21-752: Algebraic Topology

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For the latest version, visit

https://thefundamentaltheor3m.github.io/AlgTopNotes/main.pdf.

For comments, errors or suggestions, please raise an issue/make a pull request at

https://github.com/thefundamentaltheor3m/AlgTopNotes.

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Course Introduction and Overview

For the purposes of intuition, let's try and think about the concept of geometry. Broadly speaking, we can define geometry to be the study of isometries and their invariants - to talk about distance, we only need to be in a metric space. Topology is a 'squishy' notion of geometry, where instead of distance preservation, we talk about *nearness* preservation.

Already we lose some information. Isometries are automatically injective; moreover, the inverse of the corestriction of an isometry to its image is also an isometry. So distance-preservation is 'invertible' in some sense of the word. Nearness-preservation (which most of us would call "continuity"), however, is famously not (guaranteed to be) invertible. Topology is thus 'more interesting'.

A non-exhaustive list of things we might do with (algebraic) topology:

- **Solve equations implicitly:** for example we may have results analogous to the intermediate value theorem (IVT). In the example of IVT analogues, we can see IVT as "working" because of path-connectedness of the unit interval. Say we want an analogous statement for when maps $f : D^2 \to \mathbb{R}^2$ (with D^2 the disk in $\mathbb{R}^2$) have a 0 - in this instance, we may need a "higher" notion of connectivity...

- **Homotopy:** Just as we can talk about 'higher' notions of connectivity, we can also talk about 'higher' notions of *homotopy*. We are already familiar with questions of path homotopies: can I continuously deform one path to another? We can generalise this in the following manner. If there are two homotopies between two paths, we can ask whether there is a way to continuously deform one *homotopy* to the other. We are all adults here and know how to do induction, so I'm sure we all know where this is going.

- **Local-to-Global:** Do local solutions glue together to give global solutions? Algebraic topology gives us tools that track this.

- **Makes precise ideas such as "testing into/out of an object":** for example, say we want to look at sequences in a metric space X, which are just maps from $\mathbb{N}$ to X. Sequences essentially tell us all we would want to know about continuity in a metric space - so here our "test object" is $\mathbb{N}$, and studying the sequences (i.e. the maps from $\mathbb{N}$ to X) can tell us about the metric space. In topological spaces, maybe we want to look at all continuous maps from n-simplices into our space - in this instance we would have a different "test object" for each dimension, and the test object for dimension n will be the n-simplex $\Delta_n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1\}$. We can use tools from homological algebra to study maps from the n-simplex into the space. The properties we deduce using these methods will tell us a great deal about the space. Moreover, n-simplices tell us about n-fold homotopies (ie, 2-simplices tell us about homotopies of paths; 3-simplices tell us about homotopies of homotopies; and so on).

  As a categorical aside, it turns out that we can define a category Δ known as the **simplex category**, whose objects are finite linear orders (labelled $[n] = \{0 < 1 < \cdots < n\}$) and whose morphisms are weakly order-preserving maps (ie, if $x < y$ then $f(x) \le f(y)$). We can show that the operation taking any $[n]$ to the n-simplex Δ_n defines a contravariant functor $K : \Delta \to \mathbf{Set}$. Spaces and categories admit a "common generalisation" in simplicial sets; this area is called infinity category theory.

We will spend the first few weeks talking about fundamental groups and how to compute them. We will then move onto 'higher dimensional stuff'.

Chapter 1

The Fundamental Group

Our broad objective will be to *associate invariants to spaces*. Let's begin by putting together a 'wishlist' of things we would want such 'invariants' to obey.

- We want our invariants to be *invariant under deformations*: if we 'do something' to a space ('deform' it), we don't want that to affect our invariant.

- We want our invariants to be *functorial*: an invariant describing spaces should describe groups if we move from spaces to group in a nice way.

