q-TASEP From the Duality Approach

by

Wang Kunzhen

Submitted to the Department of Mathematics, Faculty of Science
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Author . . . .
Wang Kunzhen
Department of Computer Science, School of Computing
March, 2016

Supervised by . . . .
Wang Dong
Assistant Professor
Thesis Supervisor

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Abstract

In this thesis, I studied the model q-TASEP from the duality approach with main reference to the two papers [1] and [11]. An ansatz explicit formula for the q-Laplace transform of the probability mass function describing the position of a particular particle at any time in the future will be provided in form of the nested contour integral. After this, I worked to transform a generating function of the nested contour integrals to the form of a Fredholm determinant using both the Mellin-Barnes type approach and the Cauchy-type approach. The thesis is then concluded with an asymptotic analysis on the behaviour of the rescaled fluctuation of a particular particle around the macroscopic approximation of the particle position.

Thesis Supervisor: Wang Dong
Title: Assistant Professor

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Contents

1 Introduction **11**

1.1 Introduction to q-TASEP 12

1.2 Initial data for q-TASEP 13

1.3 q-TASEP for a few particles 14

1.3.1 q-TASEP with 1 particle 14

1.3.2 q-TASEP with 2 particles 15

1.4 Introduction to q-TAZRP 18

2 Duality and Nested Contour Integral **21**

2.1 Duality 21

2.2 Evolution Equation Systems 24

2.3 Nested Contour Integral Solution 27

3 Fredholm Determinants **33**

3.1 Fredholm Determinants 33

3.2 Mellin-Barnes type determinants 35

3.2.1 Mellin-Barnes type transformation 35

3.2.2 Transformation to Fredholm determinants 42

3.2.3 Application to q-TASEP 47

3.3 Cauchy type Fredholm determinants 49

3.3.1 Cauchy type transformation 50

3.3.2 Transformation to Fredholm determinants 52

3.3.3 Application to q-TASEP 54

3.4 q Laplace Transform and Inversion Formula 57
3.5 q-TASEP with half stationary initial data 58

4 Tracy-Widom Asymptotics 59
4.1 Tracy-Widom distribution 59
4.2 Reformulation of Mellin-Barnes type kernel 60
4.3 Integration Contours 63
4.4 Asymptotics of the Fredholm determinant 66
4.5 Distribution of the rescaled fluctuation 69

5 Conclusion 73

List of Figures

1-1 An illustration of q-TASEP 12

1-2 An illustration of q-TAZRP 19

4-1 The contours $\tilde{C}_\varphi$(left), C_φ(right) and $\tilde{D}_w$ 63

4-2 Contour replacement of D_W 65

4-3 Contour localization and transformation of C_φ and D_W 67

Chapter 1

Introduction

In this thesis I studied a 1-dimensional interacting particle system model called q-TASEP. It's one of the most well studied particle model in this field that has important application in the physical world. For example, the model can be used to study mass transport, traffic flow and queuing behaviour etc. The article is concerned with the asymptotic behavior of the particles and is going to provide an explicit formulat describing the behavior. The approach starts from the duality between the q-TASEP model and another model called q-TAZRP.

We are going to define the two model, q-TASEP and q-TAZRP in this chapter, and show some calculations of the q-TASEP from the elementary approach. After this we define duality and prove the duality between q-TASEP and q-TAZRP in *Chapter 2*. Then we provide an ansatz explicit formula for the q-Laplace transform of the probability mass function describing the position of a particular particle at any time in the future in form of a nested contour integral. In *Chapter 3* we work to transform a generating function of the nested contour integral we have obtained into the form of a Fredholm determinant, using two different approaches. Lastly in *Chapter 4* we perform asymptotic analysis to the Fredholm determinant and conclude that the rescaled fluctuation of the particle around the macroscopic approximation of its position follows a GUE Tracy-Widom distribution.

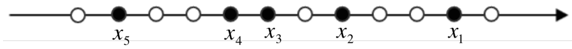

Figure 1-1: An illustration of q-TASEP. The jump rate of the particle x_2 is given by $a_2(1-q^2)$, while that of the particle x_4 is 0.

1.1 Introduction to q-TASEP

q-TASEP, short for q-deformed totally asymmetric simple exclusion process, refers to a continuous time, discrete space Markov Process $\vec{x}(t)$ describing the dynamics of the following interacting particle system:

- Particles denoted by $\{x_1, x_2, x_3, ...\}$ occupy sites of $\mathbb{Z}$ exclusively with positions at time t denoted as $x_i(t)$. The particles are ordered in such a way that $x_i(t) < x_j(t)$ for $i > j$.
- Particles jump to the right by one spot ($x_i(t)$ increases by 1) with jump rate given by $a_i(1-q^{x_{i-1}(t)-x_i(t)+1})$ for $i \geq 2$, where $q \in [0,1)$, $a_i > 0$. The jump rate for the particle x_1 is defined to be a_1.
- All jumps occur indenpendently of each other according to exponential clocks with parameter 1.

It's worth noting that the ordering of the particles remains unchanged throughout the process. Note also that in the definition of q-TASEP, numbering of the particles starts with 1. However, to facilitate our discussion, we could have added a virtual partical x_0 with $x_0 = \infty$ so that the jump rate of x_1 is also equal to $a_1(1-q^{x_0(t)-x_1(t)+1}) = a_1$.

It can be easily seen that dynamics of the right most N particles, i.e., particle x_1, x_2, ..., x_N, is independent from those to the left of them, i.e., x_{N+1}, x_{N+2}, ... Therefore, from now onwards, we will be focusing on the study of q-TASEP with N particles, with configuration $x_1(t) > x_2(t) > \cdots > x_N(t)$. We could then define our state space as

$$X^N = \{\vec{x} = (x_0, x_1, \ldots, x_N) \in \{\infty\} \times \mathbb{Z}^N : \infty = x_0 > x_1 > \cdots > x_N\}.$$

For q-TASEP with N particles, the infinitesimal generator of $\vec{x}(t)$ acting on suitable function $f : X^N \to \mathbb{R}$, denoted by $L^{q-TASEP} f$, is given by

$$(L^{q-TASEP} f)(\vec{x}) = \sum_{i=1}^{N} a_i (1 - q^{x_{i-1} - x_i - 1})(f(\vec{x}_i^+) - f(\vec{x})),$$

where $\vec{x}_i^+$ denotes the configuration of the q-TASEP with particle x_i jumps to the right by 1 position, i.e., $\vec{x}_i^+ = (x_0, x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_N)$. This follows from the fact that for t small, there is only 1 chance for one of the N particles to jump according its own jump rate.

1.2 Initial data for q-TASEP

For a q-TASEP to be well defined, additional to the dynamics of q-TASEP described in Section 1.1, we also need an initial configuration of the process, called initial data. In this paper, we focus on two types of initial data, namely, the step initial data and the half stationary initial data.

Step initial data is defined as the configuration of $x_i(0) = -i$ for $1 \leq i \leq N$. For half stationary initial data, we define q-Geometric distribution.

DEFINITION 1.1. For $\alpha \in [0, 1)$, we say that a random variable X follows the q-Geometric distribution with parameter α [written as $X \sim qGeo(\alpha)$] if

$$\mathbb{P}(X = k) = (\alpha; q)_\infty \frac{\alpha^k}{(q; q)_k},$$

where $(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \ldots (1 - aq^{n-1})$ and $(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \ldots$

Let $X_i \sim qGeo(\alpha/a_i)$ for $1 \leq i \leq N$ be N independent q-Geometric random variables. Then the half stationary initial condition is defined recursively by first setting $x_1(0) = -1 + X_1$ and letting $x_i(0) = -1 + x_{i-1}(0) + X_i$ for $i > 1$. Note that when $\alpha = 0$, the step initial data is recovered.

1.3 q-TASEP for a few particles

The main goal for the study of q-TASEP in this thesis is to identify an exact formula for the distribution of the particle positions, $\mathbb{P}(x_N(t) = m)$ for $m \in \mathbb{Z}$ with step and half stationary initial data. In this section, we provide some intuitions of how this problem can be approached via elementary methods for q-TASEP of 1 and 2 particles, thereby illustrating how the complexity grows for N large.

1.3.1 q-TASEP with 1 particle

In this section we focus on the q-TASEP with only 1 particle denoted by x_1 starting at position $x_1(0) = 0$. Let T_i, $i = 1, 2, \dots$ be independent and identically distributed exponential random variables with parameter 1 denoting the time between the $(i-1)th$ and the ith jump. For $m \in \mathbb{Z}_{\geq 0}$, define

$$a_m = \mathbb{P}(x_1(t) \leq m) = \mathbb{P}(\sum_{i=1}^{m+1} T_i > t)$$

such that $\mathbb{P}(x_1(t) = m) = a_m - a_{m-1}$ for $m \geq 1$ and $\mathbb{P}(x_1(t) = 0) = a_0 = e^{-t}$. Since $T_i \sim exp(1)$, $\sum_{i=1}^{m} T_i \sim Erlang(m, 1)$ and therfore, by the density function of Erlang distribution,

$$1 - a_m = \int_0^t \frac{s^m}{m!} e^{-s} ds.$$

PROPOSITION 1.1. *For q-TASEP with one particle x_1 with jump rate parameter $a_1 = 1$ and initial profile at $x_1(0) = -1$, we have that $\mathbb{E}[x_1(t)+1] = t$ and $\mathbb{E}[q^{x_1(t)+1}] = e^{(q-1)t}$.*

Proof. From the notations above, we have that $\mathbb{P}(x_1(t) = m) = c_m - c_{m-1}$, where c_m is defined by $c_m = 1 - \int_0^t \frac{s^m}{m!} e^{-s} ds$. Therefore, we have

$$\begin{aligned}
\mathbb{E}[x_1(t)] &= \lim_{n\to\infty} ((1 - c_0) + (1 - c_1) + ... + (1 - c_{n-1}) - n(1 - c_n)) \\
&= \lim_{n\to\infty} (\int_0^t e^{-s} (\sum_{i=0}^{n-1} \frac{s^i}{i!}) ds) + \lim_{n\to\infty} (n \times \int_0^t \frac{s^n}{n!} e^{-s} ds)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t e^{-s} \lim_{n\to\infty} (\sum_{i=0}^{n-1} \frac{s^i}{i!}) ds + 0 \\
&= \int_0^t e^{-s} e^s dt \\
&= t
\end{aligned}
$$

Moreover, $\mathbb{E}[q^{x_0(t)}]$ can be computed as below:

$$
\begin{aligned}
\mathbb{E}[q^{x_0(t)}] &= \lim_{n\to\infty} (q^0 c_0 + \sum_{i=1}^{n} q^i (c_i - c_{i-1})) \\
&= \lim_{n\to\infty} (-(1-q)(\sum_{i=0}^{n-1} q^i (1-a_i)) - q^n (1-c_n) + 1) \\
&= -(1-q) \lim_{n\to\infty} (\int_0^t e^{-s} \sum_{i=0}^{n-1} q^i \frac{s^i}{i!} ds) - \lim_{n\to\infty} \int_0^t q^n \frac{s^n}{n!} ds + 1 \\
&= -(1-q) \int_0^t e^{-s} \lim_{n\to\infty} \sum_{i=0}^{n-1} q^i \frac{s^i}{i!} ds - \lim_{n\to\infty} \int_0^t q^n \frac{s^n}{n!} ds + 1 \\
&= -(1-q) \int_0^t e^{-s} e^{qs} ds - \lim_{n\to\infty} \int_0^t q^n \frac{s^n}{n!} ds + 1 \\
&= e^{(q-1)t}.
\end{aligned}
$$

□

1.3.2 q-TASEP with 2 particles

Elementary methods similar to the one adopted for the one-particle system were applied for the 2 particles system, and it was realized that the computation grew so complicated that nearly no conclusions was reached. Therefore, a different approach, namely the Komogorov Equation, was used to carry out the computation. Although no final conclusion was reached either at the end, some interesting results were found.

Let us consider the two particles labeled as x_1 and x_2 starting at the initial positions that $x_1(0) = 0$ and $x_2(0) = a << 0$. Let $\mathbb{P}_{k,j}(t) = \mathbb{P}(x_1(t) = k, x_2(t) = j)$. We begin by giving the Komogorov Equation for the problem:

$$\frac{d}{dt}\mathbb{P}_{k,j}(t) = -\mathbb{P}_{k,j}(t) - (1-q^{k-j-1})\mathbb{P}_{k,j}(t) + (1-q^{k-j})\mathbb{P}_{k,j-1}(t) + \mathbb{P}_{k-1,j}(t)$$

with base cases $\mathbb{P}_{k,j}(t) = 0$ if $j < a$ or $k < 0$ or $j \geq k$.

We will be applying induction on k and j separately, starting by iterating through $k = 0, 1, 2, \ldots$. The result is stated as a proposition.

PROPOSITION 1.2. *With the notations and setting introduced above, for $m \geq 1$*

$$\mathbb{P}_{m,a}(t) = \frac{1}{2\pi i}\oint_{allpoles} \frac{e^{(-2+z)t}}{(z-q^{-a-1})(z-q^{-a})...(z-q^{-a+m-1})}dz.$$

Proof. We will not be showing a complete proof here. Rather, we give some calculations for some base cases to check that the proposition is correct and provide some intuitions.

Case 1: For $k = 0, j = a < 0$, we have

$$\frac{d}{dt}\mathbb{P}_{0,a}(t) + (2-q^{-a-1})\mathbb{P}_{0,a}(t) = 0$$

Solving this equation gives

$$\mathbb{P}_{0,a}(t) = ce^{-(2-q^{-a-1})t}$$

Notice that $\mathbb{P}_{0,a}(0) = 1$, then $c = 1$. Therefore

$$\mathbb{P}_{0,a}(t) = e^{-(2-q^{-a-1})t} = \frac{1}{2\pi i}\oint_{allpoles} \frac{e^{(-2+z)t}}{z-q^{-a-1}}dz$$

Case 2: For $k = 1, j = a < 0$, we have

$$\frac{d}{dt}\mathbb{P}_{1,a}(t) + (2-q^{-a})\mathbb{P}_{1,a}(t) = e^{-(2-q^{-a-1})t}$$

Therefore,

$$\frac{d}{dt}[e^{(2-q^{-a})t}\mathbb{P}_{1,a}(t)] = e^{(q^{-a-1}-q^{-a})t}$$

Solving this equation we have

$$\mathbb{P}_{1,a}(t) = \frac{e^{-(2-q^{-a-1})t}}{q^{-a-1} - q^{-a}} + ce^{-(2-q^{-a})t}$$

Note that $\mathbb{P}_{1,a}(0) = 0$, we have $c = -\frac{1}{q^{-a-1}-q^{-a}}$ and therefore

$$\mathbb{P}_{1,a}(t) = \frac{1}{q^{-a-1} - q^{-a}}(e^{-(2-q^{-a-1})t} - e^{-(2-q^{-a})t})$$

In **contour integral** form, it can also be written as

$$\mathbb{P}_{1,a}(t) = \frac{1}{2\pi i}\oint_{allpoles} \frac{e^{(-2+z)t}}{(z - q^{-a-1})(z - q^{-a})}dz$$

Case 3: For $k = 2, j = a < 0$, we have

$$\frac{d}{dt}\mathbb{P}_{2,a}(t) + (2 - q^{1-a})\mathbb{P}_{2,a}(t) = \mathbb{P}_{1,a}(t)$$

Solving this equation gives

$$\begin{aligned}
\mathbb{P}_{2,a}(t) &= \frac{e^{(-2+q^{-a-1})t}}{(q^{-a-1} - q^{-a})(q^{-a-1} - q^{1-a})} - \frac{e^{(-2+q^{-a})t}}{(q^{-a-1} - q^{-a})(q^{-a} - q^{1-a})} \\
&\quad + \frac{e^{(-2+q^{-a+1})t}}{(q^{-a-1} - q^{-a+1})(q^{-a} - q^{-a+1})} \\
&= \frac{1}{2\pi i}\oint_{allpoles} \frac{e^{(-2+z)t}}{(z - q^{-a-1})(z - q^{-a})(z - q^{-a+1})}dz
\end{aligned}$$

□

Next, we investigate the cases $k = 0, j = a, (a + 1), \ldots, -1$. The result is stated as the following proposition.

PROPOSITION 1.3. *With the notations and setting introduced above, for* $k = 0, j = a + n$, $0 \leq n < a$, *we have*

$$\mathbb{P}_{0,a+n}(t) = \frac{(1-q^{-a-1})(1-q^{-a-2})...(1-q^{-a-n})}{2\pi i} \times \oint_{allpoles} \frac{e^{(-2+z)t}}{(z-q^{-a-1})(z-q^{-a-2})...(z-q^{-a-n-1})} dz.$$

Proof. Again, we will only be providing some intuitions regarding the equality. For $k = 0, j = a + 1$, we have

$$\frac{d}{dt}\mathbb{P}_{0,a+1}(t) + (2 - q^{-a-2})\mathbb{P}_{0,a+1}(t) = (1 - q^{-a-1})\mathbb{P}_{0,a}(t)$$

Solving this equation gives

$$\begin{aligned}
\mathbb{P}_{0,a+1}(t) &= \frac{1-q^{-a-1}}{q^{-a-1}-q^{-a-2}}(e^{-(2-q^{-a-1})t} - e^{-(2-q^{-a-2})t}) \\
&= (1-q^{-a-1})\frac{1}{2\pi i}\oint_{allpoles} \frac{e^{(-2+z)t}}{(z-q^{-a-1})(z-q^{-a-2})} dz
\end{aligned}$$

□

More general application of induction on both k and j together was carried out and didn't reach any elegant results as the ones mentioned above. Therefore, the elementary calculation for the q-TASEP model was stopped and the focus was then turned to studying the duality approach. However, such calculations demonstrate how fast the complexity grows with the number of particles considered, and they were also important as they provided essential fundamental understanding of the model.

