

MATH70062: Lie Algebras

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Chapter 1

An Introduction to the Theory of Lie Algebras

While many of the definitions and constructions we shall see in this course can easily be adapted to any field, we will work over \mathbb{C} for simplicity, unless otherwise stated.

1.1 Important Definitions and First Examples

We will begin by defining the fundamental objects of study in this course. We will then provide some examples of these objects and discuss means of constructing them.

1.1.1 Algebras

We begin by recalling the notion of a bilinear map.

Definition 1.1.1 (Bilinear Map). Let V and W be vector spaces. We say that a map $f : V \times W \rightarrow \mathbb{C}$ is **bilinear** if it is linear in each argument. That is, for all $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{C}$, we have

$$f(v + v', w) = f(v, w) + f(v', w)$$

$$f(v, w + w') = f(v, w) + f(v, w')$$

$$f(\lambda v, w) = \lambda f(v, w) = f(v, \lambda w)$$

We will be particularly interested in bilinear maps from a vector space to itself.

Definition 1.1.2 (Algebra). An **algebra** is a vector space A equipped with a bilinear map $\cdot : A \times A \rightarrow A$.

Convention. Given any algebra A , we will often refer to the corresponding bilinear map \cdot as the **multiplication** map of A , and denote $\cdot(x, y)$ as simply $x \cdot y$ or even xy (where the definition of \cdot is clear from the context) for any $x, y \in A$.

There are many different kinds of algebras. We will be particularly interested in Lie algebras and associative algebras.

Definition 1.1.3 (Associative Algebras). We say that an algebra A is **associative** if the multiplication map \cdot is associative. That is, for all $x, y, z \in A$, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

We have all seen associative algebras before.

Example 1.1.4 (The Matrix Algebra). The set $M_n(\mathbb{C})$ of $n \times n$ matrices over \mathbb{C} forms an associative algebra under matrix multiplication, known as the Matrix Algebra.

We will come back to associative algebras soon enough. We will now define the main object of study in this module.

Definition 1.1.5 (Lie Algebras). A **Lie algebra** is an algebra L whose bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ satisfies the following properties:

1. For all $x \in L$, we have $[x, x] = 0$.
2. For all $x, y, z \in L$, we have

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (1.1.1)$$

Such a bilinear map $[\cdot, \cdot]$ is known as a **Lie Bracket**, and (1.1.1) is known as the **Jacobi**

Identity.

Remark. We immediately notice that the first condition (over not just \mathbb{C} but any field) implies the fact that

$$[x, y] = -[y, x] \quad (1.1.2)$$

One simply needs to apply bilinearity and the first condition to evaluate $[x + y, x + y]$. This argument reverses nicely as well, but only over fields of characteristic $\neq 2$.

One may recall that the $[\cdot, \cdot]$ notation is often used in group theory to denote the **commutator** of two elements. The reason why the same notation is used for the Lie bracket is the following.

Lemma 1.1.6. *Let A be an associative algebra. Then, the commutator map $[x, y] = xy - yx$ is a Lie bracket on A .*

Proof. Clearly, $[x, x] = xx - xx = 0$ for all $x \in A$. We now show that $[\cdot, \cdot]$ satisfies (1.1.1): for all $x, y, z \in A$, we have

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] \\ &= 6xyz - 6xyz = 0 \end{aligned}$$

where we skip over some of the intermediate computations because they are tedious and uninteresting. □

Lemma 1.1.6 gives us a large class of examples of Lie algebras. One of the most important of these is the following.

Example 1.1.7 (General Linear Lie Algebra). For all $n \in \mathbb{N}$, the set of all $n \times n$ matrices forms a Lie algebra under the commutator bracket: this follows immediately from applying Lemma 1.1.6 to Example 1.1.4. We call this the **General Linear Lie Algebra**, denoted $\mathfrak{gl}(n)$.

Convention. We will denote by $M_n(\mathbb{C})$ the set of all $n \times n$ matrices, viewed (interchangeably) as a *set*, a *vector space* or an *associative algebra*. When viewing it as a *Lie algebra under the commutator bracket*, we will adopt the notation $\mathfrak{gl}(n, \mathbb{C})$, where \mathbb{C} can be replaced by any field. We will usually abbreviate this to $\mathfrak{gl}(n)$, because we will primarily work over \mathbb{C} .

Lastly, we will define the notion of an abelian Lie algebra.

Definition 1.1.8 (Abelian Lie Algebra). A Lie algebra A is said to be **abelian** if for all $x \in A$, we have $[x, x] = 0$.

The reason for this terminology is that if A is an associative algebra whose multiplication map is commutative, then its commutator bracket is identically zero, making the corresponding Lie algebra abelian.

Example 1.1.9. Clearly, $\mathfrak{gl}(1)$ is abelian: for all $x, y \in \mathfrak{gl}(1) = \mathbb{C}$, we have $xy - yx = 0$.

We will now define subalgebras and homomorphisms of algebras, which will allow us to construct more examples of algebras (Lie and otherwise).

1.1.2 Subalgebras and Homomorphisms

As with objects in any category, we have subobjects and morphisms. We will define these over general algebras and apply them to get more examples of Lie algebras.

Definition 1.1.10 (Subalgebras). Let A be a vector space. A **subalgebra** of A is a subspace $B \subseteq A$ such that B is closed under the multiplication map of A . That is, for all $x, y \in B$, we have $x \cdot y \in B$.

Convention. Given an algebra A and a subset $B \subseteq A$, we will denote the statement that B is a subalgebra of A by $B \leq A$.

Definition 1.1.11 (Homomorphisms). Let A and B be algebras. A **homomorphism** $\phi : A \rightarrow B$ is a linear map that respects the multiplication maps of A and B . That is, for all $x, y \in A$, we have

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y)$$

Convention. We will define Lie subalgebras to be subalgebras with respect to the algebra structure given by the Lie bracket, and we will define Lie algebra homomorphisms to be homomorphisms that respect the Lie bracket (ie, that are algebra homomorphisms with respect to the algebra structure given by the Lie bracket).

We have the following unsurprising result.

Lemma 1.1.12. *Let A and B be algebras, and let $\phi : A \rightarrow B$ be a homomorphism. Then,*

1. $\text{im}(\phi) \leq B$
2. $\text{ker}(\phi) \leq A$

Proof. These are standard results, but we will prove them for completeness.

1. Fix $x, y \in \text{im}(\phi)$. Then, there exist $a, b \in A$ such that $\phi(a) = x$ and $\phi(b) = y$. Since ϕ is a homomorphism, we have

$$x \cdot y = \phi(a) \cdot \phi(b) = \phi(a \cdot b) \in \text{im}(\phi)$$

so $\text{im}(\phi)$ is closed under the multiplication map of B .

2. Let $x, y \in \text{ker}(\phi)$. Then, we have

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) = 0 \cdot 0 = 0$$

where the last equality follows from the fact that \cdot is bilinear. Therefore, $x \cdot y \in \text{ker}(\phi)$, and $\text{ker}(\phi)$ is closed under the multiplication map of A .

□

This allows us to construct another matrix Lie algebra.

Example 1.1.13 (The Special Linear Lie Algebra). For all $n \in \mathbb{N}$, consider the trace map $\text{Tr} : \mathfrak{gl}(n) \rightarrow \mathfrak{gl}(1)$. This is a (Lie) algebra homomorphism: for all $A, B \in \mathfrak{gl}(n)$,

$$\text{Tr}([A, B]) = \text{Tr}(AB - BA) = \text{Tr}(AB) - \text{Tr}(BA) = 0 = [\text{Tr}(A), \text{Tr}(B)]$$

because the Lie algebra $\mathfrak{gl}(1)$ is abelian (see Example 1.1.9). By Lemma 1.1.12, its kernel, the set of all $n \times n$ matrices of trace zero, is a Lie subalgebra of $\mathfrak{gl}(n)$. We call this the **Special Linear Lie Algebra**, denoted $\mathfrak{sl}(n)$.

