Problem Set 4

Isaac Clark

November 3, 2025

Problem 1. *Show that* $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ *converges to* $a \in A \iff$ *for all* $U \subseteq A$ *which are open in* A *and contain* a*, there exists some* $N \in \mathbb{N}$ *such that* $a_n \in U$ *for all* $n \geq N$*.*

Problem 2. *If* $f : X \to Y$ *is a homeomorphism and* $A \subseteq X$*, then* $f : A \to f(A)$ *is a homeomorphism.*

Problem 3. $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous* $\iff$ $f \circ \iota_A$ *is continuous for all* $\varnothing \neq A \subsetneq X$*.*

Problem 4. *If* $f : A \to B$ *is continuous and* $B \subseteq \mathbb{R}^m$*, then* $f : A \to \mathbb{R}^m$ *is continuous.*

Problem 5. *Show that if* $A \subseteq \mathbb{R}^n$ *is sequentially compact and* $A \cong B$ *then* B *is sequentially compact. Deduce that* $[0, 1]$ *is not homeomorphic to* $\mathbb{R}$ *or* $(0, 1]$*.*

Problem 6. *Suppose that* $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ *converges to* $a \in A$*, where* A *is discrete. What can you say about the* a_n*? What can you say about Cauchy sequences in* A*?*

Problem 7. *Prove the dyadic rationals* $\mathbb{Z}[2^{-1}] = \{n2^{-m} \mid n \in \mathbb{Z}\ m \in \mathbb{N}\}$ *are dense.*

Problem 8. *Suppose that* $f : X \to Y$ *is a homeomorphism. Show that the restriction of* f *to* $X \setminus \{x\}$ *is a homeomorphism onto its image for any* $x \in X$*.*

Problem 9. *Show that,*

(a) $\mathbb{R} \cong \Delta$*, where* $\Delta = \{(x, x) \mid x \in \mathbb{R}\}$*.*

(b) $\mathbb{R} \cong \mathbb{R}^+$*.*

(c) $A \cong B$ *if* A, B *are discrete and* $|A| = |B|$*.*