Solution Manual 1

Isaac Clark

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Problem 1. ***(1.2)*** *Fix $k \in \mathbb{N}$. Let $\{-1,1\}^k$ be the set of all k-tuples in $\{-1,1\}$. Show that, for any $x_0, x_1, ..., x_k \in \mathbb{R}^n$, we have,*

$$\sum_{a \in \{-1,1\}} \left| x_0 + \sum_{i=1}^{k} a_i x_i \right|^2 = 2^k \sum_{i=0}^{k} |x_i|^2$$

Proof. We proceed by way of induction. The $k = 0$ case is clear, since

$$\sum_{a \in \{-1,1\}^0} \left| x_0 + \sum_{i=1}^{0} a_i x_i \right|^2 = |x_0|^2 = 2^0 \sum_{i=0}^{0} |x_i|^2$$

The $k = 1$ case is also clear, since

$$\begin{aligned}
\sum_{a \in \{-1,1\}^1} \left| x_0 + \sum_{i=1}^{1} a_i x_i \right|^2 &= |x_0 + x_1|^2 + |x_0 - x_1|^2 \\
&= \langle x_0 + x_1, x_0 + x_1 \rangle + \langle x_0 - x_1, x_0 - x_1 \rangle \\
&= 2\langle x_0, x_0 \rangle + 2\langle x_1, x_1 \rangle + 2\langle x_0, x_1 \rangle - 2\langle x_0, x_1 \rangle \\
&= 2|x_0|^2 + 2|x_1|^2 \\
&= 2^1 \sum_{i=0}^{1} |x_i|^2
\end{aligned}$$

So suppose the claim is true for $k \geq 1$. Then,

$$\begin{aligned}
2^{k+1}\sum_{i=0}^{k+1} &= 2\left(2^k|x_{k+1}|^2 + 2^k\sum_{i=0}^{k}|x_i|^2\right)\\
&= \sum_{a\in\{-1,1\}^k} 2\left|x_0 + \sum_{i=1}^{k} a_i x_i\right|^2 + 2^k\cdot 2|x_{k+1}|^2\\
&= \sum_{a\in\{-1,1\}^k}\left(2\left|x_0 + \sum_{i=1}^{k} a_i x_i\right|^2 + 2|x_{k+1}|^2\right) \text{ , since } |\{-1,1\}^k| = 2^k\\
&= \sum_{a\in\{-1,1\}^k}\left(\left|x_0 + \sum_{i=1}^{k} a_i x_i + x_{k+1}\right|^2 + \left|x_0 + \sum_{i=1}^{k} a_i x_i - x_{k+1}\right|^2\right)\\
&= \sum_{b\in\{-1,1\}^{k+1}}\left|x_0 + \sum_{i=1}^{k+1} b_i x_i\right|^2
\end{aligned}$$

And so we have the inductive step, and hence the claim. □

Problem 2. ***(1.7)*** *Suppose $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ are sequences with, for all $n \in \mathbb{N}$,*

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

Define, for each $n \in \mathbb{N}$, $I_n = [a_n, b_n]$. Show that there exists some $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$.

Proof. By the Monotone Convergence theorem, $a^* := \lim_{n\to\infty} a_n$ and $b^* := \lim_{n\to\infty} b_n$ exist as the a_n and b_n are monotone increasing and decreasing respectively, and are bounded above by b_0 and below by a_0 respectively. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, $a^* \leq b^*$. Further, for all $n \in \mathbb{N}$,

$$a_n \leq a^* \leq b^* \leq b_n$$

And so $[a^*, b^*] \subseteq \bigcap_{i=0}^{\infty} I_n$, whence it is non-empty. □

Problem 3. ***(1.12)*** *Suppose $\{x_n\}_{n\in\mathbb{Z}^+} \subseteq \mathbb{R}$ is a bounded sequence. Define $\{c_n\}_{n\in\mathbb{Z}^+}$ by $c_n = \left(\sum_{k=0}^{n} x_k\right)/n$. Show that,*

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty} c_n \leq \limsup_{n\to\infty} x_n$$

Deduce that if $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} c_n$ exists and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} c_n$. Also show that, while the sequence $x_n = (-1)^n$ doesn't converge, the Cesàro sums converge to 0. Also, find a sequence whose Cesàro sums don't converge.

