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DEPARTMENT OF MATHEMATICS

MSCI RESEARCH PROJECT

Viazovska's Magic Function in Dimension 8: A Formalisation in Lean

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Submitted in partial fulfillment of the requirements for the MSci in Mathematics at Imperial College London

February 13, 2025

Abstract

Hi

Acknowledgments

Plagiarism statement

The work contained in this thesis is my own work unless otherwise stated.

Signature: Sidharth Hariharan
Date: February 13, 2025

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Chapter 1

Introduction

On 5 July, 2022, in Helsinki, Finland, the International Mathematical Union announced the names of the four mathematicians who were to be awarded the Fields Medal, the most coveted prize in the world of mathematics: Hugo Duminil-Copin, June Huh, James Maynard and Maryna Viazovska. Duminil-Copin, Huh and Maynard received this most prestigious honour for making several outstanding contributions to their specific fields of expertise—respectively, statistical physics, geometric combinatorics, and analytic number theory. Viazovska, on the other hand, received the Fields Medal for more interdisciplinary achievements. Arguably the most remarkable of these was her solution to the sphere packing problem in dimension 8 [1]. It is difficult to place her solution in a specific mathematical field: what makes it so revolutionary is that it uses insights from Fourier analysis and the theory of modular forms to construct a special function—the Magic Function—that, in combination with a previous result by Cohn and Elkies [2], proves that the E_8 lattice packing is the densest possible sphere packing in $\mathbb{R}^8$. Very shortly afterwards, Cohn, Kumar, Miller, Radchenko and Viazovska were able to use similar ideas to prove that the Leech lattice packing is the densest possible sphere packing in $\mathbb{R}^{24}$ [3].

Before Viazovska's remarkable breakthrough, the optimal sphere packing density was only known in dimensions 1, 2 and 3 [4]. Furthermore, Thomas Hales' solution in dimension 3 [5] was lengthy and involved extensive computer-assisted calculations; in contrast, Viazovska's proof in dimension 8 is elegant and concise. Even before Viazovska was awarded the Fields Medal, her work received wide acclaim from eminent mathematicians across the world: Peter Sarnak described it as "stunningly simple, as all great things are," and Akshay Venkatesh remarked that her Magic Function is very likely "part of some richer story" that connects to other areas of mathematics and physics [6]. Viazovska's work is a truly remarkable achievement in modern mathematics, with its elegance coming from the manner in which the many pieces of the puzzle fit perfectly together. One of the goals of this project is to offer a detailed exposition of one of those pieces: the construction of the so-called 'Magic Function' in dimension 8.

1.1 The Sphere Packing Problem

The Sphere Packing problem is a classical optimsation problem in mathematics. The problem can be formulated as follows.

Problem 1.1.1 (The Sphere Packing Problem in Dimensinon n). *Given some $n \in \mathbb{N}$, what is the densest possible non-overlapping arrangement of n-spheres of equal radius in $\mathbb{R}^n$?*

Despite its rather straightforward formulation, Problem 1.1.1 is notoriously difficult to solve. Indeed, one obvious question that arises when one looks at the problem statement is how one might define the concept of density. It turns out that the definition is slightly unwieldy, though introducing a periodicity assumption on the sphere packing whose density one wishes to find considerably simplifies this problem.

A key challenge in solving the sphere packing problem in dimension n is the fact that proceeding inductively is not always helpful: 'stacking' the optimal n-dimensional sphere packing onto itself is not guaranteed to yield the optimal sphere packing in $n+1$ dimensions. [4]. In fact, this appraoch is known to fail in dimensions as low as 10 [7]. This is not obvious, not least because the approach does, in fact, succeed in the visualisable dimensions of 1, 2 and 3.

The 1-dimensional case is uninteresting. Visually, one can easily see that the densest possible arrangement of disjoint intervals of the form $(-r, r)$ on the real line consists of intervals centred at all points $2rm$ for $m \in \mathbb{Z}$. Indeed, one can fix r to be $\frac{1}{2}$ by rescaling the real line. The optimal packing therefore consists of open intervals of unit length centred at points on the lattice $\mathbb{Z} \subset \mathbb{R}$.

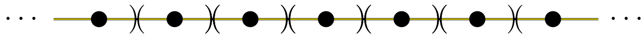


Figure 1.1: The $\mathbb{Z}$ lattice packing in dimension 1.

Rescaling gives us a powerful—if straightforward—simplification of the sphere packing problem where we can fix the radius of the spheres to a convenient value. Indeed, we only mention rescaling explicitly because it needs to be explicitly dealt with when formalising the problem. We will resume this discussion in . For now, we will take for granted the fact that rescaling does not affect the density of a sphere packing, meaning that we can talk about optimal sphere packings without worrying about the radius of the spheres in question. Bearing in mind that the spheres must all have the same radius, as per the statement of Problem 1.1.1, we will henceforth describe sphere packings simply by describing the points at which the spheres are centred.