1.4 Introduction to q-TAZRP

Next, we introduce another Markov process called q-TAZRP, short for q-deformed totally asymmetric zero range process. q-TAZRP on an interval $\{0, 1, ..., N\}$ is a

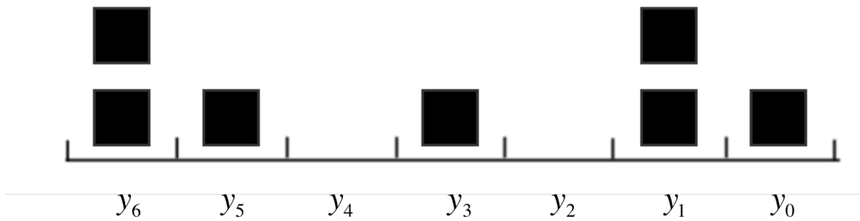

Figure 1-2: An illustration of q-TAZRP. Rate function for site y_1 is given by $a_1(1-q^2)$.

continuous time, discrete space Markov process $\vec{y}(t)$ with state space

$$Y^N = (\mathbb{Z}_{\geq 0})^{N+1}$$

subject to the condition of $\sum_{i=0}^{N} y_i(t) = \sum_{i=0}^{N} y_i(0)\ \forall\ t \in \mathbb{R}_{\geq 0}$. Dynamics of a q-TAZRP is defined as the following:

- For each $i \in \{1, 2, \ldots, N\}$, $y_i(t)$ decreases by 1 and $y_{i-1}(t)$ increases by 1 simultaneously according to a rate function given by $g_i(k) = a_i(1 - q^{y_i(t)})$.
- All changes occur independently for each site according to exponential clocks.
- y_0 is always non-decreasing.

A natural interpretation of the q-TAZRP is given in Figure 1-2, where a total of $\sum_{i=0}^{N} y_i(0)$ particles occupy the N sites ordered in such a way that y_i is to the left of y_{i-1} for $i = 1, 2, \ldots, N$. The changes in the definition then refer to a jump of the top particle at site y_i to the site y_{i-1}. Note then that no particle leaves site 0.

It can be shown that the infinitesimal generator of $\vec{y}(t)$ acting on a suitable function $h : Y^N \to \mathbb{R}$, denoted as $L^{q-TAZRP}h$, is given by

$$(L^{q-TAZRP}h)(\vec{y}) = \sum_{i=1}^{N} a_i(1 - q^{y_i})(h(\vec{y}^{i,i-1}) - h(\vec{y})),$$

where $\vec{y}^{i,i-1}$ represents the configuration of $(y_0, \ldots, y_{i-2}, y_{i-1} + 1, y_i - 1, y_{i+1}, \ldots, y_N)$.

Lastly, we present another notation, $\vec{n} \in W^k_{>0}$, for the states of a q-TAZRP, where $W^k_{>0} = \{(n_1, n_2, \ldots, n_k) \in (\mathbb{Z}_{>0})^k : n_1 \geq n_2 \geq \cdots \geq n_k \geq 0\}$. The vector $\vec{n}(\vec{y})$ for any state $\vec{y}$ of the q-TAZRP is given by

$$y_i = |\{n_j : n_j = i\}| .$$

The intuition is that while y_i in $\vec{y}$ denotes the number of particles at site i, n_j in $\vec{n}$ denotes the positioin of Particle j ordered according to decreasing site locations. Take the configuration in Figure 1-2 for example. Its representation in $\vec{y}$ is given by $\vec{y} = (1, 2, 0, 1, 0, 1, 2)$, while in $\vec{n}$ it is represented as $\vec{n} = (6, 6, 5, 3, 1, 1, 0)$.

Chapter 2

Duality and Nested Contour Integral

In this chapter, we recall the general definition of duality between two Markov process defined in [9] and provide some remarks on the definition. We then prove the duality between q-TASEP and q-TAZRP and conclude by giving an ansatz solution for q-TASEP with step initial data as well as half stationary initial data in the form of nested contour integrals.

2.1 Duality

We begin by recalling the general definition of duality from $Definition 3.1$ of [9].

DEFINITION 2.1. Suppose $x(t)$ and $y(t)$ are two independent Markov processes with state space X and Y respectively, and let $H(x, y)$ be a bounded measurable function on $X \times Y$. Then the processes $x(t)$ and $y(t)$ are said to be **dual** to one another with respect to H if

$$\mathbb{E}^x[H(x(t), y)] = \mathbb{E}^y[H(x, y(t))]$$

for all $x \in X$ and $y \in Y$. Here $\mathbb{E}^x$ is the expectation taken with respect to the process $x(t)$ with initial data $x(0) = x$ and similarly for $\mathbb{E}^y$.

Note that with duality, we can easily derive that the infinitesimal generator Ω^x and Ω^y of the processes $x(t)$ and $y(t)$, respectively, acting on the function $H(x, y)$ are equivalent, i.e.,

$$\begin{aligned}(\Omega^x H(\cdot, y))(x) &= \lim_{t\to 0} \frac{\mathbb{E}^x[H(x(t), y)] - H(x,y)}{t} \\ &= \lim_{t\to 0} \frac{\mathbb{E}^y[H(x, y(t))] - H(x,y)}{t} \\ &= (\Omega^x H(x, \cdot))(y)\end{aligned}$$

In fact, the reverse is also true under some weak conditions [4]. That is, given that the infinitesimal generator of two Markov processes $x(t)$ and $y(t)$ acting on function $H(x, y)$ are equal, we can derive that the two processes are dual with respect to the function $H(x, y)$.

THEOREM 2.1. *The q-TASEP, $\vec{x}(t)$ with state space X^N and particle jump rate parameters $a_i > 0$, and the q-TAZRP, $\vec{y}(t)$, with state space Y^N and rate functions $g_i(k) = a_i(1 - q^k)$ are dual with respect to*

$$H(\vec{x}, \vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}.$$

REMAEK 2.1. It is noted that in the definition, $x_0 = \infty$ as we have discussed before. Therefore, for any $\vec{y}$ with $y_0 > 0$, $H(\vec{x}, \vec{y}) = 0$.

LEMMA 2.1.1. *Assume q-TASEP $\vec{x}(t)$ with initial condition $\vec{x}$ and q-TAZRP $\vec{y}(t)$ with initial condition $\vec{y}$. Then*

$$(L^{q-TASEP} H(\cdot, \vec{y}))(\vec{x}) = (L^{q-TAZRP} H(\vec{x}, \cdot))(\vec{y})$$

.

Proof. By definition,

$$\begin{aligned}(L^{q-TASEP} H(\cdot, \vec{y}))(\vec{x}) &= \sum_{i=1}^{N} a_i(1 - q^{x_{i-1}-x_i-1})(H(\vec{x}_i^+, \vec{y}) - H(\vec{x}, \vec{y})) \\ &= \sum_{i=1}^{N} a_i(1 - q^{x_{i-1}-x_i-1})(\prod_{j=0, j\neq i}^{N} q^{(x_j+j)y_j}(q^{(x_i+1+i)y_i} - q^{(x_i+i)y_i}))\end{aligned}$$

$$= \sum_{i=1}^{N} a_i(1 - q^{x_{i-1}-x_i-1})(q^{y_i} - 1)H(\vec{x}, \vec{y}) \tag{2.1}$$

Similarly, for q-TAZRP we also have the following

$$\begin{aligned}
(L^{q-TAZRP} H(\vec{x}, \cdot))(\vec{y}) &= \sum_{i=1}^{N} a_i(1 - q^{y_i})(H(\vec{x}, \vec{y}^{i,i-1}) - H(\vec{x}, \vec{y})) \\
&= \sum_{i=1}^{N} a_i(1 - q^{y_i}) \prod_{j=0, j\neq i, i-1}^{N} q^{(x_j+j)y_j} \\
&\quad \times (q^{(x_i+i)(y_i-1)} q^{(x_{i-1}+i-1)(y_{i-1}+1)} - q^{(x_i+i)y_i} q^{(x_{i-1}+i-1)y_{i-1}}) \\
&= \sum_{i=1}^{N} a_i(1 - q^{y_i})(q^{x_{i-1}-x_i-1} - 1)H(\vec{x}, \vec{y}) \qquad (2.2)
\end{aligned}$$

Comparing Equation (2.1) and (2.2), we conclude that

$$(L^{q-TASEP} H(\cdot, \vec{y}))(\vec{x}) = (L^{q-TAZRP} H(\vec{x}, \cdot))(\vec{y}).$$

□

Before we continue to the proof of Theorem 2.1, we define a system of ordinary differential equation:

DEFINITION 2.2. We say that $h(t; \vec{y}) : \mathbb{R}_+ \times Y^N \to \mathbb{R}$ solves the ***true evolution equation system*** with initial data $h_0(\vec{y})$ if:

(1) For all $\vec{y} \in Y^N$ and $t \in \mathbb{R}_+$,

$$\frac{d}{dt} h(t; \vec{y}) = L^{q-TAZRP} h(t; \vec{y});$$

(2) For all $\vec{y} \in Y^N$ such that $y_0 > 0$, $h(t; \vec{y}) = 0$ for all $t \in \mathbb{R}_+$;

(3) For all $\vec{y} \in Y^N$, $h(0; \vec{y}) = h_0(\vec{y})$.

REMAEK 2.2. It is remarked that the existence and uniqueness of solutions to the ***true evolution equation system*** is assured. This follows from elementary ODE theories since the system reduces to a finite number of differential equations.

Proof of Theorem 2.1: We prove the theorem by showing that both $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$ and $\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$ is a solution to the ***true evolution equation system*** as defined in Definition 2.2. Then by uniqueness, duality follows.

That $\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$ is a solution to the ***true evolution equation system*** is clear from the fact that

$$\frac{d}{dt}\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))] = \lim_{t\to 0}\frac{\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))] - H(\vec{x},\vec{y})}{t} = L^{q-TAZRP}\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))],$$

with initial and boundary conditions easily checked.

It then suffices to show that $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$ is a solution to the ***true evolution equation system***. Since the initial and boundary conditions can be easily checked, we only need to show that $\frac{d}{dt}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] = L^{q-TAZRP}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$ as below:

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] &= L^{q-TASEP}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] \quad &&\text{(by definition)}\\
&= \mathbb{E}^{\vec{x}}[L^{q-TASEP}H(\vec{x}(t),\vec{y})] \quad &&\text{(by commutativity)}\\
&= \mathbb{E}^{\vec{x}}[L^{q-TAZRP}H(\vec{x}(t),\vec{y})] \quad &&\text{(by Lemma 2.1.1)}\\
&= L^{q-TAZRP}\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})] \quad &&\text{(Since } L^{q-TAZRP} \text{ acts on } \vec{y}\text{)}
\end{aligned}$$

□

Using the duality, in order to find $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$, it suffices to find $\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$. Note that the difference between the two expression is that the former is evaluating an expectation in the context of a q-TASEP, while the latter is evaluating the expectation in the context of a q-TAZRP. In what follows, we are going to show how to evaluate the expectation $\mathbb{E}^{\vec{y}}[H(\vec{x},\vec{y}(t))]$ in q-TAZRP using the *free evolution equation system*. With this, we then are able to get an explicit formula for $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$.

2.2 Evolution Equation Systems

In this section, we are going to define another evolution equation systems, namely, the *free evolution equation system* and show that a solution to the *free evolution equation* also solves the *true evolution equation system* with certain initial data.

Before moving on to the definition of the *free evolution equation system*, we introduce a notion ∇_i defined by $\nabla_i f(\vec{n}) = f(\vec{n}_i^-) - f(\vec{n})$ for $\vec{n} = (n_1, \ldots, n_k)$, where $\vec{n}_i^- = (n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k)$.

DEFINITION 2.3. Given $\vec{x} \in X^N$, the state space of a q-TASEP, we say that $u(t;\vec{n}) : \mathbb{R}_+ \times (\mathbb{Z}_{\geq 0})^k \to \mathbb{R}$ solves the ***free evolution equation system*** with $k-1$ boundary conditions if:

(1) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ and $t \in \mathbb{R}$,

$$\frac{d}{dt} u(t;\vec{n}) = (1-q) \sum_{i=1}^{k} a_{n_i} \nabla_i u(t;\vec{n});$$

(2) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ such that for some $i \in \{1, ..., k-1\}$, $n_i = n_{i+1}$,

$$\nabla_i u(t;\vec{n}) = q \nabla_{i+1} u(t;\vec{n});$$

(3) For all $\vec{n} \in (\mathbb{Z}_{\geq 0})^k$ such that $n_k = 0$, $u(t;\vec{n}) = 0$ for all $t \in \mathbb{R}_+$;

(4) For all $\vec{n} \in W^k_{>0}$, $u(0,\vec{n}) = H(\vec{x}, \vec{y}(\vec{n}))$.

It's not clear up to now that a solution exists for the *free evolution equation system*. However, we are going to show that if one exists, then it must also solve the *true evolution equation system* with the corresponding initial data. This is stated as a proposition below:

PROPOSITION 2.1. *If $u(t;\vec{n}) : \mathbb{R}_+ \times (\mathbb{Z}_{\geq 0})^k \to \mathbb{R}$ is a solution to the* free evolution equation system *with $\vec{x} \in X^N$, then for all $\vec{y} \in Y^N$ such that $\sum_{i=1}^N y_i = k$,*

$$\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{y})] = u(t; \vec{n}(\vec{y})).$$

REMAEK 2.3. It should be noted that $u(t;\vec{n}(\vec{y}))$ is a restriction of the solution $u(t;\vec{n})$ with domain $\mathbb{R}_+ \times (\mathbb{Z}_{\geq 0})^k$ to the domain $\mathbb{R}_+ \times W^k_{>0}$. Therefore, we are not claiming the truthness of the inverse because we are not sure whether the inverse extension would be possible for any given $\vec{y} \in Y^N$.

Proof. Recall that $\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{y})]$ is the solution to the *true evolution equation system* with initial data $H(\vec{x}, \vec{y})$. In order to show that $\mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{y})] = u(t; \vec{n}(\vec{y}))$, we only need to show that for any solution $u(t; \vec{n})$ of the *free evolution equation system*, when restricted to the domain $\mathbb{R}_+ \times W^k_{>0}$, it also solves the *true evolution equation system* with initial data $H(\vec{x}, \vec{y}(\vec{n}))$, where $\sum_{i=1}^N y_i = k$. It then suffices to check $v(t; \vec{y}) = u(t; \vec{n}(\vec{y}))$ against the three conditions in Definition 2.2.

(1) For all $\vec{y} \in Y^N$ and $t \in \mathbb{R}^+$, we want to show that

$$(1-q)\sum_{i=1}^{k} a_{n_i} \nabla_i u(t; \vec{n}) = \frac{d}{dt} u(t; \vec{n}(\vec{y})) = \frac{d}{dt} v(t; \vec{y}) = \sum_{i=1}^{N} a_i (1-q^{y_i})(v(t; \vec{y}^{i,i-1}) - v(t; \vec{y})).$$

For this purpose, we claim that for each site i with y_i number of particles $n_{k_1} = n_{k_2} = \cdots = n_{k_{y_i}} = i$,

$$(1-q)\sum_{j=k_1}^{k_{y_i}} a_i \nabla_j u(t; \vec{n}) = a_i(1-q^{y_i})(v(t; \vec{y}^{i,i-1}) - v(t; \vec{y})).$$

Then by summing up over i, all sites, we get the desired equality.

For notational simplicity, we only prove the claim for $i = N$, and other sites work similarly. That is, we want to show that

$$(1-q)\sum_{j=1}^{y_N} a_N \nabla_j u(t; \vec{n}) = a_N(1-q^{y_N}) \nabla_{y_N} u(t; \vec{n}).$$

It follows from Condition (2) of Definition 2.3 that

$$\begin{aligned}
(1-q)\sum_{j=1}^{y_N} a_N \nabla_j u(t; \vec{n}) &= (1-q)\sum_{j=1}^{y_N} a_N q^{y_N - j} \nabla_{y_N} y(t; \vec{n}) \\
&= (1-q) a_N \nabla_{y_N} y(t; \vec{n}) \sum_{j=1}^{y_N} q^{y_N - j} \\
&= a_N (1-q^{y_N}) \nabla_{y_N} u(t; \vec{n}).
\end{aligned}$$

(2) Assume that $\vec{y} \in Y^N$ with $\sum_{i=1}^{N} y_i = k$ such that $y_0 > 0$. Then we would have $\vec{n}(\vec{y})$ satisfying $n_k = 0$. From Definition 2.3 Condition (3), it follows that $v(t;\vec{y}) = u(t;\vec{n}(\vec{y})) = 0$.

(3) For all $\vec{y} \in Y^N$, $v(0;\vec{y}) = u(0;\vec{n}(\vec{y})) = H(\vec{x},\vec{y})$.

$\square$

2.3 Nested Contour Integral Solution

As discussed in Section 2.2, existence of solutions to the *free evolution equation system* is not clear in general. However, in this section, we are going to provide solutions to the *free evolution equation system* with step and half stationary initial data in the form of nested contour integral.

