MATH70063: Algebra 4

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Imperial College London - Spring 2025

Contents

Chapter 1

A Crash Course on Category Theory

The objective of this course is to develop a comprehensive toolkit to study a very broad class of mathematical objects, such as abelian groups, modules over commutative rings, vector spaces, and more. Broadly speaking, these objects are all **categories**, and the toolkit we will develop will apply to a specific kind of categories known as **abelian categories**.

Before we develop the homological algebra toolkit, we will need to develop a basic understanding of category theory. This will be the primary focus of this chapter. In so doing, we will be able to understand the theory of abelian categories and understand the broader context in which the theory of homological algebra is applicable.

1.1 Important Fundamentals

The basic idea of category theory is to reason collectively with large classes of mathematical objects. It is often useful to talk about a 'set of all sets' or a 'set of all groups'. Unfortunately, if we reason about these naïvely, we run the risk of running into paradoxes, such as Russell's paradox. Category theory provides a way to reason about these large classes of objects without running into these paradoxes.

In this module, we will not be too precise about what constitutes a *class*; this is actually a very important choice in category theory, and the fact that we will not be precise about this makes our treatment of the subject fundamentally imprecise. Nevertheless, our treatment will be rigorous enough for the purposes of studying homological algebra.

With this disclaimer in mind, we are ready to begin.

1.1.1 Objects and Morphisms

Definition 1.1.1 (Category). A **category** consists of the following data.

1. A **class** of **objects**, denoted $|\mathcal{C}|$.
2. For each pair of objects $A, B \in |\mathcal{C}|$, a **class** of **morphisms** from A to B, denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$. This class can be empty.
3. For each object $A \in |\mathcal{C}|$, a **distinguished morphism** $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$, known as the **identity morphism on** A.
4. For each ordered triple of objects $A, B, C \in |\mathcal{C}|$, a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C) : (f, g) \mapsto g \circ f$$

such that

(i) $\circ$ is associative, ie, for all objects $A, B, C, D \in |\mathcal{C}|$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

(ii) For all objects $A, B \in |\mathcal{C}|$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

There are numerous examples of categories, some familiar and some unfamiliar.

Example 1.1.2 (Sets). We can define a category **Set** whose objects are sets, whose morphisms are maps between sets, and in which the composition map is the standard composition of funtions.

There is a slightly less familiar example that is of a computer scientific flavour.

Example 1.1.3 (Pre-Orders). Let X be any set. Let $\leq$ be a pre-order on X, ie, a binary relation that is reflexive, antisymmetric and transitive. We can define a category $\langle X, \leq \rangle$ such that

1. The objects of $\langle X, \leq \rangle$ are single-element sets containing the elements of X. Ie, if $X = \mathbb{N}$, then $|X| = \{\{0\}, \{1\}, \{2\}, \ldots\}$. In particular, $|X|$ is a set—in fact, a set that is in bijection with X.
2. For any $\{x\}, \{y\} \in |X|$, there can be a unique function $f : \{x\} \to \{y\}$ (in the category of sets). In the category $\langle X, \leq \rangle$, we say define

$$\mathrm{Hom}_{\langle X, \leq \rangle}(\{x\}, \{y\}) = \begin{cases} \text{the sole function } \{x\} \to \{y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

3. We know that $\leq$ is reflexive, so we have the identity morphism from any $\{x\}$ to itself.
4. Composition of functions is the standard set-theoretic composition of functions:
   (i) This is sensible because composition of functions and pre-orders are both transitive.
   (ii) This is associative because the set-theoretic composition of functions is associative.
   (iii) There does exist an identity on every object with respect to this composition operation because the pre-order is reflexive.

Finally, a more abstract example.

Example 1.1.4 (Monoids). Let M be a monoid with with operation $\times$ and identity e. We can view M as a category $\mathcal{C}$ with the following data.

1. There is only one object in this category. This object can be anything. We denote it $\star$. Ie, we have $|\mathcal{C}| = \{\star\}$.
2. We can allow M to act on $\star$ syntactically. This means that we associate to any $x \in M$ a map $\star \to \star$, which we denote by an arrow from $\star$ to itself. When we say this action is syntactic, we mean that we do not distinguish actions by their *effects*, ie, we do not view these actions of elements of M as maps from $\star$ to $\star$ (because in that case, we would need there to be enough maps from $\star$ to $\star$ to account for all elements of M). Instead, we view these actions as *labels* on the arrows from $\star$ to itself.
3. The identity morphism on $\star$ is the action of the identity element $e \in M$.
4. Composition of morphisms is given by the monoid operation $\times$. This is associative

because the monoid operation is associative, and the identity morphism is an identity with respect to this composition because it is associated with the identity element of the monoid.