Remark. In Example 1.1.13, we have indirectly shown that

$$\text{im}([\cdot, \cdot]) = [\mathfrak{gl}(n), \mathfrak{gl}(n)] \subseteq \mathfrak{sl}(n)$$

because of the unique property of the trace that $\text{Tr}(AB) = \text{Tr}(BA)$ for any $A, B \in \mathfrak{gl}(n)$.

The very natural relationship between associative and Lie algebra structures given by Lemma 1.1.6 gives us an elegant criterion for proving that a subspace is a subalgebra of a Lie algebra whose Lie bracket is the commutator of an associative bilinear map.

Proposition 1.1.14. *Let (A, \cdot_A) be an associative algebra and let (B, \cdot_B) be a subalgebra of A . Denoting by $(A, [\cdot, \cdot]_A)$ the Lie algebra whose Lie bracket is the commutator of the multiplication map of A and by $(B, [\cdot, \cdot]_B)$ the Lie algebra whose Lie bracket is the commutator of the multiplication map of B , we have $B' \leq A'$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} (A, \cdot_A) & \longrightarrow & (A, [\cdot, \cdot]_A) \\ \uparrow \text{Associative Subalgebra} & & \uparrow \text{Lie Subalgebra} \\ (B, \cdot_B) & \longrightarrow & (B, [\cdot, \cdot]_B) \end{array} \quad (1.1.3)$$

Proof. First, observe that $[\cdot, \cdot]_B = [\cdot, \cdot]_A|_B$ (ie, the Lie bracket obtained from \cdot_B agrees with the

one obtained from \cdot_A on B): for all $T_1, T_2 \in B$,

$$[T_1, T_2]_B = T_1 \cdot_B T_2 - T_2 \cdot_B T_1 = T_1 \cdot_A T_2 - T_2 \cdot_A T_1 = [T_1, T_2]_A$$

Therefore, since B is closed under $[\cdot, \cdot]_B$ (which, by definition, is a map from $B \times B$ to B), B must be closed under $[\cdot, \cdot]_A$. \square

This allows us to construct more examples still.

Example 1.1.15 (The Upper-Triangular Lie Algebra). For $n \in \mathbb{N}$, we define the **Upper-Triangular Lie Algebra** to be the set of all $n \times n$ upper-triangular matrices (with respect to some predetermined basis), denoted $\mathfrak{t}(n)$. Given that the product of upper-triangular matrices is upper-triangular, $\mathfrak{t}(n)$ forms an associative subalgebra of $M_n(\mathbb{C})$, and therefore, a Lie subalgebra of $\mathfrak{gl}(n)$.

1.1.3 Ideals

Throughout this subsection, we will denote by L an arbitrary Lie algebra.

Definition 1.1.16 (Ideal). We say that $I \subseteq L$ is an **ideal** of L , denoted $I \trianglelefteq L$, if I is a linear subspace of L and $[x, y] \in I$ for all $x \in L$ and $y \in I$.

Convention. We will use the notation $[I, L]$ to denote the subspace of L spanned by all elements of the form $[i, \ell]$ for $i \in I$ and $\ell \in L$.

Remark. We could equivalently require that $[I, L] \leq L$ in the definition of an ideal instead of requiring that $[x, y] \in I$ for all $x \in L$ and $y \in I$. Similarly, we can observe that it doesn't matter whether we require $[x, y] \in I$ or $[y, x] \in I$ because of (1.1.2) and bilinearity.

Example 1.1.17 (Trivial Ideals). Given any Lie algebra L , both $\{0\}$ and L are ideals of L .

In certain respects, despite their name, ideals of Lie algebras are more like normal subgroups of a group than they are like ideals of a ring.

Lemma 1.1.18. *Any ideal $I \trianglelefteq L$ is also a subalgebra of L .*

Proof. This is clear from Definition 1.1.16. □

Lemma 1.1.19. *For any Lie algebra K and homomorphism $\phi : L \rightarrow K$, we have $\ker(\phi) \trianglelefteq L$.*

Proof. From Lemma 1.1.12, we know that $\ker(\phi)$ is a linear subspace of L . We now need to show that $[x, y] \in \ker(\phi)$ for all $x \in L$ and $y \in \ker(\phi)$. To that end, fix $x \in \ker(\phi)$ and $y \in L$. Then,

$$\phi([x, y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0$$

proving that $[x, y] \in \ker(\phi)$ as required. □

We come back to the theme of the Lie bracket being some sort of ‘commutator’ when we define the notion of the centre of a Lie algebra: the terminology and notation match those from group theory, where the centre consists of elements that commute with every other element of the group (making its commutator with every other element the identity).

Definition 1.1.20 (The Centre of a Lie Algebra). We define the **centre** of L to be

$$Z(L) := \{x \in L \mid \forall y \in L, [x, y] = 0\}$$

Lemma 1.1.21. *$Z(L)$ is an ideal of L .*

Proof. The fact that $Z(L)$ is a subspace of L follows from the fact that $[\cdot, \cdot]$ is bilinear. Now, fix $x \in Z(L)$ and $y \in L$. Clearly, $[x, y] = 0$, and it is easily seen that $0 \in Z(L)$. □

Example 1.1.22. For all $n \in \mathbb{N}$,

$$Z(\mathfrak{gl}(n)) = \{A \in \mathfrak{gl}(n) \mid \exists \lambda \in \mathbb{C} \text{ s.t. } A = \lambda I\}$$

Proof. Let $S := \{A \in \mathfrak{gl}(n) \mid \exists \lambda \in \mathbb{C} \text{ s.t. } A = \lambda I\}$. It is clear that $S \subseteq Z(\mathfrak{gl}(n))$. Now, fix $A \in Z(\mathfrak{gl}(n))$. Then, for all $B \in \mathfrak{gl}(n)$, we have that $[A, B] = AB - BA = 0$. In particular, this implies that A commutes with all the elementary matrices E_{ij} , which are the matrices

with a 1 in the ij -th position and 0 elsewhere. Therefore, A must be a diagonal matrix. \square

It turns out that ideals are well-behaved under several operations.

Proposition 1.1.23 (The Behaviour of Ideals). *Let $I, J \trianglelefteq L$. Then,*

1. $I + J \trianglelefteq L$
2. $I \cap J \trianglelefteq L$
3. $[I, J] := \text{Span}(\{[i, j] \mid i \in I, j \in J\}) \trianglelefteq L$
4. *If $I + J = J$, then $I \subseteq J$.*

Proof. sorry

\square

The abelian case is particularly nice.

Proposition 1.1.24 (Ideals of an Abelian Lie Algebra). *Let L be abelian. Then, every sub-vector space of L is an ideal of L .*

Proof. Let I be a sub-vector space of L . Then, for all $x \in L$ and $y \in I$, we have $[x, y] = 0$. Since I is a subspace, we must have $0 \in I$, proving that I is an ideal of L . \square

We end by defining a special kind of ideal, which will become rather important.

Definition 1.1.25 (Derived Subalgebra). The **derived subalgebra** of L , denoted L' , is the ideal (and subalgebra) $[L, L]$.

Note that L' is, indeed, an ideal, by the third property proven in Proposition [1.1.23](#).

Convention. Though L' is an ideal, we will often refer to it as either the **derived subalgebra** or the **commutator subalgebra** of L . Indeed, Lemma [1.1.18](#) tells us that this is a reasonable, if not the most completely descriptive, thing to do.

