuous. We have also seen that the Cauchy-Riemann equations (CR) are satisfied at $(0,0)$, trivially. Therefore $f'(0)$ exists and

$$f'(0) = u_x(0,0) + i\,v_x(0,0) = 0.$$

(2) Consider $f(z) = (z + i\bar{z})^2$, we noted that $u(x,y) = 0$ and $v(x,y) = 2(x+y)^2$. We have seen that the only point where $f(z)$ can be differentiable are $z = x + iy \in \mathbf{C}$ such that $y = \operatorname{Im} z = -\operatorname{Re} z = -x$. That is, at points of the form $(x,-x)$. The partial derivatives in a neighbourhood of $(x,-x)$ are

$$\begin{aligned} u_x &= 0 & v_x &= 4(x+y) \\ u_y &= 0 & v_y &= 4(x+y) \end{aligned}$$

which clearly exist and are continuous. Note the Cauchy-Riemann equations (CR) are satisfied at $(x,-x)$ trivially, since

$$u_x(x,-x) = u_y(x,-x) = v_x(x,-x) = v_y(x,-x) = 0.$$

Therefore $f'(z)$ exists, for $z = x - ix$, and

$$f'(z) = u_x(x,-x) + i\,v_x(x,-x) = 0.$$

(3) The reason Example 7.3 doesn't contradict Theorem 7.4 is because, u_x, in particular, is not continuous at $(0,0)$. Note that we have

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$, we compute $u_x(x,y)$ using the quotient rule, while we have already computed $u_x(0,0) = 1$ in Example 7.3, giving us

$$u_x(x,y) = \begin{cases} \dfrac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Suppose $u_x(x,y)$ is continuous at $(0,0)$, then we have

$$\lim_{(x,y)\to(0,0)} u_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} = u_x(0,0) = 1$$

Restricting the limit along the y-axis, where $x = 0$, we get

$$1 = \lim_{(0,y)\to(0,0)} \frac{-3y^4}{(y^2)^2} = \lim_{y\to 0} \frac{-3y^4}{y^4} = -3,$$

a contradiction. Hence, $u_x(x,y)$ is not continuous at $(0,0)$.

Example 7.6 (Complex Exponential)**.** Define, for any $z = x + iy \in \mathbf{C}$

$$\exp(z) = e^z := e^x e^{iy} = e^x(\cos y + i \sin y)$$

the *complex exponential function*. Note that e^x is the usual real exponential and e^{iy} is given by Euler's formula (Definition 2.7). Here,

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y$$

We then see that

$$\begin{aligned} u_x &= e^x \cos y = v_y, \\ v_x &= -e^x \sin y = -u_y; \end{aligned}$$

so exp satisfies the Cauchy-Riemann equations (CR) everywhere. Furthermore, u_x, u_y, v_x and v_y are everywhere defined and continuous. Hence exp is everywhere complex-differentiable, an *entire* function. Furthermore $\exp(z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = \exp(z)$.

Discussion 7.7 (Polar Cauchy-Riemann Equations)**.** Recall that if the domain of a function f is contained in $\mathbf{C}^*$ or restricted to within $\mathbf{C}^*$, one can express in polar coordinates at $z = re^{i\theta}$ as

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\, v(r,\theta)$$

Then, the Cauchy-Riemann equations (CR) at a point (r_0, θ_0) can be expressed in polar coordinates, Polar Cauchy-Riemann Equations (see Problem 7.3)

$$\begin{aligned} ru_r &= v_\theta \\ u_\theta &= -rv_r \end{aligned} \tag{Polar CR}$$

and a differentiable function at $z_0 = r_0 e^{i\theta_0}$ is then expressed as

$$f'(z_0) = f'(r_0 e^{i\theta_0}) = e^{-i\theta_0}(u_r(r_0, \theta_0) + i\, v_r(r_0, \theta_0)).$$

Example 7.8. Consider the function

$$f(z) = f(re^{i\theta}) = \sqrt{r}\, e^{i\frac{\theta}{2}},$$

where $r > 0$ and $-\pi < \theta < \pi$. This is the function that outputs the principal square root of z. We compute $f'(z)$ at $z = re^{i\theta}$ using the polar form of Theorem 7.4. We first note that

$$f(z) = \underbrace{\sqrt{r} \cos\left(\frac{\theta}{2}\right)}_{u(r,\theta)} + i \underbrace{\sqrt{r} \sin\left(\frac{\theta}{2}\right)}_{v(r,\theta)}$$

Now, we compute

$$\begin{aligned} ru_r &= r\frac{1}{2\sqrt{r}} \cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{r}}{2} \cos\left(\frac{\theta}{2}\right) = v_\theta \\ u_\theta &= -\frac{\sqrt{r}}{2} \sin\left(\frac{\theta}{2}\right) = -r\frac{1}{2\sqrt{r}} \sin\left(\frac{\theta}{2}\right) = -rv_r \end{aligned}$$

Clearly the first order partial derivatives exist everywhere and the Polar Cauchy-Riemann equations (Polar CR) are also satisfied everywhere. Hence $f'(z)$ exists and

$$\begin{aligned}
f'(z) &= e^{-i\theta}(u_r(r,\theta) + i\,v_r(r,\theta)) \\
&= e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\,\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right) \\
&= \frac{e^{-i\theta}}{2\sqrt{r}}\left(\cos\left(\frac{\theta}{2}\right) + i\,\sin\left(\frac{\theta}{2}\right)\right) \\
&= \frac{1}{2\sqrt{r}} \cdot e^{-i\theta} \cdot e^{i\frac{\theta}{2}} \\
&= \frac{1}{2\sqrt{r}e^{i\frac{\theta}{2}}} \\
&= \frac{1}{2f(z)}
\end{aligned}$$

7.1. Problems

Problem 7.1. Define

$$f(z) = \begin{cases} 0 & \text{if } \operatorname{Re}(z)\cdot\operatorname{Im}(z) = 0, \\ 1 & \text{if } \operatorname{Re}(z)\cdot\operatorname{Im}(z) \neq 0 \end{cases}.$$

Show that f satisfies the Cauchy–Riemann equation at $z = 0$, yet f is not diferentiable at $z = 0$.

Problem 7.2. Show that when $f(z) = x^3 + i(1-y)^3$, it makes sense to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

Problem 7.3. Show that $f'(z)$ does not exist at any point if

(a) $f(z) = z - \overline{z}$

(b) $f(z) = 2x + ixy^2$

Problem 7.4. Show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$

(b) $f(z) = e^{-x}e^{-iy}$

Problem 7.5. Let $f : G \to \mathbf{C}$ be a function, such that $G \subseteq \mathbf{C}^*$, then we can write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y) \quad \text{or} \quad f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

Using the fact that $x = r\cos\theta$ and $y = r\sin\theta$ and the chain rule from calculus, write u_r and u_θ in terms of u_x and u_y. Assuming f is differentiable, rewrite the CR-equations and $f'(z)$ in terms of u_r and u_θ.

Problem 7.6. Prove that the function

$$f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$$

is differentiable when $r > 0$ and $0 < \theta < 2\pi$, and find $f'(z)$ in terms of $f(z)$.

8. Lecture 8 (4/21)

Holomorphic Functions

Definition 8.1 (Holomorphic Functions)**.** A function f is *holomorphic on an open set* U if $f'(z)$ exists for every $z \in U$.

We say f is *holomorphic at a point* z_0 if it holomorphic on some open disk $D_\varepsilon(z_0)$ for an $\varepsilon > 0$. We say f is holomorphic if it is holomorphic at every point in its domain.

A function that is holomorphic on all of $\mathbf{C}$ is said to be entire.

Example 8.2.

(1) $f(z) = \dfrac{1}{z}$ is holomorphic on any open set not containing 0, in particular on $\mathbf{C}^*$.

(2) $f(z) = |z|^2$ is nowhere holomorphic since we have already seen that f is only complex-differentiable at $z = 0$ and at no other point.

(3) Polynomials are entire.

(4) $f(z) = \bar{z}$ is nowhere holomorphic, since it's nowhere differentiable.

Discussion 8.3. Let G be a domain (open and connected subset of $\mathbf{C}$). We know several necessary and sufficient conditions for $f = u + iv$ to be holomorphic on G.

(Necessary)
(1) f is continuous on G.
(2) Cauchy-Riemann equations (CR) are satisfied on G.

(Sufficient)
(1) First order partial derivatives of u and v exist and continuous on G, and the Cauchy-Riemann equations (CR) are satisfied on G.
(2) Differentiation Laws. If f and g are holomorphic on G, then so are $f + g$, fg and f/g (if $g \neq 0$ on G).
(3) Composition of holomorphic functions is holomorphic.

Theorem 8.4 (Sufficient Condition for Constantness)**.** *Suppose G is a domain and $f'(z) = 0$ for all $z \in G$. Then $f(z)$ is constant on G.*

Proof. Write $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$, so we have

$$0 = f'(z) = u_x + iv_x = v_y - i\,u_y$$

Therefore $u_x = u_y = 0$ and $v_x = v_y = 0$. We consider points $p, q \in G$ such that there's a line segment L in G connecting them. Let $\vec{w} = (a, b)$ be a unit vector parallel to L, then the directional derivative of u along L is

$$(\operatorname{grad} u) \cdot \vec{w} = au_x + bu_y = 0.$$

So, u is constant along L. Since G is a domain, any two points can be connected by a polygon line. Applying the above argument along constituent line segments, we see that u has the same value along the endpoints of any polygon line. This shows that u is constant on G, say $u(x, y) = c$. A similar argument works for v, giving us $v(x, y) = d$. Hence

$$f(z) = c + id,$$

that is, f is constant. □

Theorem 8.4 has many interesting consequences.

Proposition 8.5. *Suppose f and $\bar{f}$ are holomorphic on a domain G. Then f is constant on G.*

Proof. We write

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$$

$$\bar{f}(z) = \overline{f(x + iy)} = u(x, y) - i\,v(x, y)$$

Since f and $\bar{f}$ are holomorphic, they satisfy the Cauchy-Riemann equations (CR)

$$\text{for } f\text{:} \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\text{for } \bar{f}\text{:} \quad \begin{cases} u_x = (-v)_y = -v_y \\ u_y = -(-v)_x = v_x \end{cases}$$

This gives us $v_y = -v_y$ and $v_x = -v_x$, and therefore $u_x = v_x = 0$. Hence $f'(z) = u_x + i\,v_x = 0$, giving us that f is constant by Theorem 8.4. □

Corollary 8.6. *Suppose f is holomorphic on a domain G and always real-valued. Then f is constant on G.*

Proof. Since f is always real-valued, we have $f = \bar{f}$. Therefore $\bar{f}$ is holomorphic on G as well, and hence f is constant by Proposition 8.5. □

Corollary 8.7. *Suppose f is holomorphic on a domain G and $|f|$ is constant on it. Then f is also constant on G.*

Proof. By assumption $|f(z)| = c$, for all $z \in G$, for some $c \in \mathbf{C}$. This gives us

$$f(z)\overline{f(z)} = |f(z)|^2 = c^2 \qquad (*)$$

Suppose $c = 0$, then $|f(z)| = 0$ and therefore $f(z) = 0$. Suppose $c \neq 0$, then necessarily $f(z) \neq 0$ for every $z \in G$ by $(*)$. Hence

$$\overline{f(z)} = \frac{c^2}{f(z)},$$

and thus $\bar{f}$ is holomorphic. Therefore both f and $\bar{f}$ are holomorphic and hence f is constant by Proposition 8.5. □

Example 8.8. We apply Corollary 8.7 to $f(z) = \dfrac{\overline{z}}{z}$ to conclude that it's not holomorphic.

We first note that, for any $z \in \mathbf{C}$,

$$|f(z)| = \left|\frac{\overline{z}}{z}\right| = \frac{|\overline{z}|}{|z|} = 1;$$

that is, $|f|$ is constant. Suppose f was holomorphic on $\mathbf{C}$ (this argument can be specialised to any domain G), then f would be a holomorphic function such that $|f|$ is constant. Therefore, by Corollary 8.7, f is constant on $\mathbf{C}$. That's a contradiction, since f is non-constant, as $f(1) = 1$ and $f(i) = -1$.

Example 8.9 (in-class)**.** Is the function $f(z) = \operatorname{Re} z$ holomorphic?

Answer. Note that $f(z) = \operatorname{Re} z$ is a real-valued function, for any $z \in \mathbf{C}$. Suppose f was holomorphic on $\mathbf{C}$ (this argument can be specialised to any domain G), then f would be a holomorphic function such that f is always real-valued. Therefore, by Corollary 8.6, f is constant on $\mathbf{C}$. That's a contradiction, since f is non-constant, as $f(1) = 1$ and $f(i) = 0$. □

We now discuss a large class of holomorphic functions, which are complex versions of functions you may have seen in your Calculus classes

The Exponential Function

Definition 8.10 (The Exponential Function)**.** The (complex) exponential function e^z (or $\exp(z)$) is defined on all of $\mathbf{C}$ as follows

$$e^z := e^{\operatorname{Re} z} e^{i \operatorname{Im} z} = e^{\operatorname{Re} z}(\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z)).$$

That is, writing $z = x + iy$, we have

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Since $x \in \mathbf{R}$, e^x is the usual real exponential function, while e^{iy} is given by Euler's formula.

Furthermore, the definitions give us $\overline{e^z} = e^{\overline{z}}$.

Note that when $z = x \in \mathbf{R}$, we have $e^z = e^x$, since then $\operatorname{Im} z = 0$.

Proposition 8.11 (Properties of the Exponential)**.** *Consider $z, w \in \mathbf{C}$.*

(1) $|e^z| = e^{\operatorname{Re} z}$ *and* $\arg e^z = \{\operatorname{Im} z + 2k\pi \ : \ k \in \mathbf{Z}\}$.

(2) $e^{z+w} = e^z e^w$.

(3) $e^{z-w} = \dfrac{e^z}{e^w}$.

(4) e^z *is entire, and* $(e^z)' = e^z$.

(5) e^z is periodic: $e^{z+2k\pi i} = e^z$ for all $k \in \mathbf{Z}$.

Proof.

(1) Write $z = x + iy$, then $|e^z| = |e^x| |\cos x + i \sin x| = |e^x|$. Which tells us

$$\arg e^z = \{y + 2k\pi \ : \ k \in \mathbf{Z}\}.$$

(2) Write $z = x + iy$ and $w = u + iv$, then

$$\begin{aligned} e^{z+w} &= e^{(x+u)+i(y+v)} \\ &= e^{x+u} e^{i(y+v)} \\ &= e^x e^u e^{iy} e^{iv} \\ &= e^x e^{iy} e^u e^{iv} \\ &= e^z e^w \end{aligned}$$

(3) From (2) we get $e^{z-w} e^w = e^z$.

(4) This was seen in Example 7.6.

(5) From (2) we have $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$.

$\square$

8.1. Problems

Problem 8.1. Let $f = u + iv$ be a complex-valued function defined on an open set $G \subseteq \mathbf{C}$. Suppose that the first-order partial derivatives of $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ exist and are continuous on G.

(a) Recall that if $z = x + iy$, then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Treat $f = f(x, y)$ as a function in two real-variables, and *formally* apply the chain rule in Calculus to obtain the expressions

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

(b) Define $\dfrac{\partial f}{\partial x} := \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$, and similarly for $\dfrac{\partial f}{\partial y}$.

Prove that f is holomorphic on G if and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$.

(c) (i) If f is holomorphic on G, prove that $f'(z) = \dfrac{\partial f}{\partial z}$.

(ii) The *Jacobian* of $(x, y) \mapsto (u(x, y), v(x, y))$ is the determinant of the matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

If f is holomorphic on G, prove that the Jacobian equals $|f'(z)|^2 \geqslant 0$.

Problem 8.2. Suppose f is entire and can be written as

$$f(z) = u(x) + i\,v(y),$$

that is, the real part of f depends only on $x = \operatorname{Re}(z)$ and the imaginary part of f depends only on $y = \operatorname{Im}(z)$.

Prove that $f(z) = az + b$ for some $a \in \mathbf{R}$ and $b \in \mathbf{C}$.

Problem 8.3. Suppose f is entire, with real and imaginary parts u and v satisfying

$$u(x, y)\, v(x, y) = 3$$

for all $z = x + iy$. Show that f is constant.

Problem 8.4. Prove that, if $G \subseteq \mathbf{C}$ is a domain and $f : G \to \mathbf{C}$ is a complex-valued function with $f''(z)$ defined and equal to 0 for all $z \in G$, then $f(z) = az + b$ for some $a, b \in \mathbf{C}$.

Problem 8.5. Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$

(b) $\exp\left(\dfrac{2+\pi}{4}\right) = \sqrt{\dfrac{e}{2}}(1+i)$

(c) $\exp(z + \pi i) = -\exp z$.

Problem 8.6. Prove that

(a) $f(z) = \exp \overline{z}$ is nowhere holomorphic.

(b) $f(z) = \exp z^2$ is entire. What is its derivative?

Problem 8.7. Show that

(a) $\left|\exp(2z + i) + exp(iz^2)\right| \leqslant e^{2x} + e^{-2xy}$.

(b) $\left|\exp(z^2)\right| \leqslant \exp(|z|^2)$.

(c) $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

Problem 8.8. Find all values of z such that

(a) $\exp z = -2$

(b) $\exp z = 1 + i\sqrt{3}$

(c) $\exp(2z - 1) = 1$.

Problem 8.9. Find all solutions to the equation $e^{2z} - 2ie^z = 1$.

Problem 8.10. Let $G \subseteq \mathbf{C}^*$ be an open set and let f be a function that is continuous on G with the property

$$e^{f(z)} = z, \quad z \in G.$$

Show that f is holomorphic on G.

Remark 8.12. This shows that a *continuously* defined logarithm on an open set is immediately holomorphic.

9. Lecture 9 (4/26)

The Logarithmic Function

Discussion 9.1. The complex logarithmic function arises, just the like the usual real logarithmic function, from trying to solve the following equation for w

$$e^w = z \quad (z \neq 0)$$

Write $z = re^{i\theta}$ and $w = u + iv$, then

$$e^u e^{iv} = e^w = z = re^{i\theta}.$$

So, $e^u = r$, giving us $u = \ln r = \ln |z|$, and $v = \theta + 2k\pi$ for some $k \in \mathbf{Z}$, that is the possible values of v are exactly $\arg z = \operatorname{Arg} z + 2k\pi,\ k \in \mathbf{Z}$.

Therefore,

$$\begin{aligned} w &= \ln |z| + i \arg(z) \\ &= \ln |z| + i \operatorname{Arg}(z) + 2k\pi i,\ k \in \mathbf{Z} \end{aligned}$$

Essentially, w is not unique, as v is not unique. This is to be expected, since e^z is not injective as it is periodic.

Multiple functions satisfy the equation we considered, which we package into a *multi-valued function* using $\arg z$.

Definition 9.2 (The Logarithmic Function)**.** We define the logarithmic function $\log z$ for any $z \neq 0$, following the discussion above, as

$$\log z := \ln |z| + i \arg(z)$$

Note that $\log z$ is not really a function but a *multi-valued function*, as $\arg z$ is not single-valued.

The principal logarithm, denoted $\operatorname{Log} z$, is defined by taking the principal argument of z

$$\operatorname{Log} z := \ln |z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z \leqslant \pi$$

The principal branch of log is a single-valued function.

Proposition 9.3 (Properties of the Logarithm)**.** *Consider $z \in \mathbf{C}$.*

(1) $e^{\log z} = z$.

(2) $\log e^z = z + 2k\pi i,\ k \in \mathbf{Z}$.

(3) $\log z = \operatorname{Log} z + 2k\pi i,\ k \in \mathbf{Z}$.

(4) If $z = x \in \mathbf{R}_{>0}$, then $\operatorname{Log} z = \ln x$.

Proof.

(1) Note that

$$\begin{aligned} e^{\log z} &= e^{\ln|z|+i\arg z} \\ &= e^{\ln|z|}e^{i(\operatorname{Arg} z+2k\pi)},\ k\in\mathbf{Z} \\ &= e^{\ln|z|}e^{i\operatorname{Arg} z}e^{2k\pi i},\ k\in\mathbf{Z} \\ &= |z|\,e^{i\operatorname{Arg} z} \\ &= z \end{aligned}$$

(2) Note that

$$\begin{aligned} \log e^z &= \ln|e^z| + i\arg(e^z) \\ &= \ln e^{\operatorname{Re} z} + i(\operatorname{Im} z + 2k\pi),\ k\in\mathbf{Z} \\ &= \operatorname{Re} z + i\operatorname{Im} z + 2k\pi i,\ k\in\mathbf{Z} \\ &= z + 2k\pi i,\ k\in\mathbf{Z} \end{aligned}$$

(3) Note that

$$\begin{aligned} \log z &= \log e^{\operatorname{Log} z},\ \text{by (1)} \\ &= \operatorname{Log} z + 2k\pi i,\ k\in\mathbf{Z},\ \text{by (2)} \end{aligned}$$

(4) Note that if $z = x \in \mathbf{R}_{>0}$, then $\operatorname{Arg} z = 0$, therefore

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z = \ln x.$$

□

Example 9.4.

(1)
$$\begin{aligned} \log(1+i\sqrt{3}) &= \ln\left|1+i\sqrt{3}\right| + i\arg(1+i\sqrt{3}) \\ &= \ln 2 + i\left(\frac{\pi}{3}+2k\pi\right),\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log}(1+i\sqrt{3}) = \ln 2 + \frac{\pi i}{3}$$

(2)
$$\begin{aligned} \log 1 &= \ln|1| + i\arg 1 \\ &= 0 + i\,(0+2k\pi),\ k\in\mathbf{Z} \\ &= 2k\pi i,\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log} 1 = 0$$

(3)
$$\begin{aligned} \log -1 &= \ln|-1| + i\arg 1 \\ &= \ln 1 + i\,(\pi+2k\pi),\ k\in\mathbf{Z} \\ &= (2k+1)\pi i,\ k\in\mathbf{Z} \end{aligned}$$

$$\operatorname{Log} -1 = \pi i$$

(4) Familiar properties of logarithms that you know may not hold.

(a) $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$

$$\begin{aligned}
\operatorname{Log}(-1+i)^2 = \operatorname{Log}(-2i) &= \ln|-2i| + i\operatorname{Arg}(-2i) \\
&= \ln 2 + i\left(-\frac{\pi}{2}\right) \\
&= \ln 2 - \frac{\pi i}{2} \\
2\operatorname{Log}(-1+i) &= 2\ln|-1+i| + 2i\arg(-1+i) \\
&= 2\ln\sqrt{2} + 2i\left(\frac{3\pi}{4}\right) \\
&= \ln 2 + \frac{3\pi i}{2}
\end{aligned}$$

(b) $\log i^2 \neq 2\log i$

$$\log i^2 = \log -1 = (2k+1)\pi i,\ k \in \mathbf{Z}$$

$$\begin{aligned}
2\log i &= 2\ln|i| + 2i\arg i \\
&= 0 + 2i\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbf{Z} \\
&= (4k+1)\pi i,\ k \in \mathbf{Z}
\end{aligned}$$

Proposition 9.5. *For all z, $w \in \mathbf{C}^*$*

(1) $\log zw = \log z + \log w$ *(2)* $\log w^{-1} = -\log w$

One treats this as an equality of sets. (1) and (2) also gives you $\log z/w = \log z - \log w$.

Proof.

(1) We have

$$\begin{aligned}
\log z + \log w &= \ln|z| + i\arg z + \ln|w| + i\arg w \\
&= \ln|z|\,|w| + i(\arg z + \arg w) \\
&= \ln|zw| + i\arg zw, \text{ by Proposition 3.1 (1)} \\
&= \log zw
\end{aligned}$$

(2) We have

$$\begin{aligned}
\log w^{-1} &= \ln|w^{-1}| + i\arg w^{-1} \\
&= \ln|w|^{-1} + i(-\arg w), \text{ by Proposition 3.1 (2)} \\
&= -(\ln|w| + i\arg w) \\
&= -\log w
\end{aligned}$$

This statement does not hold if we replace $\log z$ with $\operatorname{Log} z$. □

Definition 9.6 (Branch of a Multi-Valued Functions)**.** A branch of a multi-valued function f is a single-valued function F such that

- F is holomorphic on some domain G; and
- F assigned to each $z \in G$ precisely one value $F(z)$ of $f(z)$.

A portion of a line or curve in the complex plane is called a branch cut for f if a branch f is defined on its complement. A point belonging to *every* branch cut of f is a branch point.

Proposition 9.7 (Branches of log)**.** *Let $\alpha \in \mathbf{R}$. The function*

$$L_\alpha(z) = L_\alpha(re^{i\theta}) = \ln r + i\theta, \quad \alpha < \theta < \alpha + 2\pi$$

is a branch of $f(z) = \log z$. Note that $\operatorname{Re} L_\alpha = u(r,\theta) = \ln r$ and $\operatorname{Im} L_\alpha = v(r,\theta) = \theta$.

Proof. We first remark that if we were to define L_α also on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, as one notes that $\lim_{\theta\to\alpha^-} \theta = \alpha$ but $\lim_{\theta\to\alpha^+} \theta \neq \alpha$ as the points close to the ray to the right have arguments near $\alpha + 2\pi$.

It is clear that $L_\alpha(z)$ is single-valued and, for each z, $L_\alpha(z)$ is a value of $\log z$. We need to show L_α is holomorphic. Note that $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ have continuous partial derivatives on the domain of definition

$$u_r = \frac{1}{r} \qquad\qquad v_r = 0$$

$$u_\theta = 0 \qquad\qquad v_\theta = 1$$

Clearly, the Polar Cauchy Riemann equations (Polar CR) are satisfied, and therefore L_α is holomorphic. In fact,

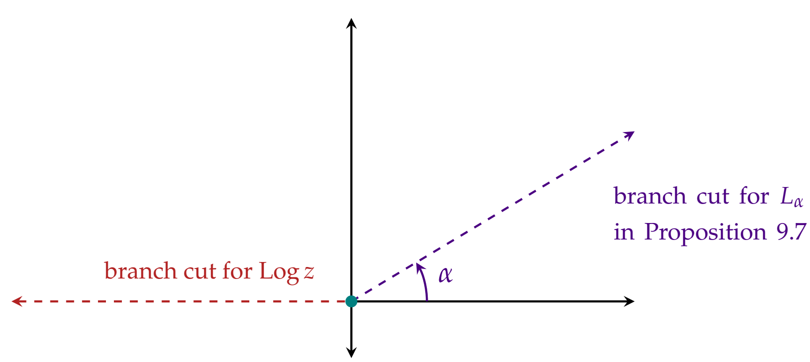
$$L'_\alpha(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{z}$$

In particular, $\operatorname{Log} z$ for those z such that $-\pi < \operatorname{Arg} z < \pi$ is a branch of $\log z$, called the principal branch of the logarithm and

$$(\operatorname{Log} z)' = \frac{1}{z}$$

□

Remark 9.8. The branch cut for $\log z$ in Proposition 9.7 is the ray $r > 0$, $\theta = \alpha$



The branch cut for $\operatorname{Log} z$ is the ray $r > 0$, $\theta = \pi$, i.e., the negative real axis. The origin is a branch point of $\log z$.

Example 9.9 (Integer Powers and Roots)**.** The logarithmic function can be used to compute integer powers and roots (as previously seen and defined).

(1) $z^n = e^{n\log z}$

(2) $z^{1/n} = e^{\frac{\log z}{n}}$

Proof. We note that

$$\begin{aligned} e^{n\log z} &= e^{n(\ln|z|+i\arg z)} \\ &= e^{n\ln|z|}\cdot e^{in\arg z} \\ &= |z|^n\cdot(e^{i\arg z})^n \\ &= (|z|\,e^{i\arg z})^n \\ &= z^n \end{aligned} \qquad \begin{aligned} e^{\frac{\log z}{n}} &= e^{\frac{1}{n}(\ln|z|+i\arg z)} \\ &= e^{\frac{1}{n}\ln|z|}\cdot e^{i\left(\frac{\arg z}{n}\right)} \\ &= e^{\frac{1}{n}\ln|z|}\cdot e^{i\left(\frac{\operatorname{Arg} z+2k\pi}{n}\right)} \\ &= \sqrt[n]{|z|}\cdot e^{i\left(\frac{\operatorname{Arg} z+2k\pi}{n}\right)} \\ &= z^{1/n} \end{aligned}$$

Recall that z^n is single-valued, but $z^{1/n}$ is multi-valued, as complex numbers have n distinct n^{th} roots (Proposition 3.6). In fact, using the the principal logarithm, the complex number

$$e^{\frac{\operatorname{Log} z}{n}}$$

gives the principal n^{th} root of z. $\square$

Power and Exponential Functions

Definition 9.10 (Power Function)**.** The power function z^α for a fixed $c \in \mathbf{C}$ is the *multi-valued* function

$$z^c := e^{c\log z}, \quad z \neq 0$$

Proposition 9.11 (Branches of z^c)**.** *A branch of z^c is determined by specifying a branch of* $\log z$

$$\log z = \ln|z| + i\arg z, \quad z \neq 0,\ \alpha < \arg z < \alpha + 2\pi$$

Moreover,

$$(z^c)' = cz^{c-1},$$

whenever $z \neq 0,\ \alpha < \arg z < \alpha + 2\pi$.