Add cross-reference

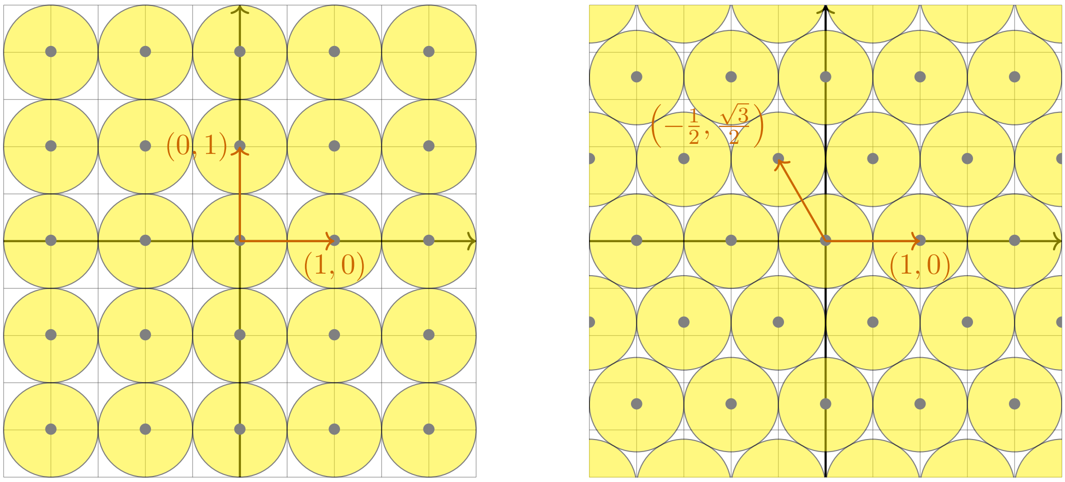
The sphere packing problem in dimension 2, also known as the circle packing problem, turns out to be more interesting. A reasonable strategy for finding the densest packing is to 'stack' the $\mathbb{Z}$ lattice packing from dimension 1 onto itself in some manner, but the question remains as to exactly how this should be done. 'Stacking' it onto itself would involve extending the lattice $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{R}^2$ into a lattice in $\mathbb{R}^2$ by extending the $\mathbb{R}$-basis $\{(1,0)\}$ of $\mathbb{R}$ (viewed as a subspace of $\mathbb{R}^2$) to an $\mathbb{R}$-basis of $\mathbb{R}^2$, and taking its $\mathbb{Z}$-span.

One natural way of doing this is to extend the lattice $\mathbb{Z} \subset \mathbb{R}$ to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ consisting of points with integer coordinates. This corresponds to the natural extension of $\{(1,0)\}$ to the standard $\mathbb{R}$-basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$. See Figure 1.2a.

Unfortunately, this packing turns out to be sub-optimal . A better candidate is the A_2 lattice packing, corresponding to the extension of $\{(1,0)\}$ to the A_2 root basis $\left\{(1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}$. See Figure 1.2b. This packing is sometimes referred to as the *honeycomb packing* due to the fact that every circle has six neighbours, whose centres form the vertices of a regular hexagon.

Do we want to add the density?

It is well-known that the honeycomb packing is optimal in $\mathbb{R}^2$. What this means is that no circle packing has a density greater than that of the honeycomb packing. The original proof of this fact is attributed to Thue [8], and it is sometimes referred to in the literature as *Thue's Theorem*. Several other mathematicians have since constructed proofs of Thue's Theorem. One approach based on an idea of Rogers's that does not require particularly sophisticated mathematical tools was outlined by Hales in [9].



(a) The $\mathbb{Z}^2$ lattice packing.

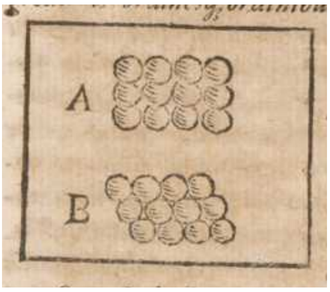
(b) The A_2 lattice packing.

Figure 1.2: Circle packings covering the square $\{(x, y) \subset \mathbb{R}^2 \mid -2.5 \leq x, y \leq 2.5\}$.

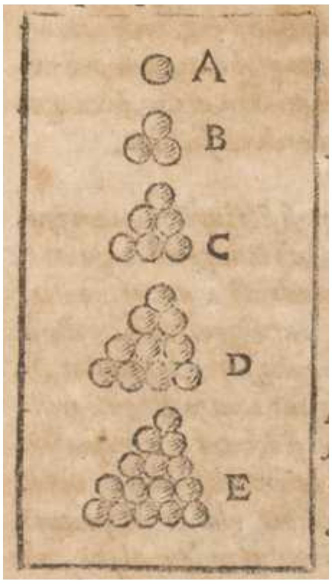
While we do not offer an exposition of Hales's or any other proof of Thue's Theorem, we include a small discussion of the sphere packing problem in two dimensions to offer some intuition as to why the A_2 packing is optimal. For simplicity, we will work under the assumption that the optimal packing in $\mathbb{R}^2$ is some extension of the $\mathbb{Z}$ lattice packing in $\mathbb{R}$, as defined above. We use the strategy of stacking $\mathbb{Z}$ packings on top of each other 'row by row', shifting rows around till their density cannot be further increased.

