Problem Set 3

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Throughout this problem set, let (X, d) be a metric space.

Problem 1. *Suppose* $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *is a sequence in* X *which has the property that every subsequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{x_n\}_{n\in\mathbb{N}}$ *admits a subsubsequence* $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ *which converges to* $x \in X$*. Show that* $x_n \to x$*.*

Problem 2. *Find all of the limit points of* $A = \{(-1)^n + n^{-1} \mid n \in \mathbb{Z}\}$*.*

Problem 3. *We say that* $a \in A \subseteq \mathbb{R}^n$ *is isolated if there exists an open neighbourhood* U *of* a *such that* $U \cap A = \{a\}$*. Show that* $\overline{A}$ *may be written as the disjoint union of the limit points of* A *and the isolated points of* A*.*

Problem 4. *Suppose* $A \subseteq \mathbb{R}^n$*. Prove that* A'*, the set of limit points of* A*, is closed.*

Problem 5. *Prove or disprove:* $(A \cup B)' = A' \cup B'$*.*

Problem 6. *For* $A \subseteq \mathbb{R}^n$ *and* $x \in \mathbb{R}^n$*, we define* $d(x, A) = \inf_{a\in A} ||x - a||$*. For* $n \in \mathbb{Z}^+$*, let* $A_n = \{x \in \mathbb{R}^n \mid d(x, A) < n^{-1}\}$*. Show that,* $A' = \bigcap_{n=1}^{\infty} \overline{A_n} \setminus A$

Problem 7. *Suppose* $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^n$ *is such that* $||y_{n+1} - y_n|| \leq 2^{-n}$*. Show that the* y_n *converge.*

Problem 8. *Let* $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^n$ *be such that* $x_{2n} \to L_1$ *and* $x_{2n+1} \to L_2$*. Show that the* x_n *converge if and only if* $L_1 = L_2$*.*

Problem 9. *Suppose* $\limsup_{n\to\infty} ||x_n|| = 0$*. Prove* $x_n \to 0$*.*

1 Bonus topic: double sequences

Definition. A double sequence in X is a function $a : \mathbb{N} \times \mathbb{N} \to X$ typically denoted by $(m, n) \mapsto a_{m,n}$. We say that the double limit of $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ is $a \in X$ if, for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n, m > N$ we have $d(a_{m,n}, a) < \varepsilon$. We say that a double sequence is double Cauchy if for all $\varepsilon > 0$ there exist some $N \in \mathbb{N}$ such that for all $n, n', m, m' > N$, we have $d(a_{m,n}, a_{m',n'}) < \varepsilon$.

Problem 10. *Suppose $\{a_{m,n}\}_{n,m\in\mathbb{N}} \subseteq X$ is a double sequence with double limit a. Then,*

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} = a = \lim_{m\to\infty} \lim_{n\to\infty} a_{m,n}$$

Let $a_{m,n} = 1$ if $m = n$ and 0 otherwise. Show that each iterated limit exists, and that they are equal, but that the double limit does not.

Problem 11. *Suppose X is complete. Show that,*

(1) If $\{a_{m,n}\}_{n,m\in\mathbb{N}} \subseteq X$ has a double limit then it is double Cauchy.

(2) If $\{a_{m,n}\}_{n,m\in\mathbb{N}} \subseteq X$ is double Cauchy, then it has a double limit.

Problem 12. *Suppose X is complete. Show that if $d(a_{m,n}, a_{m,n+1}), d(a_{m,n}, d_{m+1,n}) \to 0$ as $n, m \to \infty$, then $\{a_{m,n}\}_{n,m\in\mathbb{N}}$ has a double limit.*

Problem 13. *Suppose $\{a_{m,n}\}_{n,m\in\mathbb{N}} \subseteq X$ is such that (1) for fixed $m \in \mathbb{N}$, there exists some $x_m \in X$ and $N_m \in \mathbb{N}$ such that $a_{m,n} = x_m$ for all $n > N_m$ and (2) for fixed $n \in \mathbb{N}$, $\lim_{m\to\infty} a_{m,n}$ exists. Prove that $\{a_{m,n}\}_{n,m\in\mathbb{N}}$ has a double limit.*

Problem 14. *Suppose now that $X \subseteq \mathbb{R}^n$ is closed and bounded. Show that for all $\{a_{m,n}\}_{n,m\in\mathbb{N}} \subseteq X$, there exists a sequence $m_k = n_k$ such that $\lim_{k\to\infty} a_{m_k,n_k}$ exists.*