Proof. We only need to verify that z^c is holomorphic, once a branch of $\log z$ has been specified. Since $z^c = e^{c\log z}$ is a composition of two holomorphic functions, z^c itself is holomorphic. Moreover, by the chain rule

$$\begin{aligned} (z^c)' = (e^{c\log z})' &= e^{c\log z}(c\log z)' \\ &= e^{c\log z}\cdot\frac{c}{z} \\ &= c\cdot\frac{e^{c\log z}}{e^{\log z}} = c\cdot e^{(c-1)\log z} = cz^{c-1} \end{aligned}$$

$\square$

Discussion 9.12. The principal branch of z^c is defined by specifying the principal branch $\operatorname{Log} z$ of $\log z$. The principal branch of z^c reduces to the usual power function when $z = x \in \mathbf{R}$.

9.1. Problems

Problem 9.1. Find the all possible values of

(a) $\log(-5)$

(b) $\log(-2+2i)$

(c) $\log(\sqrt{2}+i\sqrt{6})$

(d) $\log(-ei)$

(e) $\log(1+i)$

(f) $\log(-\sqrt{3}+i)$

Problem 9.2. Compute

(a) $\operatorname{Log}(6-6i)$

(b) $\operatorname{Log}(-e^2)$

(c) $\operatorname{Log}(-12+5i)$

(d) $\operatorname{Log}((1+i\sqrt{3})^5)$

(e) $\operatorname{Log}(3-4i)$

(f) $\operatorname{Log}((1+i)^4)$

Problem 9.3.

(a) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

(b) Show that for any two non-zero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i,$$

where $N \in \{0, \pm 1\}$.

Problem 9.4. Example 9.4 (4) tells us that it's not necessarily true that $\log z^n = n \log z$, for $n \in \mathbf{Z}_{>0}$. Writing $z = re^{i \operatorname{Arg} z}$, show that, where $n \in \mathbf{Z}_{>0}$

$$\log(z^{1/n}) = \frac{1}{n} \ln r + i \left(\frac{\operatorname{Arg} z + 2(pn+k)\pi}{n} \right), \quad k = 0, \ldots, n-1.$$

Now, after writing

$$\frac{1}{n} \log z = \frac{1}{n} \ln r + i \left(\frac{\operatorname{Arg} z + 2qz}{n} \right), \quad q \in \mathbf{Z},$$

show that we have equality of sets

$$\log(z^{1/n}) = \frac{1}{n} \log z$$

Problem 9.5. Find a domain in which the given function f is holomorphic; then find the derivative f'.

(a) $f(z) = 3z^2 - e^{2iz} + i \operatorname{Log} z$

(b) $f(z) = (z+1) \operatorname{Log} z$

(c) $f(z) = \dfrac{\operatorname{Log}(2z - i)}{z^2 + 1}$

(d) $f(z) = \operatorname{Log}(z^2 + 1)$

10. Lecture 10 (4/28)

Definition 10.1 (Exponential Function with Base c)**.** The exponential function with base c, where $c \in \mathbf{C}^*$, is the *multi-valued* function

$$c^z := e^{z \log c}$$

Discussion 10.2. Once a branch of $\log z$ has been specified, c^z is an entire function. In that case, using chain rule we have

$$\begin{aligned}(c^z)' = (e^{z \log c})' &= e^{z \log c}(z \log c)' \\ &= e^{c \log z} \cdot \log c \\ &= c^z \log c\end{aligned}$$

What happens if we take $c = e$? Specifying the principal branch $\operatorname{Log} z$ we see

$$e^z = e^{z \operatorname{Log} e} = e^{z(\ln e + i \operatorname{Arg} e)} = e^{z(1+0)} = e^z$$

Example 10.3.

(1) We compute

$$\begin{aligned} i^i &= e^{i \log i} \\ &= e^{i(\ln|i| + i \arg i)} \\ &= e^{i(\ln 1 + i(\frac{\pi}{2} + 2k\pi))},\ k \in \mathbf{Z} \\ &= e^{i^2(\frac{\pi}{2} + 2k\pi)},\ k \in \mathbf{Z} \\ &= e^{-\frac{\pi}{2}} e^{-2k\pi},\ k \in \mathbf{Z}\end{aligned}$$

(2) We compute

$$\begin{aligned} (-1)^{\frac{1}{\pi}} &= e^{\frac{1}{\pi} \log -1} \\ &= e^{\frac{1}{\pi}(\ln|-1| + i \arg -1)} \\ &= e^{\frac{1}{\pi}(\ln 1 + i(\pi + 2k\pi))},\ k \in \mathbf{Z} \\ &= e^{\frac{1}{\pi}(\pi i(2k+1))},\ k \in \mathbf{Z} \\ &= e^{i(2k+1)},\ k \in \mathbf{Z}\end{aligned}$$

Trigonometric Functions

Discussion 10.4. Recall that for any $z \in \mathbf{C}$,

$$\operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}$$

Therefore, for $x \in \mathbf{R}$,

$$\begin{aligned}\cos x &= \operatorname{Re}(e^{ix}) \\ &= \frac{e^{ix} + \overline{e^{ix}}}{2} \\ &= \frac{e^{ix} + e^{-ix}}{2}\end{aligned} \qquad \begin{aligned}\sin x &= \operatorname{Im}(e^{ix}) \\ &= \frac{e^{ix} - \overline{e^{ix}}}{2i} \\ &= \frac{e^{ix} - e^{-ix}}{2i}\end{aligned}$$

This suggests a way to extend the domain of definition of sine and cosine functions to all of $\mathbf{C}$.

Definition 10.5 (Sine and Cosine)**.** The (complex) sine and cosine functions are defined as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

respectively. Moreover, this gives us $e^{iz} = \cos z + i \sin z$. And our calculations above tell us that these functions reduce to the usual sine and cosine for $z = x \in \mathbf{R}$.

Proposition 10.6 (Holomorphicity of sin and cos)**.**

(1) $\sin z$ *and* $\cos z$ *are entire.*

(2) $(\sin z)' = \cos z$ *and* $(\cos z)' = -\sin z$*.*

Proof.

(1) Since $\sin z$ and $\cos z$ are linear combinations of entire functions, they themselves are entire functions.

(2) We note that

$$\begin{aligned} (\sin z)' &= \frac{(e^{iz})' - (e^{-iz})'}{2i} \\ &= \frac{ie^{iz} - (-i)e^{-iz}}{2i} \\ &= \frac{ie^{iz} + ie^{-iz}}{2i} \\ &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \cos z \end{aligned} \qquad \begin{aligned} (\cos z)' &= \frac{(e^{iz})' + (e^{-iz})'}{2} \\ &= \frac{ie^{iz} - ie^{-iz}}{2} \\ &= i \cdot \frac{e^{iz} - e^{-iz}}{2} \\ &= -\frac{e^{iz} - e^{-iz}}{2i} \\ &= -\sin z \end{aligned}$$

□

Discussion 10.7 (Trigonometric Identities)**.** Various familiar identities hold, here are a few.

(1) $\sin(-z) = -\sin z$

(2) $\cos(-z) = \cos z$

(3) $\sin(z + w) = \sin z \cos w + \cos z \sin w$

(4) $\cos(z + w) = \cos z \cos w - \sin z \sin w$

(5) $\sin(z + 2\pi) = \sin z$

(6) $\cos(z + 2\pi) = \cos z$

(7) $\sin(\pi/2 - z) = \cos z$

(8) $\sin^2 z + \cos^2 z = 1$

To define other trigonometric functions, we need to understand the zeros of $\sin z$ and $\cos z$.

Theorem 10.8 (Zeros of Sine and Cosine)**.** *The zeros of* $\sin z$ *and* $\cos z$ *are precisely the zeros of sine and cosine functions in a real variable:*

$$\begin{aligned} \sin z = 0 \quad &\textit{if and only if} \quad z = k\pi,\ k \in \mathbf{Z} \\ \cos z = 0 \quad &\textit{if and only if} \quad z = k\pi + \frac{\pi}{2},\ k \in \mathbf{Z} \end{aligned}$$

Proof. We immediately note that

$$\sin z = \sin k\pi = 0 \quad \text{and} \quad \cos z = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$$

since the inputs are real numbers and sine and cosine reduce to the usual real sine and cosine for real inputs.

Conversely, suppose

$$\frac{e^{iz} - e^{-iz}}{2i} = \sin z = 0,$$

this gives us $e^{iz} = e^{-iz}$, and therefore $e^{2iz} = 1$. Applying log gives us

$$2iz + 2m\pi i = 2n\pi i, \quad \text{for } m, n \in \mathbf{Z}$$

by Proposition 9.3 (2) and Example 9.4 (2). Giving us $z = (n - m)\pi = k\pi$ for any $k \in \mathbf{Z}$.

Suppose $\cos z = 0$. By Discussion 10.7 (1) and (7), we have

$$\sin\left(z - \frac{\pi}{2}\right) = -\cos z = 0$$

Hence, $z - \frac{\pi}{2} = k\pi,\ k \in \mathbf{Z}$. □

Definition 10.9 (Other Trigonometric Functions)**.** The (complex) tangent, cotangent, secant and cosecant functions are defined in terms of sine and cosine.

$$\tan z := \frac{\sin z}{\cos z}, \quad z \neq k\pi + \frac{\pi}{2} \qquad \sec z := \frac{1}{\cos z}, \quad z \neq k\pi + \frac{\pi}{2}$$

$$\cot z := \frac{\cos z}{\sin z}, \quad z \neq k\pi \qquad \csc z := \frac{1}{\sin z}, \quad z \neq k\pi$$

These functions are entire in their stated domains of definition since $\sin z$ and $\cos z$ are. They also all reduce to the usual real trigonometric functions when z is real, since $\sin z$ and $\cos z$ do. The derivatives are exactly as expected.

PART III. INTEGRATION

We now want to develop a theory of integration of complex-valued functions in a single complex variable. Integrals will be defined over suitable curves (contours) in the complex plane. This theory of integration is a surprisingly powerful tool in the study of holomorphic functions.

Using this theory, we will obtain powerful characterisations of holomorphic functions. Roughly speaking we will prove the following: let G be a domain and $f : G \to \mathbf{C}$ a function. The following are equivalent.

(1) f is holomorphic on G.

(2) For all $n \in \mathbf{Z}_{>0}$, $f^{(n)}$ exists and is holomorphic on G.

(3) In each *simply connected* subdomain D of G, there exists a holomorphic function $F : D \to \mathbf{C}$ such that $F' = f|_D$.

(4) f is continuous on G and
$$\int_C f(z)\, dz = 0$$
for every *contour* C lying in a *simply connected* subdomain.

(5) If C is a *simple closed contour* in G and z_0 is interior to C, then
$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Additionally, as an application of the theory, we will prove

- *Liouville's theorem*. Every bounded holomorphic function is constant.
- *Fundamental Theorem of Algebra*. Every polynomial of degree $n \geqslant 1$ has atleast one complex root.

Derivatives of Functions of a Real-variable

To define an integral of a complex-valued functions in a single complex variable, we need to understand how to differentiate a complex-valued function in a single real variable
$$\gamma : [a, b] \to \mathbf{C},$$
where $[a, b] \subseteq \mathbf{R}$.

Definition 10.10. For $\gamma : [a, b] \to \mathbf{C}$, writing $\gamma(t) = u(t) + i\, v(t)$, where $u,\ v : [a, b] \to \mathbf{R}$, we define the *derivative* of γ to be
$$\gamma'(t) = u'(t) + iv'(t),$$
provided that $u'(t)$ and $v'(t)$ exist. In this case, we say γ is differentiable.

Proposition 10.11. *Suppose $\gamma_1(t) = u_1(t) + iv_1(t)$ and $\gamma_2(t) = u_2(t) + iv_2(t)$ are differentiable, then*

(1) $(\gamma_1 + \gamma_2)'(t) = \gamma_1'(t) + \gamma_2'(t)$

(2) $(\gamma_1\gamma_2)'(t) = \gamma_1'(t)\gamma_2(t) + \gamma_1(t)\gamma_2'(t)$

Proof.

(1)
$$\begin{aligned} (\gamma_1 + \gamma_2)' &= ((u_1 + u_2) + i(v_1 + v_2))' \\ &= (u_1 + u_2)' + i(v_1 + v_2)' \\ &= (u_1' + u_2') + i(v_1' + v_2') \\ &= (u_1' + iv_1') + (u_2' + iv_2') \\ &= \gamma_1' + \gamma_2' \end{aligned}$$

(2) $$\begin{aligned}(\gamma_1\gamma_2)' &= ((u_1+iv_1)(u_2+iv_2))' \\ &= ((u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1))' \\ &= (u_1u_2 - v_1v_2)' + i(u_1v_2 + u_2v_1)' \\ &= (u_1u_2)' - (v_1v_2)' + i(u_1v_2)' + i(u_2v_1)' \\ &= (u_1'u_2 + u_1u_2') - (v_1'v_2 + v_1v_2') + i(u_1'v_2 + u_1v_2') + i(u_2'v_1 + u_2v_1') \\ &= (u_1'u_2 - v_1'v_2) + i(u_1'v_2 + u_2v_1') + (u_1u_2' - v_1v_2') + i(u_1v_2' + u_2'v_1) \\ &= (u_1' + iv_1')(u_2 + iv_2) + (u_1 + iv_1)(u_2' + iv_2') \\ &= \gamma_1'\gamma_2 + \gamma_1\gamma_2'\end{aligned}$$

Hence, $(\gamma_1\gamma_2)' = \gamma_1'\gamma_2 + \gamma_1\gamma_2'$. □

Example 10.12. We will often encounter the function $\gamma : [a,b] \to \mathbf{C}$, where

$$\gamma(t) = e^{z_0 t}, \quad z_0 \in \mathbf{C}$$

Let's compute $\gamma'(t)$, for which we first need to express it as $u(t) + iv(t)$. Let $z_0 = x_0 + iy_0$,

$$\begin{aligned}\gamma(t) = e^{z_0 t} &= e^{(x_0+iy_0)t} \\ &= e^{x_0 t + iy_0 t} \\ &= e^{x_0 t}e^{iy_0 t} = e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t))\end{aligned}$$

Therefore, $u(t) = e^{x_0 t}\cos(y_0 t)$ and $v(t) = e^{x_0 t}\sin(y_0 t)$. We note,

$$\begin{aligned}u'(t) &= (e^{x_0 t})'(\cos(y_0 t)) + (e^{x_0 t})(\cos(y_0 t))' \\ &= x_0 e^{x_0 t}\cos(y_0 t) - y_0 e^{x_0 t}\sin(y_0 t)\end{aligned} \qquad \begin{aligned}v'(t) &= (e^{x_0 t})'(\sin(y_0 t)) + (e^{x_0 t})(\sin(y_0 t))' \\ &= x_0 e^{x_0 t}\sin(y_0 t) + y_0 e^{x_0 t}\cos(y_0 t)\end{aligned}$$

Hence,

$$\begin{aligned}\gamma'(t) = u'(t) + iv'(t) &= x_0 e^{x_0 t}\cos(y_0 t) - y_0 e^{x_0 t}\sin(y_0 t) + ix_0 e^{x_0 t}\sin(y_0 t) + iy_0 e^{x_0 t}\cos(y_0 t) \\ &= x_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) + iy_0 e^{x_0 t}(\cos(y_0 t) + i\sin(y_0 t)) \\ &= (x_0 e^{x_0 t} + iy_0 e^{x_0 t})(\cos(y_0 t) + i\sin(y_0 t)) \\ &= (x_0 + iy_0)e^{x_0 t}e^{iy_0 t} \\ &= z_0 e^{z_0 t}\end{aligned}$$

To summarise, for $\gamma(t) = e^{z_0 t}$, we have $\gamma'(t) = z_0 e^{z_0 t}$.

10.1. Problems

Problem 10.1. Find the all possible values of

(a) $(-1)^{3i}$

(b) $3^{2i/\pi}$

(c) $(1+i)^{1-i}$

(d) $(1+i\sqrt{3})^{i}$

(e) $(-i)^{i}$

(f) $(ei)^{\sqrt{2}}$

(g) $(-1)^{1/\pi}$

(h) $i^{i/\pi}$

Problem 10.2. Compute the principal value of the given complex powers.

(a) $(-1)^{3i}$

(b) $3^{2i/\pi}$

(c) 2^{4i}

(d) $(1+i\sqrt{3})^{3i}$

(e) $i^{i/\pi}$

(f) $(1+i)^{2-i}$

(g) $\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i}$

(h) $(1-i)^{4i}$

Problem 10.3.

(a) Verify that $(z^{\alpha})^{n} = z^{n\alpha}$ for $z \neq 0$ and $n \in \mathbf{Z}$.

(b) Find a counterexample to the statement: $(z^{\alpha})^{\beta} = z^{\alpha\beta}$, where $z \neq 0$ and $\alpha, \beta \in \mathbf{C}$.

Problem 10.4. Let z^{α} represent the principal value of the complex power. Find the derivative of the given function at the given point.

(a) $z^{3/2}; \quad z = 1+i$

(b) $z^{1+i}; \quad z = 1+i\sqrt{3}$

(c) $z^{2i}; \quad z = i$

(d) $z^{\sqrt{2}}; \quad z = -i$

Problem 10.5. Let $z \in \mathbf{C}$.

(a) Prove that $|1^{z}|$ is single-valued if and only if $\operatorname{Im} z = 0$.

(b) Find a necessary and sufficient condition for $|i^{iz}|$ to be single-valued.

(c) Find a counterexample to the statement: 1^{z} is single-valued if and only if $\operatorname{Im} z = 0$.

Problem 10.6. Express the value of the given trigonometric function in the form $x + iy$.

(a) $\sin(4i)$

(b) $\cos(-3i)$

(c) $\cos(2-4i)$

(d) $\sin\left(\frac{\pi}{4} + i\right)$

(e) $\tan(2i)$

(f) $\cot(\pi + 2i)$

(g) $\sec\left(\frac{\pi}{2} - i\right)$

(h) $\csc(1+i)$

Problem 10.7. Find all complex values z satisfying the given equation.

(a) $\sin z = i$

(b) $\cos z = 4$

(c) $\sin z = \cos z$

(d) $\cos z = i\,\sin z$

Problem 10.8. Prove the properties stated in Discussion 10.7.

Problem 10.9.

(a) Prove that $\overline{\cos z} = \cos \overline{z}$.

(b) What is $\operatorname{Re}\cos z$ and $\operatorname{Im}\cos z$?

(c) Using the identity $e^{iz} = \cos z + i \sin z$, prove $\overline{\sin z} = \sin \overline{z}$ and find $\operatorname{Re}\sin z$ and $\operatorname{Im}\sin z$.

11. Lecture 11 (5/03)

$$\boxed{\text{Integral of } \gamma : [a,b] \to \mathbf{C}}$$

Definition 11.1 (Definite Integral of γ)**.** Consider a function $\gamma : [a,b] \to \mathbf{C}$ with

$$\gamma(t) = u(t) + iv(t),$$

where u, $v : [a,b] \to \mathbf{R}$. The definite integral of γ is defined as

$$\int_a^b \gamma(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

provided the integrals of u and v exist.

Improper integrals can be defined in a similar manner.

Example 11.2. We illustrate this definition by integrating $\gamma(t) = e^{it}$ on $[0,\pi]$.

$$\begin{aligned}\int_0^\pi e^{it}\,dt &= \int_0^\pi \cos t\,dt + \int_0^\pi \sin t\,dt \\ &= \Big[\sin t\Big]_0^\pi + i\Big[-\cos t\Big]_0^\pi \\ &= (\sin\pi - \sin 0) + i(-\cos\pi + \cos 0) = 2i\end{aligned}$$

Definition 11.3 (Piecewise Continuity)**.** A function $u : [a,b] \to \mathbf{R}$ is piecewise continuous on $[a,b]$ if it is continuous on $[a,b]$ except at a finite number of points, where despite its discontinuity on those points, both one sided limits exist.

We call $\gamma(t) = u(t) + iv(t)$ *piecewise continuous* if both u and v are.

Remark 11.4. The existence of the integrals

$$\int_a^b u(t)\,dt \quad \text{and} \quad \int_a^b v(t)\,dt$$

is guaranteed when γ is piecewise continuous.

Proposition 11.5 (Properties of the Integral of γ)**.** *Suppose γ and γ_1 are piecewise continuous on $[a,b]$, then*

(1) $\displaystyle\int_a^b z_0\gamma(t)\,dt = z_0\int_a^b \gamma(t)\,dt$, *for any $z_0 \in \mathbf{C}$.*

(2) $\int_a^b \gamma(t) + \gamma_1(t)\, dt = \int_a^b \gamma(t)\, dt + \int_a^b \gamma_1(t)\, dt.$

(3) $\int_a^b \gamma(t)\, dt = \int_a^c \gamma(t)\, dt + \int_c^b \gamma(t)\, dt$, *for any* $c \in [a, b]$.

(4) $\int_b^a \gamma(t)\, dt = -\int_a^b \gamma(t)\, dt.$

Proof. These properties follow from the properties of regular real integrals applied to the real and imaginary part of γ and γ_1. $\square$

Proposition 11.6 (Extension of Fundamental Theorem of Calculus)**.** *Suppose that* $\gamma(t) = u(t) + iv(t)$ *is continuous on* $[a, b]$ *and* $\Gamma(t) = U(t) + iV(t)$ *is differentiable such that* $\Gamma'(t) = \gamma(t)$ *on* $[a, b]$. *Then*

$$\int_a^b \gamma(t)\, dt = \Gamma(b) - \Gamma(a)$$

Proof. By assumption $\Gamma' = \gamma$, therefore $U'(t) = u(t)$ and $V'(t) = v(t)$, therefore

$$\begin{aligned}
\int_a^b \gamma(t)\, dt &= \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt \\
&= U(b) - U(a) + i(V(b) - V(a)), \text{ by the Fundamental Theorem of Calculus} \\
&= U(b) + iV(b) - (U(a) + iV(a)) \\
&= \Gamma(b) - \Gamma(a)
\end{aligned}$$

$\square$

Example 11.7. We use this proposition to integrate e^{it} on $[0, \pi]$. For this, we first note that

$$\left(\frac{e^{it}}{i}\right)' = \frac{1}{i}\left(e^{it}\right)' = \frac{i}{i} e^{it} = e^{it}.$$

Therefore,

$$\begin{aligned}
\int_0^\pi e^{it}\, dt = \left[\frac{e^{it}}{i}\right]_0^\pi &= \left[-ie^{it}\right]_0^\pi \\
&= -ie^{i\pi} + ie^{i\cdot 0} \\
&= i + i = 2i
\end{aligned}$$

Contours

So far, we have only defined the integral of a complex-valued function in a single real variable over an interval. Integrals of complex-valued functions in a single complex variable are defined over suitable curves in the complex plane called *contours*.

Definition 11.8 (Arcs)**.**

(1) An arc, or curve, is a collection of points

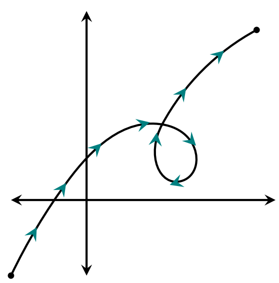
$$C = \{z(t) \ : \ t \in [a,b]\},$$

where $z(t) = x(t) + iy(t)$ and $x, y : [a,b] \to \mathbf{R}$ are continuous functions. The function $z(t)$ is called a parametrization of C.

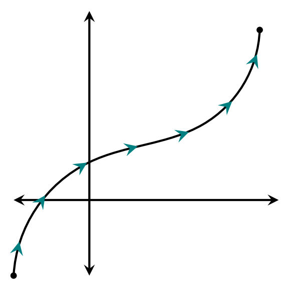
(2) An arc (or curve) C is called simple or a Jordan arc if it does not cross itself, which is equivalent to saying the function $z(t)$ is injective; that is, if $z(t_1) = z(t_2)$ then $t_1 = t_2$.

(3) If C is simple except for the fact that $z(a) = z(b)$, then C is called a simple closed curve or a Jordan curve.

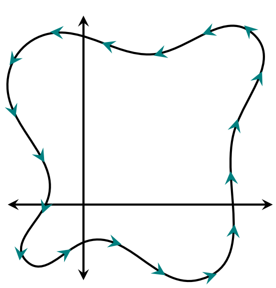
(4) A simple closed curve is positively oriented if it is transversed counter-clockwise as t increases from a to b. It is called negatively oriented if it is transversed clockwise.



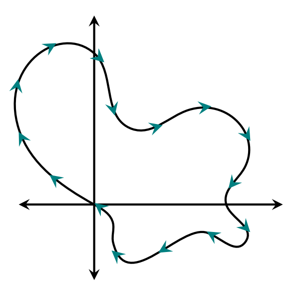
a not simple arc



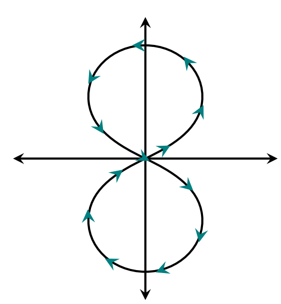
a simple arc



a simple closed curve with positive orientation



a simple closed curve with negative orientation

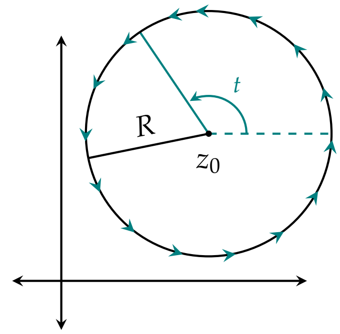


a not simple closed non-orientable curve

Example 11.9. The most frequently encountered arcs and curves are line segments and circles.

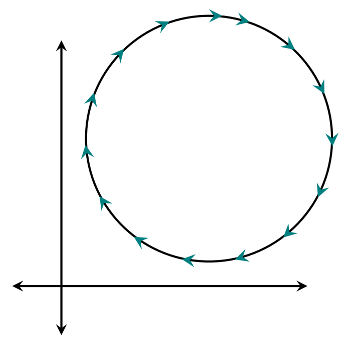
(1) The circle of radius R centered at z_0 with positive orientation has as a parametrisation

$$z(t) = z_0 + Re^{it}, \quad t \in [0, 2\pi]$$



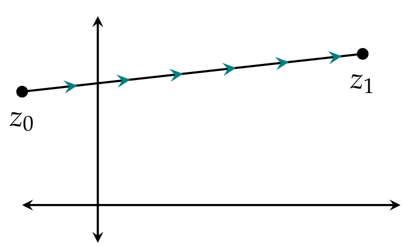
(2) The circle of radius R centered at z_0 with negative orientation has as a parametrisation

$$z(t) = z_0 + Re^{-it}, \quad t \in [0, 2\pi]$$



(3) The line segment from z_0 to z_1 in $\mathbf{C}$ has as a parametrisation

$$z(t) = z_0 + (z_1 - z_0)t = (1-t)z_0 + tz_1, \quad t \in [0, 1]$$



Definition 11.10 (Reparametrisation of an arc)**.** Suppose an arc C is parametrised by $z : [a, b] \to \mathbf{C}$. A map

$$w : [c, d] \to \mathbf{C}$$

is called an orientation-preserving reparametrisation of C if there exists a surjective function

$$\phi : [c, d] \to [a, b]$$

with continuous derivative such that $\phi(c) = a$ (preserves initial point), $\phi(d) = b$ (preserves final point), $\phi'(s) > 0$ and $w(s) = z(\phi(s))$ (w and z trace out the same arc C).

Example 11.11. Note that $z(t) = e^{it}$ for $t \in [0, 2\pi]$ is a parametrisation of the unit circle. Now, consider

$$w : [0, \pi] \to \mathbf{C},\ s \mapsto e^{2is},$$

this is, in fact, an orientation-preserving reparametrisation of the unit circle. To conclude this, we produce the following surjective map

$$\phi : [0, \pi] \to [0, 2\pi],\ s \mapsto 2s,$$

we note that $\phi(0) = 0$ and $\phi(\pi) = 2\pi$, furthermore $\phi'(s) = 2 > 0$ which is clearly continuous. Lastly, $z(\phi(s)) = z(2s) = e^{2is} = w(s)$.

Remark 11.12. Suppose an arc C is parametrised by $z : [a, b] \to \mathbf{C}$, a map $w : [c, d] \to \mathbf{C}$ is called an orientation-reversing reparametrisation of C if there exists a surjective function

$$\psi : [c, d] \to [a, b]$$

with continuous derivative such that $\psi(c) = b$ and $\psi(d) = b$ (swaps initial and final points), $\psi'(s) < 0$ and $w(s) = z(\psi(s))$ (w and z trace out the same arc C).

Consider the unit circle, which has parametrisation $z(t) = e^{it},\ t \in [0, 2\pi]$. Then $w(t) = e^{-it}$ for $0 \leqslant t \leqslant 2\pi$ is an orientation-reversing parametrisation. To see this, we consider the surjective function

$$\psi : [0, 2\pi] \to [0, 2\pi],\ s \mapsto 2\pi - s;$$

we note that $\psi(0) = 2\pi$ and $\psi(2\pi) = 0$, furthermore $\psi'(s) = -1 < 0$ and

$$z(\psi(s)) = z(2\pi - s) = e^{2\pi i - is} = e^{-is} = w(s),$$

since $e^{2\pi i} = 1$.

