

MATH-314: Representation Theory of Finite Groups

Lecturer: Aluna Rizzoli

Scribe: Sidharth Hariharan

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Contents

1	An Introduction to the Theory of Representations of Groups	3
1.1	Important Definitions	4
1.1.1	What is a Representation?	4
1.1.2	Morphisms of Representations	6
1.1.3	Subrepresentations	8
1.1.4	Irreducibility	9
1.1.5	Lifting Representations	11
1.2	Invariant Constructions	13
1.2.1	Direct Sums of Representations	13
1.2.2	Complementary Subrepresentations	16
1.2.3	Maschke's Theorem	19
1.2.4	G -Invariant Inner-Products	24
1.2.5	Tensor Products of Representations	25
1.3	Group Algebras and Modules	26
1.3.1	Preliminaries	26
1.3.2	Schur's Lemmas	29
1.3.3	Representations of Finite Abelian Groups over \mathbb{C}	33
2	Character Theory	37
2.1	The Theory of Irreducible Characters	37
2.1.1	On Central Functions	38
2.1.2	Introduction to Character Theory	40
2.1.3	The Orthogonality Theorem	44

2.1.4	Understanding Irreducible Characters	48
2.2	Character Tables	54
2.2.1	The Orthogonality Relations	54
2.2.2	Constructing Character Tables	56
2.3	Integrality	60
2.3.1	The Algebraic Integers	60
2.3.2	The n th Roots of Unity	63
3	Applications to the Study of Groups	69
3.1	Important Classes of Finite Groups	69
3.1.1	p -Groups	69
3.1.2	Simple Groups	70
3.2	Burnside's Theorem	70
3.2.1	The Algebraic Numbers	70
3.2.2	The Three Versions of Burnside's Theorem	72
3.2.3	Applications of Burnside's Theorem	74
3.3	Products, Products, Products	74
3.3.1	The Multiplicative Properties of (Irreducible) Characters	74
3.3.2	Tensor Products and Direct Products	76
3.4	The Character Theory of Subgroups	78
3.4.1	Restriction	78
3.4.2	Induction	82
3.4.3	The Relationship between Induction and Restriction	85

Chapter 1

An Introduction to the Theory of Representations of Groups

As I understand it, the fundamental idea behind Representation Theory is to study the actions of groups on vector spaces. While arbitrary vector spaces over arbitrary fields might not have naturally visualisable geometric properties, representations of groups in the ones that do can greatly illustrate the nature of these groups, especially to individuals like myself who delight in (somewhat literally) *seeing* mathematics come alive.

A key motivating example in the study of representation theory would be the representations of Dihedral groups over \mathbb{R}^2 . It is very natural to (at least informally) view the Dihedral group D_n of order $2n$ as the group of symmetries of the regular n -gon; in other words, elements of D_n have natural actions on a regular n -gon that preserve its structure. For instance, D_4 contains an element that rotates a square clockwise by 90° , an action under which the square is, of course, invariant.



If one were to now plot this square in \mathbb{R}^2 , then action of the same element on the square can be

extended to an orthogonal transformation of \mathbb{R}^2 that maps the x -axis to the y -axis and vice-versa, but in a manner preserving orientation (ie, that *rotates the plane clockwise by 90°*). In a similar fashion, one can extend the actions of all dihedral groups D_n to actions on the entirety of \mathbb{R}^2 . More precisely, to every element of a dihedral group, one can ascribe a specific *matrix* that transforms \mathbb{R}^2 in a manner preserving the regular n -gon.

This motivates the formal definition of a representation.

1.1 Important Definitions

1.1.1 What is a Representation?

It turns out that representations can be defined quite broadly, sidestepping the geometric niceties (or are they constraints?) of Euclidean spaces.

Definition 1.1.1 (Group Representation). Let G be a group. A representation of G is a pair (V, ρ) of a vector space V and a group homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Here, $\text{GL}(V)$ refers to the **General Linear** group over V , consisting of all vector space automorphisms of V equipped with the binary operation of composition.

Definition 1.1.2 (Degree of a Representation). Let G be a group and let (V, ρ) be a representation of G . We define the degree of V to be the dimension of V over its base field.

There exist innumerable examples of representations throughout mathematics. Below, we give some important ones.

Example 1.1.3 (Important Classes of Representations).

1. The trivial representation. Let G be a group and V be any vector space. The map $\rho : G \rightarrow \text{GL}(V) : g \mapsto \text{id}_V$ is a representation.
2. The zero representation. Let G be a group and let $V = \{0\}$ be the zero vector space over an arbitrary field K . The trivial representation over V is known as the zero representation.

3. The sign representation. Let $G = S_n$, the symmetric group on n elements, and let $V = K$, a field. Then, $\text{GL}(V) = K^\times$, the multiplicative group of K . Denoting by ξ the canonical morphism from \mathbb{Z} to K , the map

$$\rho : G \rightarrow \text{GL}(V) : \sigma \mapsto \xi(\text{sgn}(\sigma))$$

is a representation, where $\text{sgn} : G \rightarrow \{-1, 1\}$ denotes the sign homomorphism.

4. Permutation representations. Let G be a group acting on a finite set X , and let $V = K[X]$, the free vector space (over some field K) generated by X . Consider a K -basis $\{e_x \in V : x \in X\}$ of V . Then, the map $\rho : G \rightarrow \text{GL}(V)$ given by

$$\rho(g)(e_x) = e_{g(x)}$$

is a representation.

5. The regular representation. Let G be a *finite* group. The permutation representation corresponding to the canonical action of G on itself by left-multiplication gives a representation of G over $K[G]$, the free vector space generated by G (as a set) over any field K .

Non-Example 1.1.4. Let G be a group and let V be a nonzero vector space over an arbitrary field. The map $g \mapsto 0 : G \rightarrow (V \rightarrow V)$ is not a representation because the zero map $0 : V \rightarrow V$ is not invertible.

A useful perspective to adopt is that a representation is merely an action of a group on a vector space. And, just as faithful actions are an important class of actions, it will, later on, turn out to be important to have a corresponding notion for representations as well.

Definition 1.1.5 (Faithfulness). Let G be a group and let (V, ρ) be a representation of G . We say (V, ρ) is faithful if $\ker(\rho)$ is trivial.

In the next subsection, we begin to develop the theory of morphisms of representations, which will be crucial to the study of interactions and relationships between representations.

1.1.2 Morphisms of Representations

Definition 1.1.6 (Homomorphism of Representations). Let G be a group and let (V, ρ) and (V', ρ') be two representations of G . A homomorphism of representations $T : V \rightarrow V'$ is a linear map $T : V \rightarrow V'$ such that $\forall g \in G$,

$$T \circ \rho(g) = \rho'(g) \circ T \quad (1.1.1)$$

or equivalently, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ T \downarrow & & \downarrow T \\ V' & \xrightarrow{\rho'(g)} & V' \end{array} \quad (1.1.2)$$

Such a map T is said to be G -linear.

Remark. The term G -linear comes from the fact that a homomorphism of representations satisfies the property that $T(g(v)) = g(T(v))$, where the notation $g(\cdot)$ represents the action of some $g \in G$, encoded by a representation. In this sense, T is somehow “linear over G ”.

A natural way to define two representations to be equal, or ‘isomorphic,’ is as follows.

Definition 1.1.7 (Equivalence of Representations). Let G be a group and let (V, ρ) and (V', ρ') be two representations of G . We say that (V, ρ) and (V', ρ') are equivalent, denoted $(V, \rho) \sim (V', \rho')$, if there exists a homomorphism $T : (V, \rho) \rightarrow (V', \rho')$ that is invertible as a linear map—ie, that gives a linear isomorphism between V and V' .

Representations of the same group over the same vector space need not be equivalent.

Example 1.1.8 (Non-Equivalent Representations of the Klein 4-Group). Let $G = C_2 \times C_2$ be the Klein 4-group (where $C_2 = \langle x \rangle$ is the cyclic group of order 2). Let $\alpha = (x, 1)$ and $\beta = (1, x)$. Together, they generate G .

Now, let K be a field. Consider a degree 1 representation $\rho : G \rightarrow K^\times$. We know that $\rho(G)$ must be a subgroup of K^\times such that $|\rho(G)| \in \{1, 2, 4\}$. If $\text{char}(K) = 2$, then ρ must be

the trivial representation, since $2 \nmid |K^\times|$. Else, all four maps ρ satisfying

$$(\rho(\alpha), \rho(\beta)) = (\pm 1, \pm 1)$$

give *non-equivalent* representations of G in K^\times . In particular, we see the non-equivalence because K^\times is commutative.

The point of morphisms of representations is to be able to move from one vector space to another without losing the structural information captured by the representation. This is precisely illustrated in (1.1.2).

Example 1.1.9 (Representations of Cyclic Groups over \mathbb{R}^2 and \mathbb{R}^3). Consider the cyclic group $C_n = \langle g \rangle$ of order n . Let $V = \mathbb{R}^2, V' = \mathbb{R}^3$. Together with the respective maps

$$\begin{aligned} \rho : G \rightarrow \text{GL}(\mathbb{R}^2) : g^m &\mapsto \begin{bmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{bmatrix} \\ \rho' : G \rightarrow \text{GL}(\mathbb{R}^3) : g^m &\mapsto \begin{bmatrix} \cos(2\pi/m) & -\sin(2\pi/m) & 0 \\ \sin(2\pi/m) & \cos(2\pi/m) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

they give representations of C_n . Consider now the inclusion $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ whose matrix with respect to the standard bases of \mathbb{R}^2 and \mathbb{R}^3 is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. One can see that T gives a map from (V, ρ) to (V', ρ') . Indeed, the corestriction of T to its image is a linear isomorphism, which gives an equivalence between (V, ρ) and $(T(V), \rho)$, where we restrict the domains of each $\rho(g^m)$ to $T(V)$.

The above example leads to an interesting question. Can we think of one representation as being “contained” in another?

It turns out that we can.

1.1.3 Subrepresentations

We have the objects; we have the morphisms. It is only natural to think about what the subobjects would be in the context of group representations. And if Example 1.1.9 is any indication, they involve something more than just an inclusion. There is some structural property of a sub-vector space of a representation that makes it *compatible* with the representation structure. In the case of Example 1.1.9, for instance, this is the fact that the representation ρ' acted only “horizontally”—ie, “parallel” to the subspace $T(V)$.

More generally, it turns out that the property we really require a subspace to have in order to be ‘compatible’ with the representation structure is the following.

Definition 1.1.10 (*G*-Invariance). Let G be a group and let (V, ρ) be a representation of G . We say that a sub-vector space $W \leq V$ is G -invariant if for all $w \in W$ and $g \in G$,

$$\rho(g)(w) \in W$$

In other words, W is G -invariant if W is $\rho(g)$ -invariant for every $g \in G$.

One can make the following observation. Let G be a group, (V, ρ) a representation of G , and $W \leq V$ a G -invariant subspace. Then, $\forall g \in G$, $\rho(g) \in \text{GL}(W)$. That is, $\rho(g)$ is a linear automorphism of W whose inverse, $\rho(g^{-1})$, is *also* a linear automorphism of W . This then leads to the following definition of a subrepresentation.

Definition 1.1.11 (Subrepresentation). Let G be a group and let (V, ρ) be a representation of G . A subrepresentation of V is a pair $(W, \rho|_W)$ consisting of a G -invariant subspace $W \leq V$ and the map

$$\rho|_W : G \rightarrow \text{GL}(W) : g \mapsto \rho(g)|_W$$

It is very important to note that the map $\rho|_W$ is *not actually a restriction of ρ to a specific domain*. Rather, it is a map that restricts the domain of $\rho(g)$ for every $g \in G$.

One can also observe easily that a subrepresentation is given uniquely by a G -invariant subspace. Hence, we will often abuse notation and not distinguish between the pair $(W, \rho|_W)$ (which is actually

a representation) and simply W (which is merely a subspace).

Example 1.1.12. Let G be a finite group and K a field. Consider the regular representation $\rho : G \rightarrow K[G]$. Let $\{e_g : g \in G\}$ denote a basis of $K[G]$. Then, the subspace $W := \text{Span}\left(\sum_{g \in G} e_g\right)$ is G -invariant.

It turns out that morphisms of representations also give us subrepresentations.

Proposition 1.1.13. *Let G be a group and let (V, ρ) be a representation of G . Let $T : (V, \rho) \rightarrow (V, \rho)$ be a homomorphism of representations. Then, the subspaces $\ker(T)$ and $\text{im}(T)$ of V are G -invariant.*

Proof. Fix $g \in G$ and $v \in \ker(T)$. We know $T(\rho(g)(v)) = \rho(g)(T(v))$. Since $T(v) = 0$, $T(\rho(g)(v)) = 0$. Hence, $\rho(g)(v) \in \ker(T)$, proving that $\ker(T)$ is G -invariant.

Now, fix $w \in \text{im}(T)$. Then, $w = T(u)$ for some $u \in V$. Clearly, $\rho(g)(w) = \rho(g)(T(u)) = T(\rho(g)(u)) \in \text{im}(T)$, proving that $\text{im}(T)$ is G -invariant as well. \square

1.1.4 Irreducibility

Having discussed the subobjects of representations (namely, subrepresentation), it is only natural to wish to describe whether a representation ever contains a nontrivial subrepresentation. I say "nontrivial" because any representation naturally admits two (uninteresting) subrepresentations: the trivial representation and itself.

Akin to the definition of simple groups, where we answer a similar question, we have the following definition that captures this idea.

Definition 1.1.14 (Irreducibility). Let G be a group and (V, ρ) a nonzero representation of G . We say (V, ρ) is irreducible if V contains no proper, nonzero G -invariant subspaces.

In similar fashion, we say a nonzero representation is reducible if it is not irreducible.

Given that MATH-314 focuses on *finite* groups, the following result is quite useful.

Proposition 1.1.15. *Let G be a group and let (V, ρ) be a representation of G . If G is finite and (V, ρ) is irreducible, then V is finite-dimensional.*

Proof. Since (V, ρ) is irreducible, in particular, $V \not\supseteq \{0\}$ —ie, $\exists v \in V$ such that $v \neq 0$. Let $W := \text{Span}(\{\rho(g)(v) : g \in G\})$. Since $0 \neq v \in W$, W is a nonzero subspace of V . Furthermore, since G is finite, W is finite-dimensional. We show that W is, in fact, G -invariant. Then, since V is irreducible, W could not possibly be a proper subspace of V , meaning that $W = V$, making V finite-dimensional as well.

Fix $h \in G$, and consider an arbitrary element $w = \sum_{g \in G} \lambda_g \rho(g)(v) \in W$. Then,

$$\begin{aligned} \rho(h)(w) &= \sum_{g \in G} \lambda_g \rho(h)(\rho(g)(v)) \\ &= \sum_{g \in G} \lambda_g (\rho(h) \circ \rho(g))(v) \\ &= \sum_{g \in G} \lambda_g \rho(hg)(v) \in W \end{aligned}$$

proving that W is $\rho(h)$ -invariant for every $h \in G$, making it a G -invariant subspace of V . Therefore, as argued above, $W = V$, proving that V is finite-dimensional. \square

Example 1.1.16 (Simple Examples of Irreducible Representations).

1. Any representation of degree 1 is irreducible.

Proof. A vector space of dimension 1 has no proper, nonzero subspaces— G -invariant or otherwise. Hence, a representation of degree 1 *must* be irreducible \square

2. Let K be a field. The trivial embedding $\text{SL}(n, K) \hookrightarrow \text{GL}(n, K)$ gives an irreducible representation of $\text{SL}(n, K)$ over K^n .

Proof. Assume $n > 1$ (else, the result follows from the previous point). For the sake of contradiction, suppose there exists a nonzero, $\text{SL}(n, K)$ -invariant subspace W of K^n having dimension $m < n$. Let $\mathcal{B} = \{e_1, \dots, e_m\}$ be a basis of W , extending to a basis $\overline{\mathcal{B}} = \{e_1, \dots, e_m, e_{m+1}, \dots, e_n\}$ of V . Consider the linear map $T \in \text{SL}(n, K)$

having matrix

$$[T]_{\bar{B}} = \begin{bmatrix} & & & (-1)^{n+1} \\ & & \ddots & \\ & & -1 & \\ 1 & & & \end{bmatrix}$$

with respect to \bar{B} . Clearly, $T(e_1) = e_n$, even though $e_1 \in W$ and $e_n \notin W$, contradicting the $\text{SL}(n, K)$ -invariance of W . \square

Non-Example 1.1.17. Let G be a finite group and K a field. Consider the regular representation $(K[G], \rho)$. In the notation of Example 1.1.12, we know that $W := \text{Span}\left(\sum_{g \in G} e_g\right)$ is G -invariant. If $|G| > 1$, then W is a proper subspace of $K[G]$, as it has dimension 1 (whereas $K[G]$ has dimension $|G|$). Furthermore, W is nonzero. Hence, $(K[G], \rho)$ is not irreducible (unless $|G| = 1$, in which case it follows from the first point of Example 1.1.16 that $(K[G], \rho)$ is irreducible).

We also have the following interesting criterion for irreducibility of representations of finite groups over \mathbb{C} .

Lemma 1.1.18. *Let G be a finite group and let (\mathbb{C}^2, ρ) be a representation of G over \mathbb{C} . If there exist $g, h \in G$ such that $\rho(g)$ and $\rho(h)$ do not commute, then (\mathbb{C}^2, ρ) is irreducible.*

Proof. We argue by contraposition. Suppose that (\mathbb{C}^2, ρ) is reducible. Let $W \leq \mathbb{C}^2$ denote a proper, nonzero, G -invariant subspace. Clearly, $\dim(W) = 1$. This means that as a \mathbb{C} -vector space, $W \cong \mathbb{C}$. Indeed, $\text{GL}(W) \cong \mathbb{C}^\times$. Seeing as \mathbb{C}^\times is abelian, there cannot possibly exist such a g and h . So, if they do exist, then (\mathbb{C}^2, ρ) must be irreducible. \square

1.1.5 Lifting Representations

In this short subsection, we will investigate the lifting property of representations, which will allow us to construct representations of groups from those of their factor groups. For the purposes of this subsection, let G be any group, $N \trianglelefteq G$ a normal subgroup of G , H the factor group G/N

and $\pi : G \twoheadrightarrow H$ the canonical surjection.

Definition 1.1.19 (Lifting of Representations). Let (V, ρ) be a representation of H . We define a lift of ρ to be the map $\rho' = \rho \circ \pi : G \rightarrow \text{GL}(V)$, where $\pi : G \twoheadrightarrow G/N$ is the canonical surjection. Diagrammatically,

$$\begin{array}{ccc} G & \xrightarrow{\rho'} & \text{GL}(V) \\ \pi \downarrow & \nearrow \rho & \\ H & & \end{array} \quad (1.1.3)$$

The lift of a representation has several useful properties.

Proposition 1.1.20. *Let (V, ρ) be a representation of H . Then, (V, ρ') is a representation of G , where $\rho' = \rho \circ \pi$ is a lift of ρ . Furthermore, (V, ρ) is irreducible if and only if (V, ρ') is.*

Proof. As a composition of homomorphisms, ρ' is a homomorphism from G to $\text{GL}(V)$, making (V, ρ') a representation of G . We now show that (V, ρ) is irreducible iff (V, ρ') is.

(\implies) Assume (V, ρ) is irreducible. Suppose V contains a proper, nonzero subspace W that is invariant under the action of G via ρ' . Then, by the construction of ρ' , we know that W must be invariant under the action of $H = \pi(G)$ via ρ . However, this would make W H -invariant, contradicting the irreducibility of (V, ρ) . Hence, (V, ρ') cannot contain any proper, nonzero subrepresentations.

(\impliedby) Assume (V, ρ') is irreducible. Suppose V contains a proper, nonzero subspace W that is invariant under the action of H via ρ . Then, W must be invariant under the action of G via $\rho' = \rho \circ \pi$, because π maps any $g \in G$ to an element of H , making W invariant under the action via ρ of $\pi(g) \in H$. However, this would make W G -invariant, contradicting the irreducibility of (V, ρ') . Hence, (V, ρ) cannot contain any proper, nonzero subrepresentations.

□

We can do even better when $N = [G, G]$, the commutator subgroup of G .

Corollary 1.1.21. *Let k be a field. Then, there is a one-to-one correspondence between degree*

1 representations of G over k and those of H .

Proof. Degree 1 representations of G and H are precisely group homomorphisms from G and H to k^\times . Seeing as k^\times is abelian and H is the abelianisation of G , any homomorphism from G to k^\times must factor uniquely through H . This gives us a map from degree 1 representations of G to degree 1 representations of H . The inverse of this map is precisely the lift of any degree 1 representation of H to one of G , as described in Proposition 1.1.20. These are the maps that take the solid arrows to the dashed arrows (brown to brown, pink to pink) in the following diagrams:



□

These properties will prove especially useful when we study character theory. Specifically, they will help us determine the values of irreducible characters of certain large groups using those of much smaller factor groups. For now, though, we close this subsection and move on to studying the compatibility of linear algebraic constructions with representation structures.

1.2 Invariant Constructions

In this section, we briefly examine how ordinary linear algebraic constructions can interact with representations. We are particularly interested in the notion of *invariance*, wherein a linear algebraic construction respects the structure of any representations involved.