THEOREM 2.2. *Fix $q \in (0,1)$, $a_i > 0$ for $i \geq 1$ and let $\vec{n} = (n_1, n_2, \ldots, n_k)$. The* free evolution equation system *as defined in Section 2.2 is solved by the following:*

(1) For step initial data,

$$u(t;\vec{n}) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \right) e^{(q-1)t z_j} \frac{dz_j}{z_j}, \tag{2.3}$$

where the integration contour for z_A contains $\{q z_B\}_{B>A}$ and all a_m's but not 0.

(2) For half stationary initial data with parameter $\alpha > 0$,

$$u(t;\vec{n}) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \right) e^{(q-1)t z_j} \frac{dz_j}{z_j - \alpha/q}, \tag{2.4}$$

where the integration contour for z_A contains $\{q z_B\}_{B>A}$ and all a_m's but not α/q.

COROLLARY 2.2.1. *For q-TASEP with step initial data and $\vec{n} \in W^k_{>0}$,*

$$\mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}+n_j}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j}\right) e^{(q-1)tz_j} \frac{dz_j}{z_j},$$

where the integration contour for z_A contains $\{qz_B\}_{B>A}$ and all $a_m's$ but not 0.

Proof of Corollary 2.2.1. Noting that the right hand side is simply a restriction of Equation (2.3), and that $u(t;\vec{n})$ is equal to $\mathbb{E}^{\vec{x}}[H(\vec{x}(t),\vec{y})]$, it then suffices to show that $\mathbb{E}[H(\vec{x}(t),\vec{y})] = \mathbb{E}\left[\prod_{i=1}^{N} q^{(x_i(t)+i)y_i}\right] = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}+n_j}\right]$, where the expectation is taken with the step initial data. Recall that $y_i = |\{n_j : n_j = i\}|$, therefore

$$\prod_{i=1}^{N} q^{(x_i+i)y_i} = \prod_{i=1}^{N} \prod_{j=n_{i,1}}^{n_{i,1}+y_i-1} q^{x_{n_j}+n_j} = \prod_{j=1}^{k} q^{x_{n_j}+n_j},$$

where $n_{i,1} = \min\{j : n_j = i\}$. □

Proof of Theorem 2.2. We first prove the case for step initial data. That is, we need to show that the function $u(t;\vec{n})$ as defined satisfies the four conditions in Definition 2.3, referred to as (1) - (4).

(1) Let

$$f(t;\vec{n}) = \prod_{j=1}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j}\right) e^{(q-1)tz_j}$$

such that

$$u(t;\vec{n}) = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} f(t;\vec{n}) \prod_{j=1}^{k} \frac{dz_j}{z_j}.$$

Note that

$$\frac{d}{dt} f(t;\vec{n}) = (q-1)\left(\sum_{i=1}^{k} z_i\right) f(t;\vec{n})$$

and that

$$\nabla_i f(t;\vec{n}) = \prod_{j=1, j\neq i}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j}\right) e^{(q-1)tz_j} \prod_{m=1}^{n_i - 1} \frac{a_m}{a_m - z_i} \left(1 - \frac{a_{n_i}}{a_{n_i} - z_i}\right)$$

$$
\begin{aligned}
&= \prod_{j=1, j\neq i}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \right) e^{(q-1)tz_j} \prod_{m=1}^{n_i - 1} \frac{a_m}{a_m - z_i} \left(-\frac{z_i}{a_{n_i} - z_i} \right) \\
&= -\frac{z_i}{a_{n_i}} f(t;\vec{n}) \qquad (2.5)
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt} f(t;\vec{n}) = (1-q) \sum_{i=1}^{k} a_{n_i} \nabla_i f(t;\vec{n}).
$$

Since $f(t;\vec{n})$ is the only component in $u(t;\vec{n})$ that is affected by the operators, by linearity, condition (1) follows.

(2) Without loss of generality, assume $n_1 = n_2$.

From Equation (2.5), we have that $\nabla_i u(t;\vec{n}) = -\frac{z_i}{a_{n_i}} u(t;\vec{n})$ and hence

$$
\begin{aligned}
&(\nabla_1 - q\nabla_2) u(t;\vec{n}) \\
&= -\frac{z_1 - q z_2}{a_{n_1}} u(t;\vec{n}) \\
&= \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \left(-\frac{z_1 - q z_2}{a_{n_1}} \right) \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} f(t;\vec{n}) \prod_{j=1}^{k} \frac{dz_j}{z_j} \\
&= \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \left(-\frac{z_1 - z_2}{a_{n_1}} \right) \prod_{A < B, (A,B) \neq (1,2)} \frac{z_A - z_B}{z_A - q z_B} f(t;\vec{n}) \prod_{j=1}^{k} \frac{dz_j}{z_j}
\end{aligned}
$$

From the last equality, the contours of z_1 and z_2 can be deformed to be the same without encountering any poles (since the term $z_1 - q z_2$ in the denominator has been canceled). Notice also that in the integrand the terms involving z_1 and the terms z_2 are separated into identical parts, and hence the integral can be re-written as

$$
(\nabla_1 - q\nabla_2) u(t;\vec{n}) = \int \int (z_1 - z_2) G(z_1) G(z_2) dz_1 dz_2,
$$

where $G(z)$ incorporates the integration involving $z_3, z_4, \ldots, z_k$. Since the contour of z_1 and z_2 are the same, we have that

$$
\begin{aligned}
&\int\int (z_1 - z_2)G(z_1)G(z_2)dz_1dz_2 \\
&= \int z_1G(z_1)dz_1 \times \int G(z_2)dz_2 - \int z_2G(z_2)dz_2 \times \int G(z_1)dz_1 \\
&= 0.
\end{aligned}
$$

That is, $\nabla_1 u(t;\vec{n}) = q\nabla_2 u(t;\vec{n})$.

(3) Notice that poles corresponding to z_k can only occur at the term $\prod_{m=1}^{n_k} \frac{a_m}{a_m - z_k}$. When $n_k = 0$, the term vanishes and hence the integration with respect to z_k also vanishes by Cauchy's integral theorem. Therefore, $u(t;\vec{n}) = 0$ when $n_k = 0$ follows.

(4) For step initial data, we have

$$
u(0,\vec{n}) = H(\vec{x}(0), \vec{y}(\vec{n})) = \prod_{i=0}^{N} q^{(x_i(0)+i)y_i} = 1
$$

for all $\vec{n} \in W^k_{>0}$ since $x_i(0) = -i$. It then suffices to show that $u(0,\vec{n}) = 1$ for all $\vec{n} \in W^k_{>0}$.

We show this by directly evaluating the integrals using the technique of contour deformation. Consider first the contour for z_1. It can be expanded to infinity and the only poles it can possibly encounter in this process are $z_1 = 0$ and $z_1 = \infty$. However, $z_1 = \infty$ is not a pole because as $z_1 \to \infty$, the integrand does not go ∞ because of the decay in $\frac{a_m}{a_m - z_1}$. Therefore, the only pole encountered is $z_1 = 0$ and the contribution is calculated to be

$$
-q^{-(k-1)} \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^{k-1}} \times \int \cdots \int \prod_{2 \le A < B \le k} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=2}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \right) \frac{dz_j}{z_j}.
$$

Continuing this process for $z_2, \ldots, z_k$ will bring down a factor of $-q^{-(k-2)}, \ldots, -q^0$ respectively, from which we conclude that $u(0;\vec{n}) = 1$ as desired.

Next, we show the case for half stationary initial data. Condition $(1)-(3)$ follows in exactly the same way as that for step initial data. Therefore, we only need to show that Condition (4) follows. To show this, we need a lemma.

LEMMA 2.2.1. *Fix $r \geq 1$. If X is a q-Geometric random variable with parameter $\alpha \in [0,1)$, then*

$$\mathbb{E}\left[q^{-rX}\right] = \prod_{i=1}^{r} \frac{1}{1-\alpha/q^i}$$

so long as $\alpha q^{-r} < 1$; and otherwise the expectation is infinite.

Proof of Lemma 2.2.1. Recall the q-Binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\left[q^{-rX}\right] &= (\alpha;q)_\infty \sum_{k=0}^{\infty} \frac{(\alpha/q^r)^k}{(q;q)_k} = (\alpha;q)_\infty \sum_{k=0}^{\infty} \frac{(0;q)_k}{(q;q)_k}(\alpha/q^r)^k \\ &= \frac{(\alpha;q)_\infty}{(\alpha q^{-r};q)_\infty} = \prod_{i=1}^{r} \frac{1}{1-\alpha/q^i}\end{aligned}$$

$\square$

For half-stationary initial data, we have that

$$\prod_{i=1}^{k} q^{x_{n_i}(0)+n_i} = \prod_{i=1}^{k} q^{-\sum_{m=1}^{n_i} X_m} = \prod_{i=1}^{k} \prod_{m=n_{i+1}+1}^{n_i} q^{-iX_m},$$

where X_i's are independent q-Geometric random variables. By taking expectation and using the independence, we have that

$$\mathbb{E}[H(\vec{x};\vec{y}(\vec{n})] = \prod_{i=1}^{k} \prod_{m=n_{i+1}+1}^{n_i} \mathbb{E}\left[q^{-iX_m}\right] = \prod_{i=1}^{k} \prod_{m=n_{i+1}+1}^{n_i} \prod_{j=1}^{i} \frac{a_m}{a_m - \alpha/q^j}.$$

It then suffices to show that $u(0;\vec{n}) = \mathbb{E}[H(\vec{x};\vec{y}(\vec{n})]$. We prove this by directly evaluating the nested contour integrals at $t=0$ using the same technique as in that for step initial data. Consider first the contour for z_1 and expand it to infinity. The only pole encountered is $z_1 = \alpha/q$. Evaluating the residue, we get that

$$u(0;\vec{n}) = \prod_{m=1}^{n_1} \frac{a_m}{a_m - \alpha/q} \times \frac{(-1)^{k-1} q^{(k-1)(k-2)/2}}{(2\pi i)^{k-1}} \times$$
$$\int \cdots \int \prod_{2 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=2}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j} \right) \frac{dz_j}{z_j - \alpha/q^2}$$

Continuing this process for $z_2, \ldots, z_k$, we get the equality as desired. $\square$

Chapter 3

Fredholm Determinants

In last chapter, we have derived a nested contour integral expression for $\mathbb{E}[\prod_{j=1}^{k} q^{x_{n_j}+n_j}]$ using some ODE systems. In this chapter we introduce Fredholm Determinants and try to transform the nested contour integrals into the form of a Fredholm determinants. However, from now onwards, we will only be focusing on the distribution of a single particle $x_n(t)$, since the joint distribution of an arbitrary collection of particles $x_{n_1}(t), \ldots, x_{n_l}(t)$ will involve some further complexities that will bring challenges into getting a relatively simple expression. To do this, we apply Corollary 2.2.1 with $\vec{n} = (n, n, \ldots, n)$, which results in $\mathbb{E}[\prod_{j=1}^{k} q^{x_{n_j}+n_j}] = q^{kn}\mathbb{E}[q^{kx_n(t)}]$.

3.1 Fredholm Determinants

We begin by quoting the definition of a Fredholm determinant from Definition 3.2.6 in [2].

DEFINITION 3.1. Fix a Hilbert space $L^2(X, \mu)$ where X is a measurable space and μ is a measure on X. When $X = \Gamma$, a simple smooth contour in $\mathbb{C}$, we write $L^2(\Gamma)$ where μ is understood to be the path measure along Γ divided by $2\pi i$. When X is the product of a discrete set D and a contour Γ, μ is understood to be the product of the counting measure on D and the path measure along Γ divided by $2\pi i$. Let K be an integral operator acting on $f(\cdot) \in L^2(X, \mu)$ by

$$(Kf)(x) = \int_X K(x,y) f(y) d\mu(y).$$

$K(x, y)$ is called the kernel of K. A formal Fredholm determinant expansion of $I + K$ is a formal series written as

$$\det(I+K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X det[K(x_i, x_j)]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$

If the above series is absolutely convergent, then we call this a numerical Fredholm determinant expansion.

In order to transform a nested contour integrals of similar forms to the ones given in *Theorem 2.2*, the idea is to deform the nested contours so that they coincide. However, there are two types of deformation, namely the Mellin-Barnes type and the Cauchy type. The strategy for the Mellin-Barnes type approach is to deform the contours in such a way that all of the poles corresponding to $z_A = qz_B$ for $A < B$ that were previously outside the contour for z_A are now enclosed in the contour for z_A. This is, in some sense, shrinking the contours. The Cauchy type approach does the opporsite that it enlarges the contours so that they coincide. Poles at $z_k = 0, k = 1, 2, \ldots$ are added to each of the contours in this process.

In the following sections, we introduce the two approaches in details and see how each of them can be applied to q-TASEP. But before that, we first identify the exact form of nested contours to be manipulated by Definition 3.1 in [2].

DEFINITION 3.2. For a meromorphic function $f(z)$ and $k \geq 1$ let $\mathbb{A}$ to be a fixed set of poles of f (not including 0) and assume that $q^m\mathbb{A}$ is disjoint from $\mathbb{A}$ for all $m \geq 1$. Define

$$\mu_k = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - qz_B} \prod_{i=1}^k f(z_i) \frac{dz_i}{z_i},$$

where the integration contour for z_A contains $\{qz_B\}_{B>A}$, the fixed set of poles $\mathbb{A}$ of

$f(z)$ but not 0 or any other poles.

Notice that when we take $f(z) = \left(\prod_{m=1}^{n} \frac{a_m}{a_m - z}\right) e^{(q-1)tz}$, then the ansatz solution of the free evolution equation with step initial data in *Theorem 2.2* is recovered:

$$u(t;\vec{n}) = \mu_k.$$

3.2 Mellin-Barnes type determinants

3.2.1 Mellin-Barnes type transformation

First, we introduce some notations to be used. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, we write $\lambda \vdash k$ if $\sum_i \lambda_i = k$ and $\lambda = 1^{m_1} 2^{m_2} \ldots$ if i appears m_i times in λ. Let $l(\lambda) = \sum_i m_i$ denote the number of non-zero elements of λ.

Recall that in Mellin-Barnes type determinants, the nested contours are deformed to coincide so that the contour for z_A now encircles poles at $z_A = qz_B$ for $A < B$. By this strategy, we transform the μ_k as defined in Definition 3.2 into the following form with the contours shrinked to coincide. The result is given in the following proposition.

PROPOSITION 3.1. *Define*

$$\gamma_k = k_q! \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \ldots}} \frac{1}{m_1! m_2! \ldots} \frac{(1-q)^k}{(2\pi i)^{l(\lambda)}}$$
$$\times \int \ldots \int \det \left[\frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{l(\lambda)} \times \prod_{j=1}^{l(\lambda)} f(w_j) f(q w_j) \ldots f(q^{\lambda_j - 1} w_j) dw_j \qquad (3.1)$$

where the w_j contours contain the same fixed set of singularities $\mathbb{A}$ of $f(z)$ but no other poles and $k_q! = \prod_{i=1}^{k} \frac{1-q^i}{1-q} = \frac{(q;q)_k}{(1-q)^k}$. Then for all $k \in \mathbb{Z}$,

$$\gamma_k = \mu_k$$

REMAEK 3.1. Note that in the definition of γ_k the contours have been deformed to coincide with each other containing the same fixed set of singularities.

Before moving on to the proof, we discuss a transformation of the RHS of (3.1). In γ_k, for each $\lambda \vdash k$ fixed, we have the following set of variables $\{w_1, w_2, \dots, w_{l(\lambda)}\}$. In order to evaluate the integral, we need to sum up all combinations of possible poles of each for the w_i's. From definition of the contour $C_{\mathbb{A}}$, we know that the contours for all of the w_i's contain the same fixed set of poles $\mathbb{A}$ and no other poles. Therefore, we need to choose from the set $\mathbb{A}$ all possible combinations of $l(\lambda)$ elements and sum up the corresponding residues.

Notice that for a particular λ with some $\lambda_i = \lambda_j$, interchanging the poles for w_i and w_j will result in identical residue. Since we know from the definition of m_i that there are m_i of λ_i for each i, we can cancel out the prefactor of $\frac{1}{m_1!m_2!\dots}$ and replace the summation over all possible subsets of $\mathbb{A}$ of size $l(\lambda)$ by the summation over the collection of subsets of $\mathbb{A}$ of size $m_1, m_2, \dots$ respectively, where $\sum m_i = l(\lambda)$ and $\sum i m_i = k$. Denote these subsets by $S_{\lambda,1}, S_{\lambda,2}, \dots$, and therefore we have $|S_{\lambda,1}| = m_1$, $|S_{\lambda,1}| = m_2$, etc.

Also, because of the determinant, zero contribution would occur if identical poles were chosen for different variable $w_i \neq w_j$ since it would result in two identical columns in the determinant. Therefore, we only need to consider the case that each pair of the S_i's does not have any elements in common.

