21-752: Algebraic Topology

Lecturer: Florian Frick

Scribes: Sidharth Hariharan, Teresa Pollard

Carnegie Mellon University - Spring 2026

This version of the notes was last compiled on January 16, 2026.

For the latest version, visit

https://thefundamentaltheor3m.github.io/AlgTopNotes/main.pdf.

For comments, errors or suggestions, please raise an issue/make a pull request at

https://github.com/thefundamentaltheor3m/AlgTopNotes.

Contents

Course Introduction and Overview 2

1 The Fundamental Group 4

1.1 The Setup 4

1.1.1 Paths and Homotopies 5

1.1.2 Groupoids 7

1.1.3 The Fundamental Group(oid) 8

1.1.4 Basic Properties of the Fundamental Group 9

2 Another Chapter 13

2.1 Introducing the Main Object of Study in this Chapter 13

2.2 Another Section 13

Bibliography 16

Course Introduction and Overview

For the purposes of intuition, let's try and think about the concept of geometry. Broadly speaking, we can define geometry to be the study of isometries and their invariants - to talk about distance, we only need to be in a metric space. Topology is a 'squishy' notion of geometry, where instead of distance preservation, we talk about *nearness* preservation.

Already we lose some information. Isometries are automatically injective; moreover, the inverse of the corestriction of an isometry to its image is also an isometry. So distance-preservation is 'invertible' in some sense of the word. Nearness-preservation (which most of us would call "continuity"), however, is famously not (guaranteed to be) invertible. Topology is thus 'more interesting'.

A non-exhaustive list of things we might do with (algebraic) topology:

- **Solve equations implicitly:** for example we may have results analogous to the intermediate value theorem (IVT). In the example of IVT analogues, we can see IVT as "working" because of path-connectedness of the unit interval. Say we want an analogous statement for when maps $f : D^2 \to \mathbb{R}^2$ (with D^2 the disk in $\mathbb{R}^2$) have a 0 - in this instance, we may need a "higher" notion of connectivity...

- **Homotopy:** Just as we can talk about 'higher' notions of connectivity, we can also talk about 'higher' notions of *homotopy*. We are already familiar with questions of path homotopies: can I continuously deform one path to another? We can generalise this in the following manner. If there are two homotopies between two paths, we can ask whether there is a way to continuously deform one *homotopy* to the other. We are all adults here and know how to do induction, so I'm sure we all know where this is going.

- **Local-to-Global:** Do local solutions glue together to give global solutions? Algebraic topology gives us tools that track this.

- **Makes precise ideas such as "testing into/out of an object":** for example, say we want to look at sequences in a metric space X, which are just maps from $\mathbb{N}$ to X. Sequences essentially tell us all we would want to know about continuity in a metric space - so here our "test object" is $\mathbb{N}$, and studying the sequences (i.e. the maps from $\mathbb{N}$ to X) can tell us about the metric space. In topological spaces, maybe we want to look at all continuous maps from n-simplices into our space - in this instance we would have a different "test object" for each dimension, and the test object for dimension n will be the n-simplex $\Delta_n = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1\}$. We can use tools from homological algebra to study maps from the n-simplex into the space. The properties we deduce using these methods will tell us a great deal about the space. Moreover, n-simplices tell us about n-fold homotopies (ie, 2-simplices tell us about homotopies of paths; 3-simplices tell us about homotopies of homotopies; and so on).

  As a categorical aside, it turns out that we can define a category Δ known as the **simplex category**, whose objects are finite linear orders (labelled $[n] = \{0 < 1 < \cdots < n\}$) and whose morphisms are weakly order-preserving maps (ie, if $x < y$ then $f(x) \leq f(y)$). We can show that the operation taking any $[n]$ to the n-simplex Δ_n defines a contravariant functor $K : \Delta \to$ **Set**. Spaces and categories admit a "common generalisation" in simplicial sets; this area is called infinity category theory.

We will spend the first few weeks talking about fundamental groups and how to compute them. We will then move onto 'higher dimensional stuff'.

Chapter 1

The Fundamental Group

Our broad objective will be to *associate invariants to spaces*. Let's begin by putting together a 'wishlist' of things we would want such 'invariants' to obey.

- We want our invariants to be *invariant under deformations*: if we 'do something' to a space ('deform' it), we don't want that to affect our invariant.
- We want our invariants to be *functorial*: an invariant describing spaces should describe groups if we move from spaces to group in a nice way.
- We want our invariants to *completely classify spaces* (possibly up to deformation): in other words, we want a 'converse' to invariance under deformations, where we know that if our invariant is unchanged by some transformation, then that transformation must be a compatible one (a 'deformation').
- We would (ideally) want our definition to be *easily computable*.