1.2.1 Direct Sums of Representations

The most elementary operation we can think about when we have two objects is *putting them together*. One of the most meaningful ways of doing so in the context of linear algebra is the direct sum of two vector spaces. It turns out that this extends rather naturally to representations.

Definition 1.2.1 (The Direct Sum of Two Representations). Let G be a group and let (V, ρ) and (V', ρ') be representations of G . We define the direct sum of (V, ρ) and (V', ρ') to be the pair $(V \oplus V', \rho \oplus \rho')$, where $V \oplus V'$ is the direct sum of V and V' as vector spaces and $\rho \oplus \rho' : G \rightarrow \text{GL}(V \oplus V')$ maps every $g \in G$ to the map

$$(\rho \oplus \rho')(g)(v \oplus v') = \rho(g)(v) \oplus \rho'(g)(v') \in \text{GL}(V)$$

Proposition 1.2.2. Let G be a group and let (V, ρ) and (V', ρ') be representations of G .

1. The direct sum $(V \oplus V', \rho \oplus \rho')$ of (V, ρ) and (V', ρ') is, indeed, a representation of G .
2. V and V' are G -invariant subspaces¹ of $V \oplus V'$.

Proof.

1. Fix $g, h \in G$. For all $v \oplus v' \in V \oplus V'$,

$$\begin{aligned} (\rho \oplus \rho')(gh)(v \oplus v') &= \rho(gh)(v) \oplus \rho'(gh)(v') \\ &= \rho(g)(\rho(h)(v)) \oplus \rho'(g)(\rho'(h)(v')) \\ &= (\rho \oplus \rho')(g)((\rho \oplus \rho')(h)(v \oplus v')) \end{aligned}$$

proving that $\rho \oplus \rho'$ is multiplicative. Then, for any $g \in G$, $(\rho \oplus \rho')(g)$ has inverse $(\rho \oplus \rho')(g^{-1})$.

Hence, $\rho \oplus \rho'$ is a homomorphism from G to $\text{GL}(V \oplus V')$.

2. Fix $g \in G$ and $v \in V$. Clearly, $(\rho \oplus \rho')(g)(v) = \rho(g)(v)$. Since $\rho(g) \in \text{GL}(V)$, it follows that $\rho(g)(v) \in V$. The proof that V' is G -invariant is identical.

□

The above proposition gives us another reason to consider the direct sum to be an “invariant” construction: while it enriches both the vector space structure and the representation structure of a summand by adding another representation into the mix, it does not take anything away from the constructions that already exist.

With direct sums, we also have similar notions to reducibility.

¹Technically, isomorphic to the subspaces $V \oplus \{0\}$ and $\{0\} \oplus V'$, but we overlook such distinctions.

Definition 1.2.3 (Indecomposability). A nonzero representation is said to be indecomposable if it is inexpressible as a direct sum of two proper, nonzero subrepresentations.

Nonzero representations that are not indecomposable are said to be decomposable.

We have a natural relationship between irreducibility and indecomposability.

Proposition 1.2.4. *Let G be a group and let (V, ρ) be a representation of G . If (V, ρ) is irreducible, then it is indecomposable.*

Proof. If $V = \{0\}$, then the result is vacuously true. If $V \neq \{0\}$, then if it is decomposable, it contains a proper, nonzero, G -invariant subspace, making it reducible. \square

Example 1.2.5. Let $C_2 = \{1, a\}$ be the cyclic group of order 2, and let (V, ρ) be the regular representation of C_2 over a field K .

1. Let $K = \mathbb{C}$. Then, let $W_1 := \text{Span}(e_1 + e_a)$ and $W_2 := \text{Span}(e_1 - e_a)$. It is obvious that $W_1 \oplus W_2 = V$, as vector spaces. Furthermore, W_1 and W_2 are both C_2 -invariant. Hence, $W_1 \oplus W_2 = V$ as representations, too. One would say that the regular representation of C_2 over \mathbb{C} is decomposable.
2. Let $K = \mathbb{F}_2$. Then, $V = \{0, e_1, e_a, e_1 + e_a\}$. Observe that $W := \{0, e_1 + e_a\}$ is a proper, C_2 -invariant subspace of V . Hence, (V, ρ) is reducible. However, W is the *only* such subspace, meaning (V, ρ) cannot be expressed as a direct sum of *two* proper, nonzero subrepresentations. Therefore, (V, ρ) is indecomposable.

The $K = \mathbb{F}_2$ case in the above example demonstrates an important fact: *the converse of Proposition 1.2.4 is not, in general, true.* It is natural to wonder if we can, at least, prove a *partial* converse. It turns out that we can: the converse is true under a specific condition that is far from obvious. We do not yet have the tools we need to prove this result, but we will build a chain of implications, culminating in a famous theorem proved by Heinrich Maschke.

Finally, just like in other areas of mathematics where we encounter the word “irreducible,” in the context of representation theory, too, we have a notion of decomposition into irreducibles.

Definition 1.2.6 (Complete Reducibility). A representation is said to be completely reducible if it is expressible as a direct sum of irreducible (sub)representations.

It is clear that complete reducibility is related to irreducibility: any irreducible representation is trivially a direct sum of one irreducible representation. It turns out that complete reducibility is also related to indecomposability.

Proposition 1.2.7. *Let G be a finite group and let (V, ρ) be a finite-dimensional representation of G . Let K be a field. All indecomposable, finite-dimensional representations of G over K are irreducible if and only if every finite-dimensional representation of G over K is completely reducible.*

Proof.

(\implies) Let (V, ρ) be a finite-dimensional representation of G over K . If (V, ρ) is indecomposable, it is irreducible, making it completely reducible. Else, (V, ρ) is decomposable, making it a direct sum of two proper, nonzero subrepresentations. We can construct an inductive argument (on $\dim(V)$) to show that each of these subrepresentations is completely reducible. We can then conclude that (V, ρ) is completely reducible.

(\impliedby) Let (V, ρ) be a finite-dimensional, indecomposable representation of G over K . By assumption, (V, ρ) is expressible as a direct sum of irreducible representations. Since (V, ρ) is indecomposable, none of the summands can be proper or nonzero. Therefore, (V, ρ) must be a direct sum of *one* irreducible representation—namely, itself.

□

It will turn out that we can use complete reducibility to fill in an important gap in the converse of Proposition 1.2.4. To do so, we will need to better understand the theory of complete reducibility. We will do so by examining the notion of complementary subrepresentations.

1.2.2 Complementary Subrepresentations

It is a well-known fact from Linear Algebra that for any finite-dimensional vector space V , for any subspace $W \leq V$, there exists a *complementary* subspace $W' \leq V$ such that $W \oplus W' = V$. As it

turns out, we can define a notion of complementarity for representations, too.

Definition 1.2.8 (Complementary Subrepresentation). Let G be a group and let (V, ρ) be a representation of G . Let $(W, \rho|_W)$ be a subrepresentation of (V, ρ) . A complementary subrepresentation of $(W, \rho|_W)$ is a subrepresentation $(U, \rho|_U)$ such that $V = U \oplus W$.

This notion of complementarity is, indeed, compatible with the notion of direct sums of representations.

Proposition 1.2.9. *Let G be a group and let (V, ρ) be a representation of G . Let $(W, \rho|_W)$ and $(U, \rho|_U)$ be complementary subrepresentations. Then, their direct sum $(V, \rho|_W \oplus \rho|_U)$ is equivalent to (V, ρ) as a representation of G .*

Proof. It suffices to show that $\rho = \rho|_W \oplus \rho|_U$. Then, the identity map would give an equivalence of representations. Indeed, every $v \in V$ is expressible uniquely as a direct sum $w \oplus u$ for some $w \in W$ and $u \in U$. So, for all $g \in G$,

$$\begin{aligned} \rho(g)(v) &= \rho(g)(w \oplus u) \\ &= \rho(g)(w) \oplus \rho(g)(u) \\ &= \rho|_W(g)(w) \oplus \rho|_U(g)(u) \\ &= (\rho|_W \oplus \rho|_U)(g)(w \oplus u) \end{aligned}$$

where the sum in the second equality is direct because W and U are $\rho(g)$ -invariant. □

We now recall an important result from Linear Algebra.

Definition 1.2.10 (Projection). Let V be a vector space and let $T : V \rightarrow V$ be linear. Observe that we have the following equivalence:

$$T^2 = T \iff \forall w \in \text{im}(T), T(w) = w \tag{1.2.1}$$

If T satisfies either one of the above conditions, T is said to be a projection.

We do not prove (1.2.1), but we do prove the following lemma, which will prove to be useful.

Lemma 1.2.11. *Let V be a vector space. For all projections $T : V \rightarrow V$, $V = \ker(T) \oplus \operatorname{im}(T)$.*

Proof. Let $T : V \rightarrow V$ be a projection. We then have the following.

$\operatorname{im}(T) \cap \ker(T) = \{0\}$: Fix $w \in \operatorname{im}(T) \cap \ker(T)$. Since $w \in \operatorname{im}(T)$, $\exists v \in V$ such that $w = T(v)$.

Furthermore, since $w \in \ker(T)$, $T(w) = 0$. Since $w = T(v)$, this is equivalent to saying that $T(T(v)) = 0$. But, by (1.2.1), $T(T(v)) = T(v)$. Hence, $T(v) = 0$. Then, since $T(v) = w$, it follows that $w = 0$.

$V = \ker(T) + \operatorname{im}(T)$: Fix $v \in V$. We write $v = T(v) + (v - T(v))$. Clearly, $T(v) \in \operatorname{im}(T)$.

Further, $T(v - T(v)) = T(v) - T(v) = 0$. Hence, $v - T(v) \in \ker(T)$.

Therefore, we do, indeed, have $V = \ker(T) \oplus \operatorname{im}(T)$. □

It turns out that this gives us an important criterion for decomposability.

Corollary 1.2.12. *Let G be a group and let (V, ρ) be a representation of G . If $T : (V, \rho) \rightarrow (V, \rho)$ is a G -linear projection, then $V = \ker(T) \oplus \operatorname{im}(T)$ is a direct sum of subrepresentations.*

Proof. The result follows immediately from Lemma 1.2.11 and Proposition 1.1.13. □

One also has a converse criterion for G -linearity.

Proposition 1.2.13. *Let G be a group and let (V, ρ) be a representation of G , and let $T : V \rightarrow V$ be a projection. If $\ker(T)$ and $\operatorname{im}(T)$ are both G -invariant, then T is G -linear.*

Proof. Since T is a projection, we know that $V = \ker(T) \oplus \operatorname{im}(T)$. Now, fix $g \in G$ and $v \in V$. We know v can uniquely be expressed as $u + w$, where $u \in \ker(T)$ and $w \in \operatorname{im}(T)$. Then,

$$\begin{aligned} T(\rho(g)(v)) &= T\left(\underbrace{\rho(g)(u)}_{\in \ker(T)} + \underbrace{\rho(g)(w)}_{\in \operatorname{im}(T)}\right) \\ &= \rho(g)(w) \\ &= \rho(g)(T(v)) \end{aligned}$$

proving that T is, indeed, G -linear. □

Example 1.2.14. Consider the situation in Example 1.1.9. As we discussed briefly at the beginning of Subsection 1.1.3, we can view (V, ρ) as a subrepresentation of (V', ρ') . Now, consider the linear map $S : V' \rightarrow V' : (x, y, z) \mapsto (x, y, 0)$, where (x, y, z) are coordinates with respect to the standard basis. This is clearly a projection operator with image V , the (x, y) plane, and kernel the z -axis. These are both clearly G -invariant, making S a G -linear projection.

Finally, we relate complementary subrepresentations and complete reducibility, which makes it clear why we are so interested in complementary subrepresentations.

Proposition 1.2.15. *Let G be a finite group and let (V, ρ) be a finite-dimensional, non-zero representation of G . Assume that every subrepresentation of (V, ρ) admits a complementary subrepresentation. Then, (V, ρ) is completely reducible.*

Proof. We argue by induction on $\dim(V)$. If $\dim(V) = 1$, then (V, ρ) is irreducible, making it completely reducible. Now, assume that the statement holds for all representations of degree less than $\dim(V)$. If V is irreducible, we are done. Else, there exists a proper, nonzero, G -invariant subspace $W \leq V$. By assumption, W admits a complementary subrepresentation $U \leq V$. By induction, W and U are completely reducible. Hence, $V = W \oplus U$ is completely reducible. \square

1.2.3 Maschke's Theorem

Given the theme of this section—namely, understanding the compatibility of ordinary linear-algebraic constructions with representation structures—one might wonder under what conditions (if any) we have the existence of a complementary subrepresentations. The answer lies in Maschke's Theorem, which is the first major result of the course.

Theorem 1.2.16 (Maschke's Theorem). *Let G be a finite group, K a field such that $\text{char}(K) \nmid |G|$, and (V, ρ) a representation of G over K . Then, any subrepresentation of V admits a complementary subrepresentation.*

Proof. Let $W \leq V$ be G -invariant. The idea is to construct a G -linear map from V to V with

image W . Then, by Corollary 1.2.12, its kernel would give a complementary subrepresentation.

From Linear Algebra, we know that W admits a complementary (but not necessarily G -invariant) subspace $U \leq V$. Then, every $v \in V$ can uniquely be expressed as a sum $u + w$, where $u \in U$ and $w \in W$. Define $T : V \rightarrow V : u + w \mapsto w$. Clearly, T is a projection operator with image W and kernel U .

If T were G -linear, we would be done with the proof; unfortunately, T does not have to be G -linear. We therefore “convert” T into a G -linear projection $S : V \rightarrow V$ by *averaging over G* . Specifically, define

$$S := \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ T \circ \rho(g)^{-1} \quad (1.2.2)$$

which is well-defined because $|G| \neq 0$ in K . We then show the following.

S is a projection with image W . Fix $v \in V$ and express it as $u + w$ for a unique $u \in U$ and $w \in W$. Then, for all $g \in G$,

- $T(\rho(g)^{-1}(v)) \in W$ because T is a projection with image W .
- $\rho(g)(T(\rho(g)^{-1}(v))) \in W$ because $T(\rho(g)^{-1}(v)) \in W$ and W is G -invariant.

Combined with the fact that W is closed under addition, this proves that $\text{im}(S) \subseteq W$. Conversely, for all $w \in W$ and $g \in G$,

- $(\rho(g)^{-1})(w) = \rho(g^{-1})(w) \in W$ because W is G -invariant.
- $T(\rho(g^{-1})(w)) \in W$ because $\rho(g^{-1})(w) \in W$ and W is T -invariant.
- $\rho(g)(T(\rho(g^{-1})(w))) \in W$ because W is G -invariant.

Combined, again, with the fact that W is closed under addition, this proves that $W \subseteq \text{im}(S)$. Therefore, we have that $W = \text{im}(S)$.

Finally, since $T|_W = \text{id}_W$, we have that $\forall w \in \text{im}(S) = W$,

$$S(w) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \left(T \left(\underbrace{\rho(g)^{-1}(w)}_{\in W} \right) \right)$$

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ \rho(g)^{-1})(w) \\
&= \frac{1}{|G|} \sum_{g \in G} w = w
\end{aligned}$$

proving that S is, indeed, a projection.

S is G -linear. Fix $v \in V$ and $h \in G$. We have

$$\begin{aligned}
S(\rho(h)(v)) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ T \circ \rho(g)^{-1})(\rho(h)(v)) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ T \circ \rho(g^{-1}h))(v)
\end{aligned}$$

We now perform a change of variables. Observe that the map $g \mapsto h^{-1}g : G \rightarrow G$ is an automorphism. Hence, writing $g' = h^{-1}g$, we have

$$\begin{aligned}
S(\rho(h)(v)) &= \frac{1}{|G|} \sum_{g' \in G} (\rho(hg') \circ T \circ \rho((g')^{-1}))(v) \\
&= \rho(h) \left(\frac{1}{|G|} \sum_{g' \in G} (\rho(g') \circ T \circ \rho(g')^{-1}) \right) (v) \\
&= \rho(h)(S(v))
\end{aligned}$$

proving that S is, indeed, G -linear.

Therefore, by Corollary 1.2.12, $\ker(S)$ is a complementary subrepresentation of W . □

We can relate the Maschke condition to complete reducibility.

Corollary 1.2.17. *Let G be a finite group, K a field such that $\text{char}(K) \nmid |G|$. Then, every finite-dimensional representation of G over K is completely reducible.*

Proof. The result follows immediately from Proposition 1.2.15 and Theorem 1.2.16. □

We can now prove the promised partial converse of Proposition 1.2.4.

Corollary 1.2.18. *Let G be a finite group. Let K be a field such that $\text{char}(K)$ does not divide $|G|$. Then, all indecomposable, finite-dimensional representations of G over K are irreducible.*

Proof. Let (V, ρ) be a finite-dimensional representation of G over K . Corollary 1.2.17 tells us that (V, ρ) is completely reducible. We can then apply Proposition 1.2.7 to conclude that if (V, ρ) is indecomposable, it is irreducible. \square

We note that both hypotheses of Maschke's Theorem—namely, that G is a finite group and that $\text{char}(K) \nmid |G|$ —are essential.

Non-Example 1.2.19 (When $\text{char}(K) \mid |G|$). Let $G = \langle a \rangle$ be a cyclic group of prime order p . Let $K = \mathbb{F}_p$ and $V = K^2$. Define $\rho : G \rightarrow \text{GL}(2, \mathbb{F}_p)$ by

$$\rho(a^r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \quad \text{for } r \in \mathbb{F}_p.$$

1. (V, ρ) is a representation of G over \mathbb{F}_p .

Proof. We merely need to verify that ρ is, indeed, a homomorphism of groups. Fix $r, s \in \mathbb{F}_p$. Then,

$$\rho(a^r) \circ \rho(a^s) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s+r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r+s \\ 0 & 1 \end{bmatrix} = \rho(a^{r+s})$$

\square

2. (V, ρ) is reducible.

Proof. Clearly, the subspace $W := \mathbb{F}_p \oplus \{0\}$ is proper and nonzero. Further, for all $a^r \in G$ and $x \in \mathbb{F}_p$, we have

$$\rho(a^r)(x \oplus 0) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in W$$

showing not only that W is G -invariant but also that the corresponding subrepresentation is trivial. \square

3. W admits no complementary subrepresentation.

Proof. We will show that W is the only G -invariant subspace of dimension 1. Let $U \leq V$ be a G -invariant subspace of dimension 1. Then, U is spanned by a vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$ for some $x, y \in \mathbb{F}_p$. Since U is G -invariant, in particular, there must exist

some $\lambda \in \mathbb{F}_p$ such that

$$\rho(a^1)(u) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

For y to be equal to λy , we must have that $\lambda = 1$. This would then imply that $x = x + y$, requiring that y be 0. Therefore, U must be equal to W . Hence, W cannot admit a complementary subrepresentation, as such a subrepresentation would have to have degree 1, which would make it equal to W . \square

This illustrates that when the hypothesis of Maschke's Theorem (ie, that $\text{char}(K) \nmid |G|$) fails, so too might the conclusion. Indeed, the argument above also shows (V, ρ) to be indecomposable, as any decomposition would need to involve two distinct G -invariant subspaces of dimension 1. This shows that the conclusion of Corollary 1.2.18 can also fail when $\text{char}(K) \mid |G|$. Finally, if (V, ρ) is indecomposable and reducible, it cannot be completely reducible, demonstrating that the conclusion of Corollary 1.2.17 *also* fails when $\text{char}(K) \mid |G|$.

It turns out that Maschke's Theorem also has a *converse*.

Theorem 1.2.20 (Converse of Maschke's Theorem). *Let G be a finite group such that every finite-dimensional representation of G over some field K is completely reducible. Then, $\text{char}(K) \nmid |G|$.*

Proof. Consider the regular representation $(K[G], \rho)$ of G over K , with basis $\mathcal{B} = \{e_g : g \in G\}$. The idea is to take advantage of the G -invariant properties of \mathcal{B} .

Consider the subspace

$$W := \left\{ \sum_{g \in G} \alpha_g e_g : \sum_{g \in G} \alpha_g = 0 \right\}$$

of dimension $\dim(V) - 1$. It turns out that W is G -invariant: for all $\sum_{g \in G} \alpha_g e_g \in W$ and $h \in G$, we have

$$\rho(h) \left(\sum_{g \in G} \alpha_g e_g \right) = \sum_{g \in G} \alpha_g e_{hg} \in W$$

(where the sum of the coefficients α_g is still zero). Then, by assumption, $\exists U \leq V$ that is both G -invariant and complementary to W . This means that U must be of dimension 1, and is hence the span of a single vector $u \in U$.

