Handout 1

Isaac Clark

October 20, 2025

Throughout, let for $x \in \mathbb{R}^n$, let $||x|| = \sqrt{\sum_{i=1}^n x_i^2}$ be the norm of x. For $x, y \in \mathbb{R}^n$, let $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ be the dot product of x with y. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, let $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid ||x - y|| < \varepsilon\}$. When it is clear that A is a subset of $\mathbb{R}^n$, then we will tend to neglect to mention the ambient space, i.e. rather than saying that $U \subseteq \mathbb{R}^n$ is open in $\mathbb{R}^n$, we will simply say that it is open.

1 Open, closed sets and continuous functions

Definition 1.1. *We say that $U \subseteq \mathbb{R}^n$ is open if, $\forall u \in U$ $\exists \varepsilon > 0$ such that $B_\varepsilon(u) \subseteq U$.*

Proposition 1.1. *Given $U_i \subseteq \mathbb{R}^n$ an open set for each $i \in I$ for some indexing set I,*

1. *$\varnothing, \mathbb{R}^n$ are open.*
2. *$\bigcup_{i \in I} U_i$ is open.*
3. *$U_i \cap U_j$ is open for any $i, j \in I$.*

Proof. Exercise. See PS2. $\square$

Proposition 1.2. *The following sets are open,*

(a) *$B_r(x)$ for any $x \in \mathbb{R}^n$ and $r > 0$.*

(b) *$P = \{x \in \mathbb{R}^n \mid x_i > 0$ for all $1 \leq i \leq n\}$.*

(c) *$\bigcup_{n \in \mathbb{Z}^+} B_{2^{-n}}(n)$.*

Remark. (c) is unbounded but has a finite volume, per the geometric series formula.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$ be given. Fix $y \in B_r(x)$ and put $\varepsilon = r - ||x - y||$. Now, $\varepsilon > 0$ since $||x - y|| < r$ by supposition that $y \in B_r(x)$. And, if $z \in B_\varepsilon(y)$ then

$||y - z|| < r - ||x - y||$, and so we have, by the triangle inequality,

$$||x - z|| \leq ||x - y|| + ||y - z|| < ||x - y|| + r - ||x - y|| = r$$

Whence $z \in B_r(x)$ by definition, and thus $B_\varepsilon(y) \subseteq B_r(x)$. Since $y \in B_r(x)$ was arbitrary, we have that $B_r(x)$ is open. Hence (a).

Suppose $x \in P$. Let $\varepsilon = 2^{-1} \min_{1 \leq i \leq n} x_i$. Since each $x_i > 0$, $\varepsilon > 0$. Suppose $y \in B_\varepsilon(x)$. Then, for any $1 \leq i \leq n$,

$$|x_i - y_i|^2 \leq \sum_{i=1}^{n} |x_i - y_i|^2 = ||x - y||^2 < \varepsilon^2$$

And so, for each $1 \leq i \leq n$, we have that $|x_i - y_i| < \varepsilon$ and thus,

$$y_i > x_i - \varepsilon = x_i - 2^{-1} \min_{1 \leq i \leq n} x_i \geq (1 - 2^{-1}) \min_{1 \leq i \leq n} x_i > 0$$

And so $y \in P$ and so $B_\varepsilon(x) \subseteq P$ and so P is open. Hence (b).

By (a), each $B_{2^{-n}}(n)$ is open, and so by **Proposition 1.1** the union over $\mathbb{Z}^+$ is also open. Hence (c). □

Definition 1.2. *We say $C \subseteq \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus C$ is open.*

Note. If $C \subseteq \mathbb{R}^n$ is closed, then $\mathbb{R}^n \setminus C = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus U) = U$ for some $U \subseteq \mathbb{R}^n$ open. Hence, the compliment of closed is open and vice versa.

Proposition 1.3. *Given $C_i \subseteq \mathbb{R}^n$ a closed set for each $i \in I$ for some indexing set I,*

1. *$\varnothing, \mathbb{R}^n$ are closed.*
2. *$\bigcap_{i \in I} C_i$ is closed.*
3. *$C_i \cup C_j$ is closed for any $i, j \in I$.*

Proof. Exercise. See PS2. □

Proposition 1.4. *The following sets are open,*

(a) $\overline{B_r}(x) = \{y \in \mathbb{R}^n \mid ||x - y|| \leq r\}$ *for any* $x \in \mathbb{R}^n$ *and* $r \geq 0$.

(b) $N_{\geq 0} = \{x \in \mathbb{R}^n \mid x_i \leq 0$ *for some* $1 \leq i \leq n\}$.