1.1.4 Quotients

We now define the notion of a quotient (Lie) algebra. For the remainder of this subsection, let L be a Lie algebra and I an arbitrary ideal of L . Given that we already have a notion of L/I —recall that I is a subspace of L , meaning we can take the quotient in a linear algebraic sense—it seems only natural to attempt to define a Lie bracket on this vector space. It turns out that the definition of an ideal allows us to do this in a very natural way.

Proposition 1.1.26. *Consider the vector space L/I . The map $[\cdot, \cdot] : L/I \times L/I \rightarrow L/I$ given by*

$$[x + I, y + I] := [x, y] + I \quad (1.1.4)$$

for all $x, y \in L$ is a Lie bracket on L/I .

Proof. We begin by showing that the Lie bracket on L/I is well-defined. Fix $x, x', y, y' \in L$ with $x - x' = i \in I$ and $y - y' = j \in I$, so that $x + I = x' + I$ and $y + I = y' + I$. Then,

$$\begin{aligned} [x, y] - [x', y'] &= [x' + i, y' + j] - [x', y'] \\ &= [\cancel{x'}, y'] + [i, y'] + [x', j] + [i, j] - [\cancel{x'}, y'] \\ &= [i, y'] + [x', j] + [i, j] \in I \end{aligned}$$

because I is an ideal, proving that $[x, y] + I = [x', y'] + I$, making the choice of representative irrelevant and the bracket on L/I well-defined.

From the definition of $[\cdot, \cdot]$ on L/I , it is clear that $[x + I, x + I] = 0$ for all $x \in L$. Now, for all $x, y, z \in L$, notice that

$$[x + I, [y + I, z + I]] = [x + I, [y, z] + I] = [x, [y, z]] + I$$

The Jacobi identity follows immediately. □

Definition 1.1.27 (Quotient Algebra). The **quotient algebra** of L with respect to I is the vector space L/I equipped with the bracket defined in (1.1.4), which we showed to be

a Lie bracket in Proposition 1.1.26 above.

Example 1.1.28 (Quotienting by the Derived Subalgebra). The quotient of L by L' is always an abelian Lie algebra.

The centre is particularly well-behaved under taking quotients, a fact we will use when studying a class of Lie algebras called *nilpotent* Lie algebras.

Proposition 1.1.29. *Let $\phi : L \twoheadrightarrow L/Z(L)$ be the quotient epimorphism. Then, $\phi(Z(L)) = Z(\phi(L)) = Z(L/Z(L))$.*

Indeed, we can show that the map $x \mapsto x + I : L \rightarrow L/I$ is a Lie algebra homomorphism. More generally, we have the following results.

1.1.5 Isomorphism Theorems

Our favourite isomorphism theorems do, indeed, hold in the category of Lie algebras. Throughout this subsection, let L be a Lie algebra.

Theorem 1.1.30 (First Isomorphism Theorem). *Let K be a Lie algebra and $\phi : L \rightarrow K$ a Lie algebra homomorphism. Then,*

$$L/\ker(\phi) \cong \text{im}(\phi) \quad (1.1.5)$$

Theorem 1.1.31 (Second Isomorphism Theorem). *Let $I, J \trianglelefteq L$. Then,*

$$I + J/I \cong J/I \cap J \quad (1.1.6)$$

We also have a correspondence between ideals of L and ideals of L/I .

Theorem 1.1.32 (The Correspondence Theorem). *Let $I \trianglelefteq L$. Then, there is a one-to-one correspondence between the ideals of L containing I and the ideals of L/I . I.e., there is a*

bijection

$$\{J \trianglelefteq L \mid J \supseteq I\} \longleftrightarrow \{J \trianglelefteq L/I\} \quad (1.1.7)$$

Proof.

□

Note that each of the sets in (1.1.32) is partially ordered by inclusion.

1.1.6 Adjoints

Throughout this subsection, V will refer to a finite-dimensional vector space.

We begin with a general Lie algebra construction.

Definition 1.1.33 (General Linear Lie Algebra over an Arbitrary Vector Space). We define the **General Linear Lie Algebra over V** to be the set of all linear maps from V to V , viewed as a Lie algebra under the commutator bracket. We denote it $\mathfrak{gl}(V)$.

That this is, indeed, a Lie algebra should come as no surprise. Given that this construction is well-defined over *any* vector space, we can, in particular, apply it to Lie algebras.

For the remainder of this subsection, let L denote an arbitrary Lie algebra. It turns out that we can define a rather nice map that relates L with $\mathfrak{gl}(L)$: the adjoint.

Definition 1.1.34 (Adjoint Map). To every $x \in L$, we can associate the linear map

$$\text{ad}(x) : L \rightarrow L : y \mapsto [x, y]$$

We call this map the **adjoint map** associated to x .

Proposition 1.1.35. *The adjoint map $\text{ad} : L \rightarrow \mathfrak{gl}(L)$ is a Lie algebra homomorphism.*

Proof. That ad is linear follows from the fact that $[\cdot, \cdot]$ is bilinear. Now, fix $x, y \in L$, and consider the map $\text{ad}([x, y]) \in \mathfrak{gl}(L)$. We need to show that

$$\text{ad}([x, y]) = \text{ad}(x) \text{ad}(y) - \text{ad}(y) \text{ad}(x)$$

because the Lie bracket on $\mathfrak{gl}(L)$ is the commutator with respect to composition of linear maps. To that end, fix $z \in L$. Then,

$$\begin{aligned} (\text{ad}(x) \text{ad}(y) - \text{ad}(y) \text{ad}(x))(z) &= \text{ad}(x)(\text{ad}(y)(z)) - \text{ad}(y)(\text{ad}(x)(z)) \\ &= \text{ad}(x)([y, z]) - \text{ad}(y)([x, z]) \\ &= [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [y, [z, x]] && \text{(by (1.1.2))} \\ &= -[z, [x, y]] && \text{(by the Jacobi Identity)} \\ &= [[x, y], z] \\ &= \text{ad}([x, y])(z) \end{aligned}$$

□

Furthermore, we make the following observation:

Lemma 1.1.36. $\ker(\text{ad}) = Z(L)$

Proof. This is immediate. We only state the result to highlight it. □

1.1.7 Derivations

Throughout this subsection, let A be an arbitrary algebra with multiplication \cdot .

Definition 1.1.37. We say that a linear map $D : A \rightarrow A$ is a **derivation** if it satisfies the Leibniz rule, ie, if

$$D(x \cdot y) = x \cdot D(y) + D(x) \cdot y \tag{1.1.8}$$

for all $x, y \in A$.

Convention. We will denote the set of all derivations of an algebra A by $\text{Der}(A)$.

Recall that since A is a vector space, $\mathfrak{gl}(A)$ is a Lie algebra with respect to the commutator bracket (cf. Definition 1.1.33). It turns out there is a relationship between $\text{Der}(A)$ and $\mathfrak{gl}(A)$.

Proposition 1.1.38. $\text{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proof. That $\text{Der}(A)$ is a subspace of $\mathfrak{gl}(A)$ is not too difficult to show: it is clear that the zero map satisfies (1.1.8), and it readily follows from the bilinearity of \cdot that $\text{Der}(A)$ is closed under addition and scalar multiplication.

We now need to show that $\text{Der}(A)$ is closed under the commutator bracket. Fix $D, E \in \text{Der}(A)$. We need to show that $[D, E] = DE - ED$ satisfies (1.1.8). Indeed, for all $x, y \in A$,

$$\begin{aligned} (DE - ED)(x \cdot y) &= D(E(x \cdot y)) - E(D(x \cdot y)) \\ &= D(x \cdot E(y) + E(x) \cdot y) - E(x \cdot D(y) + D(x) \cdot y) \end{aligned}$$

which can be simplified, if tediously, to the desired form. □

Most readers will have encountered derivations before. We give below a classic example (over \mathbb{R} , for the first time so far) that the reader is sure to recognise.