- We want our invariants to *completely classify spaces* (possibly up to deformation): in other words, we want a 'converse' to invariance under deformations, where we know that if our invariant is unchanged by some transformation, then that transformation must be a compatible one (a 'deformation').

- We would (ideally) want our definition to be *easily computable*.

Now that we have a list of things we'd like to do, why don't we go ahead and... do them?!

1.1   The Setup

Before we go any further, we will state something (that is par for any course on topology).

> **Convention.** Unless otherwise specified,
>
> - All maps in this course will be assumed to be continuous.
>
> - The (uppercase) letters X, Y and Z will denote topological spaces.

1.1.1   Paths and Homotopies

We begin by defining the notion of 'deformation' we talked about so handwavily just a few moments ago.

> **Definition 1.1.1** (Homotopy of Continuous Maps). Let X and Y be topological spaces and let $f, g : X \to Y$ be (continuous) functions. A **homotopy from f to g** is a (continuous) map $H : X \times I \to Y$ such that for all $x \in X$,
>
> $$H(x, 0) = f(x) \qquad \text{and} \qquad H(x, 1) = g(x)$$
>
> If a homotopy exists between f and g, we say they are **homotopic**, and we write $f \simeq g$.

It is not hard to see that homotopy (or more precisely, the property of being homotopic) is an equivalence relation on the set of continuous functions from one space to another. In particular, it doesn't matter whether we say 'f is homotopic to g' or 'g is homotopic to f'.

> **Definition 1.1.2** (Paths and Loops). A **path** in a topological space X is a continuous map $\gamma : [0, 1] \to X$. A path γ is a **loop** if $\gamma(0) = \gamma(1)$.

Since life is too short, we will adopt the following notation.

> **Notation.** Unless otherwise specified, the symbol I will denote the unit interval $[0, 1]$.

There are obviously many examples of paths. In fact, there are many examples of homotopic paths.

> **Example 1.1.3.** Any two paths in $\mathbb{R}^n$ (with the same endpoints) are homotopic. For example, if $\alpha, \beta : I \to \mathbb{R}^n$ are paths with the same endpoints, then
>
> $$H : I \times I \to \mathbb{R}^n : (s, t) \mapsto t \cdot \alpha(s) + (1 - t) \cdot \beta(s)$$

> is a homotopy between the paths.

Really, it only makes sense to talk about homotopies between paths whose endpoints are the same. That being said, our definition of homotopies is permissive enough that this is not necessarily true of a homotopy. Therefore, we will introduce separate notation for what it means for paths with the same endpoints to be homotopic (notation that emphasises that their endpoints are the same).

> **Notation.** If α and β with the same endpoints, we will write
>
> $$\alpha \simeq)\partial I \beta$$
>
> to indicate that α and β are homotopic.

It turns out that paths can be composed, provided they are 'compatible'.

Proposition 1.1.4 (Composing Paths). *Fix points $x_0, x_1, x_2 \in X$. If γ is a path from x_0 to x_1 and δ is a path from x_1 to x_2, then the function*

$$\varepsilon : [0,1] \to X : t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path between x_0 and x_2.

Proof. It is clear that ε is a well-defined function: at the point $t = \frac{1}{2}$,

$$\gamma(2t) = \gamma(1) = x_1 = \delta(0) = \delta(2t-1)$$

So indeed the function is well-defined.

All we need to show is that ε is continuous. But this is true because of a homework exercise from General Topology in which we showed that if we have a space that is a union of two closed sets then we can continuous 'glue' together continuous functions that agree on the overlap of these closed sets. So ε is the 'gluing' of γ of δ, and is hence continuous. □

Definition 1.1.5 (Composition of Paths). Fix points $x_0, x_1, x_2 \in X$. If γ is a path from x_0 to x_1 and δ is a path from x_1 to x_2, then the function $\varepsilon : [0, 1] \to X$ defined as above is called the **composition** of γ and δ.

We will denote it $\delta \star \gamma$, $\delta * \gamma$, $\delta \cdot \gamma$ or simply $\delta\gamma$.