(c) *Put* $C_0 = [0,1]$. *Recursively define* C_{n+1} *by subdividing each interval of* C_n *into three pieces and deleting the middle piece, but keeping the endpoints. For example,* $C_1 = [0,1/3] \cup [2/3,1]$. *Let* $\mathcal{C} = \bigcap_{n\in\mathbb{N}} C_n$.

Proof. (a) is left as an exercise. *Hint*: use PS2Q3.

$x \in \mathbb{R}^n \setminus N_{\geq 0}$ if and only if there does not exist some $1 \leq i \leq n$ such that $x_i \leq 0$, i.e $x_i > 0$ for every $1 \leq i \leq$. Thus, $\mathbb{R}^n \setminus N_{\geq 0} = P$, which is open per **Proposition 1.2**, and so $N_{\geq 0}$ is closed. Hence (b).

We formalize the construction as follows. For each $[a,b] \subseteq \mathbb{R}$, define,

$$S[a,b] = \left[a, a + \frac{b-a}{3}\right] \cup \left[a + \frac{2(b-a)}{3}, b\right]$$

By applying (a) to $x = 2^{-1}(a+b)$ and $r = 2^{-1}(b-a)$ we see that each interval of the form $[a,b]$ is closed. By **Proposition 1.3** $S[a,b]$ is closed as the union of two closed sets. Then, if we extend S as follows,

$$S\big([a,b] \cup [c,d]\big) = S[a,b] \cup S[c,d]$$

The above construction of the C_n yields $C_n = S^n C_0$. By induction, we have that $S^n C_0$ is closed for every $n \in \mathbb{N}$, and so each C_n is closed. Putting $I = \mathbb{N}$ and applying **Proposition 1.3** we have that $\mathcal{C} = \bigcap_{i\in I} C_i$ is closed. Hence (c). □

Definition 1.3. *We say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is "*$\varepsilon - \delta$ *continuous" if*

$$\forall a \in \mathbb{R}^n\ \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } ||x - a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$$

Definition 1.4. *We say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is "sequentially continuous" if*

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ whenever } \lim_{n\to\infty} x_n = x$$

Definition 1.5. *We say that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is "preimage continuous" if*

$$f^{-1}(U) \subseteq \mathbb{R}^n \text{ is open whenever } U \subseteq \mathbb{R}^m \text{ is open}$$

Proposition 1.5. *The following are* $\varepsilon - \delta$*, sequentially, and preimage continuous,*

1. $f : \mathbb{R}^n \to \mathbb{R}^m$ *given by* $f(x) = c$ *for some* $c \in \mathbb{R}^m$ *a constant.*

2. $f : \mathbb{R}^n \to \mathbb{R}^n$ *given by* $f(x) = x$.
3. $\nu : \mathbb{R}^n \to \mathbb{R}$ *given by* $\nu(x) = ||x||$.
4. $\varphi : \mathbb{R}^n \to \mathbb{R}$ *given by* $\varphi(x) = \langle x, y \rangle$ *for some* $y \in \mathbb{R}^n$ *a constant.*
5. $\pi_i : \mathbb{R}^n \to \mathbb{R}$ *given by* $\pi_i(x) = x_i$ *for some* $1 \leq i \leq n$ *fixed.*
6. $f : \mathbb{R} \to \mathbb{R}^n$ *given by* $f(x) = (x, 0, 0, ..., 0)$.

Proof. 3, 4, and 5 are left as exercises (see PS1Q3, PS1Q4 and PS2Q12 respectively).

Fix $c \in \mathbb{R}^m$ and let $f(x) = c$ for all $x \in \mathbb{R}^n$. Then, $||f(x) - f(a)|| = ||c - c|| = 0 < \varepsilon$ for any $\varepsilon > 0$ and for any $x, a \in \mathbb{R}^n$, and so f is $\varepsilon - \delta$ continuous. Suppose $x_n \to x$ as $n \to \infty$, then $f(x_n)$ is the constant sequence c and so converges to c, and $f(x) = c$, so we also have that f is sequentially continuous. Now, let $U \subseteq \mathbb{R}^m$ be open. If $c \in U$, then $f^{-1}(U) = \mathbb{R}^n$ and if $c \notin U$ then $f^{-1}(U) = \varnothing$, each of which are open per **Proposition 1.1**, and so f is preimage continuous. Hence 1.

Now let $f(x) = x$ for all $x \in \mathbb{R}^n$. Then, if $a \in \mathbb{R}^n$, and $\varepsilon > 0$, put $\delta = \varepsilon$. Then, if $||x - a|| < \delta$ then $||f(x) - f(a)|| = ||x - a|| < \delta = \varepsilon$, and so f is $\varepsilon - \delta$ continuous. There is nothing to show for sequential continuity. Now, let $U \subseteq \mathbb{R}^n$ be open, then $f^{-1}(U) = U$ is open, whence f is preimage continuous. Hence 2.

