# Handout 3

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## 1 The subspace topology

Given $A \subseteq \mathbb{R}^n$, we want to define a family of subsets of $A$, say $\tau_A$, such that,

1. $\varnothing, A \in \tau_A$.
2. $\tau_A$ is closed under arbitrary unions and finite intersections.
3. $(f \circ \iota)^{-1}(U) \in \tau_A$ whenever $U \subseteq \mathbb{R}^m$ is open and $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

Turns out that $\tau_A = \{U \cap A \mid U \subseteq \mathbb{R}^n \text{ open}\}$ is the unique family of sets with these properties. One may check, $\varnothing = \varnothing \cap A$ and $A = \mathbb{R}^n \cap A$, and so $\varnothing, A \in \tau_A$. Further,

$$\bigcup_{i \in I} (U_i \cap A) = \left( \bigcup_{i \in I} U_i \right) \cap A$$
$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A$$

And so $\tau_A$ is closed under arbitrary unions and finite intersections. Finally,

$$\begin{aligned} \iota^{-1}(U) &= \{x \in A \mid \iota(x) \in U\} \\ &= \{x \in \mathbb{R}^n \mid x \in A\} \cap \{x \in \mathbb{R}^n \mid x \in U\} \\ &= U \cap A \end{aligned}$$

$$\begin{aligned} (f \circ \iota)^{-1}(U) &= \iota^{-1}(f^{-1}(U)) \\ &= f^{-1}(U) \cap A \end{aligned}$$

And so $(f \circ \iota)^{-1}(U)$ is open whever $U \subseteq \mathbb{R}^m$ is open. I leave it as an exercise to adjust the definition of $\varepsilon - \delta$ and sequential continuity for $f$ defined on a subset of $\mathbb{R}^n$, and to check that the definitions agree.

**Definition 1.1.** *We say that $V \subseteq A$ is open in $A$ if $V = U \cap A$ for some open $U \subseteq \mathbb{R}^n$.*

**Definition 1.2.** *We say that $C \subseteq A$ is closed in $A$ if $A \setminus C$ is open in $A$.*

**Definition 1.3.** *We say that $f : A \to B$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, if $f^{-1}(U)$ is open in $A$ whenever $U$ is open in $B$.*

**Definition 1.4.** *We say that $f : A \to B$ is a homeomorphism if $f$ is bijective, $f$ is continuous, and $f^{-1}$ is continuous.*

**Definition 1.5.** *We say that $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are homeomorphic if there exists a homeomorphism $f : A \to B$. In this case, we write $A \cong B$.*

**Proposition 1.1.** *Let $f : A \to B$ be bijective. Then, $f$ is open $\iff$ $f^{-1}$ is continuous.*

*Proof.* We observe that since $f$ is bijective, for each $y \in B$ there exists a unique $x_y \in A$ such that $f(x_y) = y$. So,

$$(f^{-1})^{-1}(U) = \{y \in B \mid f^{-1}(y) \in U\} = \{f(x_y) \in B \mid x_y \in U\} = f(U)$$

And thus $(f^{-1})^{-1}(U)$ is open whenever $f(U)$ is open, and so we have the claim. $\square$

**Proposition 1.2.** *The following are homeomorphic to $(0, 1)$,*

*(a)* $A = (a, b)$

*(b)* $B = (a, \infty)$

*(c)* $C = (-\infty, a)$

*(d)* $D = \mathbb{R}$

*Proof.* For (a), put $f(x) = a + (b - a)x$.

For (b), consider that $g(x) = a^{-1}x$ exhibits $(a, \infty) \cong (1, \infty)$. And that $h(x) = x^{-1}$ exhibits $(1, \infty) \cong (0, 1)$, and so $g \circ h$ is as desired.

For (c), consider that $p(x) = -x + 2a$ exhibits $(-\infty, a) \cong (a, \infty)$, which is homeomorphic to $(0, 1)$ per (b), so the composition is as desired.

For (d), consider that, per (a), $(0, 1) \cong (-\pi/2, \pi/2)$, and then that $q(x) = \tan(x)$ exhibits $(-\pi/2, \pi/2) \cong \mathbb{R}$, and so the composition is as desired. $\square$

**Proposition 1.3.** *The relation $A \sim B \iff A \cong B$ is an equivalence relation.*

*Proof.* $A \sim A$ via $f(x) = x$. If $A \sim B$ via $g$, then $B \sim A$ via $g^{-1}$. And the composition of homeomorphims is a homeomorphism. $\square$

**Proposition 1.4.** *$\{a_n\}_{n\in\mathbb{N}} \subseteq A$ converges to $a \in A \iff$ for all $U \subseteq A$ which are open in $A$ and contain $a$, there exists some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq N$.*