By the above strategy, we can replace the summation over λ in (3.1) by a summation over disjoint subsets of $\mathbb{A}$ of size m_1, m_2, etc, where we only require that $m_1 + 2m_2 + \cdots = k$. (That $\sum m_i = l(\lambda)$ is removed because we are now summing over all λ.) Similarly, denote these subsets by $S_1, S_2, \dots$ respectively and write

$$S = (\underbrace{b_1, \dots, b_{m_1}}_{S_1}, \underbrace{b_{m_1+1}, \dots, b_{m_1+m_2}}_{S_2}, b_{m_1+m_2+1}, \dots),$$

where $b_i \in S_1$ for $1 \le i \le m_1$, $b_i \in S_2$ for $m_1 < i \le m_2$ and so on. Denote the set of all such collections of subsets S to be $\mathcal{S}$. Then by evaluating the corresponding residues (3.1) can be transformed to

$$\gamma_k = \sum_{S\in\mathcal{S}} k_q!(1-q)^k \prod_{j=1}^{m_1+m_2+\dots} Res_{w=b_j} f(w)f(qb_j)\dots f(q^{\lambda(b_j)}b_j) det \left[\frac{1}{b_i q^{\lambda(b_i)} - b_j}\right]_{i,j=1}^{m_1+m_2+\dots} \tag{3.2}$$

To illustrate what has been discussed above, we take $\mathbb{A} = \{a_1, a_2, a_3\}$ and $k = 3$ for an example. In this case, all possible $\lambda's$ are $(3), (2,1), (1,1,1)$. Prior to the transformation, the summation over λ, if we take into account the effect that assigning equal poles to two different variables will result in zero contribution, would have been the following:

(1) For $\lambda = (3)$, there are only one variable w_1 and we have three poles for it, namely,

$$(w_1) = (a_1), (a_2), (a_3);$$

(2) For $\lambda = (2,1)$, there are two variables w_1 and w_2. We have the following all possible assignments of poles:

$$(w_1, w_2) = (a_1, a_2), (a_1, a_3), (a_2, a_1), (a_2, a_3), (a_3, a_1), (a_3, a_2);$$

(3) For $\lambda = (1,1,1)$, there are three variables w_1, w_2, w_3. We have the following all possible assignments of poles:

$$(w_1, w_2, w_3) = (a_1, a_2, a_3), (a_1, a_3, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2), (a_3, a_2, a_1).$$

However, notice that all the assignments in (3) actually results in the same residue because of the determinant, we can combine the case of (3) to be $3! = 6$ times the residue at $(w_1, w_2, w_3) = (a_1, a_2, a_3)$. The factor 6 cancels out the prefactor of $\frac{1}{3!0!\dots} = \frac{1}{6}$.

Also, in this case, the set $\mathcal{S}$ is

$$\mathcal{S} = \{ \underbrace{\{a_1\}, \{a_2\}, \{a_3\}}_{m_3=1, m_i=0 \text{ for } i\neq 3},$$

$$\underbrace{\{a_1,a_2\},\{a_1,a_3\},\{a_2,a_1\},\{a_2,a_3\},\{a_3,a_1\},\{a_3,a_2\}}_{m_1=1,m_2=1,m_i=0 \text{ for } i\neq 1,2},$$
$$\underbrace{\{a_1,a_2,a_3\}}_{m_1=3,m_i=0 \text{ for } i\neq 1}\quad\}.$$

We therefore have the identity for the two types of summation.

With this transformation, we are now ready to present the proof of the proposition.

Proof of Proposition 3.1. The proof is by induction on k. For $k=1$, by definition,

$$\mu_1 = -\frac{1}{2\pi i}\int f(z_1)\frac{dz_1}{z_1}.$$

Moreover, the only λ is $\lambda=(1,0)$ and γ_k reduces to

$$\frac{1-q}{2\pi i}\int\frac{1}{qw_1-w_1}f(w_1)dw_1 = -\frac{1}{2\pi i}\int f(w_1)\frac{dw_1}{w_1}.$$

Comparing the two expression, we get that $\mu_1=\gamma_1$.

For the induction step, let $k\in\mathbb{Z}_+$ and assume that $\mu_{k-1}=\gamma_{k-1}$. Let J denote an index set of $\mathbb{A}$ ($J=\{1,\dots,|\mathbb{A}|\}$) and recall the definition of μ_k in Definition 3.2 that

$$\mu_k = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k}\int\cdots\int\prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-qz_B}\prod_{i=1}^{k}f(z_i)\frac{dz_i}{z_i},$$

we evaluate the integral over z_k using residue calculus to get

$$\begin{aligned}\mu_k &= (-q^{k-1})\sum_{j\in J}\frac{Res_{z=a_j}f(z)}{a_j}(-1)^{k-1}\frac{q^{(k-1)(k-2)/2}}{(2\pi i)^{k-1}}\\ &\times\int\cdots\int\prod_{1\le A<B\le k-1}\frac{z_A-z_B}{z_A-qz_B}\prod_{i=1}^{k-1}\frac{z_i-a_j}{z_i-qa_j}\frac{f(z_i)}{z_i}dz_i.\end{aligned}\tag{3.3}$$

Notice that the integral on the right hand side of (3.3) is an $k-1$ fold nested contour integral, to which we can apply, for each j, our induction assumption with $\tilde{f}_j(z)=$

$f(z)\frac{z-a_j}{z-qa_j}$ and the new sets of poles $\tilde{\mathbb{A}}_j = (\mathbb{A} \setminus \{a_j\}) \cup \{qa_j\}$. The resulting form is

$$\mu_k = (-q^{k-1}) \sum_{j \in J} \frac{Res_{z=a_j} f(z)}{a_j} \times (k-1)_q! \sum_{\substack{\lambda \vdash k-1 \\ \lambda = 1^{\tilde{m}_1} 2^{\tilde{m}_2} \dots}} \frac{1}{\tilde{m}_1! \tilde{m}_2! \dots} \frac{(1-q)^{k-1}}{(2\pi i)^{l(\lambda)}}$$
$$\times \int \dots \int det \left[\frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{l(\lambda)} \times \prod_{t=1}^{l(\lambda)} \tilde{f}_j(w_t) \tilde{f}_j(qw_t) \dots \tilde{f}_j(q^{\lambda_t - 1} w_t) dw_t \quad (3.4)$$

From the transformation discussed just before the proof, we can see that (3.4) can be transformed to

$$\mu_k = (-q^{k-1})(k-1)_q!(1-q)^{k-1} \sum_{j \in J} \sum_{\tilde{S} \in \tilde{\mathcal{S}}_j} \frac{1}{a_j} det \left[\frac{1}{\tilde{b}_i q^{\tilde{\lambda}(\tilde{b}_i)} - \tilde{b}_l} \right]_{\tilde{b}_i, \tilde{b}_l \in \tilde{S}}$$
$$\times Res(\tilde{S}) \times Res_{w=a_j} f(w), \quad (3.5)$$

where for $\tilde{S}$ such that $qa_j \in \tilde{S}$,

$$Res(\tilde{S}) = \prod_{\tilde{b} \in \tilde{S} \setminus \{qa_j\}} Res_{w=\tilde{b}} \tilde{f}_j(w) \tilde{f}_j(q\tilde{b}) \dots \tilde{f}_j(q^{\tilde{\lambda}(\tilde{b})-1} \tilde{b})$$
$$\times Res_{w=qa_j} \tilde{f}_j(w) \tilde{f}_j(q^2 a_j) \dots \tilde{f}_j(q^{\tilde{\lambda}(qa_j)} a_j) \quad (3.6)$$

and otherwise

$$Res(\tilde{S}) = \prod_{\tilde{b} \in \tilde{S}} Res_{w=\tilde{b}} \tilde{f}_j(w) \tilde{f}_j(q\tilde{b}) \dots \tilde{f}_j(q^{\tilde{\lambda}(\tilde{b})-1} \tilde{b}),$$

where $\tilde{\mathcal{S}}_j$ denotes the set of all collections $\tilde{S}$ of subsets $\tilde{S}_{j,1}, \tilde{S}_{j,2}, \dots$ of $\tilde{\mathbb{A}}_j$, where $\tilde{S}_{j,k}$ is of size $\tilde{m}_k$ and $\tilde{m}_1 + 2\tilde{m}_2 + \cdots = k-1$. Also, we reorder the elements in $\tilde{S}$ to be $\tilde{S} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{\tilde{m}_1}, \dots)$ so that the first $\tilde{m}_1$ elements belong to $\tilde{S}_{j,1}$ etc.

Note that for $\tilde{b} \neq a_j$ such that $\tilde{\lambda}(\tilde{b}) = \lambda$,

$$Res_{w=\tilde{b}} \tilde{f}_j(w) \tilde{f}_j(q\tilde{b}) \dots \tilde{f}_j(q^{\lambda-1} \tilde{b})$$
$$= \frac{\tilde{b} - a_j}{\tilde{b} - qa_j} \frac{q\tilde{b} - a_j}{q\tilde{b} - qa_j} \dots \frac{q^{\lambda-1}\tilde{b} - a_j}{q^{\lambda-1}\tilde{b} - qa_j} Res_{w=\tilde{b}} f(w) f(q\tilde{b}) \dots f(q^{\lambda-1}\tilde{b})$$

$$= \frac{q^{\lambda-1}\tilde{b} - a_j}{\tilde{b} - qa_j} q^{1-\lambda} Res_{w=\tilde{b}} f(w) f(q\tilde{b}) \dots f(q^{\lambda-1}\tilde{b}) \tag{3.7}$$

and

$$\begin{aligned}
&Res_{w=qa_j} \tilde{f}_j(w) \tilde{f}_j(q^2 a_j) \dots \tilde{f}_j(q^{\tilde{\lambda}(qa_j)} a_j) \\
&= (q-1)a_j \frac{q^2 a_j - a_j}{q^2 a_j - qa_j} \dots \frac{q^{\tilde{\lambda}(qa_j)} a_j - a_j}{q^{\tilde{\lambda}(qa_j)} a_j - qa_j} f(qa_j) \dots f(q^{\tilde{\lambda}(qa_j)} a_j) \\
&= a_j (q^{\tilde{\lambda}(qa_j)} - 1) q^{2-\tilde{\lambda}(qa_j)} f(qa_j) \dots f(q^{\tilde{\lambda}(qa_j)} a_j)
\end{aligned} \tag{3.8}$$

Applying (3.7) and (3.8) to (3.6), we get for $\tilde{b} \neq a_j$ such that $\tilde{\lambda}(\tilde{b}) = \lambda$,

$$\begin{aligned}
Res(\tilde{S}) &= \prod_{\tilde{b} \in \tilde{S} \setminus \{qa_j\}} \frac{q^{\tilde{\lambda}(\tilde{b})-1}\tilde{b} - a_j}{\tilde{b} - qa_j} q^{1-\tilde{\lambda}(\tilde{b})} Res_{w=\tilde{b}} f(w) f(q\tilde{b}) \dots f(q^{\tilde{\lambda}(\tilde{b})-1}\tilde{b}) \\
&\quad \times a_j (q^{\tilde{\lambda}(qa_j)} - 1) q^{2-\tilde{\lambda}(qa_j)} f(qa_j) \dots f(q^{\tilde{\lambda}(qa_j)} a_j) \\
&= \prod_{l \neq j} \frac{q^{\tilde{\lambda}(b_l)-1} b_l - a_j}{b_l - qa_j} q^{1-\tilde{\lambda}(b_l)} Res_{w=b_l} f(w) f(qb_l) \dots f(q^{\tilde{\lambda}(b_l)-1} b_l) \\
&\quad \times a_j (q^{\tilde{\lambda}(qa_j)} - 1) q^{2-\tilde{\lambda}(qa_j)} f(qa_j) \dots f(q^{\tilde{\lambda}(qa_j)} a_j)
\end{aligned} \tag{3.9}$$

For convenience purpose, for each $\tilde{S} = (\tilde{S}_{j,1}, \tilde{S}_{j,2}, \dots) \in \tilde{\mathcal{S}}_j$, we map it to a new set $S_j = (S_{j,1}, S_{j,2}, \dots)$ in the following manner:
If $qa_j \in \tilde{S}_{j,l}$ for some l, then

$$S_{j,l} = \tilde{S}_{j,l} \setminus \{qa_j\}, S_{j,l+1} = \tilde{S}_{j,l+1} \cup \{a_j\}, S_{j,m} = \tilde{S}_{j,m} \text{ for all } m \neq l, l+1,$$

Otherwise,

$$S_{j,1} = \tilde{S}_{j,1} \cup \{a_j\}, S_m = \tilde{S}_m \text{ for all } m > 1.$$

With the new set S_j, we can then write $Res_{w=a_j} f(w) Res(\tilde{S})$ in the following form

$$Res_{w=a_j} f(w) Res(\tilde{S}) = a_j (q^{\tilde{\lambda}(qa_j)} - 1) q^{2-\tilde{\lambda}(qa_j)} \prod_{l \neq j} \frac{q^{\tilde{\lambda}(b_l)-1} b_l - a_j}{b_l - qa_j} q^{1-\tilde{\lambda}(b_l)}$$

$$\times \prod_{b\in S_{j,1}\cup S_{j,2}\dots} Res_{w=b} f(w) f(qb)\dots f(q^{\tilde{\lambda}(b)-1}b) \tag{3.10}$$

Notice that (3.10) always holds regardless of whether qa_j is in $\tilde{S}$, and that the union of all such S_j for each j is just $\mathcal{S}$ for γ_k. Therefore, μ_k in (3.5) can be re-written as

$$\mu_k = \sum_{S\in\mathcal{S}} (k-1)_q!(1-q)^{k-1}(-q^{k-1}) \prod_{j=1}^{m_1+m_2+\dots} Res_{w=b_j} f(w) f(qb_j)\dots f(q^{\lambda(b_j)-1}b_j)$$
$$\times \sum_{j=1}^{m_1+m_2+\dots} (q^{\lambda(b_j)-1}-1)q^{2-\lambda(b_j)} \prod_{l\neq j} \frac{q^{\lambda(b_l)-1}b_l - b_j}{b_l - qb_j} q^{1-\lambda(b_l)} det\left[\frac{1}{b_i q^{\lambda(b_i)} - b_l q^{\delta_{l,j}}}\right]_{i,l=1}^{m_1+m_2+\dots},$$

where $\delta_{l,j} = 1$ if $l = j$ and 0 otherwise.

Hence, in order to complete the proof, what we need to show is $\mu_k = \gamma_k$. After tidying up the expressions, we are only left to show that for each $S \in \mathcal{S}$,

$$\sum_{j=1}^{m_1+m_2+\dots} (q-q^{\lambda_j}) \prod_{l\neq j} \frac{qb_j - q^{\lambda_l}b_l}{qb_j - b_l} det\left[\frac{1}{b_i q^{\lambda_i} - b_l q^{\delta_{l,j}}}\right]_{i,l=1}^{m_1+m_2+\dots}$$
$$= (1-q^{\sum\lambda_i}) det\left[\frac{1}{b_i q^{\lambda_i} - b_l}\right]_{i,l=1}^{m_1+m_2+\dots}, \tag{3.11}$$

where $\lambda_i = \lambda(b_i)$. Recall the Cauchy determinant

$$det\left[\frac{1}{x_i + y_i}\right] = \frac{V(x_i)V(x_j)}{\prod_{i,j}(x_i + x_j)},$$

where $V(x_i) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. Using the Cauchy determinant we obtain that (3.11) is equivalent to

$$\sum_{j=1}^{m_1+m_2+\dots} (q-q^{\lambda_j}) \prod_{l\neq j} \frac{qb_j - q^{\lambda_l}b_l}{qb_j - b_l} \frac{V(b_i q^{\lambda_i})V(b_l q^{\delta_{l,j}})}{\prod_{i,l}(b_l q^{\delta_{l,j}} - b_i q^{\lambda_i})} = (1-q^{\sum\lambda_i}) \frac{V(b_i q^{\lambda_i})V(b_l)}{\prod_{i,l}(b_l - b_i q^{\lambda_i})}$$

After cancelling out some terms, we are left with

$$\sum_{j\geq 1} \frac{\prod_{l\geq 1}(b_j - b_l q^{\lambda_l})}{\prod_{l\neq j}(b_j - b_l)} \frac{1}{b_j} = 1 - q^{\sum\lambda_i} \tag{3.12}$$

Take

$$g(z) = \prod_{l\geq 1} \frac{z - b_l q^{\lambda_l}}{z - b_l} \frac{1}{z}.$$

Then the LHS of (3.12) is just the sum of the residues of the function $g(z)$ at the points $z = b_l$ for $l \geq 1$. Moreover, notice that $Res_{z=0} g(z) = q^{\sum \lambda_l}$ and $Res_{z=\infty} g(z) = 1$ so the RHS is just the difference of the residues of the function $g(z)$ at the point $z = \infty$ and $z = 0$. Recalling the fact that for f holomorphic in $\mathbb{C}$ except for isolated singularities $a_1, \ldots, a_n$, then

$$Res_{z=\infty} f(z) = \sum_{k=1}^{n} Res_{z=a_k} f(z),$$

we complete the proof. □

With *Proposition 3.1*, we have successfully transformed the nested contours so that they can coincide with each other. That is, while the pervious contour for z_A contains poles at $\mathbb{A}$ and $\{qz_B\}_{B>A}$, current contour for w_A only contain poles at $\mathbb{A}$. Therefore, the contours for w_i, $i = 1, \ldots, k$ can now be chosen to be the same contour, denoted as $C_{\mathbb{A}}$ that contains poles only at $\mathbb{A}$ and no other poles.

3.2.2 Transformation to Fredholm determinants

Through out the chapter, the complex power function z^s for $s \in \mathbb{C}$ is defined with respect to a branch cut along $z \in \mathbb{R}^-$.

