Handout 2

Isaac Clark

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It will be convenient to introduce a bit of nomenclature. We say that $U \subseteq \mathbb{R}^n$ is an open neighbourhood of $x \in \mathbb{R}^n$ if U is open and $x \in U$. We begin with a few prepatory results.

Lemma 0.1. *If $A \subseteq B \subseteq \mathbb{R}^n$, then $\overline{A} \subseteq \overline{B}$.*

Proof. Per Handout 1, $x \in \overline{A}$ if and only if every open neighbourhood of x intersects A non-trivially, i.e. $\exists a \in A \cap U$ for all U an open neighbourhood of x. But then $a \in A$ implies $a \in B$ and so $\exists a \in B \cap U$ for all U an open neighbourhood of x, and so $x \in \overline{B}$, again per Handout 1. Hence $\overline{A} \subseteq \overline{B}$. $\square$

Proposition 0.1. *If $|A| < \infty$ then A is closed.*

Proof. We proceed by induction on $|A|$. If $|A| = 0$ then $A = \varnothing$ and we are done. If $|A| = 1$, then $A = \{a\}$ for some a. Now suppose $y \in \mathbb{R}^n \setminus A$. Then $y \neq a$, so $||y - a|| > 0$, and so, putting $\varepsilon = ||y - a||/2$, we have that $U = B_\varepsilon(y)$ is an open neighbourhood of y which does not contain a, so $U \cap A = \varnothing$, so $U \subseteq \mathbb{R}^n \setminus A$. This exhibits $\mathbb{R}^n \setminus A$ as open, and thus A as closed. Now suppose that every A with $|A| \leq k$ for some $k \in \mathbb{Z}^+$ is closed. Then, if $|A| = k + 1$, we may write $A = B \cup \{a\}$, for some $|B| = k$. Then, by the inductive hypothesis, B and $\{a\}$ are closed, so A is the finite union of closed, which is closed per Handout 1. This completes the inductive step and thus the proof. $\square$

1 Limit points

Suppose $A \subseteq \mathbb{R}^n$.

Definition 1.1. *We say that $x \in \mathbb{R}^n$ is a limit point of A if every open neighbourhood of x intersects A at a point other than itself.*

Proposition 1.1. *x is a limit point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Proof. Per Handout 1, $x \in \overline{A \setminus \{x\}}$ if and only if every open neighbourhood of x intersects $A \setminus \{x\}$, but this means that every open neighbourhood intersects A at a point other than itself. $\square$

Theorem 1.1. *Let A' be the set of all limit points of A. Then, $\overline{A} = A \cup A'$.*

Proof. "$\subseteq$" Suppose $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$, and we are done. If $x \notin A$, then $A \setminus \{x\} = A$, and so, per **Proposition 1.1**, since $x \in \overline{A} = \overline{A \setminus \{x\}}$, we have that $x \in A'$, and so $x \in A \cup A'$. In either case, we have that $x \in A \cup A'$.

"$\supseteq$" Suppose $x \in A \cup A'$. If $x \in A$, then, since per Handout 1, $A \subseteq \overline{A}$, we have $x \in \overline{A}$. If $x \in A'$, then per **Proposition 1.1**, $x \in \overline{A \setminus \{x\}}$. But $A \setminus \{x\} \subseteq A$, so by **Lemma 0.1**, $\overline{A \setminus \{x\}} \subseteq \overline{A}$, and so $x \in \overline{A}$. In either case, we have that $x \in \overline{A}$.

Taking "$\subseteq$" and "$\supseteq$" together, we have set equality, and thus the claim. $\square$

Proposition 1.2. $0 \in \mathbb{R}$ *is:*

(a) NOT a limit point of $\{0\}$.

(b) a limit point of $[0, 1]$.

(c) NOT a limit point of $[1, \infty)$.

(d) a limit point of $\mathbb{Q}$.

(e) NOT a limit point of $\mathbb{Z}$.