Definition 11.13 (Arc length and Smooth arcs)**.**

(1) If C is parametrised by $z(t) = x(t) + iy(t)$ and $x'(t)$, $y'(t)$ exist and are continuous on $[a, b]$, then C is called a differentiable arc.

(2) The arc length of such a differentiable arc C is

$$L(C) = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

(3) A differentiable curve parametrised by $z(t)$ is called smooth if $z'(t) \neq 0$ on $[a, b]$.

11.1. Problems

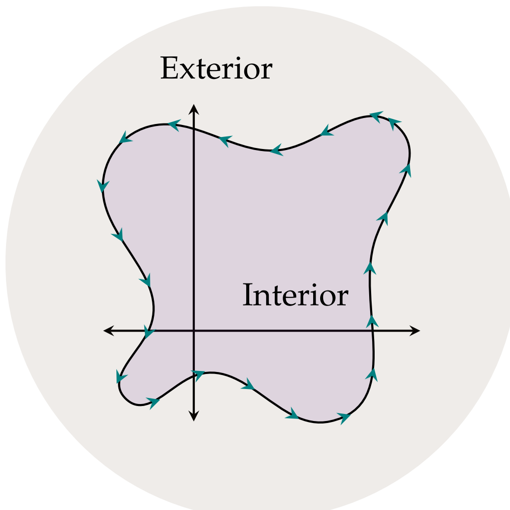
To be added

12. Lecture 12 (5/05)

Definition 12.1 (Contours)**.** A contour is an arc consisting of a finite number of smooth arcs joined end to end.

A simple closed contour is a contour that does not cross itself except that the initial and final points are the same.

Discussion 12.2 (Jordan Curve Theorem)**.** A deep theorem known as the *Jordan Curve theorem* tells us that every simple closed contour C is the boundary of two distinct domains called the interior of C, which is bounded, and the exterior of C, which is unbounded.



The theorem is geometrically evident but the proof is not easy. We will assume its truth so that we can refer to the interior of a simple closed contour.

Contour Integration

Definition 12.3 (Contour Integral)**.** Suppose $f : G \to \mathbf{C}$ is a complex function and C is a contour lying in G. If $z(t)$, $t \in [a, b]$, is a parametrisation of C and $f(z(t))$ is piecewise continuous, then the contour integral of f over C is

$$\int_C f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt$$

Remark 12.4. Since C is a contour, $z'(t)$ is piecewise continuous and therefore the above integral exists.

Proposition 12.5 (Integral is Parametrisation-independent)**.** *Suppose $z : [a, b] \to \mathbf{C}$ parametrises C and $w : [c, d] \to \mathbf{C}$ is an orientation-preserving reparametrisation of C, then*

$$\int_C f(z)\, dz = \int_C f(w)\, dw$$

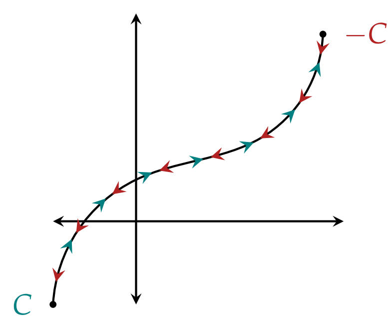
Proof. By definition of an orientation-preserving reparametrisation, there exists a surjective map $\phi : [c, d] \to [a, b]$ such that $\phi(c) = a$, $\phi(d) = b$, $\phi'(s) > 0$ and $w(s) = \phi(z(s))$. Then

$$\begin{aligned}
\int_C f(w)\, dw &= \int_c^d f(w(s))\, w'(s)\, ds \\
&= \int_c^d f(z(\phi(s)))\, \phi'(z(s))\, z'(s)\, ds, \text{ apply chain rule to } w(s) = \phi(z(s)) \\
&= \int_a^b f(z(t))\, z'(t)\, dt, \text{ set } t = \phi(s) \\
&= \int_C f(z)\, dz
\end{aligned}$$

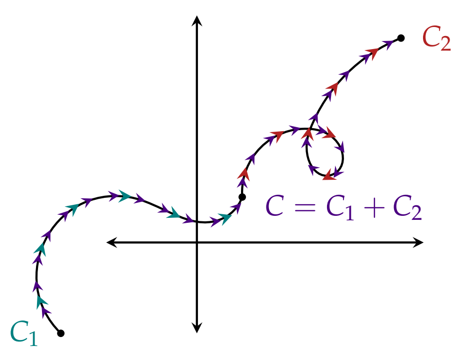
□

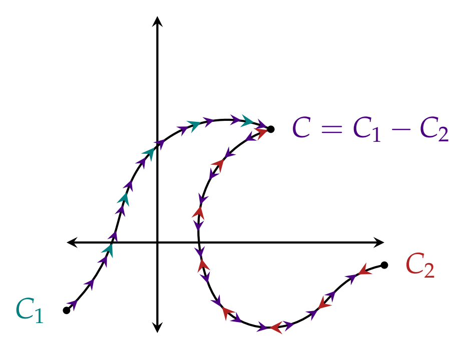
Discussion 12.6 (Notation for Contours)**.**

(1) Suppose C is a contour, then $-C$ denotes the same set of points as C but with opposite orientation. If $z : [a, b] \to \mathbf{C}$ is a parametrisation of C, then $w : [-b, -a] \to \mathbf{C}$ defined as $w(t) := z(-t)$ is a parametrisation of $-C$.



(2) If C_1 is a contour from z_1 to z_2 and C_2 is a contour from z_2 to z_3, then their sum $C = C_1 + C_2$ is the contour obtained by transversing C_1 and then C_2.





If C_1 and C_2 have the same final point, then we can consider the sum of C_1 and $-C_2$ and is written as $C_1 - C_2 := C_1 + (-C_2)$.

Proposition 12.7 (Properties of Contour Integral). *Assume f, g are piecewise continuous on the contours we consider below.*

(1) $\displaystyle\int_C z_0 f(z)\, dz = z_0 \int_C f(z)\, dz$, *for any* $z_0 \in \mathbf{C}$.

(2) $\displaystyle\int_C f(z) + g(z)\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$.

(3) $\displaystyle\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$.

(4) $\displaystyle\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$ *if* $C = C_1 + C_2$.

Proof.

(1) Suppose C is parametrised by $z : [a, b] \to \mathbf{C}$

$$\begin{aligned}\int_C z_0 f(z)\, dz &= \int_C z_0 f(z(t))\, z'(t)\, dz \\ &= z_0 \int_a^b f(z(t))\, z'(t)\, dz, \text{ by Proposition 11.5 (1)} \\ &= z_0 \int_C f(z)\, dz\end{aligned}$$

(2) This will follow from Proposition 11.5 (2).

(3) Suppose C is parametrised by $z : [a, b] \to \mathbf{C}$, then, as we note before, a parametrisation of $-C$ is $w : [-b, -a] \to \mathbf{C}$ where $w(t) = z(-t)$. Then

$$\begin{aligned}\int_{-C} f(w)\, dw &= \int_{-b}^{-a} f(w(t))\, w'(t)\, dt \\ &= -\int_{-b}^{-a} f(z(-t))\, z'(-t)\, dt, \text{ apply chain rule to } w(t) = z(-t) \\ &= -\int_{-a}^{-b} f(z(-t))\, z'(-t)\, dt, \text{ by Proposition 11.5 (4)} \\ &= \int_a^b f(z(s))\, z'(s)\, ds, \text{ set } s = -t \\ &= \int_C f(z)\, dz\end{aligned}$$

(4) We leave this as an exercise (Problem **??**) for the motivated student.

□

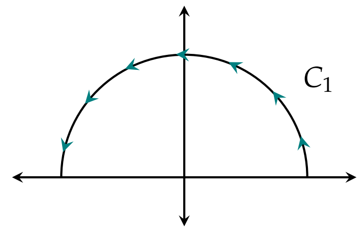
Example 12.8.

(1) Integrate $f(z) = \dfrac{1}{z}$ over the following contours:

- C_1: upper semicircle of the unit circle, from 1 to -1.
- C_2: lower semicircle of the unit circle, from 1 to -1.
- C_3: $C_1 - C_2$.

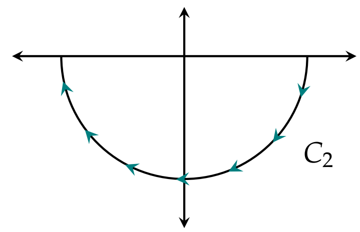
For C_1, parametrise C_1 as $z(t) = e^{it}$, $0 \leqslant t \leqslant \pi$. Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^\pi \frac{1}{e^{it}}\, ie^{it}\, dt = i\int_0^\pi dt = \pi i$$



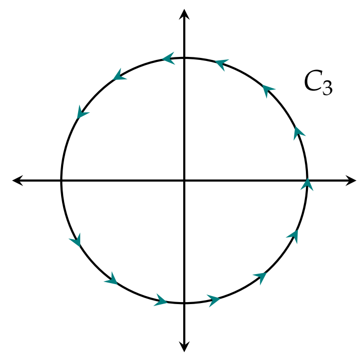
For C_1, parametrise C_1 as $z(t) = e^{-it}$, $0 \leqslant t \leqslant \pi$. Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^\pi \frac{1}{e^{-it}}\, (-ie^{-it})\, dt = -i\int_0^\pi dt = -\pi i$$



For C_1, parametrise C_1 as $z(t) = e^{-it}$, $0 \leqslant t \leqslant \pi$. Then

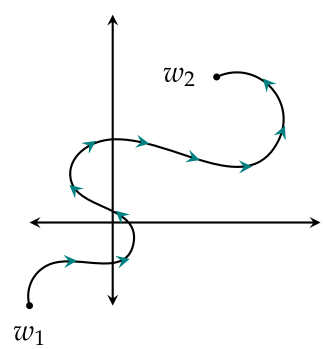
$$\begin{aligned}\int_{C_3} \frac{1}{z}\, dz = \int_{C_1 - C_2} \frac{1}{z}\, dz &= \int_{C_1} \frac{1}{z}\, dz + \int_{-C_2} \frac{1}{z}\, dz \\ &= \int_{C_1} \frac{1}{z}\, dz - \int_{C_2} \frac{1}{z}\, dz \\ &= \pi i - (-\pi i) \\ &= 2\pi i\end{aligned}$$



This example shows that the integral may depend on the path taken and not just on the endpoints. Also, the integral over a closed contour may be non-zero.

(2) Integrate $f(z) = z$ over *any* contour C connecting a point w_1 to a point w_2.

First, suppose C is a smooth arc joining w_1 and w_2 with parametrisation $z : [a, b] \to \mathbf{C}$.



Since,

$$\left(\frac{z(t)^2}{2}\right)' = \frac{z'(t)z(t) + z(t)z'(t)}{2} = z(t)z'(t).$$

Therefore,

$$\begin{aligned}\int_C f(z)\,dz = \int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt \\ &= \frac{z(b)^2}{2} - \frac{z(a)^2}{2}, \text{ by Proposition 11.6} \\ &= \frac{w_2^2 - w_1^2}{2}\end{aligned}$$

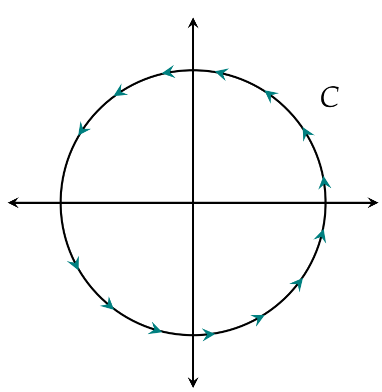
Now, if C is a contour, we can write $C = C_1 + \cdots + C_n$, where C_i is a smooth arc joining z_i to z_{i+1} with $z_1 = w_1$ and $z_{n+1} = w_2$. Then,

$$\begin{aligned}\int_C z\,dz &= \sum_{i=1}^{n} \int_{C_i} z\,dz, \text{ by Proposition 12.7 (4)} \\ &= \sum_{i=1}^{n} \frac{z_{i+1}^2 - z_i^2}{2} \\ &= \frac{z_{n+1}^2 - z_1^2}{2} \\ &= \frac{w_2^2 - w_1^2}{2}\end{aligned}$$

This example shows that some integrals do depend only on the end points and not the path taken. Also, for any contour C is closed, that is, when $w_2 = w_1$, we have shown hence that

$$\int_C z\,dz = 0.$$

(3) Integrate $f(z) = z^m \overline{z}^n$, for $m, n \in \mathbf{Z}$, over the unit circle C.



Parametrise C as $z(t) = e^{it}$, $0 \leqslant t \leqslant 2\pi$. Then,

$$
\begin{aligned}
\int_C f(z)\,dz &= \int_C z^m \overline{z}^n\,dz \\
&= \int_0^{2\pi} (e^{it})^m (\overline{e^{it}})^n\, ie^{it}\,dt \\
&= i\int_0^{2\pi} (e^{it})^m (e^{-it})^n\, e^{it}\,dt \\
&= i\int_0^{2\pi} e^{imt}e^{-int}e^{it}\,dt \\
&= i\int_0^{2\pi} e^{(m-n+1)it}\,dt
\end{aligned}
$$

Case I. $m = n - 1$

$$
\int_C f(z)\,dz = i\int_0^{2\pi} e^{(m-n+1)it} = i\int_0^{2\pi} dt = 2\pi i
$$

Case II. $m \neq n - 1$

$$
\begin{aligned}
\int_C f(z)\,dz = i\int_0^{2\pi} e^{(m-n+1)it} &= i\left[\frac{e^{(m-n+1)it}}{i(m-n+1)}\right]_0^{2\pi} \\
&= \frac{1}{m-n+1}\left(e^{2(m-n+1)\pi i} - e^0\right) \\
&= \frac{1}{m-n+1}\,(1-1) \\
&= 0
\end{aligned}
$$

12.1. Problems

To be added

13. Lecture 13 (5/10)

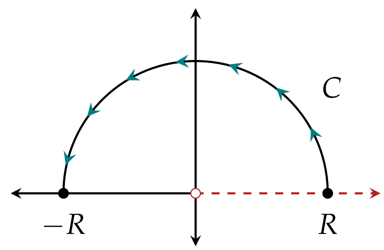
Some examples involving a branch of a multi-valued function.

Example 13.1.

(4) Integrate the branch of square root

$$f(z) = z^{1/2} = e^{(1/2)\log z}, \quad |z| > 0,\ 0 < \arg z < 2\pi$$

over the contour



$$C:\ z(t) = Re^{it}, \quad R > 0,\ 0 \leqslant t \leqslant \pi$$

Note that $f(z)$ is not defined at the initial point $z = R$ of the contour C as $\arg R = 0$, that is, $f(z(t))$ is not defined for $t = 0$. The integral

$$\int_C f(z)\,dz = \int_0^{\pi} f(z(t))\,z'(t)\,dt$$

nevertheless exists as the integrand $f(z(t))\,z'(t)$ is piecewise continuous on $[0,\pi]$. To see this, we note that for $0 < t \leqslant \pi$

$$\begin{aligned} f(z(t))\,z'(t) = e^{(1/2)\log Re^{it}} Rie^{it} &= iRe^{(\ln R + it)/2}e^{it} \\ &= iR(R^{1/2}e^{it/2})e^{it} \\ &= iR^{3/2}e^{3it/2} \\ &= iR^{3/2}\left(\cos\frac{3t}{2} + i\sin\frac{3t}{2}\right) = R^{3/2}\left(-\sin\frac{3t}{2} + i\cos\frac{3t}{2}\right) \end{aligned}$$

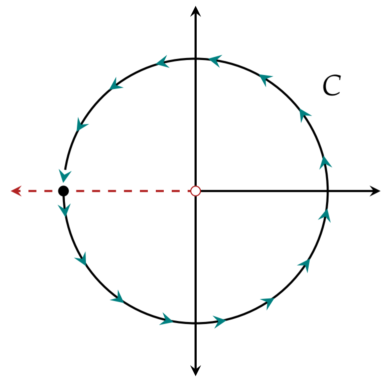
The right hand limits of the real and imaginary parts of $f(z(t))\,z'(t)$ at $t = 0$ exist, and equal 0 and $R^{3/2}$. Therefore, $f(z(t))\,z'(t)$ is continuous on $[0,\pi]$ with its value at $t = 0$ defined as $iR^{3/2}$. Hence,

$$\begin{aligned} \int_C f(z)\,dz = \int_0^{\pi} f(z(t))\,z'(t)\,dt = \int_0^{\pi} iR^{3/2}e^{3it/2}\,dt & \\ &= iR^{3/2}\int_0^{\pi} e^{3it/2}\,dt \\ &= iR^{3/2}\left[\frac{e^{3it/2}}{3i/2}\right]_0^{\pi} \\ &= \frac{2}{3}R^{3/2}\left(e^{3\pi i/2} - e^0\right) = \frac{2}{3}R^{3/2}(-i-1) = -\frac{2}{3}R^{3/2}(1+i) \end{aligned}$$

(5) Integrate the principal branch of

$$f(z) = z^{i-1} = e^{(i-1)\,\mathrm{Log}\,z}, \quad |z| > 0,\ -\pi < \mathrm{Arg}\,z < \pi$$

over the contour



$$C:\ z(t) = e^{it}, \quad R > 0,\ -\pi \leqslant t \leqslant \pi$$

Since the curve crosses the branc cut, we need to check if integrand $f(z(t))\, z'(t)$ is piecewise continuous on $[-\pi, \pi]$. To see this, we note that for $-\pi < t \leqslant \pi$

$$\begin{aligned} f(z(t))\, z'(t) = e^{(i-1)\,\mathrm{Log}\, e^{it}} ie^{it} &= ie^{(i-1)(\ln 1 + it)} e^{it} \\ &= ie^{(i-1)it} e^{it} \\ &= ie^{(i-1)it+it} = ie^{i^2 t} = ie^{-t} \end{aligned}$$

The right hand limits of the real and imaginary parts of $f(z(t))\, z'(t)$ at $t = \pi$ exist, and equal 0 and $e^{-\pi}$. Therefore, $f(z(t))\, z'(t)$ is continuous on $[-\pi, \pi]$ with its value at $t = -\pi$ defined as $ie^{-\pi}$. Hence,

$$\begin{aligned} \int_C f(z)\, dz = \int_{-\pi}^{\pi} f(z(t))\, z'(t)\, dt = \int_{-\pi}^{\pi} ie^{-t}\, dt \\ &= i \int_{-\pi}^{\pi} e^{-t}\, dt \\ &= i\Big[-e^{-t}\Big]_{-\pi}^{\pi} \\ &= i\left(-e^{-\pi} - (-e^{-(-\pi)})\right) \\ &= i\left(e^{\pi} - e^{-\pi}\right) \end{aligned}$$

Estimating Contour Integrals

Lemma 13.2 (Triangle Inequality for Integrals). *Suppose $\gamma : [a, b] \to \mathbf{C}$ is piecewise continuous. Then*

$$\left| \int_a^b \gamma(t)\, dt \right| \leqslant \int_a^b |\gamma(t)|\, dt$$

Proof. Let's first assume

$$\int_a^b \gamma(t)\, dt = 0,$$

then the lemma holds as $|\gamma(t)| \geqslant 0$ for all $t \in [a, b]$ and so its integral is non-negative. Otherwise, let

$$r_0 e^{it_0} = \int_a^b \gamma(t)\, dt \neq 0.$$

Then,

$$\begin{aligned}
\left|\int_a^b \gamma(t)\, dt\right| = |r_0 e^{it_0}| = r_0 = \operatorname{Re} r_0 &= \operatorname{Re}(r_0 e^{it_0} e^{-it_0}) \\
&= \operatorname{Re}\left(e^{-it_0} \int_a^b \gamma(t)\, dt\right) \\
&= \operatorname{Re}\left(\int_a^b e^{-it_0}\gamma(t)\, dt\right) \\
&= \int_a^b \operatorname{Re}(e^{-it_0}\gamma(t))\, dt \\
&\leqslant \int_a^b |e^{-it_0}\gamma(t)|\, dt, \text{ using Discussion 1.10} \\
&= \int_a^b |e^{-it_0}|\, |\gamma(t)|\, dt \\
&= \int_a^b |\gamma(t)|\, dt
\end{aligned}$$

□

Theorem 13.3 (Bound for Contour Integrals)**.** *Suppose that C is a contour of length L and f is piecewise continuous on C. Then*

$$\left|\int_C f(z)\, dz\right| \leqslant \max_{z\in C} |f(z)| \cdot L(C)$$

Proof. Suppose $z : [a, b] \to \mathbf{C}$ parametrises C. By assumption $f(z(t))$ is piecewise continuous on $[a, b]$. Hence, $\max_{z\in C} |f(z)| = \max_{t\in[a,b]} |f(z(t))|$ is finite as $f(z(t))$ is continuous on a closed and bounded interval. Thus,

$$\begin{aligned}
\left|\int_C f(z)\, dz\right| &= \left|\int_a^b f(z(t))\, z'(t)\, dz\right| \\
&\leqslant \int_a^b \left|f(z(t))\, z'(t)\right|\, dz, \text{ by Lemma 13.2} \\
&= \int_a^b |f(z(t))|\, \left|z'(t)\right|\, dz \\
&\leqslant \int_a^b \max_{t\in[a,b]} |f(z(t))|\, \left|z'(t)\right|\, dz \\
&= \max_{t\in[a,b]} |f(z(t))| \int_a^b \left|z'(t)\right|\, dz = \max_{t\in[a,b]} |f(z(t))| \cdot L(C) = \max_{z\in C} |f(z)| \cdot L(C)
\end{aligned}$$

□

Example 13.4.

(1) Finding a bound for

$$\int_C \frac{z^2+1}{z^3+2}\,dz,$$

where C is the semicircle $z(t) = 2e^{it}$, $0 \leqslant t \leqslant \pi$.

All we need to find is an $M > 0$ such that, for all $z \in C$

$$\left|\frac{z^2+1}{z^2+3}\right| \leqslant M, \qquad \text{because then} \quad \max_{z\in C}\left|\frac{z^2+1}{z^2+3}\right| \leqslant M$$

Suppose $z \in C$, then $|z| = 2$, and therefore

$$|z^2+1| \leqslant |z|^2 + 1 = 5;$$

also,

$$|z^3+2| \geqslant |\,|z|^3 - 2| = |2^3 - 2| = 6.$$

Together, we get, for any $z \in C$

$$\left|\frac{z^2+1}{z^2+3}\right| \leqslant \frac{5}{6}.$$

Hence,

$$\left|\int_C \frac{z^2+1}{z^3+2}\,dz\right| \leqslant \max_{z\in C}\left|\frac{z^2+1}{z^2+3}\right| \cdot L(C) \leqslant \frac{5}{6}\cdot L(C) = \frac{5}{6}\cdot 2\pi = \frac{5\pi}{3}$$

(2) Show that

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\,dz = 0,$$

where C_R is the semicircle $z(t) = Re^{it}$, $0 \leqslant t \leqslant 2\pi$. Note that $L(C) = 2\pi R$.

Let $z \in C_R$, then $|z| = R$, and therefore

$$|z^2+z| \leqslant |z|^2 + z = R^2 + R;$$

also,

$$|z^4+2z^2+1| \geqslant |(z^2+1)| = |z^2+1|^2 \geqslant ||z|^2-1|^2 = |R^2-1|^2 = (R^2-1)^2.$$

Together, we get, for any $z \in C$ and $R > 1$

$$\left|\frac{z^2+z}{z^4+2z^2+1}\right| \leqslant \frac{R^2+R}{(R^2-1)^2}.$$

Hence,

$$\left|\int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\,dz\right| \leqslant \frac{R^2+R}{(R^2-1)^2}\cdot 2\pi R \to 0, \text{ as } R \to \infty$$

Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\,dz = 0,$$

by the Sandwich theorem.

Example 13.5 (in-class)**.** Finding a bound for

$$\int_C \frac{z^2-1}{z^4+2}\,dz,$$

where C is the sector $z(t)=5e^{it},\ \pi/4 \leqslant t \leqslant 3\pi/4$.

Answer. Let's first compute $L(C)$. We first note that $z'(t)=i5e^{it}$, therefore,

$$\begin{aligned} L(C) &= \int_{\pi/4}^{3\pi/4} \left|z'(t)\right|\,dt \\ &= \int_{\pi/4}^{3\pi/4} \left|5ie^{it}\right|\,dt \\ &= \int_{\pi/4}^{3\pi/4} 5\,dt \\ &= 5\int_{\pi/4}^{3\pi/4} dt \\ &= 5\left(\frac{3\pi}{4}-\frac{\pi}{4}\right) \\ &= \frac{5\pi}{2} \end{aligned}$$

Now, suppose $z\in C$, then $|z|=5$, and therefore

$$|z^2-1| \leqslant \left|z^2\right| + |-1| = |z|^2+1 = 26;$$

also,

$$|z^4+2| \geqslant \left|\,\left|z^4\right| - |2|\,\right| = \left|\,|z|^4-2\right| = 623.$$

Together, we get, for any $z\in C$

$$\left|\frac{z^2-1}{z^4+2}\right| \leqslant \frac{26}{623}, \quad \text{hence } \max_{z\in C}\left|\frac{z^2-1}{z^4+2}\right| \leqslant \frac{26}{623}$$

Hence,

$$\left|\int_C \frac{z^2-1}{z^4+2}\,dz\right| \leqslant \max_{z\in C}\left|\frac{z^2-1}{z^4+2}\right| \cdot L(C) \leqslant \frac{26}{623}\cdot L(C) = \frac{26}{623}\cdot\frac{5\pi}{2} = \frac{65\pi}{623}$$

□

13.1. Problems

To be added

14. Lecture 14 (5/12)

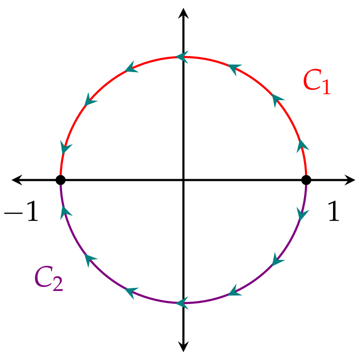
Antiderivatives & Fundamental Theorem of Contour Integrals

Discussion 14.1. Suppose C is a contour joining z_1 to z_2. In general, the value of the integral

$$\int_C f(z)\, dz$$

depends on C. For example, we have seen that

$$\int_{C_1} \frac{1}{z}\, dz = \pi i \quad \text{and} \quad \int_{C_2} \frac{1}{z}\, dz = -\pi i$$



But on the other hand we have also seen that

$$\int_C z\, dz = \frac{z_2^2 - z_1^2}{2}$$

for any contour C with initial point z_1 and end point z_2.

The difference between these functions turns out to be that $f(z) = z$ has an antiderivative on $\mathbf{C}$ while $g(z) = 1/z$ does not any domain containing C_1 and C_2.

Definition 14.2 (Antiderivative)**.** Suppose that f is a continuous function on a domain G. Any holomorphic function $F : G \to \mathbf{C}$ is called an antiderivative of f if $F'(z) = f(z)$ for every $z \in G$.

Definition 14.3 (Independence of Path)**.** Let $f : G \to \mathbf{C}$ be a continuous function on a domain G and fix $z_1, z_2 \in G$. If

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

for any pair of contours C_1 and C_2 joining z_1 to z_2, then the integral of f from z_1 to z_2 is *independent of path* and we denote the unique value by

$$\int_{z_1}^{z_2} f(z)\, dz.$$

So, for instance, we would write

$$\int_{z_1}^{z_1} z\, dz = \frac{z_2^2 - z_1^2}{2},$$

since we have already proved the integral of $f(z) = z$ from z_1 to z_2, for any $z_1, z_2 \in \mathbf{C}$, is independent of path.

Theorem 14.4 (Fundamental Theorem of Contour Integrals)**.** *Suppose f is continuous on a domain G. The following are equivalent.*

(1) f has an antiderivative $F : G \to \mathbf{C}$.

(2) For all $z_1, z_2 \in G$, the integral of f from z_1 to z_2 are independent of path.

(3) If C is any closed contour lying in G, then

$$\int_C f(z)\, dz = 0$$

If any of these conditions hold, then the unique value of the integral in (2) is given as

$$\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1)$$

where F is the antiderivative given in (1).

Proof.

(1) $\Rightarrow$ (2) Suppose f has an antiderivative $F : G \to \mathbf{C}$. Let $z_1, z_2 \in G$ and let C be any contour with initial point z_1 to z_2 and lying in G.