We study the action of G on U . Fix $h \in G$, and write $u = \sum_{g \in G} \beta_g e_g$ for $\beta_g \in K$. Then,

$$\rho(h)(u) - u = \sum_{g \in G} \underbrace{\beta_g e_{hg} - \beta_g e_g}_{\in W}$$

meaning that $\rho(h)(u) - u \in W$. But, $\rho(h)(u) - u \in U$ as well. Since $U \cap W = \{0\}$, this means that $\rho(h)(u) = u$ for all $h \in G$. Hence, the action of G on U is *trivial*. Therefore, for all $x \in G$,

$$\sum_{g \in G} \beta_g e_{hg} = \sum_{g \in G} \beta_g e_g$$

Comparing coefficients, we conclude that $\beta_{h^{-1}g} = \beta_g$ for all $h, g \in G$. Letting $h = g$, we get, in particular, that $\forall g \in G, \beta_g = \beta_1$. Therefore, $u = \beta_1 \sum_{g \in G} e_g$. This, in particular, implies that $u' := \sum_{g \in G} e_g \notin W$, because otherwise, $u = \beta_1 u'$ would also lie in W , which it does not. Therefore, the sum of the coordinates of u' with respect to \mathcal{B} cannot be zero. But, this sum is nothing but the cardinality of G (or rather, its image in the canonical map $\mathbb{Z} \rightarrow K$). Since this is nonzero, it must be that $\text{char}(K) \nmid |G|$, as required. \square

1.2.4 G -Invariant Inner-Products

It turns out that we also have a notion of inner-products being compatible with representation structures.

Definition 1.2.21 (G -Invariant Inner-Product). Let G be a group and let (V, ρ) be a representation of G over \mathbb{C} such that V admits an inner-product $\langle \cdot, \cdot \rangle$. We say that $\langle \cdot, \cdot \rangle$ is G -invariant if $\forall g \in G$ and $\forall x, y \in V$,

$$\langle x, y \rangle = \langle \rho(g)(x), \rho(g)(y) \rangle$$

Equivalently, $\langle \cdot, \cdot \rangle$ is G -invariant if $\text{im}(\rho) \subseteq U(V)$, ie, if, for every $g \in G$, $\rho(g)$ is a *unitary* \mathbb{C} -linear map from V to V .

Intrinsic to the notion of an inner-product is that of orthogonality. In the following proposition, we

understand the significance of G -invariance in the context of subrepresentations.

Proposition 1.2.22. *Let G be a group and let (V, ρ) be a representation of G over \mathbb{C} of finite dimension. Let $\langle \cdot, \cdot \rangle$ be a G -invariant inner-product on V . Then, the orthogonal complement of any G -invariant subspace of V is also G -invariant.*

Proof. Let $W \leq V$ be G -invariant, and denote by W^\perp its orthogonal complement. Fix $g \in G$ and $w \in W^\perp$. To show that $\rho(g)(w) \in W^\perp$, we show it is orthogonal to every $v \in W$.

Fix $v \in W$. Then, since $\langle \cdot, \cdot \rangle$ is G -invariant,

$$\begin{aligned} \langle v, \rho(g)(w) \rangle &= \langle \rho(g^{-1})(v), \rho(g^{-1}g)(w) \rangle \\ &= \langle \rho(g^{-1})(v), w \rangle \end{aligned}$$

Since W is G -invariant, $\rho(g^{-1})(v) \in W$. Since $w \in W^\perp$, we may conclude that $\langle \rho(g^{-1})(v), w \rangle = 0$. Therefore, $\langle v, \rho(g)(w) \rangle = 0$ as well, proving that $\rho(g)(w) \in W^\perp$. \square

Corollary 1.2.23. *Let G be a group and let (V, ρ) be a representation of G over \mathbb{C} . If V is finite dimensional and admits a G -invariant inner-product, then V is completely reducible.*

Proof. If V is finite dimensional and admits a G -invariant inner-product, then by Proposition 1.2.22, for any $W \leq V$ G -invariant, W^\perp is G -invariant as well. Since $W \oplus W^\perp = V$ and both W and W^\perp are finite-dimensional, we can prove the result using similar reasoning to what we used to prove Corollary 1.2.17. \square

1.2.5 Tensor Products of Representations

We begin by recalling the notion of a tensor product of two finite-dimensional vector spaces.

Definition 1.2.24 (Tensor Product of Vector Spaces). Let K be a field and let V, W be finite-dimensional vector spaces over K with bases $\{v_i\}$ and $\{w_j\}$ respectively. The tensor product of V and W , denoted $V \otimes_K W$, is a vector space of dimension $\dim(V) \dim(W)$ with basis

$$\{v_i \otimes w_j : 1 \leq i \leq \dim(V), 1 \leq j \leq \dim(W)\} \quad (1.2.3)$$

such that the map

$$\otimes_K : V \times W \rightarrow V \otimes_K W : (v, w) \mapsto v \otimes_K w$$

is bilinear.

Remark. Sometimes, when the base field (or ring) is clear from the context, we omit the subscript in the notation for the tensor product.

In similar fashion to Definition 1.2.1, we can extend this definition to representations.

Definition 1.2.25 (Tensor Product of Representations). Let G be a finite group and let (V, ρ) and (W, ρ') be representations of G over K . The tensor product of (V, ρ) and (W, ρ') , denoted $(V, \rho) \otimes (W, \rho')$, is a representation of G over K with underlying set $V \otimes_K W$ and action

$$(\rho \otimes \rho')(g)(v \otimes w) = \rho(g)(v) \otimes \rho'(g)(w) \in V \otimes W$$

We will see more of the properties of Tensor Products as we go along, primarily in Section 3.3. We do not go into more detail here as the multiplicative properties of the tensor product are best appreciated if considered alongside a well-developed theory of characters. Until then, the invariant construction whose properties we will explore in detail shall be the direct sum.

1.3 Group Algebras and Modules

In this section, we study an important class of field algebras, namely, group algebras, and an important class of modules over said algebras, namely, group modules.

1.3.1 Preliminaries

Definition 1.3.1 (Group Algebra). Let G be a finite group and let K be a field. The group algebra KG is the K -algebra obtained by endowing the free K -vector space $K[G]$ generated

by G (as a set) with the multiplication

$$\left(\sum_{g \in G} \alpha_g e_g \right) \cdot \left(\sum_{g \in G} \beta_g e_g \right) := \sum_{g \in G} \sum_{h \in G} \alpha_g \beta_h e_{gh}$$

Remark.

1. It is easy to verify that KG is, indeed, a K -algebra, with the multiplicative identity given by $1 \cdot e_1$ (where $1 \in K$ is the identity in K and e_1 is the basis vector corresponding to $1 \in G$, the identity in G).
2. The map $g \mapsto e_g : G \rightarrow KG$ gives a trivial embedding of G in KG .
3. Going forward, for ease of notation, we will denote each basis element e_g as simply g . In particular, we will denote the identity as simply 1 , slightly abusing notation by omitting the distinction between $1 \in K$ and $1 \in G$.

We have a similar notion of group modules.

Definition 1.3.2 (Group Module). Let G be a group and let V be a vector space over a field K . We say that V is a KG -module if we can define a multiplication $g \cdot v$ for some $g \in G$ and $v \in V$ that satisfies the following conditions for all $u, v \in V$, $g, h \in G$ and $\lambda \in K$:

1. $g \cdot v \in V$
2. $(gh) \cdot v = g \cdot (h \cdot v)$
3. $1 \cdot v = v$
4. $g \cdot (\lambda v) = \lambda(g \cdot v)$
5. $g \cdot (u + v) = g \cdot u + g \cdot v$

As expected, we have the following result. The proof merely involves checking several conditions, so we omit it here.

Proposition 1.3.3. *Let G be a group and let V be a vector space over a field K . If V is a*

KG -module with multiplication \cdot (as per Definition 1.3.2), then for $v \in V$, the multiplication

$$\underbrace{\left(\sum_{g \in G} \lambda_g e_g \right)}_{\in KG} \cdot v := \sum_{g \in G} \lambda_g \underbrace{(g \cdot v)}_{\substack{K\text{-vector space multiplication} \\ KG\text{-module multiplication}}}$$

endows V with a module structure over $K[G]$.

It turns out that we can move from modules to representations and vice-versa quite easily.

Proposition 1.3.4. *Let G be a group and let V be a vector space over a field K .*

1. *If $\rho : G \rightarrow \text{GL}(V)$ gives a representation of G , then V is a KG -module with multiplication given by $g \cdot v = \rho(g)(v)$ for all $g \in G$ and $v \in V$.*
2. *If V is a KG -module with multiplication \cdot , the map $\rho : G \rightarrow \text{GL}(V)$ given by $\rho(g)(v) := g \cdot v$ is a representation.*

While we do not prove this proposition either, we will mention that this underscores the importance of the language of KG -modules in representation theory. As commutative algebra—specifically, the theory of modules—has been well-studied, we can use those tools to understand representations of groups. Table 1.1 relates notions about group modules and representations.

KG -Modules	Representations
Simple	Irreducible
Semi-Simple	Completely Irreducible
Submodule	Subrepresentation
Viewing KG as a KG -Module	The Regular Representation
Isomorphism	Equivalence of Representations
Dimension (as a K -vector space)	Degree

Table 1.1: A ‘dictionary’ to translate between the language of KG -modules and that of representations of G over K .

We illustrate the above equivalence by stating Maschke’s Theorem in the language of KG -Modules.

Lemma 1.3.5 (Maschke’s Theorem, Module Version). *Let G be a finite group, K a field whose characteristic does not divide the order of G . Then, any KG -Module V is semi-simple.*

1.3.2 Schur's Lemmas

In this subsection, we explore several versions of an important result by Schur. The main result consists of two cases, the first of which shall (by convention) deal with the non-isomorphic case and the second of which shall deal with the isomorphic case. We will refer to them as Schur's First and Second Lemmas respectively. We begin by stating them in their most general form.

Theorem 1.3.6 (Schur's Lemmas for Rings). *Let A be a ring and let S, T be simple A -modules.*

1. *If S and T are non-isomorphic, then $\text{Hom}_A(S, T) = \{0\}$.*
2. *If S and T are isomorphic, then $\text{Hom}_A(S, T)$ is a division ring.*

Proof. We rely on the fact that for all $\phi \in \text{Hom}_A(S, T)$, $\ker(\phi) \leq S$ and $\text{im}(\phi) \leq T$.

1. Let S and T be non-isomorphic. Fix $\phi \in \text{Hom}_A(S, T)$. Since S is simple, we must have that $\ker(\phi) \in \{\{0\}, S\}$. If $\ker(\phi) = \{0\}$, then $\text{im}(\phi) = T$, meaning $S \cong T$, a contradiction.
2. Let $\phi \in \text{Hom}_A(S, T) \setminus \{0\}$. Then, $\ker(\phi) \neq S$, meaning that $\ker(\phi) = \{0\}$. Then, $\text{im}(\phi) = T$, making ϕ an isomorphism. In particular, this means that ϕ admits an inverse, making $\text{Hom}_A(S, T)$ a division ring.

□

It turns out we can do a bit better when dealing with a specific class of rings, namely, algebras over fields.

Theorem 1.3.7 (Schur's Lemmas for Algebras). *Let K be an algebraically closed field and A a K -algebra. Let S and T be simple A -modules.*

1. *If $S \not\cong T$, then $\text{Hom}_A(S, T) = \{0\}$.*
2. *If $S \cong T$, then $K \cong \text{Hom}_A(S, T)$ via the map $\alpha \mapsto \alpha \cdot \text{id}$.*

Proof.

1. As before.

2. We do not distinguish S and T in this proof.

Fix $\phi \in \text{Hom}_A(S, S)$. Then, ϕ can be viewed as an element of $M_n(K)$, where $n = \dim(S)$. Since K is algebraically closed, ϕ admits an eigenvalue $\lambda \in K$. Now, consider the map $\phi - \lambda \text{id} \in \text{Hom}_A(S, S)$. Clearly, $\ker(\phi - \lambda \text{id}) \neq \{0\}$, since it contains all eigenvectors with eigenvalue λ . Since S is simple, it must be that $\ker(\phi - \lambda \text{id}) = S$, meaning $\phi - \lambda \text{id} = 0$. In other words, $\phi = \lambda \text{id}$.

□

We also have a converse (of sorts) when working with algebras.

Theorem 1.3.8 (Converse of Schur's Lemma for Algebras). *Let K be a field, A a K -algebra and M a semi-simple A -module. If $\text{Hom}_A(M, M) \cong K$, then M is simple.*

Proof. Assume that $\text{Hom}_A(M, M) \cong K$. As M is semi-simple, M admits a decomposition $M = \bigoplus_{i \in \mathcal{I}} N_i$, where for all $i \in \mathcal{I}$, N_i is a simple A -module. We will proceed by proving that $|\mathcal{I}| = 1$ and that $M \cong N_i$ for the unique $i \in \mathcal{I}$.

Suppose $|\mathcal{I}| > 1$, ie, that there exist distinct $i, j \in \mathcal{I}$. Theorem 1.3.7 tells us that $\text{Hom}_A(N_i, N_i) \cong \text{Hom}_A(N_j, N_j) \cong K$. Consider the maps $\phi_i \in \text{Hom}_A(N_i, N_i)$ and $\phi_j \in \text{Hom}_A(N_j, N_j)$. We know that $\exists \lambda_i, \lambda_j \in K$ such that $\phi_i = \lambda_i \cdot \text{id}_{N_i}$ and $\phi_j = \lambda_j \cdot \text{id}_{N_j}$. Indeed, we can choose ϕ_i and ϕ_j so that $\lambda_i \neq \lambda_j$. Then, we can construct a map $\phi \in \text{Hom}_A(M, M)$ such that $\phi|_{N_i} = \phi_i$ and $\phi|_{N_j} = \phi_j$.² By assumption, ϕ must be of the form $\lambda \cdot \text{id}_M$ for some $\lambda \in K$. But, this implies that $\lambda = \lambda_i = \lambda_j$, a contradiction. Therefore, $|\mathcal{I}| = 1$ and $M \cong N_i$ for the unique $i \in \mathcal{I}$. □

Theorem 1.3.9 (Schur's Lemmas for Finite Groups, over \mathbb{C}). *Let G be a finite group and let S and T be simple $\mathbb{C}G$ modules that are finite-dimensional (as vector spaces) over K , with associated representations $\rho_S : G \rightarrow \text{GL}(S)$ and $\rho_T : G \rightarrow \text{GL}(T)$. Let $f : S \rightarrow T$ be*

²For example, if $M = N_i \oplus N_j$, then take $\phi = \phi_i \oplus \phi_j$. Else, add the identity maps on the other summands.

an arbitrary \mathbb{C} -linear map. Define the map

$$\hat{f} := \frac{1}{|G|} \sum_{g \in G} \rho_T(g) \circ f \circ \rho_S(g^{-1}) \quad (1.3.1)$$

Then, \hat{f} is, in fact, a $\mathbb{C}G$ -module homomorphism. Furthermore, we have the following:

1. If $S \not\cong T$, then \hat{f} is identically zero.
2. If $S \cong T$, then

$$\hat{f} = \frac{1}{\dim(S)} \operatorname{Tr}(f) \cdot \operatorname{id}_S$$

Proof. We only really need to show that $\hat{f} \in \operatorname{Hom}_{\mathbb{C}G}(S, T)$. The rest follows relatively naturally from Theorem 1.3.7 (Schur's Lemmas for Algebras), as we shall see. So, fix $h \in G$. Observe that

$$\begin{aligned} \rho_T(h) \circ \hat{f} &= \rho_T(h) \left(\frac{1}{|G|} \sum_{g \in G} \rho_T(g) \circ f \circ \rho_S(g^{-1}) \right) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_T(hg) \circ f \circ \rho_S(g^{-1}) \circ \rho_S(h^{-1}) \circ \rho_S(h) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho_T(hg) \circ f \circ \rho_S(g^{-1}h^{-1})) \circ \rho_S(h) \\ &= \hat{f} \circ \rho_S(h) \end{aligned}$$

proving that \hat{f} is, indeed, a homomorphism of $\mathbb{C}G$ -modules. Then,

1. If $S \not\cong T$, then by Schur's First Lemma for Algebras, \hat{f} is identically 0.
2. If $S \cong T$, then by Schur's Second Lemma for Algebras, \hat{f} is a scalar multiple of the identity. Write $\hat{f} = \lambda \cdot \operatorname{id}_S$. Clearly, we have that $\operatorname{Tr}(\hat{f}) = \lambda \dim(S)$. But, by the definition of \hat{f} , we have that

$$\begin{aligned} \operatorname{Tr}(\hat{f}) &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(\rho_T(g) \circ f \circ \rho_S(g^{-1})) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(f \circ \rho_T(g) \circ \rho_S(g^{-1})) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(f) = \operatorname{Tr}(f) \end{aligned}$$

where we can apply the Trace Theorem to the composition $\rho_T(g) \circ f \circ \rho_S(g^{-1})$ to get the second inequality because $S \cong T$, meaning that we can, indeed, change the order of composition and get a ‘sensible’ linear map, as all dimensions agree. Furthermore, since $S \cong T$, we have, in particular, that $\rho_S = \rho_T$, which gives us the third equality.

Seeing as the trace of a linear map is unique, we must have that $\lambda \cdot \dim(S) = \text{Tr}(f) \iff \lambda = \frac{1}{\dim(S)} \text{Tr}(f)$ (we know $\dim(S) \neq 0$ because S is simple). This then tells us that \hat{f} is of the desired form.

□

We can simplify the above a little when f is a $\mathbb{C}G$ -module homomorphism.

Corollary 1.3.10. *Let G be a finite group and let S and T be simple $\mathbb{C}G$ modules that are finite-dimensional (as vector spaces) over K , with associated representations $\rho_S : G \rightarrow \text{GL}(S)$ and $\rho_T : G \rightarrow \text{GL}(T)$. Let $f : S \rightarrow T$ be a $\mathbb{C}G$ -module homomorphism. Then,*

1. *If $S \not\cong T$, then f is identically zero.*
2. *If $S \cong T$, then*

$$f = \frac{1}{\dim(S)} \text{Tr}(f) \cdot \text{id}_S$$

Proof. We simply show that $f = \hat{f}$, where \hat{f} is as defined in (1.3.1): for all $s \in S$,

$$\begin{aligned} \hat{f} &= \frac{1}{|G|} \sum_{g \in G} \rho_T(g) \circ f \circ \rho_S(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_T(g) \circ \rho_T(g^{-1}) \circ f \\ &= \frac{1}{|G|} \sum_{g \in G} f = f \end{aligned}$$

where the second equality follows from the fact that f is a $\mathbb{C}G$ -module homomorphism, and therefore, a homomorphism of representations—cf. (1.1.1). We can now simply apply Theorem 1.3.9 to get the desired result. □

Schur's Lemmas (especially the version thereof pertaining to complex representations of finite

groups) will prove to be incredibly useful. In the next subsection, we will give a direct application to the classification of representations of finite abelian groups; however, this is not the last that we will be seeing of them.

1.3.3 Representations of Finite Abelian Groups over \mathbb{C}

It is natural to wonder what the purpose was of studying group algebras and modules. It turns out that one of the reasons the correspondence between representations and group modules is so powerful is that it allows the application of Schur's Lemmas to representation theoretic problems. For instance, in the following Lemma, we classify all irreducible representations of finite abelian groups over \mathbb{C} .

Lemma 1.3.11. *Let G be a finite abelian group. Then, all irreducible $\mathbb{C}G$ -modules are of dimension 1. Equivalently, all irreducible representations of G over \mathbb{C} are of degree 1.*

Proof. Let V be an irreducible $\mathbb{C}G$ -module. Since G is abelian, for all $g, h \in G$ and $v \in V$, $(gh) \cdot v = (hg) \cdot v$. Therefore, for some fixed $h \in G$, the following map is $\mathbb{C}G$ -linear:

$$\phi_h : V \rightarrow V : v \mapsto h \cdot v$$

By Theorem 1.3.9, we know that $\exists \lambda_h \in \mathbb{C}$ such that $\widehat{\phi_h} = \phi_h = \lambda_h \cdot \text{id}_V$. Hence, any subspace of V must be a $\mathbb{C}G$ -submodule. Therefore, the only way for V to be irreducible is for it to *admit* no proper, nonzero subspaces in the first place, which would necessitate that $\dim(V) = 1$. \square

Remark. It goes without saying that any degree 1 representation of a group is irreducible. Hence, the above Lemma effectively tells us that irreducibility is equivalent to having degree 1 when working with finite abelian groups over \mathbb{C} .

We now have a complete classification of irreducible representations of cyclic groups over \mathbb{C} .

Corollary 1.3.12. *Let $G = C_n = \langle a \rangle$ be the cyclic group of order n . Then, there are precisely n irreducible representations of G over \mathbb{C} .*

Proof. Let G be a group and let (V, ρ) be a representation of G . We know, from Lemma 1.3.11, that $V \cong \mathbb{C}$ and hence, $\text{GL}(V) \cong \mathbb{C}^\times$. Now, let $x := \rho(a)$. It must be that $x^n = 1$, making x an

n th root of unity. In other words, $\exists 1 \leq k \leq n$ such that $x = e^{\frac{2\pi i}{k}}$. Therefore, there are precisely n possible choices of x , each giving a different representation. \square

Combining this with the structure theorem for finite abelian groups, we can classify all irreducible representations thereof over \mathbb{C} .

Theorem 1.3.13. *Let G be a finite abelian group. Then, there are precisely $|G|$ irreducible representations of G over \mathbb{C} .*

Proof. Let $G = C_{n_1} \times \cdots \times C_{n_r}$ be the decomposition of G into cyclic groups of orders n_i . We know that G has generators $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the i th coordinate. There are precisely n_i choices of where to send e_i , namely, the n_i th roots of unity. Given that group homomorphisms are uniquely defined by where they map the generators, the degree 1 representations are those mapping the e_i s to each possible n_i th root of unity. Hence, there are $n_1 \times \cdots \times n_r = |G|$ irreducible representations of G over \mathbb{C} . \square

Remark. We will see later on that the sum of the squares of the dimensions of all of the irreducible representations of a group will need to equal its order (cf. (2.1.5)). This is another way to see that there must be exactly $|G|$ irreducible complex representations of a finite abelian group G . Specifically, this works because of Lemma 1.3.11.