Now let $f : \mathbb{R} \to \mathbb{R}^n$ be given by $f(x) = (x, 0, 0, ..., 0)$. We observe that $||f(x)|| = |x|$, and so taking $\delta = \varepsilon$ and arguing similar to the above yields that f is $\varepsilon - \delta$ continuous. Suppose $x_n \to x$ as $n \to \infty$. Then, let $\varepsilon > 0$ be given and pick $N \in \mathbb{N}$ such that $n \geq N \implies ||x_n - x|| < \varepsilon$. Then, if $n \geq N$, then

$$||f(x_n) - f(x)|| = ||f(x_n - x)|| = ||x_n - x|| < \varepsilon$$

And so f is sequentially continuous. Now, let $U \subseteq \mathbb{R}^n$ be open. Then, suppose $x \in f^{-1}(U)$, then $(x, 0, 0, ..., 0) \in U$. Since U is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon\big((x, 0, 0, ..., 0)\big) \subseteq U$. Then, $(y, 0, 0, ..., 0) \in U$ for all $y \in (x - \varepsilon, x + \varepsilon)$, whence $(x - \varepsilon, x + \varepsilon) \subseteq f^{-1}(U)$, so applying PS2Q3 we have that $f^{-1}(U)$ is open, and therefore that f is preimage continuous. Hence 6. $\square$

Remark. It will turn out that there is no use distinguishing between these different "flavours" of continuity, as all are equivalent; however it is still educational to prove that a function is continuous in each of these ways, as oftentimes one definition will be the easiest to work with, a distinction only possible when one has practice will all of them.

2 Closure, interior

Definition 2.1. *Given $A \subseteq \mathbb{R}^n$, we define the clousre of A, $\overline{A} = \bigcap_{C \supseteq A} C$ to be the intersection of all closed sets which contain A.*

Proposition 2.1. *$\overline{A}$ is closed for any $A \subseteq \mathbb{R}^n$.*

Proof. Let $I = \{C \subseteq \mathbb{R}^n \mid C \text{ is closed and } C \supseteq A\}$ and put $U_i = i$ for each $i \in I$, then apply **Proposition 1.3**. □

Proposition 2.2. *$A \subseteq \overline{A}$ for any $A \subseteq \mathbb{R}^n$.*

Proof. This is straightforward,

$$\overline{A} = \bigcap_{\substack{C \text{ closed} \\ C \supseteq A}} C \supseteq \bigcap_{\substack{C \text{ closed} \\ C \supseteq A}} A = A$$

Where the $\supseteq$ follows because we require $C \supseteq A$ in the intersection. □

Theorem 2.1. *$\overline{A}$ is precisely the set of $x \in \mathbb{R}^n$ such that*

$$\forall U \subseteq \mathbb{R} \text{ open}, x \in U \implies U \cap A \neq \varnothing$$

Proof. We will show the contrapositive for $\subseteq$. Suppose $x \in \mathbb{R}^n$ is such that there exists some $U \subseteq \mathbb{R}^n$ open such that $x \in U$ and $U \cap A = \varnothing$. Then $C = \mathbb{R}^n \setminus U$ is closed and contains A, and so $C \supseteq \overline{A}$. In particular, if $x \in \overline{A}$ then $x \in C$, a contradiction, as $x \in U$. Hence we have that $x \notin \overline{A}$ and thus $\subseteq$.

We will also show the contrapositive for $\supseteq$. Suppose $x \notin \overline{A}$, then $x \in \mathbb{R}^n \setminus \overline{A}$. By **Proposition 2.1**, $\overline{A}$ is closed and so $U = \mathbb{R}^n \setminus \overline{A}$ is open. Then, $U \cap \overline{A} = \varnothing$ by definition and so $U \cap A = \varnothing$ in particular. Clearly $x \in U$ as well. Thus, $\supseteq$.

Taken together, we have the claim. □

Proposition 2.3. *$A \subseteq \mathbb{R}^n$ is closed $\iff$ $\overline{A} = A$.*

Proof. We always have $A \subseteq \overline{A}$ by **Proposition 2.2**, so it suffices to show that A is closed if and only if $\overline{A} \subseteq A$. Now, if A is closed, since $\overline{A} \subseteq C$ for any $C \supseteq A$ which is closed, and certianly $A \supseteq A$, we have $\overline{A} \subseteq A$. Conversely, if $\overline{A} = A$ then A is closed since $\overline{A}$ is closed by **Proposition 2.1**. Hence, we have the claim. □

A similar, though less useful, notion to that of closure is interior.

