Problem Set 2

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Problem 1. *Let I be a set. For each $i \in I$, let $U_i \subseteq \mathbb{R}^n$ be an open set. Suppose also that $U, V \subseteq \mathbb{R}^n$ are open. Show that,*

1. *$\varnothing, \mathbb{R}^n$ are open.*
2. *$U \cap V$ is open.*
3. *$\bigcup_{i \in I} U_i$ is open.*

Problem 2. *Replacing each occurence of open with closed in* ***Problem 1****, show that,*

1. *$\varnothing, \mathbb{R}^n$ are closed.*
2. *$U \cup V$ is closed.*
3. *$\bigcap_{i \in I} U_i$ is closed.*

Problem 3. *Suppose $U \subseteq \mathbb{R}^n$ is such that for all $u \in U$, there exists an open set $A \subseteq U$ such that $x \in A$. Then, U is open.*

Problem 4. *A subset $D \subseteq \mathbb{R}$ is said to be dense if for all $U \subseteq \mathbb{R}$ non-empty and open, we have $U \cap D \neq \varnothing$. Fix $x, y \in \mathbb{R}$ with $x < y$. Show that,*

1. *There exists some $q \in \mathbb{Q}$ such that $x < q < y$.*
2. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*
3. *$U \subseteq \mathbb{R}$ is open $\iff$ for all $u \in U$ there exists some $\varepsilon \in \mathbb{Q}^+$ such that $B_\varepsilon(u) \subseteq U$.*
4. *Every open set is the countable union of open balls.*

Hint: for (1), apply the Archimedian property to $(y - x)^{-1}$. Then, it suffices to show that (1) $\implies$ (2), that (2) $\implies$ (3) and then use all of these plus ***Problem 3*** *for (4).*

Problem 5. *Fix $a, b \in \mathbb{R}$ with $a < b$. Show that the following are open,*

1. (a, b)
2. (a, ∞)
3. $(-\infty, a)$

Note: there are at least 3 different proofs of this, so one mustn't settle for just one!

Problem 6. *Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a bounded, monotone decreasing sequence. Show that $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}}\{x_n\}$.*

Problem 7. *Recall that for $A \subseteq \mathbb{R}^n$, we defined*

$$\partial A = \{x \in \mathbb{R}^n \ : \ \forall U \subseteq \mathbb{R}^n \text{ open with } x \in U,\ U \cap A \neq \varnothing,\ U \cap (\mathbb{R}^n \setminus A)\}$$

Show that,

1. $\partial A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$.
2. $int(A) \cap \partial A = \varnothing$.
3. $\partial A = \varnothing \iff A$ *is clopen.*

Problem 8. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $\sup f(A) = f(\sup A)$ for all bounded, non-empty $A \subseteq \mathbb{R}$. What can you deduce about f?*

Problem 9. *Find the boundary, interior, and closure of each of the following subsets of $\mathbb{R}^2$.*

(a) $\{(x, y) \ : \ y = 0\}$.

(b) $\{(x, y) \ : \ x > 0 \text{ and } y \neq 0\}$.

(c) $\{(x, y) \ : \ x \in \mathbb{Q}\}$.

(d) $\{(x, y) \ : \ x \neq 0 \text{ and } y \leq x^{-1}\}$.

Problem 10. *For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $||x|| = \sqrt{x \cdot x}$ be the usual norm, let $||x||_1 = \sum_{i=1}^n |x_i|$ be the 1-norm, and let $||x||_\infty = \max_{1\leq i\leq n} |x_i|$ be the max-norm. Prove that, for all $x \in \mathbb{R}^n$,*

$$||x||_\infty \leq ||x||_1 \leq ||x|| \leq n||x||_\infty$$

Deduce that if $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, then $U \times V \subseteq \mathbb{R}^{n+m}$ is open.

Problem 11. *Show that,*

1. *For fixed* $\alpha \in \mathbb{R}$*, the map* $m_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ *given by* $m_\alpha(x) = \alpha x$ *is continuous.*
2. *The map* $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *given by* $+(x, y) = x + y$ *is continuous.*
3. *Linear combinations of continuous functions are continuous.*
4. *The map* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *given by* $d(x, y) = ||x - y||$ *is continuous.*

Problem 12. *We say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is open if* $f(U)$ *is open whenever* U *is open. For* $1 \leq i \leq n$*, let* $\pi_i : \mathbb{R}^n \to \mathbb{R}$ *be the map* $x \mapsto x_i$*. Show that* π_i *is both continuous and open.*