Proof. Per **Proposition 1.1**, x is a limit point of $\{0\}$ if and only if $x \in \overline{\{0\} \setminus \{x\}}$. But $\{0\} \setminus \{x\}$ is either $\varnothing$, if $x = 0$, or $\{0\}$, if $x \neq 0$. In particular, for $x = 0$, $\overline{\{0\} \setminus \{0\}} = \varnothing$, which does not contain 0, so 0 is not a limit point of $\{0\}$. Hence (a). In fact, $\{0\}$ contains no limit points.

For (b) and (d), consider that every U an open neighbourhood of 0 contains $B_{1/n}(0)$ for some $n \in \mathbb{Z}^+$, which contains, in particular $1/2n$, which is a member of $[0, 1]$ and $\mathbb{Q}$, and so by definition 0 is a limit point of $[0, 1]$ and $\mathbb{Q}$. Hence (b) and (d).

For (c) and (e), put $U = B_{1/2}(0)$. Then, U is an open neighbourhood of 0, but if $y \in B_{1/2}(0)$ then $|y| < 1/2$, but no integer besides 0 or element of $[1, \infty)$ satisfies this, whence $0 \notin \overline{\mathbb{Z} \setminus \{0\}}$ and $0 \notin \overline{[1, \infty) \setminus \{0\}}$, so 0 is not a limit point of $\mathbb{Z}$ nor $[1, \infty)$, per **Proposition 1.1**. $\square$

Proposition 1.3. *A is closed if and only if $A' \subseteq A$.*

Proof. Per Handout 1, A is closed if and only if $A = \overline{A}$. Per **Theorem 1.1**, $\overline{A} = A \cup A'$. So A is closed if and only if $A = A \cup A'$ if and only if $A' \subseteq A$. $\square$

Theorem 1.2. *x is a limit point of A if and only if every open neighbourhood of x contains infinitely many points of A.*

Proof. If an open neighbourhood of x contains infinitely many points of A, then it certainly contains a point of A other than x, and so, since this criteria is satsified for all open neighbourhoods of x, we have that x is a limit point of A.

Conversely, suppose for a contradiction that x is a limit point of A such that there is an open neighbourhood of x, say U, which only contains finitely many points of A. Then U only contains finitely many points of $A \setminus \{x\}$. Write $\{x_1, ..., x_m\} = U \cap (A \setminus \{x\})$. Then, $\mathbb{R}^n \setminus \{x_1, ..., x_m\}$ is the compliment of a closed set by **Lemma 0.1**, and so is open. But then $U \cap (\mathbb{R}^n \setminus \{x_1, ..., x_m\})$ is an open (as the finite intersection of open, per Handout 1) neighbourhood of x (as $x \in U$ and $x \neq x_i$ for each i) which contains no points of $A \setminus \{x\}$, which contradicts that $x \in \overline{\mathbb{R}^n \setminus \{x\}}$, and thus that x is a limit point of A, per **Proposition 1.1**. So, every open neighbourhood of a limit point contains infinitely many points of A. $\square$

2 Sequences, revisited

One may notice that up until now, we have primarily considered constructions that make use of the normed vector space structure of $\mathbb{R}^n$, without paying much mind to underlying structure of $\mathbb{R}$ itself. This begs the question as to why we are considering $\mathbb{R}^n$ at all, instead of $\mathbb{Q}^n$, which enjoys many of the same properties, i.e. it is a vector space (over $\mathbb{Q}$) and can be endowed with a norm in the same way $\mathbb{R}^n$ is. There are a number of reasons to work with $\mathbb{R}^n$ instead of $\mathbb{Q}^n$, perhaps the most elementary reason is that the class of convergent sequences in $\mathbb{Q}$ is very small; a notion we seek in this section to (among other things) formalize.