From Figure 1.2, one can convince oneself with relative ease that the A_2 packing is denser than the $\mathbb{Z}^2$ packing. This makes it denser than any packing that is sparser than the $\mathbb{Z}^2$ packing. In particular, we only need to improve the $\mathbb{Z}^2$ packing to construct the optimal packing. Since vertical shifts only push rows further apart, rendering the packing sparser, it suffices to consider horizontal row shifts. In the $\mathbb{Z}^2$ packing, each circle in a given row is in contact with only one sphere from the row below. A natural improvement is to shift rows about so each circle is in contact with *two* circles from the row below. This results in the A_2 lattice packing. This packing cannot be improved because it is impossible for a circle to be in contact with *three* circles from the row below due to the separation between circles in the one-dimension $\mathbb{Z}$ packing. See Figure 1.3a.

While this is far from a rigorous argument, this approach illustrates why we should not be surprised that the A_2 sphere packing is optimal: the key observation is that the A_2 packing maximises the number of neighbours a circle can have. Admittedly, it is neither obvious why this optimises density nor why the optimal packing involves stacking the $\mathbb{Z}$ packing repeatedly on itself, not least because we have yet to formally define the density of a sphere packing. We merely reassure the reader, at this stage, that the definitions and characteristics of the sphere packing problem in $\mathbb{R}^2$ strongly agree with visual intuition. With this, we close our discussion.



(a)

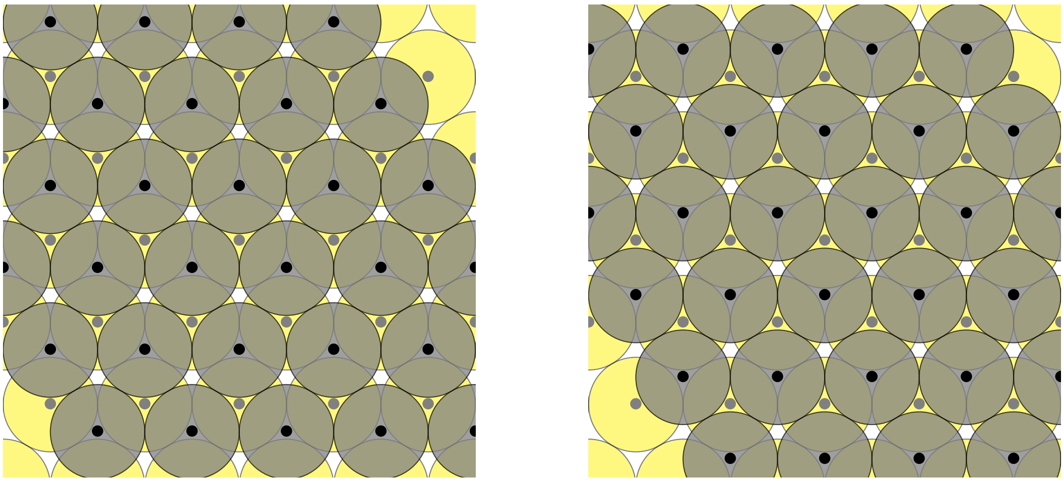


(b)

Figure 1.3: Diagrams from an essay written by Johannes Kepler in Latin in 1611 [10].

In dimension 3, too, it is tempting to replicate this strategy: we can attempt to stack the A_2 packing on top of itself, in layers instead of rows, in such a manner as maximises the number of neighbours a sphere can have. From trial and error, it appears to be the case that a sphere cannot be in contact with more than three neighbours from the layer below. This suggests that the optimal sphere packing in dimension 3 is given by stacking honeycomb arrangements on top of each other with spheres in each layer being nestled in the gaps between three spheres in the layer below.

As it turns out, unlike dimension 2, a characterisation in terms of the number of neighbours in the layer below does not describe a unique packing. In $\mathbb{R}^3$, spheres are so large that it is not possible to stack honeycomb arrangements on top of each other such that *all* gaps between spheres in one layer are occupied by spheres in the next. There is no unique stacking of honeycomb layers such that each sphere has exactly three neighbours in the layer below: in different stackings, the spheres in a layer might fill a different arrangement of gaps between spheres in the layer below, as shown in Figure 1.4. One can construct many different sphere packings in $\mathbb{R}^3$, all of which are as dense as possible, by varying how successive layers are placed. For instance, the sphere packing obtained by successively repeating the arrangement in Figure 1.4a and that obtained by alternating between the arrangements in Figure 1.4a and Figure 1.4b are globally different, despite having the same density and identical layers. The former is referred to as the *face-centred cubic packing* and the latter is referred to as the *hexagonal close-packing*.



