Tutorial 8 solutions - MAT267

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Consider the second-order ODE

$$\alpha(x)u'' + p(x)u' + q(x)u = r(x) \ , \qquad x \in [a, b] \subseteq \mathbb{R}$$

where the coefficient functions $\alpha(x), p(x), q(x), r(x)$ are assumed to be continuous real-valued functions on $I = [a, b]$. If $\alpha(x) \equiv 1$, then the above equation is said to be in normal form, so we get

$$u'' + p(x)u' + q(x)u = r(x) \ , \qquad x \in [a, b]$$

Provided $\alpha(x) \neq 0$ on $[a, b]$, we may divide by $\alpha(x)$ to obtain an equivalent ODE in normal form. We will mostly focus on the homogeneous case:

$$u'' + p(x)u' + q(x)u = 0 \ , \qquad x \in [a, b]$$

We can then transform the above into a system by introducing $w = u'$ and apply the existence and uniqueness theorem to this system to obtain the existence of a unique solution to the above, given a initial condition.

Definition 1. The Wronskian of two differentiable functions $f(x), g(x)$ is:

$$W(f, g, x) = f(x)g'(x) - f'(x)g(x) = \det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}$$

Clearly, if f, g are linearly dependent differentiable functions, the Wronskian vanishes identically.

Theorem 2. If f, g are linearly independent solutions to the homogeneous ODE, the Wronskian never vanishes.

Note: we are not given nor required to know a proof for this.

Theorem 3. If f, g are linearly independent solutions of the homogeneous ODE, then $f(x)$ must vanish at one point between any two successive zeros of $g(x)$. In other words, the zeroes of $f(x)$ and $g(x)$ occur alternately.

Proof: Suppose $x_1 < x_2$ are successive zeroes of g. Then, we may compute:

$$W(f, g, x_1) = f(x_1)g'(x_1)$$

$$W(f, g, x_2) = f(x_2)g'(x_2)$$

Further, $g'(x_1) \cdot g'(x_2) \leq 0$, since if not, then $g'(x_1), g'(x_2)$ would both be non-zero with the same sign, so, for example, g would "pass through" each zero from below, but, by the intermediate value theorem, g would then have to obtain a zero in between x_1, x_2, which contradicts our assumption that they are successive zeros of $g(x)$. Now assume, for a contradiction, that $f \neq 0$ on $[x_1, x_2]$, then, again by the intermediate value theorem, $f < 0$ or $f > 0$ on $[x_1, x_2]$. In either case we have $W(f, g, x_1) \cdot W(f, g, x_2) \leq 0$. If $W(f, g, x_1) \cdot W(f, g, x_2) = 0$ then one of $W(f, g, x_1), W(f, g, x_2)$ must be zero, which would contradict that $f(x), g(x)$ are linearly independent. If $W(f, g, x_1) \cdot W(f, g, x_2) < 0$ then one of $W(f, g, x_1), W(f, g, x_2)$ is positive and the other negative, but, by definition, the Wronskian is simply a polynomial in f, f', g, g' all continuous functions, the map $x \mapsto W(f, g, x)$ is continuous, so again by the intermediate value theorem, there is some $x^* \in (x_1, x_2)$ such that $W(f, g, x^*) = 0$, but this would again contradict that $f(x), g(x)$ are linearly independent. Thus, there must be at least one point, $x_f \in (x_1, x_2)$ such that $f(x_f) = 0$, as desired. $\square$

Theorem 4 (Sturn comparison theorem). Let $f(x), g(x)$ be non-trivial solutions of the ODES

$$u'' + p(x)u = 0$$

$$v'' + q(x)v = 0$$

respectively, where $p(x) \geq q(x)$. Then $f(x)$ vanishes at least once between any two zeroes of $g(x)$, unless $p \equiv q$ and f is a constant multiple of g.