Definition 2.1. *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying:*

1. *For all $x, y \in X$, $d(x, y) \geq 0$, with equality if and only if $x = y$.*
2. *For all $x, y \in X$, $d(x, y) = d(y, x)$.*
3. *For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.*

Remark. A pair (X, d) of a set and a metric on the set X is called a metric space.

Definition 2.2. *Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence in X (i.e. a map from $\mathbb{N}$ to X). Then, we say that x_n converges to x for $x \in X$, if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x, x_n) < \varepsilon$. In this case we often write $x_n \to x$.*

Remark. If $X = \mathbb{R}^n$ and $d(x, y) = ||x - y||$ then one can check that d is a metric, and then **Definition 2.2** is precisely the $\varepsilon - N$ definition of convergence.

Remark. Importantly, we require the limit of a convergent sequence in X to lie in X.

Definition 2.3. *Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence. We say that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$.*

Proposition 2.1. *Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges. Then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.*

Proof. Essentially the same as Eitan's proof for the case of $\mathbb{R}$. □

Definition 2.4. *We say X is complete if every Cauchy sequence in X converges.*

Proposition 2.2. *$\mathbb{Q}$ is not complete under the metric $d(x, y) = |x - y|$.*

Proof. Put $x_0 = 2$. For $n \geq 0$, put $x_{n+1} = \big(x_n + 2/x_n\big)/2$. By induction, we have that each $x_n \in \mathbb{Q}$. Suppose $\lim_{n\to\infty} x_n$ exists, and put $x := \lim_{n\to\infty} x_n$. Then,

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{2}{\lim_{n\to\infty} x_n}\right) = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

Rearranging we have $x^2 = 2$, but no rational number squares to 2, so $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$.

However, we may observe that, by AM-GM,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{x_n \frac{2}{x_n}} = \sqrt{2}$$

And so $x_n \geq \sqrt{2}$ for all $n \in \mathbb{N}$. Further,

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n = \frac{2 - x_n^2}{2x_n} \leq \frac{2 - (\sqrt{2})^2}{2x_n} = 0$$

So $x_{n+1} \leq x_n$, whence $\{x_n\}_{n\in\mathbb{N}}$ is a monotone-decreasing sequence which is bounded below, so by the Monotone Convergence theorem, we have that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Thus, since convergent implies Cauchy, we have that the $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Thus, there exists a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Hence $\mathbb{Q}$ is not complete. $\square$

Remark. The procedure in the proof of **Proposition 2.2** is called the Newton-Heron algorithm, and can be used more generally to construct sequences which converge to a root, r, of a twice continuously differentiable function, f, under the condition that $f'(r) \neq 0$, and x_0 is sufficiently close to r.

Proposition 2.3. $(0,1)$ *is not complete under the metric* $d(x,y) = |x-y|$.

Proof. Put $x_n = (n+2)^{-1}$ for each $n \in \mathbb{N}$. Then, $x_n \in (0,1)$ for each n. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ converges in $(0,1)$, and put $x = \lim_{n\to\infty} x_n$. Since the x_n are strictly decreasing, the limit must be less than or equal to all of the $x_n = (n+2)^{-1}$. Then, if $x \in (0,1)$ then $x^{-1} \in \mathbb{R}$ and so there exists some $m \in \mathbb{N}$ such that $x^{-1} < m$. But then $m^{-1} < x$, a contradiction. Thus, $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $(0,1)$. However, for $\varepsilon > 0$, $2\varepsilon^{-1} \in \mathbb{R}$, and so there exists some $N \in \mathbb{N}$ such that $2\varepsilon^{-1} < N$, but then $N^{-1} < \varepsilon/2$, and so for $n, m \geq N$, $|n^{-1} - m^{-1}| \leq n^{-1} + m^{-1} \leq 2N^{-1} < \varepsilon$, so $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(0,1)$. Hence, there is a Cauchy sequence in $(0,1)$ which does not converge in $(0,1)$, and so $(0,1)$ is not complete. $\square$

We now move to prove that $\mathbb{R}$ is complete, which we will deduce using the Monotone Convergence Theorem, which in turn comes from the Supremum Axiom. This will essentially say that the broadest class of sequences in $\mathbb{R}$ converge (since every convergent sequence is Cauchy, the class of convergent sequences is a subset of the Cauchy sequences). There are other ways to deduce the completeness of $\mathbb{R}$ using alternate constructions, which we may explore later on, time permitting.