(a) **(b)**

Figure 1.4: Two different ways of stacking the honeycomb packing on itself.

The face-centred cubic packing can be visualised by thinking about how spheres can be arranged tetrahedrally. For some n, let $T_n := n\,(n+1)\,/2$ be the nth triangular number[1]. Begin by arranging T_n spheres in a triangular formation. On top of this layer, arrange T_{n-1} spheres in a triangular formation, such that each sphere is nestled in the gaps between the spheres in the layer below, as in Figure 1.4a. Continue in this manner till there is only one ball left to be arranged. This leads to an arrangement in which the number of spheres is the nth tetrahedral number[2]. A key characteristic of the face-centred cubic packing is that it consists of such arrangements. In fact, in a 1611 essay whose title has been translated from Latin as *The Six-Cornered Snowflake* [10], it was asserted by Johannes Kepler that spheres cannot be more tightly packed together than they are in a tetrahedral arrangement. This assertion was accompanied by an illustration: see Figure 1.3b. For over three centuries, this assertion remained unproven,

[1]See OEIS A000217
[2]See OEIS A000292

and was referred to as the *Kepler Conjecture*. It was only in 2005 that a paper proving the Kepler Conjecture, written by Thomas Hales, was published [5].

The complexity of the sphere packing problem in dimension 3 is illustrated not only by the time elapsed between Kepler's original assertion and a proof being published but also by the length of Hales's paper. Indeed, in an expository account of his proof published in 2000, five years before the publication of the full paper in the Annals, Hales recounted how a jury of twelve referees, despite having been in deliberation for over a year, had yet to make a "thorough, independent check of the computer code" he had written to perform the elaborate calculations on which "every aspect of [his proof] is based" [9]. In January 2003, at the Joint Math Meetings in Baltimore, USA, Hales announced that he intended to formally verify his proof [11]. The paper authored by Hales and his collaborators on their successful formalisation of his argument was only published in 2017. Therefore, not only did the Kepler Conjecture take close to 400 years to solve, but it took nearly two decades to eliminate any doubt as to the correctness of the solution.

At first glance, this appears to set a dangerous precedent for the sphere packing problem in other dimensions. It well might, for there is much we do not understand about the behaviour of spheres in high dimensions. In the words of Henry Cohn, "each dimension has its own idiosyncracies and charm" [7]. That being said, in the specific cases of dimensions 8 and 24, this turns out to work in our favour.

The solutions in dimensions 8 and 24 are products of the same recipe, which consists primarily of two ingredients. The first is a linear programming bound from a 2003 paper by Henry Cohn and Noam Elkies [2, Theorem 3.1] on all sphere packing densities in $\mathbb{R}^n$. The second is the remarkable insight that the theory of modular forms can be used to obtain tight bounds in dimensions 8 and 24, equal to the densities of the E_8 and Leech lattice packings respectively.

The applicability of the theory of modular forms comes from the formulation of Cohn and Elkies's theorem: if a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies certain conditions, then *all* sphere packings in $\mathbb{R}^n$ are bounded above by a quantity that depends on f. The trick is therefore not only to find a function satisfying the Cohn-Elkies conditions but to find one for which the Cohn-Elkies bound also corresponds to the density of some sphere packing in $\mathbb{R}^n$. Viazovska's groundbreaking contribution was constructing such a function in dimension 8 using the theory of modular forms. The function is often referred to as the Magic Function, a term we shall adopt in this project because it befits the nature of Viazovska's achievement. A similar approach was used in dimension 24 by Cohn, Kumar, Miller, Radchenko and Viazovska.

As tempting as it is to continue our discussion on the sphere packing problem, this project does consist of two parts: a mathematical examination of the construction of Viazovska's Magic Function in dimension 8 and a formalisation thereof. We will therefore pause this discussion and take a detour into the world of formalisation, which will offer context for the second part of this project.

1.2 The Formalisation Movement

While Hales announced his intent to formally verify his proof of the Kepler Conjecture in 2003, it was not till 2006, after Hales's solution appeared in the *Annals*, that a formal description of Hales's formalisation project was published. Of his motivations, Hales wrote:

> *In response to the lingering doubt about the correctness of the proof, at the beginning of 2003, I launched the* Flyspeck *project, whose aim is a complete formal verification*

> *of the Kepler Conjecture. In truth, my motivations for the project are far more complex than a simple hope of removing residual doubt from the minds of few referees. Indeed, I see formal methods as fundamental to the long-term growth of mathematics.* [12]

Formal theorem proving was not unheard of in 2006. Interactive theorem provers, such as Coq and PRL, have existed since the 1980s. However, it was still a relatively young field, and the amount of mathematics that had been formalised was limited. Hales's project was immensely ambitious, and the fact that it succeeded, despite taking over a decade, is impressive.