Proof: Let $x_1 < x_2$ be two successive zeroes of g. Assume, for a contradiction, that $f(x) \neq 0$ for all $x \in [x_1, x_2]$. Without loss of generality (by replacing f with $-f$ and/or g with $-g$ as necessary) we may assume that $f > 0$ and

$g > 0$ on (x_1, x_2). We may compute, under these assumptions, that:

$$W(f, g, x_1) = f(x_1)g'(x_1) \geq 0$$

$$W(f, g, x_2) = f(x_2)g'(x_2) \leq 0$$

Next,

$$\begin{aligned}
\frac{d}{dx}W(f, g, x) &= \frac{d}{dx}\Big(f(x)g'(x) - f'(x)g(x)\Big) \\
&= f(x)g''(x) - f''(x)g(x) \\
&= f(x)\big(-q(x)g(x)\big) - \big(p(x)f(x)\big)g(x) \\
&= f(x)g(x)\Big(p(x) - q(x)\Big)
\end{aligned}$$

Since $p(x) \geq q(x)$ and $f, g > 0$ all by assumption, we have $\dfrac{d}{dx}W(f, g, x) \geq 0$ and therefore that the map $x \mapsto W(f, g, x)$ is non-decreasing. In particular, $x_1 < x_2 \implies W(f, g, x_1) \leq W(f, g, x_2)$. But, as remarked above, $W(f, g, x_2) \leq 0 \leq W(f, g, x_1)$, so taken together $W(f, g, x_1) = W(f, g, x_2) = 0$. Again using that $W(f, g, x)$ is non-decreasing in x, $W(f, g, x) \equiv 0$ on $[x_1, x_2]$, and so, f, g are linearly dependent by **Theorem 2**. Further, $\dfrac{d}{dx}W(f, g, x) = 0$ on $[x_1, x_2]$. Since we assumed $f, g > 0$ and thus that $f(x)g(x) > 0$ on (x_1, x_2), it also follows that $p \equiv q$ on (x_1, x_2), which finishes the proof. $\square$

Application 1. Suppose that $q(x) \leq 0$. Then no non-trivial solution of $u'' + q(x)u = 0$ can have more than one zero.

Proof: First, $v \equiv 1$ is a solution to $v'' = 0$. So, if u is a solution to $u'' + q(x)u = 0$ with $x_1 < x_2$ zeroes, then by **Theorem 4**, v must vanish at some $x^* \in (x_1, x_2)$, which, since $v \equiv 1$, is clearly a contradiction. Thus, u has at most one zero. $\square$

Application 2: Suppose $q(x) \geq k^2 > 0$. Then any solution of $u'' + q(x)u = 0$ must vanish between any two successive zeroes of any given function $A\cos\big(k(x - x_1)\big)$ and hence in any interval of length π/k.

Proof: First, $v(x) = A\cos\big(k(x - x_1)\big)$ is a solution to $v'' + k^2v = 0$. So, if u is a solution to $u'' + q(x)u = 0$, then u must vanish in $(x_1, x_1 + \pi/k)$ by

Theorem 4. Allowing x_1 to vary in $\mathbb{R}$ then yields that u must also vanish on any interval of length π/k, as desired. $\square$

Recall from Tutorial 7 that the Bessel differential equation of order n is:

$$u'' + \frac{1}{x}u' + \left(1 - \frac{n^2}{x^2}\right)u = 0$$

whose solution is the so-called Bessel function of order n. We may then perform the substitution $w = u\sqrt{x}$ to obtain the equivalent ODE:

$$w'' + \left(1 - \frac{4n^2 - 1}{4x^2}\right)w = 0$$

whose solutions vanish whenever u does, for $x \neq 0$, as is apparent from our substitution.

Application 3: Every interval of length π of the positive x-axis contains at least 1 zero of any solution of the Bessel ODE of order zero, and at most 1 zero of any non-trivial solution of the Bessel ODE of order $n > 1/2$.

Proof: If $n = 0$, then $q(x) = 1 - (4n^2 - 1)/4x^2 \equiv 1 \geq 1^2 > 0$, so, by **Application 2**, if u solves $u'' + q(x)u = 0$, i.e. the Bessel ODE of order 0, then it must vanish on every interval of length π. If $n > 1/2$, then $q(x) = 1 - (4n^2 - 1)/4x^2 \leq 0$, so, by **Application 1**, any solution to $u'' + q(x)u = 0$, i.e. any non-trivial solution to the Bessel ODE of order n, can have at most one zero. $\square$