Problem 13. *Show that* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = \sqrt{|x|}$ *is continuous.*

Problem 14. *Suppose* $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^n$ *is a sequence. Then,*

$$\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} ||x_n|| = 0$$

Problem 15. *Suppose* $A \subseteq \mathbb{R}^n$ *is not closed. Find a function* $f : A \to \mathbb{R}$ *which is continuous and unbounded.*

Hint: Try $f(x) = ||y - x||^{-1}$ *for a suitable choice of* y*.*

Problem 16. *We say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is finite-to-one if* $|f^{-1}(\{y\})| < \infty \ \forall y \in \mathbb{R}^m$*. In this case, is* f *necessarily continuous?*

Hint: Consider $f(x) = x$ *if* $x \in \mathbb{Q}$ *and* $f(x) = x + 1$ *if* $x \notin \mathbb{Q}$*.*

Problem 17. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *has the property that* $|f(x) - f(y)| \leq C|x - y|^\alpha$ *for all* $x, y \in \mathbb{R}^n$ *and some constants* $C \in \mathbb{R}$ *and* $\alpha > 1$*. Prove that* f *is constant.*

Hint: For $x, y \in \mathbb{R}$ *with* $x < y$*, divide* $[x, y]$ *into* N *subintervals, construct a telescoping sum, and then take absolute values to show that* $|f(y) - f(x)| \leq C|y - x|^\alpha N^{1-\alpha}$*.*

Problem 18. *Let* $\mathcal{F} = \{f_i : \mathbb{R} \to \mathbb{R}\}_{i\in I}$ *be a family of continuous functions. Let* $V(\mathcal{F})$ *be the set of points for which the* f_i *simultaneously vanish, i.e.* $f_i(x) = 0$*. Then,*

1. $V(\mathcal{F})$ *is closed.*
2. *If* C *is closed, then there is a continuous function* f *such that* $C = V(\{f\})$*.*

Hint: Consider $f(x) = e^{(x-a)^{-1}(x-b)^{-1}}$ *for* $x \in (a,b)$ *and* $f(x) = 0$ *otherwise.*

Problem 19. *Suppose* $C_1, C_2 \subseteq \mathbb{R}^n$ *are disjoint closed sets. Find* $U_1, U_2 \subseteq \mathbb{R}^n$ *disjoint open sets such that* $C_1 \subseteq U_1$ *and* $C_2 \subseteq U_2$.

Problem 20. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence. Show,*

$$\lim_{n\to\infty} x_n = x \iff \forall U \subseteq \mathbb{R}^n \text{ open with } x \in U,\ \exists N \in \mathbb{N} \text{ such that } n > N \implies x_n \in U$$

Problem 21. *Show that if* $A \subseteq B \subseteq \mathbb{R}^n$ *then* $\overline{A} \subseteq \overline{B}$. *Further, if for each* $i \in I$, $U_i \subseteq \mathbb{R}^n$ *is open, we have that* $\overline{\bigcup_{i\in I} U_i} \supseteq \bigcup_{i\in I} \overline{U_i}$, *with equality when* I *is finite.*

Problem 22. *Show that* $\Delta = \{(x,x) \ : \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{2n}$ *is closed.*

Problem 23. *Is it true that, if* $U \subseteq \mathbb{R}$ *is open, we have* $U = int(\overline{U})$*?*

Problem 24. *Fix* $x, y > 0$. *Define two sequences by* $a_1 = \max\{x,y\}$, $g_1 = \min\{x,y\}$, $a_{n+1} = (a_n + g_n)/2$, *and* $g_{n+1} = \sqrt{a_n g_n}$. *Prove that,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} g_n$$

Hint: First prove the AM-GM inequality.

Problem 25. *Suppose* $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous functions such that* $f(x) = g(x)$ *when* $x \in D$, *where* D *is dense in* $\mathbb{R}$. *Prove that* $f(x) = g(x)$ *for all* $x \in \mathbb{R}$.

Problem 26. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous, show that the graph of* f, *denoted by* $\Gamma(f) = \{(x, f(x)) \ : \ x \in \mathbb{R}^n\}$ *is closed.*

Problem 27. *Similarly to the definiton of an open map, we say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is a closed map if* $f(C)$ *is closed when* C *is closed. Show that the map* $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ *as defined in* ***Problem 12*** *is NOT closed.*

Hint: Consider $\{(x,y) \in \mathbb{R}^2 \ : \ xy = 1\}$.

Problem 28. *Show that the map* $M(x,y) = \max\{x,y\}$ *is continuous.*