Example 1.1.39. The space $C^\infty(\mathbb{R})$ of smooth $\mathbb{R} \rightarrow \mathbb{R}$ functions is an \mathbb{R} -algebra under pointwise addition and multiplication. The differentiation map $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) : f \mapsto f'$ is easily seen to be a derivation.

We have also encountered a slightly more sophisticated derivation. For the remainder of this subsection, let L be an arbitrary Lie algebra.

Proposition 1.1.40. For all $x \in L$, the adjoint map $\text{ad}(x) : L \rightarrow L : y \mapsto [x, y]$ associated with x is a derivation.

Proof. We already know that $\text{ad}(x) \in \mathfrak{gl}(L)$. It only remains to show that $\text{ad}(x)$ satisfies (1.1.8)

with respect to $[\cdot, \cdot]$. To that end, fix $y, z \in L$. Then, we have that

$$\begin{aligned} \operatorname{ad}(x)([y, z]) &= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] \\ &= [y, [x, z]] + [[x, y], z] \\ &= [y, \operatorname{ad}(x)(z)] + [\operatorname{ad}(x)(y), z] \end{aligned}$$

as required. □

Abbreviating the set $\{\operatorname{ad}(x) \mid x \in L\}$ of all adjoint maps on L to $\operatorname{ad}(L)$, we have the following chain of Lie subalgebras:

Lemma 1.1.41. $\operatorname{ad}(L) \leq \operatorname{Der}(L) \leq \mathfrak{gl}(L)$

Proof. sorry □

1.1.8 Structure Constants

Fix $n \in \mathbb{N}$, and let L be an n -dimensional Lie algebra. Consider the \mathbb{C} -basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of L . Given the fundamentally linear algebraic nature of Lie algebras, it is natural to study what happens when we apply the Lie bracket to elements of \mathcal{B} .

Definition 1.1.42 (Structure Constants). Fix $i, j \in \{1, \dots, n\}$. We know that there exist unique constants $s_{ij1}, s_{ij2}, \dots, s_{ijn}$ such that

$$[e_i, e_j] = \sum_{k=1}^n s_{ijk} e_k$$

We call the scalars $\{s_{ijk}\}_{1 \leq i, j, k \leq n}$ the **structure constants** of L with respect to \mathcal{B} .

1.1.9 Direct Sums

In this subsection, we briefly describe the theory of the direct sum of two Lie algebras. Let L_1 and L_2 be arbitrary Lie algebras. Just as we did in Proposition 1.1.26, we will define a Lie bracket on

the vector space $L_1 \oplus L_2$, and define the Lie algebra direct sum of L_1 and L_2 to be this vector space equipped with this bracket.

Proposition 1.1.43. *Define the map $[\cdot, \cdot] : (L_1 \oplus L_2) \times (L_1 \oplus L_2) \rightarrow (L_1 \oplus L_2)$ given by*

$$[x_1 \oplus x_2, y_1 \oplus y_2] := [x_1, y_1] \oplus [x_2, y_2] \quad (1.1.9)$$

for all $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. Then, $[\cdot, \cdot]$ is a Lie bracket on $L_1 \oplus L_2$.

Proof. **sorry**

□

Definition 1.1.44 (Direct Sum). The **direct sum** of L_1 and L_2 is the vector space $L_1 \oplus L_2$ equipped with the bracket defined in (1.1.9), which we showed to be a Lie bracket in Proposition 1.1.43 above.

We can repeat this definition successively to define the direct sum of any finite number of Lie algebras. We will not explore this idea in any more detail and will take it for granted.

1.2 Lie Algebras of Dimension ≤ 3

It turns out that we do not need any particularly sophisticated machinery to classify all Lie algebras of dimension less than or equal to 3.

1.2.1 Abelian Lie Algebras and Lie Algebras of Dimension 1

We begin with a simple observation about abelian Lie algebras.

Proposition 1.2.1. *Fix $n \in \mathbb{N}$. Then, any abelian Lie algebra of dimension n is isomorphic to \mathbb{C}^n with the zero bracket.*

Proof. Let L be a Lie algebra of dimension n . We know there exists a \mathbb{C} -linear isomorphism $\phi : L \rightarrow \mathbb{C}^n$. It follows immediately that for any $x, y \in L$,

$$\phi([x, y]) = \phi(0) = 0 = [\phi(x), \phi(y)]$$

A similar argument will show that $\phi^{-1} : \mathbb{C}^n \rightarrow L$, viewed as a linear map, is a Lie algebra homomorphism as well, proving that $L \cong \mathbb{C}^n$. \square

The classification of Lie algebras in 1 dimension is then straightforward. We will begin by a rather strong but straightforward result on one-dimensional subspaces of Lie algebras.

Proposition 1.2.2. *Let L be a Lie algebra. Any 1-dimensional subspace of L is an abelian Lie subalgebra.*

Proof. Let K be a sub-vector space of dimension 1. We know any \mathbb{C} -basis of K consists of a single, nonzero element. Consider such a basis element x . For any $y_1, y_2 \in L$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $y_1 = \lambda_1 x$ and $y_2 = \lambda_2 x$. Then,

$$[y_1, y_2] = [\lambda_1 x_1, \lambda_2 x_2] = \lambda_1 \lambda_2 [x, x] = 0$$

proving that $[\cdot, \cdot] = 0$. Since K is a subspace, $0 \in K$, proving that K is a Lie subalgebra. \square

The classification of Lie algebras of dimension 1 is then immediate.

Corollary 1.2.3. *Any Lie algebra of dimension 1 is abelian, isomorphic to \mathbb{C} equipped with the zero bracket.*

Proof. Let L be a Lie algebra of dimension 1. That L is abelian follows from applying Proposition 1.2.2 to L viewed as a subspace of itself. The isomorphism then follows immediately from Proposition 1.2.1. \square

We can now turn our attention to the slightly more non-trivial problem of classifying non-abelian Lie algebras of dimension 2 and 3.

1.2.2 Lie Algebras of Dimension 2

From Proposition 1.2.1, we already know that there is only one abelian Lie algebra of dimension 2. The question remains, how many non-abelian Lie algebras of dimension 2 are there?

We begin by giving an example.

Example 1.2.4 (A Two-Dimensional Non-Abelian Lie Algebra). Consider the set

$$\mathfrak{t}_2 := \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{C} \right\} = \text{Span} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \subseteq \mathfrak{gl}(2)$$

Clearly, \mathfrak{t}_2 is a linear subspace of $\mathfrak{gl}(2)$. Furthermore, One can show that

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

proving that \mathfrak{t}_2 is closed under the commutator bracket. It follows that \mathfrak{t}_2 is a Lie subalgebra of $\mathfrak{gl}(2)$, and therefore, a 2-dimensional Lie algebra in its own right.

The reason we are interested in the above example will become clear, and we will reserve the notation \mathfrak{t}_2 for this particular Lie algebra. For the remainder of this section, denote by L an arbitrary non-abelian Lie subalgebra of dimension 2.

We will begin by describing the derived subalgebra L' (cf. Definition 1.1.25) of L .

Lemma 1.2.5. *For any \mathbb{C} -basis $\{u, v\}$ of L , we have that $L' = \text{Span}([u, v])$.*

Proof. Let $\{u, v\}$ be a basis of L . Define $x := [u, v]$. Since L is non-abelian, $x \neq 0$, making $X := \text{Span}(x)$ a 1-dimensional subspace of L . Seeing as $L' = [L, L] = \text{Span}(\{[x, y] \mid x, y \in L\})$, it is clear that $L' \supseteq X$. It remains to show that $L' \subseteq X$.

