Handout 4

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1 Connectedness

Definition 1.1. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is disconnected if there exist $\varnothing \neq A, B \subseteq X$ disjoint, open (in X) sets such that $X = A \cup B$. Such A, B are called a disconnection of X. Equivalently, X is disconnected if there exists a proper clopen subset of X.*

Definition 1.2. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is connected if it is not disconnected.*

Definition 1.3. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is path-connected if for every $x, x' \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = x'$. Such an f is called a path from x to x' in X.*

Theorem 1.1. $[0, 1]$ *is connected.*

Proof. Suppose, for a contradiction, that $[0, 1]$ is disconnected. Let $A, B \subseteq [0, 1]$ be a disconnection of $[0, 1]$. We may assume without loss of generality that $0 \in A$. Let $s = \sup A$ and observe that $0 \leq s \leq 1$. So, since $[0, 1] = A \cup B$, either $s \in A$ or $s \in B$. If $s \in A$, then since A is open, there exists some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq A$. But then $s + \varepsilon/2 \in A$ with $s < s + \varepsilon/2$, contradicting that s is an upper bound of A. If $s \in B$, then, since B is open, there exists some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq B$. But, since $s \in \overline{A}$, there must be some point $a \in A$ such that $a \in (s - \varepsilon, s + \varepsilon)$, contradicting that A and B are disjoint. Hence, $[0, 1]$ is connected. $\square$

Theorem 1.2. *Suppose $X \subseteq \mathbb{R}^n$ is (path-)connected and $f : X \to \mathbb{R}^m$ is continuous. Then $f(X)$ is (path-)connected.*

Proof. Suppose X is path-connected. Given $y, y' \in f(X)$, pick x, x' such that $f(x) = y$ and $f(x') = y'$. Let $\gamma : [0, 1] \to X$ be a path connecting x and x' in X. Then, $\gamma' = f \circ \gamma : [0, 1] \to f(X)$ is continuous and satisifes $\gamma'(0) = f(\gamma(0)) = f(x) = y$ and

$\gamma'(1) = y'$. Thus, there is a path from y to y' for any $y, y' \in f(X)$, and so $f(X)$ is path-connected.

For the second part, we will show the contrapositive. Suppose $f(X)$ is disconnected. Let A, B be a disconnection of $f(X)$. Then, consider $f^{-1}(A)$ and $f^{-1}(B)$. Each are open, since f is continuous, they are disjoint, since A, B are disjoint, and their union is X, and so they exhibit a disconnection of X. $\qquad\square$

Proposition 1.1. *X is disconnected $\iff$ there exists a non-constant, continuous function $f : X \to \{0, 1\}$. Equivalently, X is connected $\iff$ every continuous function $f : X \to \{0, 1\}$ is constant.*

Proof. If $f : X \to \{0, 1\}$ is non-constant and continuous, then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are a disconnection of X. If X is disconnected, let A, B be a disconnection of X, and define $f(x) = \chi_A(x)$, i.e. $f(x) = 1$ if $x \in A$ and $f(x) = 0$ otherwise. Then, one may check that the only open sets in $\{0, 1\}$ are $\varnothing, \{0\}, \{1\}, \{0, 1\}$, each of which are, by construction, open in X. Since $A \neq X$, f is not constant, and so is as desired. $\qquad\square$

Proposition 1.2. *Let $\{X_i\}_{i \in I}$ be a family of (path-)connected sets. Then, if we further have that $\bigcap_{i \in I} X_i \neq \varnothing$ then $\bigcup_{i \in I} X_i$ is (path-)connected.*

Proof. Pick some $x \in \bigcap_{i \in I} X_i$ and let $C \subseteq \bigcup_{i \in I} X_i$ be a clopen neighbourhood of x. Each $X_i \cap C$ is clopen and non-empty in X_i and so, since each X_i is connected, $X_i \subseteq C$ for every $i \in I$. So, $\bigcup_{i \in I} X_i \subseteq C$, and so we have equality and thus that $\bigcup_{i \in I} X_i$ contains no non-trivial clopen sets, and so is connected. $\qquad\square$

Theorem 1.3. *If X is path-connected, then it is also connected.*

Proof. Fix $x_0 \in X$. Then, for each $x \in X$, let $\gamma_x : [0, 1] \to X$ be a path from x_0 to x in X. Then, $\bigcup_{x \in X} \gamma_x([0, 1]) \subseteq X$ since each image is contained in X. Also, $X \subseteq \gamma_x([0, 1])$ since $x \in \gamma_x([0, 1])$ for each $x \in X$. Further, $x_0 \in \gamma_x([0, 1])$ for every $x \in X$, and so $\bigcap_{x \in X} \gamma_x([0, 1]) \neq \varnothing$, and, since each γ_x is continuous by **Theorem 1.2** and $[0, 1]$ is connected by **Theorem 1.1**, $\gamma_x([0, 1])$ is connected for each $x \in X$. Taken all together, by **Proposition 1.2**, $X = \bigcup_{x \in X} \gamma_x([0, 1])$ is connected. $\qquad\square$