This tells us a little something about representations of any finite group over \mathbb{C} .

Corollary 1.3.14. *Let G be a finite group. The number of irreducible representations of G over \mathbb{C} is precisely $|G| / |[G, G]|$.*

Proof. We know, from Corollary 1.1.21 that the irreducible complex representations of G are in bijection with those of its abelianisation $G/[G, G]$. Given that G is finite, so is its abelianisation. Therefore, by Theorem 1.3.13, the number of such irreducible representations is the cardinality of its abelianisation. \square

Lemma 1.3.11 is also useful in the study of arbitrary complex representations over \mathbb{C} of finite groups.

Proposition 1.3.15. *Let G be a finite group and let (V, ρ) be a representation of G over \mathbb{C} . For all $g \in G$, there is a basis of V with respect to which $\rho(g)$ has matrix $\text{diag}(\epsilon_1, \dots, \epsilon_n)$, with $\epsilon_i^{\text{ord}(g)} = 1$ for all $1 \leq i \leq n$.*

Proof. Fix $g \in G$, and consider the representation $\rho' : \langle g \rangle \rightarrow \text{GL}(V)$ given by $\rho' = \rho|_{\langle g \rangle}$. Then, ρ' is a representation of a finite abelian group.

By Maschke's Theorem, $\rho' = \sigma_1 \oplus \dots \oplus \sigma_k$ for irreducible subrepresentations $\sigma_1, \dots, \sigma_m$ of $\langle g \rangle$. By Lemma 1.3.11, we know that $\deg(\sigma_i) = 1$ for each i , and hence, that $m = n$. Picking \mathcal{B} to be the basis corresponding to this decomposition of ρ' , we get that the matrix of ρ with respect to \mathcal{B} is, indeed, of the desired form. \square

As it turns out, we can combine the theory developed here with the Structure Theorem for Finite Abelian Groups to get an interesting result.

Lemma 1.3.16. *Let G be a finite abelian group, expressed as a product $C_{n_1} \times \dots \times C_{n_r}$ of cyclic groups C_{n_i} of order $n_i > 1$. Then, G has a faithful representation of degree r over \mathbb{C} .*

Proof. Consider the space $V = \mathbb{C}^r = \mathbb{C}_1 \oplus \dots \oplus \mathbb{C}_r$ (where each \mathbb{C}_i is the one-dimensional subspace spanned by the i th element of some chosen \mathbb{C} -basis for V). Let $C_{n_i} = \langle g_i \rangle$ and denote by e_i the corresponding generators $(1, \dots, 1, g_i, 1, \dots, 1)$ of G . Define the map

$$\rho : G \rightarrow \mathbb{C} : e_i \mapsto R_i$$

where R_i is the rotation by $2\pi/n_i$ acting on the subspace $\mathbb{C}_i \cong \mathbb{C}$. In other words, with respect to the isomorphism $\text{GL}(\mathbb{C}_i) \cong \mathbb{C}^\times$, the map R_i corresponds to $\exp(2\pi/n_i)$.³

ρ has the following effect on group elements: for two group elements acting on the same component of V , ρ maps their product to the composition of their associated rotations, and for elements acting on different components, ρ combines their componentwise actions into a single action across two components. Therefore, ρ is a group homomorphism, and hence, (V, ρ) is a representation of G

³To be perfectly precise, e_i is mapped not to R_i but to the image of R_i in the inclusion $\text{GL}(\mathbb{C}_i) \rightarrow \text{GL}(V)$ that extends R_i by acting as the identity on components other than i and as R_i on component i . We use the word 'component' to refer to a one-dimensional direct summand \mathbb{C}_j of V .

over \mathbb{C} .

We now show that ρ is injective. If $g \in G$ acts identically on all of V , it must, in particular, act identically on each component. But, the action of g on each \mathbb{C}_j is merely the action of the j th component of g on \mathbb{C}_j . It is easy to see that the componentwise actions of ρ on V are all faithful, meaning that each component of g is the identity in its respective cyclic group. Therefore, g must be the identity in G , making ρ a faithful representation. \square

Now that we have classified all complex representations of finite abelian groups, we can move on to complex representations of arbitrary finite groups. Unfortunately, this is easier said than done: the machinery we have thus far developed is not enough. In the next chapter, we will explore the theory of **characters**, which will allow us to understand complex representations of finite groups in much greater detail.

Chapter 2

Character Theory

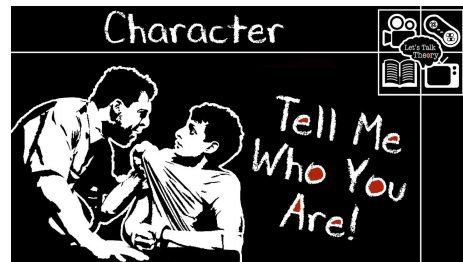


Figure 2.1: The thumbnail of a YouTube video titled "What is Character Theory? | Let's Talk Theory" by Dapper Mr. Tom. The video has nothing to do with mathematics.

In this chapter, we study an important type of functions from groups to fields known as characters. As we shall see, characters encode several useful properties of a group, and have been used extensively to prove several results about finite groups, (the representations of) which are the main object of study in this course. One of the reasons characters are useful to understand representation structures is that they are *class functions*. That is, they encode information not about an individual element of a group but about its conjugacy class, making them good indicators of *structural* and *behavioural* properties. In particular, the character of a representation is independent of the choice of basis of the associated vector space.

Notation. Throughout this chapter, we denote by G an arbitrary finite group.

2.1 The Theory of Irreducible Characters

In this section, we give an overview of the theory of irreducible characters. Our approach focuses extensively on the notion of orthogonality with respect to an inner product we shall soon define. An interesting result we will go on to prove is that irreducible characters play an important role in a broader class of functions on groups, telling us that representation theory can be used to study not only groups themselves but also functions thereof.

2.1.1 On Central Functions

As we all know, a group can contain several elements that all behave similarly—take, for instance, similar matrices in any general linear group. This is why we have the notion of *conjugacy classes*, which allow us to study the elements of a group in terms of their actions or behaviours.

The purpose of character theory is to understand the structural and behavioural properties of a group, without focusing on syntactic particularities. This is why we define the notion of a *class function*.

Definition 2.1.1 (Class Function). Let X be any set. A function $f : G \rightarrow X$ is said to be a class function if $f(x) = f(g^{-1}xg)$ for all $x, g \in G$ —in other words, if it is constant on all conjugacy classes of G .

In this section, we will primarily be focusing on complex representations. Therefore, the following definition is useful.

Definition 2.1.2 (Central Function). We say $f : G \rightarrow \mathbb{C}$ is central if it is a class function. In other words, central functions are precisely \mathbb{C} -valued class functions.

We are already familiar with several examples of central functions.

Example 2.1.3 (Examples of Central Functions). The following are all central:

1. The order function $h \mapsto \text{ord}(h) : H \rightarrow \mathbb{N} \subset \mathbb{C}$ for any group H
2. The determinant function $A \mapsto \det(A) : \text{GL}(n, \mathbb{C}) \rightarrow \mathbb{C}$ for any $n \in \mathbb{N}$
3. The trace function $A \mapsto \text{Tr}(A) : \text{GL}(n, \mathbb{C}) \rightarrow \mathbb{C}$ for any $n \in \mathbb{N}$

The last two examples are, in particular, compatible with representations of G over \mathbb{C} . This will prove to be important when we define the character of a representation. Before doing so, however, we will need to outline the structure of the inner-product space of central functions on G . First, some notation.

Notation. We define

1. $\mathcal{F}(G, \mathbb{C}) := \{f : G \rightarrow \mathbb{C}\}$
2. $\mathcal{F}_C(G, \mathbb{C}) := \{f \in \mathcal{F}(G, \mathbb{C}) : f \text{ is central}\}$
3. $\delta_g(x)$ to be the indicator function (for $g, x \in G$).

We have the following natural result.

Proposition 2.1.4 (\mathbb{C} -Vector Space Structure of $\mathcal{F}(G, \mathbb{C})$ and $\mathcal{F}_C(G, \mathbb{C})$).

1. $\mathcal{F}(G, \mathbb{C})$ is a \mathbb{C} -vector space of dimension $|G|$, with basis $\{\delta_g : g \in G\}$.
2. $\mathcal{F}_C(G, \mathbb{C})$ is a subspace of $\mathcal{F}(G, \mathbb{C})$ of dimension equal to the number of conjugacy classes of G , with basis $\{\delta_g : g \text{ uniquely represents a conjugacy class of } G\}$.

Proof. The model we choose for this proof is that of k^n being isomorphic to the set of functions from $\{1, \dots, n\}$ to k (for any field k). Under this model, the indicator functions on $\{1, \dots, n\}$ correspond to the standard basis of k^n .

1. Since G is finite, this is immediate from the model described above.
2. That $\mathcal{F}_C(G, \mathbb{C})$ is a subspace follows from the fact that the sum of two central functions is central, as is any scalar multiple of a central function. Furthermore, since central functions are uniquely determined by their values on each conjugacy class, $\dim(\mathcal{F}_C(G, \mathbb{C}))$ is the number of conjugacy classes of G .

I also offer a more formal argument here for the dimensionality, as I found it instructive to think about it this way. Let C_1, \dots, C_r be the conjugacy classes of G . Write $C_i = \{g_{i_1}, g_{i_2}, \dots, g_{i_{l_i}}\}$. Let $w_i := \sum_{j=1}^{l_i} \delta_{g_{i_j}}$. It is easy to see that the w_i are linearly independent: any linear combination of all the w_i is, in particular, a linear combination of δ_g as g ranges over G , with each δ_g appearing exactly once. Furthermore, the w_i span $\mathcal{F}_C(G, \mathbb{C})$: their span is clearly contained in $\mathcal{F}_C(G, \mathbb{C})$, and every element of $\mathcal{F}_C(G, \mathbb{C})$ must be constant on any given conjugacy class, meaning that the coordinates of conjugate components must be the same.

□

It turns out that there is also a natural inner-product on $\mathcal{F}(G, \mathbb{C})$.

Definition 2.1.5 (Standard Inner-Product on $\mathcal{F}(G, \mathbb{C})$). Define the function $\langle \cdot, \cdot \rangle : \mathcal{F}(G, \mathbb{C}) \times \mathcal{F}(G, \mathbb{C}) \rightarrow \mathbb{C}$ by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \quad (2.1.1)$$

It is easy enough to show that the $\langle \cdot, \cdot \rangle$ is, indeed, an inner-product on $\mathcal{F}(G, \mathbb{C})$. We will also use it as an inner-product on $\mathcal{F}_C(G, \mathbb{C})$.

An obvious orthogonal basis for $\mathcal{F}_C(G, \mathbb{C})$ is that of the indicator functions of representatives of distinct conjugacy classes. However, in this paradigm, the set S of conjugacy classes of G merely acts as an *index* set for the standard basis of $\mathbb{C}^{|S|}$. I.e, we can infer nothing about S beyond its size, a quantity that does not tell us about the actual *group* structure of G (for instance, for any prime p , the decidedly non-isomorphic abelian groups $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ both have exactly p^2 conjugacy classes). It turns out that we can find a much better orthogonal (indeed, *orthonormal*) basis for $\mathcal{F}_C(G, \mathbb{C})$ in the world of *characters*, a class of central functions closely related to the group structure of G .

2.1.2 Introduction to Character Theory

In this subsection, we introduce a central function that is compatible with the notion of a representation, namely, the *character*. As we saw in Example 2.1.3, both the trace and the determinant would be good options, but the convention is to define the character in terms of the trace. A heuristic reason is that this way, given characters associated to two representations, their sum will give the character of the direct sum, and their product that of the tensor product of the underlying representations. There are deeper reasons too, which will become clearer as we progress.

Definition 2.1.6 (Character). Let V be a $\mathbb{C}G$ -module. The character of V is the map $\chi_V : G \rightarrow \mathbb{C}$ given by $\chi_V(g) = \text{Tr}(\rho(g))$, where ρ is the representation associated to V .

Immediately, we are able to “import” the following definitions from Chapter 1.

Definition 2.1.7 (Irreducibility). We say a character χ_V is irreducible if the associated represen-

tation (V, ρ) is irreducible over \mathbb{C} .

Definition 2.1.8 (Degree). We define the degree of a character to be that of its associated representation.

Characters have several important properties, which we list below. We will make extensive use of these properties for the remainder of this chapter.

Proposition 2.1.9 (On the Behaviour of Characters). *Let V be a $\mathbb{C}G$ -module, with associated representation ρ .*

1. For all $\mathbb{C}G$ -modules W , $\chi_{V \oplus W} = \chi_V + \chi_W$
2. For all $\mathbb{C}G$ -modules W , $V \cong W \implies \chi_V = \chi_W$
3. $\dim(V) = \chi_V(1)$
4. For all $g \in G$, $\chi_V(g)$ is a sum of d th roots of unity, where $d = \text{ord}(g)$.
5. For all $g \in G$, $|\chi_V(g)| \leq \dim(V)$, with equality if $g \in \ker(\rho)$.
6. For all $g \in G$, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$
7. For all $\mathbb{C}G$ -modules W , $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Proof. We only prove the more nontrivial of the above results in detail; we merely offer sketches for the rest.

1. This follows from the fact that the trace of a direct sum is the sum of the traces.
2. This follows from the invariance of the trace under change of basis.
3. This follows from the fact that the trace of the identity is the dimension of the space.
4. For any $g \in G$ of order d , $\rho(g)^d = \text{id}_V$. Hence, the minimal polynomial of $\rho(g)$ divides $X^d - 1 \in \mathbb{C}[X]$, which has distinct roots in \mathbb{C} . This means that the eigenvalues of $\rho(g)$ are d th roots of unity. Putting $\rho(g)$ in Jordan Normal Form, we see that its trace is a sum of d th roots of unity.¹²
5. We prove the inequality using the diagonalisation from Proposition 1.3.15. Fix $g \in G$, and write $\rho(g) = \text{diag}(\varepsilon_1, \dots, \varepsilon_{\dim(V)})$ in an appropriate basis, where each ε_i is a complex

¹It is not necessarily a sum of *all* d th roots of unity, or even *distinct* d th roots of unity.

²One could also prove this result using Proposition 1.3.15.

$\dim(V)$ -th root of unity. Then,

$$|\chi_V(g)| = \left| \text{Tr} \left(\begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_{\dim(V)} \end{bmatrix} \right) \right| = \left| \sum_{i=1}^{\dim(V)} \varepsilon_i \right| \leq \sum_{i=1}^{\dim(V)} |\varepsilon_i| = \dim(V)$$

It is obvious that if $g \in \ker(\rho)$, then $|\chi_V(g)| = |\text{Tr}(\text{id}_V)| = \dim(V)$.

6. Fix $g \in G$. Diagonalise $\rho(g)$ as $\text{diag}(\varepsilon_1, \dots, \varepsilon_{\dim(V)})$, as per Proposition 1.3.15. We know that $\rho(g^{-1}) = \rho(g)^{-1}$; hence, if we write $\rho(g^{-1}) = \text{diag}(\varepsilon'_1, \dots, \varepsilon'_{\dim(V)})$, it must be that for all $1 \leq i \leq \dim(V)$, $\varepsilon'_i = \varepsilon_i^{-1} = \bar{\varepsilon}_i$ because ε_i is a complex $\text{ord}(g)$ -th root of unity. Taking the trace of both sides then gives us the result.
7. Let $B = \{v_1, \dots, v_n\}$ and $B' = \{w_1, \dots, w_m\}$ be bases of V and W respectively. Denote by ρ' the representation associated to W , and fix an arbitrary $g \in G$. Let $(a_{ij})_{1 \leq i, j \leq n}$ be the matrix of $\rho(g)$ with respect to B , and let $(b_{kl})_{1 \leq k, l \leq m}$ be that of $\rho'(g)$ with respect to B' . For all $1 \leq j \leq n$ and $1 \leq l \leq m$, we have that

$$\begin{aligned} (\rho \otimes \rho')(g)(v_j \otimes w_l) &= \rho(g)(v_j) \otimes \rho'(g)(w_l) \\ &= \left(\sum_{i=1}^n a_{ij} v_i \right) \otimes \left(\sum_{k=1}^m b_{kl} w_k \right) \\ &= \sum_{i=1}^n \sum_{k=1}^m a_{ij} b_{kl} v_i \otimes w_k \end{aligned}$$

Given that $\mathcal{B} := \{v_j \otimes w_l : v_j \in B, w_l \in B'\}$ is a basis of $V \otimes W$, the trace of $(\rho \otimes \rho')(g)$ is the sum of all of the diagonal elements of its matrix with respect to \mathcal{B} . The diagonal elements are precisely the $(v_j \otimes w_l)$ -components of $(\rho \otimes \rho')(g)(v_j \otimes w_l)$, which we can deduce from the above expression to be $a_{jj}b_{ll}$. Therefore,

$$\chi_{V \otimes W}(g) = \text{Tr}((\rho \otimes \rho')(g)) = \sum_{j=1}^n \sum_{l=1}^m a_{jj} b_{ll} = \left(\sum_{j=1}^n a_{jj} \right) \left(\sum_{l=1}^m b_{ll} \right) = \chi_V(g) \cdot \chi_W(g)$$

□

We now give a few examples of characters of representations. We begin with characters that exist for any G .

Example 2.1.10 (The Trivial Character). The trivial character of G is the character associated to the trivial representation on \mathbb{C} —in other words, it is the character of \mathbb{C} viewed as a $\mathbb{C}G$ -module where every element of G acts as the identity. Often denoted χ_1 , the trivial character is identically equal to 1.

Example 2.1.11 (The Regular Character). The regular character of G is the character associated to its regular representation (V, ρ) . It is usually denoted χ_{reg} . We can exploit how closely the regular representation is tied to the structure of G to explicitly compute the values of χ_{reg} . Write $G = \{g_1 = 1, g_2, \dots, g_r\}$. Then, for any $g \in G$, the ij th entry of $\rho(g)$ (with respect to the basis $\{e_{g_i}\}_{i=1}^r$) is 1 if $g \cdot g_j = g_i$ and 0 otherwise. In particular, since the only element of G that has any fixed points is g_1 , the diagonal elements of $\rho(g_i)$ are all 0 for all $i \neq 1$. On the contrary, $\rho(g_1)$ is the identity. Hence, for all $g \in G$,

$$\chi_{\text{reg}}(g) = \text{Tr}(\rho(g)) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.1.2)$$

Next, we look at more specific examples.

Example 2.1.12 (The Dihedral Group of Order 8). Let $G = D_8$, the dihedral group of order 8. Consider the presentation

$$G = \langle a, b \mid a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

Let $\rho : G \rightarrow \text{GL}(2, \mathbb{C})$ be a representation of G over \mathbb{C} given by

$$\rho(a) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The associated character χ_V then takes on the following values:

g	1	a	a^2	a^3	b	ab	a^2b	a^3b
$\rho(g)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\chi_V(g)$	2	0	-2	0	0	0	0	0

One can show the conjugacy classes of D_8 to be precisely $\{1\}$, $\{a, a^3\}$, $\{a^2\}$, $\{b, a^2b\}$, and $\{ab, a^3b\}$. From the table above, it is clear that conjugate elements do, indeed, have the same character value. Therefore, for the sake of brevity, one typically only lists the values of a character corresponding to the representatives of its conjugacy classes.

Example 2.1.13 (The Cyclic Group of Order 3). Let $G = C_3 = \langle a \rangle$ be the cyclic group of order 3. By Corollary 1.3.12, we know that G admits exactly three irreducible representations over \mathbb{C} , all of degree 1, each corresponding to choice of 3rd root of unity to which to send a . Denote these by ρ_1 , ρ_2 and ρ_3 , so that $\rho_j(a) = e^{\frac{2\pi i(j-1)}{3}}$ for $j = 1, 2, 3$. Denote their corresponding (irreducible) characters χ_1 , χ_2 and χ_3 . Since the trace map is simply the constant map on a space of dimension 1, we have the following values for the characters:

g	1	a	a^2
$\chi_1(g)$	1	1	1
$\chi_2(g)$	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
$\chi_3(g)$	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$

In the above example, we see that the characters of the irreducible representations of G form an orthonormal system with respect to the standard inner-product (2.1.1)—for instance, $\chi_1 \neq \chi_2$, and clearly,

$$\begin{aligned}
 \langle \chi_1, \chi_2 \rangle &= \frac{1}{3} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} \\
 &= \frac{1}{3} \left(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}} \right) \\
 &= \frac{1}{3} \left(1 + e^{\frac{4\pi i}{3}} + e^{\frac{2\pi i}{3}} \right) = 0
 \end{aligned}$$

whereas $\langle \chi_1, \chi_1 \rangle = 1$. In similar fashion, one can check the remaining 7 cases to show that $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for $i, j = 1, 2, 3$. It turns out that this is merely one instance of a more general orthogonality phenomenon. We will understand this better in the next subsection.