Definition 2.2. *Given $A \subseteq \mathbb{R}^n$, we define the interior of A, int $A = \bigcup_{U \subseteq A} U$ to be the union of all open sets contained in A.*

Proposition 2.4. *int A is open for any $A \subseteq \mathbb{R}^n$.*

Proof. Exercise. □

Proposition 2.5. *int $A \subseteq A$ for any $A \subseteq \mathbb{R}^n$.*

Proof. Exercise. □

Theorem 2.2. *int A is precisely the $x \in \mathbb{R}^n$ such that $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.*

Proof. Exercise. □

Corollary. int $A = A$ if and only if A is open.

3 Continuous functions, revisited

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then, the following are equivalent,*

1. *$f^{-1}(C) \subseteq \mathbb{R}^n$ is closed whenever $C \subseteq \mathbb{R}^m$ is closed.*
2. *f is "preimage" continuous.*
3. *$f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq \mathbb{R}^n$.*
4. *f is "sequentially" continuous.*
5. *f is "$\varepsilon - \delta$" continuous.*
6. *For all $a \in \mathbb{R}^n$ and every $V \subseteq \mathbb{R}^m$ open with $f(a) \in V$, there exists some $U \subseteq \mathbb{R}^n$ open such that $a \in U$ and $f(U) \subseteq V$.*

Proof. "(1) $\Longrightarrow$ (2)" Suppose f satisfies (1) and that $U \subseteq \mathbb{R}^m$ is open, then $\mathbb{R}^m \setminus U$ is closed, so $f^{-1}(\mathbb{R}^m \setminus U)$ is closed. But $f^{-1}(\mathbb{R}^m \setminus U) = f^{-1}(\mathbb{R}^m) \setminus f^{-1}(U) = \mathbb{R}^n \setminus f^{-1}(U)$. Hence, $f^{-1}(U)$ is open. So, (1) $\Longrightarrow$ (2).

"(2) $\Longrightarrow$ (3)" Suppose f satisfies (2) and $y \in f(\overline{A})$. Then there exists some $x \in \overline{A}$ such that $f(x) = y$. Since $x \in \overline{A}$, by **Theorem 2.1**, every $U \subseteq \mathbb{R}^n$ which contains x intersects A non-trivially. Then, given $V \subseteq \mathbb{R}^m$ an open set containing y, $f^{-1}(V)$ is open and contains x, so $f^{-1}(V) \cap A \neq \varnothing$. Taking the image of each side under f, we have $\varnothing \neq f(f^{-1}(V)) \cap f(A) \subseteq V \cap f(A)$, and so $f(A)$ intersects V non-trivially. Since V was arbitrary, we have that $y \in \overline{f(A)}$, and thus that $f(\overline{A}) \subseteq \overline{f(A)}$. So (2) $\Longrightarrow$ (3).

We will prove "(3) $\implies$ (4)" when we have more technology.

"(4) $\implies$ (5)" We will show the contrapositive. Suppose f satisfies not (5), i.e. there exists some $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists some point $x \in \mathbb{R}^n$ such that $||x - a|| < \delta$ but $||f(x) - f(a)|| \geq \varepsilon_0$. For $n \in \mathbb{Z}^+$, put $\delta_n = 1/n$. Then, for each δ_n, there is some $x_n \in \mathbb{R}^n$ such that $||x_n - a|| < \delta_n$ but $||f(x_n) - f(a)|| \geq \varepsilon_0$. But then the $\{x_n\}_{n\in\mathbb{Z}^+}$ is a sequence which converges to a for which $f(x_n) \not\to f(a)$, and so f is not sequentially continuous at a.

"(5) $\implies$ (6)" Suppose f satisfies (5), fix $a \in \mathbb{R}^n$, and let $V \subseteq \mathbb{R}^m$ be open set which contains $f(a)$. Then, there exists some $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq V$. By definition, there is some $\delta > 0$ such that $||x - a|| < \delta \implies ||f(x) - f(a)|| < \varepsilon$. Then $f(B_\delta(a)) \subseteq B_\varepsilon(f(a)) \subseteq V$, so $U = B_\delta(a)$ is as desired. Hence (5) $\implies$ (6).

We will prove "(6) $\implies$ (1)" when we have more technology.

Taken together, we have

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6) \implies (1)$$

And thus equivalence thereof. $\square$

Remark. If one is willing to prove a few more implications, there is a way to show equivalence of the above using only our current results, which may be a good exercise.