Theorem 1.4. *There exists $X \subseteq \mathbb{R}^2$ such that X is connected but not path-connected.*

Proof. Let $X = \{(x, \sin(1/x)) \mid x > 0\} \cup \{(0, 0)\}$. The rest is an exercise. *Hint:* Let $f : [0, 1] \to X$ be any continuous function with $f(0) = (0, 0)$ and consider the $\sup\{t \in [0, 1] \mid f([0, t]) = \{(0, 0)\}\}$. $\qquad\square$

Theorem 1.5. *If X, Y are (path-)connected then $X \times Y$ is (path-)connected.*

Proof. Fix $y \in Y$. For $x \in X$, let $U_x = (\{x\} \times Y) \cup (X \times \{y\})$. Note that each U_x is connected by **Theorem 1.2**, since it is the image of Y, X under $y' \mapsto (x, y')$ and $x' \mapsto (x', y)$ respectively, and $(x, y) \in (\{x\} \times Y) \cap (X \times \{y\})$ so we conclude by **Proposition 1.2**. Clearly $X \times Y = \bigcup_{x \in X} U_x$ and $\varnothing \neq X \times \{y\} \subseteq \bigcap_{x \in X} U_x$, so again by **Proposition 1.2**, $X \times Y$ is connected. $\square$

Theorem 1.6. *If X is connected and $X \subseteq Y \subseteq \overline{X}$ then Y is connected.*

Proof. Let $f : Y \to \{0, 1\}$ be a continuous function. Then $f|_X = f \circ \iota_X$ is continuous, and so, since X is connected, by **Proposition 1.1**, it is constant. Since $Y \subseteq \overline{X}$, we have that X is dense in Y, and so that f is also constant, whence, again by **Proposition 1.1**, Y is disconnected. $\square$

Proposition 1.3. *The following are (path-)connected:*

1. *Intervals (i.e. $(a, b), [a, b)$, etc.).*

2. *Rays (i.e. (a, ∞), etc.).*

3. $\mathbb{R}^n$.

4. $B_r(x)$.

Proof. For (1), by **Theorem 1.2** and **Theorem 1.1** we have that any interval of the form $[a, b]$ is connected. Then, $(0, 1] = \bigcup_{n \in \mathbb{Z}^+} [1/n, 1]$ and so, by **Proposition 1.2**, it is connected. We may construct each other unit interval type in a similar manner, and then use **Theorem 1.2**.

For (2), pick $N > a$, then $(a, \infty) = \bigcup_{n \geq N} (a, n)$, and so we may apply **Proposition 1.2**. The other cases follow similarly.

For (3), write $\mathbb{R} = \bigcup_{n \in \mathbb{Z}^+} (-n, n)$ and use **Proposition 1.2**. Then proceed by induction and use **Theorem 1.5**.

For (4), observe that if $a, b \in B_r(0)$ then $t \mapsto (1 - t)a + tb$ exhibits a path from a to b in $B_r(0)$, so by **Theorem 1.3**, it is connected. Then, $B_r(x)$ is simply the image of $B_r(0)$ under $y \mapsto y + x$, so apply **Theorem 1.2**. $\square$

Definition 1.4. *We say that $\varnothing \neq X \subseteq \mathbb{R}^n$ is totally disconnected if the only connected subsets of X are the singletons.*

Proposition 1.4. *The following are totally disconnected:*

1. *Discrete sets.*

2. $\mathbb{Q}$.

3. *The Cantor set, see Handout 1.*

Proof. For (1), simply observe that discrete sets have only isolated points, so each may be covered by some $B_\varepsilon(x)$. Then, if the set is a singleton, it is clearly connected. If the set contains more than 1 point, fix one, and consider the disconnection given by the ball about the fixed point and the union of the balls about each other point.

(2) follows from the density of $\mathbb{R} \setminus \mathbb{Q}$.

For (3), the idea is that for any two distinct points in the Cantor set will eventually belong to different sub-intervals, and so these respective closed intervals exhibit a disconnection. *Exercise:* Make this formal. $\square$

Definition 1.5. *We say that $C \subseteq \mathbb{R}^n$ is convex if for all $a, b \in C$, we have that $(1-t)a + tb \in C$ for every $t \in [0, 1]$.*

Theorem 1.7. *Suppose $X \subseteq \mathbb{R}$. Then the following are equivalent,*

1. *X is connected.*

2. *X is convex.*

3. *X is path-connected.*

Proof. (1) $\implies$ (2). Suppose $a, b \in X$ and $a < t < b$. Then, $X \cap (-\infty, t)$ and $X \cap (t, \infty)$ are open in X and so, if $t \notin X$ then this would exhibit a disconnection of X, which would be a contradiction, hence $t \in X$, and so X is convex.

(2) $\implies$ (3). $t \mapsto (1-t)a + tb$ is a path from a to b in X by definition.

(3) $\implies$ (1). This is **Theorem 1.3**. $\square$

Corollary. If $f : X \to \mathbb{R}$ is continuous and X is connected then f attains all values between any two points in the image, i.e. the Intermediate Value theorem.