There is something prophetic about Hales's "far more complex" motivations for launching the Flyspeck project. The field of formal theorem proving has grown rapidly in the last decade, and interactive theorem provers like Lean are slowly making their way into mainstream mathematics. An excellent example of this is the formal verification of Gowers, Green, Manners and Tao's proof of Marton's Conjecture [13], which was formally verified in Lean in just three weeks. In particular, their proof was formally verified *before* their paper was submitted for publication. The paper is set to appear in the *Annals*.

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There are many advantages of formal theorem proving. One advantage is the fact that formally proved theorems are verified by a proof assistant. When code written in proof assistants is compiled, if there are no errors, then the proof can be thought of as being 'correct', in the sense of being consistent with the axioms of the proof assistant.

Finish

1.3 The Scope of this Project

In November 2023, I had the privilege of meeting Maryna Viazovska while pursuing an exchange programme at the Swiss Federal Institute of Technology, Lausanne, where she is based. We began discussing formalising her solution to the sphere packing problem in 8 dimensions, and soon initiated a collaboration with Christopher Birkbeck, Seewoo Lee, and Gareth Ma, with invaluable assistance from Kevin Buzzard, Utensil Song, and Patrick Massot. On 31 May 2024, Viazovska formally announced at the ICMS workshop *Formalisation of Mathematics: Workshop for Women and Mathematicians of Minority Gender* that we would be attempting to formalise her groundbreaking paper.

Viazovska's original paper [1] is divided into five sections. The first section introduces sphere packings and develops basic theory; the second discusses the Cohn-Elkies linear programming bounds [2, Theorem 3.1]; the third offers some background on the theory of modular forms; the fourth constructs two radial, Schwartz Fourier eigenfunctions with double zeroes at almost all points on the E_8 lattice; and finally, the fifth uses these eigenfunctions to construct the "Magic Function", a Schwartz function that satisfies the conditions of Cohn and Elkies's theorem to give an upper bound for all sphere packings in $\mathbb{R}^8$ that is equal to the density of the E_8 packing. The first two sections were formalised collaboratively in July and August 2024, and the third section is actively being worked on by Birkbeck and Lee. This project focuses on formalising the fourth and fifth sections of Viazovska's paper. The code written for this section is primarily my own, and I have credited the contributions of others where appropriate.

The primary objective of this thesis is to offer a mathematical exposition of the fourth and fifth sections of Viazovska's original paper and to provide an account of the formalisation process.

Say where we are with the formalisation before submitting.

Chapter 2

The Sphere Packing Problem in Dimension 8

The purpose of this chapter is to conduct a detailed examination of Viazovska's original paper solving the sphere packing problem in dimension 8 [1] and the formalisation blueprint [14]. While we have already seen the high-level idea in Section 1.1, in this chapter, we will take a closer look at the mathematical details.

We will begin by providing precise mathematical definitions for sphere packings, densities, and the sphere packing constant. We will then discuss the linear programming bound conceived by Cohn and Elkies [2, Theorem 3.1] (or, more precisely, the slight modification thereof that is more directly applicable: see [1, Theorem 2] and [14, Theorem 5.1]). Finally, we will include a small discussion on the theory of modular forms and establish its relevance to the subsequent chapters of this thesis, which will focus on the construction of the 'Magic Function' (denoted as g in [1, Theorem 3]).

In each section, we will not only mathematical details from the sources outlined above but also an overview of the choices made and challenges encountered when formalising these notions. While this chapter primarily concerns the contents of the first three sections of [1], which are not within the scope of this project, there is an undeniable relevance of both the informal and the formal definitions and results. It is therefore necessary to include a detailed treatment thereof before we can construct g and prove it satisfies the desired 'Magic' properties.

2.1 Preliminaries

Before we begin defining things formally, we must include a small disclaimer about the terminology we have been using—and will continue to use—in this project. While Problem 1.1.1 is usually referred to as the *sphere* packing problem, a sphere is not usually thought to have an interior. Typically, in any metric space X with metric d, the *sphere* of radius $r \geq 0$ centred at $x \in X$ is defined to be $\{y \in X \mid d(x, y) = r\}$. In other words, the sphere consists only of a surface. In contrast, the sphere packing problem involves packing *solid balls*. One can see why, in [9], Hales opines that a more proper term for the problem would be the *ball packing problem*. Nevertheless, in this project, we will continue to use the standard terminology, but we include this disclaimer so the reader bears in mind two things: first, that we will often mean 'ball' when we use the word 'sphere', and second, that we work with balls instead of spheres in Lean. We will also mention that it is convenient to require that the balls in question be open, so that the condition that spheres cannot overlap but merely touch tangentially can be shortened to that of disjointedness. We introduce notation.