It suffices to show that $\{[x, y] \mid x, y \in L\} \subseteq X$. To that end, fix $a, b \in L$. We know there exist $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$ such that $a = \lambda_1 u + \mu_1 v$ and $b = \lambda_2 u + \mu_2 v$. Then,

$$\begin{aligned} [a, b] &= [\lambda_1 u + \mu_1 v, \lambda_2 u + \mu_2 v] \\ &= \lambda_1 \lambda_2 \underbrace{[u, u]}_{=0} + \lambda_1 \mu_2 [u, v] + \mu_1 \lambda_2 [v, u] + \mu_1 \mu_2 \underbrace{[v, v]}_{=0} \\ &= (\lambda_1 \mu_2 - \mu_1 \lambda_2) [u, v] \in X \end{aligned}$$

as required. □

This tells us, in particular, that the span of the commutator of any basis of L is an ideal. We now

have everything we need to describe L .

Proposition 1.2.6. L is isomorphic to \mathfrak{t}_2 .

Proof. It suffices to show that L admits a basis $\{x, y\}$ such that $[x, y] = y$, as this will readily yield the right structure constants.¹

Let $\{u, v\}$ be an arbitrary \mathbb{C} -basis of L . Let $y := [u, v]$. Since L is non-abelian, $y \neq 0$. Therefore, there exists some $z \in L \setminus \{0\}$ that is linearly independent of y . Since $\text{Span}(y) = L' \trianglelefteq L$, we know that $[z, y] \in L'$. In particular, $\exists \lambda \in \mathbb{C}$ such that $[z, y] = \lambda y$. Furthermore, since y and z are linearly independent and L is non-abelian, $\lambda \neq 0$. So, define $x := \lambda^{-1}z$. Then, x is still linearly independent of y , making $\{x, y\}$ a basis of L , and $[x, y] = y$, as required. \square

Yes, it's true! Up to isomorphism, there is only one non-abelian Lie algebra of dimension 2. Therefore, there are only two Lie algebras of dimension 2: one non-abelian one and one abelian one.

We can now turn our attention to the classification of Lie algebras in dimension 3.

1.2.3 Lie Algebras of Dimension 3

sorry

1.3 Solvability and Nilpotency

We now begin discussing some nontrivial objects in the theory of Lie algebras. Throughout this section, L will denote an arbitrary Lie algebra.

¹Alternatively, if we can show that $[x, y] = y$, it will follow immediately that the linear isomorphism sending x to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and y to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is, indeed, a Lie algebra isomorphism.

1.3.1 Descending Series of Ideals

Definition 1.3.1 (Derived Series). The **derived series** of L is the descending series of ideals

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$$

where $L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 1$.

Definition 1.3.2 (Solvability). L is said to be **solvable** if there exists an $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Definition 1.3.3 (Lower Central Series). The **lower central series** of L is the descending series of ideals

$$L = L^0 \supseteq L^1 \supseteq L^2 \supseteq \dots$$

where $L^i := [L, L^{i-1}]$ for $i \geq 1$.

Convention. Elements of the derived series are denoted $L^{(i)}$, with parenthesised superscript indices, whereas elements of the lower central series are denoted L^i , with no parentheses around the indices.

Definition 1.3.4 (Nilpotency). L is said to be **nilpotent** if there exists an $n \in \mathbb{N}$ such that $L^n = 0$.

Indeed, there is the following relationship between solvability and nilpotency.

Lemma 1.3.5. For all $i \in \mathbb{N}$, $L^i \supseteq L^{(i)}$.

Proof. We argue by induction on i . The base case is trivial, because $L^0 = L = L^{(0)}$. Now, fix $i \in \mathbb{N}$ and assume that $L^i \supseteq L^{(i)}$. Then,

$$L^{i+1} = [L, L^i] = [L, L^{(i)}]$$

$$\begin{aligned}
&= \text{Span}\left(\left\{[\ell, x] \mid x \in L^{(i)}, \ell \in L\right\}\right) \\
&\supseteq \text{Span}\left(\left\{[\ell, x] \mid x \in L^{(i)}, \ell \in L^{(i)}\right\}\right) \\
&= [L^{(i)}, L^{(i)}] = L^{(i+1)}
\end{aligned}$$

where the inclusion on the third line follows from the fact that $L^{(i)} \subseteq L$. This completes the induction and proves the desired result for all $i \in \mathbb{N}$. \square

Corollary 1.3.6. *If L is nilpotent, then L is solvable.*

Proof. Lemma 1.3.5 tells us that for all $n \in \mathbb{N}$, $L^n = 0$ implies $L^{(n)} = 0$. Thus, if such an n exists that makes L nilpotent, the same n would also make L solvable. \square

The converse is not true. We need to develop a bit of theory to see this.

Lemma 1.3.7. *If L is nilpotent, its centre is nonzero.*

Proof. Let $n \in \mathbb{N}$ be such that

$$L = L^0 \supseteq L^1 \supseteq \dots \supseteq L^n \subsetneq L^{n+1} = 0$$

with $L^n \neq 0$. \square

1.3.2 Ideals, Quotients and Subalgebras

Throughout this subsection, let $I \trianglelefteq L$ and $K \leq L$. Recall that I is a Lie subalgebra of L (cf. Lemma 1.1.18), meaning we can impose solvability and nilpotency conditions on I as well.

Definition 1.3.8 (Solvability of Subalgebras). We say a subalgebra of L is **solvable** if it is solvable as a Lie algebra in its own right.

Proposition 1.3.9 (Solvability Conditions).

1. If L is solvable, then so is L/I .
2. If L is solvable, then so is K .

3. If I and L/I are solvable, then so is L .

Proof. Let $\phi : L \rightarrow L/I$ be the quotient homomorphism.

1. Observe that it suffices to show that $\phi(L^{(i)}) = \phi(L)^{(i)}$ for all $i \in \mathbb{N}$: if this were true, then the existence of some $n \in \mathbb{N}$ such that $L^{(n)} = 0$ would imply that

$$(L/I)^{(n)} = \phi(L)^{(n)} = \phi(L^{(n)}) = \phi(0) = 0$$

making L/I solvable whenever L is.

We will now prove that $\phi(L^{(i)}) = \phi(L)^{(i)}$ by induction on i . When $i = 0$, the result is trivial: it is true that $\phi(L) = \phi(L)$ by reflexivity. Now, fix $i \in \mathbb{N}$ and assume that $\phi(L^{(i)}) = \phi(L)^{(i)}$. Then,

$$\begin{aligned} \phi(L^{(i+1)}) &= \phi([L^{(i)}, L^{(i)}]) = \phi(\text{Span}(\{[x, y] \mid x, y \in L^{(i)}\})) \\ &= \text{Span}(\phi(\{[x, y] \mid x, y \in L^{(i)}\})) \\ &= \text{Span}(\{[\phi(x), \phi(y)] \mid x, y \in L^{(i)}\}) \\ &= [\phi(L^{(i)}), \phi(L^{(i)})] \\ &= [\phi(L)^{(i)}, \phi(L)^{(i)}] = \phi(L)^{(i+1)} \end{aligned}$$

as required.

2. It suffices to prove that for all $i \in \mathbb{N}$, $K^{(i)} \subseteq L^{(i)}$: if this were true, then the existence of some $n \in \mathbb{N}$ such that $L^{(n)} = 0$ would imply that $K^{(n)} = 0$, making K solvable whenever L is.

We will now prove that $K^{(i)} \subseteq L^{(i)}$ by induction on i . The base case is trivial, because $K^{(0)} = K \subseteq L = L^{(0)}$. Now, fix $i \in \mathbb{N}$ and assume that $K^{(i)} \subseteq L^{(i)}$. Then,

$$\begin{aligned} K^{(i+1)} &= [K^{(i)}, K^{(i)}] = \text{Span}(\{[x, y] \mid x, y \in K^{(i)}\}) \\ &\subseteq \text{Span}(\{[x, y] \mid x, y \in L^{(i)}\}) \\ &= [L^{(i)}, L^{(i)}] = L^{(i+1)} \end{aligned}$$

as required.