Proof. As the x_n are bounded, say $|x| \leq M$ we have $|c_n| \leq \sum_{i=1}^n |x_k|/n \leq \sum_{i=1}^n M/n = M$, and so the c_n are bounded, and so

$$\liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty} c_n$$

Put $L := \limsup_{n\to\infty} x_n$. Then, for all $\varepsilon > 0$, there exists an $N \geq 1$ such that for all $n \geq N$, we have $x_n \leq L + \varepsilon$. Then, for $n \geq N$,

$$c_n = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{n}\sum_{k=1}^{N-1} + \frac{1}{n}\sum_{k=N}^{n} x_k \leq \frac{1}{n}\sum_{k=1}^{N-1} x_k + \frac{(n-N+1)(L+\varepsilon)}{n}$$

For fixed ε, N is fixed, so the first terms tends to 0 and the second term tends to $L+\varepsilon$. Thus, $\limsup_{n\to\infty} c_n \leq L+\varepsilon$ for all $\varepsilon > 0$, whence $\limsup_{n\to\infty} c_n \leq L = \limsup_{n\to\infty} x_n$. Then,

$$\liminf_{n\to\infty} c_n = -\limsup_{n\to\infty}(-c_n) \geq -\limsup_{n\to\infty}(-x_n) = \liminf_{n\to\infty} x_n$$

And so taken all together, we have the first part of the claim.

Next, put $x_n = (-1)^n$. Then, if $n = 2k$ for some $k \in \mathbb{Z}^+$, then,

$$c_n = \frac{1}{n}\sum_{j=1}^{n}(-1)^j = \frac{1}{n}\left(\sum_{l=1}^{k}(-1)^{2k-1} + (-1)^{2k}\right) = \frac{1}{n}\left(\sum_{l=1}^{k} 1 - 1\right) = 0$$

If $n = 2k+1$ for some $k \in \mathbb{N}$, then,

$$c_n = \frac{1}{n}\sum_{k=1}^{n} x_n = \frac{1}{n}\sum_{k=1}^{2k}(-1)^k - \frac{1}{n} = -\frac{1}{n}$$

So, given $\varepsilon > 0$, if $n \geq 2\varepsilon^{-1}$, then, $|c_n| \leq n^{-1} < \varepsilon$, and so though the x_n do not converge, the Cesàro sums converge.

Finally, put $x_n = (-1)^{\lfloor \log_2(n) \rfloor}$. Then,

$$\begin{aligned} c_{2^{2k+1}-1} &= \frac{1}{2^{2k+1}-1}\sum_{l=1}^{2k}(-2)^l \\ &= \frac{2}{3}\cdot\frac{4^k-1}{2\cdot 4^k-1} \\ &\to \frac{1}{3}\text{ , as } k\to\infty \end{aligned}$$

$$\begin{aligned} c_{2^{2k}-1} &= \frac{1}{2^{2k}-1}\sum_{l=1}^{2k-1}(-2)^l \\ &= \frac{-1}{3}\cdot\frac{4^k+2}{4^k-1} \\ &\to \frac{-1}{3}\text{ , as } k\to\infty \end{aligned}$$

Hence c_n has two subsequences which converge to different limits, and so diverges. $\square$

Problem 4. ***(2.7)*** *Given $A \subseteq \mathbb{R}^n$. Show that,*

(1) $\partial A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$.

(2) int$(A) \cap \partial A = \varnothing$.

(3) $\partial A = \varnothing \iff A$ is clopen.

Proof. $x \in \partial A$ if and only if every open neighbourhood of x intersects A and $\mathbb{R}^n \setminus A$ non-trivially if and only if $x \in \overline{A}$ and $x \in \overline{\mathbb{R}^n \setminus A}$.

Suppose, for a contradiction, that $x \in \operatorname{int}(A) \cap \partial A$. Then, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \operatorname{int}(A) \subseteq A$. But then, by definition there exists some $y \in B_\varepsilon(x) \cap (\mathbb{R}^n \setminus A) \subseteq A \cap (\mathbb{R}^n \setminus A) = \varnothing$, a contradiction.

If A is clopen, then $\mathbb{R}^n \setminus A$ is closed, and so $\overline{A} \cap \overline{\mathbb{R}^n \setminus A} = A \cap (\mathbb{R}^n \setminus A) = \varnothing$. On the other hand, if $\partial A = \varnothing$, then every $x \in A$ has an open neighbourhood U such that either $U \subseteq A$ if $x \in A$ or $U \subseteq \mathbb{R}^n \setminus A$ if $x \notin A$, which exhibits A as clopen. $\square$

Problem 5. ***(2.10)*** *For $x \in \mathbb{R}^n$, let $||x|| = \sqrt{x \cdot x}$, let $||x||_1 = \sum_{i=1}^n |x_i|$, and let $||x||_\infty = \max_{1\le i\le n}|x_i|$. Prove that, for all $x \in \mathbb{R}^n$,*

$$||x||_\infty \le ||x||_1 \le ||x|| \le n||x||_\infty$$

Deduce that if $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, then $U \times V \subseteq \mathbb{R}^{n+m}$ is open.

