Arithmetic and Geometric Sequences

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1 Sequences: A Brief Introduction

Generally speaking, we can think of a (real) sequence as being an infinite list of real numbers that can be indexed by the natural numbers. That is, a sequence has a well-defined first element, second element, third element, and so on.

It is important to note that certain things cannot be “sequenced”: for instance, a famous argument known as **Cantor’s Diagonal Argument** shows that you cannot fit every single real number into a sequence. Indeed, it isn’t too difficult to see intuitively that if you look at the interval $[0, 1]$, you cannot possibly list down every single element. You can think of this as being due to the fact that you can have as many decimal places as you want, for example.

Anyway, the bottom line is, a sequence is a list of numbers as well as a well-defined way of enumerating them. We usually denote sequences by something like

$$(a_n)_{n=1}^{\infty}$$

consisting of elements $a_1, a_2, a_3, a_4, \ldots$ which all stand for real numbers. The letter a is used to identify the sequence—that is, it is the “name” of the sequence—and the subscripts, also called indices, are used to denote various numbers in the sequence. Formally speaking, one can think of a sequence as a function taking in natural numbers (ie, some element of $\mathbb{N} = \{1, 2, 3, \cdots\}$) and outputting real numbers. So, for every index n, there is some real number a_n associated with the sequence a.

2 Arithmetic Sequences

2.1 What is an Arithmetic Sequence?

Definition 2.1 (Arithmetic Sequence). We say that a sequence $(a_n)_{n=1}^{\infty}$ is an **arithmetic sequence** if there is some number d, known as the **common difference**, such that for all $n \in \mathbb{N}$,

$$a_{n+1} = a_n + d \tag{2.1}$$

In other words, adding d to any term in the sequence gives the next term.

Exercise 2.1 (An Equivalent Characterisation)**.** Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence. Show that the nth term a_n of the sequence is given by

$$a_n = a_1 + d \cdot (n-1) \tag{2.2}$$

2.2 Examples of Arithmetic Sequences

Arithmetic sequences appear everywhere, and we are already used to working with them, even if we may not realise it.

Example 2.2. We already know of the following arithmetic sequences:

1. The sequence natural numbers $1, 2, 3, 4, 5, 6, \dots$ form an arithmetic sequence with first term 1 and common difference 1.[1]
2. The two-times-table $2, 4, 6, 8, 10, 12, \dots$ is an arithmetic sequence with first term 2 and common difference 2. Indeed, an analogous result is true for *any* multiplication table.
3. The odd numbers $1, 3, 5, 7, 9, 11, \dots$ is also an arithmetic sequence with first term 1 and common difference 2.

Given two terms of an arithmetic sequence, it is easy to compute the common difference d by looking at the **number of differences** separating the two terms. For example:

Example 2.3. Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence with $a_2 = 5$ and $a_5 = 14$. Find the common difference of the sequence.

Solution. Since a_2 and a_5 are $5 - 2 = 3$ terms apart, adding d to a_2 three times should give a_5. In other words,

$$\begin{aligned} && 5 + 3d &= 14 \\ \iff && d &= \frac{14-5}{3} \\ && &= 3 \end{aligned}$$

[1]Formally speaking, this isn't quite true: the sequence itself is the *map* $a_n = n$ that corresponds to the trivial inclusion map from $\mathbb{N}$ to $\mathbb{R}$ that sends every element of $\mathbb{N}$ to itself. We have essentially identified each sequence with its elements, as we are more interested in those than the maps themselves.

Try the following yourself.

Exercise 2.4. Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence. In each of the following cases, find the common difference and the first term.

1. $a_5 = 10$, $a_{45} = 90$
2. $a_3 = 2$, $a_7 = 4$
3. $a_{365} = 364$, $a_{1001001} = 1001000$

2.3 Sum of n Terms of an Arithmetic Sequence

It turns out that there is a rather ingenious trick to sum the first n terms of an arithmetic sequence.

Example 2.5 (Gauss)**.** It is said that the renowned German mathematician, Carl Friedrich Gauss, came up with the following trick as a child when his teacher asked him to add the first 100 natural numbers as a punishment for misbehaving in class.

He first wrote down the numbers like so:

$$S = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$

Then, he wrote another copy below, like so:

$$\begin{array}{ccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 98 & + & 99 & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 3 & + & 2 & + & 1 \end{array}$$

Then, he added the numbers column-wise:

$$\begin{array}{ccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 98 & + & 99 & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 3 & + & 2 & + & 1 \\ \hline 2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 & + & 101 \end{array}$$

In other words, $2S = 101 \times 100$, meaning $S = 5050$.

If you pay close attention to Gauss's method, there is nothing particularly special about the first 100 natural numbers. They can just as easily be replaced by any other sequence, as we shall soon see.

Exercise 2.6. Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence and let n be any natural number. Show that for any natural number j lying between 1 and n (inclusive),

$$a_j + a_{n-j+1} = a_1 + a_n \tag{2.3}$$

We therefore have the following result:

Theorem 2.7. Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence. Let

$$S_n := \sum_{j=1}^{n} a_j \tag{2.4}$$

denote the sum of the first n terms of $(a_n)_{n=1}^{\infty}$—in other words, $S_n = a_1 + a_2 + \cdots + a_n$. Then, we have

$$S_n = \frac{n}{2}(a_1 + a_n) \tag{2.5}$$

Proof. We use the same argument as Example 2.5:

$$\begin{array}{ccccccccccccccccc}
S_n & = & a_1 & + & a_2 & + & 3 & + & \cdots & + & a_{n-2} & + & a_{n-1} & + & a_n \\
S_n & = & a_n & + & a_{n-1} & + & a_{n-2} & + & \cdots & + & a_3 & + & a_2 & + & a_1 \\
\hline
2S_n & = & (a_1 + a_n) & + & (a_1 + a_n) & + & (a_1 + a_n) & + & \cdots & + & (a_1 + a_n) & + & (a_1 + a_n) & + & (a_1 + a_n)
\end{array}$$

where the last line follows from Exercise 2.6.

Therefore,

$$\begin{aligned}
& 2S_n = n(a_1 + a_n) \\
\iff \quad & S_n = \frac{n}{2}(a_1 + a_n)
\end{aligned}$$

as required. □

Hereafter, we will discard the summation notation and instead use S_n to denote the sum of the first n terms of a given sequence.

Sometimes, we do not know what the nth term of some sequence is. Therefore, the following equivalent formula, which is easily derived, is quite useful.

Exercise 2.8 (An Equivalent Formula)**.** Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence with common difference d. Then,

$$S_n = \frac{n}{2}\left(2a_1 + d\left(n-1\right)\right) \tag{2.6}$$

It is common to be asked questions involving adding up terms of a sequence whose nth term formula is not explicitly given. In such cases, one simply needs to find the first term and common difference and apply the formula. Once one gets more comfortable with arithmetic sequences, one will start seeing patterns more intuitively, and be able to add up terms more easily. The key for this is, of course, **practice**.

Exercise 2.9. Let $(a_n)_{n=1}^{\infty}$ be an arithmetic sequence. In each of the following cases, find the sum of the first n terms.

1. $a_1 = 20$, $d = 3$, $n = 10$
2. $a_1 = 3$, $a_6 = 101$, $n = 6$
3. $a_5 = 49$, $a_7 = 51$, $n = 11$
4. $a_4 = 14$, $a_6 = 22$, $n = 15$

Hint: The formula (2.5) *can be rewritten as*

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Is this a more helpful form? Think about the quantity in parentheses.