Now that we have a list of things we'd like to do, why don't we go ahead and... do them?!

1.1 The Setup

Before we go any further, we will state something (that is par for any course on topology).

Convention. Unless otherwise specified,

- All maps in this course will be assumed to be continuous.
- The (uppercase) letters X, Y and Z will denote topological spaces.

1.1.1 Paths and Homotopies

We begin by defining the notion of 'deformation' we talked about so handwavily just a few moments ago.

Definition 1.1.1 (Homotopy of Continuous Maps). Let X and Y be topological spaces and let $f, g : X \to Y$ be (continuous) functions. A **homotopy from f to g** is a (continuous) map $H : X \times I \to Y$ such that for all $x \in X$,

$$H(x, 0) = f(x) \qquad \text{and} \qquad H(x, 1) = g(x)$$

If a homotopy exists between f and g, we say they are **homotopic**, and we write $f \simeq g$.

It is not hard to see that homotopy (or more precisely, the property of being homotopic) is an equivalence relation on the set of continuous functions from one space to another. In particular, it doesn't matter whether we say 'f is homotopic to g' or 'g is homotopic to f'.

Definition 1.1.2 (Paths and Loops). A **path** in a topological space X is a continuous map $\gamma : [0, 1] \to X$. A path γ is a **loop** if $\gamma(0) = \gamma(1)$.

Since life is too short, we will adopt the following notation.

Notation. Unless otherwise specified, the symbol I will denote the unit interval $[0, 1]$.

There are obviously many examples of paths. In fact, there are many examples of homotopic paths.

Example 1.1.3. Any two paths in $\mathbb{R}^n$ (with the same endpoints) are homotopic. For example, if $\alpha, \beta : I \to \mathbb{R}^n$ are paths with the same endpoints, then

$$H : I \times I \to \mathbb{R}^n : (s, t) \mapsto t \cdot \alpha(s) + (1 - t) \cdot \beta(s)$$

is a homotopy between the paths.

Really, it only makes sense to talk about homotopies between paths whose endpoints are the same. That being said, our definition of homotopies is permissive enough that this is not necessarily true of a homotopy. Therefore, we will introduce separate notation for what it means for paths with the same endpoints to be homotopic (notation that emphasises that their endpoints are the same).

Notation. If α and β with the same endpoints, we will write

$$\alpha \simeq_{\partial I} \beta$$

to indicate that α and β are homotopic.

It turns out that paths can be composed, provided they are 'compatible'.

Proposition 1.1.4 (Composing Paths). *Fix points $x_0, x_1, x_2 \in X$. If γ is a path from x_0 to x_1 and δ is a path from x_1 to x_2, then the function*

$$\varepsilon : [0,1] \to X : t \mapsto \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path between x_0 and x_2.

Proof. It is clear that ε is a well-defined function: at the point $t = \frac{1}{2}$,

$$\gamma(2t) = \gamma(1) = x_1 = \delta(0) = \delta(2t-1)$$

So indeed the function is well-defined.

All we need to show is that ε is continuous. But this is true because of a homework exercise from General Topology in which we showed that if we have a space that is a union of two closed sets then we can continuous 'glue' together continuous functions that agree on the overlap of these closed sets. So ε is the 'gluing' of γ of δ, and is hence continuous. □

Add ref

Definition 1.1.5 (Composition of Paths)**.** Fix points $x_0, x_1, x_2 \in X$. If γ is a path from x_0 to x_1 and δ is a path from x_1 to x_2, then the function $\varepsilon : [0, 1] \to X$ defined as above is called the **composition** of γ and δ.

We will denote it $\delta \star \gamma$, $\delta * \gamma$, $\delta \cdot \gamma$ or simply $\delta\gamma$.

Warning. We emphasise that $\delta\gamma$ means

First do γ, then do δ.

In other words, we obey the convention we normally use when we do function composition.

Warning. You know everything I just said? Be warned, I may not always heed my own warning and might write $\gamma\delta$ to mean $\delta\gamma$ sometimes. The real warning is, just use your head, typecheck that function compositions and path concatenations are all ok, and you should be fine. Just read the vibe and you should be ok.

Finally, we define the notion of an inverse path.

Definition 1.1.6 (Inverse Path)**.** Given a path $\alpha : I \to X$, we define the **inverse path** $\overline{\alpha} : I \to X$ by $\overline{\alpha}(t) := \alpha(1 - t)$ for all $t \in I$.