A specific thing that we can take $\star$ to be is the monoid M itself (ie, $|\mathcal{C}| = \{M\}$). Then, the morphisms in $\mathcal{C}$ correspond to the monoid homomorphisms $M \to M$ by (left-)multiplication by elements of M. In other words, we describe $\mathcal{C}$ by the standard action of M on itself.

In the above examples, the category **Set** stands out as being the 'largest': in the other two examples, the class of objects was actually a set. This is not true in **Set**. That being said, in all our examples, the morphisms between any two objects formed a set. We make two definitions here to capture this idea.

Definition 1.1.5 (Locally Small Categories). A category is **locally small** if the class of morphisms between any two objects is a set.

In this module, we will not study any categories that are not locally small. We next define a category that is even more restrictive.

Definition 1.1.6 (Small Category). A category is **small** if it is locally small and the class of objects is a set.

All the examples we have discussed so far are of locally small categories. **Set**, however, is not a small category, whereas pre-ordered sets and monoids are small categories.

Finally, we define a construction that flips arrows in a category.

Definition 1.1.7 (The Opposite Category). Given a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined as follows.

1. The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as the objects of $\mathcal{C}$.
2. For each pair of objects $A, B \in |\mathcal{C}|$, we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.
3. For each object $A \in |\mathcal{C}|$, the identity morphism in $\mathcal{C}^{\mathrm{op}}$ is the same as the identity morphism in $\mathcal{C}$, i.e., id_A.

4. For each ordered triple of objects $A, B, C \in |\mathcal{C}|$, the composition map in $\mathcal{C}^{\text{op}}$ is defined by reversing the order of composition in $\mathcal{C}$, i.e., for $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(B, C)$, we define $g \circ f$ in $\mathcal{C}^{\text{op}}$ to be $f \circ g$ in $\mathcal{C}$.

One can show that the above data does, indeed, form a category.

1.1.2 Properties of Morphisms

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1.1.3 Functors

It turns out that we can define a meaningful notion of mapping categories to categories.

Definition 1.1.8 (Covariant Functor). Given categories $\mathcal{C}$ and $\mathcal{D}$, a **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ associates

1. To each object $A \in |\mathcal{C}|$, an object $F(A) \in |\mathcal{D}|$.
2. To each pair of objects $A, B \in |\mathcal{C}|$, a map $F : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ such that
   (i) For all objects $A \in |\mathcal{C}|$, $F(\text{id}_A) = \text{id}_{F(A)}$.
   (ii) For all objects $A, B, C \in |\mathcal{C}|$ and morphisms $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C)$, $F(g \circ f) = F(g) \circ F(f)$.

A functor is essentially something that associates objects to objects and arrows to arrows. Covariance means that arrows are preserved. We also have a notion of functors that flip arrows.

Definition 1.1.9 (Contravariant Functor). Given categories $\mathcal{C}$ and $\mathcal{D}$, a **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$.

To some degree, we can view functors as 'structure-preserving maps' between categories, ie, as 'morphisms' between categories.

Example 1.1.10 (The Category of Small Categories). We denote by **Cat** the category whose objects are small categories and whose morphisms are functors between small categories. The identity morphism on a small category $\mathcal{C}$ is the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. The composition operation on morphisms is the composition operation on functors.

There are many examples of functors with which we are familiar.

Example 1.1.11 (Exponential Functors). Recall from **??** that monoids can be viewed as categories. Consider the monoids $(\mathbb{N}, 0, +)$ and $(\mathbb{N}, 1, \times)$. We can define a functor $F : (\mathbb{N}, 0, +) \to (\mathbb{N}, 1, \times)$ by $F(+) = \times$ and $F(n) = 2^n$. This is a functor because it preserves the monoid structure.

1.2 Natural Transformations

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1.2.1 Equivalences of Categories

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1.2.2 Adjoint Pairs

Often, categories will be related but not quite equivalent. The idea is to define a weak notion of equivalence using the concept of **adjunction**. This is a very important concept in category theory, relevant not only to mathematicians but also to computer scientists.

The way that adjunctions are expressed are in terms of functors going both ways between two categories. These functions will express an adjunction if they are an **adjoint pair**, ie, if there is a specific relationship between them.

Definition 1.2.1 (Adjoint Pair). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. sorry

Example 1.2.2 (The Curry Adjunction).

1.3 Categorical Constructions

Exercises

Here, we provide solutions to the weekly problem sheets.

Problem Sheet 1

Exercise 1.1. *Foo*

Solution. This is how you write a solution.

For the latest version of these notes, visit https://thefundamentaltheor3m.github.io/HomAlgNotes/LastLocallyCompiled.pdf. For any suggestions or corrections, please feel free to fork my repository and make a pull request.