We introduce some contours that will be used in later propositions. First we define the contour $C_{1,2,\ldots}$ to be an infinite contour negatively oriented that encloses $1, 2, \ldots$ and no poles of $f(q^s w)$ for all $s \in \mathbb{Z}$. One possible such contour is $\frac{1}{2} + i\mathbb{R}$ oriented from $\frac{1}{2} - i\infty$ to $\frac{1}{2} + i\infty$.

Moreover, for any $R > 0, d > 0$, we define the contour $D_{R,d}$ to be such that it goes by straight lines from $R - i\infty$ to $R - id$, to $1/2 - id$, to $1/2 + id$, to $R + id$ and lastly to $R + i\infty$. Refer to *Figure* **??** for an illustration of $D_{R,d}$. With $D_{R,d}$, for any $k \in \mathbb{Z}_{>0}$, we further define the contour $D_{R,d;k}$ to be as follows: let $p, \bar{p}$ (let $Im(p) > 0$) be the two points at which the contour $D_{R,d}$ and the circle centered at 0 with radius $k + 1/2$ intersect. Then $D_{R,d;k}$ is the union of the portion of $D_{R,d}$ inside the circle with reversed orientation, with the arc from $\bar{p}$ to p (oriented counterclockwise).

The following proposition writes a generating function of μ_k into the Fredholm determinants form.

PROPOSITION 3.2. *Consider μ_k as in (3.1) with closed contours $C_{\mathbb{A}}$ for $k = 1, 2, \ldots$. Then the following formal equality holds:*

$$\sum_{k \geq 0} \mu_k \frac{\zeta^k}{k_q!} = \det(I + K^1_\zeta),$$

where $K^1_\zeta : L^2(\mathbb{Z}_{>0} \times C_{\mathbb{A}}) \to L^2(\mathbb{Z}_{>0} \times C_{\mathbb{A}})$ is defined by its integral kernel

$$K^1_\zeta(n_1, w_1; n_2, w_2) = \frac{(1-q)^{n_1} \zeta^{n_1} f(w_1) f(qw_1) \ldots f(q^{n_1 - 1} w_1)}{q_{n_1} w_1 - w_2},$$

where the identity is formal. It also holds numerically with the following condition that

(1) for all $w, w' \in C_{\mathbb{A}}$ and $n \geq 1$, $|q^n w - w'|^{-1}$ is uniformly bounded above;

(2) $\exists M > 0$ constant such that for all $w \in C_{\mathbb{A}}$ and all $n \geq 0$, $|f(q^n w)| \leq M$ and $|(1-q)\zeta| < M^{-1}$.

We defer the proof of the proposition and state the immediate result continuing from *Proposition 3.2*.

PROPOSITION 3.3. *Assume $f(w) = g(w)/g(qw)$ for some function g. Then the following formal identity holds:*

$$\det(I + K^1_\zeta) = \det(I + K^2_\zeta),$$

where $\det(I + K^1_\zeta)$ is given in Proposition 3.2 and $K^2_\zeta : L^2(C_{\mathbb{A}}) \to L^2(C_{\mathbb{A}})$ is given by its integration kernel

$$K^2_\zeta(w, w') = \frac{1}{2\pi i} \int_{C_{1,2,\ldots}} \Gamma(-s)\Gamma(1+s)(-(1-q)\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds.$$

The above identity also holds numerically given the condition that $\det(I+K^1_\zeta)$ *is convergent and that* $C_{1,2,\ldots}$ *is chosen as* $D_{R,d}$ *with* $d>0$ *and* $R>0$ *such that*

$$inf_{\substack{w,w'\in C_{\mathbb{A}}\\ k\in\mathbb{Z}_{>0},s\in D_{R,d;k}}} |q^s w - w'| > 0 \text{ and } sup_{\substack{w,w'\in C_{\mathbb{A}}\\ k\in\mathbb{Z}_{>0},s\in D_{R,d;k}}} \left|\frac{g(w)}{g(q^s w)}\right| < \infty.$$

In this case the function $\det(I+K^2_\zeta)$ *of* ζ *is analytic for all* $\zeta\notin\mathbb{R}_+$*.*

Proof of Proposition 3.2. Write

$$I_{l(\lambda)}(\lambda;w;\zeta) = \frac{1}{(2\pi i)^{l(\lambda)}} det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{l(\lambda)} \prod_{j=1}^{l(\lambda)} (1-q)^{\lambda_j}\zeta^{\lambda_j} f(w_j) f(qw_j)\ldots f(q^{\lambda_j-1}w_j).$$

Then for the μ_k given in *Proposition 3.1*, we have

$$\begin{aligned}
\mu_k\frac{\zeta^k}{k_q!} &= \sum_{\substack{\lambda\vdash k\\ \lambda=1^{m_1}2^{m_2}\ldots}} \frac{1}{m_1!m_2!\ldots}\int\ldots\int\prod_{j=1}^{l(\lambda)} I_{l(\lambda)}(\lambda;w;\zeta)dw_j\\
&= \sum_{\substack{\lambda\vdash k\\ \lambda=1^{m_1}2^{m_2}\ldots}} \frac{1}{(m_1+m_2+\ldots)!}\frac{(m_1+m_2+\ldots)!}{m_1!m_2!\ldots}\int\ldots\int\prod_{j=1}^{l(\lambda)} I_{l(\lambda)}(\lambda;w;\zeta)dw_j\\
&= \sum_{l(\lambda)\geq 0}\frac{1}{l(\lambda)!}\sum_{\substack{\sum m_i=l(\lambda)\\ \sum im_i=k}} \frac{(m_1+m_2+\ldots)!}{m_1!m_2!\ldots}\int\ldots\int\prod_{j=1}^{l(\lambda)} I_{l(\lambda)}(\lambda;w;\zeta)dw_j,
\end{aligned}$$

where all the contour integrals are with respect to the contour $C_{\mathbb{A}}$.

Notice that the coefficient $\frac{(m_1+m_2+\ldots)!}{m_1!m_2!\ldots}$ is a multinomial coefficient. Therefore, the inner summation can be replaced by

$$\sum_{\substack{n=(n_1,\ldots,n_{l(\lambda)})\\ \sum n_i=k}} \int\ldots\int\prod_{j=1}^{l(\lambda)} I_{l(\lambda)}(n;w;\zeta)dw_j.$$

We can replace $l(\lambda)$ simply by L since we have got rid of the dependence on λ which is now on n. Also, summing over all $k\geq 0$ will remove the restriction on $\sum n_i = k$.

This gives us the following form

$$\sum_{k\geq 0}\mu_k\frac{\zeta^k}{k_q!}=\sum_{L\geq 0}\frac{1}{L!}\sum_{n_1,\dots,n_L\in\mathbb{Z}_{>0}}\frac{1}{(2\pi i)^L}\int\dots\int det\left[\frac{1}{q^{n_i}w_i-w_j}\right]_{i,j=1}^{L}$$
$$\times\prod_{j=1}^{L}(1-q)^{n_j}\zeta^{n_j}f(w_j)\dots f(q^{n_j-1}w_j)dw_j \tag{3.13}$$

This is exactly the definition of $\det(I+K_1^{\zeta})_{L(\mathbb{Z}_{>0}\times C_{\mathbb{A}})}$. This proves the formal equality of the two sides.

Next, we show the numerical equality also holds under the conditions given. For this, we only need to show that under the conditions, (3.13) is absolutely convergent. By condition (1), there exists a constant $B>0$ such that $\frac{1}{q^{n_i}w_i-w_j}\leq B$ for all $w_i,w_j\in C_{\mathbb{A}}$ and all $n_i\in\mathbb{Z}_{>0}$. Therefore, by Hadamard's bound,

$$\left|det\left[\frac{1}{q^{n_i}w_i-w_j}\right]_{i,j=1}^{L}\right|\leq B^L L^{L/2}.$$

Moreover, by condition (2),

$$\left|\prod_{j=1}^{L}(1-q)^{n_j}\zeta^{n_j}f(w_j)\dots f(q^{n_j-1}w_j)\right|\leq 1.$$

Therefore, we have the following inequality that

$$\left|\int\dots\int\det\left[\frac{1}{q^{n_i}w_i-w_j}\right]_{i,j=1}^{L}\prod_{j=1}^{L}(1-q)^{n_j}\zeta^{n_j}f(w_j)\dots f(q^{n_j-1}w_j)\frac{dw_j}{2\pi i}\right|\leq(BC)^L L^{L/2},$$

where C is the length of the fixed contour $C_{\mathbb{A}}$. Summing over $L\geq 0$, we get that (3.13) is uniformly bounded above by

$$\sum_{L\geq 0}\frac{1}{L!}(BC)^L L^{L/2},$$

which converges finitely due to the $L!$ in the denominator. As a result, we have the numerical identity. $\square$

Proof of Proposition 3.3. We first show the formal identity. Recall a result that for $k \in \mathbb{Z}_{>0}$,

$$Res_{z=k}\Gamma(-z)\Gamma(1+z) = (-1)^{k+1}.$$

Therefore, we have that

$$\begin{aligned} K^2_\zeta(w,w') &= \frac{1}{2\pi i}\int_{C_{1,2,\ldots}} \Gamma(-s)\Gamma(1+s)(-(1-q)\zeta)^s \frac{g(w)}{g(q^s w)}\frac{1}{q^s w - w'}ds \\ &= \sum_{n\in\mathbb{Z}_{>0}} ((1-q)\zeta)^n \frac{g(w)}{g(q^n w)}\frac{1}{q^n w - w'}. \end{aligned}$$

and hence, by definition, the Fredholm determinants $\det(I + K^2_\zeta)$ can be expanded as

$$1 + \sum_{L=1}^{\infty}\frac{1}{L!}\int \cdots \int \sum_{n_1,\ldots,n_L\geq 1} det\left[\frac{1}{q^{n_i}w_i - w_j}\right]_{i,j=1}^{L} \prod_{j=1}^{L}((1-q)\zeta)^{n_j}\frac{g(w)}{g(q^{n_j}w)}\frac{dw_j}{2\pi i}.$$

This is exactly the same as (3.13). We have thus proven the formal identity.

Next we show the numerical identity under the conditions given. This follows from the fact that the additional conditions ensure that the kernel $K^2_\zeta(w,w')$ absolutely converges so that $\det(I + K^2_\zeta)$ is well defined numerically. This can bee seen from the fact that

$$\begin{aligned} K^2_\zeta(w,w') &= \sum_{n\in\mathbb{Z}_{>0}} ((1-q)\zeta)^n \frac{g(w)}{g(q^n w)}\frac{1}{q^n w - w'} \\ &\leq \sum_{n\in\mathbb{Z}_{>0}} M^{-n} B_1 B_2 \\ &= B_1 B_2 \frac{1}{1-M}, \end{aligned}$$

where B_1, B_2 is such that $\left|\frac{g(w)}{g(q^n w)}\right| \leq B_1$ and $\left|\frac{1}{q^n w - w'}\right| \leq B_2$ for all $n \geq 0$ and $w, w' \in C_{\mathbb{A}}$. □

3.2.3 Application to q-TASEP

In Section 3.2.1 and 3.2.2, we discussed a general manipulation of nested contour integrals of the form μ_k using the Mellin-Barnes type determinant. Recall the result from Corollary 2.2.1 that for step initial data q-TASEP and $\vec{n} \in W^k_{>0}$,

$$\mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}+n_j}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \times \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{n_j} \frac{a_m}{a_m - z_j}\right) e^{(q-1)t z_j} \frac{dz_j}{z_j},$$

where the integration contour for z_A contains $\{qz_B\}_{B>A}$ and all $a'_m s$ but not 0. As remarked, if we were to take

$$f(z) = \left(\prod_{m=1}^{n} \frac{a_m}{a_m - z}\right) e^{(q-1)tz},$$

then $\mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}+n_j}\right] = \mu_k$. Further more, in order to apply *Proposition 3.3*, we need a function $g(w)$ such that $f(w) = \frac{g(w)}{g(qw)}$. In this case, we can choose

$$g(w) = \prod_{m=1}^{n} \frac{1}{(w/a_m; q)_\infty} e^{-tw}.$$

It's easy to verify that the choice of $g(w)$ is valid. Therefore, applying *Proposition 3.2* and then *Proposition 3.3* to $\mu_k = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}+n_j}\right]$, we get that for $\vec{n} \in W^k_{>0}$ and for all $t \ge 0$,

$$\sum_{k\ge 0} \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right] \frac{\zeta^k}{k_q!} = \det(I + K^2_\zeta) \tag{3.14}$$

We state the result as a theorem.

THEOREM 3.1. *Fix $0 < q < 1$ and $n \ge 1$. Consider q-TASEP with step initial data and jump rate parameters $a_i > 0$. Denote $\mathbb{A}$ to be the set of all a_i, $i = 1, \ldots, n$, namely, $\mathbb{A} = \{a_i : i = 1, \ldots, n\}$ and define $C_{\mathbb{A}}$ to be a closed contour which contains $\mathbb{A}$ and not 0 such that $w \ne q^n w'$ for any $n \ge 1$ for all $w, w' \in C_{\mathbb{A}}$. Then for all $t \ge 0$*

and $\zeta \in \{\zeta : |\zeta| < 1, \zeta \notin \mathbb{R}_+\}$, *the following characterizes the distribution of* $x_n(t)$*:*

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right] = \det(I + K_\zeta^{q-TASEP}), \tag{3.15}$$

where $\det(I + K_\zeta^{q-TASEP})$ *is the Fredholm determinant of* $K_\zeta^{q-TASEP} : L^2(C_{\mathbb{A}}) \to L^2(\mathbb{A})$*. The operator* $K_\zeta^{q-TASEP}$ *is defined in terms of its integral kernel*

$$K_\zeta^{q-TASEP}(w, w') = \frac{1}{2\pi i}\int_{C_{1,2,\dots}} \Gamma(-s)\Gamma(1+s)(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds.$$

With the additional conditions that

(1) for all $w, w' \in C_{\mathbb{A}}$ *and* $n \geq 1$, $|q^n w - w'|^{-1}$ *is uniformly bounded above;*

(2) $\exists M > 0$ *constant such that for all* $w \in C_{\mathbb{A}}$ *and all* $n \geq 0$, $|f(q^n w)| \leq M$ *and* $|(1-q)\zeta| < M^{-1}$.

(3) $C_{1,2,\dots}$ *is chosen as* $D_{R,d}$ *with* $d > 0$ *and* $R > 0$ *such that*

$$inf_{\substack{w,w' \in C_{\mathbb{A}} \\ k \in \mathbb{Z}_{>0}, s \in D_{R,d;k}}} |q^s w - w'| > 0 \text{ and } sup_{\substack{w,w' \in C_{\mathbb{A}} \\ k \in \mathbb{Z}_{>0}, s \in D_{R,d;k}}} \left|\frac{g(w)}{g(q^s w)}\right| < \infty.$$

(3.15) *also holds numerically.*

REMAEK 3.2. It's remarked that from the results above, the probability that $\mathbb{P}(x_n(t) = k)$ can be uniquely determined.

Proof. We continue from the result as stated in (3.14). If we take $\vec{n} = (n, n, \dots, n)$, then for $|\zeta| < 1$,

$$\begin{aligned}
\sum_{k \geq 0} \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right] \frac{\zeta^k}{k_q!} &= \mathbb{E}\left[\sum_{k \geq 0} (\zeta q^{x_n(t)+n})^k \frac{1}{k_q!}\right] \\
&= \mathbb{E}\left[\sum_{k \geq 0} \frac{[(1-q)\zeta q^{x_n(t)+n}]^k}{(q;q)_k}\right] \\
&= \mathbb{E}\left[\frac{1}{((1-q)\zeta q^{x_n(t)+n};q)_\infty}\right] \text{ (by the q-Binomial theorem)}
\end{aligned}$$

Therefore, (3.14) now becomes

$$\mathbb{E}\left[\frac{1}{((1-q)\zeta q^{x_n(t)+n};q)_\infty}\right] = \det(I+K_\zeta^2).$$

Absorbing the $(1-q)$ factor into ζ, we get the formal identity that

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right] = \det(I+K_\zeta^{q-TASEP}),$$

where the integral kernel of $K_\zeta^{q-TASE}(w,w')$ is defined by the integral kernel of $K_\zeta^2(w,w')$ with ζ replaced by $(1-q)\zeta$. That is,

$$K_\zeta^{q-TASEP}(w,w') = \frac{1}{2\pi i}\int_{C_{1,2,\dots}} \Gamma(-s)\Gamma(1+s)(-\zeta)^s \frac{g(w)}{g(q^s w)}\frac{1}{q^s w - w'}ds.$$

Numerical identity follows from the proof of *Proposition 3.2* and 3.3. □

REMAEK 3.3. It is remarked that the domain for ζ can actually be enlarged to be $\mathbb{C}\setminus\mathbb{R}_+$. Please refer to [2] *Theorem 3.2.11* for details.

3.3 Cauchy type Fredholm determinants

In Mellin-Barnes type Fredholm determinants, we deform the contours so that the poles at $z_A = qz_B$ for $B > A$ are encountered. In Cauchy type Fredholm determinants, however, we do the opposite, which is to deform the contours so that poles at $z_A = 0$ are encountered for all $k \in \mathbb{Z}_{>0}$. This is done in the following manner:

We first deform the contour for z_1 so that it passes through a pole at $z_1 = 0$. Then we deform the contour for z_2 so that it also passes through the pole at $z_2 = 0$ and that qz_2 is contained in the contour for z_1. After this, the contour for z_3 is deformed so that $z_3 = 0$ is encountered and that qz_3 is contained in both the contour for z_1 and z_2. The same procedure is repeated for all the rest of the contours.

