MATH70063: Algebra 4

Lecturer: Oliver Gregory

Scribe: Sidharth Hariharan

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Chapter 1

A Crash Course on Category Theory

The objective of this course is to develop a comprehensive toolkit to study a very broad class of mathematical objects, such as abelian groups, modules over commutative rings, vector spaces, and more. Broadly speaking, these objects are all **categories**, and the toolkit we will develop will apply to a specific kind of categories known as **abelian categories**.

Before we develop the homological algebra toolkit, we will need to develop a basic understanding of category theory. This will be the primary focus of this chapter. In so doing, we will be able to understand the theory of abelian categories and understand the broader context in which the theory of homological algebra is applicable.

1.1 Important Fundamentals

The basic idea of category theory is to reason collectively with large classes of mathematical objects. It is often useful to talk about a 'set of all sets' or a 'set of all groups'. Unfortunately, if we reason about these naïvely, we run the risk of running into paradoxes, such as Russell's paradox. Category theory provides a way to reason about these large classes of objects without running into these paradoxes.

In this module, we will not be too precise about what constitutes a *class*; this is actually a very important choice in category theory, and the fact that we will not be precise about this makes

our treatment of the subject fundamentally imprecise. Nevertheless, our treatment will be rigorous enough for the purposes of studying homological algebra.

With this disclaimer in mind, we are ready to begin.

1.1.1 Categories and Functors

1.1.2 Properties of Morphisms

1.2 Natural Transformations

1.3 Categorical Constructions

Bibliography

These lecture notes are based heavily on the following references:

[1] Nicolas Bourbaki. *Groupes et Algèbres de Lie. Chapitre 1*. 1st ed. Éléments de Mathématique. Édition originale publiée par Hermann, Paris, 1972. Springer Berlin, Heidelberg, 2007. ISBN: 978-3-540-35337-9. DOI: https://doi.org/10.1007/978-3-540-35337-9.

[2] Karin Erdmann and Mark J. Wildon. *Introduction to Lie Algebras*. Springer Undergraduate Mathematics Series. Springer, 2006.

[3] Anna Lachowska et al. *MATH-335: Coxeter Groups*. Lecture Notes. École Polytechnique Fédérale de Lausanne, Spring 2020.

[4] Martin Liebeck and Aleksander Horawa. *M4P46: Lie Algebras*. Lecture Notes. Imperial College London, Winter 2016.

[5] Aluna Rizzoli and Sidharth Hariharan. *MATH-314: Representation Theory of Finite Groups*. Lecture Notes. École Polytechnique Fédérale de Lausanne, Spring 2024. URL: https://github.com/thefundamentaltheor3m/RepTheoryEPFL.

For the latest version of these notes, visit https://thefundamentaltheor3m.github.io/HomAlgNotes/LastLocallyCompiled.pdf. For any suggestions or corrections, please feel free to fork my repository and make a pull request.