Problem Set 1

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Problem 1. *Suppose $x, y, z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Show that,*

(a) $(\lambda x + y) \cdot z = \lambda x \cdot z + y \cdot z$.

(b) $x \cdot y = y \cdot x$.

(c) $x \cdot x = 0 \iff x = 0$.

Problem 2. *(**Generalized parallelogram law**) Fix $k \in \mathbb{N}$. Let $\{-1, 1\}^k$ be the set of all k-tuples of $\{-1, 1\}$. Show that, for any $x_0, x_1, ..., x_k \in \mathbb{R}^n$, we have,*

$$\sum_{a \in \{-1,1\}^k} \left| x_0 + \sum_{i=1}^{k} a_i x_i \right|^2 = 2^k \sum_{i=0}^{k} |x_i|^2$$

Hint: You only need to "expand out" the $k = 1$ case.

Problem 3. *(**Reverse triangle inequality**) Show that, for any $x, y \in \mathbb{R}^n$,*

$$\big| |x| - |y| \big| \leq |x - y|$$

Deduce that the map $\nu(x) = |x|$ is continuous.

Problem 4. *Fix $x \in \mathbb{R}^n$. Show that the map $f(y) = y \cdot x$ is bounded, i.e. find some $A \in \mathbb{R}$ such that $|f(y)| \leq A|y|$ for all $y \in \mathbb{R}^n$. Deduce that f is continuous.*

Problem 5. *Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a sequence of positive real numbers. Show that,*

$$\lim_{n \to \infty} x_n = \infty \iff \lim_{n \to \infty} x_n^{-1} = 0$$

Problem 6. (How $\mathbb{N}$ sits in $\mathbb{R}$) *Recall: the supremum axiom says that if $A \subseteq \mathbb{R}$ is non-empty and bounded, then there exists a least upper bound of A, called $\sup(A)$. Assuming this for now, show that,*

1. *(Archimedian property of $\mathbb{R}$) For all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.*

2. *Find an appropriate definition of $\lfloor x \rfloor$ for $x \geq 0$ in terms of the supremum.*

Problem 7. *Suppose $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ are sequences such that, for all $n \in \mathbb{N}$,*

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

Define, for each $n \in \mathbb{N}$, $I_n = [a_n, b_n]$. Show that $\varnothing \neq \bigcap_{n=0}^{\infty} I_n$, i.e. that there exists some $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$.

Hint: Consider $\lim_{n \to \infty} a_n$ and $\lim_{n \to \infty} b_n$.

Problem 8. *Fix sets I, A, and B. For each $i \in I$, let $A_i \subseteq A$ and $B_i \subseteq B$ be subsets. We define $\bigcup_{i \in I} A_i = \{a \in A : \exists i \in I, a \in A_i\}$. Similarly, we can also define, $\bigcap_{i \in I} A_i = \{a \in A : \forall i \in I, a \in I_i\}$. Let $f : A \to B$ be a function. Show that,*

(a) *For any set C, we have $C \cap \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} \left(C \cap A_i \right)$.*

(b) *For any set C, we have $C \cup \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \left(C \cup A_i \right)$.*

(c) *For any set C, we have $C \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \left(C \setminus A_i \right)$.*

(d) *For any set C, we have $C \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} \left(C \setminus A_i \right)$.*

(e) *For any $i \in I$, $A_i \subseteq f^{-1}(f(A_i))$, with equality if f is injective.*

(f) *For any $i \in I$, $B_i \subseteq f(f^{-1}(B_i))$, with equality if f is surjective.*

(g) *$f^{-1} \left(\bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} f^{-1}(B_i)$.*

(h) *$f^{-1} \left(\bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} f^{-1}(B_i)$.*

(i) *$f \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i)$.*

(j) *$f \left(\bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} f(A_i)$, with equality if f is injective.*

(k) *Find, and prove, similar statements for subsets and set differences. What do you notice about the statements for images versus those for pre-images?*

Problem 9. *Let A, B, C be sets. Prove or disprove the following,*

(a) $A \subseteq B$ *and* $A \subseteq C \iff A \subseteq (B \cup C)$.

(b) $A \subseteq B$ *or* $A \subseteq C \iff A \subseteq (B \cup C)$.

(c) $A \subseteq B$ *and* $A \subseteq C \iff A \subseteq (B \cap C)$.

(d) $A \subseteq B$ *or* $A \subseteq C \iff A \subseteq (B \cap C)$.

(e) $A \setminus (A \setminus B) = B$.

(f) $A \setminus (B \setminus A) = A \setminus B$.

(g) $(A \cap B) \cup (A \setminus B) = A$.

Problem 10. *Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a convergent sequence. Show that, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$ then $|x_n - x_m| < \varepsilon$.*

Hint: A $\varepsilon/2$ will probably appear.

Problem 11. *Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a sequence. Recall:*

$$\limsup_{n \to \infty} x_n := \lim_{n \to \infty} \sup_{k \geq n} x_k$$

$$\liminf_{n \to \infty} x_n := \lim_{n \to \infty} \inf_{k \geq n} x_k$$

Show that, if $\{x_n\}_{n \in \mathbb{N}}$ is bounded, then $\limsup_{n \to \infty} x_n$ and $\liminf_{n \to \infty} x_n$ exist, and that

$$\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n$$

With equality if and only if $\lim_{n \to \infty} x_n$ exists; in which case, each is equal to the limit.

Problem 12. *Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is a bounded sequence. Define a new sequence $\{c_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ by $c_n := \left(\sum_{k=0}^{n} x_k \right)/n$, the "Cesàro sums" of the x_n. Show that,*

$$\liminf_{n \to \infty} x_n \leq \liminf_{n \to \infty} c_n \leq \limsup_{n \to \infty} c_n \leq \limsup_{n \to \infty} x_n$$

Deduce that if $\lim_{n \to \infty} x_n$ exists, then $\lim_{n \to \infty} c_n$ exists and $\lim_{n \to \infty} c_n = \lim_{n \to \infty} x_n$. Also show that, while the sequence $x_n = (-1)^n$ doesn't converge, the Cesàro sums converge to 0. If you're up to it (i.e. this is a very hard problem), try to find a sequence whose Cesàro sums don't converge.