Handout 4

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1 Connectedness

Definition 1.1. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is disconnected if there exist $\varnothing \neq A, B \subseteq X$ disjoint, open (in X) sets such that $X = A \cup B$. Such A, B are called a disconnection of X. Equivalently, X is disconnected if there exists a proper clopen subset of X.*

Definition 1.2. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is connected if it is not disconnected.*

Definition 1.3. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is path-connected if for every $x, x' \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = x'$. Such an f is called a path from x to x' in X.*

Theorem 1.1. $[0,1]$ *is connected.*

Proof. Suppose, for a contradiction, that $[0,1]$ is disconnected. Let $A, B \subseteq [0,1]$ be a disconnection of $[0,1]$. We may assume without loss of generality that $0 \in A$. Let $s = \sup A$ and observe that $0 \leq s \leq 1$. So, since $[0,1] = A \cup B$, either $s \in A$ or $s \in B$. If $s \in A$, then since A is open, there exists some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq A$. But then $s + \varepsilon/2 \in A$ with $s < s + \varepsilon/2$, contradicting that s is an upper bound of A. If $s \in B$, then, since B is open, there exists some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq B$. But, since $s \in \overline{A}$, there must be some point $a \in A$ such that $a \in (s - \varepsilon, s + \varepsilon)$, contradicting that A and B are disjoint. Hence, $[0,1]$ is connected. $\square$

Theorem 1.2. *Suppose $X \subseteq \mathbb{R}^n$ is (path-)connected and $f : X \to \mathbb{R}^m$ is continuous. Then $f(X)$ is (path-)connected.*

Proof. Suppose X is path-connected. Given $y, y' \in f(X)$, pick x, x' such that $f(x) = y$ and $f(x') = y'$. Let $\gamma : [0,1] \to X$ be a path connecting x and x' in X. Then, $\gamma' = f \circ \gamma : [0,1] \to f(X)$ is continuous and satisifes $\gamma'(0) = f(\gamma(0)) = f(x) = y$ and

$\gamma'(1) = y'$. Thus, there is a path from y to y' for any $y, y' \in f(X)$, and so $f(X)$ is path-connected.

For the second part, we will show the contrapositive. Suppose $f(X)$ is disconnected. Let A, B be a disconnection of $f(X)$. Then, consider $f^{-1}(A)$ and $f^{-1}(B)$. Each are open, since f is continuous, they are disjoint, since A, B are disjoint, and their union is X, and so they exhibit a disconnection of X. □

Proposition 1.1. *X is disconnected $\iff$ there exists a non-constant, continuous function $f : X \to \{0, 1\}$. Equivalently, X is connected $\iff$ every continuous function $f : X \to \{0, 1\}$ is constant.*

Proof. If $f : X \to \{0, 1\}$ is non-constant and continuous, then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are a disconnection of X. If X is disconnected, let A, B be a disconnection of X, and define $f(x) = \chi_A(x)$, i.e. $f(x) = 1$ if $x \in A$ and $f(x) = 0$ otherwise. Then, one may check that the only open sets in $\{0, 1\}$ are $\varnothing, \{0\}, \{1\}, \{0, 1\}$, each of which are, by construction, open in X. Since $A \neq X$, f is not constant, and so is as desired. □

Proposition 1.2. *Let $\{X_i\}_{i \in I}$ be a family of (path-)connected sets. Then, if we further have that $\bigcap_{i \in I} X_i \neq \varnothing$ then $\bigcup_{i \in I} X_i$ is (path-)connected.*

Proof. □

Theorem 1.3. *If X is path-connected, then it is also connected.*

Proof. Fix $x_0 \in X$. Then, for each $x \in X$, let $\gamma_x : [0, 1] \to X$ be a path from x_0 to x in X. Then, $\bigcup_{x \in X} \gamma_x([0, 1]) \subseteq X$ since each image is contained in X. Also, $X \subseteq \gamma_x([0, 1])$ since $x \in \gamma_x([0, 1])$ for each $x \in X$. Further, $x_0 \in \gamma_x([0, 1])$ for every $x \in X$, and so $\bigcap_{x \in X} \gamma_x([0, 1]) \neq \varnothing$, and, since each γ_x is continuous by **Theorem 1.2** and $[0, 1]$ is connected by **Theorem 1.1**, $\gamma_x([0, 1])$ is connected for each $x \in X$. Taken all together, by **Proposition 1.2**, $X = \bigcup_{x \in X} \gamma_x([0, 1])$ is connected. □

Theorem 1.4. *There exists $X \subseteq \mathbb{R}^2$ such that X is connected but not path-connected.*

Proof. Let $X = \{(x, \sin(1/x)) \mid x > 0\} \cup \{(0, 0)\}$. □

Theorem 1.5. *If X, Y are (path-)connected then $X \times Y$ is (path-)connected.*

Theorem 1.6. *If X is connected and $X \subseteq Y \subseteq \overline{X}$ then Y is connected.*

Proposition 1.3. *The following are (path-)connected:*

1. *Intervals (i.e. $(a, b), [a, b)$, etc.).*

2. *Rays (i.e. (a, ∞), etc.).*

3. $\mathbb{R}^n$.

4. $B_r(x)$.

Definition 1.4. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is totally disconnected if the only connected subsets of X are the singletons.*

Proposition 1.4. *The following are totally disconnected:*

1. *Discrete sets.*

2. $\mathbb{Q}$.

3. *The Cantor set, see Handout 1.*

Definition 1.5. *We say that $C \subseteq \mathbb{R}^n$ is convex if for all $a, b \in C$, we have that $(1-t)a + tb \in C$ for every $t \in [0, 1]$.*

Theorem 1.7. *Suppose $X \subseteq \mathbb{R}$. Then the following are equivalent,*

1. *X is connected.*

2. *X is convex.*

3. *X is path-connected.*

Corollary. If $f : X \to \mathbb{R}$ is continuous and X is connected then f attains all values between any two points in the image, i.e. the Intermediate Value theorem.

Definition 1.6. *We say that $f : X \to \mathbb{R}^n$ is locally constant if for any $x \in X$, there is an open neighbourhood of X, say U, on which f is constant.*

Proposition 1.5. *If $f : X \to \mathbb{R}^n$ is locally constant and X is connected, then f is constant.*