First assume C is a smooth arc parametrised by $z : [a, b] \to \mathbf{C}$; therefore, in particular, $z(a) = z_1$ and $z(b) = z_2$. Then we first note

$$(F \circ z)'(t) = F'(z(t))z'(t) = f(z(t))z'(t)$$

That is, we have found an antiderivative of $f(z(t))z'(t)$, the function $F \circ z$. Hence,

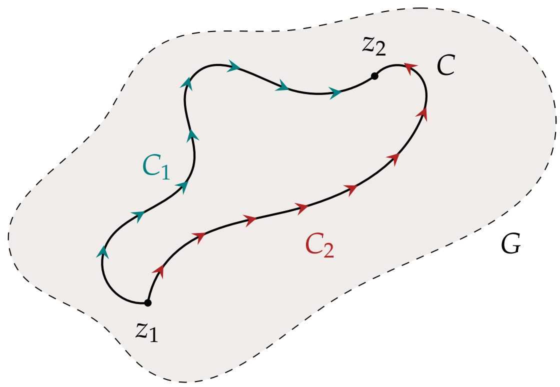
$$\begin{aligned}\int_C f(z)\, dz &= \int_a^b f(z(t))z'(t)\, dt \\ &= F(z(a)) - F(z(b)), \quad \text{by Proposition 11.6} \\ &= F(z_2) - F(z_1)\end{aligned}$$

Now, assume C is a contour; that is, we can write $C = C_1 + \cdots + C_n$, where C_i's are smooth arcs with initial point w_i and end point w_{i+1}. In particular, $w_1 = z_1$ and $w_{n+1} = z_2$. Then,

$$\begin{aligned}\int_C f(z)\, dz &= \sum_{i=1}^{n} \int_{C_i} f(z)\, dz \\ &= \sum_{i=1}^{n} F(w_{i+1}) - F(w_i) \\ &= F(w_{n+1}) - F(w_1) \\ &= F(z_2) - F(z_1)\end{aligned}$$

Since $F(z_2) - F(z_1)$ only depends on z_1 and z_2 and note the contour itself, we have proved the claim.

(2) $\Rightarrow$ (3) Let C be any closed contour lying in G, and choose two distinct point z_1 and z_2 on C. Let C_1 and C_2 be contours from z_1 to z_2 such that $C = C_1 - C_2$.



By assumption, the integral of f from z_1 to z_2 is independent of path, therefore

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

Hence,

$$\int_C f(z)\, dz = \int_{C_1 - C_2} f(z)\, dz = \int_{C_1} f(z)\, dz - \int_{C_2} f(z)\, dz = 0,$$

as claimed.

(3) $\Rightarrow$ (2) Suppose

$$\int_C f(z)\, dz = 0$$

for any closed contour C lying in G. Let $z_1, z_2 \in G$ and C_1 and C_2 are two contour with initial point z_1 and end point z_2. Then $C_1 - C_2$ is a closed contour, and therefore by assumption

$$0 = \int_{C_1 - C_2} f(z)\, dz = \int_{C_1} f(z)\, dz - \int_{C_2} f(z)\, dz$$

Hence,

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz,$$

as claimed.

(2) $\Rightarrow$ (1) Assume (2) (and also (3), since we've shown them to be equivalent). We need to show that f has an antiderivative on G. Fix any point $z_0 \in G$ and define

$$F(w) = \int_{z_0}^{w} f(z)\, dz,$$

which is well defined by (2). We need to show $F'(w) = f(w)$ for any $w \in G$. That is,

$$\lim_{h \to 0} \frac{F(w+h) - F(w)}{h} = f(w)$$

Let $\varepsilon > 0$ and consider an $z \in G$. Since f is continuous at z, we can find $\delta > 0$ such that

$$\text{if } |z - w| < \delta, \quad \text{then } |f(z) - f(w)| < \varepsilon$$

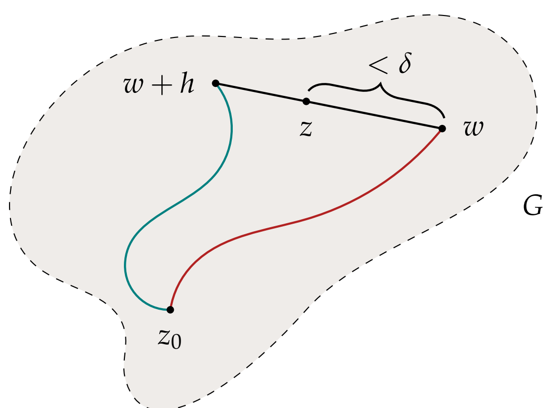
For $w \in G$, since G is a domain and so in particular an open set, we can find a $d > 0$ such that $D_d(w) \subseteq G$. Pick a $h \in \mathbf{C}$ such that $0 < |h| < \min\{d, \delta\}$. Then $0 < |h| < d$ and $0 < |h| < \delta$. In particular, $w + h \in D_d(w) \subseteq G$; then,

$$F(w+h) - F(w) = \int_{z_0}^{w+h} f(z)\, dz - \int_{z_0}^{w} f(z)\, dz = \int_{w}^{w+h} f(z)\, dz$$

Since our integrals are path-independent, we assume that the integral above is over a line segment from w to $w + h$, which lies in G, since $D_d(w)$ is convex. Also,

$$f(w) = \frac{f(w)h}{h} = \frac{1}{h} f(w) \int_{w}^{w+h} dz = \frac{1}{h} \int_{w}^{w+h} f(w)\, dz$$

Also, since $|h| < \delta$, then $|z - w| < \delta$ for any point z lying on ℓ, the line segment joining w to $w + h$. Therefore, $|f(z) - f(w)| < \varepsilon$ for any $z \in \ell$, that is, $\max_{z \in \ell} |f(z) - f(w)| < \varepsilon$.



Using the preceding computations we have

$$\begin{aligned}
\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| &= \left| \frac{1}{h} \int_{w}^{w+h} f(z)\, dz - \frac{1}{h} \int_{w}^{w+h} f(w)\, dz \right| \\
&= \frac{1}{h} \left| \int_{w}^{w+h} f(z) - f(w)\, dz \right| \\
&\leqslant \frac{1}{h} \max_{z \in \ell} |f(z) - f(w)| \cdot L(\ell) \\
&< \frac{\varepsilon}{h} \cdot L(\ell) \\
&= \varepsilon, \quad \text{since } L(\ell) = h
\end{aligned}$$

We have shown that given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

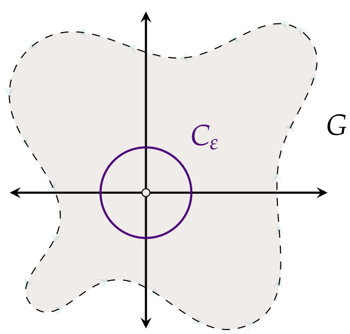
$$\text{if } |h| < \delta, \quad \text{then } \left| \frac{F(w+h) - F(w)}{h} - f(w) \right| < \varepsilon$$

That is, $F'(w) = f(w)$, for all $w \in G$.

□

Example 14.5.

(1) The function $f(z) = 1/z$ has no antiderivative on $\mathbf{C}^*$. In fact, it has no antiderivative on any domain G containing a deleted neighbourhood of 0. Take a circle $C_\varepsilon = C_\varepsilon(0)$ with radius $\varepsilon > 0$ such that it lies in our domain G.



Then,

$$\begin{aligned}\int_{C_\varepsilon} \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt \\ &= \int_0^{2\pi} i\, dt \\ &= 2\pi i\end{aligned}$$

By Theorem 14.4, $f(z)$ does not have an antiderivative on such a domain, as the integral over the closed contour C_ε was non-zero. The problem is as follows: it is true that that a branch of the logarithm $F(z) = \log z$ is such that

$$F'(z) = \frac{1}{z},$$

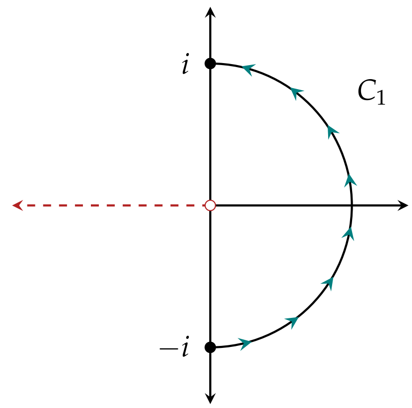
but it is only holomorphic on the complement of the branch cut. Since our domain contains a deleted neighbourhood of 0, it has a non-empty intersection with any branch cut we take, and therefore F is not holomorphic on G. This argument, in particular, holds for the domain $\mathbf{C}^*$.

(2) The function $f(z) = \cos z$ is entire on $\mathbf{C}$, so is $F(z) = \sin z$. Moreover $F'(z) = \cos z = f(z)$, so f has an antiderivative on $\mathbf{C}$. So, for instance

$$\int_0^{\pi i} \cos z\, dz = \sin \pi i - \sin 0 = \sin \pi i$$

(3) Although $f(z) = 1/z$ has no antiderivative on any domain containing a deleted neighbourhood of 0, we can integrate f over a circle C by using two different antiderivatives.

Let C_1 be parametrised by $z(t) = e^{it}$, $t \in [-\pi/2, \pi/2]$, a contour from $-i$ to i.



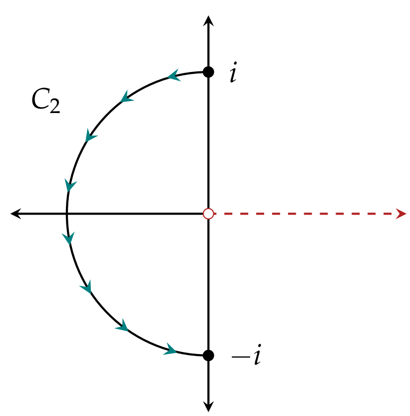
On $\mathbf{C} \setminus \mathbf{R}_{\leqslant 0}$, $f(z)$ has an antiderivative, namely the principal branch of the logarithm

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z < \pi$$

Then, by Theorem 14.4

$$\begin{aligned}\int_{C_1} \frac{1}{z}\, dz &= \operatorname{Log} i - \operatorname{Log}(-i) \\ &= (\ln|i| + i \operatorname{Arg} i) - (\ln|-i| + i \operatorname{Arg}(-i)) \\ &= \left(\ln 1 + i\frac{\pi}{2}\right) - \left(\ln 1 - i\frac{\pi}{2}\right) \\ &= \pi i\end{aligned}$$

Let C_2 be parametrised by $z(t) = e^{it}$, $t \in [\pi/2, 3\pi/2]$, a contour from i to $-i$.



On $\mathbf{C} \setminus \mathbf{R}_{\geqslant 0}$, $f(z)$ has an antiderivative, namely the following branch of the logarithm

$$\log z = \ln|z| + i \arg z, \quad 0 < \arg z < 2\pi$$

Then, by Theorem 14.4

$$\begin{aligned}
\int_{C_2} \frac{1}{z}\, dz &= \log(-i) - \log i \\
&= (\ln|i| + i\arg(-i)) - (\ln|-i| + i\arg i) \\
&= \left(\ln 1 + i\frac{3\pi}{2}\right) - \left(\ln 1 + i\frac{\pi}{2}\right) \\
&= \pi i
\end{aligned}$$

Hence,

$$\int_C \frac{1}{z}\, dz = \int_{C_1} \frac{1}{z}\, dz + \int_{C_2} \frac{1}{z}\, dz = \pi i + \pi i = 2\pi i$$

14.1. Problems

To be added

15. Lecture 15 (5/17)

Cauchy-Goursat Theorem

Discussion 15.1. The Cauchy-Goursat theorem gives a sufficient condition for the integral of a function over a simple closed curve to be zero. The theorem has powerful implications, ultimately it leads to

- The Cauchy Integral formula.
- The theory of residues for computing contour integrals.
- A method to evaluate real-valued functions in a real variable, using contour integration.

Historically, a weaker version of the theorem was first proved by Cauchy. We prove this first.

We first note the following.

(1) Contour integrals are related to line (or path) integrals. We note this by writing our function $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$ and formally writing $dz = dx + i\,dy$. Then formally,

$$\begin{aligned} \int_C f(z)\,dz &= \int_C (u+iv)(dx+idy) \\ &= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx \end{aligned}$$

(2) **Green's Theorem.** Suppose C is a simple closed contour in $\mathbf{R}^2$ and let R be the region enclosed by C and including C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on R. Then

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_R (Q_x - P_y)\,dA$$

Theorem 15.2 (Weak Cauchy Integral Theorem)**.** *Let C be a simple closed contour, and let R denote the region consisting of C and its interior. If f is holomorphic on R and f' continuous on R, then*

$$\int_C f(z)\,dz = 0.$$

Proof. If $f(z) = u(x,y) + i\,v(x,y)$ is holomorphic on R, then the Cauchy-Riemann equations hold, and so $u_x = v_y$ and $u_y = -v_x$ on R, and $f'(z) = u_x + i\,v_x = v_y - i\,u_y$.

Since f' is continuous, so are u_x, u_y, v_x and v_y. Hence,

$$\begin{aligned} \int_C f(z)\,dz &= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx \\ &= \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA, \quad \text{by Green's theorem} \\ &= 0, \quad \text{using Cauchy-Riemann equations} \end{aligned}$$

□

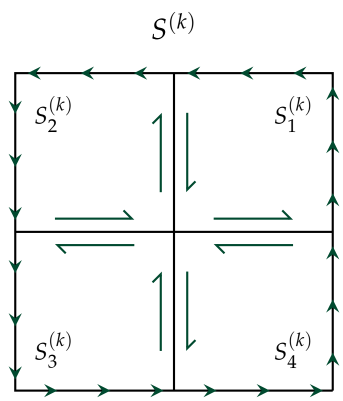
Goursat was the first to prove that the assumption on the continuity of f' can be omitted. This turns out to be essential for the theory of holomorphic functions. The problem is that it may be difficult to prove the derivative of holomorphic function is continuous.

Theorem 15.3 (Cauchy-Goursat Theorem)**.** *Let C be a simple closed contour, and let R denote the region consisting of C and its interior. If f is holomorphic on R, then*

$$\int_C f(z)\,dz = 0.$$

Proof (skipped in class). **For simplicity, we will assume C is a square.** The idea is to "divide and conquer". We break the curve into a finite number of smaller squares on which we can estimate the integral. We first construct a sequence of positively oriented curves $S^{(k)}$, each of which is the boundary of a square region $R^{(k)}$.

To begin with, set $S^{(0)} = C$. Then, inductively, after the first k squares have been chosen, we define $(k+1)^{\text{th}}$ square as follows. Divide $S^{(k)}$ into four congruent squares with positive orientation: $S_1^{(k)}$, $S_2^{(k)}$, $S_3^{(k)}$, $S_4^{(k)}$.



Note that the integral of f along the shared boundaries of these squares cancel. Hence,

$$\sum_{i=1}^{4} \int_{S_i^{(k)}} f(z)\,dz = \int_{S^{(k)}} f(z)\,dz$$

We choose $S^{(k+1)}$ to be one of the squares $S_j^{(k)}$ such that

$$\left|\int_{S^{(k+1)}} f(z)\,dz\right| = \left|\int_{S_j^{(k)}} f(z)\,dz\right| = \left|\max_{i=1}^{4} \int_{S_i^{(k)}} f(z)\,dz\right|$$

At this point, we have a sequence $S^{(0)}, \ldots, S^{(k)}, \ldots$. Note that, by triangle inequality

$$\left|\int_{S^{(k)}} f(z)\,dz\right| \leqslant \sum_{i=1}^{4} \left|\int_{S_i^{(k)}} f(z)\,dz\right| \leqslant 4\left|\int_{S^{(k+1)}} f(z)\,dz\right|$$

So, inductively we get

$$\left|\int_C f(z)\,dz\right| = \left|\int_{S^{(0)}} f(z)\,dz\right| \leqslant 4^n \left|\int_{S^{(n)}} f(z)\,dz\right| \qquad (*)$$

We record some more facts. Denote by $d^{(n)}$ the length of the diagonal of the n^{th} square $S^{(n)}$ and denote by $P^{(n)}$ its perimeter. Then,

$$d^{(n)} = \frac{1}{2^n} \cdot d^{(0)}$$

$$p^{(n)} = \frac{1}{2^n} \cdot p^{(0)}$$

Also, $d^{(n)}$, $p^{(n)} \to 0$, as $n \to \infty$.

Next, consider the associated sequence of regions

$$R = R^{(0)} \supseteq R^{(1)} \supseteq \cdots \supseteq R^{(k)} \supseteq \cdots .$$

Each $R^{(k)}$ is compact (closed and bounded) and hence, using a fact from topology, there exists a unique point

$$z_0 \in \bigcap_{i \geqslant 0} R^{(i)}.$$

Since $z_0 \in R^{(0)} = R$, f is holomorphic at z_0. So, we define the following function on R

$$\psi(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

and we note

$$\lim_{z \to z_0} \psi(z) = f'(z_0) - f'(z_0) = 0 = \psi(z_0),$$

and therefore ψ is continuous at z_0. We can write

$$f(z) = f(z_0) + (z - z_0)(\psi(z) + f'(z_0)) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

Note that $f(z_0)$ and $f'(z_0)(z - z_0)$ have antiderivatives on $\mathbf{C}$, hence, by Theorem 14.4, we have

$$\begin{aligned} \int_{S^{(n)}} f(z)\, dz &= \int_{S^{(n)}} f(z_0)\, dz + \int_{S^{(n)}} f'(z_0)(z - z_0)\, dz + \int_{S^{(n)}} \psi(z)(z - z_0)\, dz \\ &= 0 + 0 + \int_{S^{(n)}} \psi(z)(z - z_0)\, dz \\ &= \int_{S^{(n)}} \psi(z)(z - z_0)\, dz \end{aligned}$$

Consider $\varepsilon > 0$. Since ψ is continuous at z_0 with $\psi(z_0) = 0$, choose $\delta > 0$ such that

$$\text{if } |z - z_0| < \delta, \quad \text{then } |\psi(z)| < \varepsilon$$

Since $d^{(n)} \to 0$, as $n \to \infty$, we choose an $N \in \mathbf{Z}_{>0}$ such that $|d^{(n)}| < \delta$ for every $n \geqslant N$. Thus, if $z \in S^{(N)}$, then $|z - z_0| < |d^{(N)}| < \delta$ and therefore $|\psi(z)| < \varepsilon$ for every $z \in S^{(N)}$. Hence,

$$\max_{z \in S^{(N)}} |z - z_0| < d^{(N)} \quad \text{and} \quad \max_{z \in S^{(N)}} |\psi(z)| < \varepsilon$$

Hence, we obtain

$$\begin{aligned}
\left|\int_{S^{(N)}} f(z)\, dz\right| &= \left|\int_{S^{(N)}} \psi(z)(z-z_0)\, dz\right| \\
&\leqslant \max_{z\in S^{(N)}} |\psi(z)|\,|z-z_0| \cdot L(S^{(N)}) \\
&< \varepsilon \cdot d^{(N)} \cdot L(S^{(N)}) \\
&= d^{(N)} p^{(N)} \varepsilon \\
&= \frac{1}{4^N} d^{(0)} p^{(0)} \varepsilon
\end{aligned}$$

By $(*)$, we have

$$\begin{aligned}
\left|\int_C f(z)\, dz\right| \leqslant 4^N \left|\int_{S^{(N)}} f(z)\, dz\right| \\
< 4^N \cdot \frac{1}{4^N} d^{(0)} p^{(0)} \varepsilon \\
= d^{(0)} p^{(0)} \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we necessarily get that

$$\left|\int_C f(z)\, dz\right| \leqslant 0$$

Thus,

$$\int_C f(z)\, dz = 0$$

□

Simply Connected Domains

Definition 15.4 (Simply Connected Domain)**.** A domain G is called simply connected if it has the following property: if C is any simple closed contour lying in G and z is interior to C, then $z \in G$.

Intuitively, a simply connected domain is a domain that has no "holes".

Open disks, complex plane, interior of any simple closed contour etc. are all examples of simply connected domains. While deleted open disks, $\mathbf{C} \setminus \{p\}$ etc. are examples of non-simply connected domains.

A result similar to Theorem 15.3 holds for closed contours, not necessarily simple, provided they lie in a simply connected domain.

Theorem 15.5 (Cauchy-Goursat Theorem for Simply Connected Domain)**.** *Suppose f is holomorphic on a simply connected G. If C is any closed contour lying in G, then*

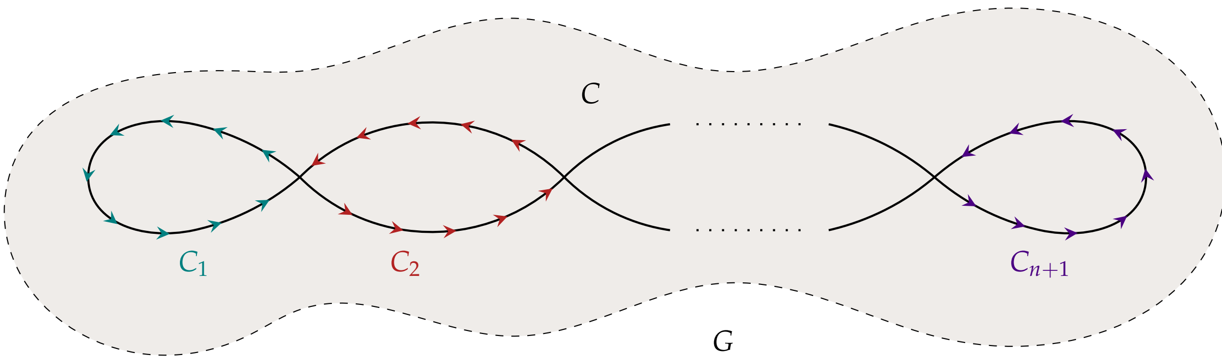
$$\int_C f(z)\, dz = 0.$$

Proof. We are presented with two cases: C has finitely many self-intersections, or infinitely many self-intersections. Let's focus on the first cases, where the proof is a consequence of Theorem 15.3.

Suppose C has n-many self-intersections, then those points of self-intersections allow us to write

$$C = C_1 + C_2 + \cdots + C_{n+1},$$

where each C_i is a simple closed contour that all, necessarily, lie in G.



Therefore f is holomorphic at each point interior of and on C_i, hence by Theorem 15.3 we get

$$\int_{C_i} f(z)\, dz = 0$$

Finally, we have

$$\int_C f(z)\, dz = \sum_{i=1}^{n} \int_{C_i} f(z)\, dz = 0$$

as claimed.

The proof in the case the contour has infinitely many self-intersections is subtle , so we assume validity without a proof. □

Corollary 15.6 (Antiderivatives of Holomorphic Functions). *If f is holomorphic on a simply connected domain G, then f has an antiderivative on G.*

Proof. By Theorem 15.5,

$$\int_C f(z)\, dz = 0$$

for any closed contour C lying in G. By Theorem 14.4, this is equivalent to f having an antiderivative on G. □

Corollary 15.7 (Entire Functions have Antiderivatives). *Suppose f is entire, then f has an antiderivative on $\mathbf{C}$ which is necessarily also entire.*

Proof. $\mathbf{C}$ is simply connected, the result follows from Corollary 15.6. □

Multiply Connected Domains

Definition 15.8 (Multiply Connected Domain). A domain G is called multiply connected if it not simply connected.

We can generalise Theorem 15.5 to a multiply connected domain with finitely many holes.

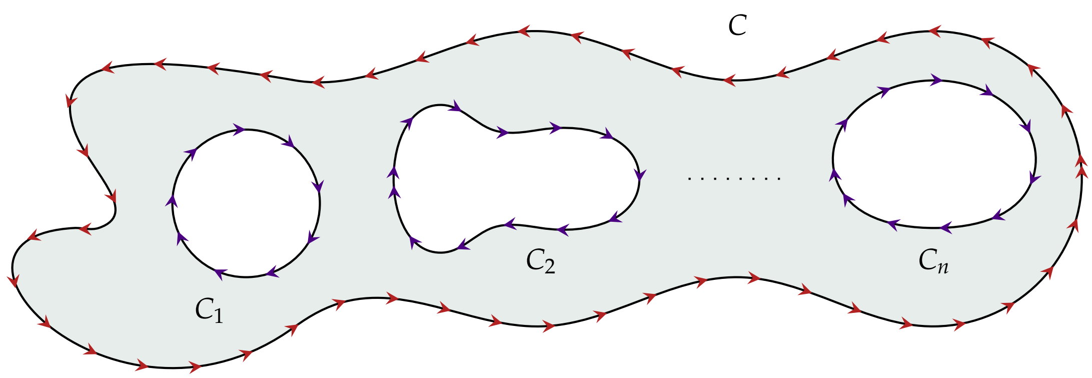
Theorem 15.9 (Generalised Cauchy-Goursat Theorem). *Suppose that*

(1) C is a simple closed positively oriented contour.

(2) $C_1, \ldots, C_n$ are simple closed negatively oriented contours enclosing regions $R_1, \ldots, R_n$. Further assume that the regions are pairwise disjoint and interior to C.

If f is holomorphic on each contour and the region consisting of all points interior to C but exterior to each C_i, then

$$\int_C f(z)\, dz + \sum_{i=1}^{n} \int_{C_i} f(z)\, dz = 0$$



Proof. We prove this using induction.

Base Case. $n = 1$. Assume C and C_1 are contour satisfying the hypotheses. Let z_1, z_2 be points on C while w_1, w_2 be points on C_1. Join z_1 to w_1 with a polygon line L_1, and also join z_2 to w_2 with a polygon line L_2.

Define contour Γ_1 and Γ_2 as follows.

Γ_1: Start with z_1 and follow to w_1 along L_1, then w_1 to w_2 along C_1 (we'll call this C_{11}), then w_2 to z_2 along L_2, and finally z_2 to z_1 along C (we'll call this C'). So,

$$\Gamma_1 = L_1 + C_{11} + L_2 + C'$$

Γ_2: Start with z_2 and follow to w_2 along $-L_2$, then w_2 to w_1 along C_1 (we'll call this C_{12}), then w_1 to z_1 along $-L_1$, and finally z_1 to z_2 along C (we'll call this C''). So,

$$\Gamma_2 = -L_2 + C_{12} - L_1 + C''$$

We note $C' + C'' = C$ and $C_{11} + C_{12} = C$.

insert image

Then f is holomorphic in the interior of and on the simple closed curves Γ_1 and Γ_2, so by Theorem 15.3 we have

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz = 0$$

So, this gives us

$$\begin{aligned}
0 &= \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz \\
&= \left(\int_{L_1} f(z)\,dz + \int_{C_{11}} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C'} f(z)\,dz\right) \\
&\quad + \left(-\int_{L_2} f(z)\,dz + \int_{C_{12}} f(z)\,dz - \int_{L_1} f(z)\,dz + \int_{C''} f(z)\,dz\right) \\
&= \int_{C'} f(z)\,dz + \int_{C''} f(z)\,dz + \int_{C_{11}} f(z)\,dz + \int_{C_{12}} f(z)\,dz \\
&= \int_{C} f(z)\,dz + \int_{C_1} f(z)\,dz
\end{aligned}$$

Inductive Step. Assume the statement holds for $n = k$, that is

$$\int_C f(z)\,dz + \sum_{i=1}^{k} \int_{C_i} f(z)\,dz = 0$$

for any k-many contours satisfying the hypotheses.

Now, let $C_1, \ldots, C_k, C_{k+1}$ be any $k+1$-many contours. Introduce a polygon line L that separates $C_1, \ldots, C_k$ from C_{k+1}, say with end points z_1 and z_2. We define Γ_1 and Γ_2 as follows.

Γ_1: Start with z_1 and follow to z_2 along C (we'll call this C'), then z_2 to z_1 along $-L$. So,

$$\Gamma_1 = C' - L$$

Γ_2: Start with z_1 and follow to z_2 along L, then z_2 to z_1 along C (we'll call this C''). So,

$$\Gamma_2 = C'' + L$$

We note $C' + C'' = C$.

insert image

We note that

$$\begin{aligned}
\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz &= \left(\int_{C'} f(z)\,dz - \int_{L} f(z)\,dz\right) + \left(\int_{C''} f(z)\,dz + \int_{L} f(z)\,dz\right) \\
&= \int_{C'} f(z)\,dz + \int_{C''} f(z)\,dz \\
&= \int_{C} f(z)\,dz \qquad (\dagger)
\end{aligned}$$

By the inductive hypothesis

$$\int_{\Gamma_1} f(z)\,dz + \sum_{i=1}^{k} \int_{C_i} f(z)\,dz = 0 \tag{1}$$

and by the computation in the base case we have

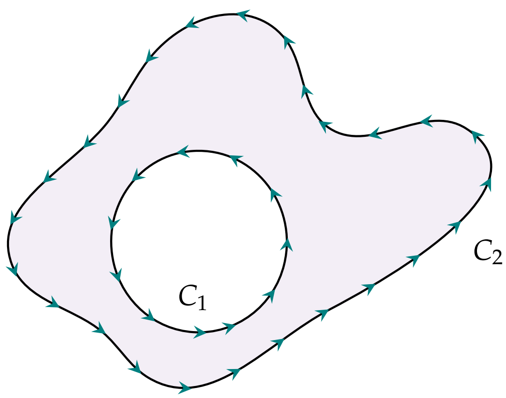
$$\int_{\Gamma_2} f(z)\,dz + \int_{C_{k+1}} f(z)\,dz = 0 \tag{2}$$

Adding (1) and (2) and using (†) we have

$$0 = \int_{\Gamma_1} f(z)\,dz + \sum_{i=1}^{k} \int_{C_i} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{C_{k+1}} f(z)\,dz = \int_{C} f(z)\,dz + \sum_{i=1}^{k+1} \int_{C_i} f(z)\,dz$$

Thus, we have proved our result using the principle of mathematical induction. □

Corollary 15.10 (Principle of Deformation of Paths)**.** *Suppose C_1 and C_2 are positively oriented simple closed contours with C_1 interior to C_2.*



If f is holomorphic on the region consisting of C_1 and C_2 and all the points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Proof. Applying Theorem 15.9 to C_2 and $-C_1$, we get

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0.$$

Therefore,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

□

Among other things, the principle of deformation of paths is useful for integrating over complicated contours. Often, we can just replace this contour with a circle.