Notation. For some $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $r > 0$, we denote

$$B_d(x, r) := \left\{ y \in \mathbb{R}^d \;\middle|\; \|x - y\| < r \right\}$$

We organise this section into three subsections. The first defines fundamental notions about sphere packings. The second introduces the properties of two important, and closely related, classes of sphere packings, namely, lattice packings and periodic packings. The third subsection studies the most important sphere packing for our project: the E_8 lattice packing.

2.1.1 Sphere Packing Fundamentals

We begin by defining a sphere packing. As we have stated, we want sphere packings to consist of disjoint spheres of the same radius. Given that lying on the interior of a certain sphere corresponds to being within some distance from its centre, we can capture this notion of disjointedness by imposing a separation condition on the set of centres of the sphere packing.

Definition 2.1.1 (Sphere Packing)**.** Fix $d \in \mathbb{N}$ and $X \subset \mathbb{R}^d$. Assume that there exists a real number $r > 0$, known as the **separation radius**, such that

$$\|x - y\| \geq r$$

for all distinct $x, y \in X$. We define the **sphere packing with centres at** X to be

$$\mathcal{P}(X) := \bigcup_{x \in X} B_d(x, r)$$

Note that the assumption that a separation radius exists is very important.

Non-Example 2.1.2. Let $d = 1$ and $X = \mathbb{R}$. Consider the set

$$\bigcup_{x \in \mathbb{R}} B_1(x, r) = \bigcup_{x \in \mathbb{R}} (x - r, x + r)$$

For any $r > 0$, the above union is all of $\mathbb{R}$. However, it does not make sense to construct a sphere packing whose set of centres is the entirety of $\mathbb{R}$, as this would involve spheres overlapping. It is precisely to avoid such constructions that we impose the condition that r be a separation radius on the set of centres.

Since all the information about a sphere packing is encoded in its set of centres and the corresponding separation radius (which must exist in order for the set of centres to be a valid set of centres for a sphere packing), we decided that a sphere packing would be formalised purely as a set of centres with a valid separation, and that a separate definition would be made to obtain the open balls that constitute the packing. We packaged the data of

- the set of centres

- the separation radius
- the (automatically checked) condition that the separation radius is positive
- the condition that the set of centres is, indeed, separated by this radius

into a `structure` called `SpherePacking`: see [15, `SpherePacking.Basic.SpherePacking`].

Is this an acceptable way of citing the documentation? Should the repo be made public?

We now define finite density, an indicator of how much of a bounded region of space a sphere packing covers.

Definition 2.1.3 (Finite Density). Let $\mathcal{P}$ be a sphere packing. For all $R > 0$, define the **finite density** to be

$$\Delta_{\mathcal{P}}(R) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0, R))}{\mathrm{Vol}(B_d(0, R))}$$

where Vol is the Lebesgue measure on $\mathbb{R}^d$.

Finite density is a somewhat local notion, in that it expresses sphere how closely packed spheres are in a bounded region. The sphere packing problem, on the other hand, examines the notion of closeness on a more global level. While taking the limit of finite densities as the radius of the bounding region approaches infinity might seem like a natural way to define density, it is not obvious that this limit always exists. Therefore, we define density to be the limit superior instead.

Definition 2.1.4 (Density). Let $\mathcal{P}$ be a sphere packing. Define the **density** of $\mathcal{P}$ to be

$$\Delta(\mathcal{P}) := \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R)$$

where $\Delta_{\mathcal{P}}(R)$ is the finite density of $\mathcal{P}$, as defined in Definition 2.1.3.

The sphere packing problem asks for the sphere packing that achieves the highest possible density. We can be formal about the notion of the highest possible density.

Definition 2.1.5 (Sphere Packing Constant). The **sphere packing constant** in $\mathbb{R}^d$, for any $d > 0$, is defined to be

$$\Delta_d := \sup\left(\left\{\Delta_{\mathcal{P}} \;\middle|\; \mathcal{P} \text{ is a sphere packing in } \mathbb{R}^d\right\}\right)$$

There is a trivial upper-bound on sphere packing density.

Lemma 2.1.6. *For any sphere packing $\mathcal{P}$ and $R > 0$, we have that $\Delta_{\mathcal{P}}(R) \leq 1$.*

Proof. This is an immediate consequence of the fact that $\mathcal{P} \cap B_d(0, R) \subseteq B_d(0, R)$. □

This immediately gives us the following basic facts.

Corollary 2.1.7. *For any sphere packing $\mathcal{P}$, we have that $\Delta_{\mathcal{P}} \leq 1$.*

Corollary 2.1.8. *For any $d \in \mathbb{N}$, $\Delta_d \leq 1$*

This is not a very good upper-bound. However, it tells us that the sphere packing constant in any number of dimensions is a finite real number in the interval $(0, 1]$. There is still some work to be done before we can give better bounds on the sphere packing constant. Furthermore, it is unclear whether the sphere packing constant actually is, for a general d, the density of a sphere packing in $\mathbb{R}^d$. Nevertheless, this is a good starting point.