3. Let $m \in \mathbb{N}$ be such that $I^{(m)} = 0$ and let $n \in \mathbb{N}$ be such that $(L/I)^{(n)} = 0$. It suffices to prove that for all $i, j \in \mathbb{N}$, $(L^{(i)})^{(j)} = L^{(i+j)}$: if this were true, then the fact that

$$\phi(L^{(n)}) = (L/I)^{(n)} = 0$$

would immediately imply that $L^{(n)} \subseteq \ker(\phi) = I$, from which it would follow that $(L^{(n)})^{(m)} = 0$, and therefore, that $L^{(n+m)} = 0$, making L solvable whenever I and L/I are.

We will now prove that $(L^{(i)})^{(j)} = L^{(i+j)}$ by letting i be arbitrary and performing induction on j . The base case is trivial, because $(L^{(i)})^{(0)} = L^{(i)}$. Now, fix $j \in \mathbb{N}$ and assume that $(L^{(i)})^{(j)} = L^{(i+j)}$. Then,

$$(L^{(i)})^{(j+1)} = [(L^{(i)})^{(j)}, (L^{(i)})^{(j)}] = [L^{(i+j)}, L^{(i+j)}] = L^{(i+j+1)}$$

as required.

□

We have similar results for nilpotency.

Definition 1.3.10 (Nilpotency of Subalgebras). We say a subalgebra of L is **nilpotent** if it is solvable as a Lie algebra in its own right.

Proposition 1.3.11 (Nilpotency Conditions).

1. If L is nilpotent, then so is L/I .
2. If L is nilpotent, then so is K .

We will not prove these results here, as they are very similar to the corresponding results for solvability. We will, however, mention that the reason why we do not have a nilpotency condition for L when I and L/I are nilpotent is that it is not, in general, true that $(L^i)^j = L^{i+j}$ for $i, j \in \mathbb{N}$, as we can easily see from the following counterexample.

Counterexample 1.3.12. sorry

We will end the discussion on solvability and nilpotency by saying a bit about the derived subalgebra. We first make a general observation.

Lemma 1.3.13. *If $L = L'$, then for all $i \in \mathbb{N}$, $L^{(i)} = L^{(1)} = L' = L$.*

Proof. We argue by induction on i . The base case is trivial, because $L^{(0)} = L$. Now, fix $i \in \mathbb{N}$ and assume that $L^{(i)} = L$. Then,

$$L^{(i+1)} = [L^{(i)}, L^{(i)}] = [L, L] = L'$$

Furthermore, it is clear that $L^{(1)} = L'$, and, by assumption, $L' = L$. This completes the induction and proves the desired result for all $i \in \mathbb{N}$. \square

There is an immediate consequence.

Corollary 1.3.14. *If $L \neq 0$ and L is solvable, then $L' < L$.*

Proof. We argue by contraposition. If $L' = L$, then we know that $L^{(i)} = L$ for all $i \in \mathbb{N}$. In particular, since L is nonzero, none of the $L^{(i)}$ can be zero. L is therefore not solvable. \square

We will be more interested the following, somewhat less immediate consequence that comes from combining applying the Correspondence Theorem to the ideals of quotient spaces of solvable Lie algebras.

Proposition 1.3.15. *If L is solvable, there exists an ideal $I \trianglelefteq L$ of codimension 1.*

Proof. Consider the quotient Lie algebra $K := L/L'$. We know that $0 < K$, because $K = 0$ would imply that $L = L'$, which is impossible because L is solvable, as shown in Corollary 1.3.14. Therefore, K contains a subspace W of codimension 1. Since K is abelian, Proposition 1.1.24 tells us that W is an ideal of K . Theorem 1.1.32 tells us that the preimage V of W under the quotient epimorphism is an ideal of L that contains L' . Simple arithmetic and dimension results from linear algebra then tell us

$$\dim(V) = \dim(W) + \dim(L') = (\dim(L) - \dim(L') - 1) + \dim(L') = \dim(L) - 1$$

□

1.3.3 The Radical Ideal

Throughout this subsection, we will assume that L is finite-dimensional.

We begin with a basic result about the sums of ideals.

Lemma 1.3.16. *Let $I, J \trianglelefteq L$. If I and J are solvable, then so is $I + J \trianglelefteq L$.*

Proof. sorry

□

Corollary 1.3.17. *There exists a solvable ideal of L that contains all other solvable ideals of L .*

Proof. Let R be a solvable ideal of L of maximal dimension.² Now, fix any $I \trianglelefteq L$. Lemma 1.3.16 tells us that $I + R$ is a solvable ideal of L . But, since R is of maximal dimension, we know that $\dim(I + R) \leq \dim(R)$. Therefore, we must have that $I + R = R$. The fourth point in Proposition 1.1.23 then tells us that $I \subseteq R$, as required. □

This solvable ideal has a name.

Definition 1.3.18 (Radical Ideal). The **radical ideal** of L is the solvable ideal of L that contains all other solvable ideals of L , which we know exists from Corollary 1.3.17.

We can now define what it means for a Lie algebra to be semi-simple. We will be very interested in this class of Lie algebras going forward.

Definition 1.3.19 (Semi-Simplicity). We say that L is **semi-simple** if its radical ideal is the 0 ideal.

²When we say maximal dimension, we mean that the dimension of R is the largest possible dimension such that a solvable ideal of that dimension exists. This is well-defined because L is finite-dimensional, and the dimension of any ideal of L is necessarily $\leq \dim(L)$.

Our aim for this module will be to classify all semi-simple Lie algebras. We will do this by first classifying all solvable Lie algebras and then using that classification to classify all semi-simple Lie algebras. We will need a *lot* more machinery before we can do this, but we will get there eventually.

We also have a notion of simplicity, which is no different from what we would expect in groups.

Definition 1.3.20 (Simplicity). We say that L is **simple** if it has no nontrivial ideals.

1.3.4 Ascending Series of Ideals

We will end by talking about ascending series of ideals and their corresponding quotients. Throughout this subsection, L will denote an arbitrary Lie algebra.

Definition 1.3.21 (Ascending Central Series). We say that an increasing chain of ideals

$$0 \subseteq L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$$

is an **ascending central series** of L if $L_1 = Z(L)$ and for all $i \in \mathbb{N}$, we have

1. $L_i \trianglelefteq L$ with quotient map $g_i : L \twoheadrightarrow L/L_i$
2. $L_{i+1} = g_i^{-1}(Z(L/L_i))$

Convention. We will use subscripted L_i s to denote elements of the ascending central series, in contrast to superscripts used for the descending central series.

We now have an equivalent criterion for nilpotency.

Proposition 1.3.22. L is nilpotent if and only if $L_n = 0$ for some $n \in \mathbb{N}$.

Proof.

(\implies) One can show by induction on n that if $L^n = 0$, then $L_n = 0$. Then, if $\exists n \in \mathbb{N}$ such that $L^n = 0$, ie, if L is nilpotent, then $L_n = 0$ as well. **sorry**

(\impliedby) **sorry**

□

We end with another counterexample that shows that solvable Lie algebras need not be nilpotent.

Counterexample 1.3.23. For all n , $\mathfrak{t}(n)$ is solvable but not nilpotent.

Proof that $\mathfrak{t}(n)$ is solvable. First, observe that $[\mathfrak{t}(n), \mathfrak{t}(n)] = \mathfrak{t}(n)' = \mathfrak{u}(n)$. By **sorry**, we know that $\mathfrak{u}(n)$ is nilpotent. Therefore, $\mathfrak{u}(n)$ is solvable. Furthermore, $\mathfrak{t}(n)/\mathfrak{t}(n)'$ is abelian, making it solvable by **sorry**. Therefore, by Proposition 1.3.9, $\mathfrak{t}(n)$ is solvable. □

Proof that $\mathfrak{t}(n)$ is not nilpotent. □

1.4 Subalgebras of $\mathfrak{gl}(n)$

We now turn our attention to the structure of subalgebras of $\mathfrak{gl}(n)$ for some fixed $n \in \mathbb{N}$. We will begin by developing some more general theory, following which we will prove important theorems about the structure of such subalgebras.