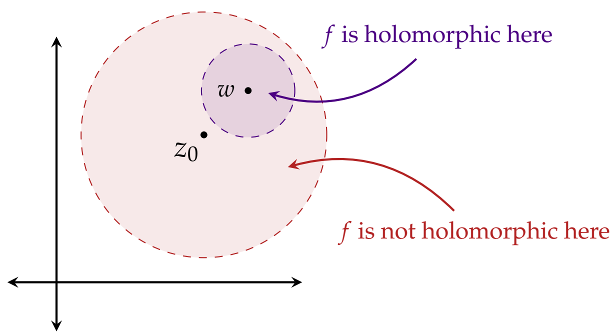
Example 15.11. Let C be any simple closed contour whose interior contains 0. We show that

$$\int_C \frac{1}{z}\, dz = 2\pi i.$$

Since 0 is interior to C, we can choose an $\varepsilon > 0$ small enough such that $C_\varepsilon = C_\varepsilon(0)$ is contained in the interior of C. The region containing C and C_ε and points between them does not contain 0, so $1/z$ is holomorphic there. By Corollary 15.10,

$$\begin{aligned}\int_C f(z)\, dz &= \int_{C_\varepsilon} f(z)\, dz \\ &= \int_0^{2\pi} \frac{1}{\varepsilon e^{it}}\, i e^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i\end{aligned}$$

Definition 15.12 (Singularities)**.** Suppose f is not holomorphic at z_0, but every neighbourhood of z_0 contains a point at which f is holomorphic, then z_0 is called a singular point (or singularity) *of* f.



Example 15.13.

(1) $f(z) = \dfrac{1}{z}$ has a singularity at 0.

(2) $f(z) = |z|^2$ has no singular points, as f is only differentiable at 0 but is nowhere holomorphic.

(3) $f(z) = \dfrac{z^2+3}{(z+1)(z^2+5)}$ has singularities at those z where

$$(z+1)(z^2+5) = 0.$$

That is, at -1, $i\sqrt{5}$ and $-i\sqrt{5}$.

Remark 15.14. More generally, the generalised Generalised Cauchy-Goursat Theorem (Theorem 15.9) and its Corollary 15.10 provide a technique for integrating functions over contours whose interior contains singularities of that function. The idea is to introduce small circles around the singular points, and apply the theorem (or corollary). It is usually easy to integrate over a circle.

15.1. Problems

To be added

16. Lecture 16 (5/19)

Cauchy's Integral Formula

Discussion 16.1. Cauchy's Integral Formula is a remarkable theorem. It asserts that if a function is holomorphic inside and on C, a simple closed contour, then its values interior to C are completely determined by its values on C.

Theorem 16.2 (Cauchy's Integral Formula)**.** *Let C be a simple closed contour, with positive orientation, and let f be a function that is holomorphic at all points on and interior to C. The for any $z_0 \in \text{int}(C)$, we have*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

Proof. Our strategy is to show that for all $\varepsilon > 0$, we get

$$\left| \int_C \frac{f(z)}{z - z_0}\, dz - f(z) \cdot 2\pi i \right| < \varepsilon$$

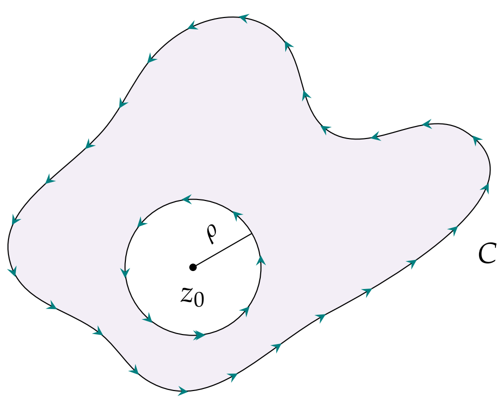
because then

$$\int_C \frac{f(z)}{z - z_0}\, dz - f(z) \cdot 2\pi i = 0$$

Let $\varepsilon > 0$, and since, by assumption, f is holomorphic on z_0, it's continuous on z_0. So, there exists a $\delta > 0$ such that

$$\text{if } \; |z - z_0| < \delta, \quad \text{then } \; |f(z) - f(z_0)| < \frac{\varepsilon}{2\pi}$$

Let $\rho > 0$ be small enough such that the circle $C_\rho = C_\rho(z_0)$ centered at z_0 of radius ρ lies in the interior of C; assume C_ρ has positive orientation.



We may assume $\rho < \delta$, then for every point $z \in C_\rho$, since $|z - z_0| = \rho < \delta$, we have

$$|f(z) - f(z_0)| < \frac{\varepsilon}{2\pi}, \quad \text{therefore } \; \max_{z \in C_\rho} |f(z) - f(z_0)| < \frac{\varepsilon}{2\pi}$$

Now, note that

$$\frac{f(z)}{z - z_0}$$

is holomorphic on the region consisting of C, C_ρ and all points that are interior to C but exterior to C_ρ. So, by Corollary 15.10, we have

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

and then

$$\begin{aligned}
\left|\int_C \frac{f(z)}{z-z_0}\,dz - f(z)\cdot 2\pi i\right| &= \left|\int_{C_\rho} \frac{f(z)}{z-z_0} - f(z)\cdot 2\pi i\right| \\
&= \left|\int_{C_\rho} \frac{f(z)}{z-z_0} - f(z)\int_{C_\rho} \frac{1}{z-z_0}\,dz\right| \\
&= \left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \\
&\leqslant \max_{z\in C_\rho}\left|\frac{f(z)-f(z_0)}{z-z_0}\right|\cdot L(C_\rho) \\
&= \max_{z\in C_\rho}\frac{|f(z)-f(z_0)|}{\rho}\cdot(2\pi\rho) \\
&= \frac{1}{\rho}\max_{z\in C_\rho}|f(z)-f(z_0)|\cdot(2\pi\rho) \\
&< \frac{\varepsilon}{2\pi}\cdot 2\pi \\
&= \varepsilon
\end{aligned}$$

and the claim follows. □

Among other things, Cauchy's Integral formula is useful for computing integrals.

Example 16.3.

(1) Let's compute $\int_C \frac{\cos z}{z(z^2+2)}\,dz$, where C is the unit circle, positively oriented.

Consider

$$f(z) = \frac{\cos z}{z^2+2}$$

Then f is holomorphic on all points on and interior to C, as they don't include $\pm 2i$ and 0 is in the interior or C. Therefore, by Cauchy's Integral Formula (Theorem 16.2) we have

$$\int_C \frac{\cos z}{z(z^2+2)}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i\cdot f(0) = \pi i.$$

(2) Let's compute $\int_C \frac{e^{z^2}}{z-1}\,dz$, where C is a positively oriented circle with radius 2.

Consider $f(z) = e^{z^2}$, then f is entire, and therefore holomorphic on all points on and interior to C. Therefore, by Cauchy's Integral Formula (Theorem 16.2) we have

$$\int_C \frac{e^{z^2}}{z-1}\, dz = 2\pi i f(1) = 2\pi i e.$$

(3) Let's compute $\displaystyle\int_C \frac{z^2+1}{z^2-1}\, dz = \int_C \frac{z^2+1}{(z-1)(z+1)}\, dz$, where C is as follows

The contour C is not simple but it can be decomposed as a sum of simple closed contours $C = C_1 - C_2$

So,

$$\int_C \frac{z^2+1}{(z-1)(z+1)}\, dz = \int_{C_1} \frac{z^2+1}{(z-1)(z+1)}\, dz - \int_{C_2} \frac{z^2+1}{(z-1)(z+1)}\, dz$$

For C_1, consider

$$f(z) = \frac{z^2+1}{z+1},$$

then f is holomorphic on all points on and interior to C_1, as they don't include -1, and 1 is in the interior or C_1. Therefore by Cauchy's Integral Formula (Theorem 16.2) we have

$$\int_{C_1} \frac{z^2+1}{(z-1)(z+1)}\, dz = \int_{C_1} \frac{f(z)}{z-1}\, dz = 2\pi i \cdot f(1) = 2\pi i$$

For C_2, consider

$$g(z) = \frac{z^2+1}{z-1},$$

then g is holomorphic on all points on and interior to C_2, as they don't include 1, and -1 is in the interior or C_2. Therefore by Cauchy's Integral Formula (Theorem 16.2) we have

$$\int_{C_2} \frac{z^2+1}{(z-1)(z+1)}\, dz = \int_{C_2} \frac{g(z)}{z+1}\, dz = 2\pi i \cdot g(-1) = -2\pi i$$

Hence,

$$\int_C \frac{z^2+1}{(z-1)(z+1)}\, dz = \int_{C_1} \frac{z^2+1}{(z-1)(z+1)}\, dz - \int_{C_2} \frac{z^2+1}{(z-1)(z+1)}\, dz = 2\pi i + 2\pi i = 4\pi i.$$

Theorem 16.4 (Generalised Cauchy's Integral Formula)**.** *Let C be a simple closed contour, with positive orientation, and let f be a function that is holomorphic at all points on and interior to C. The for any $z_0 \in \operatorname{int}(C)$, we have that $f^{(n)}(z_0)$ exists and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Proof. We prove by induction, with the base case $n = 0$ being just Theorem 16.2. Assume the statements holds for $n = k$, we need to prove that

$$f^{(k+1)}(z_0) := \lim_{h\to 0} \frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} = \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{(k+1)+1}}\, dz$$

We assume $|h|$ is small enough such that $z + h \in \text{int}(C)$, then by the inductive hypothesis

$$f^{(k)}(z_0+h) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-(z_0+h))^{k+1}}\, dz$$

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

Recall the algebraic identity, that for a, $b \in \mathbf{C}$ we have

$$a^{k+1} - b^{k+1} = (a-b)(a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k),$$

We will apply this to $a = \dfrac{1}{z-z_0-h}$ and $b = \dfrac{1}{z-z_0}$, and we also note $\lim_{h\to 0} a = b$. Then,

$$\begin{aligned}
&\lim_{h\to 0} \frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} \\
&= \lim_{h\to 0} \frac{k!}{2\pi i} \int_C \frac{f(z)}{h} \left(\frac{1}{(z-z_0-h)^{k+1}} - \frac{1}{(z-z_0)^{k+1}} \right) dz \\
&= \lim_{h\to 0} \frac{k!}{2\pi i} \int_C \frac{f(z)}{h} \left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0} \right) (a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k)\, dz \\
&= \lim_{h\to 0} \frac{k!}{2\pi i} \int_C \frac{f(z)}{h} \left(\frac{h}{(z-z_0-h)(z-z_0)} \right) (a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k)\, dz \\
&= \lim_{h\to 0} \frac{k!}{2\pi i} \int_C \left(\frac{f(z)}{(z-z_0-h)(z-z_0)} \right) (a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k)\, dz \\
&= \frac{k!}{2\pi i} \int_C \lim_{h\to 0} \left(\frac{f(z)}{(z-z_0-h)(z-z_0)} \right) (a^k + a^{k-1}b + \cdots + ab^{k-1} + b^k)\, dz \\
&= \frac{k!}{2\pi i} \int_C \left(\frac{f(z)}{(z-z_0)^2} \right) (b^k + b^{k-1}b + \cdots + b \cdot b^{k-1} + b^k)\, dz \\
&= \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} \cdot (k+1) \cdot b^k\, dz \\
&= \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} \cdot \frac{1}{(z-z_0)^k}\, dz \\
&= \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{(k+1)+1}}\, dz
\end{aligned}$$

Thus we have our result by the principle of mathematical induction. $\square$

Example 16.5. Compute the integral

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz$$

where C is any simple closed positively oriented contour whose interior contains 0 and $0 \leqslant k \leqslant n$.

Let $f(z) = (1+z)^n$, since f is entire, f is holomorphic on all points on and interior to C. Since 0 is in the interior of C, then generalised Cauchy's Integral formula (Theorem 16.4) gives us

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{k!}\left(\frac{k!}{2\pi i}\int_C \frac{(1+z)^n}{(z-0)^{k+1}}\,dz\right) = \frac{1}{k!}\cdot f^{(k)}(0)$$

We have,

$$f^{(k)}(z) = n(n-1)\cdots(n-(k-1))(1+z)^{n-k},$$

and therefore

$$f^{(k)}(0) = n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$$

Hence,

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{k!}\cdot f^{(k)}(0) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Theorem 16.6 (Derivatives of Holomorphic functions are Holomorphic)**.** *Suppose that f is holomorphic at $z_0 \in \mathbf{C}$, then for all $n \in \mathbf{Z}_{>0}$, $f^{(n)}$ is also holomorphic at z_0.*

Proof. Suppose f is holomorphic at $z_0 \in \mathbf{C}$. Choose an open disk $D_\varepsilon(z_0)$ on which f is differentiable. To conclude f' exists and is holomorphic at z_0, it's enough to find a neighbourhood of z_0 where $f''(w)$ exists for all w in that neighbourhood. Let C be the positive oriented circle of radius $\varepsilon/2$ centered at z_0, then f is holomorphic on all points on and interior to C. So, by generalised Cauchy's Integral formula (Theorem 16.4),

$$f''(w) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-w)^3}\,dz$$

for any w in the interior of C. Thus, f' is differentiable in the open set $D_{\varepsilon/2}(z_0)$, and hence f' is holomorphic at z_0. Induction then gives us that $f^{(n)}$ is holomorphic at z_0 for any $n \in \mathbf{Z}_{>0}$. $\square$

Corollary 16.7. *If $f(z) = u(x,y) + iv(x,y)$ is holomorphic at $z = x+iy$, then u and v have continuous partial derivatives of all orders at (x,y).*

16.1. Problems

To be added

17. Lecture 17 (5/24)

Theorem 17.1 (Morera's Theorem)**.** *Suppose f is continuous on a domain G. If*

$$\int_C f(z)\,dz = 0$$

for every closed contour G, then f is holomorphic on G.

Proof. By Theorem 14.4, there exists a holomorphic function $F : G \to \mathbf{C}$ such that $F'(z) = f(z)$ for all $z \in G$. But by Theorem 16.6, F' is holomorphic on G, and therefore so is f. $\square$

Remark 17.2. When G is simply connected, Morera's theorem (Theorem 17.1) is just the converse of Cauchy-Goursat Theorem for simply connected domains (Theorem 15.5).

Theorem 17.3 (Cauchy's Inequalities)**.** *Suppose that f is holomorphic on all points on and interior to $C_R = C_R(z_0)$, a positively oriented circle of radius R centered at some $z_0 \in \mathbf{C}$. Then,*

$$|f^{(n)}(z_0)| \leqslant \frac{n!}{R^n} \max_{z \in C_R(z_0)} |f(z)|$$

Proof. By Theorem 16.4,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Hence,

$$\begin{aligned} |f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right| &= \frac{n!}{2\pi i} \left| \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right| \\ &\leqslant \frac{n!}{2\pi i} \max_{z \in C_R} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \cdot L(C_R) \\ &= \frac{n!}{2\pi i} \max_{z \in C_R} \frac{|f(z)|}{R^{n+1}} \cdot 2\pi R \\ &= \frac{n!}{R^n} \max_{z \in C_R} |f(z)| \end{aligned}$$

$\square$

Liouville's Theorem and the Fundamental Theorem of Algebra

As an application, we will prove that every non-constant polynomial with complex coefficients has a root in $\mathbf{C}$. In the language of algebra, we will provide a proof for the fact that $\mathbf{C}$ is *algebraically closed*. Thus, the the statement is "purely algebraic" while no "purely algebraic" proof exists. The proof relies on the following wonderful theorem.

Theorem 17.4 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

Proof. We show that $f'(z) = 0$ for all $z \in \mathbf{C}$, then it follows that f is constant since $\mathbf{C}$ is a domain by Theorem 8.4.

Consider any $z_0 \in \mathbf{C}$. Since f is bounded, we can find a $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbf{C}$. Let $C_R(z_0)$ be a circle of radius R centered at z_0, then f is holomorphic at all points on and interior to $C_R(z_0)$. Hence, by Theorem 17.3,

$$\begin{aligned} |f'(z_0)| &\leqslant \frac{1}{R} \max_{z \in C_R(z_0)} |f(z)| \\ &\leqslant \frac{M}{R} \to 0, \ \text{ as } R \to \infty \end{aligned}$$

Thus $|f'(z_0)| = 0$, giving us $f'(z_0) = 0$. Since z_0 was arbitrary, the result follows. □

Theorem 17.5 (Fundamental Theorem of Algebra)**.** *For any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ where $a_n \neq 0$, $a_i \in \mathbf{C}$ and $n \geqslant 1$, there exists an $\alpha \in \mathbf{C}$ such that $p(\alpha) = 0$. That is, every non-constant polynomial $p(z)$ has at least one root in $\mathbf{C}$.*

Proof. Suppose otherwise that $p(z)$ has no root in $\mathbf{C}$, then $p(z) \neq 0$ for every $z \in \mathbf{C}$. Hence $1/p(z)$ is an entire function. We show that $1/p(z)$ is bounded.

For a non-zero $z \in \mathbf{C}$, consider the complex number

$$w_z := \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$$

Note that $p(z) = (w_z + a_n)\, z^n$, and by triangle inequality we have

$$|w_z| \leqslant \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|} = \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}}$$

For each $0 \leqslant k \leqslant n-1$ we note that $\dfrac{|a_k|}{|z|^{n-k}} \to 0$ as $z \to \infty$.

Then, for $\varepsilon = \dfrac{|a_n|}{2n} > 0$, we can find an $R > 0$ such that whenever $|z| > R$, we get

$$\frac{|a_k|}{|z|^{n-k}} = \left| \frac{|a_k|}{|z|^{n-k}} - 0 \right| < \varepsilon = \frac{|a_n|}{2n}$$

for any $k = 0, \ldots, n-1$. This then gives us

$$|w_z| \leqslant \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} < \sum_{k=0}^{n-1} \frac{|a_n|}{2n} = n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}$$

Now, by the reverse triangle inequality we have

$$|a_n + w_z| \geqslant \big||a_n| - |w_z|\big| > \left| |a_n| - \frac{|a_n|}{2} \right| = \frac{|a_n|}{2}$$

Thus,

$$\begin{aligned} |p(z)| &= |(w_z + a_n)\, z^n| \\ &= |w_z + a_n|\, |z^n| > \frac{|a_n|}{2} R^n \end{aligned}$$

Therefore, for any $z \in \mathbf{C}$ such that $|z| > R$, we have

$$\left| \frac{1}{p(z)} \right| \leqslant \frac{2}{R^n\, |a_n|}$$

So, $1/p(z)$ is bounded outside the closed disk $\overline{D}_R(0)$.

Now, the closed disk $\overline{D}_R(0)$ is compact (closed and bounded) and $1/p(z)$ is continuous on $\overline{D}_R(0)$. Hence $1/p(z)$ is bounded on $\overline{D}_R(0)$ by Theorem 6.7.

Thus, $1/p(z)$ is bounded on all of $\mathbf{C}$. Hence, by Theorem 17.4, $1/p(z)$ is constant, and therefore so is $p(z)$. We have arrived a contradiction, since $p(z)$ was non-constant by assumption. $\square$

Lemma 17.6 (Maximum Modulus Principle)**.** *Suppose that $|f(z)| \leqslant |f(z_0)|$ at each point z in a neighbourhood $D_\varepsilon(z_0)$ where f is holomorphic. Then $f(z) = f(z_0)$ on $D_\varepsilon(z_0)$. That is, if a holomorphic function on an open disk achieves its maximum on it, then it is constant on the open disk.*

Proof. Let $z_1 \in D_\varepsilon(z_0)$ such that $z_1 \neq z_0$. Set $\rho := |z_1 - z_0| > 0$, and consider $C_\rho = C_\rho(z_0)$, the circle of radius $\rho > 0$ centered at z_0, which is interior to $D_\varepsilon(z_0)$. We parametrise C_ρ as $z(t) = z_0 + \rho e^{it}$ for $0 \leqslant t \leqslant 2\pi$.

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By Theorem 16.2,

$$\begin{aligned} |f(z_0)| = \left| \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz \right| &= \frac{1}{2\pi} \left| \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz \right| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}}\, i\rho e^{it}\, dt \right| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt \right| \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt, \quad \text{by assumption} \\ &\leqslant |f(z_0)| \end{aligned}$$

This tells us that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt \qquad (\dagger)$$

Since, $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\ dt$. Rewriting (†), we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{it})|\ dt = 0$$

By assumption $|f(z_0)| - |f(z_0 + \rho e^{it})| \geqslant 0$; suppose $|f(z_0)| - |f(z_0 + \rho e^{it})| > 0$, then necessarily

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{it})|\ dt > 0 \tag{$*$}$$

since the integrand in ($*$) is continuous in the variable t, giving us a contradiction. Thus,

$$|f(z_0)| - |f(z_0 + \rho e^{it})| = 0$$

Therefore $|f(z)| = |f(z_0)|$ for every $z \in C_\rho(z_0)$. Varying the radius $\rho > 0$, we may then obtain $|f(z)| = |f(z_0)|$ for every $z \in D_\varepsilon(z_0)$.

Thus, $|f|$ is a holomorphic function on $D_\varepsilon(z_0)$, and thus by Corollary 8.7, we have f is constant on $D_\varepsilon(z_0)$ and $f(z) = f(z_0)$ for every $z \in D_\varepsilon(z_0)$. □

Part IV. Series: Road to Residue Calculus

Discussion 17.7. We now begin discussing series of complex numbers; we assume results about real series from calculus. Among other things, this consideration leads to the following results.

(1) A function f that is holomorphic on a disk $D_R(z_0)$ has a convergent power series expansion on that disk

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

such functions are called (complex) analytic. Conversely, every power series (analytic function) $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is holomorphic. That is, a function is holomorphic if and only if it's analytic.

(2) A function that's analytic on an annulus $R_1 < |z - z_0| < R_2$ has a convergent series expansion on the annulus

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}$$

with coefficients

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{k+1}}\, dz \quad \text{and} \quad b_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{-k+1}}$$

where C is any positively oriented simple closed contour in the annulus and surrounding z_0.

In fact, (2) provides a method for computing integrals over contours that surround a singular point! Suppose that f has a singularity at z_0, but is holomorphic everywhere else on a deleted neighbourhood $D_R(z_0) \setminus \{z_0\}$. Then f is holomorphic on the annulus $0 < |z - z_0| < R$. If C is any simple closed contour positive oriented around z_0 and lying inside the annulus, then according to (2),

$$\int_C f(z)\,dz = 2\pi i \cdot b_1.$$

In other words, we can compute the contour integral of f about a singularity just by computing the coefficient b_1 in the series expansion of f. This is the beginning of *Calculus of Residues*.

17.1. Problems

To be added

18. Lecture 18 (5/26)

Sequences & Series

Definition 18.1 (Sequences)**.** A sequence of complex numbers is a complex-valued function z whose domain is the set of positive integers. We write $z_n = z(n)$ for the value of the function z at n. We think of values occurring in a "sequential" order

$$z_1, z_2, \dots, z_n, \dots$$

and we usually denote the sequence as $(z_n)_n$.

Definition 18.2 (Limit of a Sequence)**.** A sequence $(z_n)_n$ has a limit $L \in \mathbf{C}$ if for all $\varepsilon > 0$, there exists an $N \in \mathbf{Z}_{>0}$ such that

$$|z - z_n| < \varepsilon, \text{ whenever } n \geqslant N$$

A sequence that has a limit is called convergent and we write

$$\lim_{n\to\infty} z_n = L \text{ or } z_n \to L$$

while a sequence with no limit is called divergent.

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Proposition 18.3.

(1) The limit of a convergent sequence is unique.

(2) For a sequence $(z_n)_n$, we write each term as $z_n = x_n + iy_n$, and extract two real sequences $(x_n)_n$ and $(y_n)_n$. Then,

$$x_n + iy_n \to x + iy \quad \textit{if and only if} \quad x_n \to x \textit{ and } y_n \to y$$

Proof. The proof of (1) is similar to that of Theorem 5.2, and that of (2) is similar to the proof of Theorem 5.5. □

Example 18.4. We show that

$$\lim_{n\to\infty} -1 + i\frac{(-1)^n}{n^2}$$

Proposition 18.3 tells us that

$$\begin{aligned}\lim_{n\to\infty} -1 + i\frac{(-1)^n}{n^2} &= \lim_{n\to\infty} -1 + i\lim_{n\to\infty}\frac{(-1)^n}{n^2} \\ &= -1 + i\lim_{n\to\infty}\frac{(-1)^n}{n^2}\end{aligned}$$

So, let's show

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$$

For any $\varepsilon > 0$, choose an $N > 1/\sqrt{\varepsilon}$, then for any $n \geqslant N$ we get

$$\left|\frac{(-1)^n}{n} - 0\right| = \left|\frac{(-1)^n}{n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n^2} \leqslant \frac{1}{N^2} < \varepsilon$$

Thus,

$$\lim_{n\to\infty} -1 + i\,\frac{(-1)^n}{n^2}$$

and hence,

$$\lim_{n\to\infty} -1 + i\,\frac{(-1)^n}{n^2}$$

Definition 18.5 (Series)**.** A series of complex numbers is a *symbol*

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_n + \cdots$$

associated to the sequence $(z_n)_n$ of complex numbers. A series has an associated sequence of partial sums

$$s_n = \sum_{k=1}^{n} z_k = \underbrace{z_1 + z_2 + \cdots + z_n}_{\text{sum of the first } n \text{ terms}}$$

A series is convergent if $(s_n)_n$ is convergent, this case we write

$$\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k$$

and the limit $\lim_{n\to\infty} s_n$ is called the sum of the series. A series that does not converge is said to be divergent.

Proposition 18.6. *Suppose that $(z_n)_n$ is a sequence with $z_n = x_n + iy_n$. Then,*

$$\sum_{k=1}^{\infty} x_k + iy_k = x + iy \quad \textit{if and only if} \quad \sum_{k=1}^{\infty} x_k = x \;\; \textit{and} \;\; \sum_{k=1}^{\infty} y_k = y$$

Proof. This is just Proposition 18.3 (2) applied to the sequences of partial sums. □

Remark 18.7. According to Proposition 18.6, we can write

$$\sum_{k=1}^{\infty} x_k + iy_k = \sum_{k=1}^{\infty} x_k + i\sum_{k=1}^{\infty} y_k$$

provided the series on the left *or* the two on the right converge.

Proposition 18.8 (Test for Divergence)**.** *If $\sum_{k=1}^{\infty} z_k$ converges, then $z_n \to 0$.*

Proof. Write $z_n = x_n + iy_n$, then by Proposition 18.6, the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ converge. As series of real numbers, we know that $x_n \to 0$ and $y_n \to 0$. Therefore,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$$

□

Corollary 18.9. *If $\sum_{k=1}^{\infty} z_k$ converges, then there exists an $M > 0$ such that $|z_n| \leqslant M$ for all n. That is, the sequence $(z_n)_n$ is* bounded.

Proof. If $\sum_{k=1}^{\infty} z_k$ converges, then by Proposition 18.8, $z_n \to 0$. Then, we can find an N such that $|z_n| \leqslant 1$ for every $n \geqslant N$. Set,

$$M = \max\{1, |z_1|, \ldots, |z_{N-1}|\},$$

then $|z_n| \leqslant M$, for every n. □

Definition 18.10. A series $\sum_{k=1}^{\infty} z_k$ is absolutely convergent if the series $\sum_{k=1}^{\infty} |z_k|$ of real numbers converges.

Corollary 18.11 (Absolutely Convergent Series converge)**.** *If $\sum_{k=1}^{\infty} z_k$ is absolutely convergent, then it is convergent.*

Proof. By assumption, $\sum_{k=1}^{\infty} |z_k|$ converges. Note that $|x_n| \leqslant |z_n|$ and $|y_n| \leqslant |z_n|$ for every n. By the comparison test (from calculus), the series

$$\sum_{k=1}^{\infty} |x_k| \quad \text{and} \quad \sum_{k=1}^{\infty} |y_k|$$

converge. Hence, the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ absolutely converge, and thus converge (a result from calculus). By Proposition 18.6, we conclude that $\sum_{k=1}^{\infty} z_k$ converges. □

Definition 18.12 (Remainder of a Convergent Series)**.** Suppose $\sum_{k=1}^{\infty} z_k$ is a convergent series and S is its sum. Then n^{th} remainder of the series is the complex number

$$\begin{aligned}\rho_n = S - s_n &= S - \sum_{k=1}^{n} z_k \\ &= \sum_{k=1}^{\infty} z_k - \sum_{k=1}^{n} z_k\end{aligned}$$

The remainder provides a convenient way to prove that $\sum_{k=1}^{\infty} z_k = S$, as we note that

$$\sum_{k=1}^{\infty} z_k = S \quad \text{if and only if} \quad \rho_n \to 0,$$

as $|S - s_n| = |\rho_n - 0|$.