2.1.3 The Orthogonality Theorem

In this subsection, we dive into the heart of Character Theory, proving one of the most fundamental results in the field: the Orthogonality Theorem. This theorem generalises the orthogonality that we observed in Example 2.1.13, and has several important consequences, most notably the fact

that the irreducible characters of a group form an orthonormal basis for the space of class functions on that group, a result we will see in the next subsection.

First, we state and prove a useful lemma that helps us understand the *matrices* of irreducible representations (taken, as per convention, with respect to the standard basis of \mathbb{C}^n). The point of this is that matrices are more computation-friendly.

Lemma 2.1.14. *Let $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$ and $\rho' : G \rightarrow \mathrm{GL}(m, \mathbb{C})$ be irreducible representations of G over \mathbb{C} . Fix $j, s \in \{1, \dots, m\}$ and $r, i \in \{1, \dots, n\}$.*

1. *If ρ and ρ' are not equivalent, then*

$$\frac{1}{|G|} \sum_{g \in G} [\rho(g)]_{ri} [\rho'(g^{-1})]_{js} = 0$$

2. *If ρ and ρ' are equivalent, then*

$$\frac{1}{|G|} \sum_{g \in G} [\rho(g)]_{ri} [\rho'(g^{-1})]_{js} = \begin{cases} \frac{1}{n} & \text{if } i = j \text{ and } r = s \\ 0 & \text{otherwise} \end{cases}$$

where we use the notation $[T]_{ij}$ to refer to the ij th entry of the matrix of any $T \in \mathrm{GL}(n, \mathbb{C})$ with respect to the standard basis of \mathbb{C}^n .

Proof. Let $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ be the two simple $\mathbb{C}G$ -modules corresponding to ρ and ρ' respectively. The idea is to define a linear map that will allow us to use Schur's Lemmas—specifically, those pertaining to finite groups over \mathbb{C} (see Theorem 1.3.9).

Let $\phi_{ij} : W \rightarrow V$ be the \mathbb{C} -linear map given by the $n \times m$ matrix (with respect to the standard bases of V and W) with ij th entry 1 and all other entries 0. In other words, let ϕ_{ij} denote the ij th element of the standard basis of the space of linear maps from V to W . Define

$$\widehat{\phi_{ij}} := \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \phi_{ij} \circ \rho'(g^{-1})$$

By Theorem 1.3.9, $\widehat{\phi_{ij}}$ is a $\mathbb{C}G$ -module homomorphism from W to V . Indeed, we have that

$$[\widehat{\phi_{ij}}]_{rs} = \left[\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \phi_{ij} \circ \rho'(g^{-1}) \right]_{rs}$$

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} [\rho(g) \circ \phi_{ij} \circ \rho'(g^{-1})]_{rs} \\
&= \sum_{k=1}^n \sum_{l=1}^m \frac{1}{|G|} \sum_{g \in G} [\rho(g)]_{rk} [\phi_{ij}]_{kl} [\rho'(g^{-1})]_{ls} \\
&= \frac{1}{|G|} \sum_{g \in G} [\rho(g)]_{ri} [\rho'(g^{-1})]_{js}
\end{aligned}$$

1. If $W \not\cong V$, we have $\widehat{\phi_{ij}} = 0$. In particular,

$$\frac{1}{|G|} \sum_{g \in G} [\rho(g)]_{ri} [\rho'(g^{-1})]_{js} = [\widehat{\phi_{ij}}]_{rs} = 0$$

2. Similarly, if $W \cong V$, we have $m = n$. Now, by Theorem 1.3.9, we know that

$$\widehat{\phi_{ij}} = \frac{1}{n} \text{Tr}(\phi_{ij}) \cdot \text{id}_V = \frac{1}{n} \delta_{ij} \cdot \text{id}_V$$

We then have that

$$[\rho(g)]_{ri} [\rho'(g^{-1})]_{js} = [\widehat{\phi_{ij}}]_{rs} = \begin{cases} \frac{1}{n} & \text{if } i = j \text{ and } r = s \\ 0 & \text{otherwise} \end{cases}$$

□

Remark. As per Dr. Rizzoli, on the exam, it's more important to know the idea of such a proof than the specifics of *which index goes where*.

It turns out that by simply unfolding a few definitions and playing with the order of summations, we can use the above lemma to prove the Orthogonality Theorem.

Theorem 2.1.15 (Orthogonality Theorem). *Let S, T be irreducible $\mathbb{C}G$ -modules.*

1. *If $S \not\cong T$, then $\langle \chi_S, \chi_T \rangle = 0$.*

2. *If $S \cong T$, then $\langle \chi_S, \chi_T \rangle = 1$.*

In other words, irreducible characters form an orthogonal system.

Proof. Let $P : G \rightarrow \text{GL}(n, \mathbb{C})$ and $Q : G \rightarrow \text{GL}(m, \mathbb{C})$ be the representations corresponding to S

and T . We know that

$$\begin{aligned}
 \langle \chi_S, \chi_T \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_S(g) \chi_T(g^{-1}) \\
 &= \frac{1}{|G|} \sum_{g \in G} \text{Tr}(P(g)) \text{Tr}(Q(g^{-1})) \\
 &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^n [P(g)]_{ii} \right) \left(\sum_{j=1}^n [Q(g^{-1})]_{jj} \right) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \frac{1}{|G|} \sum_{g \in G} [P(g)]_{ii} [Q(g^{-1})]_{jj}
 \end{aligned}$$

We can now use Lemma 2.1.14 to evaluate this sum.

1. If $S \not\cong T$, then

$$\frac{1}{|G|} \sum_{g \in G} [P(g)]_{ii} [Q(g^{-1})]_{jj} = 0$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence,

$$\langle \chi_S, \chi_T \rangle = \sum_{i=1}^n \sum_{j=1}^m 0 = 0$$

2. If $S \cong T$, then first of all, $n = m$. Indeed,

$$\frac{1}{|G|} \sum_{g \in G} [P(g)]_{ii} [Q(g^{-1})]_{jj} = \begin{cases} \frac{1}{n} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for all $1 \leq i, j \leq n$. Hence,

$$\langle \chi_S, \chi_T \rangle = \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n} \delta_{ij} = \sum_{i=1}^n \frac{1}{n} = 1$$

□

We have the following important corollary.

Corollary 2.1.16. *Up to isomorphism, there are finitely many irreducible $\mathbb{C}G$ -modules.*

Proof. Suppose there exist infinitely many, pairwise nonisomorphic $\mathbb{C}G$ -modules. Then, the (in-

finite) subset of $\mathcal{F}(G, \mathbb{C})$ consisting of all of their characters would be pairwise orthogonal (by Theorem 2.1.15). Since orthogonality implies linear independence, this set would need to be contained in a basis of $\mathcal{F}(G, \mathbb{C})$. However, by Proposition 2.1.4, such a basis would need to be finite, leading to a clear contradiction. Hence, there can only exist finitely many $\mathbb{C}G$ -modules (up to isomorphism). \square

2.1.4 Understanding Irreducible Characters

In this subsection, we will explore the properties of irreducible characters in more detail. In particular, we will see that they form a basis for the space of class functions on G . This is a very important result, which illustrates the power of character theory in understanding the properties of a group. We will begin by introducing some notation and stating a few basic properties of irreducible characters that will be useful in the proof of the main result.

Notation (Irreducible Characters). Denote by $\text{Irr}(G)$ the subset of $\mathcal{F}_C(G, \mathbb{C})$ consisting of irreducible characters of G .

The Orthogonality Theorem tells us the following facts about Irreducible Characters.

Proposition 2.1.17 (On the Behaviour of Irreducible Characters).

1. $\text{Irr}(G)$ is a linearly independent set. In particular, $|\text{Irr}(G)| \leq |G|$.
2. Let $V = V_1 \oplus \cdots \oplus V_r$ be a $\mathbb{C}G$ module, with V_i simple for $1 \leq i \leq r$. For any simple $\mathbb{C}G$ module S , the number of V_i s isomorphic to S , known as the multiplicity of S , is given by $\langle \chi_V, \chi_S \rangle$.
3. Let V, V' be simple $\mathbb{C}G$ -modules. Then, $V \cong V' \iff \chi_V = \chi_{V'}$.
4. A $\mathbb{C}G$ -module V is simple iff $\langle \chi_V, \chi_V \rangle = 1$.

Proof.

1. The linear independence follows immediately from the fact that $\text{Irr}(G)$ form an orthonormal system. The inequality follows from the fact that all central functions are class functions: they agree for all conjugate elements of G . This means that $\dim(\mathcal{F}_C(G, \mathbb{C}))$ is simply the number of conjugacy classes of G . Since $\dim(\mathcal{F}_C(G, \mathbb{C}))$ must be at least $|\text{Irr}(G)|$ and the

number of conjugacy classes of G must be at most $|G|$, we have the desired result.

2. We know, from Proposition 2.1.9, that $\chi_V = \chi_{V_1} + \cdots + \chi_{V_r}$. So,

$$\begin{aligned} \langle \chi_V, \chi_S \rangle &= \langle \chi_{V_1} + \cdots + \chi_{V_r}, \chi_S \rangle \\ &= \sum_{i=1}^r \langle \chi_{V_i}, \chi_S \rangle \end{aligned}$$

Observe that for $1 \leq i \leq r$, $\langle \chi_{V_i}, \chi_S \rangle = 1$ if $V_i \cong S$ and 0 otherwise. Hence, the only nonzero terms in the summation are given by those i for which $V_i \cong S$. Since all of those terms are 1, the sum must evaluate to the multiplicity of S .

3. (\implies) This is true regardless of the simplicity of V and V' (cf. Proposition 2.1.9).

(\impliedby) This turns out to be a straightforward application of the previous part. Let $W = V \oplus V'$. Then, the multiplicity of V is given by

$$\langle \chi_W, \chi_V \rangle = \langle \chi_V + \chi_{V'}, \chi_V \rangle = \langle \chi_V, \chi_V \rangle + \langle \chi_{V'}, \chi_V \rangle = 2 \langle \chi_V, \chi_V \rangle = 2$$

This means that there are two terms in the decomposition $W = V \oplus V'$ that are isomorphic to V , which is only possible if $V \cong V'$.

4. (\implies) This follows immediately from the Orthogonality Theorem.

(\impliedby) Let $\text{Irr}(G) = \{\chi_1, \dots, \chi_r\}$, with corresponding simple $\mathbb{C}G$ -modules $\{V_1, \dots, V_r\}$. Denoting by a_i the multiplicity of each V_i in V , we have that

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_r^{\oplus a_r} \tag{2.1.3}$$

where we use the notation $V_i^{\oplus a_i}$ to mean $\underbrace{V_i \oplus \cdots \oplus V_i}_{a_i \text{ times}}$, allowing $a_i = 0$.

We then have

$$\begin{aligned} \chi_V &= \sum_{i=1}^r a_i \chi_i \\ \implies \langle \chi_V, \chi_V \rangle &= \sum_{i=1}^r a_i^2 \end{aligned}$$

by the Orthogonality Theorem. Since $\langle \chi_V, \chi_V \rangle = 1$, it must be that only one of the

a_i s is nonzero, and equal to 1. Hence, $V = V_i$ for this i , making V simple.

□

Example 2.1.18 (The Regular Character: A Deep Dive). We know, from Corollary 1.2.17, that $\mathbb{C}G$ is semi-simple when viewed as a module over itself. Let V_1, \dots, V_r be all of the irreducible $\mathbb{C}G$ -modules (we know there are finitely many of them by Corollary 2.1.16). Denote by χ_i the irreducible character corresponding to V_i . In the notation of (2.1.3), we may write

$$\mathbb{C}G = V_1^{\oplus a_1} \oplus \dots \oplus V_r^{\oplus a_r}$$

Then, by Proposition 2.1.9, we have that

$$\chi_{\text{reg}} = a_1 \chi_1 + \dots + a_r \chi_r$$

where χ_{reg} is the character corresponding to the regular representation (ie, to $\mathbb{C}G$). As we have just seen in Proposition 2.1.17,

$$\begin{aligned} a_i = \langle \chi_{\text{reg}}, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{\text{reg}}(g) \overline{\chi_i(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} |G| \delta_1(g) \overline{\chi_i(g)} \\ &= \chi_i(1) = \dim(V_i) \end{aligned}$$

This tells us that the multiplicities of the simple constituents of $\mathbb{C}G$ are precisely their dimensions (as \mathbb{C} -vector spaces). This allows us to express χ_{reg} in the following manner:

$$\chi_{\text{reg}} = \dim(V_1) \chi_1 + \dots + \dim(V_r) \chi_r = \chi_1(1) \chi_1 + \dots + \chi_r(1) \chi_r \quad (2.1.4)$$

In particular, we can evaluate both sides of (2.1.4) at 1 to get the following nice expression for $|G|$ in terms of the elements of $\text{Irr}(G)$:

$$|G| = \dim(\mathbb{C}G) = \chi_{\text{reg}}(1) = \chi_1(1)^2 + \dots + \chi_r(1)^2 \quad (2.1.5)$$

We will use (2.1.5) extensively when constructing Character Tables.

We now prove one of the most important theorems in this course.

Theorem 2.1.19. *$\text{Irr}(G)$ is a basis for $\mathcal{F}_C(G, \mathbb{C})$.*

Proof. As $\text{Irr}(G)$ is linearly independent, we only need to show that it spans $\mathcal{F}_C(G, \mathbb{C})$. To that end, let $W = \text{Span}(\text{Irr}(G)) \leq \mathcal{F}_C(G, \mathbb{C})$, with orthogonal complement W^\perp with respect to the inner-product (2.1.1). Since $V = W \oplus W^\perp$, if we can show that $W^\perp = \{0\}$, we would have that $V = W$, proving the desired result.

Fix $f \in W^\perp$, and consider the element $\hat{f} \in \mathbb{C}G$ given by

$$\hat{f} = \sum_{g \in G} \overline{f(g)} \cdot g$$

Our approach will be to consider the action of \hat{f} by scalar multiplication on simple $\mathbb{C}G$ -modules. We will show this to be a $\mathbb{C}G$ -module homomorphism, compute its trace, and apply the Orthogonality Theorem to show that f must be identically zero.

First, we show that $\hat{f} \in Z(\mathbb{C}G)$ —ie, that \hat{f} commutes (multiplicatively) with all elements of $\mathbb{C}G$. To show this, it suffices to show that $h^{-1}\hat{f}h = \hat{f}$ for all $h \in G$ (ie, that \hat{f} commutes with all elements of G). To that end, fix $h \in G$. Then,

$$\begin{aligned} h^{-1}\hat{f}h &= \sum_{g \in G} \overline{f(g)} \cdot h^{-1}gh \\ &= \sum_{g \in G} \overline{f(h^{-1}gh)} \cdot h^{-1}gh \\ &= \hat{f} \end{aligned}$$

where the first equality follows from the fact that $\overline{f(g)} \in \mathbb{C}$, thereby commuting with all elements of g in $\mathbb{C}G$; the second equality from the fact that f is central (recall that $W, W^\perp \leq \mathcal{F}_C(G, \mathbb{C})$); and last equality from a change of variables in the summation, replacing g with hgh^{-1} . The fact that $\hat{f} \in Z(\mathbb{C}G)$ will prove useful in the next step of the proof.

Now, let S be any simple $\mathbb{C}G$ -module. Consider the \mathbb{C} -linear map $\phi : S \rightarrow S$ given by

$$\phi(v) = \hat{f} \cdot v = \sum_{g \in G} \overline{f(g)} \cdot (g \cdot v)$$

where $g \cdot v$ refers to the action of g on S given by its $\mathbb{C}G$ -module structure³. In the notation of (1.3.1), consider the map $\hat{\phi} : S \rightarrow S$. Since $\hat{f} \in Z(\mathbb{C}G)$, we have that for all $v \in V$,

$$\begin{aligned}\hat{\phi}(v) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \phi(g^{-1} \cdot v) \\ &= \frac{1}{|G|} \sum_{g \in G} g \hat{f} g^{-1} \cdot v \\ &= \frac{1}{|G|} \sum_{g \in G} \hat{f} \cdot v \\ &= \hat{f} \cdot v = \phi(v)\end{aligned}$$

It follows from Theorem 1.3.9 that

$$\phi = \hat{\phi} = \frac{\text{Tr}(\phi)}{\dim(S)} \cdot \text{id}$$

Our next step will be to show that ϕ is identically zero. We will do this by computing its trace and showing it to be zero.⁴ Denote by ρ_S the (irreducible) representation corresponding to S and by χ_S its corresponding character. Then, as per the definition of ϕ , we have that

$$\begin{aligned}\text{Tr}(\phi) &= \text{Tr} \left(\sum_{g \in G} \overline{f(g)} \cdot \rho_S(g) \right) \\ &= \sum_{g \in G} \overline{f(g)} \cdot \text{Tr}(\rho_S(g)) \\ &= \sum_{g \in G} \overline{f(g)} \cdot \chi_S(g)\end{aligned}$$

Observe that the above expression is merely $|G|$ times the inner-product of f and χ_S (cf. (2.1.1)). Since $f \in W^\perp$ and $\chi_S \in W$, we must have that

$$\text{Tr}(\phi) = |G| \langle \chi_S, f \rangle = 0$$

proving that in fact, $\phi = 0$.

³ie, if ρ_S is the representation corresponding to S , then $g \cdot v$ corresponds to $\rho_S(g)(v)$

⁴Indeed, a trace zero map is not, in general, the zero map; this is where the computation involving Schur's Lemmas plays a crucial role. As we have shown ϕ to be given by the action of scalar multiplication by its trace (divided by some nonzero factor, which is irrelevant for our purposes), if we can show its trace to be zero, we will have shown that ϕ itself is identically zero.

It remains to show that $f = 0$. To that end, consider $V = \mathbb{C}G$ as a $\mathbb{C}G$ -module⁵. Let

$$\mathbb{C}G = S_1 \oplus \cdots \oplus S_r$$

be its decomposition into simple $\mathbb{C}G$ -modules. By the above computations, we know that the maps $\phi_i : S_i \rightarrow S_i$ given by $\phi_i(v) = \hat{f} \cdot v$ are all zero. In particular, for i such that $1 \in S_i$ (with 1 being the multiplicative identity of $\mathbb{C}G$), we have that

$$\begin{aligned} 0 = \phi_i(1) &= \hat{f} \cdot 1 = \hat{f} \\ &= \sum_{g \in G} \overline{f(g)} \cdot g \end{aligned}$$

Since $G \subset \mathbb{C}G$ is a \mathbb{C} -basis of $\mathbb{C}G$, in particular, the elements of G are linearly independent over \mathbb{C} . This means that the only way the above sum can be zero is if $\overline{f(g)} = 0$ for all $g \in G$. Since f was arbitrary, we have that $W^\perp = \{0\}$, proving the desired result. \square

Remark. The above proof illustrates the power of several concepts we have learnt so far.

1. The $\mathbb{C}G$ -module-formulation of a representation allowed us to simplify notation by constructing \hat{f} as a (central) element of $\mathbb{C}G$, making it correspond effectively to a (central) element of $\text{GL}(V)$ for *any* representation V of G .
2. The semi-simplicity of $\mathbb{C}G$ as a $\mathbb{C}G$ -module (ie, the complete reducibility of the regular representation⁶) allowed us to apply a result on simple $\mathbb{C}G$ -modules to study the properties of G itself. In particular, by reducing to the case of simple $\mathbb{C}G$ -modules, we were able to apply Schur's Lemmas.
3. Schur's Lemmas allowed us to express ϕ in a more convenient form, whereby we could simplify the problem of showing ϕ to be zero to that of showing its trace to be zero.

⁵Recall that this corresponds to the regular representation.

⁶Indeed, Corollary 1.2.17 tells us that *any* $\mathbb{C}G$ -module is semi-simple because $\text{char}(\mathbb{C}) = 0$ cannot divide the order of a group (which cannot be zero).

2.2 Character Tables

In Example 2.1.13, we constructed a table consisting of the character values of each conjugacy class of the cyclic group of order 3 (in this case, each class consisted of only one element—but this is not always the case). Our approach in Example 2.1.13 was direct computation, which, while straightforward, is not always practical—especially when dealing larger groups. In this section, we will introduce a more systematic way of constructing such tables using the orthogonality properties of irreducible characters.

Notation. For the purposes of this section, let C_1, \dots, C_k be the conjugacy classes of G with representatives g_1, \dots, g_k respectively. Recall that by the Orbit-Stabiliser Theorem,

$$|C_i| = \frac{|G|}{|C_G(g_i)|}$$

where $C_G(\cdot)$ refers to the centraliser^a of an element in G . Now, let $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$.

^aRecall that the centraliser of a group element is the set of all elements that commute with it.

2.2.1 The Orthogonality Relations

Definition 2.2.1 (Character Table). A character table is a $k \times k$ table whose columns are the conjugacy classes (or their representatives) and whose rows are the irreducible characters.

	g_1	\cdots	g_k
χ_1	$\chi_1(g_1)$	\cdots	$\chi_1(g_k)$
\vdots	\vdots	\ddots	\vdots
χ_k	$\chi_k(g_1)$	\cdots	$\chi_k(g_k)$

In other words, it is a table whose (i, j) th entry is $\chi_i(g_j)$.