3.3.1 Cauchy type transformation

We identify the form after the deformation as in the following definition.

DEFINITION 3.3. For a meromorphic function $f(z)$ and $k \geq 1$ set $\mathbb{A}$ to be a fixed set of poles of f (not including 0) and assume that $q^m\mathbb{A}$ is disjoint from $\mathbb{A}$ for all $m \geq 1$. Define

$$\tilde{\mu}_k = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - qz_B} \prod_{i=1}^{k} f(z_i)\frac{dz_i}{z_i},$$

where the integration contour for z_A contains $\{qz_B\}_{B>A}$, the fixed set of poles $\mathbb{A}$ of $f(z)$ and 0 but no other poles.

REMAEK 3.4. Note that the definitions of μ_k and $\tilde{\mu}_k$ only differ by the inclusion of poles at $z_k = 0$ for $\tilde{\mu}_k$.

We provide the relationship between μ_k and $\tilde{\mu}_k$ in the following proposition.

PROPOSITION 3.4. *Assume $f(0) = 1$. Then*

$$\tilde{\mu}_k = (-1)^k q^{k(k-1)/2} \sum_{j=0}^{k} \binom{k}{j}_{q^{-1}} (-1)^j q^{-j(j-1)/2}\mu_j,$$

where $\binom{n}{k}_q = \frac{n_q!}{k_q!(n-k)_q!}$.

Proof. Define a triangular array of integrals $\nu_{j,k}$ by

$$\nu_{j,k} = \frac{1}{(2\pi i)^{k+j}} \int \cdots \int \prod_{1\leq A<B\leq k+j} \frac{z_A - z_B}{z_A - qz_B} \prod_{m=1}^{k+j} f(z_m)\frac{dz_m}{z_m},$$

where the contours for $z_1, \ldots, z_k$ contains 0 and those for $z_{k+1}, \ldots, z_{k+j}$ do not contain 0. Then we can recover μ_k and $\tilde{\mu}_k$ by the relationship

$$\mu_j = (-1)^j q^{j(j+1)/2}\nu_{j,0}, \quad \tilde{\mu}_k = (-1)^k q^{k(k-1)/2}\nu_{0,k}.$$

If we try to deform the contour of z_k to encounter only the pole at $z_k = 0$ so that it does not contain 0, we get that

$$\begin{aligned}\nu_{j,k} &= \nu_{j+1,k-1} + Res_{z_k=0}\left(\prod_{k<B\le k+j}\frac{z_k - z_B}{z_k - qz_B}\frac{f(z_k)}{z_k}\right)\nu_{j,k-1}\\ &= \nu_{j+1,k-1} + q^{-j}\nu_{j,k-1}\end{aligned}$$

The recursion can be solved to find that

$$\nu_{0,k} = \sum_{j=0}^{k}\binom{k}{j}_{q^{-1}}\nu_{j,0}.$$

Therefore,

$$\tilde{\mu}_k = (-1)^k q^{k(k-1)/2}\sum_{j=0}^{k}\binom{k}{j}_{q^{-1}}(-1)^j q^{-j(j-1)/2}\mu_j.$$

□

If after the Cauchy type deformation the new contours can be further deformed to coincide at the contour $\tilde{C}_{\mathbb{A}}$ without passing through any poles, then we have the following result:

PROPOSITION 3.5. *If the contours for $z_k, k \ge 1$ in the definition of $\tilde{\mu}_k$ can be deformed to coincide at a contour $\tilde{C}_{\mathbb{A}}$ without passing through any poles, then*

$$\tilde{\mu}_k = \frac{k_q!}{k!}\frac{(1-q^{-1})^k}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\frac{1}{w_i q^{-1} - w_j}\right]_{i,j=1}^{k}\prod_{j=1}^{k} f(w_j)dw_j.$$

Proof. For the proof we quote a result from [8], *III, (1.4)* that

$$\sum_{\sigma\in S_k}\prod_{1\le A<B\le k}\frac{z_{\sigma(A)} - z_{\sigma(B)}}{z_{\sigma(A)} - qz_{\sigma(B)}} = c_{q,k}z_1\dots z_k det\left[\frac{1}{z_i q^{-1} - z_j}\right]_{i,j=1}^{k},$$

where $c_{q,k} = (q^{-1}-1)(q^{-2}-1)\dots(q^{-k}-1)$. Observe now that the $z_k's$ are on the same contour, and hence interchanging z_i and z_j for any i, j will not change the value of $\tilde{\mu}_k$. With this and applying the symmetrization equality to $\tilde{\mu}_k$ we get

$$\begin{aligned}\tilde{\mu}_k &= \frac{1}{k!}\sum_{\sigma\in S_k}\frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}}\prod_{1\le A<B\le k}\frac{z_{\sigma(A)}-z_{\sigma(B)}}{z_{\sigma(A)}-qz_{\sigma(B)}}\prod_{i=1}^{k}f(z_i)\frac{dz_i}{z_i}\\ &= \frac{1}{k!}\frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} c_{q,k}det\left[\frac{1}{w_i q^{-1}-w_j}\right]_{i,j=1}^{k}\prod_{j=1}^{k}f(w_j)dw_j.\end{aligned}$$

Therefore, it suffices to show that

$$k_q!(1-q^{-1})^k = (-1)^k q^{k(k-1)/2}c_{q,k} = (-1)^k q^{k(k-1)/2}(q^{-1}-1)(q^{-2}-1)\dots(q^{-k}-1).$$

This follows by

$$\begin{aligned}k_q!(1-q^{-1})^k &= \frac{(1-q)(1-q^2)\dots(1-q^k)}{(1-q)^k}(1-q^{-1})^k\\ &= (1-q)(1-q^2)\dots(1-q^k)\frac{(1-q^{-1})^k}{(1-q)^k}\\ &= q^{k(k+1)/2}(q^{-1}-1)(q^{-2}-1)\dots(q^{-k}-1)q^{-k}(-1)^k\\ &= (-1)^k q^{k(k-1)/2}(q^{-1}-1)(q^{-2}-1)\dots(q^{-k}-1)\end{aligned}$$

□

3.3.2 Transformation to Fredholm determinants

In this section we are going to take the form of $\tilde{\mu}_k$ as given in *Proposition 3.5* and write a generating function of the $\tilde{\mu}_k$ into the form of a Fredholm determinants, similar to that in *Proposition 3.3*. We state the key results in the following two propositions.

PROPOSITION 3.6. *If the contours for $z_k, k \ge 1$ in the definition of $\tilde{\mu}_k$ can be deformed to coincide at a contour $\tilde{C}_{\mathbb{A}}$ without passing through any poles, then using the form of tildeμ_k as given in Proposition 3.5, we have the following formal identity that*

$$\sum_{k\ge 0}\tilde{\mu}_k\frac{\zeta^k}{k_q!} = \det(I+\zeta\tilde{K}^1),$$

where $\det(I+\tilde{K}^1)$ *is the formal Fredholm determinant expansion of* $\tilde{K}^1 : L^2(\tilde{C}_{\mathbb{A}}) \to L^2(\tilde{C}_{\mathbb{A}})$ *defined in terms of its integral kernel*

$$\tilde{K}^1(w,w') = (1-q)\frac{f(w)}{qw'-w}.$$

The identity also holds numerically given that the left-hand side converges absolutely and the right-hand side operator $\tilde{K}^1$ *is trace-class.*

PROPOSITION 3.7. *Let* $\det(I+\tilde{K}^1)$ *be defined as in Proposition 3.6 and* $\det(I+\tilde{K}^2)$ *be defined by its Fredholm determinant expansion with* $\tilde{K}^2 : L^2(\tilde{C}_{\mathbb{A}}) \to L^2(\tilde{C}_{\mathbb{A}})$ *defined by its integral kernel*

$$\tilde{K}^2(w,w') = (1-q)\frac{f(w)}{qw-w'}.$$

Then we have the following formal equality:

$$\det(I+\tilde{K}^1) = \det(I+\tilde{K}^2).$$

Numerical identity also holds if $\tilde{K}^1$ *and* $\tilde{K}^2$ *are both trace-class.*

Proof of Proposition 3.6.

$$\begin{aligned}
\sum_{k\geq 0}\tilde{\mu}_k\frac{\zeta^k}{k_q!} &= \sum_{k\geq 0}\frac{\zeta^k}{k!}\frac{(1-q^{-1})^k}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\frac{1}{w_iq^{-1}-w_j}\right]_{i,j=1}^k\prod_{j=1}^k f(w_j)dw_j \\
&= \sum_{k\geq 0}\frac{1}{k!}\frac{1}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\frac{1}{w_iq^{-1}-w_j}\right]_{i,j=1}^k\prod_{j=1}^k\left(\zeta(1-q^{-1})f(w_j)\right)dw_j \\
&= \sum_{k\geq 0}\frac{1}{k!}\frac{1}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\zeta(1-q^{-1})\frac{f(w_i)}{w_iq^{-1}-w_j}\right]_{i,j=1}^k\prod_{j=1}^k dw_j \\
&= \sum_{k\geq 0}\frac{1}{k!}\frac{1}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\zeta(1-q)\frac{f(w_i)}{qw_j-w_i}\right]_{i,j=1}^k\prod_{j=1}^k dw_j \\
&= \det(I+\zeta\tilde{K}^1).
\end{aligned}$$

Numerical identity follows from the fact that if $\tilde{K}^1$ is trace class then $\det(I+\zeta\tilde{K}^1)$ is absolutely convergent. $\square$

Proof of Proposition 3.7. By definition of Fredholm determinant expansion, we have that

$$\begin{aligned}
\det(I+\zeta\tilde{K}^1) &= \sum_{k\geq 0}\frac{1}{k!}\frac{1}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\zeta(1-q)\frac{f(w_i)}{qw_j-w_i}\right]_{i,j=1}^k\prod_{j=1}^k dw_j \\
&= \sum_{k\geq 0}\frac{1}{k!}\frac{1}{(2\pi i)^k}\int_{\tilde{C}_{\mathbb{A}}}\cdots\int_{\tilde{C}_{\mathbb{A}}} det\left[\zeta(1-q)\frac{f(w_j)}{qw_j-w_i}\right]_{i,j=1}^k\prod_{j=1}^k dw_j \\
&= \det(I+\zeta\tilde{K}^2).
\end{aligned}$$

Numerical identity follows from the fact that if $\tilde{K}^1$ and $\tilde{K}^2$ are trace class then $\det(I+\zeta\tilde{K}^1)$ and $\det(I+\zeta\tilde{K}^2)$ are absolutely convergent. $\square$

We call Fredholm determinants of the form $\det(I+\tilde{K}^2)$ Cauchy type.

3.3.3 Application to q-TASEP

In this section, we apply what we have discussed so far for the Cauchy type Fredholm determinant to q-TASEP with step initial data. We state the result as a theorem.

THEOREM 3.2. *Fix $0<q<1$, $n\geq 1$ and $a_1,\dots,a_n$ such that for all i, $a_i>0$. Denote $\mathbb{A}$ to be the set defined as $\mathbb{A}=\{a_i : i=1,\dots,n\}$. Consider q-TASEP with step initial data and jump rate parameters a_i. Let $x_n(t)$ be the location of particle n at time t. Then for all $\zeta\in\{\zeta\in\mathbb{C}:|\zeta|<1\}$,*

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right]=\frac{1}{(\zeta;q)_\infty}\det(I+\zeta\tilde{K}^{q-TASEP}),$$

where $\det(I+\zeta\tilde{K}^{q-TASEP})$ *is the Fredholm determinant of $\tilde{K}^{q-TASEP}: L^2(\tilde{C}_{\mathbb{A}})\to L^2(\tilde{C}_{\mathbb{A}})$ defined in terms of its integral kernel*

$$\tilde{K}^{q-TASEP}(w,w')=\frac{f(w)}{qw'-w},$$

with

$$f(z) = \left(\prod_{m=1}^{n} \frac{a_m}{a_m - z} \right) e^{(q-1)tz}.$$

The equality holds numerically if the contour $\tilde{C}_{\mathbb{A}}$ is chosen to be a simple close contour such that it strictly contains 0 *and every ray from* 0 *crosses $\tilde{C}_{\mathbb{A}}$ exactly once, and such that it contains $\mathbb{A}$, the fixed set of singularities of $f(w)$.*

REMAEK 3.5. Such a contour $\tilde{C}_{\mathbb{A}}$ for numerical identity is called star-shaped with respect to the point 0.

REMAEK 3.6. It's remarked that the domain of ζ can be extended to $\mathbb{C} \setminus \{q^{-i}\}_{i \in \mathbb{Z}_+}$. Please refer to [2] *Theorem 3.2.16* for further details.

Proof. We first show the formal identity. By the same observation as in Section 3.2.3 that for q-TASEP with step initial data, we identify that if we take

$$f(z) = \left(\prod_{m=1}^{n} \frac{a_m}{a_m - z} \right) e^{(q-1)tz},$$

then $\mathbb{E}\left[q^{kx_n(t)+kn}\right] = \mu_k$. Therefore, by Propositionn 3.4,

$$\begin{aligned}
\tilde{\mu}_k &= (-1)^k q^{k(k-1)/2} \sum_{j=0}^{k} \binom{k}{j}_{q^{-1}} (-1)^j q^{-j(j-1)/2} \mu_j \\
&= (-1)^k q^{k(k-1)/2} \sum_{j=0}^{k} \binom{k}{j}_{q^{-1}} (-1)^j q^{-j(j-1)/2} \mathbb{E}\left[q^{jx_n(t)+jn}\right] \\
&= (-1)^k q^{k(k-1)/2} \mathbb{E}\left[\sum_{j=0}^{k} \binom{k}{j}_{q^{-1}} (-1)^j q^{-j(j-1)/2} q^{jx_n(t)+jn} \right]
\end{aligned}$$

From [6] Corollary 10.2.2.c, we have that for any x and q,

$$\sum_{k=0}^{n} \binom{n}{k}_{q} (-1)^k q^{k(k-1)/2} x^k = (x;q)_n.$$

Then

$$\tilde{\mu}_k = (-1)^k q^{k(k-1)/2} \mathbb{E}\left[(q^{x_n(t)+n}; q^{-1})_k\right] = \mathbb{E}\left[q^{kx_n(t)}(q^{-x_n(t)}; q)_k\right].$$

Therefore, for $|\zeta| < 1$, $|\zeta q^{x_n(t)}| < 1$ and hence

$$\begin{aligned}\sum_{k\geq 0} \tilde{\mu}_k \frac{(\zeta/(1-q))^k}{k_q!} &= \mathbb{E}\left[\sum_{k\geq 0} \frac{(q^{-x_n(t)};q)_k}{(q;q)_k} (\zeta q^{x_n(t)})^k\right] \\ &= \mathbb{E}\left[\frac{(\zeta;q)_\infty}{(\zeta q^{x_n(t)};q)_\infty}\right] \text{ by the q-Binomial theorem} \end{aligned} \tag{3.16}$$

Applying *Proposition 3.6* and then *Proposition 3.7* to $\sum_{k\geq 0} \tilde{\mu}_k \frac{(\zeta/(1-q))^k}{k_q!}$ with ζ replaced by $\zeta/(1-q)$, we have

$$\sum_{k\geq 0} \tilde{\mu}_k \frac{(\zeta/(1-q))^k}{k_q!} = \det(I + \tilde{K}^{q-TASEP}), \tag{3.17}$$

where $\det(I + \tilde{K}^{q-TASEP})$ is exactly as desired.

Comparing (3.16) and (3.17), we have the formal identity that

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)};q)_\infty}\right] = \frac{\det(I + \tilde{K}^{q-TASEP})}{(\zeta;q)_\infty}.$$

For the numerical identity, we quote the following lemma from [10] *Page 345*:

LEMMA 3.2.1. *An operator K acting on $L^2(\Gamma)$ for a simple smooth contour Γ in $\mathbb{C}$ with integral kernel $K(x,y)$ is* trace-class *if $K(x,y) : \Gamma^2 \to \mathbb{R}$ is continuous as well as $K_y(x,y)$ is continuous in y. Here $K_y(x,y)$ is the partial derivative with respect to y along the contour Γ.*

For the star-shaped contour as chosen, we have that $\tilde{K}^{q-TASEP}$ is trace-class by Lemma 3.2.1, and therefore $\det(I + \tilde{K}^{q-TASEP})$ is absolutely convergent. Moreover, since

$$\begin{aligned}\mathbb{E}\left[\frac{(\zeta;q)_\infty}{(\zeta q^{x_n(t)};q)_\infty}\right] &= \mathbb{E}\left[(\zeta;q)_{x_n(t)}\right] \\ &= \sum_{i=1}^{\infty} \mathbb{P}(x_n(t) = i)(1-\zeta)(1-\zeta q)\dots(1-\zeta q^{i-1}),\end{aligned}$$

and the summands are uniformly converging to non-trivial limits in any fixed neighborhood of ζ, we have that $\mathbb{E}\left[\frac{(\zeta;q)_\infty}{(\zeta q^{x_n(t)};q)_\infty}\right]$ is uniformly convergent. Therefore, the numerical identity also holds for the star-shaped contour chosen. $\square$

3.4 q Laplace Transform and Inversion Formula

In the last two sections, we have successfully identify the exact forms in terms of Fredholm determinants for $\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right]$. In fact, this is called the q-Laplace transform of the function $f(m) = \mathbb{P}(x_n(t) = m)$. We quote the following definition from [2] *Section 3.1.1.*

DEFINITION 3.4. For a function $f(n) \in \ell(\{0, 1, 2, \dots\})$, the q-Laplace transform of $f(n)$ is defined to be

$$\hat{f}^q(z) = \sum_{n\geq 0} \frac{f(n)}{(zq^n;q)_\infty}.$$

From [2] *Proposition 3.1.1* we can also recover the function $f(n)$ from its q-Laplace transform with the following inversion formula:

PROPOSITION 3.8. *One may recover a function $f(n) \in \ell(\{0, 1, 2, \dots\})$ from its q-Laplace transform $\hat{f}^q(z)$ with $z \in \mathbb{C} \setminus \{q^{-k}\}_{k\geq 0}$ via*

$$f(n) = -q^n \frac{1}{2\pi i} \int_{C_n} (q^{n+1}z;q)_\infty \hat{f}^q(z) dz,$$

where C_n is any positively oriented contour which encircles poles at $z = q^{-M}$ for $0 \leq M \leq n$.