We will now take a brief detour into the world of groupoids.

1.1.2 Groupoids

Here, we will assume some familiarity with Category Theory. For a quick and dirty introduction to the subject, read sorry.

Add refs

Recall that we can view a group as a (small) category, with the morphisms being its action on itself.

Example 1.1.7 (A Group as a (Small) Category)**.** Fix a group G. We can turn it into a small category with one single object $\star$ and one morphism from $\star$ to itself corresponding to each group element g.

We define a groupoid to be a generalisation of the above example that allows for more than one object.

Definition 1.1.8 (Groupoid). A **groupoid** is a small category in which every morphism is invertible.

Comparing Definition 1.1.8 and Example 1.1.7, the following is evident.

Proposition 1.1.9. *A group is a groupoid with only one object.*

1.1.3 The Fundamental Group(oid)

Fix a topological space X. We ask ourselves the following question: can we associate a groupoid to X? As it turns out, we can.

Definition 1.1.10. The **fundamental groupoid of** X, denoted $\pi_1(X)$, is the category with the following data:

- The object set is X.
- For each $x, y \in X$, define the set of morphisms $\mathrm{Hom}(x, y)$ to be the set of all paths from x to y, up to homotopy.
- For $x, y, z \in X$, $[\gamma] \in \mathrm{Hom}(x, y)$ and $[\delta] \in \mathrm{Hom}(y, z)$, define $[\delta] \circ [\gamma]$ to be $[\delta \star \gamma]$.
- For $x, y \in X$ and $[\gamma] \in \mathrm{Hom}(x, y)$, define the inverse operation $[\gamma]^{-1}$ to be $[\overline{\gamma}]$.

We have not really shown that this composition is associative (and of course it is not something that should be shown *after* the definition but rather something that should be shown *before*) but we'll do it now.

Lemma 1.1.11. *Let $\phi : I \to I$ be a path such that $\phi(0) = 0$ and $\phi(1) = 1$. For any path $\gamma : I \to X$, $\gamma \circ \phi \simeq_{\partial I} \gamma$.*

Proof. The function

$$H : I \times I \to X : (s, t) \mapsto \gamma(st + (1 - t)\phi(s))$$

is an explicit homotopy. □

Corollary 1.1.12. *For all paths $\alpha, \beta, \gamma : I \to X$ that are compatible,*

$$\gamma \star (\beta \star \alpha) \simeq_{\partial I} (\gamma \star \beta) \star \alpha$$

Thus, concatenation gives rise to a well-defined (associative) composition on homotopy classes, thus, on the fundamental groupoid.

Recall that a group is a groupoid with a single object. So, it stands to reason that the fundamental groupoid, except when we look at just one point in our space (and paths from that point to itself) rather than *all* points in our space (and *all* paths), should be a group. This is precisely the fundamental group.

Definition 1.1.13 (The Fundamental Group). Fix a point $x_0 \in X$. The **fundamental group of X based at x_0** is the set

$$\pi_1(X; x_0) = \mathrm{Hom}_{\pi_1(X)}(x_0, x_0)$$

Some fundamental groups are easy to compute and some are difficult.

Example 1.1.14. $\pi_1(\mathbb{R}^n)$ is trivial.

1.1.4 Basic Properties of the Fundamental Group

Theorem 1.1.15. *For any space X and basepoint $x_0 \in X$, $\pi_1(X, x_0)$ is a group under the composition operation $[\gamma] \cdot [\gamma'] = [\gamma \star \gamma']$.*

Proof. This should just follow trivially from the definition (because we did 'check' that $\pi_1(X)$ is a groupoid), but let's play the game and check the axioms anyway.

- Identity: The identity element is the loop $c_{x_0} : I \to X : t \mapsto x_0$. Indeed, $c_{x_0} \star \gamma \simeq_{\partial I} \gamma \simeq_{\partial I}$

$\gamma \star c_{x_0}$ via

$$\Phi : I \to I : t \mapsto \begin{cases} 2t & \text{if } t \leq \frac{1}{2} \\ 1 & \text{if } t \geq \frac{1}{2} \end{cases}$$

- Inverses: Recall that we defined the inverse of a path γ to be the path with the time parameter going in the opposite direction - $\overline{\gamma}(t) := \gamma(1-t)$. It's not hard to show that for all loops γ based at x_0, $\gamma \star \overline{\gamma} \simeq_{\partial I} \overline{\gamma} \star \gamma$.