*Proof.* Exercise. See PS4. □

**Theorem 1.1.** *Suppose $f : X \to \mathbb{R}^n$ is such that there are closed (in $X$) subsets $A, B \subseteq X$ such that $X = A \cup B$ and $f \circ \iota_A$ and $f \circ \iota_B$ are continuous. Then $f$ is continuous.*

*Proof.* Let $C \subseteq \mathbb{R}^n$ be closed. Then, since $f \circ \iota_A$ and $f \circ \iota_B$ are continuous, $f^{-1}(C) \cap A$ and $f^{-1}(C) \cap B$ are closed in $A$ and $B$ respectively. Since $A, B$ themselves are closed in $X$, $f^{-1}(C) \cap A$ and $f^{-1}(C) \cap B$ are closed in $X$. And so their union, $f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$ is closed in $X$. Hence, $f$ is continuous. □

**Definition 1.6.** *We say that $A \subseteq \mathbb{R}^n$ is discrete if $\tau_A = \mathcal{P}(A)$.*

**Proposition 1.5.** *$A$ is discrete if and only if for all $a \in A$, there exists $U \subseteq \mathbb{R}^n$ an open neighbourhood of $a$ such that $U \cap A = \{a\}$.*

*Proof.* We may first observe that $A$ is discrete if and only if $\{a\}$ is open in $A$ for all $a \in A$, as every subset of $A$ is the union of singletons. Then, $\{a\}$ is open if and only if there exists some $U \subseteq \mathbb{R}^n$ open such that $\{a\} = U \cap A$. □

**Proposition 1.6.** *If $A \cong B$ and $A$ is discrete, then $B$ is discrete.*

*Proof.* Let $f : A \to B$ be a homeomorphism. By **Proposition 1.1**, $f$ is open. For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$. Then, $\{b\} = f(\{a\})$ is open in $B$. So by **Proposition 1.5**, $B$ is discrete. □

**Proposition 1.7.** *We have,*

*(1) $\mathbb{Z}$ is discrete.*

*(2) Finite sets are discrete.*

*(3) $\mathbb{Q}$ is not discrete.*

*(4) $\{1/n \mid n \in \mathbb{Z}^+\}$ is discrete.*

*(5) $\{1/n \mid n \in \mathbb{Z}^+\} \cup \{0\}$ is not discrete.*

*Proof.* We argue via **Proposition 1.5**.

For (1), $B_{1/2}(n) \cap \mathbb{Z} = \{n\}$.

For (2), let $F \subseteq \mathbb{R}^n$ be finite, and pick $\varepsilon > 0$ such that $\varepsilon < 2 \min_{(x,y) \in F^2} ||x - y||$. Then $B_\varepsilon(x) \cap F = \{x\}$ for any $x \in F$.

For (3), note that if $U, C \subseteq \mathbb{R}$ are open and closed respectively, $U \setminus C$ is open. Thus, every open subset of $\mathbb{R}$ intersects $\mathbb{Q}$ at at least two points.

For (4), simply observe that $(n+2)^{-1} < (n+1)^{-1} < (n)^{-1}$ for all $n \in \mathbb{Z}^+$, and so adjacent points can be separated by open sets.

For (5), by the Archimedean property, every open neighbourhood of 0 contains $n^{-1}$ for some $n \in \mathbb{Z}^+$. □

**Proposition 1.8.** *If $A$ is discrete, then $A$ is countable.*

*Proof.* Since $\mathbb{Q}^{n+1}$ is countable, we may enumerate the set of all open balls with centers in $\mathbb{Q}^n$ and rational radii, say by $j \mapsto B_j$. Then, for each $a \in A$, by **Proposition 1.5** there exists some $U \subseteq \mathbb{R}^n$ open such that $A \cap U = \{a\}$. But, by density we can find some $j$ such that $a \in B_j \subseteq U$. So we define a map $A \to \mathbb{N}$ which sends every point to the least $j \in \mathbb{N}$ such that $B_j$ is as desired, which exhibits an injection from $A$ to $\mathbb{N}$, whence $A$ is countable. □

**Proposition 1.9.** *If $f : X \to Y$ is a homeomorphism and $A \subseteq X$, then $f : A \to f(A)$ is a homeomorphism.*

*Proof.* Exercise. See PS4. □

**Proposition 1.10.** *$f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f \circ \iota_A$ is continuous for all $\varnothing \neq A \subsetneq X$.*

*Proof.* See PS4. □

**Proposition 1.11.** *If $f : A \to B$ is continuous and $B \subseteq \mathbb{R}^m$, then $f : A \to \mathbb{R}^m$ is continuous.*

*Proof.* See PS4. □