Using the inversion formula given in the above proposition, we are able to recover $\mathbb{P}(x_n(t) = m)$ from

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)+n};q)_\infty}\right].$$

However, since $f(m) = \mathbb{P}(x_n(t) = m)$ is supported on $m = -n, -(n-1), \dots$, in order to apply *Proposition 3.8*, we need to consider $f(m)$ shifted to the right by n and the contours C_N will also have to be chosen so that instead of encircling poles at $z = q^{-M}$ for $0 \leq M \leq N$ it has to now encircle poles at $z = q^{-M}$ for $-n \leq M \leq N - n$. With these modification, we have the following results after directly applying the results of *Theorem 3.1* and *Theorem 3.2* to *Proposition 3.8*:

For Mellin-Barnes type Fredholm determinants,

PROPOSITION 3.9. *Under the assumptions of Theorem 3.1, we have*

$$\mathbb{P}(x_n(t)=m)=\frac{-q^n}{2\pi i}\int_{C_m}(\zeta q^{n+1};q)_\infty \det(I+\tilde{K}_\zeta^{q-TASEP})d\zeta,$$

where the contour C_m is any positively oriented contour that encircles poles at $z = q^{-M}$, for $-n \leq M \leq m-n$.

For Cauchy type Fredholm determinants,

PROPOSITION 3.10. *Under the assumptions of Theorem 3.2, we have*

$$\mathbb{P}(x_n(t)=m)=\frac{-q^n}{2\pi i}\int_{C_m}\frac{\det(I+\tilde{K}_\zeta^{q-TASEP})}{(\zeta;q)_{n+1}}d\zeta,$$

where the contour C_m is any positively oriented contour that encircles poles at $z = q^{-M}$, for $-n \leq M \leq m-n$.

3.5 q-TASEP with half stationary initial data

We do not have similar results in general for the case of half stationary initial data, because for any $\alpha > 0$ fixed, when k gets so large that $\alpha > q^k$, the expectation of $\mathbb{E}\left[q^{kx_n(y)}\right]$ tends to infinity because of *Lemma 2.2.1*. Therefore, when forming a generating function of this, we will not be able to get a converging series.

Chapter 4

Tracy-Widom Asymptotics

In the last chapter we've determined explicitly the formula for $\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)};q)_\infty}\right]$. This chapter will continue from this and perform asymptotic analysis for q-TASEP. The final result is that the fluctuation of the position of $X_N(t)$ will be following the GUE Tracy-Widom distribution. For simplicity, we assume through out the chapter that the jump rate parameters we are considering are $a_i = 1$ for all i, and that $q \in (0, 1)$, $\theta > 0$ is fixed.

4.1 Tracy-Widom distribution

We begin by providing a basic introduction to the GUE Tracy-Widom distribution. First the definition of the GUE Tracy-Widom distribution is quoted from [7] *Definition 3 (1)*.

DEFINITION 4.1. The GUE Tracy-Widom distribution is defined as

$$F_{GUE}(r) = det(I - K_{Ai})_{L^2(r,\infty)},$$

where K_{Ai} is the Airy kernel that has integral representations

$$K_{Ai}(\eta, \eta') = \frac{1}{(2\pi i)^2} \int_{e^{-\frac{2\pi i}{3}}\infty}^{e^{\frac{2\pi i}{3}}\infty} dw \int_{e^{-\frac{\pi i}{3}}\infty}^{e^{\frac{\pi i}{3}}\infty} dz \frac{e^{z^3/3 - z\eta}}{e^{w^3/3 - w\eta'}} \frac{1}{z - w},$$

where the contours for z and w do not intersect.

REMAEK 4.1. It's remarked that the Airy kernel defined above ie equivalent to

$$K_{Ai}(\eta,\eta') = \int_{\mathbb{R}_+} d\lambda Ai(\eta+\lambda)Ai(\eta'+\lambda),$$

where $Ai(x)$ is the Airy function defined as

$$Ai(\eta) = \frac{1}{2\pi i}\int_C \exp(\frac{z^3}{3} - z\eta)dz,$$

where C is any contour starting at the point at infinity with argument $-\frac{\pi}{2}$ and ending at the point at infinity with argument $\frac{\pi}{2}$. Please refer to [5] for more details on the equality

4.2 Reformulation of Mellin-Barnes type kernel

We provide the definitions of some parameters used in this section.

DEFINITION 4.2. For fix $q \in (0,1)$ and $\theta > 0$, let the q-gamma function be defined as

$$\Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty}$$

and the q-digamma function be defined as

$$\Psi_q(z) = \frac{d}{dz}\log\Gamma_q(z) = -\log(1-q) + \log_q \sum_{n=1}^{\infty}\frac{q^{n+z}}{1-q^{n+z}}.$$

Furthermore, we introduce the following parameters

$$\kappa = \kappa(q,\theta) = \frac{\Psi_q'(\theta)}{(\log q)^2 q^\theta} = \sum_{n=0}^{\infty}\frac{q^n}{(1-q^{n+\theta})^2}; \tag{4.1}$$

$$f = f(q,\theta) = \frac{\Psi_q'(\theta)}{(\log q)^2} - \frac{\Psi_q'(\theta)}{\log q} - \frac{\log(1-q)}{\log q}; \tag{4.2}$$

$$\chi = \chi(q,\theta) = \frac{\Psi_q'(\theta)\log q - \Psi_q''(\theta)}{2}. \tag{4.3}$$

DEFINITION 4.3. For any $c, x \in \mathbb{R}$, we define the following parameters

$$\tau(N,c) = \kappa N + cq^{-\theta}N^{2/3} \tag{4.4}$$

$$p(N,c) = (f-1)N + cN^{2/3} - c^2\frac{(\log q)^3}{4\chi}N^{1/3} \tag{4.5}$$

The rescaled fluctuations, $\xi_N(c)$, of the N^{th} particle at time $\tau(N,c)$ around $p(N,c)$ is defined to be

$$\xi_N(c) = \frac{X_N(\tau(N,c)) - p(N,c)}{\chi^{1/3}(\log q)^{-1}N^{1/3}}. \tag{4.6}$$

REMAEK 4.2. It's remarked here that $p(N,c)$ is the macroscopic approximation of the particle's position at time $\tau(N,c)$. However, we will not pursue further here.

From *Theorem 3.1*, we have that

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_N(t)+N};q)_\infty}\right] = det(I + K_\zeta^{q-TASEP}), \tag{4.7}$$

where $det(I + K_\zeta^{q-TASEP})$ is the Fredholm determinant of $K_\zeta^{q-TASEP} : L^2(C_{\mathbb{A}}) \to L^2(C_{\mathbb{A}})$ and the operator $K_\zeta^{q-TASEP}$ is defined in terms of its integral kernel

$$K_\zeta^{q-TASEP}(w,w') = \frac{1}{2\pi i}\int_{C_{1,2,\ldots}} \Gamma(-s)\Gamma(1+s)(-\zeta)^s \left(\frac{(q^s w;q)_\infty}{(w;q)_\infty}\right)^N \frac{e^{tw(q^s-1)}}{q^s w - w'}ds.$$

The main goal is this section is to perform some transformation for both sides of (4.7). For the right-hand side, we introduce the following change of variables

$$w = q^W, w' = q^{W'}, s + W = Z. \tag{4.8}$$

Notice the Euler's Reflection formula that $\Gamma(-s)\Gamma(1+s) = \frac{\pi}{sin(-\pi s)}$, the kernel can be transformed to be

$$\begin{aligned}&\hat{K}_\zeta^{q-TASEP}(w,w')\\ &= \frac{q^W \log q}{2\pi i}\int_{D_{R,d}^W} \frac{\pi}{sin(\pi(W-Z))}\frac{(-\zeta)^Z}{(-\zeta)^W}\frac{\exp(tq^Z + N\log(q^Z;q)_\infty)}{\exp(tq^W + N\log(q^W;q)_\infty)}\frac{dZ}{q^Z - q^{W'}},\end{aligned} \tag{4.9}$$

where $D^W_{R,d}$ is the contour $D_{R,d}$ shifted by W.

For the left-hand side, we first choose the parameter ζ in $\mathbb{E}\left[\frac{1}{(\zeta q^{X_N(t)+N};q)_\infty}\right]$ to be

$$\zeta = -q^{-fN - cN^{2/3} + \beta_x \frac{N^{1/3}}{\log q}} \in \mathbb{C} \setminus \mathbb{R}_+, \tag{4.10}$$

$$\beta_x = c^2 \frac{(\log q)^4}{4\chi} - \chi^{1/3} x.$$

Notice that

$$\begin{aligned} -q^{\frac{\chi^{1/3}}{\log q} N^{1/3}(\xi_N - x)} &= -q^{\frac{\chi^{1/3}}{\log q} N^{1/3}\left(\frac{X_N(\tau(N,c)) - p(N,c)}{\chi^{1/3}(\log q)^{-1} N^{1/3}} - x\right)} \\ &= -q^{X_N(\tau) - p(N,c) - \frac{\chi^{1/3}}{\log q} N^{1/3} x} \\ &= -q^{X_N(\tau) - (f-1)N - cN^{2/3} + c^2 \frac{(\log q)^3}{4\chi} N^{1/3} - \frac{\chi^{1/3}}{\log q} N^{1/3} x} \\ &= \zeta q^{X_N(\tau)+N}. \end{aligned}$$

Therefore, we have the following equality that

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_N(\tau)+N};q)_\infty}\right] = \mathbb{E}\left[\frac{1}{(-q^{\frac{\chi^{1/3}}{\log q} N^{1/3}(\xi_N - x)};q)_\infty}\right] \tag{4.11}$$

Combining (4.7), (4.9) and (4.11), we have

$$\mathbb{E}\left[\frac{1}{(-q^{\frac{\chi^{1/3}}{\log q} N^{1/3}(\xi_N - x)};q)_\infty}\right] = det(I + \hat{K}^{q-TASEP}_\zeta)_{L^2(\hat{C}_{\mathbb{A}})}, \tag{4.12}$$

where

$$\begin{aligned} &\hat{K}^{q-TASEP}_\zeta(W,W') \\ &= \frac{q^W \log q}{2\pi i} \int_{D^W_{R,d}} \frac{\pi}{sin(\pi(W-Z))} \frac{(-\zeta)^Z}{(-\zeta)^W} \frac{\exp(tq^Z + N\log(q^Z;q)_\infty)}{\exp(tq^W + N\log(q^W;q)_\infty)} \frac{dZ}{q^Z - q^{W'}}, \end{aligned} \tag{4.13}$$

where ζ is chosen as (4.10), given that both sides converge absolutely.

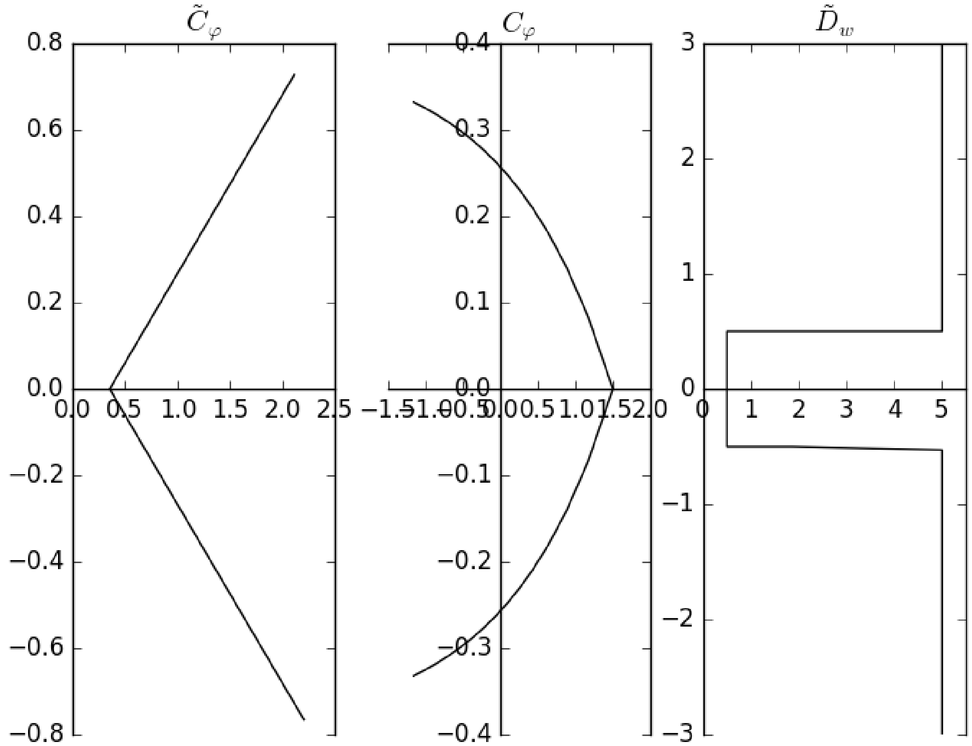

Figure 4-1: An illustration of the contours $\tilde{C}_\varphi$(left), C_φ(middle) for the parameter $q = 0.5$ and $\theta = 1.5$ and $\tilde{D}_{R,d}$(right) for the parameter $R = 5, d = 0.5$

4.3 Integration Contours

In this section, we give some concrete contours of $C_{\mathbb{A}}$ and $D_{R,d}$ that will ensure the convergence of the Fredholm determinants given in (4.7). We choose these contours also to facilitate our analysis later. We begin by defining some of the contours and then justify the convergence.

DEFINITION 4.4. (1) Fix $q \in (0, 1)$ and $\theta > 0$. For arbitrary but fix $\varphi \in (0, \pi/4]$, we define the contour $\tilde{C}_\varphi$ to be

$$\tilde{C}_\varphi = \{q^\theta + e^{i\varphi sgn(y)}|y| : y \in \mathbb{R}\}.$$

(2) Let C_φ be the image of $\tilde{C}_\varphi$ under the mapping of $x \to \log_q(x)$. That is,

$$C_\varphi = \{\log_q(q^\theta + e^{i\varphi sgn(y)}|y|) : y \in \mathbb{R}\}.$$

(3) For every $w \in \tilde{C}_\varphi$, define the contour $\tilde{D}_w$ to be $D_{R,d}$ as given in *Section 3.2.2* that it goes by straight lines from $R - i\infty$ to $R - id$, to $1/2 - id$, to $1/2 + id$, to $R + id$, and lastly to $R + i\infty$, with $R, d > 0$ chosen such that the following holds:

   (a) $arg(w(q^s - 1)) \in (\pi/2 + b, 3\pi/2 - b)$ for $b = \pi/4 - \varphi/2$;

   (b) $q^s w$ stays to the left of $\tilde{C}_\varphi$ for every $s \in \tilde{D}_w$.

(4) Lastly, for every $W \in C_\varphi$, we define the contour D_W to be such that it is the contour $\tilde{D}_{q^W}$ shifted by W. If we let $R, d > 0$ be chosen for the contour $\tilde{D}_{q^W}$ such that the two conditions above are satisfied, then D_W is defined by the straight lines going from $R + Re(W) - i\infty$ to $R + Re(W) + i(Im(W) - d)$, to $1/2 + Re(W) + i(Im(W) - d)$, to $1/2 + Re(W) + i(Im(W) + d)$, to $R + Re(W) + i(Im(W) + d)$ and to $R + Re(W) + i\infty$.

Please see *Figure 4-1* for an illustration of the contours defined.

PROPOSITION 4.1. *Fix $q \in (0, 1)$, $\theta > 0$ and $\varphi \in (0, \pi/4]$. Let C_φ be defined as given in Definition 4.4 (1). For every $W \in C_\varphi$ and the choice of R, d such that the two conditions in Definition 4.4 (3) are satisfied with $w = q^W$. Then*

$$Re(W) + R > \theta.$$

REMAEK 4.3. An immediate result from the proposition is that there exists a $\sigma > 0$ such that $\theta + \sigma = R + Re(W)$. Moreover, the parameter $d > 0$ can be chosen to be so small such that D_W and C_φ do no intersect.

