21-752: Algebraic Topology

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Carnegie Mellon University - Spring 2026

This version of the notes was last compiled on January 13, 2026.

For the latest version, visit

https://thefundamentaltheor3m.github.io/AlgTopNotes/main.pdf.

For comments, errors or suggestions, please raise an issue/make a pull request at

https://github.com/thefundamentaltheor3m/AlgTopNotes.

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Course Introduction and Overview

For the purposes of intuition, let's try and think about the concept of geometry. Broadly speaking, we can define geometry to be the study of isometries and their invariants - to talk about distance, we only need to be in a metric space. Topology is a 'squishy' notion of geometry, where instead of distance preservation, we talk about *nearness* preservation.

Already we lose some information. Isometries are automatically injective; moreover, the inverse of the corestriction of an isometry to its image is also an isometry. So distance-preservation is 'invertible' in some sense of the word. Nearness-preservation (which most of us would call "continuity"), however, is famously not (guaranteed to be) invertible. Topology is thus 'more interesting'.

A non-exhaustive list of things we might do with (algebraic) topology:

- **Solve equations implicitly:** for example we may have results analogous to the intermediate value theorem (IVT). In the example of IVT analogues, we can see IVT as "working" because of path-connectedness of the unit interval. Say we want an analogous statement for when maps $f : D^2 \to \mathbb{R}^2$ (with D^2 the disk in $\mathbb{R}^2$) have a 0 - in this instance, we may need a "higher" notion of connectivity...

- **Homotopy:** Just as we can talk about 'higher' notions of connectivity, we can also talk about 'higher' notions of *homotopy*. We are already familiar with questions of path homotopies: can I continuously deform one path to another? We can generalise this in the following manner. If there are two homotopies between two paths, we can ask whether there is a way to continuously deform one *homotopy* to the other. We are all adults here and know how to do induction, so I'm sure we all know where this is going.

- **Local-to-Global:** Do local solutions glue together to give global solutions? Algebraic topology gives us tools that track this.

- **Makes precise ideas such as "testing into/out of an object":** for example, say we want to look at sequences in a metric space X, which are just maps from $\mathbb{N}$ to X. Sequences essentially tell us all we would want to know about continuity in a metric space - so here our "test object" is $\mathbb{N}$, and studying the sequences (i.e. the maps from $\mathbb{N}$ to X) can tell us about the metric space. In topological spaces, maybe we want to look at all continuous maps from n-simplices into our space - in this instance we would have a different "test object" for each dimension, and the test object for dimension n will be the n-simplex $\Delta_n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1\}$. We can use tools from homological algebra to study maps from the n-simplex into the space. The properties we deduce using these methods will tell us a great deal about the space. Moreover, n-simplices tell us about n-fold homotopies (ie, 2-simplices tell us about homotopies of paths; 3-simplices tell us about homotopies of homotopies; and so on).

  As a categorical aside, it turns out that we can define a category Δ known as the **simplex category**, whose objects are finite linear orders (labelled $[n] = \{0 < 1 < \cdots < n\}$) and whose morphisms are weakly order-preserving maps (ie, if $x < y$ then $f(x) \leq f(y)$). We can show that the operation taking any $[n]$ to the n-simplex Δ_n defines a contravariant functor $K : \Delta \to \mathbf{Set}$. Spaces and categories admit a "common generalisation" in simplicial sets; this area is called infinity category theory.

We will spend the first few weeks talking about fundamental groups and how to compute them. We will then move onto 'higher dimensional stuff'.

Chapter 1

An Introduction to the Theory of Introductions

1.1 Important Definitions and First Examples

Definition 1.1.1 (Definitions). A definition is a way of defining a thing.

We will now see an example.

Example 1.1.2 (A Definition). You literally just saw one...

1.2 Another Section

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Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, `\lipsum` time. Boy do I love LaTeX!

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