Lemma 2.1. *Suppose* $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *are convergent sequences which satisfy* $x_n \leq y_n$ *for all* $n \in \mathbb{N}$. *Then,* $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then, since $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are convergent, say $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$ $|x_n - x| < \varepsilon$ and for all $n \geq N_1$, $|y_n - y| < \varepsilon$. Then, for all $n \geq N = \max\{N_1, N_2\}$, we have $x - \varepsilon < x_n < x + \varepsilon$ and $y - \varepsilon < y_n < y + \varepsilon$. Then,

$$x < x_n + \varepsilon \leq y_n + \varepsilon < y + 2\varepsilon$$

for all $\varepsilon > 0$. In particular, if $x > y$, then $(x-y)/2 > 0$, and so putting $\varepsilon = (x-y)/2$,

$$x < y + 2\varepsilon = y + (x - y) = x$$

A contradiction. Hence $x \leq y$, which finishes the proof. $\square$

Definition 2.5. *Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a bounded sequence. We define,*

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} \ x_k$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} \ x_k$$

Proposition 2.4. $\limsup$ *and* $\liminf$ *are well-defined. And,*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

Further, if $\{x_n\}_{n\in\mathbb{N}}$ converges, then,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, say $|x_n| \leq M$, any subset of it is also bounded, and so, in particular $\sup_{k\geq n} x_k$ and $\inf_{k\geq n} x_k$ each exist for all $k \in \mathbb{N}$. Moreover,

$$-M \leq \inf_{k\geq n} x_k \leq \inf_{k\geq n+1} x_k \leq x_{n+1} \leq \sup_{k\geq n+1} x_k \leq \sup_{k\geq n} x_k \leq M$$

In particular, $y_n = \sup_{k\geq n} x_k$ is a monotone decreasing sequence which is bounded below by $-M$, and $z_n = \inf_{k\geq n} x_k$ is a monotone increasing sequence which is bounded above by M, whence, by the Monotone Convergence theorem, we have that $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ converge, and thus that the $\limsup$ and $\liminf$ exist. Finally, since $\inf_{k\geq n} x_k \leq \sup_{k\geq n} x_k$ for all $n \in \mathbb{N}$, by **Lemma 2.1**, we have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Now suppose that $\lim_{n\to\infty} x_n$ exists and put $L = \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$ be given. Then, by definition, there exists some $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$. Rearranging, we have, $L - \varepsilon < x_n < L + \varepsilon$ for all $n \geq N$. In particular,

$$L - \varepsilon < \inf_{k\geq n} x_k \leq \sup_{k\geq n} x_k < L + \varepsilon$$

for any $n \geq N$. Thus, for any $n \geq N$,

$$\left| L - \inf_{k\geq n} x_k \right| < \varepsilon \qquad \left| L - \sup_{k\geq n} x_k \right| < \varepsilon$$

And so $\liminf_{n\to\infty} x_n = L = \limsup_{n\to\infty}$, which finishes the proof. $\square$

Lemma 2.2. *Suppose $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ are sequences such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ converge, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, and $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. Then, $\{y_n\}_{n\in\mathbb{N}}$ converges, and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$$

Proof. Put $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$. Then, for given $\varepsilon > 0$, there exists some $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n - L| < \varepsilon$ and for all $n \geq N_2$, $|z_n - L| < \varepsilon$. Then, for $n \geq N = \max\{N_1, N_2\}$, we have,

