21-800: Advanced Topics in Logic

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Contents

1 An Introduction to the Theory of Weak Containment 2

1.1 Local Convergence of Graphs 2

2 Another Chapter 5

2.1 Introducing the Main Object of Study in this Chapter 5

2.2 Another Section 5

Bibliography 8

Chapter 1

An Introduction to the Theory of Weak Containment

The main topic of study in this course is the notion of weak containment. We will borrow ideas from combinatorics, descriptive set theory, and dynamical systems.

1.1 Local Convergence of Graphs

Throughout this section, fix a natural number $d \in \mathbb{N}$.

Definition 1.1.1. For $d \in \mathbb{N}$, define

$$\mathbb{G}_0(d) := \{\text{connected rooted graphs with degree } \leq d\}/\simeq$$

where we quotient out by isomorphisms of rooted graphs.

We can define a metric on $\mathbb{G}_0(d)$ in the following manner: for graphs G and H (or isomorphism classes thereof), define

$$d((G,v),(H,u)) = 2^{-\max\left\{n\in\mathbb{N} \,\middle|\, B_n^G(v)\simeq B_n^H(u)\right\}}$$

It is possible to show that $\mathbb{G}_0(d)$ is a compact metric space.

Definition 1.1.2 (Locally Cauchy Sequence of Graphs)**.** We say a sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ (with all degrees $\leq d$) is **locally Cauchy** (or **Benjamini-Schramm Cauchy** if for all $r \in \mathbb{N}$, the distribution of a random r-ball in G_n converges weakly.

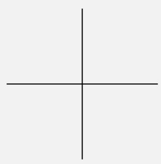
Because our ambient space $\mathbb{G}_0$ is compact, this is equivalent to saying that for every $(G, u) \in \mathbb{G}_0(d)$, the sequence of probabilities

$$\mathbb{P}\left(B_r^{G_n}(\mathbf{v}) \simeq (G, u)\right)$$

converges, where v is a uniformly random vertex from G_n.

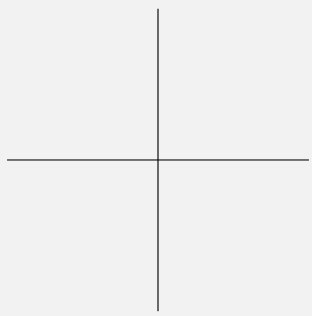
Example 1.1.3 (The Discrete Torus)**.** Consider the group $\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^2$. We can visualise this as a 'discrete torus' in the same way we identify the torus ${}^{\mathbb{R}}/_{\mathbb{Z}}$ with $S^1 \times S^1$: instead of crossing a circle with a circle, we cross a circle with n points with a circle with n points.

When $n = 3$ and $r = 1$, the ball of radius 1 (at any basepoint) looks like a plus, except all points are joined three-dimensionally to some point lying beyond the vertex of the plus.



When $n > 3$ and $r = 1$, the ball just looks like a plus. `sorry`

When $n > 5$ and $r = 2$, the ball looks like a plus with a square at the centre.



More generally, for any fixed r, for all large enough n, every r-ball in $\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^2$ is isomorphic to $B_n^{\mathbb{Z}^2}(v)$.

Example 1.1.4 (Path Graphs). Let $G_n = P_n$, the path graph with n vertices. `sorry`

Lemma 1.1.5. *If G is a random d-regular graph on n vertices, the probability that v is contained in a simple cycle of length $< k$ is at most $O\left(\frac{k}{\log(n)}\right)$.*

So if G_n is a random regular graph on $> n$ vertices, for each r, the probability that $B_r^{G_n}(\mathbf{v})$ contains a cycle almost surely tends to 0.

Example 1.1.6 (T_3). The sequence $G_n = B_n^{T_3}(v)$ is locally Cauchy, and the limit of each r-ball has more than one graph in its support.

Definition 1.1.7. We say that G_n converges to μ locally (where μ is a probability measure on $\mathbb{G}_0(d)$) if for every $r \in \mathbb{N}$, $(H, u) \in \mathbb{G}_0(d)$.

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, `\lipsum` time. Boy do I love LaTeX!

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