Power Series

Definition 18.13 (Power Series)**.** A power series is a series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

where $(a_n)_n$ is a sequence, $z_0 \in \mathbf{C}$ is fixed, and z is any complex number in a prescribed region in $\mathbf{C}$. The associated sum, partial sum and remainders depend on z, and are denoted $S(z), s_n(z)$ and $\rho_n(z)$ respectively.

Example 18.14. We show that the *geometric series* $\sum_{k=0}^{\infty} az^k$ is convergent when $|z| < 1$. In fact,

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = \frac{a}{1-z} \quad (|z| < 1)$$

We compute the remainder

$$\begin{aligned} \rho_n(z) &= \frac{a}{1-z} - s_n(z) \\ &= \frac{a}{1-z} - \sum_{k=0}^{n-1} az^k \\ &= \frac{a}{1-z} - a\left(\frac{1-z^n}{1-z}\right) \\ &= a\left(\frac{z^n}{1-z}\right) \end{aligned}$$

Hence,

$$|\rho_n(z)| = \frac{|a|}{|1-z|} \cdot |z|^n$$

Note that this sequence of real numbers converges to 0 if $|z| < 1$ and diverges otherwise. Hence,

$$\lim_{n\to\infty} \rho_n(z) = \begin{cases} 0 & \text{if } |z| < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$$

Theorem 18.15 (Taylor's Theorem)**.** *Suppose that f is holomorphic on an open disk $D_R(z_0)$. Then at each $z \in D_R(z_0)$, $f(z)$ has a convergent power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

with coefficients

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

The series expansion of f guaranteed by the theorem is called the Taylor series of f about z_0.

18.1. Problems

To be added

19. Lecture 19 (5/31)

Proof of Theorem 18.15 (Taylor's Theorem). First, let's assume that $z_0 = 0$, so f is holomorphic on $D_R(0)$. Let $z \in D_R(0)$, and write $r_z = |z|$. Let r_0 be a real number such that $r_z < r_0 < R$, and C_0 the circle of radius r_0 centered at 0.

Since z is now in the interior of C_0, by Cauchy's Integral formula (Theorem 16.2), we have

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(w)}{w-z}\,dw$$

For any positive integer n, and any $a \in \mathbf{C}$ we have the formula

$$\begin{aligned} 1 + a + \cdots + a^{n-1} &= \frac{1-a^n}{1-a} \\ &= \frac{1}{1-a} - \frac{a^n}{1-a^n} \end{aligned}$$

$$\frac{1}{1-a} = \frac{a^n}{1-a^n} + \sum_{k=0}^{n-1} a^k$$

Using this, we write

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w}\left(\frac{1}{1-z/w}\right) \\ &= \frac{1}{w}\left(\frac{(z/w)^n}{1-z/w} + \sum_{k=0}^{n-1}\left(\frac{z}{w}\right)^k\right) \\ &= \frac{z^n}{w^n(w-z)} + \sum_{k=0}^{n-1}\frac{z^k}{w^{k+1}} \end{aligned} \tag{$*$}$$

Let's now compute the remainder,

$$\begin{aligned} \rho_n(z) &= f(z) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}z^k \\ &= \frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w-z}\,dw - \sum_{k=0}^{n-1}\frac{z^k}{k!}\frac{k!}{2\pi i}\int_{C_0}\frac{f(w)}{(w-0)^{k+1}}\,dw, \quad \text{by Theorem 16.4} \\ &= \frac{1}{2\pi i}\int_{C_0}\frac{f(w)}{w-z}\,dw - \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_{C_0}\frac{z^k}{w^{k+1}}f(w)\,dw \\ &= \frac{1}{2\pi i}\int_{C_0} f(w)\left(\frac{1}{w-z} - \sum_{k=0}^{n-1}\frac{z^k}{w^{k+1}}\right)dw \\ &= \frac{1}{2\pi i}\int_{C_0} f(w)\frac{z^n}{w^n(w-z)}\,dw, \quad \text{by } (*) \end{aligned}$$

Let's use this to prove $\rho_n(z) \to 0$. We have,

$$\begin{aligned}
|\rho_n(z)| &= \frac{1}{2\pi} \left| \int_{C_0} f(w) \, \frac{z^n}{w^n(w-z)} \, dw \right| \\
&\leqslant \frac{1}{2\pi} \max_{w \in C_0} \left| f(w) \, \frac{z^n}{w^n(w-z)} \right| \cdot L(C_0) \\
&= r_0 \cdot \max_{w \in C_0} |f(w)| \, \frac{|z|^n}{|w|^n \, |w-z|} \\
&= \frac{r_z^n r_0}{r_0^n} \cdot \max_{w \in C_0} \frac{|f(w)|}{|w-z|} \\
&\leqslant \frac{r_z^n r_0}{r_0^n} \cdot \max_{w \in C_0} \frac{|f(w)|}{||w|-|z||}, \quad \text{by reverse triangle inequality} \\
&= \frac{r_z^n r_0}{r_0^n(r_0 - r_z)} \cdot \max_{w \in C_0} |f(w)| \\
&= \frac{M r_0}{(r_0 - r_z)} \left(\frac{r_z}{r_0} \right)^n, \quad \text{where } M = \max_{w \in C_0} |f(w)|
\end{aligned}$$

Note that, since $\dfrac{r_z}{r_0} < 1$, we have $\dfrac{M r_0}{(r_0 - r_z)} \left(\dfrac{r_z}{r_0} \right)^n \to 0$, and hence

$$\lim_{n \to \infty} \rho_n(z) = 0$$

Thus, we have proved the claim for $z_0 = 0$.

Now, assume that $z_0 \neq 0$, so f is holomorphic on the disk $D_R(z_0)$. Then $g(z) := f(z + z_0)$ is holomorphic on $D_R(0)$. Therefore, by our arguments above, we have

$$f(z + z_0) = g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \, z^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \, z^k$$

Replacing z with $z - z_0$ gets us

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \, (z - z_0)^k$$

□

The Taylor series of f about $z_0 = 0$ is commonly referred to as a Maclaurin series.

Example 19.1 (Maclaurin series of Elementary Functions)**.** We will derive the following Maclaurin series expansions of the most common elementary functions. We will frequently use them to compute Maclaurin and Taylor series expansions of other functions. You should try and remember them!

(1) $\dfrac{1}{1-z} = \sum_{k=0}^{\infty} z^k$, for $|z| < 1$

(2) $e^z = \sum_{k=0}^{\infty} \dfrac{z^k}{k!}$, for $|z| < \infty$

(3) $\sin z = \sum_{k=0}^{\infty} (-1)^k \dfrac{z^{2k+1}}{(2k+1)!}$, for $|z| < \infty$

(4) $\cos z = \sum_{k=0}^{\infty} (-1)^k \dfrac{z^{2k}}{(2k)!}$, for $|z| < \infty$

Answer.

(1) Let $f(z) = 1/(1-z)$, then f has a singularity at $z = 1$. So, f is holomorphic on the open disk $D_1(0)$. By Theorem 18.15, f has a Maclaurin series on this disk. One can show inductively that for any k we have,

$$f^{(k)}(z) = \frac{k!}{(1-z)^{k+1}}$$

Therefore $f^{(k)}(0) = k!$, and hence

$$\frac{1}{1-z} = f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} z^k$$

(2) Since $f(z) = e^z$ is entire, it has a Maclaurin series everywhere, by Theorem 18.15. We have,

$$f^{(k)}(0) = e^0 = 1$$

Hence,

$$e^z = f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

(3) We have,

$$\begin{aligned}
\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) &= \frac{1}{2i}\left(\sum_{k=0}^{\infty} \frac{i^k z^k}{k!} - \sum_{k=0}^{\infty} \frac{(-i)^k z^k}{k!}\right) \\
&= \frac{1}{2i} \sum_{k=0}^{\infty} \frac{i^k z^k}{k!} \left(1 - (-1)^k\right) \\
&= \frac{1}{2i} \sum_{k=0}^{\infty} \frac{i^{2k+1} z^{2k+1}}{(2k+1)!} \cdot 2 \\
&= \frac{1}{2i} \sum_{k=0}^{\infty} i^{2k} \frac{z^{2k+1}}{(2k+1)!} \cdot (2i) \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}
\end{aligned}$$

(4) We can differentiate a power series term by term (a fact that requires a proof, and we haven't given one yet). So,

$$\begin{aligned}\cos z = (\sin z)' &= \sum_{k=0}^{\infty} \left((-1)^k \frac{z^{2k+1}}{(2k+1)!} \right)' \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)\, z^{2k}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}\end{aligned}$$

$\square$

Remark 19.2. The power series in Example 19.1 are the usual Maclaurin series for these functions when $z = x$ is real. This provides additional justification that we chose the correct definitions when we extended these elementary functions to the complex plane.

Example 19.3. We use power series in Example 19.1 to compute Maclaurin or Taylor series expansions of other functions.

(a) Maclaurin series of $f(z) = \dfrac{1}{1+z}$. We have,

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{k=0}^{\infty} (-z)^k = \sum_{k=0}^{\infty} (-1)^k z^k$$

for $|z| = |-z| < 1$.

(b) Taylor series of $f(z) = \dfrac{1}{1-z}$ about $z_0 = i$. We have,

$$\begin{aligned}\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} &= \frac{1}{1-i} \left(\frac{1}{1 - \dfrac{z-i}{1-i}} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{z-i}{1-i} \right)^k, \text{ for } \left| \frac{z-i}{1-i} \right| < 1 \\ &= \sum_{k=0}^{\infty} \frac{(z-i)^k}{(1-i)^{k+1}}\end{aligned}$$

for $|z-i| < |1-i| = \sqrt{2}$.

(b) Maclaurin series of $f(z) = z^2 e^{2z}$. We have,

$$\begin{aligned} z^2 e^{2z} &= z^2 \sum_{k=0}^{\infty} \frac{(2z)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{2^k\, z^{k+2}}{k!} \\ &= \sum_{k=2}^{\infty} \frac{2^{k-2}\, z^k}{(k-2)!} \end{aligned}$$

for $|z - i| < |1 - i| = \sqrt{2}$.

19.1. Problems

To be added

20. Lecture 20 (6/02)

Laurent Series

Remark 20.1. When f is not holomorphic, Theorem 18.15 cannot be applied. However, we can often find a series expansion of f that involves *negative* powers of $(z - z_0)$.

Example 20.2.

(1) $f(z) = \dfrac{e^{-z}}{z^2}$. The function is not holomorphic at $z_0 = 0$.

So we look for a series expansion involving powers of z. We have,

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\sum_{k=0}^{\infty}\frac{(-z)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{k-2}}{k!}$$

$$= \frac{1}{z^2} - \frac{1}{z} + \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(k+2)!}$$

for $0 < |z| < \infty$.

(2) $f(z) = \dfrac{1+2z^2}{z^3+z^5}$. We have,

$$\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) = \frac{1}{z^3}\left(\frac{2(1+z^2)-1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \sum_{k=0}^{\infty}(-z^2)^k\right), \quad \text{for } 0 < |z| < 1$$

$$= \frac{2}{z^3} - \frac{1}{z^3}\sum_{k=0}^{\infty}(-1)^k z^{2k}$$

$$= \frac{2}{z^3} - \sum_{k=0}^{\infty}(-1)^k z^{2k-3}$$

$$= \frac{2}{z^3} - \frac{1}{z^3} + \frac{1}{z} - \sum_{k=2}^{\infty}(-1)^k z^{2k-3}$$

$$= \frac{1}{z^3} + \frac{1}{z} - \sum_{k=2}^{\infty}(-1)^k z^{2k-3}$$

(3) $f(z) = \dfrac{e^z}{(1+z)^2}$. The singularity is at $z_0 = -1$, so we want powers of $(1+z)$. We have,

$$\begin{aligned}
\frac{e^z}{(1+z)^2} = \frac{e^{z+1}}{e(1+z)^2} &= \frac{1}{e(z+1)^2}\sum_{k=0}^{\infty}\frac{(z+1)^k}{k!}, \quad \text{for } 0 < |z+1| < \infty \\
&= \frac{1}{e}\sum_{k=0}^{\infty}\frac{(z+1)^{k-2}}{k!} \\
&= \frac{1}{e}\left(\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{k=0}^{\infty}\frac{(z+1)^k}{(k+2)!}\right)
\end{aligned}$$

Theorem 20.3 (Laurent's Theorem)**.** *Suppose that f is holomorphic on an annulus $R_1 < |z-z_0| < R_2$. Then f has a* Laurent series *expansion on that annulus*

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty}\frac{a_{-k}}{(z-z_0)^k}, \quad \textit{for } R_1 < |z-z_0| < R_2$$

with coefficients given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n \in \mathbf{Z}$$

where C is a positively oriented simple closed contour in the annulus whose interior contains z_0.

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In particular,

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz$$

Proof. First, let's assume that $z_0 = 0$. Let z be such that $R_1 < |z| < R_2$. Let C_1 and C_2 be circles, with positive orientation, with radii r_1 and r_2 respectively such that

$$R_1 < r_1 < |z| < r_2 < R_2,$$

and such the contour C lies in the interior of C_1 but exterior of C_2, that is, between C_1 and C_2. Hence, by Corollary 15.10, we can assume

$$a_k = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-z_0)^{k+1}}\,dz, \text{ for } k \geqslant 0 \quad \text{and} \quad a_{-k} = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-z_0)^{-k+1}}\,dz, \text{ for } k \geqslant 1$$

Also let $\varepsilon > 0$ be such that $C_\varepsilon = C_\varepsilon(z)$ lies in between C_1 and C_2.

Before computing the remainder, we note that by Theorem 15.9 (Generalised Cauchy-Goursat Theorem), we have

$$\int_{C_2}\frac{f(w)}{w-z}\,dw - \int_{C_1}\frac{f(w)}{w-z}\,dw - \int_{C_\varepsilon}\frac{f(w)}{w-z}\,dw = 0 \tag{1}$$

Furthermore, by Theorem 16.2 (Cauchy's Integral formula)

$$\int_{C_\varepsilon}\frac{f(w)}{w-z}\,dw = 2\pi i \cdot f(z) \tag{2}$$

(1) and (2) gives us

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{z-w}\,dw$$

Also recall from the Proof of Theorem 18.15 (Taylor's Theorem)

$$\frac{1}{w-z} = \sum_{k=0}^{n-1} \frac{z^k}{w^{k+1}} + \frac{z^n}{w^n(w-z)}$$

$$\begin{aligned}\frac{1}{z-w} &= \sum_{k=0}^{n-1} \frac{w^k}{z^{k+1}} + \frac{w^n}{z^n(z-w)}\\ &= \sum_{k=1}^{n} \frac{w^{k-1}}{z^k} + \frac{w^n}{z^n(z-w)}\end{aligned}$$

Now,

$$\begin{aligned}\rho_n(z) &= f(z) - \sum_{k=0}^{n-1} a_k(z-0)^k - \sum_{k=1}^{n} \frac{a_{-k}}{(z-0)^k}\\
&= \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{z-w}\,dw\\
&\quad - \sum_{k=0}^{n-1} z^k \cdot \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w^{k+1}}\,dw - \sum_{k=1}^{n} z^{-k} \cdot \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w^{-k+1}}\,dw\\
&= \frac{1}{2\pi i}\int_{C_2} f(w)\left(\frac{1}{w-z} - \sum_{k=0}^{n-1} \frac{z^k}{w^{k+1}}\right)dw + \frac{1}{2\pi i}\int_{C_1} f(w)\left(\frac{1}{z-w} - \sum_{k=1}^{n} \frac{z^{-k}}{w^{-k+1}}\right)dw\\
&= \frac{1}{2\pi i}\int_{C_2} f(w)\left(\frac{1}{w-z} - \sum_{k=0}^{n-1} \frac{z^k}{w^{k+1}}\right)dw + \frac{1}{2\pi i}\int_{C_1} f(w)\left(\frac{1}{z-w} - \sum_{k=1}^{n} \frac{w^{k-1}}{z^k}\right)dw\\
&= \frac{1}{2\pi i}\int_{C_2} f(w)\,\frac{z^n}{w^n(w-z)}\,dw + \frac{1}{2\pi i}\int_{C_2} f(w)\,\frac{w^n}{z^n(z-w)}\,dw\end{aligned}$$

Therefore, by triangle inequality

$$|\rho_n(z)| \leqslant \frac{1}{2\pi}\left|\int_{C_2} f(w)\,\frac{z^n}{w^n(w-z)}\,dw\right| + \frac{1}{2\pi}\left|\int_{C_2} f(w)\,\frac{w^n}{z^n(z-w)}\,dw\right|$$

We can then show that the right hand side converges to 0 as $n \to \infty$, similar to what we did in the proof of Theorem 18.15. This proves our claim for $z_0 = 0$.

Suppose now that $z_0 \neq 0$, and f is a function that satisfies the hypotheses of the theorem. Define $g(z) := f(z+z_0)$; since f is holomorphic on $R_1 < |z-z_0| < R_2$, we get that g is holomorphic on $R_1 < |z| < R_2$. Therefore, by our arguments above, we have

$$g(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k}$$

with

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{n+1}}\, dz, \quad n \in \mathbf{Z}$$

where Γ is the contour obtained from C after translating by z_0. To finish the proof, we simply replace $g(z)$ by $f(z+z_0)$, and then z by $z - z_0$. $\square$

Example 20.4. Laurent series are rarely found by using the integral expressions (in fact, it's the other way around, we use the Laurent series expansion to compute the integral expressions). We usually find them by making use of the six Maclaurin series from Example 19.1.

(1) $f(z) = \dfrac{1}{z(1+z^2)}$.

The singularities are at 0, $\pm i$. So, the function is holomorphic on $0 < |z| < 1$. Therefore, by Theorem 20.3, f has a Laurent series expansion on this deleted neighbourhood. We have,

$$\begin{aligned}
\frac{1}{z(1+z^2)} &= \frac{1}{z}\left(\frac{1}{1+z^2}\right) \\
&= \frac{1}{z} \cdot \sum_{k=0}^{\infty} (-z^2)^k \\
&= \sum_{k=0}^{\infty} (-1)^k z^{2k-1} \\
&= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k z^{2k-1} \\
&= \frac{1}{z} + \sum_{k=0}^{\infty} (-1)^{k+1} z^{2k+1}
\end{aligned}$$

Note that we have $a_{-1} = 1$, and thus,

$$\int_C \frac{1}{z(1+z^2)}\, dz = \int_C f(z)\, dz = 2\pi i \cdot a_{-1} = 2\pi i,$$

where C is any positively oriented simple closed contour about 0 in the deleted neighbourhood $0 < |z| < 1$.

(2) $f(z) = e^{1/z}$.

The singularity is at 0. So, the function is holomorphic on $0 < |z| < \infty$, that is, $\mathbf{C}^*$. Therefore, by Theorem 20.3, f has a Laurent series expansion on $\mathbf{C}^*$. We have,

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{(1/z)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k}$$

Note that we have $a_{-1} = 1$, and thus,

$$\int_C e^{1/z}\,dz = \int_C f(z)\,dz = 2\pi i \cdot a_{-1} = 2\pi i,$$

where C is any positively oriented simple closed contour about 0.

(3) $f(z) = \dfrac{z+1}{z-1}$.

The singularity is at 1. Therefore, by Theorem 18.15, f has a Taylor expansion on the disk $|z| < 1$, and a Laurent series expansion on $1 < |z| < \infty$, by Theorem 20.3.

- On $|z| < 1$

$$\begin{aligned} \frac{z+1}{z-1} &= -(1+z)\cdot\frac{1}{1-z} \\ &= -(1+z)\cdot\sum_{k=0}^{\infty} z^k \\ &= -\sum_{k=0}^{\infty} z^k - \sum_{k=0}^{\infty} z^{k+1} \\ &= -1 - 2\sum_{k=0}^{\infty} z^{k+1} \\ &= -1 - 2\sum_{k=1}^{\infty} z^k \end{aligned}$$

- On $1 < |z| < \infty$, since $|1/z| < 1$, we have

$$\begin{aligned} \frac{z+1}{z-1} = \frac{1+1/z}{1-1/z} &= \frac{1}{1-1/z} + \frac{1/z}{1-1/z} \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{z}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{z^k} + \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} \\ &= 1 + 2\sum_{k=0}^{\infty}\frac{1}{z^{k+1}} \\ &= 1 + 2\sum_{k=1}^{\infty}\frac{1}{z^k} \end{aligned}$$

(4) $f(z) = \dfrac{1}{(z-z_0)^{n+1}}$.

The singularity is at z_0, and therefore $f(z)$ is holomorphic on the punctured complex plane $0 < |z - z_0| < \infty$. In fact, $f(z)$ is its own Laurent series expansion. We will compute

$$\int_C \frac{1}{(z-z_0)^{(n+1)-m}}\,dz$$

By Theorem 20.3,

$$\begin{aligned} a_{-(m+1)} &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-(m+1)+1}}\, dz \\ &= \frac{1}{2\pi i} \int_C \frac{1}{(z-z_0)^{(n+1)-m}}\, dz \end{aligned}$$

But from the Laurent series expansion of f itself, we note that

$$a_{-(m+1)} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Giving us,

$$\int_C \frac{1}{(z-z_0)^{(n+1)-m}}\, dz = \begin{cases} 2\pi i & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

20.1. Problems

To be added

21. (Possible) Lecture 21

Absolute & Uniform Convergence

Theorem 21.1 (Power Series Converge Absolutely)**.** *If a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges for $z = w \neq z_0$, then it converges absolutely on the disk $D_R(z_0)$, where $R = |w - z_0|$.*

Proof. By assumption, the series $\sum_{k=0}^{\infty} a_k(w-z_0)^k$ converges, then by Corollary 18.9 we can find an $M > 0$ such that $\left|a_k(w-z_0)^k\right| \leqslant M$ for every k. Consider any $z \in D_R(z_0)$, then $|z - z_0| < R = |w - z_0|$. Define,

$$\rho = \frac{|z-z_0|}{|w-z_0|} < 1$$

Then,

$$\begin{aligned} |a_k(z-z_0)^k| &= |a_k(w-z_0)^k| \cdot \left|\frac{z-z_0}{w-z_0}\right|^k \\ &= |a_k(w-z_0)^k| \cdot \rho^k \\ &\leqslant M\rho^k \end{aligned}$$

Note that the series $\sum_{k=0}^{\infty} M\rho^k$ converges since $\rho < 1$, and hence $\sum_{k=0}^{\infty} \left|a_k(z-z_0)^k\right|$ by the comparison test from calculus. Therefore the series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely. $\square$

Remark 21.2. Theorem 21.1 asserts that if a series converges at a point $w \neq z_0$, then it also converges on a disk, the largest such disk is called the disk of convergence and its radius is called the radius of convergence. Necessarily, a series does not converge outside its disk of convergence.

Definition 21.3 (Uniform Convergence of Series)**.** Consider a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ with disk of convergence $D_R(z_0)$. Let G be a region in the disk, then we say that the series converges uniformly if for any $\varepsilon > 0$, there exists an $N \in \mathbf{Z}_{>0}$ such that

$$\text{for every } z \in G, \text{ we have } |\rho_n(z)| < \varepsilon, \text{ whenever } n \geqslant N$$

In particular, our choice of N *only* depends on ε and is independent of z; that is, we can choose an N that works uniformly for all $z \in G$.

Theorem 21.4 (Power Series are Uniformly Convergent)**.** *Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series of with disk of convergence $D_R(z_0)$, and consider any $w \in D_R(z_0)$. Then, the power series converges uniformly on the closed disk $\overline{D}_r(z_0)$, where $r = |w - z_0|$.*

Proof. By Theorem 21.1, the series

$$\sum_{k=0}^{\infty} |a_k(w-z_0)^k|$$

converges. Let's consider the remainders

$$\rho_n(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k - \sum_{k=0}^{n-1} a_k(z-z_0)^k = \sum_{k=n}^{\infty} a_k(z-z_0)^k = \lim_{m\to\infty}\sum_{k=n}^{m} a_k(z-z_0)^k$$

$$\sigma_n = \sum_{k=0}^{\infty} |a_k(w-z_0)^k| - \sum_{k=0}^{n-1} |a_k(w-z_0)^k| = \sum_{k=n}^{\infty} |a_k(w-z_0)^k| = \lim_{m\to\infty}\sum_{k=n}^{m} |a_k(w-z_0)^k|$$

For any $z \in \overline{D}_r(z_0)$, we have $|z-z_0| \leqslant r = |w-z_0|$, and hence,

$$\begin{aligned}
|\rho_n(z)| = \lim_{m\to\infty}\left|\sum_{k=n}^{m} a_k(z-z_0)^k\right| &\leqslant \lim_{m\to\infty}\sum_{k=n}^{m} |a_k|\,|z-z_0|^k \\
&\leqslant \lim_{m\to\infty}\sum_{k=n}^{m} |a_k|\,|w-z_0|^k \\
&= \sigma_n
\end{aligned}$$

Since $\sum_{k=0}^{\infty}\left|a_k(w-z_0)^k\right|$ converges, we have $\sigma_n \to 0$. Therefore, for any $\varepsilon > 0$ we can find a positive integer N such that $|\sigma_n| < \varepsilon$ whenever $n \geqslant N$. By our computations above, we therefore get $|\rho_n(z)| < |\sigma_n| < \varepsilon$, for any $z \in \overline{D}_r(z_0)$ and $n \geqslant N$. Thus, our series converges absolutely in $\overline{D}_r(z_0)$. □

Theorem 21.5 (Power Series are Continuous)**.** *A power series*

$$S(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

is a continuous function on its disk of convergence.

Proof. Let $D_R(z_0)$ be disk of convergence of our power series, and consider any $w \in D_R(z_0)$. Since the power series converges uniformly, we can find, for any $\varepsilon > 0$, a positive integer N such that

$$|\rho_n(z)| < \frac{\varepsilon}{3}$$

for any $n \geqslant N$ and $z \in D_r(z_0)$, where $r = |w-z_0|$. Also, since $s_n(z)$ is a polynomial for each n, it is a continuous function. In particular, $s_{N+1}(z)$ is a continuous function. Therefore for our ε, we can find a $\delta > 0$ such that

$$\text{if } |z-w| < \delta, \quad \text{then } |s_{N+1}(z) - s_{N+1}(w)| < \frac{\varepsilon}{3}$$

So, for any z such that $|z-w| < \delta$, we get

$$\begin{aligned}
|S(z)-S(w)| &= |s_{N+1}(z) + \rho_{N+1}(z) - (s_{N+1}(w) - \rho_{N+1}(w))| \\
&\leqslant |s_{N+1}(z) - s_{N+1}(w)| + |\rho_{N+1}(z)| + |\rho_{N+1}(w)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Thus $S(z)$ is continuous. □

Theorem 21.6 (Integrating Power Series)**.** *Let C be any contour interior to the disk of convergence of the power series*

$$S(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Let $g(z)$ be any function that is continuous on C, then

$$\int_C g(z) S(z)\, dz = \sum_{k=0}^{\infty} a_k \int_C g(z)(z - z_0)^k\, dz$$

Proof. We have our usual remainder

$$\rho_n(z) = S(z) - s_n(z) = S(z) - \sum_{k=0}^{n-1} a_k (z - z_0)^k.$$

We now consider the following remainder

$$\begin{aligned}
\sigma_n(z) &= \int_C g(z) S(z)\, dz - \sum_{k=0}^{n-1} a_k \int_C g(z)(z - z_0)^k\, dz \\
&= \int_C g(z) \left(S(z) - \sum_{k=0}^{n-1} a_k (z - z_0)^k \right) dz \\
&= \int_C g(z) \rho_n(z)\, dz
\end{aligned}$$

Let $\varepsilon > 0$. Since g is continuous on C, it's bounded on C, therefore we can find an $M > 0$ such that $|g(z)| \leqslant M$ for every $z \in C$.

Since $S(z)$ is uniformly convergent on its disk of convergence $D_R(z_0)$, we can find a positive integer N such that

$$|\rho_n(z)| < \frac{\varepsilon}{M \cdot L(C)}$$

for any $n \geqslant N$ and $z \in D_r(z_0)$. Then,

$$\begin{aligned}
|\sigma_n(z)| &= \left| \int_C g(z) \rho_n(z)\, dz \right| \\
&\leqslant \max_{z \in C} |g(z)|\, \rho_n(z) \cdot L(C) \\
&< M \cdot \frac{\varepsilon}{M \cdot L(C)} \cdot L(C) \\
&= \varepsilon
\end{aligned}$$

Thus $\sigma_n(z) \to 0$. $\square$

Corollary 21.7 (Power Series are Holomorphic). *A power series*

$$S(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is a holomorphic function on its disk of convergence.

Proof. We have that power series are continuous by Theorem 21.5. Let C be any closed contour lying in the interior of its disk of convergence. Then, by Theorem 21.6

$$\begin{aligned} \int_C S(z)\, dz &= \sum_{k=0}^{\infty} a_k \int_C (z - z_0)^k\, dz \\ &= \sum_{k=0}^{\infty} a_k \cdot 0, \quad \text{since } (z - z_0)^k \text{ has an anti-derivative} \\ &= 0 \end{aligned}$$

Thus, by Morera's theorem (Theorem 17.1), $S(z)$ is holomorphic. □

Example 21.8. The function

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is entire.