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

Thus, the y_n are convergent and $\lim_{n\to\infty} y_n = L$. □

Theorem 2.1. $\mathbb{R}$ *is complete.*

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is Cauchy. By definition, there exists some $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m \geq N$. In particular,

$$|x_n| - |x_N| \leq \big||x_n| - |x_N|\big| \leq |x_n - x_N| < 1$$

And so $|x_n| < |x_N| + 1$ for all $n \geq N$. Thus, $|x_n| \leq \max\{|x_1|, ..., |x_{N-1}|, |x_N| + 1\}$. So the $\{x_n\}_{n\in\mathbb{N}}$ are bounded. Thus, by **Proposition 2.4**, we have that $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ exist. Now, since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, given $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that, for all $k \geq n \geq N$, we have $x_n - \varepsilon < x_k < x_n + \varepsilon$. Then,

$$x_n - \varepsilon \leq \inf_{k\geq n} x_k \qquad\qquad \sup_{k\geq n} x_k \leq x_n + \varepsilon$$

And so, $\sup_{k\geq n} x_k - \inf_{k\geq n} x_k \leq 2\varepsilon$. Whence $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. Then, since $\inf_{k\geq n} x_k \leq x_n \leq \sup_{k\geq n} x_k$, by **Lemma 2.2**, we have that the $\{x_n\}_{n\in\mathbb{N}}$ are convergent. Thus, every Cauchy sequence is convergent. Hence $\mathbb{R}$ is complete. □

Remark. Notice that the proof of **Theorem 2.1** and its constituent parts heavily uses the Monotone Convergence theorem, which is essentially a rebranding of the Supremum Axiom. In fact, if one instead assumes that $\mathbb{R}$ satisfies the Monotone Convergence theorem, then one is able to deduce the Supremum Axiom handily. Further, where this breaks down for $\mathbb{Q}$ is that not every set of rationals has a least upper bound, our example corresponds to the set $\{q \in \mathbb{Q} \mid q^2 < 2\}$ having no least upper bound in $\mathbb{Q}$.

Theorem 2.2. $\mathbb{R}^d$*, under* $d(x, y) = ||x - y||$*, is complete.*

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ is Cauchy. Let $\delta > 0$ be arbitrary. Then, pick $N \in \mathbb{N}$ such that $||x_n - x_m|| < \delta$ for all $n, m \geq N$. Then, as was shown in PS2Q10, we have $|\pi_i(x_n - x_m)| \leq ||x_n - x_m||_\infty \leq ||x_n - x_m|| < \delta$ for all $n, m \geq N$ and for each $1 \leq i \leq d$, where π_i is the map which sends a vector to its i-th component. Thus, each $\{\pi_i(x_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$, and so by **Theorem 2.1** each converge to some $a_i \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Then, for each $1 \leq i \leq d$, there exists some N_i such that $|\pi_i(x_n) - a_i| < \varepsilon/\sqrt{d}$. Let $a = (a_1, .., a_d)$. Now, for $n \geq N = \max\{N_1, ..., N_d\}$,

$$||x_n - a||^2 = \sum_{i=1}^{d} |\pi_i(x_n) - a_i|^2 < \sum_{i=1}^{d} \frac{\varepsilon^2}{d} = \varepsilon^2$$

And so $||x_n - a|| < \varepsilon$, whence $\lim_{n\to\infty} x_n = a$, and every Cauchy sequence in $\mathbb{R}^d$ converges. Hence $\mathbb{R}^d$ is complete. □

Proposition 2.5. *Let* $C \subseteq \mathbb{R}^d$ *be closed. Suppose* $\{x_n\}_{n\in\mathbb{N}} \subseteq C$ *is a sequence in* C *which converges to some* $x \in \mathbb{R}^d$*. Then,* $x \in C$*.*