The table in Example 2.1.13 is an example of a character table.

It turns out that we have a certain orthogonality property as we fix two rows and sum across the columns of a character table.

Proposition 2.2.2 (First Orthogonality Relation). *Given $1 \leq r, s \leq k$, we have*

$$\frac{1}{|G|} \sum_{j=1}^k |C_j| \chi_r(g_j) \chi_s(g_j^{-1}) = \delta_{rs}$$

Proof. The proof is quite straightforward. It merely involves applying the centrality of characters to the definition of the inner-product and concluding using the Orthogonality Theorem:

$$\begin{aligned} \delta_{rs} &= \langle \chi_r, \chi_s \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_r(g) \chi_s(g^{-1}) \\ &= \frac{1}{|G|} \sum_{j=1}^k \left(\sum_{g \in C_j} \chi_r(g) \chi_s(g^{-1}) \right) \\ &= \frac{1}{|G|} \sum_{j=1}^k |C_j| \chi_r(g_j) \chi_s(g_j^{-1}) \end{aligned}$$

where the last inequality follows from the fact that characters are central (ie, they are equal for all elements of a conjugacy class). \square

Remark. Note that the result of Proposition 2.2.2 can be equivalently phrased in the following manner: given $1 \leq r, s \leq k$,

$$\sum_{j=1}^k \frac{1}{|C_G(g_j)|} \chi_r(g_j) \chi_s(g_j^{-1}) = \delta_{rs} \quad (2.2.1)$$

Indeed, we have a similar result about fixing two columns and summing down the rows.

Proposition 2.2.3 (Second Orthogonality Relation). *Given $1 \leq r, s \leq k$, we have*

$$\sum_{i=1}^k \chi_i(g_r) \chi_i(g_s^{-1}) = \begin{cases} 0 & \text{if } r \neq s \\ |C_G(g_r)| & \text{if } r = s \end{cases}$$

Proof. Let A be the matrix $(A_{ij})_{1 \leq i, j \leq k}$, where $A_{ij} := \chi_i(g_j)$. Similarly, let B be the matrix

$(B_{ij})_{1 \leq i, j \leq k}$, where $B_{ij} := \frac{|C_i|}{|G|} \chi_j(g_i^{-1})$. Then, writing $AB = ((AB)_{pq})_{1 \leq p, q \leq k}$, we have

$$\begin{aligned}
 (AB)_{pq} &= \sum_{l=1}^k A_{pl} B_{lq} \\
 &= \frac{1}{|G|} \sum_{l=1}^k |C_l| \chi_p(g_l) \chi_q(g_l^{-1}) \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_p(g) \chi_q(g^{-1}) \\
 &= \langle \chi_p, \chi_q \rangle = \delta_{pq}
 \end{aligned}$$

with the third inequality holding because all characters are class functions, meaning that terms in summation on the third line such that g is conjugate to g_l will be repeated $|C_l|$ times.

The above computation proves that $AB = I$, the identity matrix. This, in particular, means that $BA = I$ as well. In particular, writing $BA = ((BA)_{pq})_{1 \leq p, q \leq k}$, we see that

$$\begin{aligned}
 \delta_{pq} = (BA)_{pq} &= \sum_{l=1}^k B_{pl} A_{lq} \\
 &= \sum_{l=1}^k \frac{|C_p|}{|G|} \chi_l(g_p^{-1}) \chi_l(g_q) \\
 &= \sum_{j=1}^k \frac{1}{|C_G(g_r)|} \chi_i(g_r) \chi_i(g_s^{-1}) \tag{2.2.2}
 \end{aligned}$$

A simple rearrangement then yields the desired result. \square

Remark. Observe that (2.2.2) bears a striking similarity to (2.2.1). The only difference is that in (2.2.1), we fix the character and sum over the conjugacy classes (ie, fix the row and sum across the columns), while in (2.2.2), we fix the conjugacy class and sum over the characters (ie, fix the column and sum across the rows).

In the next subsection, we will demonstrate how we can use these orthogonality relations, in combination with other results—most notably, (2.1.5)—to construct character tables.

2.2.2 Constructing Character Tables

First, we construct the character table of S_3 using relatively unsophisticated machinery.

Example 2.2.4 (S_3). S_3 has precisely three conjugacy classes C_1 , C_2 and C_3 with representatives $g_1 = 1$, $g_2 = (12)$ and $g_3 = (123)$. We then have the following character table for S_3 :

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$\text{Span}(e_1 - e_2, e_2 - e_3)$ - cf. Coxeter

Here, χ_1 corresponds to the trivial representation (of degree 1), χ_2 to the sign representation (of degree 1), and χ_3 to the subrepresentation $W = \text{Span}(e_1 - e_2, e_2 - e_3)$ (of degree 2) of the permutation representation on \mathbb{C}^3 . Note that all of these representations are irreducible, the first two because their degrees are 1, and the third because we can show $\langle \chi_W, \chi_W \rangle$ to be equal to 1 (cf. the fourth point of Proposition 2.1.17).

The maxim of ‘the larger the object, the more involved the computations’ holds no less true for character tables. To construct the character table of S_4 , we will need to use two ingredients that we did not require to construct the character table of S_3 . The first consists of the Orthogonality Relations, and the second of the following result on lifting characters of quotients, which we will be able to apply to lift characters of S_3 to characters of S_4 .

Proposition 2.2.5 (Lifting of Characters). *Let $N \trianglelefteq G$. If $\chi : G/N \rightarrow \mathbb{C}$ is a character of G/N , then the map $\chi' : G \rightarrow \mathbb{C} : g \mapsto \chi(gN)$ is a character of G . In particular, χ is irreducible if and only if χ' is irreducible.*

Proof. The result follows immediately from Proposition 1.1.20. □

Example 2.2.6 (S_4). Observe that the conjugacy classes C_i , $i = 1, \dots, 5$, of S_4 correspond precisely to the various possible cycle shapes. They therefore have representatives $g_1 = 1$, $g_2 = (12)$, $g_3 = (123)$, $g_4 = (12)(34)$, and $g_5 = (1234)$. The conjugacy classes have sizes 1, 6, 8, 3 and 6 respectively. As there are five conjugacy classes, there are, in particular, exactly five irreducible characters of S_4 . We can compute the values of four of them with relative ease, but for the fifth, we will, for now, fill in the table with placeholder variables:

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	2	0	-1	2	0
χ_5	a	b	c	d	e

We now describe each character and how the above values were computed. In the case of χ_5 , we will solve for a , b , c , d and e using the Orthogonality Relations.

1. χ_1 is the trivial character. It takes the value 1 on all elements of S_4 .
2. χ_2 is the sign character^a. It takes the value 1 on all even permutations and -1 on all odd permutations—indeed, it is exactly equal to the sign homomorphism.
3. χ_3 is the deleted permutation character given by the subrepresentation of the permutation representation^b of S_4 corresponding to $\text{Span}(e_1 - e_2, e_2 - e_3, e_3 - e_4)$, an S_4 -invariant subspace of \mathbb{C}^4 . The values it takes can be computed by taking the traces of the orthogonal matrices corresponding to the action of S_4 on this subspace.
4. χ_4 is given by lifting a character of S_3 . Consider

$$N = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$$

One can show that $S_4/N \cong S_3$. We can then apply Proposition 2.2.5, taking advantage of the irreducibility equivalence, to lift the three irreducible characters from Example 2.2.4 to irreducible characters of S_4 . The lifts of the first two are χ_1 and χ_2 , so we choose χ_4 to be a lift of the third.

5. χ_5 is the regular character. We can solve for a, b, c, d, e using the Orthogonality Relations. From (2.1.5), we know that

$$1 + 1 + 9 + 4 + a^2 = 15 + a^2 = 24 = |S_4|$$

from which we can conclude that $a = \pm 3$. Seeing as $\chi_5(1)$ must correspond to the dimension of a representation, a must be positive. So, $a = 3$. Then, applying the Second Orthogonality Relation (Proposition 2.2.3) to the first two columns,

$$1 - 1 + 3 + 0 + ab = 3 + 3b = 0$$

from which we can conclude that $b = -1$. In similar fashion, we can apply the Second Orthogonality Relation to the second and third columns to solve for c :

$$1 - 1 + 0 + 0 - c = 0 - c = 0$$

giving $c = 0$. To solve for d , it does not make sense to apply the Second Orthogonality Relation to the third and fourth columns, because $cd = 0$, but we can certainly apply it to the *second* and fourth columns:

$$1 - 1 - 1 + 0 - d = -1 - d = 0$$

giving $d = -1$. Finally, we can apply the Second Orthogonality Relation to the fourth and fifth columns to solve for e :

$$1 - 1 + 1 + 0 - e = 1 - e = 0$$

giving $e = 1$.

Thus, the character table of S_4 is as follows:

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	2	0	-1	2	0
χ_5	3	-1	0	-1	1

We can double-check our computations for the regular character by applying the First Orthogonality Relation (Proposition 2.2.3) to the first and last rows (ie, by showing $\langle \chi_1, \chi_5 \rangle = 0$): clearly,

$$\begin{aligned} \frac{1}{|G|} \sum_{i=1}^5 |C_i| \chi_1(g) \chi_5(g^{-1}) &= \frac{1}{24} (1 \cdot \bar{a} + 6 \cdot \bar{b} + 8 \cdot \bar{c} + 3 \cdot \bar{d} + 6 \cdot \bar{e}) \\ &= \frac{1}{24} (3 - 6 + 0 - 3 + 6) = 0 \end{aligned}$$

We do not prove here that the conjugacy classes have sizes 1, 6, 8, 3 and 6 respectively, but this can be done by direct computation.

^aie, that corresponding to the sign representation

^bie, that corresponding to its action on \mathbb{C}^4 by permuting the standard basis

It turns out that we can glean even more information about groups from their characters, for which we will need the properties of the algebraic integers.

2.3 Integrality

In this section, we exploit our understanding of the complex numbers to gain insight into the characters of finite groups.

2.3.1 The Algebraic Integers

First, we recall the notion of integrality over an integral domain.

Definition 2.3.1 (Integrality). Let R be an integral domain, and $S \supset R$ an extension. We say that $s \in S$ is integral over R if one of the following equivalent conditions is satisfied:

- s is a root of a monic polynomial in $R[X]$.
- The minimal polynomial of s over $\text{Frac}(R)$ is actually in $R[X]$.

We now define what it means for a complex number to be an algebraic integer.

Definition 2.3.2 (Algebraic Integer). We say that a number $\alpha \in \mathbb{C}$ is an algebraic integer if it is integral over \mathbb{Z} .

One can show that the set of all algebraic integers forms a ring. Indeed, this is nothing but the integral closure of \mathbb{Z} in \mathbb{C} . This has an important consequence as far as character theory is concerned.

Proposition 2.3.3. *For all characters χ of G and $g \in G$, $\chi(g)$ is an algebraic integer.*

Proof. Fix $g \in G$ and a character χ . Let $d = \text{ord}(g)$. The fourth point of Proposition 2.1.9 tells us that $\chi(g)$ is a sum of d th roots of unity, which, as roots of $X^d - 1 \in \mathbb{Z}[X]$, are certainly algebraic integers. Thus, as a sum of algebraic integers, $\chi(g)$ is an algebraic integer. \square

We now proceed to prove several results on the conjugacy classes of G . We begin by defining a

simple way to obtain a central element of $\mathbb{C}G$ from a conjugacy class of G .

Notation. Given a conjugacy class C of G , we define

$$\widehat{C} := \sum_{g \in C} g \quad (2.3.1)$$

as an element of $\mathbb{C}G$.

Lemma 2.3.4. *For any conjugacy class C of G , $\widehat{C} \in Z(\mathbb{C}G)$.*

Proof. It is clear that \widehat{C} commutes with every element of $\mathbb{C} \subset \mathbb{C}G$. We proceed to show that they also commute with elements of $G \subset \mathbb{C}G$. Let $g \in C$ be a representative of C . We know that every term in the sum in (2.3.1) (ie, every element of C) is of the form hgh^{-1} for some $h \in H$. For some $j \in G$, we have that

$$j(hgh^{-1}) = \left(\underbrace{jhgh^{-1}j^{-1}}_{\in C} \right) j$$

Hence, left-multiplication by j acts as a permutation of C . Therefore, $j\widehat{C} = \widehat{C}j$. \square

Lemma 2.3.5. *Let $g \in G$ and C be the conjugacy class of g . Let S be a simple $\mathbb{C}G$ -module with associated character χ_S . The action of \widehat{C} on S (as an element of $\mathbb{C}G$) can be simplified to one of scalar multiplication by a complex number λ :*

$$\widehat{C} \cdot s = \lambda s$$

where $\lambda = |C| \frac{\chi_S(g)}{\chi_S(1)}$.

Proof. Define a function $\phi : S \rightarrow S : s \mapsto \widehat{C} \cdot s$. Since $\widehat{C} \in Z(\mathbb{C}G)$, we have that $\forall x \in G$, $x \cdot \phi(s) = \phi(x \cdot s)$. This makes ϕ a $\mathbb{C}G$ -module homomorphism. Then, by Corollary 1.3.10, we know that $\phi = \lambda \text{id}$, where $\lambda = \text{Tr}(\phi) / \dim(S)$. Note that $\dim(S) = \chi_S(1)$. Further, if we write $\phi_h : S \rightarrow S : s \mapsto h \cdot s$ for $h \in C$, then we can express ϕ as $\sum_{h \in C} \phi_h$. Then,

$$\text{Tr}(\phi) = \sum_{h \in C} \text{Tr}(\phi_h)$$

$$\begin{aligned}
&= \sum_{h \in G} \chi_s(h) \\
&= |C| \chi_s(g)
\end{aligned}$$

where g was our chosen representative of C . Therefore, $\lambda = |C| \frac{\chi_s(g)}{\chi_s(1)}$. \square

Lemma 2.3.6. *Let $G = \{g_1, \dots, g_n\}$. Define $r := \sum_{i=1}^n \alpha_i g_i$ for some $\alpha_i \in \mathbb{Z}$. Suppose that $\exists \lambda, v \in \mathbb{C}G \setminus \{0\}$ such that $rv = \lambda v$. Then, λ is an algebraic integer.*

Proof. For all $1 \leq j \leq n$,

$$r g_j = \sum_{i=1}^n \alpha_i g_i g_j = \sum_{i=1}^n \alpha_{ij} g_i$$

where $(\alpha_{ij})_{1 \leq i, j \leq n}$ is a permutation of the α_i induced by the action of g_j . This means, in particular, that $\alpha_{ij} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$. Suppose $\exists \lambda, v \in \mathbb{C}G \setminus \{0\}$ such that $rv = \lambda v$. Writing $v = \sum_{j=1}^n \mu_j g_j$ for $\mu_j \in \mathbb{C}$, we can see that

$$rv = \sum_{j=1}^n \mu_j r g_j = \sum_{j=1}^n \mu_j \sum_{i=1}^n \alpha_{ij} g_i = \sum_{i=1}^n g_i \sum_{j=1}^n \alpha_{ij} \mu_j = \sum_{i=1}^n g_i \lambda \mu_i = \lambda v$$

This tells us that $\lambda \mu_i = \sum_{j=1}^n \alpha_{ij} \mu_j$. In matrix form,

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \lambda \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

This makes λ is an eigenvalue of the matrix $A := (\alpha_{ij})_{1 \leq i, j \leq n} \in M_{n \times n}(\mathbb{Z})$, and thereby, a (complex) root of the characteristic polynomial of A . This polynomial is monic, with coefficients in \mathbb{Z} , making λ integral over \mathbb{Z} , and thereby, an algebraic integer. \square

The point of the above Lemmas is the following Corollary.

Corollary 2.3.7. *For any $\chi \in \text{Irr}(G)$ and $g \in G$, the quantity $\lambda = \frac{|G|}{|C_G(g)|} \frac{\chi(g)}{\chi(1)} = |C| \frac{\chi(g)}{\chi(1)}$ is an algebraic integer, where C is the conjugacy class of g .*

Proof. Lemma 2.3.5 tells us that $s \mapsto \lambda s$ is, indeed, a $\mathbb{C}G$ -module homomorphism. We can then

apply Lemma 2.3.6 to λ and any $s \in S \setminus \{0\}$ to conclude that λ is an algebraic integer. \square

We can now show that the dimension of any irreducible representation of G must divide $|G|$.

Theorem 2.3.8. *For all $\chi \in \text{Irr}(G)$, $\chi(1) \mid |G|$.*

Proof. It is a well-known fact that the integral closure of \mathbb{Z} in \mathbb{Q} is \mathbb{Z} itself⁷. Hence, if we can show that $\frac{|G|}{\chi(1)}$ is an algebraic integer, the fact that it is also rational will guarantee that it is an integer. We exploit the fact that $\langle \chi, \chi \rangle = 1$ to write

$$\begin{aligned} \frac{|G|}{\chi(1)} &= \frac{|G|}{\chi(1)} \langle \chi, \chi \rangle \\ &= \frac{|G|}{\chi(1)} \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi(g^{-1}) \end{aligned}$$

If G has conjugacy classes C_1, \dots, C_r with representatives g_1, \dots, g_r , then we can write

$$\frac{|G|}{\chi(1)} = \sum_{i=1}^r \frac{|C_i|}{\chi(1)} \chi(g_i) \chi(g_i^{-1})$$

From Corollary 2.3.7 and Proposition 2.3.3, we know that both $\frac{|C_i|}{\chi(1)} \chi(g_i)$ and $\chi(g_i^{-1})$ are algebraic integers. Hence, we can conclude that $\frac{|G|}{\chi(1)}$ is an algebraic integer, and therefore, an integer. \square

Corollary 2.3.7 also gives us another way to prove the specialisation of Proposition 2.3.3 to irreducible characters.

Corollary 2.3.9. *For any $\chi \in \text{Irr}(G)$, $\chi(g)$ is an algebraic integer for all $g \in G$.*

Proof. Fix $g \in G$. We know that $\frac{|G|}{|C_G(g)|} \frac{\chi(g)}{\chi(1)}$ is an algebraic integer. Furthermore, we know that $\frac{|G|}{|C_G(g)| \cdot \chi(1)} \in \mathbb{Q}^\times$, and that the integral closure of \mathbb{Z} in \mathbb{C} (ie, the set of all algebraic integers) is a ring. Hence, $\chi(g) = \left(\frac{|G|}{|C_G(g)| \cdot \chi(1)} \right)^{-1} \frac{|G|}{|C_G(g)|} \frac{\chi(g)}{\chi(1)}$ is an algebraic integer. \square

2.3.2 The n th Roots of Unity

We begin by stating a number theoretic result.

⁷ie, the only elements of \mathbb{Q} that are integral over \mathbb{Z} are the integers themselves.

Lemma 2.3.10. *Let w be an n th root of unity in \mathbb{C} . Then,*

$$\sum_{\substack{1 \leq i \leq n \\ (i,n)=1}} w^i \in \mathbb{Z}$$

Proof. EXERCISE. Hint: induction on n . □

Proposition 2.3.11. *Let $g \in G$ be of order n . Suppose that g^i is conjugate to g for all $1 \leq i \leq n$ such that $(i, n) = 1$. Then, for all characters χ of G , $\chi(g)$ is an integer.*

Proof. Let χ be a character of G , with associated representation (V, ρ) of degree m over \mathbb{C} . We know that $\chi(g) = \sum_{i=1}^m w_i$, where each w_i is an n th root of unity (cf. the fourth point of Proposition 2.1.9). Indeed, the idempotent linear map $\rho(g)$ is diagonalisable, with matrix $\text{diag}(w_1, \dots, w_m)$ with respect to an appropriate basis (cf. Proposition 1.3.15). Hence, we have that $\rho(g^i) = \text{diag}(w_1^i, \dots, w_m^i)$, meaning that $\chi(g^i) = \sum_{j=1}^m w_j^i$. But this is nothing but $\sum_{1 \leq i \leq m, (i,n)=1} w^i$, by assumption. Hence, $\chi(g)$ is an integer, by Lemma 2.3.10. □

Lemma 2.3.12. *Fix $g \in G$, and let p be a prime number. Then, $\exists! x, y \in G$ such that*

1. $g = xy = yx$
2. $\text{ord}(x) = p^k$ for some $k \in \mathbb{N}$
3. $(\text{ord}(y), p) = 1$

Proof. Let $\text{ord}(g) = up^v$, where $(u, v) = 1$. By Bézout's Lemma, we know that $\exists a, b \in \mathbb{Z}$ such that $au + bp^v = 1$. Then, we can define $x = g^{au}$ and $y = g^{bp^v}$. This satisfies all the conditions:

1. $g = xy = g^{au} g^{bp^v} = g^{bp^v} g^{au} = yx$.
2. $x^{p^v} = 1$.
3. $y^v = 1$.

One can also show x and y to be the only elements of G satisfying these conditions. □

Definition 2.3.13 (The p - p' -parts of g). For any $g \in G$, call x and y from Lemma 2.3.12 the p - and p' -parts of g respectively.

