Problem Set 7

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Problem 1. *Is there an uncountable collection of pairwise disjoint, open subsets of* $\mathbb{R}$*?*

Problem 2. *Suppose* $A, B \subseteq \mathbb{R}^n$*. Show that* $\overline{A \cup B} = \overline{A} \cup \overline{B}$*.*

Problem 3. *Show that* $D \subseteq \mathbb{R}^n$ *is dense if and only if* $\overline{D} = \mathbb{R}^n$*.*

Problem 4. *Is it true that the intersection of dense sets is dense?*

Problem 5. *Suppose* $A \subseteq \mathbb{R}^n$*. Show that* $x \in int(A)$ *if and only if there exists an open neighbourhood,* U*, of* x *which lies in* A*.*

Problem 6. *Suppose* $A \subseteq \mathbb{R}^n$*. Show that*

1. $A = int(A) \cup \partial(A)$.
2. $\overline{A} = A \cup \partial(A)$.
3. $int(A) = A \setminus \partial(A)$.

Problem 7. *For* $A \subseteq \mathbb{R}^n$*, let* $A^{\perp} = \mathbb{R}^n \setminus \overline{A}$*. Show that* $int(\overline{A}) = A \iff (A^{\perp})^{\perp} = A$*.*

Problem 8. *Suppose* $D \subseteq \mathbb{R}^n$ *is dense. Is* $D \setminus \{x_1, ..., x_m\}$ *also dense?*

Problem 9. *Is the intersection of dense sets also dense? What about the intersection of dense, open sets? What about countably infinite intersections of dense, open sets?*

Problem 10. *Show that the map* $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *given by* $(x, y) \mapsto xy$ *is continuous.*

Problem 11. *Suppose* $X \subseteq \mathbb{R}^n$ *and* $Y \subseteq \mathbb{R}^m$*. Let* $f : X \to Y$ *be a bijection. Show that the following are equivalent:*

1. f^{-1} *is continuous.*
2. f *is open.*
3. f *is closed.*

Problem 12. *Suppose* $A \subseteq \mathbb{R}^n$ *is closed. Show that there is a map* $\chi : \mathbb{R}^n \to \mathbb{R}$ *with:*

1. χ *is continuous.*
2. $\chi(x) = 0 \iff x \in A$.

Problem 13. *Let* $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. *Suppose* $f : S^1 \to \mathbb{R}$ *is continuous. Show that there exists* $z \in S^1$ *such that* $f(x, y) = f(-x, -y)$.

Problem 14. *Show that any open subset,* U, *of* $\mathbb{R}^n$ *may be written as the countable union of connected, open subsets of* U. *Show this fails if* U *is not open.*

Problem 15. *Show that every polynomial with odd degree has a root.*

Problem 16. *Show that a linear transformation from* $\mathbb{R}^n \to \mathbb{R}^m$ *is continuous if and only if there exists* $M \geq 0$ *such that* $||Tv|| \leq M||v||$. *Deduce that every linear transformation is continuous.*

Problem 17. *Suppose* $K \subseteq \mathbb{R}^n$ *is a non-empty compact subset, then there exists points in* K *which achieve the minimum and maximum distance from* K *to the origin. Show that if* $C \subseteq \mathbb{R}^n$ *is closed, then there exists a point in* C *which achieves the minimum distance to the origin, but there needn't be a point which attains the maximal distance.*

Problem 18. *Consider the isomorphism from* $\mathbb{R}^{n^2} \to M_n(\mathbb{R})$ *which associates a* n^2 *tuple of real numbers to a real-valued* $n \times n$ *matrix. Show that:*

1. *The maps* $\mathbb{R}^{n^2} \to \mathbb{R}$ *given by* $M \mapsto \det(M)$ *and* $M \mapsto tr(A)$ *are continuous.*
2. *The set of invertible matrices are open and dense in* $\mathbb{R}^{n^2}$. *Hint: For* B *invertible, consider* $p(t) = \det\big((1-t)A + tB\big)$.

Problem 19. *Show that* $\mathbb{R}^n \setminus A$ *is path-connected for* $|A| < \infty$ *and* $n \geq 2$.

Problem 20. *Find a subspace* $X \subseteq \mathbb{R}^n$ *and a subset* $A \subseteq X$ *which is closed in* X *and bounded, but is not compact. Why doesn't this contradict Heine-Borel?*

Problem 21. *Show that* $X \subseteq \mathbb{R}^n$ *is connected if and only if* $X \times \mathbb{R}$ *is connected.*

Definition 1. *Suppose* $X \subseteq \mathbb{R}^n$. *We say that* $\mathcal{B}$ *is a basis for the topology on* X *if:*

1. *If* $B_1, B_2 \in \mathcal{B}$ *then* $B_1 \cap B_2 = \bigcup_{i\in I} B_i$ *for some* I *and* $B_i \in \mathcal{B}$.
2. *Every open subset of* X *is equal to* $\bigcup_{i\in I} B_i$ *for some* I *and* $B_i \in \mathcal{B}$.

Problem 22. *Show that* $\mathcal{B} = \{(a, b) \mid a < b \in \mathbb{R}\}$ *and* $\mathcal{B}' = \{(a, b) \mid a < b \in \mathbb{Q}\}$ *are each a basis for the topology on* $\mathbb{R}$.

Problem 23. *Show that* $\mathcal{B} = \{\{x\} \mid x \in X\}$ *is a basis for a topology on* X *if and only if* X *is discrete. If* X *is finite, is there a basis with fewer than* $|X|$ *elements?*