1.4.1 Induced Actions on Quotients by Invariant Subspaces

We will begin by recalling the linear algebraic theory of invariant subspaces and adapt that theory to the context of Lie algebras. Throughout this subsection, we will fix a subset $L \subseteq \mathfrak{gl}(n)$ and a subspace $U \leq \mathbb{C}^n$.

Definition 1.4.1 (Invariance). We say that U is **L -invariant** if for all $T \in L$ and $v \in U$, we have $T(v) \in U$.

The action of L on \mathbb{C}^n induces a natural action on the quotient space of \mathbb{C}^n by U .

Definition 1.4.2 (Induced Action). To each $T \in L$, we can associate the linear map $\bar{T} : \mathbb{C}^n/U \rightarrow \mathbb{C}^n/U$ defined by

$$\bar{T}(v + U) = T(v) + U \quad (1.4.1)$$

for all $v + U \in \mathbb{C}^n/U$. We will refer to the map $T \mapsto \bar{T}$ as the **induced action** of L on \mathbb{C}^n/U .

Indeed, when L is a Lie subalgebra of $\mathfrak{gl}(n)$, we can go one step further.

Proposition 1.4.3. *If L is a Lie subalgebra of $\mathfrak{gl}(n)$, the induced action map $\Phi : L \rightarrow \mathfrak{gl}(\mathbb{C}^n/U) : T \mapsto \bar{T}$ is a Lie algebra homomorphism.*

Proof. sorry

□

1.4.2 Linear Algebraic and Lie Algebraic Nilpotency

Recall the following definition from linear algebra.

Definition 1.4.4 (Nilpotency of Elements). We say that $x \in \mathfrak{gl}(n)$ is **nilpotent** if there exists an $m \in \mathbb{N}$ such that $x^m = 0$.

We can extend this to sub-vector spaces.

Definition 1.4.5 (Nilpotency of Subspaces). We say a sub-vector space $N \leq \mathfrak{gl}(n)$ is **nilpotent** if every element of N is nilpotent.

We can say something about the adjoint of a nilpotent element.

Lemma 1.4.6. *Let $x \in \mathfrak{gl}(n)$ be nilpotent. Then, $\text{ad}(x) \in \mathfrak{gl}(\mathfrak{gl}(n))$ is nilpotent as well.*

Proof. We need to show that there exists an $m \in \mathbb{N}$ such that the map we get by successively composing the adjoint map $\text{ad}(x)$ m times is identically zero.

Fix $y \in \mathfrak{gl}(n)$. Then,

$$\begin{aligned}\text{ad}(x)(y) &= [x, y] = xy - yx \\ \text{ad}(x)^2(y) &= [x, [x, y]] = x[x, y] - [x, y]x = x^2y - xyx - xyx + yx^2 \\ \text{ad}(x)^3(y) &= [x, [x, [x, y]]] = x^3y + \cdots + xyx^2 - yx^3\end{aligned}$$

More generally, one can show that

$$\operatorname{ad}(x)^m(y) = \sum_{i=0}^m \lambda_{i,m} x^i y x^{m-i}$$

for all $m \in \mathbb{N}$ and some $\lambda_{i,m} \in \mathbb{Z}$. In particular, since all powers of x beyond some m are zero, we have that $\operatorname{ad}(x)^m(y) = 0$ for all $y \in \mathfrak{gl}(n)$. \square

We have an important relationship between linear algebraic and lie algebraic nilpotency of a Lie subalgebra.

Theorem 1.4.7 (Engel's Theorem). *Let N be a Lie subalgebra of $\mathfrak{gl}(n)$. If N is nilpotent as a sub-vector space of $\mathfrak{gl}(n)$, then there exists a basis of \mathbb{C}^n with respect to which every element of N is upper-triangular.*

Before proving Engel's Theorem, we will state and prove the following Corollary that underscores the significance of this result.

Corollary 1.4.8. *Any nilpotent sub-vector space of $\mathfrak{gl}(n)$ is also nilpotent as a Lie subalgebra.*

Proof. Let N be a nilpotent sub-vector space of $\mathfrak{gl}(n)$. By Engel's Theorem, there exists a basis of \mathbb{C}^n with respect to which every element of N is upper-triangular. In particular, they must all have zeros on the diagonal, because they are nilpotent: they are of the form

$$\begin{bmatrix} 0 & & * \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$$

sorry \square

For the remainder of this subsection, we will focus on proving Engel's Theorem. We will fix a nilpotent subspace $N \leq \mathfrak{gl}(n)$. The high-level idea is to perform induction on $\dim(L)$ and draw a

parallel with the proof of the Jordan Canonical Form theorem³. We will first show that it suffices to show that a certain distinguished vector exists, following which we will show that it does.

For the remainder of this subsection, we will denote the **simultaneous kernel** of all elements of N by

$$U_n := \{v \in \mathbb{C}^n \mid \forall T \in N, T(v) = 0\} = \bigcap_{T \in N} \ker(T) \quad (1.4.2)$$

As an intersection of sub-vector spaces, U_n is a subspace of \mathbb{C}^n . Furthermore, U_n (and, by extension, all of its subspaces) are N -invariant: for all $T \in N$ and $v \in U_n$, we have $T(v) = 0 \in U_n$.

We are now ready to reduce the proof of Engel's Theorem to showing that all the elements of T have a common eigenvector with eigenvalue 0—or, equivalently, to showing that U_n is nonzero.

Lemma 1.4.9. *If U_n contains a nonzero element, then there exists a basis of \mathbb{C}^n with respect to which every element of N is upper-triangular.*

Proof. We argue by induction on n . When $n = 1$, the result is trivial: every element of N (and of $\mathfrak{gl}(n) = \mathfrak{gl}(1)$) is upper-triangular, so the fact that $U_n = 0$ is not a problem. Now, suppose that there exists a nonzero element $v \in U_n$, ie, such that $T(v) = 0$ for all $T \in N$. Let $V = \mathbb{C}^n / \text{Span}(v)$. Since $\text{Span}(v) \leq U_n$ and U_n is N -invariant, we know that $\text{Span}(v)$ is N -invariant as well, allowing us to develop the machinery developed in Section 1.4.1 that tells us about the Lie algebraic properties of $\mathfrak{gl}(V)$.

From Proposition 1.4.3, we know that the map that takes $T \in N$ to its induced action on V is a Lie algebra homomorphism. Therefore, by Lemma 1.1.12, its image is a subalgebra of $\mathfrak{gl}(V)$. Indeed, the elements of this subalgebra consists of nilpotent elements: for any $T \in \mathfrak{gl}(V)$, we know there exists some $m \in \mathbb{N}$ such that $T^m = 0$, and the same m will work for the induced action \bar{T} of T on the quotient space: for any $x + \text{Span}(v) \in \mathbb{C}^n / \text{Span}(v)$,

$$\bar{T}^m(x + U) = \bar{T}^m(x) + U = 0 + U$$

³Remember, we are working over \mathbb{C} .