A great deal of basic sphere packing API in Lean was developed in July 2024 for the project to formalise Viazovska's solution in dimensino 8. The majority of the code was written by Gareth Ma, who also made significant improvements to the design choices I had made when setting up the project. The definitions and results in this section have all been formalised, and information about the code that has been written can be found in the project documentation [15, `SpherePacking.Basic.SpherePacking`].

In the next subsection, we discuss a special class of sphere packings that have periodicity properties with respect to lattices.

2.1.2 Lattice and Periodic Sphere Packings

We begin by defining lattices and briefly commenting on existing `Mathlib` API on lattices. There are primarily two ways in which lattices are defined in mathematical literature. A lattice in some Euclidean space $\mathbb{R}^n$ is either described as the $\mathbb{Z}$-span of some $\mathbb{R}$-basis of $\mathbb{R}^n$ or as a discrete, co-compact subgroup. One can borrow characteristics from both definitions to construct other equivalent definitions.

The characteristics described in both definitions do exist `Mathlib`. However, given that one of the objectives of creating a unified mathematics library is centralisation, a combination of these definitions is used as the *definition* of a `class` that we call `IsZLattice` and information about its many properties, as well as the $\mathbb{Z}$-span construction, are encoded in theorems. In particular, we have a theorem that tell us that every lattice is a free $\mathbb{Z}$-submodule, meaning it has a $\mathbb{Z}$-basis, and that this $\mathbb{Z}$-basis is actually an $\mathbb{R}$-basis of the ambient space. Furthermore, we have a result that every object constructed in that manner is a lattice. Results about types bearing the `IsZLattice` class (ie, lattices as they are defined in `Mathlib`) live in the `ZLattice` namespace, whereas results bout $\mathbb{Z}$-spans of $\mathbb{R}$-bases live in the `ZSpan` namespace.

Create citation for Mathlib.

We begin by stating the `Mathlib` definition of a lattice.

Definition 2.1.9 (Lattice)**.** A **lattice** in a Euclidean space $\mathbb{R}^n$ is a discrete $\mathbb{Z}$-submodule of $\mathbb{R}^n$ such that its $\mathbb{R}$-span contains every element in $\mathbb{R}^n$.

The `Mathlib` definition is more general, and works for any normed vector space over a normed field. Here, the word 'discrete' means that the lattice is discrete in a topological sense, meaning that the subspace topology on the lattice is precisely the discrete topology.

Definition 2.1.10 (Periodic Sphere Packing)**.** Let $\Lambda \subset \mathbb{R}^d$ be a lattice. We say a sphere packing $\mathcal{P}(X)$ with spheres centred at points in $X \subset \mathbb{R}^d$ is **periodic with respect to** Λ, or Λ**-periodic**,

$$\lambda + X = X$$

ie, for all $\lambda \in \Lambda$ and $x \in X$, we have that $\lambda + x \in X$.

We define Periodic Sphere Packings in Lean as extending the definition of Sphere Packings by creating a `structure` called `PeriodicSpherePacking` that packages the additional data of

- the lattice, viewed as a $\mathbb{Z}$-submodule of the ambient space $\mathbb{R}^d$
- the condition that the set of centres is periodic with respect to this $\mathbb{Z}$-submodule
- the (automatically checked) condition[1] that the $\mathbb{Z}$-submodule is discrete
- the (automatically checked) condition[1] that the discrete $\mathbb{Z}$-submodule is a lattice

The definition is in [15, `SpherePacking.Basic.SpherePacking`].

Lattice packings are a special class of periodic packings.

Definition 2.1.11 (Lattice Packing)**.** Let $\Lambda \subset \mathbb{R}^d$ be a lattice. The Λ **lattice packing** is the sphere packing with centres at points in Λ. Such a sphere packing admits a separation radius because Λ is discrete and is Λ-periodic because Λ is closed under addition.

In Section 2.1.3, we will briefly examine a specific lattice packing, the E_8 lattice packing.

The periodicity property of a periodic sphere packing can be exploited to derive a more convenient formula for its density.

Proposition 2.1.12. *Let $\mathcal{P}(X)$ be a sphere packing with centres at $X \subset \mathbb{R}^d$ and separation r that is periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$. We have that*

$$\Delta_{\mathcal{P}(X)} = \left| {}^{X}\!/\!{}_{\Lambda} \right| \frac{\mathrm{Vol}\left(B_d(0, \frac{r}{2})\right)}{\mathrm{Vol}\left({}^{\mathbb{R}^d}\!/\!{}_{\Lambda}\right)} \tag{2.1.1}$$

The proof is beyond the scope of this M4R project, but was formalised in Summer 2024: see `PeriodicSpherePacking.density_eq'` in [15, `SpherePacking.Basic.PeriodicPacking`].