For any $z \in \mathbf{C}$, we have

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

When $z \neq 0$, we have

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!};$$

and when $z = 0$ we also have

$$1 = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

Hence,

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

for any $z \in \mathbf{C}$, and is thus entire.

21.1. Problems

To be added

22. (Possible) Lecture 22

Theorem 22.1 (Differentiating Power Series). *A power series*

$$S(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

can be differentiated term-by-term. That is, at each point interior to its disk of convergence, we have

$$S'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

Proof. Let $D_R(z_0)$ be the disk of convergence of our power series, and consider any $z \in D_R(z_0)$. Let C be a simple closed positively oriented contour interior to $D_R(z_0)$ and surrounding z. Then, by Cauchy's Integral formula

$$\begin{aligned}
S'(z) &= \frac{1}{2\pi i} \int_C \frac{S(w)}{(w-z)^2}\, dw \\
&= \int_C g(w) S(w)\, dw, \quad \text{where } g(w) = \frac{1}{2\pi i} \cdot \frac{1}{(w-z)^2} \\
&= \sum_{k=0}^{\infty} a_k \int_C g(w)(w - z_0)^k\, dw, \quad \text{by Theorem 21.6} \\
&= \sum_{k=0}^{\infty} a_k \cdot \frac{1}{2\pi i} \int_C \frac{(w-z_0)^k}{(w-z)^2}\, dw \\
&= \sum_{k=0}^{\infty} a_k \cdot \left((w - z_0)^k \right)' \Big|_{w=z} \quad \text{by Cauchy's Integral formula} \\
&= \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}
\end{aligned}$$

□

Theorem 22.2 (Uniqueness of Taylor Series). *If a power series*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges to a function $f(z)$ on a disk $D_R(z_0)$, then it is the Taylor series of f about z_0.

Proof. It suffices to show that

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

Consider, for a fixed positive integer n

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}}$$

Let C be a circle centered at z_0 with radius $r < R$. Then, by Generalised Cauchy's Integral formula

$$\begin{aligned}
\frac{f^{(n)}(z_0)}{n!} &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \\
&= \int_C g(z) \sum_{k=0}^{\infty} a_k (z-z_0)^k \, dz, \quad \text{where } g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}} \\
&= \sum_{k=0}^{\infty} a_k \int_C g(z)(z-z_0)^k \, dz \\
&= \sum_{k=0}^{\infty} a_k \cdot \frac{1}{2\pi i} \int_C \frac{1}{(z-z_0)^{(n+1)-k}}\, dz \\
&= a_n \cdot \frac{1}{2\pi i} \int_C \frac{1}{z-z_0}\, dz + \sum_{k \neq n} a_k \cdot \frac{1}{2\pi i} \int_C \frac{1}{(z-z_0)^{(n+1)-k}}\, dz \\
&= a_n
\end{aligned}$$

using Example 20.4 (4). □

Theorem 22.3 (Uniqueness of Laurent Series). *If a series*

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

converges to a function $f(z)$ on an annulus $A_{R_1,R_2}(z_0)$, then it is the Laurent series of f on that annulus.

Proof. It is similar to the proof of Theorem 22.2. □

Discussion 22.4 (Product of Power Series). Suppose two power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k (z-z_0)^k$$

converge on a disk $D_R(z_0)$. Then f and g are holomorphic on that disk, and hence so is fg by the product rule. Hence, fg necessarily has a Taylor series on $D_R(z_0)$,

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

where

$$\begin{aligned}
c_n &= \frac{(fg)^{(n)}(z_0)}{n!} \\
&= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z_0)g^{(n-k)}(z_0), \quad \text{by the general product rule} \\
&= \frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} f^{(k)}(z_0)g^{(n-k)}(z_0) \\
&= \sum_{k=0}^{n}\frac{f^{(k)}(z_0)}{k!}\cdot\frac{g^{(n-k)}(z_0)}{(n-k)!} \\
&= \sum_{k=0}^{n} a_k b_{n-k}
\end{aligned}$$

Usually, in practice, only the first several terms are required. They can be found by formally multiplying the series like polynomials.

Example 22.5. Find the Maclaurin series for

$$\frac{\sin z}{1+z}$$

The functions $\sin z$ and $1/(1+z)$ are holomorphic on the unit disk. We have,

$$\begin{aligned}
\frac{\sin z}{1+z} &= \frac{1}{1+z}\cdot\sin z \\
&= \left(\sum_{k=0}^{\infty}(-1)^k z^k\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!}\right) \\
&= \left(1-z+z^2-\cdots\right)\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right) \\
&= \left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right) - z\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right) + z^2\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)-\cdots \\
&= \left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right) - \left(z^2-\frac{z^4}{3!}+\frac{z^6}{5!}-\cdots\right) + \left(z^3-\frac{z^5}{3!}+\frac{z^7}{5!}-\cdots\right)-\cdots \\
&= z - z^2 + \frac{5}{6}z^3 - \frac{5}{6}z^4 + \cdots
\end{aligned}$$

Discussion 22.6 (Quotient of Power Series)**.** Similarly, if f and g are holomorphic on a disk $D_R(z_0)$ and $g(z) \neq 0$ on $D_R(z_0)$, then we can write

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n$$

where

$$d_n = \frac{(f/g)^{(n)}(z_0)}{n!}$$

In practice, the coefficients can be found by formally dividing the series like polynomials.

Example 22.7. Find the Laurent series for

$$\frac{1}{\sin z}$$

on the annulus $0 < |z| < \pi$. We have,

$$\frac{1}{\sin z} = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}}$$

$$= \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$= \frac{1}{z} \cdot \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}$$

The series $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ is non-zero on the disk $|z| < \pi$, so we can perform long division of the series dividing and collect the first few terms, which gives us

$$\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 1 + \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!} \right) z^4 + \cdots$$

Therefore,

$$\begin{aligned} \frac{1}{\sin z} &= \frac{1}{z} \cdot \left(1 + \frac{1}{6} z^2 + \frac{7}{360} z^4 + \cdots \right) \\ &= \frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3 + \cdots \end{aligned}$$

22.1. Problems

To be added

23. (Possible) Lecture 23

Part V. Residue Calculus

Definition 23.1 (Isolated Singularity)**.** A singular point z_0 of a function f is called isolated if there exists a deleted disk $D_R(z_0) \setminus \{z_0\}$ on which f is holomorphic.

Example 23.2.

(1) A rational function
$$r(z) = \frac{p(z)}{q(z)}$$
has only isolated singularities: they are the zeros of $q(z)$.

(2) The principal branch of the logarithm has a singularity at 0 but it is *not isolated*. This is because any deleted disk around 0 will necessarily contain points on the branch cut $\mathbf{R}_{<0}$.

(3) $f(z) = \dfrac{1}{\sin(\pi/z)}$ has a singular point at $z = 0$, and also whenever
$$\begin{aligned} \sin\left(\frac{\pi}{z}\right) = 0 &\iff \frac{\pi}{z} = k\pi, \quad k \in \mathbf{Z} \\ &\iff z = \frac{1}{k}, \quad k \in \mathbf{Z} \end{aligned}$$
The singular point 0 is not isolated. Consider any deleted disk $D_\varepsilon(0) \setminus \{0\}$ where $\varepsilon > 0$, then we can find a positive integer n such that $0 < 1/n < \varepsilon$. So, $1/n \in D_\varepsilon(0) \setminus \{0\}$ but f is holomorphic at $1/n$.

On the other hand, the singular points $1/k$ are isolated since f is holomorphic on the delted disk $D_R(0) \setminus \{0\}$ for $R = 1/k(k+1)$.

Definition 23.3 (Isolated Singularity at ∞)**.** A function has an *isolated singularity at* ∞ if there exists an $R > 0$ such that f is holomorphic on the annulus $R < |z| < \infty$. Equivalently, if $f(1/z)$ has an isolated singularity at 0.

Definition 23.4 (Residues)**.** Let z_0 be an isolated singularity of f so that f on the annuli
$$\begin{cases} 0 < |z - z_0| < R & \text{if } z_0 \neq \infty \\ R < |z| < \infty & \text{if } z_0 = \infty \end{cases}$$
When $z_0 \neq \infty$, the residue of f at z_0 is the coefficient
$$\operatorname*{Res}_{z=z_0} f(z) := a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz$$

in the Laurent series expansion of f on the annulus $0 < |z - z_0| < R$.

When $z_0 = \infty$, the residue of f at ∞ is defined as

$$\operatorname*{Res}_{z=\infty} f(z) := \frac{1}{2\pi i} \int_{C_{R_0}} f(z)\, dz$$

where C_{R_0} is a *negatively oriented* circle centered at 0 with radius $R_0 > R$.

Example 23.5.

(1) Compute $\displaystyle\int_C \frac{e^z - 1}{z^4}\, dz$, where C is the unit circle with positive orientation.

Since 0 is an isolated singularity of $f(z) = (e^z - 1)/z^4$, and C is a contour around 0, we need to only compute $\operatorname*{Res}_{z=0} f(z)$.

The function has a Laurent series on $0 < |z| < \infty$, which is

$$\begin{aligned}
\frac{e^z - 1}{z^4} &= \frac{1}{z^4}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} - 1\right) \\
&= \frac{1}{z^4} \sum_{k=1}^{\infty} \frac{z^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{z^{k-4}}{k!} \\
&= \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \sum_{k=0}^{\infty} \frac{z^k}{(k+4)!}
\end{aligned}$$

Therefore $\operatorname*{Res}_{z=0} f(z) = \dfrac{1}{6}$, hence

$$\int_C \frac{e^z - 1}{z^4} = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z) = \frac{\pi i}{3}$$

(2) Compute $\displaystyle\int_C \cos\left(\frac{1}{z^2}\right) dz$, where C is the unit circle with positive orientation.

Since 0 is an isolated singularity of $f(z) = \cos(1/z^2)$, and C is a contour around 0, we need to only compute $\operatorname*{Res}_{z=0} f(z)$.

The function has a Laurent series on $0 < |z| < \infty$, which is

$$\begin{aligned}
\cos\left(\frac{1}{z^2}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{z^2}\right)^{2k} \\
&= 1 - \frac{1}{2z^2} + \frac{1}{24z^4} - \cdots
\end{aligned}$$

Therefore $\operatorname*{Res}_{z=0} f(z) = 0$, hence

$$\int_C \cos\left(\frac{1}{z^2}\right) dz = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z) = 0$$

(3) Compute $\displaystyle\int_C \frac{1}{z(z-2)^5}\, dz$, where C is the circle $|z-2| = 1$ with positive orientation.

Since 2 is an isolated singularity of $f(z) = 1/z(z-2)^5$, and C is a contour around 2, we need to only compute $\operatorname*{Res}_{z=2} f(z)$.

The function has a Laurent series on $0 < |z-2| < 2$, which is

$$\begin{aligned}
\frac{1}{z(z-2)^5} &= \frac{1}{(z-2)^5}\left(\frac{1}{2+(z-2)}\right) \\
&= \frac{1}{2(z-2)^5} \cdot \frac{1}{1+\dfrac{z-2}{2}} \\
&= \frac{1}{2(z-2)^5} \sum_{k=0}^{\infty} (-1)^k \left(\frac{z-2}{2}\right)^k, \quad \text{since } \left|\frac{z-2}{2}\right| < 1 \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(z-2)^{k-5}}{2^{k+1}} \\
&= \frac{1}{2z^5} - \frac{1}{4z^4} + \frac{1}{8z^3} - \frac{1}{16z^2} + \frac{1}{32z} - \sum_{k=0}^{\infty} (-1)^k \frac{(z-2)^k}{2^{k+6}}
\end{aligned}$$

Therefore $\operatorname*{Res}_{z=0} f(z) = \dfrac{1}{32}$, hence

$$\int_C \frac{1}{z(z-2)^5}\, dz = 2\pi i \cdot \operatorname*{Res}_{z=0} f(z) = \frac{\pi i}{16}$$

Theorem 23.6 (Cauchy's Residue Theorem)**.** *Let C be a positively oriented simple closed contour. If f is holomorphic everywhere on and interior to C, except at finitely many singularities $z_1, \dots, z_n$ lying interior to C, then*

$$\int_C f(z)\, dz = 2\pi i \cdot \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)$$

Proof. Since there are finitely many singularities of f interior to C, they are necessarily isolated. For each i, let C_i be a positively oriented circle centered at z_i such that

(a) the regions R_i enclosed by C_i are pairwise disjoint; and

(b) the regions R_i are interior to C, which is possibly since z_i is interior to C.

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Then f is holomorphic everywhere on C_i's and at all points that are interior to C but exterior to C_i's. Then, by the Generalised Cauchy-Goursat theorem (Theorem 15.9)

$$\begin{aligned}
\int_C f(z)\,dz &= \sum_{i=1}^{n} \int_{C_i} f(z)\,dz \\
&= \sum_{i=1}^{n} 2\pi i \cdot \operatorname*{Res}_{z=z_i} f(z) \\
&= 2\pi i \cdot \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)
\end{aligned}$$

□

Example 23.7. Compute $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$, where C is the circle $|z| = 2$ with positive orientation. The function

$$f(z) = \frac{4z-5}{z(z-1)}$$

has isolated singularities at 0 and 1, both of which lie interior to C. So, we apply the residue theorem for which we need to compute $\operatorname*{Res}_{z=0} f(z)$ and $\operatorname*{Res}_{z=1} f(z)$.

- To compute $\operatorname*{Res}_{z=0} f(z)$, we note that the function has a Laurent series on $0 < |z| < 1$, which is

$$\begin{aligned}
\frac{4z-5}{z(z-1)} &= \frac{4z-5}{z} \cdot \frac{1}{z-1} \\
&= \frac{5-4z}{z} \cdot \frac{1}{1-z} \\
&= \left(\frac{5}{z} - 4\right) \sum_{k=0}^{\infty} z^k \\
&= \frac{5}{z} \sum_{k=0}^{\infty} z^k - 4 \sum_{k=0}^{\infty} z^k \\
&= 5 \sum_{k=0}^{\infty} z^{k-1} - 4 \sum_{k=0}^{\infty} z^k \\
&= \frac{5}{z} + \sum_{k=0}^{\infty} z^k
\end{aligned}$$

  Therefore $\operatorname*{Res}_{z=0} f(z) = 5$.

- To compute $\operatorname*{Res}_{z=1} f(z)$, we note that the function has a Laurent series on $0 < |z-1| < 1$, which is

$$\begin{aligned}
\frac{4z-5}{z(z-1)} = \frac{4z-5}{z-1}\cdot\frac{1}{z} = \frac{4(z-1)-1}{z-1}\cdot\frac{1}{1+(z-1)} \\
&= \left(4-\frac{1}{z-1}\right)\sum_{k=0}^{\infty}(-1)^k(z-1)^k \\
&= 4\sum_{k=0}^{\infty}(-1)^k(z-1)^k - \frac{1}{z-1}\sum_{k=0}^{\infty}(-1)^k(z-1)^k \\
&= 4\sum_{k=0}^{\infty}(-1)^k(z-1)^k - \sum_{k=0}^{\infty}(-1)^k(z-1)^{k-1} \\
&= -\frac{1}{z-1} + 5\sum_{k=0}^{\infty}(-1)^k(z-1)^k
\end{aligned}$$

Therefore $\operatorname*{Res}_{z=1} f(z) = -1$.

Thus,

$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right) = 8\pi i$$

23.1. Problems

To be added

24. (Possible) Lecture 24

Theorem 24.1 (Residue at ∞). *If a function is holomorphic everywhere on* $\mathbf{C}$ *except at a finite number of singularities lying interior to a simple closed positively oriented contour* C*, then*

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$$

In particular,

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$$

Proof. Let C_0 be a negatively oriented circle centered at 0, whose interior contains C.

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By Principle of Deformation of Paths (Corollary 15.10),

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz$$

$$= -\int_{C_0} f(z)\,dz$$

$$= -\operatorname*{Res}_{z=\infty} f(z)$$

To compute $\operatorname*{Res}_{z=\infty} f(z)$ in terms of $\operatorname*{Res}_{z=0} f(z)$, we find the Laurent series of f on an annulus $R < |z| < \infty$ where

$$\max_{w\in C} |w| < R < \text{radius of } C_0$$

We have

$$f(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k}$$

where

$$a_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{n+1}}\,dz \quad \text{for } n \in \mathbf{Z}$$

Then,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+2}} + \sum_{k=1}^{\infty} a_{-k} z^{k-2}, \quad \text{for } 0 < \left|\frac{1}{z}\right| < \frac{1}{R}$$

Note that,

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = a_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz$$

$$= -\operatorname*{Res}_{z=\infty} f(z)$$

∎

Example 24.2 (Revisiting Example 23.7)**.** Let C be the circle $|z| = 2$ with positive orientation and $f(z) = \dfrac{4z-5}{z(z-1)}$, we can compute

$$\int_C \frac{4z-5}{z(z-1)}\,dz$$

using a single residue. $f(z)$ has isolated singularities at 0 and 1, both of which lie interior to C. We have

$$\begin{aligned}\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\left(\frac{4/z-5}{1/z(1/z-1)}\right)\\ &= \frac{4-5z}{z(1-z)}\end{aligned}$$

Since $\dfrac{1}{1-z}$ is holomorphic at 0, it has a Maclaurin series in the unit disk around 0, so

$$\begin{aligned}\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{4-5z}{z(1-z)} &= \frac{4-5z}{z}\cdot\frac{1}{1-z}\\ &= \left(\frac{4}{z}-5\right)\sum_{k=0}^{\infty} z^k\\ &= 4\sum_{k=0}^{\infty} z^{k-1} - 5\sum_{k=0}^{\infty} z^k\\ &= \frac{4}{z} - \sum_{k=0}^{\infty} z^k\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = 4 \quad \text{and} \quad \int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = 8\pi i$$

using Theorem 24.1.

Example 24.3. The function

$$f(z) = \frac{z^2}{(z-2)(z^2+3)}$$

has no antiderivative on the domain $G = \{z \in \mathbf{C} \ : \ |z| > 2\}$.

Let C be the circle of radius 3 centered at 0. Notice that all three singularities of f, which are 2, $\pm i\sqrt{3}$, lie in the interior of C. Hence, by Theorem 24.1,

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$$

We have,

$$\begin{aligned}\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\left(\frac{1/z^2}{(1/z-2)(1/z^2+3)}\right)\\ &= \frac{1}{z(1-2z)(1+3z^2)}\end{aligned}$$

The functions $\dfrac{1}{1-2z}$ and $\dfrac{1}{1+3z^2}$ are holomorphic at 0, and so have a Maclaurin series in the disk $D_{1/6}(0)$, here

$$|2z| < \frac{1}{3} < 1 \quad \text{and} \quad |3z^2| < \frac{1}{2} < 1$$

Therefore,

$$\frac{1}{1-2z} = \sum_{k=0}^{\infty} 2^k z^k \qquad\qquad \frac{1}{1+3z^2} = \sum_{k=0}^{\infty} (-3)^k z^{2k}$$

Hence the Maclaurin series of the product $\dfrac{1}{(1-2z)(1+3z^2)}$ has constant term 1, and thus

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = 1, \quad \text{therefore} \int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) = 2\pi i \neq 0$$

Hence $f(z)$ doesn't have an antiderivative on G by the Fundamental Theorem of Contour Integration.

Classifying Isolated Singularities

Discussion 24.4. Recall that if f has an isolated singularity at $z_0 \in \mathbf{C}$, then f has a Laurent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

on some annulus $0 < |z - z_0| < R$. The sum

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} = \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \cdots$$

is called the principal part of f at z_0.

We will classify isolated singularities bases on what the principal part looks like: whether it's zero, non-zero with finitely many terms, or infinitely many terms. The aim is to understand how to compute residues based on the type of singularity.

Definition 24.5 (Types of Singularities)**.** Suppose that f has an isolated singularity at $z_0 \in \mathbf{C}$ with principal part

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} = \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \cdots$$

then

- z_0 is said to be a removable singularity if the principal part of f at z_0 is zero, that is, $a_{-k} = 0$ for all $k \geqslant 1$.
- z_0 is said to be a essential singularity if the principal part of f at z_0 has infinitely many terms.

- z_0 is said to be a pole of order n if for some n we have $a_{-n} \neq 0$ and $a_{-k} = 0$ for every $k \geqslant n$. That is, the principal part has finitely many terms and is of the form

$$\frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-n}}{(z - z_0)^n}$$

A pole of order $n = 1$ is called a simple pole.

Remark 24.6 (Removable Singularity)**.** Suppose that f has a removable singularity at $z_0 \in \mathbf{C}$, then by definition we can write

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + 0$$

on some annulus $0 < |z - z_0| < R$. Let's define

$$g(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ a_0 & \text{if } z = z_0 \end{cases}$$

then

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

on the disk $|z - z_0| < R$. Therefore g is holomorphic on the disk, and agrees with f everywhere on the annulus $0 < |z - z_0| < R$. In this way, the singularity has been removed and we have obtained a holomorphic function g from f. This is the first example of *analytic continuation*.

24.1. Problems

To be added

25. (Possible) Lecture 25

Example 25.1.

(1) We've seen earlier that the function

$$g(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is entire. Consider now the function

$$f(z) = \frac{\sin z}{z}$$

we have that f has an isolated singularity at 0. We have seen that its Laurent series on the annulus $0 < |z| < \infty$ is

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!},$$

so the principal part of f at 0 is zero and thus f has a removable singularity at 0.

(2) The function

$$f(z) = \frac{1 - \cos z}{z^2}$$

also has a removable singularity at 0. The Laurent series on $0 < |z| < \infty$ is

$$\begin{aligned} \frac{1-\cos z}{z^2} &= \frac{1}{z^2}\left(1 - \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right) \\ &= -\frac{1}{z^2} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{2k-2}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+2)!} \end{aligned}$$

So the principal part of f at 0 is zero, and therefore the function has a removable singularity at 0. The entire function we obtain by "removing the singularity" is

$$g(z) = \begin{cases} \dfrac{1-\cos z}{z^2} & \text{if } z \neq 0 \\ \dfrac{1}{2} & \text{if } z = 0 \end{cases}$$

(3) The function $f(z) = e^{1/z}$ has an isolated singularity at 0. The Laurent series on $0 < |z| < \infty$ is

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^k$$

So the principal part of f at 0 has infinitely many terms, and therefore the function has an essential singularity at 0.

(4) The function

$$f(z) = \frac{1}{z^2(1-z)}$$

has an isolated singularity at 0. The Laurent series on $0 < |z| < 1$ is

$$\begin{aligned}\frac{1}{z^2(1-z)} = \frac{1}{z^2} \cdot \frac{1}{1-z} &= \frac{1}{z^2} \sum_{k=0}^{\infty} z^k \\ &= \sum_{k=0}^{\infty} z^{k-2} \\ &= \underbrace{\frac{1}{z^2} + \frac{1}{z}}_{\text{principal part}} + \sum_{k=0}^{\infty} z^k\end{aligned}$$

Clearly then f has a pole of order 2 at 0.

(5) The function

$$f(z) = \frac{z^2 + z - 2}{z+1}$$

has an isolated singularity at -1. The Laurent series on $0 < |z+1| < 1$ is

$$\begin{aligned}\frac{z^2+z-2}{z+1} &= \frac{z(z+1)-2}{z+1} \\ &= z - \frac{2}{z+1} \\ &= -\frac{2}{z+1} - 1 + (z+1)\end{aligned}$$

Clearly then f has a simple pole at -1.

Residues at Poles

The following theorem gives a characterisation of poles and provides an efficient method for computing the residue at poles.

Theorem 25.2 (Residue at Poles)**.** *Let $z_0 \in \mathbf{C}$ be an isolated singularity of f. Then the following are equivalent:*

(1) z_0 is a pole of order n.

(2) $f(z) = \dfrac{\phi(z)}{(z-z_0)^n}$ for some unique holomorphic function ϕ such that $\phi(z_0) \neq 0$.

Moreover, if (1) (or (2)) is true, the residue of f at z_0 is given as

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(n-1)}(z_0)}{(n-1)!}$$

Proof.

(1) $\Rightarrow$ (2) Let's assume z_0 is a pole of order $n \geqslant 1$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-n}}{(z-z_0)^n}$$

with $a_{-n} \neq 0$, on an annulus $0 < |z - z_0| < R$. Define

$$\phi(z) = \begin{cases} (z-z_0)^n f(z) & \text{if } z \neq z_0 \\ a_{-n} & \text{if } z = z_0 \end{cases}$$

Clearly $f(z) = \dfrac{\phi(z)}{(z-z_0)^n}$, and ϕ has the following series expansion on $|z - z_0| < R$

$$\phi(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^{k+n} + a_{-1}(z-z_0)^{n-1} + a_{-2}(z-z_0)^{n-2} + \cdots + a_{-n}$$

Hence $\phi(z)$ is holomorphic on the disk and thus at z_0, and is necessarily unique. Moreover, $\phi(z_0) = a_{-n} \neq 0$.

(2) $\Rightarrow$ (1) Assume

$$f(z) = \frac{\phi(z)}{(z-z_0)^n}$$

for some holomorphic function ϕ such that $\phi(z_0) \neq 0$. Since ϕ is holomorphic at z_0, then there's a disk $|z - z_0| < R$ on which ϕ has a Taylor expansion

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(z_0)}{k!}(z-z_0)^k$$

Hence,

$$\begin{aligned} f(z) &= \frac{\phi(z)}{(z-z_0)^n} \\ &= \frac{1}{(z-z_0)^n} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(z_0)}{k!}(z-z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{\phi^{(k)}(z_0)}{k!}(z-z_0)^{k-n} \\ &= \frac{\phi(z_0)}{(z-z_0)^n} + \frac{\phi^{(1)}(z_0)}{1!(z-z_0)^{n-1}} + \cdots + \frac{\phi^{(n-1)}(z_0)}{(n-1)!(z-z_0)} + \sum_{k=n}^{\infty} \frac{\phi^{(k)}(z_0)}{k!}(z-z_0)^{k-n} \\ &= \frac{\phi(z_0)}{(z-z_0)^n} + \frac{\phi^{(1)}(z_0)}{1!(z-z_0)^{n-1}} + \cdots + \frac{\phi^{(n-1)}(z_0)}{(n-1)!(z-z_0)} + \sum_{k=0}^{\infty} \frac{\phi^{(k+n)}(z_0)}{(k+n)!}(z-z_0)^k \end{aligned}$$

Since $\phi(z_0) \neq 0$ by assumption, this tells us that f has a pole of order n at z_0. Moreover, it's clear from the series that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(n-1)}(z_0)}{(n-1)!}$$

$\square$

Example 25.3.

(1) The function $f(z) = \dfrac{z+4}{z^2+1}$ has isolated singularities at $\pm i$.

Let's compute the residue first at i. Define

$$\phi(z) = \frac{z+4}{z+i}$$

then clearly $f(z) = \dfrac{\phi(z)}{z-i}$ and ϕ is holomorphic and non-zero at i.

By Theorem 25.2, i is a simple pole and

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{i+4}{2i}$$

For $-i$, define

$$\phi(z) = \frac{z+4}{z-i}$$

then clearly $f(z) = \dfrac{\phi(z)}{z+i}$ and ϕ is holomorphic and non-zero at $-i$.

By Theorem 25.2, $-i$ is a simple pole and

$$\operatorname*{Res}_{z=-i} f(z) = \phi(i) = \frac{i-4}{2i}$$

(2) The function $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ has an isolated singularity at i.

Let's compute the residue at i. Define $\phi(z) = z^3 + 2z$. Then clearly

$$\phi(z) = \frac{\phi(z)}{(z-i)^3}$$

and ϕ is holomorphic and non-zero at i. By Theorem 25.2, i is a pole of order 3 and

$$\operatorname*{Res}_{z=i} f(z) = \frac{\phi''(i)}{2!} = \frac{6i}{2} = 3i$$

(3) The function $f(z) = \dfrac{(\log z)^3}{z^2+1}$ has an isolated singularity at $\pm i$. Here $\log z$ is the branch

$$\log z = \ln|z| + i \arg z, \quad 0 < \arg z < 2\pi$$

Let's compute the residue at i. Define

$$\phi(z) = \frac{(\log z)^3}{z+i}$$

then clearly $f(z) = \dfrac{\phi(z)}{z-i}$ and ϕ is holomorphic and non-zero at i since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(i\pi/2)^3}{2i} = -\frac{\pi^3}{16}$$

By Theorem 25.2, i is a simple pole and

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = -\frac{\pi^3}{16}$$

Zeros of Holomorphic Functions

Definition 25.4 (Zeros of Holomorphic Functions)**.** Suppose that f is holomorphic at $z_0 \in \mathbf{C}$. We say that f has a zero of order n at z_0 if for some n we have $f^{(n)}(z_0) \neq 0$ and $f^{(k)}(z_0) = 0$ for $0 \leqslant k < n$, in particular $f(z_0) = 0$. A zero of order 1 is called a simple zero.