For the remainder of this subsection, let $n = |G|$, and let $\zeta = e^{\frac{2\pi i}{n}}$ be a primitive n th root of unity. Define $\mathbb{Z}[\zeta]$ to be the subring of \mathbb{C} generated by \mathbb{Z} and ζ . Let p be a prime number and $p\mathbb{Z}[\zeta]$ be the principal ideal of $\mathbb{Z}[\zeta]$ generated by p . By the Correspondence Theorem, the ideals of $\mathbb{Z}[\zeta]$ containing $p\mathbb{Z}[\zeta]$ are in bijection with the ideals of $\mathbb{Z}[\zeta]/p\mathbb{Z}[\zeta]$. Since the latter object is finite, it contains finitely many ideals, meaning that only finitely many of the ideals of $\mathbb{Z}[\zeta]$ contain $p\mathbb{Z}[\zeta]$. We can then look at the maximal (proper) ideal amongst these, which is a maximal ideal of $\mathbb{Z}[\zeta]$. Denote it by P . Indeed, we can show that $P \cap \mathbb{Z} = p\mathbb{Z}$.

Theorem 2.3.14. *Let $g \in G$, and let y be the p' -part of g for some p prime. For all characters χ of G , $\chi(g) - \chi(y)$ lies in the maximal ideal P defined above.*

Proof. Let $m = \text{ord}(g) = up^v$. Let $a, b \in \mathbb{Z}$ be such that $au + bp^v = 1$ (as in the proof of Lemma 2.3.12). We have that $y = g^{bp^v}$. We know that $\chi(g), \chi(y) \in \mathbb{Z}[\zeta]$.

Now, let w be an m th root of unity. Since $m|n$ (by Lagrange's Theorem), we know that $w \in \mathbb{Z}[\zeta]$. We have that $w = w^{au+bp^v}$. Raising both sides to the p^v th power, we get that $w^{p^v} = w^{aup^v} w^{bp^{2v}}$. Since $w^{up^v} = w^m = 1$, we have that $w^{p^v} = w^{bp^{2v}}$.

Now, consider the binomial expansion of $(w - w^{bp^{2v}})^{p^v}$. Some blah blah, use fact that P is a prime ideal (since it is maximal). □

This theorem has several important consequences.

Corollary 2.3.15. *Let $g \in G$ and y be the p' -part of g for some prime p . Let χ be a character of G .*

1. *If $\chi(g), \chi(y) \in \mathbb{Z}$, then $\chi(g) \equiv \chi(y) \pmod{p}$.*
2. *If $\text{ord}(g) = p^k$ for some $k \in \mathbb{N}$, then $\chi(g) \equiv \chi(1) \pmod{p}$.*

Lemma 2.3.16. *Let $g, h \in G$. Then, g is conjugate to h iff for all characters χ of G , $\chi(g) = \chi(h)$.*

Proof.

(\implies) Seen Before

(\impliedby) As we know, $\text{Irr}(G)$ is an orthonormal basis for $\mathcal{F}_C(G, \mathbb{C})$. Let f be the indicator function on g^G . Then, $f(g) = 1 \implies f(h) = 1 \implies h \in g^G$.

□

Corollary 2.3.17. *Let $g \in G$. Then, g is conjugate to g^{-1} iff for all characters χ of G , $\chi(g) \in \mathbb{R}$.*

Example 2.3.18 (Constructing the Character Table of $G = \text{PSL}(2, 7)$). First, note that $|G| = 168$ and that G has 6 conjugacy classes (and hence, 6 irreducible representations over \mathbb{C}). Denote the conjugacy classes by C_1, \dots, C_6 , with respective representatives g_1, \dots, g_6 . As per convention, $g_1 = 1$.

We begin filling the table as follows.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2						
χ_3						
χ_4						
χ_5						
χ_6						

We can then fill in the first column by applying Corollary ??.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7					
χ_3	8					
χ_4	3					
χ_5	3					
χ_6	6					

Observe that since the orders are unique for the first four columns, they contain all elements of their respective orders. We can then fill in the third column by applying the Column Orthogonality Relation and the first Lemma from today.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7		-1			
χ_3	8		0			
χ_4	3		1			
χ_5	3		1			
χ_6	6		0			

We can then fill in the fourth column using the second point of Corollary 2.3.11.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7		-1	1		
χ_3	8		0	-1		
χ_4	3		1	0		
χ_5	3		1	0		
χ_6	6		0	0		

We now have four possibilities for the second column.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7		-1	1		
χ_3	8		0	-1		
χ_4	3		1	0		
χ_5	3		1	0		
χ_6	6		0	0		

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7		-1	1		
χ_3	8		0	-1		
χ_4	3		1	0		
χ_5	3		1	0		
χ_6	6		0	0		

For the last two columns of the table, we use row orthogonality, in combination with the fact that $g_5 = g_6^{-1}$.

$C_G(g_i)$	168	8	4	3	7	7
$\text{ord}(g_i)$	1	2	4	3	7	7
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7		1	-1	0	0
χ_3	8		-1	0	1	1
χ_4			0	1	z	\bar{z}
χ_5	3		0	1	\bar{z}	z
χ_6	6		0	0	t	\bar{t}

FINAL COMMENTS. Also use the results on p groups, since G is a 7-group.

Chapter 3

Applications to the Study of Groups

Thus far, we have explored the theory of representations, and characters thereof, as ends in themselves. In this chapter, we will explore the applications of the robust theory we have constructed so far in the study of groups. This will illustrate the utility of representation theory, and, more broadly, the power of understanding groups via their actions. Throughout this chapter, we will denote by G an arbitrary finite group.

3.1 Important Classes of Finite Groups

We will begin by stating a few results on certain specific (and rather important) classes of finite groups.

3.1.1 p -Groups

Throughout this subsection, we will fix a prime number p and assume that $|G| = p^n$ for some $n \in \mathbb{N}$.

Lemma 3.1.1. *For all $\chi \in \text{Irr}(G)$, $\chi(1)$ is a power of p .*

Proof. Let $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$, with χ_1 being the trivial character. □

Corollary 3.1.2. *If $|G| = p^2$ for some p prime, then $\forall \chi \in \text{Irr}(G)$, $\chi(1) = 1$. In particular, G is*

abelian.

Proof. sorry □

3.1.2 Simple Groups

Lemma 3.1.3. *If G is simple, G does not admit an irreducible character of degree 2.*

Proof. Suppose, for contradiction, that G is simple but admits an irreducible representation $\rho : G \rightarrow \text{GL}(2, \mathbb{C})$. Since G is simple, we must have that $\ker(\rho) = \{1\}$.

If G is nonabelian, then $G' = G$. This tells us that there are no nontrivial characters of degree 1. Since the map $\rho \mapsto \det(\rho(g))$ a linear character (ie, has degree 1), we can conclude that $\det(\rho(g)) = 1$ for all $g \in G$. By Theorem 2.3.8, we know that $2 \mid |G|$. Then, by Cauchy's Theorem, $\exists g \in G$ such that $g \neq 1$ but $g^2 = 1$. This tells us that $\rho(g) = -\text{id} \in Z(G)$, which is absurd, since G is simple. □

3.2 Burnside's Theorem

3.2.1 The Algebraic Numbers

In this subsection, we investigate a generalisation of the algebraic integers, whose properties will be useful in the proof of Burnside's Theorem. We begin with the following definition.

Definition 3.2.1 (Algebraic Numbers). An algebraic number is a complex number that is a root of a non-zero polynomial in $\mathbb{Q}[X]$.

Remark (A few recollections on the notion of a minimal polynomial). For any algebraic number x ,

- There exists a *unique* monic polynomial of minimal degree over \mathbb{Q} that annihilates x . This polynomial, denoted m_x , is called the *minimal polynomial* of x .
- m_x is irreducible.
- m_x divides any polynomial over \mathbb{Q} for which x is a root.

Definition 3.2.2 (Conjugates of an Algebraic Number). Let x be an algebraic number. The conjugates of x are the roots of its minimal polynomial, m_x .

Example 3.2.3. Fix $n \in \mathbb{N}$, and let ω be an n th root of unity. The minimal polynomial of ω divides $X^n - 1$, meaning that all of its roots are also n th roots of unity, and therefore, have modulus 1.

Indeed, it turns out that if a number is an algebraic integer, then its minimal polynomial actually has integer coefficients.

Lemma 3.2.4. Let α, β be algebraic numbers, and let $r \in \mathbb{Q}$. Then,

1. The conjugates of $\alpha + \beta$ ¹ are all of the form $\alpha' + \beta'$, where α' and β' are conjugates of α and β respectively.
2. The conjugates of $r\alpha$, where $r \in \mathbb{Q}$, are all of the form $r\alpha'$, where α' is a conjugate of α .

Proof sketch. Let L denote the splitting field over \mathbb{Q} of m_α , m_β and $m_{\alpha+\beta}$. This is a normal extension of \mathbb{Q} [JUSTIFY]. Now, let $G = \text{Aut}(L/\mathbb{Q})$. One can show the action of G on the conjugates of α to be transitive [JUSTIFY]. For instance, one can show that $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\alpha')$ [relevance?]. Then, for all $g \in G$, we have that $g(\alpha + \beta) = g(\alpha) + g(\beta)$, and $g(x)$ is a conjugate of x for any $x \in L$ (by transitivity). I think the transitivity/normality comes from the fact that L is the splitting field of irreducible polynomials. \square

Lemma 3.2.5. Let G be a finite group, and let χ be a character of G . Then, $\left| \frac{\chi(g)}{\chi(1)} \right| \leq 1$, and if $0 < \left| \frac{\chi(g)}{\chi(1)} \right| < 1$, then $\chi(g)/\chi(1)$ is not an algebraic integer.

Proof. Let $\gamma = \frac{\chi(g)}{\chi(1)}$ and $d = \chi(1)$. Then, $\chi(g) = w_1 + \cdots + w_d$, where $|w_i| = 1$ for all i . This then gives us that $|\gamma| \leq 1$. Now, suppose that γ is an algebraic integer and that $|\gamma| < 1$. We show that $\chi(g) = 0$.

Let m_γ be the minimal polynomial of γ . Then, $m_\gamma(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$. Since γ is an algebraic integer, we have that $a_i \in \mathbb{Z}$ for all i . By the previous lemma, we know that the

¹It is easy to see that $\alpha + \beta$ is algebraic.

conjugates of γ are of the form $\frac{w'_1 + \dots + w'_d}{d}$, where w'_i are conjugates of w_i . Since $|w_i| = 1$, we have that $|w'_i| = 1$ for all i .

Now, let z be the product of all the conjugates of γ —ie, the product of all the roots of m_γ . Then, $|z| < 1$. Indeed, $z = a_0$, and hence, in particular, $z \in \mathbb{Z}$. Therefore, it must be that $z = 0$. Hence, it must be that $m_\gamma(X) = X$, as it is irreducible. This gives us that $\gamma = 0$, and hence, $\chi(g) = 0$. \square

3.2.2 The Three Versions of Burnside's Theorem

Theorem 3.2.6 (Burnside's Theorem for Simple Groups). *Let p be a prime number, and let $r \in \mathbb{Z}$ be greater than or equal to 1. Let G be a finite group which a conjugacy class of size p^r . Then, G is not simple.*

Proof. Let $g \in G$ be such that $|g^G| = p^r$. Note that since G contains a conjugacy class of size greater than 1, G cannot be abelian. Now, let $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. The idea is to apply column orthogonality between the g -column and the 1-column of G .

By Proposition 2.2.2, we have that

$$\begin{aligned} 0 &= \frac{1}{|G|} \sum_{i=1}^k \chi_i(g) \chi_i(1) \\ &= 1 + \frac{1}{|G|} \sum_{i=2}^k \chi_i(g) \chi_i(1) \end{aligned}$$

Dividing both sides by p , we get that

$$\sum_{i=2}^k \chi_i(g) \frac{\chi_i(1)}{p} = -\frac{1}{p}$$

Since $-1/p$ is not an algebraic integer, this implies that for some $2 \leq j \leq k$, $\chi_j(g) \frac{\chi_j(1)}{p}$ is not an algebraic integer. However, since $\chi_j(g)$ is necessarily an algebraic integer, it must be that $\chi_j(1)/p$ is not an algebraic integer. Since $\chi_j(1) \in \mathbb{Z}$ (as it is the degree of the associated representation) and p is prime, this is equivalent to saying that $p \nmid \chi_j(1)$. Since $|g^G| = p^r$, this means that $\chi_j(1)$ and $|g^G|$ are coprime.

We know, by Bézout's Lemma, that there exist integers a, b such that $a|g^G| + b\chi_j(1) = 1$. Multiplying by the character ratio $\gamma = \chi_j(g)/\chi_j(1)$, we have that

$$a|g^G|\gamma + b\chi_j(g) = \gamma$$

By Lemma 2.3.5 (make reference more precise), we know that $a|g^G|\gamma$ is an algebraic integer. This means that γ must be an algebraic integer too, giving us that $|\gamma| = 1$ (by Lemma 3.2.5).

Now, let ρ be the representation with character χ_j . Then, by Lemma 2.3.5, we have that $\rho(g) = \lambda I$, where $\lambda = |g^G|\gamma$. Finish using picture on iPad. \square

Theorem 3.2.7 (Burnside's Theorem for Groups of Order $p^a q^b$). *Let G be a finite group of order $p^a q^b$, where p, q are prime numbers and $a, b \in \mathbb{N}$. Then, G is simple only if $|G| \in \{p, q\}$.*

Proof. First, suppose that $b = 0$. Then, G is a p -group, and hence, has a nontrivial centre.² This means that G is not simple unless $|G| = p$. One can prove similarly that if $a = 0$ then $|G| = q$.

Now, suppose that $a, b > 0$. Then, G has a normal Sylow p -subgroup P of order p^a for some $a \in \mathbb{N}$. Since P is a p -group, its centre is nontrivial, and therefore, there exists some $g \in Z(P) \setminus \{1\}$. Indeed,

$$\begin{aligned} P &\leq C_G(g) \\ \implies |P| \mid |C_G(g)| &= \frac{|G|}{|g^G|} \\ \implies p^a |g^G| \mid p^a q^b \\ \implies |g^G| \mid q^b \end{aligned}$$

where we justify the first inclusion by writing it out [DO IT!].

Since G is simple, $Z(G) = \{1\}$, meaning that $|g^G| = q^r$ for some $r \geq 1$. This contradicts the

²Recall that the sizes of the conjugacy classes of a p -group are prime powers. One can then use the Class Equation to arrive at this result.

simplicity of G . [JUSTIFY!] □

Insert definition of solvable

Theorem 3.2.8 (Burnside's Theorem for Solvable Groups). *Let G be a finite group. If G has order $p^a q^b$, where p, q are prime and $a, b \geq 0$, then G is solvable.*

Proof. We argue by induction on $|G|$. If $|G| = 1$, then G is trivially solvable. Now, suppose that $|G| = p^a q^b$, where $a, b \geq 0$. If $a = 0$ or $b = 0$, then G is a p -group or a q -group, and hence, solvable.³ Else, if $a, b \geq 1$, then by Theorem 3.2.7, we know that G is not simple. It then contains a nontrivial normal subgroup N . By the Induction Hypothesis, both N and G/N are solvable, making G solvable as well. □

3.2.3 Applications of Burnside's Theorem

Lemma 3.2.9. *There are no nonabelian simple groups of order less than 30.*

3.3 Products, Products, Products

Thus far, we have seen two ways of combining representations of a group G : the direct sum and the tensor product. While we have investigated the properties of the former in detail, we have yet to explore the latter. This will be the primary focus of this section, which we have included in this chapter because we will explore the properties of tensor products of representations by relating them to those of direct products of groups. In particular, our investigation of tensor products will be from a primarily application-driven perspective.

3.3.1 The Multiplicative Properties of (Irreducible) Characters

We begin by inviting the reader to refresh their memory on what we have seen so far by taking a look at Subsection 1.2.5 and the seventh point of Proposition 2.1.9. Before we proceed any further, we establish certain results about the multiplicative properties of characters, with the

³Give/link a proof of this!

multiplication in question being that induced by the standard multiplicative structure of $\mathcal{F}_C(G, \mathbb{C})$ (which, in turn, inherits its multiplication from \mathbb{C}).

Lemma 3.3.1. *The set $R(G) := \mathbb{Z}\text{Irr}(G) \subseteq \mathcal{F}_C(G, \mathbb{C})$ is a subring of $\mathcal{F}_C(G, \mathbb{C})$.*

Proof. Observe that $R(G)$ is certainly a group under addition. We claim that it is closed under multiplication, with multiplicative identity given by the trivial character.

Let $\chi_V, \chi_W \in \text{Irr}(G)$, with associated (simple) $\mathbb{C}G$ -modules V and W . From Proposition 2.1.9, we know that their product $\chi_V \chi_W$ is the character associated to the $\mathbb{C}G$ -module $V \otimes W$. \square

Corollary 3.3.2. *The set $L(G) := \{\chi \in \text{Irr}(G) : \chi(1) = 1\}$ of degree 1 characters is a multiplicative subgroup of $R(G)$.*

Proof. Obviously, the trivial character is linear, making $L(G)$ nonempty. Further, for all $\chi_V, \chi_W \in L(G)$ (with associated $\mathbb{C}G$ -modules V and W), clearly, $\chi_V \chi_W$ is a character (namely, that of $V \otimes W$), and $(\chi_V \chi_W)(1) = 1$, proving closure under multiplication. Finally, the inverse of any character χ_V (with associated $\mathbb{C}G$ -module V) is given by the character χ_{V^*} of its dual $\mathbb{C}G$ -module V^* , which is also linear, proving the desired result. \square

Lemma 3.3.3. *Let $\chi_V, \chi_W \in \text{Irr}(G)$, with $\chi_V(1) = 1$. Then, $\chi_V \chi_W \in \text{Irr}(G)$.*

Proof. Recall, from Proposition 2.1.17, that $\chi_V \chi_W$ is irreducible iff $\langle \chi_V \chi_W, \chi_V \chi_W \rangle = 1$. Indeed, we have that

$$\begin{aligned} \langle \chi_V \chi_W, \chi_V \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \chi_W(g) \cdot \overline{\chi_V(g) \cdot \chi_W(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \chi_W(g) \cdot \overline{\chi_V(g)} \cdot \overline{\chi_W(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_W(g) \cdot \overline{\chi_W(g)} = 1 \end{aligned}$$

We justify the second equality by seeing that $\chi_V(g) \cdot \overline{\chi_V(g)} = 1$ because $\overline{\chi_V(g)} = \chi_V(g^{-1})$. Indeed, since χ_V is of degree 1, it is, in fact, *equal* to its underlying representation, making it multiplicative (ie, $\chi_V(g) \cdot \chi_V(g^{-1}) = \chi_V(gg^{-1}) = 1$). The last equality follows from the fact that

$\chi_W \in \text{Irr}(G)$. □

3.3.2 Tensor Products and Direct Products

Definition 3.3.4 (Tensor Products of Modules of Different Groups). Let K be a field, and let G and H be finite groups. Let V be a KG -module and let W be a KH -module. The tensor product $V \otimes_K W$ of V and W over K is the $K(G \times H)$ -module with scalar multiplication

$$(g, h) \cdot (v \otimes w) = (g \cdot v) \otimes (h \cdot w) \quad (3.3.1)$$

for all $g \in G$, $h \in H$, $v \in V$, and $w \in W$.

Proposition 3.3.5. Let G, H be finite groups and let V be a $\mathbb{C}G$ -module and W a $\mathbb{C}H$ -module. Then, the character $\chi_{V \otimes W}$ associated to the $\mathbb{C}(G \times H)$ -module $V \otimes W$ is given by

$$\chi_{V \otimes W}(g, h) = \chi_V(g) \chi_W(h)$$

for all $g \in G$ and $h \in H$.

Proof. We essentially consider V and W as $\mathbb{C}(G \times H)$ -modules and then apply the definition of the tensor product. Finish! □

Notation. Let G, H be finite groups and let V be a $\mathbb{C}G$ -module and W a $\mathbb{C}H$ -module. Then, the character of the tensor product $V \otimes W$ is denoted $\chi_V \times \chi_W$.

Theorem 3.3.6. $\text{Irr}(G \times H) = \{\chi \times \psi : \chi \in \text{Irr}(G), \psi \in \text{Irr}(H)\}$.

Proof. Let $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$ and let $\text{Irr}(H) = \{\psi_1, \dots, \psi_r\}$. The conjugacy classes of $G \times H$ are of the form $g^G \times h^H$ for $g \in G$ and $h \in H$ (exercise). We show that $\chi_i \times \psi_j$ is irreducible for all $1 \leq i \leq k, 1 \leq j \leq r$, with the characters being distinct (ie, if $i \neq \ell$ or $j \neq m$ then $\chi_i \times \psi_j \neq \chi_\ell \times \psi_m$).

We use the fact that a character is irreducible iff its character is 1 (insert reference). Indeed,

for all $1 \leq i, \ell \leq k$ and $1 \leq j, m \leq r$,

$$\begin{aligned} \langle \chi_i \times \psi_j, \chi_\ell \rangle &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \chi_i(g) \psi_j(h) \chi_\ell(g^{-1}) \psi_m(h^{-1}) \\ &= \left(\frac{1}{|G \times H|} \sum_{g \in G} \chi_i(g) \chi_\ell(g^{-1}) \right) \left(\sum_{h \in H} \psi_j(h) \psi_m(h^{-1}) \right) \\ &= \delta_{i\ell} \delta_{jm} \end{aligned}$$

Fill in missing computations---it's not hard. □

Example 3.3.7 (The Acyclic Group of Order 9). Let $G = H = C_3 = \langle g \rangle$, the cyclic group of order 3 generated by g , acting irreducibly on $V = W = \mathbb{R}^2$ via

$$g \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ xy \end{bmatrix}$$

In other words, let its associated representation $\rho : G \rightarrow \text{GL}(2, \mathbb{R})$ be given by the matrix

$$\rho(g) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

with respect to the standard basis.