Proof. This follows from *Condition (b)* in *Definition 4.4 (3).* □

We claim that with the choice of $C_{\mathbb{A}}$ to be $\tilde{C}_\varphi$ for any $\varphi \in (0, \pi/4]$, and the choice of $C_{1,2,\ldots}$ to be $\tilde{D}_w$ for $w \in \tilde{C}_\varphi$, we have the convergence for the right-hand side of *(4.7)*

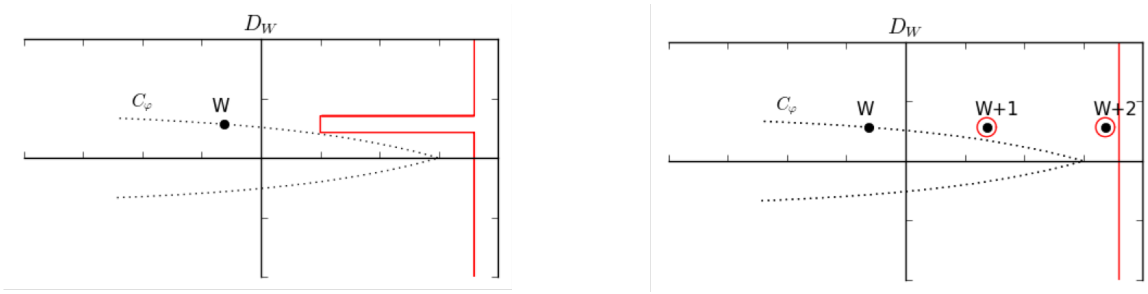

Figure 4-2: An illustration of transformation of the contour D_W mentioned in *Remark 4.4*

as desired. To justify this, we only need to justify the three conditions in *Theorem 3.1*, labeled as (1), (2), (3) respectively. For further details of the justification, please refer to *[11] Theorem (4.2)*.

Recall the change of variables (4.8), we have that the contour $\hat{C}_{\mathbb{A}}$ in (4.12) can then be chosen as C_φ and that the contour $D_{R,d}^W$ for the integral kernel $\hat{K}_\zeta^{q-TASEP}(W, W')$ can be chosen as D_W. By further re-writing the kernel in the following form, we have the following result that for $x \in \mathbb{R}$,

$$\mathbb{E}\left[\frac{1}{(-q^{\frac{\chi^{1/3}}{\log q}N^{1/3}(\xi_N - x)}; q)_\infty}\right] = det(I + K_x)_{L^2(C_\varphi)}, \tag{4.14}$$

where

$$K_x(W, W')$$
$$= \frac{q^W \log q}{2\pi i}\int_{D_W}\frac{dZ}{q^Z - q^{W'}}\frac{\pi}{sin(\pi(W - Z))}\frac{\exp(Nf_0(Z) + N^{2/3}f_1(Z) + N^{1/3}f_2(Z))}{\exp(Nf_0(W) + N^{2/3}f_1(W) + N^{1/3}f_2(W))},$$

where

$$f_0(Z) = -f(\log q)Z + \kappa q^Z + \log(q^Z; q)_\infty$$

$$f_1(Z) = -c(\log q)Z + cq^{Z-\theta}$$

$$f_2(Z) = \beta_x Z.$$

REMAEK 4.4. It's remarked that the contour D_W in the definition of K_x can be further replaced by the straight line from $R + Re(W) - i\infty$ to $R + Re(W) + i\infty$, where R is the same as in the definition of D_W and d in the definition of D_W is chose such that D_W and C_φ do no intersect, and some small circles surrounding the poles coming from the term $\frac{1}{sin(\pi(W-Z))}$, which are $W+1, W+2, \ldots, W+k_W$ and k_W denotes the total number of residues. The new contour will also be referred to as D_W. Please see *Figure 4-2* for an illustration.

4.4 Asymptotics of the Fredholm determinant

In this section, we focus on the asymptotic behaviour of the Fredholm determinant in (4.14). The main result is given in the following theorem.

THEOREM 4.1. *Let $x \in \mathbb{R}$ be fixed and choose ζ as in (4.10). Then*

$$det(I + K_x)_{L^2(C_\varphi)} \to F_{GUE}(x) \text{ as } N \to \infty,$$

where F_{GUE} is the GUE Tracy-Widom distribution function.

The intuition for the proof is that for N large we show that the Fredholm determinants $det(I + K_x)_{L^2(C_\varphi)}$ and $det(I - K_{Ai})_{L^2(x,\infty)}$ are arbitrarily close. To show how this can be achieved, we quote the following series of propositions from [11] without proof and briefly explain the ideas behind. For more precise treatment, pleass refer to the original paper [11] or [7].

PROPOSITION 4.2 (*[11] Proposition (6.3)*)**.** *For any fixed $\delta > 0$ and $\epsilon > 0$ small enough, there is an N_0 such that*

$$\left| det(I + K_x)_{L^2(C_\varphi)} - det(I + K_{x,\delta})_{L^2(C_\varphi^\delta)} \right| < \epsilon$$

for all $N > N_0$ where

$$\begin{aligned} &K_{x,\delta}(W, W') \\ &= \frac{q^W \log q}{2\pi i} \int_{D_W^\delta} \frac{dZ}{q^Z - q^{W'}} \frac{\pi}{sin(\pi(W-Z))} \frac{\exp(Nf_0(Z) + N^{2/3}f_1(Z) + N^{1/3}f_2(Z))}{\exp(Nf_0(W) + N^{2/3}f_1(W) + N^{1/3}f_2(W))} \end{aligned}$$

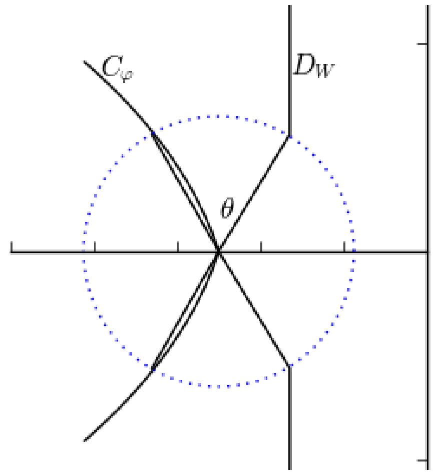

Figure 4-3: An illustration of the locally transformed contours $C_{\tilde{\varphi},\delta N^{1/3}}$ and $D_{\tilde{\varphi},\delta N^{1/3}}$

and $C^\delta_\varphi = C_\varphi \cap \{w : |w - \theta| \leq \epsilon\}$, $D^\delta_W = D_W \cap \{z : |z - \theta| \leq \delta\}$.

This proposition essentially says that the asymptotically contribution to the Fredholm determinant outside any neighbourhood of θ can be ignored. Therefore, we only need to deal with the behavior of the Fredholm determinant around a small neighborhood of θ. With this, the contour C_φ and D_W can be localized to a small neighborhood of θ (that is, C^δ_φ and D^δ_W respectively) without affecting the asymptotic behaviour of the determinant.

We then perform the following deformation of the contour C^δ_φ and D^δ_W. Using Cauchy theorem, we are able to deform the contour $C^\delta_{\tilde{\varphi}}$ to $\{\theta + |y|e^{i(\pi - \tilde{\varphi})sgn(y)} : y \in [-\delta, \delta]\}$ with some $\tilde{\varphi} \in (0, \pi/2)$ chosen such that the two contours have the same endpoint. Similarly, we can deform the straignt line $R + Re(W) + i\mathbb{R}$ in the contour D^δ_W to be $D^\delta_{\tilde{\varphi}} = \{\theta + |t|e^{i\tilde{\varphi}sgn(t)} : t \in [-\delta, \delta]\}$. Please refer to *Figure 4-3* for an example of the new contours.

After the above deformation, we introduce the following change of variables to W, W' and Z:

$$W = \theta + wN^{-1/3}, W' = \theta + w'N^{-1/3}, Z = \theta + zN^{-1/3}.$$

From the relationship, we get the corresponding contours for w, w' and z to be $C_{\tilde{\varphi},\delta N^{1/3}}$

and $D_{\tilde{\varphi},\delta N^{1/3}}$, where

$$C_{\tilde{\varphi},L} = \{|y|e^{i(\pi-\tilde{\varphi})sgn(y)} : y \in [-L, L]\}, D_{\tilde{\varphi},L} = \{|t|e^{i\tilde{\varphi}sgn(t)} : t \in [-L, L]\}$$

for $L \in (0, \infty]$. Summing up, after the change of variable, we have the following:

$$det(I + K_{x,\delta})_{L^2(C^{\delta}_{\tilde{\varphi}})} = det(I + K^N_{x,\delta})_{L^2(C_{\tilde{\varphi},\delta N^{1/3}})},$$

where

$$K^N_{x,\delta}(w, w') = N^{-1/3}K_{x,\delta N^{1/3}}(\theta + wN^{-1/3}, \theta + w'N^{-1/3}). \tag{4.15}$$

For the next step, observe the following Taylor expansion of $f_0(Z)$, $f_1(Z)$ and $f_2(Z)$ around a neighborhood of θ:

- $f_0(Z) = f_0(\theta) + \frac{\chi}{3}(Z - \theta)^3 + \mathcal{O}((Z - \theta)^4)$
- $f_1(Z) = f_1(\theta) + \frac{c(\log q)^2}{2}(Z - \theta)^2 + \mathcal{O}((Z - \theta)^3)$
- $f_2(Z) = f_2(\theta) + \beta_x(Z - \theta)$

Substituting these into (4.15), we can transform the kernel into the following explicit form as stated in the following proposition.

PROPOSITION 4.3 *([11] Proposition (6.4))*. *Let $\tilde{\varphi} \in (0, \pi/2)$ be sufficiently close to $\pi/2$ and let $\epsilon > 0$ be fixed. There is a small $\delta > 0$ and an N_0 such that for any $N > N_0$,*

$$|det(I + K^N_{x,\delta})_{L^2(C_{\varphi,\delta N^{1/3}})} - det(I + K'_{x,\delta N^{1/3}})_{L^2(C_{\tilde{\varphi},\delta N^{1/3}})}| < \epsilon,$$

where

$$K'_{x,\delta N^{1/3}}(w, w') = \frac{1}{2\pi i}\int_{D_{\tilde{\varphi},\delta N^{1/3}}} \frac{dz}{(z - w')(w - z)} \frac{e^{\chi z^3/3 + c(\log q)^2 z^2/2 + \beta_x z}}{e^{\chi w^3/3 + c(\log q)^2 w^2/2 + \beta_x w}}.$$

To ensure that $\tilde{\varphi}$ is close to $\pi/2$, we need to choose the original φ to be close to $\pi/2$. However, φ ranges originally from $(0, \pi/4]$, which prevents us from such a choice. Therefore, we need the following proposition to allow the choice of φ to be close to $\pi/2$ as desired.

PROPOSITION 4.4 (*[11] Proposition (6.2)*). *For fixed $q \in (0,1), \theta > 0$ and N large enough, the contour C_φ with $\varphi \in (0, \pi/4)$ for the kernel K_x can be extended to any $\varphi \in (0, \pi/2)$ without affecting the Fredholm determinant $det(I + K_x)_{L^2(C_\varphi)}$.*

The following proposition asserts that we can expand the bound on the contour $D_{\varphi,\delta N^{1/3}}$ to $D_{\varphi,\infty}$. Similar to *Proposition 4.2* , we are essentially saying that asymptotically the only part of the contour that contributes to the Fredholm determinant is the part around some neighborhood of 0. It is made precise by the following:

PROPOSITION 4.5 (*[11] Proposition (6.5)*). *As $N \to \infty$, we have*

$$det(I + K'_{x,\delta N^{1/3}})_{L^2(C_{\varphi,\delta N^{1/3}})} \to det(I + K'_{x,\infty})_{L^2(C_{\varphi,\infty})}.$$

And lastly, the kernel $K'_{x,\infty}$ can be reformulated to the Airy kernel via the following proposition.

PROPOSITION 4.6 (*[11] Proposition (6.6)*).

$$det(I + K'_{x,\infty})_{L^2(C_{\varphi,\infty})} = det(I - K_{Ai,x})_{L^2(\mathbb{R}_+)}.$$

By the definition of the GUE Tracy-Widom distribution introduced at the beginning of the chapter, we therefore conclude *Theorem 4.1.*

4.5 Distribution of the rescaled fluctuation

In the last section, we concluded with *Theorem 4.1* that for N large, $det(I+K_x)_{L^2(C_\varphi)} \to F_{GUE}(x)$. Therefore,

$$\mathbb{E}\left[\frac{1}{(-q^{\frac{\chi^{1/3}}{\log q}N^{1/3}(\xi_N - x)}; q)_\infty}\right] \to F_{GUE}(x) \text{ as } N \to \infty. \tag{4.16}$$

Define $f_N(y) = (-q^{\frac{\chi^{1/3}}{\log q}N^{1/3}y}; q)_\infty^{-1}$. Then we have that

$$\mathbb{E}\left[\frac{1}{(-q^{\frac{\chi^{1/3}}{\log q}N^{1/3}(\xi_N - x)}; q)_\infty}\right] = \mathbb{E}\left[f_N(\xi_N - x)\right].$$

We observe the following facts about $f_N(y)$:

- $f_N(y)$ is a mapping from $\mathbb{R}$ to $[0, 1]$.
- For each N, $f_N(y)$ is strictly decreasing on y. This is because $\log q < 0$ for $q \in (0, 1)$. Moreover, $\lim_{y\to\infty} f_N(y) = 1$ and $\lim_{y\to-\infty} f_N(y) = 0$.
- For each $\delta > 0$, on $\mathbb{R} \setminus [-\delta, \delta]$, the sequence of functions f_N converges uniformly to $\mathbb{1}(y < 0)$.

The last observation follows from the fact that $\frac{1}{1+q^{\frac{\chi^{1/3}}{\log q}N^{1/3}y+k}}$ is uniformly close to 1 if $y \in (\delta, \infty)$ and 0 if $y \in (-\infty, -\delta)$.

To continue our discussion, we quote the following lemma from *[2] Lemma 4.1.39*

.

LEMMA 4.1.1. *Consider a sequence of functions $(f_n)_{n\geq 1}$ mapping $\mathbb{R} \to [0, 1]$ such that for eaach n, $f_n(y)$ is strictly decreasing in y with a limit of 1 at $y = -\infty$ and 0 at $y = \infty$, and for each $\delta > 0$, on $\mathbb{R} \setminus [-\delta, \delta]$, f_n converges uniformly to $\mathbb{1}(y < 0)$. Consider a sequence of random variables $X_n(t)$ such that for each $r \in \mathbb{R}$,*

$$\mathbb{E}[f_n(X_n - r)] \to p(r)$$

and assume that $p(r)$ is a continuous probability distribution function. Then X_n converges weakly in distribution to a random variable X which is distributed according to $\mathbb{P}(X < r) = p(r)$.

Combining the lemma above, the observations about $f_N(y)$ and (4.16), we conclude the following theorem.

THEOREM 4.2. *Let $q \in (0,1)$ and $\theta > 0$ be fixed. For any $c, x \in \mathbb{R}$, we have that*

$$\lim_{N\to\infty} \mathbb{P}(\xi_N < x) = F_{GUE}(x),$$

where

$$\xi_N = \frac{X_N(\tau(N,c)) - p(N,c)}{\chi^{1/3}(\log q)^{-1}N^{1/3}}.$$

Proof. The proof is by applying Lemma 4.1.1 directly with the sequence of functions $f_N(y)$. $\square$

Chapter 5

Conclusion

In this thesis, the model q-TASEP is studied from the duality approach with main reference to [1] and [11]. We've arrived at a nested contour integral formulation of the q-Laplace transform of the probability density for positions of a particular particle in q-TASEP, and we also performed some asymptotic analysis to show that the rescaled fluctuation of the particle around its macroscopic position approximation follows the GUE Tracy-Widom distribution.

Main effort of the project was spent to understand the materials provided in the two papers. Along the way, almost every aspect of my knowledge in mathematics was applied somewhere. For example, real and complex analysis is one of the biggest components of the project and important to understand most of the techniques used for the analysis part. Residue calculus is also key for the contour manipulations. In addition, I've also learned a lot of new techniques in evaluating contour integrals and performing contour deformation, etc. It has brought my understanding in mathematics to a level deeper than ever before.

Lastly, I would like to conclude the thesis with my greatest thanks to my supervisor, as well as any other peers that have provided me guidance and help along the way.

Bibliography

[1] A.Borodin, I. Corwin, and T. Sasamoto, *From duality to determinants for q-TASEP and ASEP*, Annals of Probability 2014, Vol. 42, No. 6, 2314-2382, 2014, arXiv:1207.5035.

[2] A. Borodin, I. Corwin, *Macdonald processes*, Probability Theory and Related Fields, 2014, arXiv:1111.4408.

[3] A. Borodin, I. Corwin, and P.L. Ferrari, *Free energy fluctuations for directed polymers in random media in 1+1 dimention*, Comm. Pure Appl. Math(2014), 1129-1214

[4] A. Voß-Bö hme, W. Schenk and A.-K. Koellner, *On the equivalence between Liggett duality of Markov processes and the duality relation between their generators*, Markov Processes and Related Fields, 2011.

[5] Craig. A. Tracy and H. Widom, *Level-spacing Distributions and the Airy Kernel*, Comm. Math. Phys. 159 (1994), 151-174.

[6] G. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, 2000.

[7] G. Barraquand, *A phase transition for q-TASEP with a few slower particles*, Stochastic Processes and Their Applications 125 (2015), pp. 2674-2699, 2015, arXiv:1404.7409.

[8] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford University Press, New York, 1999

[9] Liggett, T. M., *Interacting Particle Systems*, Springer, Berlin, 2005, MR2108619.

[10] P. D. Lax., *Functional Analysis*, Wiley-interscience, 2002.

[11] P. L. Ferrari, B. Vető, *Tracy-Widom asymptotics for q-TASEP*, Ann. Inst. H. Poincaré Probab. Stat., 2013, arXiv:1310.2515.