- Associativity: This is precisely Corollary 1.1.12.

$\square$

Theorem 1.1.16. *For X a topological space with $x_0, x_1 \in X$, and $\alpha : I \to X$ a path from x_0 to x_1 ($\alpha(0) = x_0, \alpha(1) = x_1$), then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

There is really only one reasonable way to attempt to do this. And when you do it, surprise surprise, it works...

Proof. The desired isomorphism is given by conjugation by the path from x_0 to x_1. Explicitly, if $\alpha : I \to X$ is a path from x_0 to x_1, we can define $\Phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\Phi([\gamma]) := [\overline{\alpha} \star \gamma \star \alpha]$$

It's not very hard to show that this is a group homomorphism: given paths $[\gamma], [\gamma'] \in \pi_1(X, x_0)$, we see that

$$\begin{aligned} \Phi([\gamma] \circ [\gamma']) &= [\overline{\alpha} \star \gamma \star \gamma' \star \alpha] \\ &= [\overline{\alpha} \star \gamma \star \alpha \star \overline{\alpha} \star \gamma' \star \alpha] \\ &= [\overline{\alpha} \star \gamma \star \alpha] \circ [\overline{\alpha} \star \gamma' \star \alpha] \\ &= \Phi([\gamma]) \circ \Phi([\gamma']) \end{aligned}$$

Finally, we can define $\Phi^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by

$$\Phi^{-1}([\gamma]) := [\alpha \star \gamma \star \overline{\alpha}]$$

It's not hard to show this is a two-sided inverse. □

Finally, we show that π_1 is functorial.

Proposition 1.1.17. *Let X and Y be topological spaces. Fix a basepoint $x_0 \in X$. For any continuous function $f : X \to Y$, the map*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) : \gamma \mapsto [f \circ \gamma]$$

is a well-defined homomorphism of groups.

Proof. sorry

□

Definition 1.1.18 (Homotopy Equivalence of Spaces)**.** Let X and Y be topological spaces. If there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$, then we say X and Y are **homotopy equivalent**. We write $X \simeq Y$.

There are many well-known examples of homotopy equivalences of spaces. At some point, we will probably look at **contractible spaces**, which are spaces that are homotopic to a point.

Example 1.1.19 (Contractibility of the Unit Interval)**.** $I \simeq \{0\}$ via the maps

$$f : I \to \{0\} : t \mapsto 0 \qquad g : \{0\} \to I : 0 \mapsto 0$$

Obviously, $f \circ g = \mathrm{id}_{\{0\}}$. We only need a homotopy in the other direction. Indeed, it's not hard to show that

$$H : I \times I \to I : (s, t) \mapsto st$$

gives the homotopy $g \circ f \simeq \mathrm{id}_I$.

We can see the above example as indicative of a more general pattern - if we have spaces X and Y with $Y \subseteq X$, then it seems that X and Y should be homotopic if there are no "holes" in $X \setminus Y$.

Definition 1.1.20 (Deformation Retraction)**.** Let X be a topological space. Fix $A \subseteq X$ and let $i : A \hookrightarrow X$ denote the inclusion. A map $r : X \to Y$ is a **deformation retraction** if $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$.

See also [**EPFLTopologie**].

A special property of deformation retractions is that they induce inclusions of fundamental groups.

Convention. Going forward, unless otherwise specified, arbitrary spaces X, Y, Z etc will be assumed to be path-connected (hopefully for obvious and unproblematic reasons). We will also often omit the basepoint if we know we are working in a path-connected space.

Lemma 1.1.21. *Let X and Y be (path-connected) spaces and let $f_0, f_1 : X \to Y$ be homotopic. Then, $(f_0)_* = (f_1)_*$.*

Proof. Let $H : X \times I \to Y$ be a homotopy between f_0 and f_1. The idea is to show that for all $\gamma \in \pi_1(X)$, the path $f_1 \circ \gamma$ is homotopic to the path $f_0 \circ \gamma$. sorry □

Theorem 1.1.22. *If we have $X \simeq Y$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.*

Proof. Assume we have (continuous) functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We know show that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a homomorphism; in fact, we will show it is an *iso*morphism. It may not be true that $y_0 = f(x_0)$, but since Y is path-connected, we know we have an isomorphism between $\pi_1(Y, f(x_0))$ and $\pi_1(Y, y_0)$.

sorry □

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, `\lipsum` time. Boy do I love LaTeX!

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Visit https://thefundamentaltheor3m.github.io/AlgTopNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to the associated GitHub repository.