Problem Set 5

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Problem 1. *Suppose $X, Y \subseteq \mathbb{R}^n$ are connected. Is $\varnothing \neq X \cap Y$ also connected?*

Problem 2. *Suppose $U \subseteq \mathbb{R}^n$ is open and connected. Prove it is also path-connected.*

Problem 3. *Let $U \subseteq \mathbb{R}$ be open and $f : U \to \mathbb{R}$ be a differentiable function such that $f' \equiv 0$. Show that f is locally constant. Does this mean f is constant?*

Problem 4. *A subset $X \subseteq \mathbb{R}^n$ is called extremally disconnected if $\overline{U}$ is open in X whenever U is open in X. Then,*

1. *Show that every extremally disconnected is totally disconnected.*
2. *Show that $\mathbb{Q}$ and $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$ are not extremely disconnected.*
3. *Show that every discrete space is also extremally disconnected.*
4. *Show the converse of (3). Hint: Use (2).*

Problem 5. *Prove or disprove: if X is totally disconnected then X is disconnected.*

Problem 6. *A topological space is called hyperconnected or anti-Hausdorff, if it cannot be written as the union of any proper open sets. Find all hyperconnected subsets of $\mathbb{R}^n$. Hint: Show that any two open subsets of such a space must intersect non-trivially.*

Problem 7. *Suppose $X \cong Y$. Show that X is connected if and only if Y is connected, i.e. that connectedness is a topological invariant.*

Problem 8. *Suppose X is path-connected. Prove or disprove: $\overline{X}$ is also path-connected.*