Proof. Since C is closed, $U = \mathbb{R}^d \setminus C$ is open. Suppose, for a contradiction, that $x \notin C$, then $x \in U$, and so there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. But then, $B_\varepsilon(x) \cap C = \varnothing$, and so $||x_n - x|| \geq \varepsilon$ for all $n \in \mathbb{N}$, but then there does not exist an $N \in \mathbb{N}$ such that $||x_n - x|| < \varepsilon$ for all $n \geq N$, and so $x \neq \lim_{n\to\infty} x_n$, a contradiction. Thus, $x \in C$. □

Proposition 2.6. *If* $C \subseteq \mathbb{R}^d$ *is closed, then* C *is complete.*

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq C$ is Cauchy, then by **Theorem 2.2** we have that $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in \mathbb{R}^n$. By **Proposition 2.5**, $x \in C$, whence every Cauchy sequence in C converges in C. Hence C is complete. □

Theorem 2.3. *Suppose* $A \subseteq \mathbb{R}^d$ *and* $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ *converges. Then,* $\lim_{n\to\infty} x_n \in \overline{A}$*. Moreover, if* $y \in \overline{A}$*, then there exists a sequence* $\{y_n\}_{n\in\mathbb{N}} \subseteq A$ *which converges to* y*.*

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ converges to some $x \in \mathbb{R}^d$. Now, given $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $||x_n - x|| < \varepsilon$, i.e. $B_\varepsilon(x) \cap A \neq \varnothing$ for all $\varepsilon > 0$. Since every open neighbourhood of x contains $B_\varepsilon(x)$ for some $\varepsilon > 0$, every open neighbourhood of x intersects A nontrivially, whence, per Handout 1, $x \in \overline{A}$.

On the other hand, suppose $y \in \overline{A}$. Then, for each $n \in \mathbb{Z}^+$, since, per Handout 1, every open neigbourhood of y intersects nontrivially, we may pick $y_n \in B_{1/n}(y) \cap A$. Then,

$||y_n - y|| < n^{-1}$. And so, given $\varepsilon > 0$, by the Archimedian principle there exists some $N \in \mathbb{N}$ such that $N \geq \varepsilon^{-1}$, whence, for all $n \geq N$,

$$||y_n - y|| < n^{-1} \leq N^{-1} < \varepsilon$$

And so $y_n \to y$, as desired. $\square$

Proposition 2.7. *Suppose $A \subseteq \mathbb{R}^d$ is complete. Then, A is closed.*

Proof. Suppose $x \in \overline{A}$, then, by **Theorem 2.3**, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ which converges to x. Then, since convergent sequences are Cauchy, and A is complete, we have that the x_n converge to a point in A. Since limits are unique, we must have $x \in A$, and thus that $\overline{A} \subseteq A$, and so $\overline{A} = A$. Hence A is closed. $\square$

Remark. These results essentially tell us that we could arrive at the same topology on $\mathbb{R}^d$ by saying that the closed sets are precisely the subsets of $\mathbb{R}^d$ which are complete metric spaces, and that the open sets are those which are the compliments of closed sets. Then these abstractly satisfy the union-intersection properties we noted in Handout 1, i.e. $\mathbb{R}^d$ and $\varnothing$ are complete metric spaces, any intersection of complete metric spaces is a complete metric space, and the finite union of complete metric spaces is a complete metric space (though this last statement is perhaps a little tricky).

Our first big application of the Monotone Convergence theorem was in the proof of the Bolzano-Weierstrass theorem, which is what we now want to prove for the $\mathbb{R}^d$ case.