Just as we defined the sphere packing constant for any dimension $d \in \mathbb{N}$, we can define a *periodic* sphere packing constant in any dimension.

Definition 2.1.13 (Periodic Sphere Packing Constant)**.** For all $d \in \mathbb{N}$, define the **periodic**

[1] more precisely, the automatically inferred instance

sphere packing constant in dimension d to be

$$\Delta_d^{\text{periodic}} = \sup\Big(\Big\{\Delta_{\mathcal{P}} \;\Big|\; \mathcal{P} \text{ is a periodic sphere packing in } \mathbb{R}^d\Big\}\Big)$$

The power of periodic sphere packings is illustrated by a rather surprising fact.

Proposition 2.1.14. *For all* $d \in \mathbb{N}$*,*

$$\Delta_d = \Delta_d^{periodic}$$

We do not prove this result here, as it is beyond the scope of this M4R. A proof can be found in [2, Appendix A].

Proposition 2.1.14 tells us that finding a sphere packing that satisfies the *periodic* sphere packing constant gives us the optimal sphere packing in dimension d. We will exploit this fact in Section 2.2, where we will construct an upper bound for all sphere packing densities in dimension d by constructing an upper-bound for the periodic sphere packing constant in dimension d. When constructing this upper-bound, we will exploit the fact that periodic sphere packings admit a 'nice' density formula (cf. Proposition 2.1.12). The results we have seen about periodic sphere packings will thus greatly simplify our task of finding the optimal sphere packing in dimension 8.

We are now ready to discuss a special sphere packing in $\mathbb{R}^8$: the E_8 sphere packing.

2.1.3 The E_8 Lattice Packing

It is quite remarkable that E_8 should show up when discussing sphere packings. At its core, E_8 is an irreducible root system. It shows up in the classification of important classes of objects like irreducible Coxeter groups, crystallographic Coxeter groups, and semi-simple Lie algebras over $\mathbb{C}$. E_8 is not a classical root system but an *exceptional* root system, meaning that the geometric properties of its roots cannot be found in irreducible root systems in all dimensions.

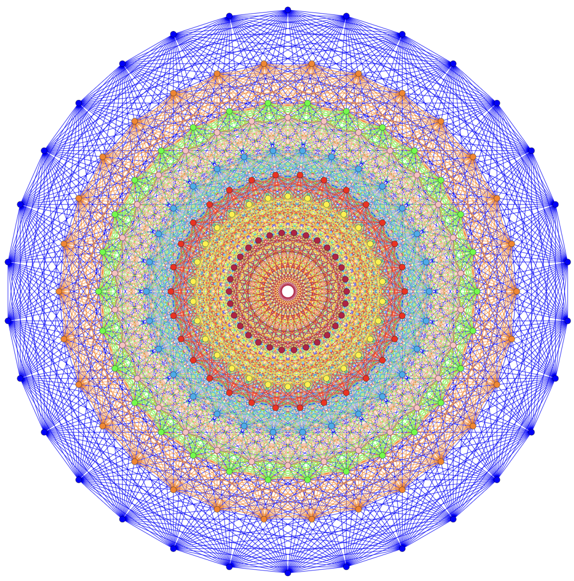

Figure 2.1: The Coxeter projection of the E_8 root system. [16]

The E_8 root system consists of 240 vectors in $\mathbb{R}^8$ that are permuted by a certain finite subgroup of the 8-dimensional orthogonal group. This group is sometimes referred to as the E_8 Coxeter group or as the Weyl group of the E_8 lattice. These roots can be divided into 8 orbits, each of which corresponds to one of the 'layers' of concentric circles in Figure 2.1. The dots in the figure correspond to projections of the roots onto a plane on which a specific type of element of the Coxeter group, known as a Coxeter element, acts as a rotation. This visualisation offers a convenient—and aesthetically pleasing—means of visualising this collection of 8-dimensional vectors and appreciating some of its symmetry.

2.2 The Cohn-Elkies Linear Programming Bounds

2.3 A Word on Modular Forms

Chapter 3

A Roadmap to Constructing the Magic Function

We mentioned, in the introduction, that the scope of this project is to construct Viazovska's Magic Function in Lean and prove that it satisfies certain specific properties, such as satisfying the hypotheses of the Cohn-Elkies Linear Programming Bound. In this chapter, we will outline the steps we will take to achieve this goal. In particular, we will list all the conditions we need to prove that the Magic Function satisfies. Our approach will be to construct the Magic Function in terms of two intermediary functions. Proving it satisfies the necessary conditions will then be a matter of proving that these intermediary functions satisfy certain properties. We will list these properties as well.

3.1 On Schwartz Functions

3.2 The Cohn-Elkies Conditions

3.3 The Desired Properties of the Magic Function

3.4 The Magician's Assistants: a and b

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