Warning. We emphasise that $\delta\gamma$ means

First do γ, then do δ.

In other words, we obey the convention we normally use when we do function composition.

Warning. You know everything I just said? Be warned, I may not always heed my own warning and might write $\gamma\delta$ to mean $\delta\gamma$ sometimes. The real warning is, just use your head, typecheck that function compositions and path concatenations are all ok, and you should be fine. Just read the vibe and you should be ok.

Finally, we define the notion of an inverse path.

Definition 1.1.6 (Inverse Path). Given a path $\alpha : I \to X$, we define the **inverse path** $\overline{\alpha} : I \to X$ by $\overline{\alpha}(t) := \alpha(1 - t)$ for all $t \in I$.

We will now take a brief detour into the world of groupoids.

1.1.2 Groupoids

Here, we will assume some familiarity with Category Theory. For a quick and dirty introduction to the subject, read `sorry`.

Add refs

Recall that we can view a group as a (small) category, with the morphisms being its action on itself.

Example 1.1.7 (A Group as a (Small) Category). Fix a group G. We can turn it into a small category with one single object $\star$ and one morphism from $\star$ to itself corresponding to each group element g.

We define a groupoid to be a generalisation of the above example that allows for more than one object.

> **Definition 1.1.8** (Groupoid). A **groupoid** is a small category in which every morphism is invertible.

Comparing Definition 1.1.8 and Example 1.1.7, the folllowing is evident.

> **Proposition 1.1.9.** *A group is a groupoid with only one object.*

1.1.3   The Fundamental Groupoid

Fix a topological space X. We ask ourselves the following question: can we associate a groupoid to X? As it turns out, we can.

> **Definition 1.1.10.** The **fundamental groupoid of** X, denoted $\pi_1(X)$, is the category with the following data:
> - The object set is X.
> - For each $x, y \in X$, define the set of morphisms $\operatorname{Hom}(x, y)$ to be the set of all paths from x to y, up to homotopy.
> - For $x, y, z \in X$, $[\gamma] \in \operatorname{Hom}(x, y)$ and $[\delta] \in \operatorname{Hom}(y, z)$, define $[\delta] \circ [\gamma]$ to be $[\delta \star \gamma]$.
> - For $x, y \in X$ and $[\gamma] \in \operatorname{Hom}(x, y)$, define the inverse operation $[\gamma]^{-1}$ to be $[\bar{\gamma}]$.

We have not really shown that this composition is associative (and of course it is not something that should be shown *after* the definition but rather something that should be shown *before*) but we'll do it now.

> **Lemma 1.1.11.** *Let $\phi : I \to I$ be a path such that $\phi(0) = 0$ and $\phi(1) = 1$. For any path $\gamma : I \to X$, $\gamma \circ \phi \simeq)\partial I\gamma$.*

Proof. The function

$$H : I \times I \to X : (s, t) \mapsto \gamma(st + (1 - t)\phi(s))$$

is an explicit homotopy.   □

> **Corollary 1.1.12.** *For all paths* $\alpha, \beta, \gamma : I \to X$ *that are compatible,*
>
> $$\gamma \star (\beta \star \alpha) \simeq)\partial I \, (\gamma \star \beta) \star \alpha$$

Thus, concatenation gives rise to a well-defined (associative) composition on homotopy classes, thus, on the fundamental groupoid.

Recall that a group is a groupoid with a single object. So, it stands to reason that the fundamental groupoid, except when we look at just one point in our space (and paths from that point to itself) rather than *all* points in our space (and *all* paths), should be a group. This is precisely the fundamental group.

> **Definition 1.1.13** (The Fundamental Group). Fix a point $x_0 \in X$. The **fundamental group of X based at** x_0 is the set
>
> $$\pi_1(X; x_0) = \mathrm{Hom}_{\pi_1(X)}(x_0, x_0)$$

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, \lipsum time. Boy do I love LaTeX!

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