Proof. Clearly $||x||_1 \leq n||x||_\infty$. By the triangle inequality, $||x|| \leq ||x||_1$. Also,

$$||x||^2 = \sum_{i=1}^{n} x_i^2 \geq ||x||_\infty^2$$

And so by transitivity, $||x||_\infty \leq ||x|| \leq ||x||_1 \leq n||x||_\infty$.

Now suppose that $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open. Suppose $(x, y) \in U \times V$, i.e. $x \in U$ and $y \in V$. Then there exists $\varepsilon_n, \varepsilon_m > 0$ such that $B_{\varepsilon_n}(x) \subseteq U$ and $B_{\varepsilon_m}(y) \subseteq V$. Put $B_r^\infty(a) = \{x \in \mathbb{R}^d \mid ||x-a||_\infty < r\}$ and likewise for $B_r^1(a)$. Put $\varepsilon = \min\{\varepsilon_n, \varepsilon_m\}$. Then, if $z \in B_\varepsilon^\infty(x, y)$ then $||(x, y) - z|| \leq n||(x, y) - z||_\infty < n\varepsilon$ and so $B_{\varepsilon/n}(x, y) \subseteq B_\varepsilon^\infty(x, y)$. Further, $||(x, y) - z||_\infty < \varepsilon$ implies $|x^{(i)} - z^{(i)}| < \varepsilon$ and $|y^{(j)} - z^{(n+j)}| < \varepsilon$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Whence $||x - (z^{(i)})_{i=1}^n|| \leq ||x - (z^{(i)})_{i=1}^n||_1 < n\varepsilon$. Likewise, $||y - (z^{(i)})_{i=n+1}^{n+m}|| < m\varepsilon$. So $(z^{(i)})_{i=1}^n \in U$ and $(z^{(i)})_{i=n+1}^{n+m} \in V$, whence $z \in U \times V$ and so $U \times V$ is open. □

Problem 6. ***(2.15)*** *Suppose $A \subseteq \mathbb{R}^n$ is not closed. Find a function $f : A \to \mathbb{R}$ which is continuous and unbounded.*

Proof. Since A is not closed, $\mathbb{R}^n \setminus A$ is not open, and so there exists some $y \in \mathbb{R}^n \setminus A$ such that $B_\varepsilon(y) \cap A \neq \varnothing$ for all $\varepsilon > 0$. Put $f(x) = ||y - x||^{-1}$. Since $y \notin A$, f is well-defined. Since $1/x$ is continuous on $(0, \infty)$, f is the composition of continuous functions and so is continuous on A. For each $n \in \mathbb{Z}^+$, we may pick some $x_n \in B_{1/n}(y) \cap A$. Then $||x_n - y||^{-1} \geq n$, and so f is unbounded. □

Problem 7. ***(2.18)*** *Let $\mathcal{F} = \{f_i : \mathbb{R} \to \mathbb{R}\}_{i \in I}$ be a family of continuous functions. Let $V(\mathcal{F})$ be the set of point for which the f_i simultaneously vanish. Then,*

1. *$V(\mathcal{F})$ is closed.*
2. *If C is closed, then there is a continuous function f such that $C = V(\{f\})$.*

Proof. For (1), $V(\mathcal{F})$ may be written as the intersection of the preimage of $\{0\}$, a closed set, under continuous functions, whence it is closed.

For (2), since every open set may be written as the countable union of open intervals, it suffices to construct a continuous function which vanishes outside of and is non-zero on a prescribed open set, since in the end we may take the sum over all such functions, as there are only countably many. Then, $f(x) = e^{(x-a)^{-1}(x-b)^{-1}}$ for $x \in (a, b)$ and 0 elsewhere is as desired, as it certainly vanishes outide of (a, b) and is non-zero on (a, b). Further, $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to b} f(x) = 0$, and so f is continuous. □

Problem 8. ***(2.25)*** *Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ when $x \in D$, where D is dense in $\mathbb{R}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.*

Proof. There are several ways to approach this, perhaps the simpliest is to consider sequences. Since D is dense, for any $x \in \mathbb{R}$ we may pick a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq D$ which converges to x. Then, since f and g are continuous and agree on D,

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

And so they are equal everywhere. □

Problem 9. ***(2.26)*** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, show that the graph of f, denoted by $\Gamma(f) = \{(x, f(x)) \mid x \in \mathbb{R}^n\}$ is closed.*

Proof. Any sequence in $\Gamma(f)$ must be of the form $\{(x_n, f(x_n))\}_{n\in\mathbb{N}}$ for some $x_n \in \mathbb{R}^n$. Then, suppose $(x_n, f(x_n)) \to (x, y)$ for some $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. (It is a good exercise to check that a sequence of vectors converges if and only if it converges pointwise). Since f is continuous, $f(x_n) \to x$, and so by uniqueness of limits $f(x) = y$, and so every convergent sequence in $\Gamma(f)$ converges to a point in $\Gamma(f)$, whence it is closed. □