Theorem 2.4. *Every bounded sequence in $\mathbb{R}$ admits a convergent subsequence.*

Proof. As shown in our first meeting. $\square$

Theorem 2.5. *Every bounded sequence in $\mathbb{R}^d$ admits a convergent subsequence.*

Proof. We proceed by way of induction on d. The base case is **Theorem 2.4**. For the inductive step, suppose we have the claim for all $d_1 < d$, for some $d \geq 2$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ be bounded, say $||x_n|| \leq M$ for all $n \in \mathbb{N}$. Then, let $x_n^{(i)}$ denote the i-th component of x_n for each $n \in \mathbb{N}$ and $1 \leq i \leq d$. Then, $|x_n^{(1)}| \leq M$ and $||(x_n^{(2)}, ..., x_n^{(d)})|| \leq M$ for all $n \in \mathbb{N}$. By the inductive hypothesis, there exists a convergent subsequence of the $(x_n^{(2)}, ..., x_n^{(d)})$, say $\lim_{j\to\infty}(x_{n_j}^{(2)}, ..., x_{n_j}^{(d)}) = a'$. Then, since subsequences of bounded sequences are bounded, by the inductive hypothesis, there is a convergent subsequence of the $x_{n_j}^{(1)}$, say $\lim_{k\to\infty} x_{n_{j_k}}^{(1)} = a''$. Since subsequences of convergent sequences converge to the same limit, $\lim_{k\to\infty}(x_{n_{j_k}}^{(2)}, ..., x_{n_{j_k}}^{(d)}) = a'$. Then, given $\varepsilon > 0$, pick $N_1, N_2 \in \mathbb{N}$ such that for all $k \geq N_1$, $||(x^{(2)}_{n_{j_k}}, ..., x^{(d)}_{n_{j_k}}) - a'|| < \varepsilon/\sqrt{2}$ and for all $k \geq N_2$, $|x^{(1)}_{n_{j_k}} - a''|| < \varepsilon/\sqrt{2}$. Then, for $k \geq N = \max\{N_1, N_2\}$,

$$||x_{n_{j_k}} - (a'', a')||^2 = ||x^{(1)}_{n_{j_k}} - a''||^2 + ||(x^{(2)}_{n_{j_k}}, ..., x^{(d)}_{n_{j_k}}) - a'||^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$$

And so the x_n admit a convergent subsequence. Since $\{x_n\}_{n\in\mathbb{N}}$ was arbitrary, we have the inductive step. Hence, we have the claim. $\square$

Definition 2.6. *We say that $A \subseteq \mathbb{R}^d$ is sequentially compact if every $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ admits a subsequence which converges in A.*

Theorem 2.6. *A subset $A \subseteq \mathbb{R}^d$ is sequentially compact $\iff$ A is closed and bounded.*

Proof. " $\implies$ " Suppose A is sequentially compact. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ be a Cauchy sequence. By definition, there is a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \subseteq A$ which converges to a point, say $x \in A$. Then, by **Theorem 2.2**, we have that $\{x_n\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}^d$, and, since the limit of a subsequence is x, the limit of the original sequence is also x, whence A is complete, and so closed per **Proposition 2.7**. Now, suppose for a contradiction that A is not bounded. Then for each $n \in \mathbb{N}$, there exists some $x_n \in A$ such that $||x_n|| \geq n$, and so we may construct a sequence in this manner. By repeating elements if necessary, we can take the x_n to be monotone increasing as well. Then, since any subsequence of a monotone increasing, unbounded sequence, is also monotone increasing and unbounded, and since such sequences cannot converge, we would have that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in A which admits no convergent subsequence, a contradiction. Thus A must be bounded.

" $\impliedby$ " Suppose A is closed and bounded. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ be any sequence. Since subsets of bounded sets are bounded, the x_n are bounded, and so by **Theorem 2.5** there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \subseteq A$. Then, by **Theorem 2.3**, the x_{n_k} converge to a point in $\overline{A} = A$, and so the x_n admit a subsequence which converges in A. Hence A is sequentially compact. $\square$

Proposition 2.8. *If $A \subseteq \mathbb{R}^n$ is sequentially compact and $B \subseteq A$ is closed, then B is sequentially compact.*

Proof. Exercise. $\square$

Proposition 2.9. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $A \subseteq \mathbb{R}^n$ is sequentially compact, then $f(A)$ is sequentially compact.*

Proof. Exercise. $\square$