Therefore, by the induction hypothesis, there exists a basis

$$\overline{B} := \{v_1 + \text{Span}(v), \dots, v_{n-1} + \text{Span}(v)\}$$

of V with respect to which every element of the image of N is upper-triangular. We can then lift this basis to a basis and this basis

$$B := \{v_1, \dots, v_{n-1}, v\}$$

of \mathbb{C}^n by adding v to it. B has the desired property that every element of N is upper-triangular with respect to it. \square

The way we will prove Engel's Theorem is to construct a sequence of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_m = \mathbb{C}^n$$

such that $N(V_i) \subseteq V_{i-1}$. We will refine this sequence so that $m = n$, ie, so that

$$\dim(V_i/V_{i-1}) = 1$$

using Proposition 1.3.15. We can then take distinguished elements from each of the quotients to form a basis of \mathbb{C}^n , and this will be the basis with respect to which every element of N is upper-triangular.

We will now show that U_n is, indeed, nonzero.

Lemma 1.4.10. *There exists a nonzero vector $v \in U_n$.*

Proof. We need to show that $T(v) = 0$ for all $T \in N$. We argue by induction on $\dim(N)$. The base case $\dim(N) = 1$ is clear: N must be the span of a single, nilpotent element, which necessarily has a (nonzero) eigenvector with eigenvalue 0. So, assume N is such that for all Lie subalgebras of $\mathfrak{gl}(n)$ of dimension less than $\dim(N)$, there exists a nonzero vector in the simultaneous kernel U_n of all elements of N .

Let $A \subseteq N$ be a maximal⁴, proper Lie subalgebra of N . Consider the map $\phi : A \rightarrow \mathfrak{gl}(N/A)$ that

⁴with respect to inclusion

maps any $g \in A$ to the map that sends every $T + A \in N/A$ to the map $[g, T] + A \in N/A$, where the map $[g, T]$ is the map $gT - Tg$.

Observe that since $\dim(\phi(A)) \leq \dim(A)$ and $\dim(A) < \dim(L)$ by the assumption that A is proper, we know that $\dim(A) < L$. Therefore, we can apply the induction hypothesis to A . sorry \square

There is also a more general formulation of Engel's Theorem over arbitrary Lie algebras.

Theorem 1.4.11 (Engel's Theorem, Second Version). *Let L be an arbitrary Lie algebra. Then, L is nilpotent if and only if for all $x \in L$, the adjoint map $\text{ad}(x)$ is nilpotent.*

Proof. Assume that L is nilpotent. Then, there exists some $m \in \mathbb{N}$ such that $L^m = 0$. Then, any composition of Lie brackets of length m is zero: for all $x, y \in L$,

$$\text{ad}(x)^n(y) = [x, [x, \dots [x, y] \dots]] = 0$$

This gives us one direction of the proof.

For the converse direction, we apply Theorem 1.4.7 (the standard formulation of Engel's Theorem) to the sorry \square

1.4.3 Weights of Lie Algebras

Throughout this subsection, let V be a finite-dimensional \mathbb{C} -vector space, L a Lie subalgebra of $\mathfrak{gl}(V)$, and $\lambda : L \rightarrow \mathbb{C}$ be an arbitrary function.

Definition 1.4.12 (Weight Space). We say the **weight space** of λ with respect to L is the space

$$V_\lambda := \{v \in V \mid \forall T \in L, T(v) = \lambda(T) \cdot v\}$$

The weight space gives us useful information about L and λ .

Lemma 1.4.13. *If V_λ is nonzero, then λ is a linear map.*

Proof. Suppose that $V_\lambda \neq 0$. Fix $S, T \in L$. We know there exists a nonzero vector $v \in V$ such that $S(v) = \lambda(S) \cdot v$ and $T(v) = \lambda(T) \cdot v$. In particular, we have that

$$\lambda(S + T) \cdot v = (S + T)(v) = S(v) + T(v) = \lambda(S) \cdot v + \lambda(T) \cdot v = (\lambda(S) + \lambda(T)) \cdot v$$

Given that $\lambda(S + T)$ and $\lambda(S) + \lambda(T)$ are both scalars, we must have that $\lambda(S + T) = \lambda(S) + \lambda(T)$. \square

Lemma 1.4.14. V_λ is a sub-vector space of \mathbb{C}^n .

Proof. sorry \square

Weight spaces are rather interesting, and in the remainder of this subsection, we will develop some theory that will prove quite useful when we prove Lie's Theorem (Theorem 1.4.16, which we shall see shortly).

Lemma 1.4.15. Let $I \trianglelefteq L$, and let $U \subseteq V$ be a weight space for I . Then, U is L -invariant, ie, $L(U) \subseteq U$.

Proof. Fix $u \in U$ and $T \in L$. Let $\mu : I \rightarrow \mathbb{C}$ be a map such that $U = I_\mu$. We want $T(u)$ to be in U as well: for any $S \in I$, we would like to have that

$$S(T(u)) = \mu(S) \cdot T(u)$$

First, observe that

$$\begin{aligned} S(T(u)) &= \text{sorry} \\ &= [S, T](u) + T(S(u)) \\ &= \mu([S, T]) \cdot u + T(\mu(S) \cdot u) \\ &= \mu([S, T]) \cdot u + \mu(S) \cdot T(u) \end{aligned}$$

If we can show that $\mu([S, T]) = 0$, we will be done.

Consider the following subspace of V :

$$W := \text{Span}(\{u, T(u), T^2(u), \dots\}) \quad (1.4.3)$$

Since $W \leq V$, which is finite-dimensional, we know W is finite-dimensional as well. Then, let $m = \dim(W)$. We know, from linear algebra, that $\{T^i(u) \mid 0 \leq i \leq m-1\}$ is a basis of W .

We will begin by showing that for all $0 \leq i \leq m-1$, there exist $\alpha_{i,1}, \dots, \alpha_{i,i-1}$ such that

$$ST^i(u) = \quad (1.4.4)$$

We will prove (1.4.4) by induction on i .

sorry

(1.4.4) tells us the following two facts.

1. W is S -invariant. sorry

2. There exists a basis of $S|_W$ with respect to which the matrix of S is
$$\begin{bmatrix} \mu(S) & & * \\ & \ddots & \\ 0 & & \mu(S) \end{bmatrix}.$$

sorry

The reason why we are interested in the above facts is that

sorry

□

1.4.4 Lie's Theorem

In this subsection, we discuss a result similar to Engel's Theorem, but for *solvable* Lie algebras instead of nilpotent ones. Throughout, we fix a **solvable** subalgebra $L \leq \mathfrak{gl}(n)$ for some $n \in \mathbb{N}$.

Theorem 1.4.16 (Lie's Theorem). *There exists a basis of \mathbb{C}^n such that every element of L is upper-triangular with respect to it.*

Before proceeding with the proof of Lie's Theorem, we will prove a corollary that underscores the significance of this result.

Corollary 1.4.17. L' is nilpotent.

Proof. Consider the adjoint map $\text{ad} : L \rightarrow \mathfrak{gl}(L)$. We know that $\text{ad}(L)$ is solvable by **sorry**; therefore, by Lie's Theorem, it is contained in $\mathfrak{t}(L)$. **sorry** \square

Our proof strategy will be to obtain some $\lambda \in L^*$, ie, a linear function $L \rightarrow \mathbb{C}$, such that \mathbb{C}^n_λ . We will repeat a simpler version of the proof of Engel's Theorem: we will perform induction on n , applying the induction hypothesis to the image of L under the quotient epimorphism $\Phi_\lambda : L \twoheadrightarrow \mathfrak{gl}(\mathbb{C}^n / \mathbb{C}^n_\lambda)$. $\Phi_\lambda(L)$ is solvable, because it is a quotient of a solvable Lie algebra. We will then be able to apply the fact that

$$\dim(\Phi_\lambda(L)) \leq \dim(\mathbb{C}^n) - \dim(\mathbb{C}^n_\lambda) < \dim(\mathbb{C}^n) = n$$

because $\mathbb{C}^n_\lambda \neq 0$, allowing us to apply the induction hypothesis on $\Phi_\lambda(L)$.

Bibliography

These lecture notes are based heavily on the following references:

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