Problem Set 6

Isaac Clark

November 23, 2025

Problem 1. *Fix* $U, V \subseteq \mathbb{R}^n$. *Let* $d(U,V) = \inf\{||x-y|| \mid x \in U,\ y \in V\}$. *Show that,*

1. $d(U,V) = 0$ *if* $\overline{U} \cap V \neq \varnothing$ *or* $U \cap \overline{V} \neq \varnothing$.
2. *If* U *is compact,* V *is closed, and* $U \cap V = \varnothing$*, then* $d(U,V) > 0$.
3. *Find closed* U, V *such that* $U \cap V = \varnothing$ *but* $d(U,V) = 0$.

Problem 2. *Recall that* $B \subseteq \mathbb{R}^n$ *is convex if the line segment joining any two points of* B *lies in* B. *Show that,*

1. *If* $B \subseteq \mathbb{R}^n$ *is convex and compact and* $x \notin B$*, then there exists an* $(n-1)$*-dimensional affine subspace of* $\mathbb{R}$ *(i.e. the image of an* $(n-1)$*-dimensional vector subspace of* $\mathbb{R}^n$ *under a translation) such the* x *and* B *lie on opposite sides of* H.
2. *If* $A \subseteq \mathbb{R}^n$ *is compact, then there is a unique convex subset* B *of* $\mathbb{R}^n$ *such that* $A \subseteq B$ *and* B *lies in any compact convex subset of* $\mathbb{R}^n$ *containing* A. *Show also that* B *is compact.* B *is called the convex hull of* A.
3. *Show that a point* x *lies in the convex hull of a compact set* A *if and only if, for every* $(n-1)$*-dimensional affine subspace* H *with* A *on one side of* H*,* x *lies on the same side.*

Problem 3. *Let* X *be a compact subset of* $\mathbb{R}^n$ *and let* $\mathcal{O}$ *be an open cover of* X. *Show that,*

1. $X \subseteq B_1 \cup \cdots B_k$ *for some* k*, where each* B_i *is a closed ball in some* $U = U_i \in \mathcal{O}$.
2. *There exists* $\varepsilon > 0$ *such that, for all* $x \in X$*,* $B_\varepsilon(x) \subseteq U$ *for some* $U \in \mathcal{O}$.

Problem 4. *Suppose* $U \subseteq \mathbb{R}^n$ *is open and* $C \subseteq U$ *is compact. Show that there exists a compact set* K *such that* C *lies in the interior of* K *and* $K \subseteq U$.

Problem 5. *Suppose* $A, B \subseteq X$ *are disjoint, compact subsets of some* $X \subseteq \mathbb{R}^n$*. Show that there exists disjoint open sets* U *and* V *such that* $A \subseteq U$ *and* $B \subseteq V$*.*

Problem 6. *Suppose* $f : X \to \mathbb{R}^m$ *is continuous and that* X *is compact. Show that* f *is a closed map. Deduce that if* f *is bijective, then it is a homeomorphism.*

Problem 7. *Suppose* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$*. Let* $A \subseteq X$ *and* $B \subseteq Y$ *be compact subsets thereof. Let* $W = U \times V$ *be an open set which contains* $A \times B$*. Show that there are open sets* $U \subseteq \mathbb{R}^n$ *and* $V \subseteq \mathbb{R}^m$ *such that* $A \times B \subseteq U \times V \subseteq W$*.*

Problem 8. *Let* $X \subseteq \mathbb{R}^n$ *be compact and let* $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ *be a sequence of open covers of* X *such that* $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ *for all* $n \in \mathbb{N}$*. Show that there is a sequence of finite subcovers, say* $\{\mathcal{O}_n^f\}_{n\in\mathbb{N}}$*, such that* $\mathcal{O}_n^f \subseteq \mathcal{O}_{n+1}^f$*.*

Problem 9. *Given* $\{X_i\}_{i\in I}$ *a family of compact sets in* $\mathbb{R}^n$*, is* $\bigcap_{i\in I} X_i$ *is compact?*

Problem 10. *Let* $\{K_n\}_{n\in\mathbb{N}}$ *be a sequence of non-empty, compact sets in* $\mathbb{R}^n$ *such that* $K_{n+1} \subseteq K_n$ *for each* $n \in \mathbb{N}$*. Show that* $\bigcap_{n\in\mathbb{N}} K_i$ *is nonempty.*

Problem 11. *Let* $\{K_i\}_{i\in I}$ *be a family of compact subsets of* $[0, 1]$ *such that for every finite subset* $F \subseteq I$*,* $\bigcap_{i\in F} K_i \neq \varnothing$*. Prove that* $\bigcap_{i\in I} K_i \neq \varnothing$*.*

Problem 12. *Show that a discrete subset is compact if and only if it is finite.*

Problem 13. *Suppose* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$ *with* Y *compact. Then,*

1. *The projection* $\pi : X \times Y \to X$ *is closed.*
2. *If* $f : X \to Y$ *is such that* $\Gamma(f)$ *is closed, then* f *is continuous.*

Problem 14. *Show that,*

1. $\Delta^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1,\ x_i \geq 0 \text{ for all } i\}$.
2. $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + xy + y^2 = 1\}$.
3. $\heartsuit = \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)\}$*, for* $a > 0$.

are compact.

Problem 15. *We say that* $A \subseteq \mathbb{R}^n$ *is Lindelöf if every open cover of* A *admits a countable subcover. Show that, in fact, every subset of* $\mathbb{R}^n$ *is Lindelöf.*

Problem 16. *We say that* $A \subseteq \mathbb{R}^n$ *is countably compact if every countable open cover of* A *has finite subcover. Show that there does not exist a subset of* $\mathbb{R}^n$ *which is countably compact but not compact. Hint: Limit points.*