A zero is isolated if there exists an $\varepsilon > 0$ such that $f(z) \neq 0$ for every $z \in D_\varepsilon(z_0) \setminus \{z_0\}$.

25.1. Problems

To be added

26. (Possible) Lecture 26

Theorem 26.1 (Characterisation of Zeros). *Suppose that f is holomorphic at $z_0 \in \mathbf{C}$. Then the following are equivalent:*

(1) z_0 is a zero of f of order n.

(2) $f(z) = (z - z_0)^n g(z)$ for some unique holomorphic function g such that $g(z_0) \neq 0$.

Proof.

(1) $\Rightarrow$ (2) Let's assume z_0 is a zero of order $n \geqslant 1$, so $f^{(n)}(z_0) \neq 0$ and $f^{(k)}(z_0) = 0$ for $0 \leqslant k < n$. Since f is holomorphic at z_0, it has a Taylor expansion on some disk $D_\varepsilon(z_0)$

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= (z - z_0)^n \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^{k-n} \\ &= (z - z_0)^n \sum_{k=0}^{\infty} \frac{f^{(k+n)}(z_0)}{(k+n)!}(z - z_0)^k \end{aligned}$$

Define

$$g(z) = \sum_{k=0}^{\infty} \frac{f^{(k+n)}(z_0)}{(k+n)!}(z - z_0)^k$$

Clearly g is holomorphic at z_0, since it converges on $D_\varepsilon(z_0)$. Moreover,

$$g(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0$$

(2) $\Rightarrow$ (1) Assume $f(z) = (z - z_0)^n g(z)$ for some holomorphic function g such that $\phi(z_0) \neq 0$. Since g is holomorphic at z_0, then there's a disk $D_\varepsilon(z_0)$ on which g has a Taylor expansion. Hence,

$$\begin{aligned} f(z) &= (z - z_0)^n g(z) \\ &= (z - z_0)^n \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z - z_0)^{k+n} \\ &= \sum_{k=n}^{\infty} \frac{g^{(k-n)}(z_0)}{(k-n)!}(z - z_0)^k \end{aligned}$$

Since Taylor expansions are unique (Theorem 22.2), the above series is the Taylor expansion of f on $D_\varepsilon(z_0)$, so

$$\frac{f^{(k)}(z_0)}{k!} = 0, \quad 0 \leqslant k < n$$

$$\frac{f^{(n)}(z_0)}{n!} = g(z_0) \neq 0$$

Therefore $f^{(n)}(z_0) \neq 0$ and $f^{(k)}(z_0) = 0$ for $0 \leqslant k < n$. Hence z_0 is a zero of f of order n. □

Example 26.2. The function $p(z) = z^3 - 1$ has a zero at 1. Define $g(z) = 1 + z + z^2$, then

$$p(z) = (z-1)(1+z+z^2) = (z-1)g(z)$$

Clearly $g(z)$ is holomorphic at 1 and $g(1) = 3 \neq 0$. So, by Theorem 26.1, p has a simple zero at 1.

Theorem 26.3 (Zeros of non-zero Holomorphic Functions)**.** *Suppose that*

(a) f is holomorphic at z_0;

(b) $f(z_0) = 0$ but f is not identically zero on any neighbourhood of z_0.

Then z_0 is an isolated zero of f.

Proof. By (a), there's a disk $D_R(z_0)$ on which we can write

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

Consider the set $S = \{m \in \mathbf{Z} \ : \ f^{(k)}(z_0) = 0, \text{for } \ 0 \leqslant k < m\}$, it is non-empty since $1 \in S$ as $f(z_0) = 0$. Note that S is either a singleton, or the set of all positive integers. The latter implies f is identically zero, which cannot be the case by (b); hence we have the former. Thus $S = \{n\}$ for some $n > 0$, so we have

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0 \quad \text{and} \quad f^{(n)}(z_0) \neq 0.$$

Thus, $f(z)$ has a zero of order n at z_0, and so

$$f(z) = (z-z_0)^n g(z)$$

for a holomorphic function $g(z)$ such that $g(z_0) \neq 0$. Since g is continuous and non-zero at z_0, there exists a disk $D_\varepsilon(z_0)$ on which $g(z) \neq 0$ for every $z \in D_\varepsilon(z_0)$. Therefore, $f(z) \neq 0$ for every $z \in D_\varepsilon(z_0) \setminus \{z_0\}$, and hence z_0 is an isolated zero of f. □

Zeros and Poles

Theorem 26.4 (Zeros and Poles). *Suppose that*

(a) $p(z)$ *and* $q(z)$ *are holomorphic at* $z_0 \in \mathbf{C}$*; and*

(b) $p(z_0) \neq 0$ *and* $q(z)$ *has a zero of order* n *at* z_0*,*

then $f(z) = \dfrac{p(z)}{q(z)}$ *has a pole of order* n *at* z_0.

Proof. Since $q(z)$ is holomorphic at z_0 and has a zero of order n at z_0, by the (proof of) preceding theorem z_0 is an isolated zero. Hence f has an isolated singularity at z_0.

Since z_0 is a zero of order n, then

$$q(z) = (z - z_0)^n g(z)$$

for a holomorphic function $g(z)$ such that $g(z_0) \neq 0$. Writing $\phi(z) = \dfrac{p(z)}{g(z)}$, we have

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_0)^n g(z)} = \frac{\phi(z)}{(z - z_0)^n}$$

Moreover, $\phi(z_0) \neq 0$ and is holomorphic at z_0 since $p(z)$ and $g(z)$ are holomorphic at z_0 and $g(z_0) \neq 0$. The claim follows. $\square$

Example 26.5. Consider

$$f(z) = \frac{1}{1 - \cos z}$$

Using Theorem 26.4, we show that f has a pole of order 2 at 0.

Let $p(z) = 1$ and $q(z) = 1 - \cos z$, clearly they're both holomorphic at 0. Moreover $p(0) = 1 \neq 0$ and $q(z)$ has a zero of order 2 since

$$q(0) = 1 - \cos 0 = 0$$

$$q'(0) = \sin 0 = 0$$

$$q''(0) = \cos 0 = 1 \neq 0$$

Therefore, by Theorem 26.4, f has a pole of order 2 at 0.

Theorem 26.6 (Residue at a Simple Pole). *Suppose that* $p(z)$ *and* $q(z)$ *are holomorphic at* $z_0 \in \mathbf{C}$*. If* $p(z_0) \neq 0$ *and* z_0 *is a simple zero of* $q(z)$*, then*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof. By Theorem 26.4, $p(z)/q(z)$ has a simple pole at z_0. As in the proof of Theorem 26.4, we first write $q(z) = (z - z_0)g(z)$ and then

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)}$$

where $\phi(z) = p(z)/g(z)$ where g is a holomorphic function with $g(z_0) \neq 0$. Therfore, by Theorem 25.2,

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{g(z_0)}$$

Since $q(z) = (z - z_0)g(z)$, we have that $q'(z) = g(z) + (z - z_0)g'(z)$ and therefore $q'(z_0) = g(z_0)$. Hence

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$\square$

Example 26.7.

(1) Consider

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Let $p(z) = \cos z$ and $q(z) = \sin z$, and $z_k = k\pi,\ k \in \mathbf{Z}$. Clearly both p and q are holomorphic at z_k as they are entire. Moreover,

$$p(z_k) = \cos k\pi = (-1)^k \neq 0$$

$$q(z_k) = \sin k\pi = 0$$

$$q'(z_k) = \cos k\pi = (-1)^k \neq 0$$

Hence, z_k is a simple pole for each $k \in \mathbf{Z}$, and

$$\operatorname*{Res}_{z=z_k} \cot z = \frac{p(z_k)}{q'(z_k)} = \frac{(-1)^k}{(-1)^k} = 1$$

Let C be the positively oriented circle of radius $k\pi + 1$ centered at 0. Then by Cauchy's Residue theorem

$$\begin{aligned}\int_C \cot z\, dz &= 2\pi i \sum_{n=-k}^{k} \operatorname*{Res}_{z=z_n} \cot z \\ &= 2\pi i(2k+1)\end{aligned}$$

(2) Consider

$$f(z) = \frac{z - \sin z}{z^2 \sin z}$$

Let $p(z) = z - \sin z$ and $q(z) = z^2 \sin z$. Clearly both p and q are holomorphic at π as they are entire. Moreover,

$$p(\pi) = \pi - \sin \pi = \pi \neq 0$$

$$q(\pi) = \pi^2 \sin \pi = 0$$

$$q'(\pi) = 2\pi \sin \pi + \pi^2 \cos \pi = -\pi^2 \neq 0$$

Hence, π is a simple pole and

$$\operatorname*{Res}_{z=\pi} \frac{z - \sin z}{z^2 \sin z} = \frac{p(\pi)}{q'(\pi)} = \frac{\pi}{-\pi^2} = -\frac{1}{\pi}$$

(3) Consider

$$f(z) = \frac{z}{z^4 + 4}$$

Let $p(z) = z$ and $q(z) = z^4 + 4$. Clearly both p and q are holomorphic at $1 + i$ as they are entire. Moreover,

$$p(1+i) = 1 + i \neq 0$$

$$q(1+i) = (1+i)^4 + 4 = 0$$

$$q'(1+i) = 4(1+i)^3 \neq 0$$

Hence, $1 + i$ is a simple pole and

$$\operatorname*{Res}_{z=1+i} \frac{z}{z^4 + 4} = \frac{p(1+i)}{q'(1+i)} = \frac{1+i}{4(1+i)^4} = \frac{1}{8i}$$

26.1. Problems

To be added

27. (Possible) Lecture 27

PART VI. APPLICATIONS OF RESIDUE CALCULUS

Discussion 27.1. We will now apply the theory of residues to compute several types of improper integrals from calculus. Additionally, we will prove

(1) *Cauchy's Argument Principle*. The winding number of the image of a curve under a *meromorphic* function depends only on the zeros and poles of that function.

(2) *Rouche's theorem*. A useful criterion for locating zeros of holomorphic functions.

Background on Improper Integrals

Definition 27.2. Suppose $f(x)$ is a real-valued function in a single real variable.

(1) If $f(x)$ is continuous on $[0, \infty)$, therefore integrable, then the improper integral of f over that interval is defined to be
$$\int_0^\infty f(x)\,dx := \lim_{R\to\infty} \int_0^R f(x)\,dx$$
If the limit exists, the integral is said to converge. It is said to diverge otherwise.

(2) If $f(x)$ is continuous on $\mathbf{R}$, then the improper integral of f over $\mathbf{R}$ is defined to be
$$\int_{-\infty}^\infty f(x)\,dx := \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$
The integral is said to converge if both the limits exist. It is said to diverge otherwise.

(3) The Cauchy Principal Value of the improper integral in (2) is the value of the limit
$$\text{P.V.} \int_{-\infty}^\infty f(x)\,dx := \lim_{R\to\infty} \int_{-R}^R f(x)\,dx$$

Lemma 27.3. *If*
$$\int_{-\infty}^\infty f(x)\,dx$$
converges, then the Cauchy principal value exists and
$$\text{P.V.} \int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx$$

Proof. Since the integral converges, the limits
$$I_1 = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx \quad \text{and} \quad I_2 = \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$

exist, i.e. $I_1, I_2 \in \mathbf{R}$. By definition, for every $\varepsilon > 0$ there exist $M_1, M_2 > 0$ such that whenever $R_1 > M_1$ and $R_2 > M_2$, then

$$\left|\int_{-R_1}^{0} f(x)\,dx - I_1\right| < \varepsilon \quad \text{and} \quad \left|\int_{0}^{R_2} f(x)\,dx - I_2\right| < \varepsilon$$

respectively.

Then, for any $R > \max\{M_1, M_2\}$ we have simultaneously

$$\left|\int_{-R}^{0} f(x)\,dx - I_1\right| < \varepsilon \quad \text{and} \quad \left|\int_{0}^{R} f(x)\,dx - I_2\right| < \varepsilon$$

That is,

$$I_1 = \lim_{R\to\infty}\int_{-R}^{0} \quad \text{and} \quad I_2 = \lim_{R\to\infty}\int_{0}^{R} f(x)\,dx$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx = I_1 + I_2 &= \lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{R} f(x)\,dx \\
&= \lim_{R\to\infty}\left(\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx\right) \\
&= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \\
&= \text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx
\end{aligned}$$

□

Remark 27.4. The converse of Lemma 27.3 is false in general: even if the Cauchy principal value exists, the integral may diverge. For example,

$$\begin{aligned}
\text{P.V.}\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx &= \lim_{R\to\infty}\int_{-R}^{R} \frac{2x}{1+x^2}\,dx \\
&= \lim_{R\to\infty}\Big[\ln(1+x^2)\Big]_{-R}^{R} \\
&= 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\,dx &= \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx \\
&= \lim_{R_1\to\infty}\Big[\ln(1+x^2)\Big]_{-R_1}^{0} + \lim_{R_2\to\infty}\Big[\ln(1+x^2)\Big]_{0}^{R_2} \\
&= \lim_{R_1\to\infty} -\ln(1+R_1^2) + \lim_{R_2\to\infty}\ln(1+R_2^2);
\end{aligned}$$

note that these limits diverge, the first to $-\infty$ and the second to $+\infty$, and thus the integral diverges.

Lemma 27.5. *Suppose $f(x)$ is an even function and continuous on $\mathbf{R}$. If*

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx$$

exists, then the improper integral exists and

$$\int_{-\infty}^{\infty} f(x)\, dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx$$

Moreover,

$$\int_{0}^{\infty} f(x)\, dx = \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx$$

Proof. We have,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\, dx &= \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\, dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\, dx \\
&= \lim_{R_1 \to \infty} \frac{1}{2} \int_{-R_1}^{R_1} f(x)\, dx + \lim_{R_2 \to \infty} \frac{1}{2} \int_{R_2}^{R_2} f(x)\, dx, \quad \text{since } f \text{ is even} \\
&= \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx + \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx \\
&= \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx
\end{aligned}$$

Along the way we also proved

$$\int_{0}^{\infty} f(x)\, dx = \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx$$

□

Improper Integrals of Rational Functions

Discussion 27.6. We make the following assumptions for this section.

(1) $f(x) = p(x)/q(x)$ is a rational function with real coefficients, and $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros.

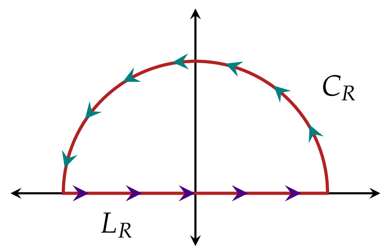
(3) $f(x)$ is an even function.

Discussion 27.7. We describe a method to compute integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx \quad \text{and} \quad \int_{0}^{\infty} \frac{p(x)}{q(x)}\, dx$$

Step 1. Treating f as a complex function, identify the singularities of f that lie above the real axis. Let's label them $z_1, \ldots, z_n$.

Step 2. Define a semicircular contour $\Gamma_R = C_R + L_R$ as follows



where C_R is the upper semicircle parametrised as $z(t) = Re^{it}$, $0 \leqslant t \leqslant \pi$, and L_R is the line segment from $-R$ to R. Where R is chosen such that $R > \max_{i=1}^{n} |z_i|$, so that z_i's lie in the interior of C.

Step 3. We apply Cauchy's Residue theorem

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \int_{\Gamma} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)$$

Parametrise L_R as $z(x) = x$, $-R \leqslant x \leqslant R$. Then,

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

Hence,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z) - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

Since $f(x)$ is even,

$$\int_{-\infty}^{\infty} f(x)\,dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx$$

Step 4. Prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\,dz = 0$$

This can always be proved if, for instance, $\deg q(z) - \deg p(z) \geqslant 2$.

Example 27.8. Compute $\displaystyle\int_0^{\infty} \frac{1}{1+x^4}\,dx$.

The singularities of $f(z) = 1/(1+z^4)$ are solutions to the equation $z^4 = -1$, which are given by, for $k = 0, 1, 2, 3$

$$\begin{aligned} z_k &= e^{i(\pi/4 + 2k\pi/4)} \\ &= e^{i\pi/4}(e^{i\pi/2})^k = \frac{i^k(1+i)}{\sqrt{2}} \end{aligned}$$

Of these four singularities, z_0 and z_1 lie above the real axis; moreover $|z_0| = |z_1| = 1$.

We integrate $f(z)$ over the semicircular contour Γ_R where $R > 1$. Using the residue theorem, we have

$$\int_{-R}^{R} \frac{1}{1+x^4}\,dx = 2\pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right) - \int_{C_R} \frac{1}{1+z^4}\,dz$$

Let $p(z) = 1$ and $q(z) = 1 + z^4$, which are clearly holomorphic. At each singularity z_k, we have $p(z_k) = 1 \neq 0$, $q(z_k) = 0$ and $q'(z_k) = 4z_k^3 \neq 0$; therefore the singularities are simple poles. Hence,

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{p(z_k)}{q'(z_k)} = \frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = -\frac{z_k}{4}$$

On C_R, we note that $|z| = R$, therefore

$$|z^4 + 1| \geqslant ||z|^4 - 1| = R^4 - 1$$

and so,

$$\begin{aligned} \left| \int_{C_R} \frac{1}{1+z^4}\,dz \right| &\leqslant L(C_R) \cdot \max_{z \in C_R} \frac{1}{|1+z^4|} \\ &\leqslant \frac{\pi R}{R^4 - 1} \to 0, \text{ as } R \to \infty \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^\infty \frac{1}{1+x^4}\,dx &= \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx \\ &= \pi i \left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right) \\ &= -\frac{\pi i}{4} (z_0 + z_1) \\ &= -\frac{\pi i}{4} \left(\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}} \right) \\ &= \frac{\pi}{2\sqrt{2}} \end{aligned}$$

27.1. Problems

To be added

28. (Possible) Lecture 28

Improper Integrals from Fourier Analysis

Discussion 28.1. We make the following assumptions for this section.

(1) $f(x) = p(x)/q(x)$ is a rational function with real coefficients, and $p(x)$ and $q(x)$ have no factors in common.

(2) $q(x)$ has no real zeros.

(3) $a > 0$ and $f(x)\sin ax$, or $f(x)\cos ax$, is an even function.

Discussion 28.2. The same steps, with minor modifications, as in Discussion 27.7 can be used to compute integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin ax\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos ax\,dx$$

We use Euler's formula to write

$$\int_{-\infty}^{\infty} f(x)\cos ax\,dx + i\int_{-\infty}^{\infty} f(x)\sin ax\,dx = \int_{-\infty}^{\infty} f(x)e^{iax}\,dx$$

We simply compute the RHS, and depending what integral we want to compute we simply take the real or imaginary part, and then take the limit.

Example 28.3. Compute $\displaystyle\int_0^{\infty} \frac{\cos 2x}{(x^2+4)^2}\,dx.$

We will integrate $f(z)e^{2iz}$, where $f(z) = 1/(z^4+4)^2$, over a semicircular contour Γ_R. Clearly $f(z)$ has a single singularity $2i$ above the real axis; moreover $|2i| = 2$, so we take $R > 2$.

Using the residue theorem, we have

$$\int_{-R}^{R} f(x)e^{2ix}\,dx = 2\pi i \cdot \operatorname*{Res}_{z=2i} f(z)e^{2iz} - \int_{C_R} f(z)e^{2iz}\,dz$$

To compute the residue, let $\phi(z) = \dfrac{e^{2iz}}{(z+2i)^2}$, so that

$$f(z)e^{2iz} = \frac{\phi(z)}{(z-2i)^2}$$

$\phi(z)$ is non-zero and holomorphic at $2i$, hence $2i$ is a pole of order 2 and

$$\begin{aligned}
\operatorname*{Res}_{z=2i} f(z)e^{2iz} = \phi'(2i) &= \left.\frac{2ie^{2iz}(z+2i)^2 - 2(z+2i)e^{2iz}}{(z+2i)^4}\right|_{z=2i} \\
&= \frac{2ie^{2i\cdot 2i}(2i+2i)^2 - 2(2i+2i)e^{2i\cdot 2i}}{(2i+2i)^4} \\
&= \frac{2ie^{-4}(4i)^2 - 2(4i)e^{-4}}{(4i)^4} \\
&= \frac{2(4i)e^{-4}(4i^2-1)}{(4i)^4} \\
&= \frac{2e^{-4}(-5)}{(4i)^3} \\
&= \frac{5}{32i}\, e^{-4}
\end{aligned}$$

On C_R, we note that $|z| = R$, therefore

$$|z^4+4| \geqslant \big||z|^4 - 4\big| = R^4 - 4 \quad \text{and} \quad |e^{2iz}| = e^{2\operatorname{Re}(iz)} = e^{-2\operatorname{Im} z} \leqslant 1$$

So,

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{2iz}\, dz\right| &\leqslant L(C_R) \cdot \max_{z\in C_R} \frac{|e^{2iz}|}{|z^4+4|} \\
&\leqslant \frac{\pi R}{R^4 - 4} \to 0, \text{ as } R \to \infty
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\, dx &= \frac{1}{2} \cdot \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2}\, dx \\
&= \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\, dx \\
&= \frac{1}{2} \lim_{R\to\infty} \operatorname{Re} \int_{-R}^{R} f(x)e^{2ix}\, dx \\
&= \frac{1}{2} \lim_{R\to\infty} \left(2\pi i \cdot \operatorname*{Res}_{z=2i} f(z)e^{2iz} - \int_{C_R} f(z)e^{2iz}\, dz\right) \\
&= \frac{1}{2} \lim_{R\to\infty} \left(\frac{5\pi}{16e^4} - \int_{C_R} f(z)e^{2iz}\, dz\right) \\
&= \frac{5\pi}{32e^4}
\end{aligned}$$

Remark 28.4. The preceding method works only if $\deg q(x) - \deg p(x) \geqslant 2$, where $f(x) = p(x)/q(x)$. Because otherwise the bound for contour integrals may not be enough to prove

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{aiz}\, dz = 0$$

In this case, we may be able to use Jordan's Lemma instead.

Lemma 28.5 (Jordan's Lemma)**.** *Assume*

(1) f is holomorphic at all points in the upper half plane (the domain where points have positive imaginary part) that are exterior to some circle $|z| = R_0$.

(2) C_R is the semicircle with $R > R_0$.

(3) There exists $M_R > 0$ such that $|f(z)| \leqslant R$ for all $z \in C_R$ and $\lim_{R\to\infty} M_R = 0$.

Then, for any $a > 0$,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{aiz}\, dz = 0$$

Proof. Skipped. □

Example 28.6. Compute $\displaystyle\int_0^\infty \frac{x \sin 2x}{x^2+3}\, dx$.

We will integrate $f(z)e^{2iz}$, where $f(z) = z/(z^2+3)$, over a semicircular contour Γ_R. Clearly $f(z)$ has a single singularity $i\sqrt{3}$ above the real axis; moreover $\left|i\sqrt{3}\right| = \sqrt{3}$, so we take $R > \sqrt{3}$.

Using the residue theorem, we have

$$\int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\, dx = \operatorname{Im} \int_{-R}^{R} f(x)e^{2ix}\, dx$$

$$= \operatorname{Im}\left(2\pi i \cdot \operatorname*{Res}_{z=i\sqrt{3}} f(z)e^{2iz} - \int_{C_R} f(z)e^{2iz}\, dz\right)$$

To compute the residue, write $p(z) = ze^{2iz}$, $q(z) = z^2+3$; both are clearly holomorphic. Note that $p(i\sqrt{3}) \neq 0$, $q(i\sqrt{3}) = 0$ and $q'(i\sqrt{3} = i2\sqrt{3} \neq 0$, so $i\sqrt{3}$ is a simple pole and therefore

$$\operatorname*{Res}_{z=i\sqrt{3}} f(z)e^{2iz} = \operatorname*{Res}_{z=i\sqrt{3}} \frac{p(z)}{q(z)} = \frac{p(i\sqrt{3})}{q'(i\sqrt{3})} = \frac{e^{-2\sqrt{3}}}{2}$$

Hence

$$\int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\, dx = \pi e^{-2\sqrt{3}} - \operatorname{Im} \int_{C_R} f(z)e^{2iz}\, dz$$

We now show

$$\lim_{R\to\infty} \operatorname{Im} \int_{C_R} f(z)e^{2iz}\, dz = 0$$

On C_R, we note that $|z| = R$, therefore

$$\left|\frac{z}{z^2+3}\right| = \frac{|z|}{|z^2+3|} \leqslant \frac{|z|}{|z|^2-3} = \frac{R}{R^2-3} =: M_R$$

Since $\lim_{R\to\infty} M_R = 0$, by Jordan's Lemma we have

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{2iz}\,dz = 0, \qquad \text{hence} \quad \lim_{R\to\infty} \operatorname{Im} \int_{C_R} f(z)e^{2iz}\,dz = 0$$

Thus,

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx = \frac{1}{2}\cdot \text{P.V.} \int_{-\infty}^\infty \frac{x\sin 2x}{x^2+3}\,dx$$

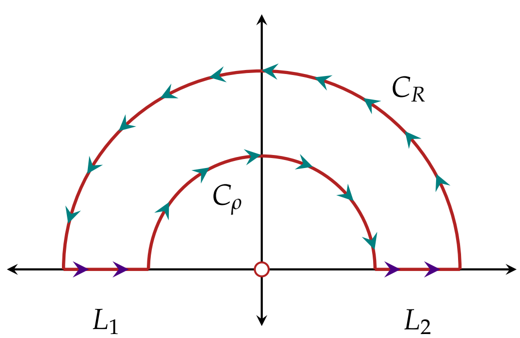
$$= \frac{1}{2}\pi e^{-2\sqrt{3}}$$

Indented Path

Discussion 28.7. Till now we've avoided functions that have singularities on the real line; how do we integrate such functions? For such an instance, *indented paths* can sometimes be used to avoid an isolated singularity or branch point that lies on the real axis. We illustrate this method using an example.

Example 28.8 (Dirichlet's Integral)**.** Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

We integrate $f(z) = \dfrac{e^{iz}}{z}$ over the contour $\Gamma = C_R + L_1 + C_\rho + L_2$



By Cauchy-Goursat

$$0 = \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz = \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz$$

Parametrise $-L_1$, L_2 as follows

$$-L_1 : z_1(x) = -x, \quad \rho \leqslant x \leqslant R$$

$$L_2 : z_2(x) = x, \quad \rho \leqslant x \leqslant R$$

Then,

$$\begin{aligned}\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz &= \int_{L_2} \frac{e^{iz}}{z}\,dz - \int_{-L_1} \frac{e^{iz}}{z}\,dz \\ &= \int_\rho^R \frac{e^{ix}}{x}\,dx - \int_\rho^R \frac{e^{-ix}}{x}\,dx \\ &= \int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\,dx \\ &= 2i \int_\rho^R \frac{\sin x}{x}\,dx\end{aligned}$$

Hence,

$$\int_\rho^R \frac{\sin x}{x}\,dx = -\frac{1}{2i}\left(\int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz\right)$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}\left(\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z}\,dz + \lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z}\,dz\right)$$

By Jordan's Lemma, for $M_R = 1/R$, we get

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z}\,dz = 0$$

We now compute $\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z}\,dz$; consider the Laurent series

$$\begin{aligned}\frac{e^{iz}}{z} &= \frac{1}{z}\sum_{k=0}^{\infty} \frac{i^k z^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{i^k z^{k-1}}{k!} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{i^k z^{k-1}}{k!} \\ &= \frac{1}{z} + g(z)\end{aligned}$$

Since $g(z)$ is a power series about 0, it is holomorphic in a neighbourhood of 0. In particular, $g(z)$ is continuous on the closed disk $\overline{D}_\varepsilon(0)$ for some $\varepsilon > 0$, and is thus bounded on this closed disk. Say, $|gz| \leqslant M$ for every $|z| \leqslant \varepsilon$. Hence if $\rho < \varepsilon$, which we can assume,

$$\begin{aligned}\left|\int_{C_\rho} g(z)\,dz\right| &\leqslant L(C_\rho)\cdot \max_{z\in C_\rho} |g(z)| \\ &\leqslant \pi\rho M \to 0, \text{ as } \rho \to 0\end{aligned}$$

Thus, parametrising $-C_\rho$ as $\gamma(t) = \rho e^{it}$, $0 \leqslant t \leqslant \pi$

$$\begin{aligned}
\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z}\,dz &= \lim_{\rho\to 0}\int_{C_\rho} \frac{1}{z}\,dz + \lim_{\rho\to 0}\int_{C_\rho} g(z)\,dz \\
&= -\lim_{\rho\to 0}\int_{-C_\rho} \frac{1}{z}\,dz + 0 \\
&= -\lim_{\rho\to 0}\int_0^{\pi} \frac{i\rho e^{it}}{\rho e^{it}}\,dt \\
&= -\lim_{\rho\to 0}\int_0^{\pi} i\,dt \\
&= -\pi i
\end{aligned}$$

Therefore

$$\int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}(-\pi i) = \frac{\pi}{2}$$

28.1. Problems

To be added

29. (Possible) Lecture 29

Argument Principle

Definition 29.1 (Meromorphic Function)**.**

29.1. Problems

To be added

References

[1] Brown, James Ward and Churchill, Ruel V.. *Complex Variables and Applications*. McGraw-Hill, 2009.

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