

Problem Set 5

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Problem 1. Suppose $X, Y \subseteq \mathbb{R}^n$ are connected. Is $\emptyset \neq X \cap Y$ also connected?

Problem 2. Suppose $U \subseteq \mathbb{R}^n$ is open and connected. Prove it is also path-connected.

Problem 3. Let $U \subseteq \mathbb{R}$ be open and $f : U \rightarrow \mathbb{R}$ be a differentiable function such that $f' \equiv 0$. Show that f is locally constant. Does this mean f is constant?

Problem 4. A subset $X \subseteq \mathbb{R}^n$ is called *extremally disconnected* if \overline{U} is open in X whenever U is open in X . Then,

1. Show that every extremally disconnected is totally disconnected.
2. Show that \mathbb{Q} and $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$ are not extremally disconnected.
3. Show that every discrete space is also extremally disconnected.
4. Show the converse of (3). Hint: Use (2).

Problem 5. Prove or disprove: if X is totally disconnected then X is disconnected.

Problem 6. A topological space is called *hyperconnected* or *anti-Hausdorff*, if it cannot be written as the union of any proper open sets. Find all hyperconnected subsets of \mathbb{R}^n .
Hint: Show that any two open subsets of such a space must intersect non-trivially.