It is useful to think about powers of characters.

Notation. Let G be a finite group and let χ be a character of G . Then, for all $n \in \mathbb{N}$, the n th power of χ is the character χ^n of G given by

$$\chi^n = \underbrace{\chi \times \cdots \times \chi}_{n \text{ times}}$$

Theorem 3.3.8. Let G be a finite group and let χ be a faithful character of G . Suppose that $|\{\chi(g) : g \in G\}| = r$. For all $\psi \in \text{Irr}(G)$, $\exists 0 \leq i \leq r-1$ such that $\langle \psi, \chi^i \rangle \neq 0$.

Proof. Let $G = \{g_1 = 1, g_2, \dots, g_r\}$. FINISH! □

3.4 The Character Theory of Subgroups

In this section, we will explore how we can understand the characters of a group using those of its subgroups and vice-versa. We will begin by discussing the latter, which we will study from the perspective of Restriction. We will then discuss the former, which we will study from the perspective of Induction.

3.4.1 Restriction

For the entirety of this subsection, let G be a finite group and let $H \leq G$.

Definition 3.4.1 (Restriction of a Representation). Let (V, ρ) be a complex representation of G with associated character χ . The restriction of (V, ρ) to H , denoted $V \downarrow H$, $\text{Res}_H^G(V)$ or V_H , is the set representation $(V, \rho|_H)$. We will denote its associated character $\chi \downarrow H$ or χ_H .

Notation. To distinguish between inner-products of central functions considered over H and those over G , we will denote the former by $\langle \cdot, \cdot \rangle_H$. In similar fashion to (2.1.1), we have, for all $\psi_1, \psi_2 \in \mathcal{F}_C(H, \mathbb{C})$,

$$\langle \psi_1, \psi_2 \rangle_H := \frac{1}{|H|} \sum_{h \in H} \psi_1(h) \overline{\psi_2(h)} \quad (3.4.1)$$

Proposition 3.4.2. Fix $\psi \in \text{Irr}(H)$. Then, there exists an irreducible character χ of G such that the inner-product $\langle \chi \downarrow H, \psi \rangle_H$ is nonzero.

Proof. Let $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. Consider the regular character χ_{reg} of G . We then have (insert reference) that for all $g \in G$,

$$\chi_{\text{reg}} = \sum_{i=1}^k \chi_i(1) \chi_i = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$$

We now show that $\langle \chi_{\text{reg}} \downarrow H, \psi \rangle_H$ is nonzero:

$$\langle \chi_{\text{reg}} \downarrow H, \psi \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi_{\text{reg}}(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} |G| \overline{\psi(1)} \neq 0$$

Considering the decomposition of χ_{reg} as a sum of irreducible characters, we see that one of the summands must have a nonzero inner-product with ψ . \square

Remark. In general, we can compute $\text{Irr}(H)$ from $\text{Irr}(G)$ quite easily if $[G : H]$ is small. However, if $[G : H]$ is large, then this can get difficult.

Proposition 3.4.3. Fix $\chi \in \text{Irr}(G)$, and write $\text{Irr}(H) = \{\psi_1, \dots, \psi_r\}$. Then, $\exists d_1, \dots, d_r \in \mathbb{N}$ such that

$$\chi \downarrow H = \sum_{i=1}^r d_i \psi_i \quad \text{and} \quad \sum_{i=1}^r d_i^2 \leq [G : H]$$

Moreover, the above inequality is an equality if and only if $\forall g \in G \setminus H$, we have $\chi(g) = 0$.

Proof. It is clear that $\exists d_1, \dots, d_r$ such that $\chi \downarrow H = \sum_{i=1}^r d_i \psi_i$. To arrive at the inequality, we consider the inner-product of $\chi \downarrow H$ with itself:

$$\begin{aligned} \langle \chi \downarrow H, \chi \downarrow H \rangle_H &= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\chi(h)} \\ &= \frac{[G : H]}{|G|} \sum_{h \in H} \chi(h) \overline{\chi(h)} \end{aligned}$$

Now, since χ is irreducible (over G), we have that

$$\begin{aligned} 1 = \langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)} \\ &= \frac{1}{|H|} \sum_{g \in H} \chi(g) \overline{\chi(g)} + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus H} \chi(g) \overline{\chi(g)}}_{=:K} \end{aligned}$$

We then have that

$$[G : H] = \sum_{i=1}^r d_i^2 + [G : H] K$$

Clearly, this quantity is greater than or equal to $\sum_{i=1}^r d_i^2$. Furthermore, $[G : H] K = 0$ iff $K = 0$, which is true iff $\chi(g) = 0$ for all $g \in G \setminus H$. \square

Proposition 3.4.4. *Assume H is a normal subgroup of G . Let V be a simple $\mathbb{C}G$ -module. Then, all simple constituents⁴ of its restriction to H have the same dimension.*

Proof. Let $U \leq V$ be a simple $\mathbb{C}H$ -submodule of V . Since H is normal, for all $g \in G$, the set gU is also a $\mathbb{C}H$ -submodule of V : indeed, for all $h \in H$, $hgU = gh'U = gU$ where $h' = g^{-1}hg$.

Now, let $W := \sum_{g \in G} gU$. Since W is a nonzero $\mathbb{C}G$ -submodule of V , and since V is simple, we have that $W = V$. Indeed, since U is simple, so is gU for every $g \in G$. This tells us that for all $g_1, g_2 \in G$,

$$g_1U \cap g_2U = \begin{cases} 0 & \text{in some cases} \\ g_1U & \text{otherwise} \end{cases}$$

as $g_1U \cap g_2U$ is a submodule of the (simple) module g_1U .⁵

Hence, the sum in the definition of W , considered only over indices i such that distinct summands have zero intersection, is direct. Therefore, all FINISH! \square

Corollary 3.4.5. *Assume H is a normal subgroup of G . Fix $\chi \in \text{Irr}(G)$. Then, all irreducible constituents of $\chi \downarrow H$ have the same degree.*

Proposition 3.4.6. *Let H be normal in G , of degree 2. Fix $\chi \in \text{Irr}(G)$. Then, $\chi \downarrow H$ is either irreducible or the sum of two irreducible characters of equal degree.*

Proof. Let $\text{Irr}(H) = \{\phi_1, \dots, \phi_r\}$. By Proposition 3.4.3, we have that $\chi \downarrow H = \sum_{i=1}^r d_i \phi_i$ for some $d_1, \dots, d_r \in \mathbb{N}$. Since $[G : H] = 2$, we have that $\sum_{i=1}^r d_i^2 \leq 2$. This implies that either $r = 1$ or $r = 2$ and $d_1 = d_2 = 1$. Refine a bit \square

Remark. Consider the situation described in the above proposition. We have that $G/H \cong C_2$, the cyclic group of order 2. By the Representation Theory of Cyclic Groups (cf. add reference), we

⁴ie, summands in its direct sum decomposition into simple modules

⁵According to Dr. Rizzoli, "there is no naïve answer" to the question of when the intersection is zero and when it is not.

have that there is a linear character λ of G/H such that for all $gH \in G/H$,

$$\lambda(gH) = \begin{cases} -1 & \text{if } g \notin H \\ 1 & \text{if } g \in H \end{cases}$$

This lifts to a character $\bar{\lambda}$ of G given by $\bar{\lambda}(g) = \lambda(gH)$. We then have that for all $\chi \in \text{Irr}(G)$, $\bar{\lambda}\chi$ is irreducible, agreeing with χ upon restriction to H .

Lemma 3.4.7. either χ or $\bar{\lambda}\chi$

Proof. We have that for all $g \in G$,

$$(\chi + \bar{\lambda}\chi)(g) = \begin{cases} 0 & \text{if } g \notin H \\ 2\chi(g) & \text{if } g \in H \end{cases}$$

Now, let χ_2 be an irreducible character of G such that $\chi_2 \downarrow H = \chi \downarrow H$. We show that χ_2 is either χ or $\bar{\lambda}\chi$. To that end, we compute the inner-product of $\chi + \bar{\lambda}\chi$ with χ_2 :

$$\begin{aligned} \langle \chi + \bar{\lambda}\chi, \chi_2 \rangle &= \frac{1}{|G|} \sum_{g \in G} (\chi(g) + \bar{\lambda}\chi(g)) \overline{\chi_2(g)} \\ &= \frac{1}{|G|} \sum_{g \in H} 2\chi(g) \overline{\chi_2(g)} \\ &= \frac{1}{|H|} \sum_{g \in H} \chi(g) \overline{\chi_2(g)} \\ &= \langle \chi \downarrow H, \chi \downarrow H \rangle_H = 1 \end{aligned}$$

□

Proposition 3.4.8. Let H have index 2 in G . Let $\chi \in \text{Irr}(G)$ be expressible as $\psi_1 + \psi_2$, for $\psi_1, \psi_2 \in \text{Irr}(H)$. Let $\phi \in \text{Irr}(G)$ be such that $\langle \phi \downarrow H, \psi_i \rangle_H \neq 0$ for $i = 1$ or $i = 2$. Then, $\phi = \chi$.

Proof. Since $[G : H] = 1 + 1$, by Proposition 3.4.3, we have that $\chi(g) = 0$ for all $g \in G \setminus H$. Computing the inner-product, we get that

$$\langle \phi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\chi(g)}$$

$$\begin{aligned}
&= \frac{1}{|H|} \sum_{h \in H} \phi(h) \overline{\chi(h)} \\
&= \frac{1}{2} \langle \phi \downarrow H, \chi \rangle_H \neq 0
\end{aligned}$$

proving that $\phi = \chi$. Need more details!

□

3.4.2 Induction

In this subsection, we will describe a process whereby we can use the representation theory of the subgroups of G to understand that of G . For the remainder of this section, let H be a subgroup of G .

The situation we will attempt to understand is the following. Given U a $\mathbb{C}H$ -submodule of $\mathbb{C}H$ (which in turn is a submodule of $\mathbb{C}G$), we will attempt to understand the $\mathbb{C}G$ -submodule of GU , which we will refer to as the *induced module* of U . We will define this more formally very soon.

First, observe that the $\mathbb{C}G$ -module $\mathbb{C}G$ is naturally a right $\mathbb{C}H$ -module with action $h(g) = gh$. Hence, taking the tensor product of $\mathbb{C}G$ with a $\mathbb{C}H$ -module over $\mathbb{C}H$ is a sensible thing to do.

Definition 3.4.9 (Induced Module). Let V be a $\mathbb{C}H$ -module. The induced module of V from H to G , denoted $\text{Ind}_H^G(V)$, V^G or $V \uparrow G$, is the $\mathbb{C}G$ -module $\mathbb{C}G \otimes_{\mathbb{C}H} V$ equipped with scalar multiplication

$$g \cdot (x \otimes v) = gx \otimes v \quad (3.4.2)$$

for all $g \in G$, $x \in \mathbb{C}G$ and $v \in V$.

We can give a more concrete description of the induced module.

Proposition 3.4.10. Let $k = [G : H]$ and let g_1, \dots, g_k be a complete set of coset representatives (so that $G = \bigsqcup_{i=1}^k g_i H$). As a vector space, we have that

$$\text{Ind}_H^G(V) = \bigoplus_{i=1}^k g_i \otimes V \quad (3.4.3)$$

where we denote by $g_i \otimes V$ the set of elements of the form $g_i \otimes v$ for $v \in V$.⁶ Furthermore, $\mathbb{C}G$ transitively permutes the summands $g_i \otimes V$ in (3.4.3) via the action

$$g \cdot (g_i \otimes v) = g_j \otimes v$$

where $g \in G$ is arbitrary and j is such that $gg_i \in g_j H$.

Proof. As a right $\mathbb{C}H$ -module, we have that

$$\begin{aligned} \mathbb{C}G &= \mathbb{C} \left(\bigsqcup_{i=1}^k g_i H \right) \\ &= \bigoplus_{i=1}^k \mathbb{C}(g_i H) \\ &= \bigoplus_{i=1}^k g_i \otimes \mathbb{C}H \end{aligned}$$

Since $\text{Ind}_H^G(V) = \mathbb{C}G \otimes_{\mathbb{C}H} V$, replacing $\mathbb{C}G$ with the above sum, we get that

$$\begin{aligned} \text{Ind}_H^G(V) &= \left(\bigoplus_{i=1}^k g_i \otimes \mathbb{C}H \right) \otimes_{\mathbb{C}H} V \\ &= \bigoplus_{i=1}^k g_i \otimes (\mathbb{C}H \otimes_{\mathbb{C}H} V) \\ &\cong \bigoplus_{i=1}^k g_i \otimes V \end{aligned}$$

as in (3.4.3), proving the first part of the proposition.

For the second part, fix $g \in G$ and $1 \leq i \leq k$. Let $1 \leq j \leq k$ and $h \in H$ be such that $gg_i = g_j h$.

We then have that

$$g(g_i \otimes V) = gg_i \otimes V = g_j h \otimes V = g_j \otimes hV = g_j \otimes V$$

demonstrating that we have the desired permutations of the summands in (3.4.3). The only thing left to show is that this action is transitive. ASK PROF TO CLARIFY TRANSITIVITY □

Corollary 3.4.11. *As a \mathbb{C} -vector space, we have that $\dim(\text{Ind}_H^G(V)) = [G : H] \cdot \dim(V)$.*

⁶Such a vector space is naturally isomorphic to V .

Example 3.4.12. The induced module of the trivial module \mathbb{C} from H to G is the permutation module of G acting on G/H by $g \cdot g_i H = g_i H$.

We now turn our attention to the characters of the induced module. We will begin by characterising the character of the induced module.

Proposition 3.4.13. *If V is a $\mathbb{C}H$ -module with character χ , the character of the induced module, denoted $\text{Ind}_H^G(\chi)$, is given by*

$$\text{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{x^{-1}gx \in H} \chi(x^{-1}gx)$$

for all $g \in G$. [Borcherds Video - 12 min mark till end](https://www.youtube.com/watch?v=_jH7OcZYooQ)
https://www.youtube.com/watch?v=_jH7OcZYooQ

Proof. Fix $g \in G$. We determine the trace of the action of g on $\text{Ind}_H^G(V)$ as follows.

Consider the decomposition $\text{Ind}_H^G(V) = \bigoplus_{i=1}^k g_i \otimes V$ (as given by (3.4.3), with $\{g_i\}_{i=1}^k$ being a set of representatives for the elements of G/H). Consider the action of g with respect to a basis given by the various bases of $g_i \otimes V$.

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If g does not fix $g_i \otimes V$, then the contribution $g_i \otimes V$ to $\text{Tr}(g)$ is zero. If g does fix $g_i \otimes V$ (or, equivalently, if $g_i^{-1}gg_i \in H$), then the contribution is given by

$$\chi(g_i^{-1}gg_i) = g(g_i \otimes v) = g_i(g_i g g_i^{-1}) \otimes v = g_i \otimes (g_i g g_i^{-1})v$$

for all $v \in V$. We can then conclude that

$$\text{Ind}_H^G(\chi)(g) = \sum_{g_i^{-1}gg_i \in H} \chi(g_i^{-1}gg_i)$$

Indeed, this sum can be extended to a sum over all $x \in G$ such that $x^{-1}gx \in H$ (except we need to divide by $|H|$ when we do so), giving us the desired result. Eh??? □

Example 3.4.14. Let $G = S_3$ and $H = \{1, (123), (132)\}$. Choose 1 and (12) as a set of coset representatives for G/H . Let χ be the character of the trivial module of H . Then,

3.4.3 The Relationship between Induction and Restriction

It is no coincidence that we investigated these phenomena in succession. As we shall see, they share an interesting relationship. This turns out to be a consequence of the following important **universal property**.

Proposition 3.4.15 (The Universal Property of the Induced Module). *Let V be a $\mathbb{C}H$ -module. For all $\mathbb{C}G$ -modules W and all $\mathbb{C}H$ -module homomorphisms $f : V \rightarrow W \downarrow H$, there exists a unique $\mathbb{C}G$ -module homomorphism $\tilde{f} : \text{Ind}_H^G(V) \rightarrow W$ such that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \downarrow H \\ \downarrow \iota & & \downarrow \iota' \\ \text{Ind}_H^G(V) & \xrightarrow{\exists! \tilde{f}} & W \end{array} \quad (3.4.4)$$

where $\iota : V \rightarrow \text{Ind}_H^G(V)$ is the map $v \mapsto 1 \otimes v$ and $\iota' : W \downarrow H \rightarrow W$ is the map sorry.

Proof. Since f is a morphism of $\mathbb{C}H$ -modules, the map $\bar{f} : \mathbb{C}G \times V \rightarrow W : (x, v) \mapsto xf(v)$ is $\mathbb{C}H$ -stable: for all $x, x' \in \mathbb{C}G$, $v, v' \in V$ and $y \in \mathbb{C}H$,

$$\bar{f}(x + x', v) = (x + x')f(v) = xf(v) + x'f(v) = \bar{f}(x, v) + \bar{f}(x', v)$$

$$\bar{f}(x, v + v') = xf(v + v') = xf(v) + xf(v') = \bar{f}(x, v) + \bar{f}(x, v')$$

$$\bar{f}(xy, v) = (xy)f(v) = xyf(v) = xf(yv) = \bar{f}(x, yv)$$

By the Universal Property of the Tensor Product, $\exists! \tilde{f} : \mathbb{C}G \otimes_{\mathbb{C}H} V \rightarrow W : x \otimes v \mapsto xf(v)$. It is clear that \tilde{f} is a morphism of $\mathbb{C}G$ -modules (is it?). Finally, one can check that the diagram commutes. □

This gives us the following important relationship between induction and restriction.

Corollary 3.4.16. *Let V be a $\mathbb{C}H$ -module and W a $\mathbb{C}H$ -module, as above. Then, there is an*

isomorphism of \mathbb{C} -vector spaces between $\text{Hom}_{\mathbb{C}H}(V, W \downarrow H)$ and $\text{Hom}_{\mathbb{C}G}(\text{Ind}_H^G(V), W)$.

Proof. Fix $\varphi \in \text{Hom}_{\mathbb{C}G}(\text{Ind}_H^G(V), W)$. Then, $\varphi \circ \iota$ is a morphism of $\mathbb{C}H$ -modules from V to $W \downarrow H$. One can use the Universal Property to show that $\phi \mapsto \phi \circ \iota$ is \mathbb{C} -linear, injective and surjective. \square

Lemma 3.4.17. Let W_1 and W_2 be $\mathbb{C}G$ -modules and with respective characters χ_1 and χ_2 . Then,

$$\dim(\text{Hom}_{\mathbb{C}G}(W_1, W_2)) = \langle \chi_1, \chi_2 \rangle$$

where we consider the dimension to be that of a \mathbb{C} -vector space.

Proof. Exercise in PS 10. TODO: place elsewhere! \square

Theorem 3.4.18 (Frobenius Reciprocity). Let V be a $\mathbb{C}H$ -module and W a $\mathbb{C}G$ -module, with characters χ_V and χ_W respectively. Then,

$$\langle \text{Ind}_H^G(\chi_V), \chi_W \rangle_G = \langle V, \text{Res}_H^G(W) \rangle_H$$

Proof. This is a direct consequence of the previous lemma and the corollary (reference):

$$\begin{aligned} \langle \text{Ind}_H^G(\chi_V), \chi_W \rangle_G &= \dim(\text{Hom}_{\mathbb{C}G}(\text{Ind}_H^G(V), W)) \\ &= \dim(\text{Hom}_{\mathbb{C}H}(V, W \downarrow H)) \\ &= \langle V, W \downarrow H \rangle_H = \langle V, \text{Res}_H^G(W) \rangle_H \end{aligned}$$

\square

Example 3.4.19 (The Quaternion Group). Let $G = Q_8$, the quaternion group of order 8, viewed as a subgroup of $\text{GL}(2, \mathbb{C})$. We know it is generated by

$$a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Let $H = \langle A \rangle \cong C_4$, the cyclic group of order 4. For $\chi \in \text{Irr}(H)$, we determine $\text{Ind}_H^G(\chi)$.

Consider the following character table for G :

	1	-1	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	1	1	-1
χ_4	1	1	1	-1	1
χ_5	2	-2	-2	0	0

The group H has 4 irreducible characters, which we denote $\eta_1, \eta_2, \eta_3, \eta_4$. Indeed, $\eta_1 = \chi_1 \downarrow H$ and $\eta_2 = \chi_3 \downarrow H = \chi_4 \downarrow H$. Furthermore, Frobenius Reciprocity tells us that χ_1 and χ_2 are both constituents of $\text{Ind}_H^G(\eta_1)$ —indeed, they are the only ones. Similarly, χ_3 and χ_4 are both constituents of $\text{Ind}_H^G(\eta_2)$. Finally, no linear character is a constituent of $\text{Ind}_H^G(\eta_3)$ or $\text{Ind}_H^G(\eta_4)$.