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Viazovska's Magic Function in Dimension 8: A Formalisation in Lean

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Abstract

Hi

Acknowledgments

Plagiarism statement

The work contained in this thesis is my own work unless otherwise stated.

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Chapter 1

Introduction

On 5 July, 2022, in Helsinki, Finland, the International Mathematical Union announced the names of the four mathematicians who were to be awarded the Fields Medal, the most coveted prize in the world of mathematics: Hugo Duminil-Copin, June Huh, James Maynard and Maryna Viazovska. Duminil-Copin, Huh and Maynard received this most prestigious honour for making several outstanding contributions to their specific fields of expertise—respectively, statistical physics, geometric combinatorics, and analytic number theory. Viazovska, on the other hand, received the Fields Medal for more interdisciplinary achievements. Arguably the most remarkable of these was her solution to the sphere packing problem in dimension 8 [1]. It is difficult to place her solution in a specific mathematical field: what makes it so revolutionary is that it uses insights from Fourier analysis and the theory of modular forms to construct a special function—the Magic Function—that, in combination with a previous result by Cohn and Elkies [2], proves that the E_8 lattice packing is the densest possible sphere packing in $\mathbb{R}^8$. Very shortly afterwards, Cohn, Kumar, Miller, Radchenko and Viazovska were able to use similar ideas to prove that the Leech lattice packing is the densest possible sphere packing in $\mathbb{R}^{24}$ [3].

Before Viazovska's remarkable breakthrough, the optimal sphere packing density was only known in dimensions 1, 2 and 3 [4]. Furthermore, Thomas Hales' solution in dimension 3 [5] was lengthy and involved extensive computer-assisted calculations; in contrast, Viazovska's proof in dimension 8 is elegant and concise. Even before Viazovska was awarded the Fields Medal, her work received wide acclaim from eminent mathematicians across the world: Peter Sarnak described it as "stunningly simple, as all great things are," and Akshay Venkatesh remarked that her Magic Function is very likely "part of some richer story" that connects to other areas of mathematics and physics [6]. Viazovska's work is a truly remarkable achievement in modern mathematics, with its elegance coming from the manner in which the many pieces of the puzzle fit perfectly together. One of the goals of this project is to offer a detailed exposition of one of those pieces: the construction of the so-called 'Magic Function' in dimension 8.

1.1 The Sphere Packing Problem

The Sphere Packing problem is a classical optimsation problem in mathematics. The problem can be formulated as follows.

Problem 1.1.1 (The Sphere Packing Problem in Dimensinon n)**.** *Given some $n \in \mathbb{N}$, what is the densest possible non-overlapping arrangement of n-spheres of equal radius in $\mathbb{R}^n$?*

Despite its rather straightforward formulation, Problem 1.1.1 is notoriously difficult to solve. Indeed, one obvious question that arises when one looks at the problem statement is how one might define the concept of density. It turns out that the definition is slightly unwieldy, though introducing a periodicity assumption on the sphere packing whose density one wishes to find considerably simplifies this problem.

A key challenge in solving the sphere packing problem in dimension n is the fact that proceeding inductively is not always helpful: 'stacking' the optimal n-dimensional sphere packing onto itself is not guaranteed to yield the optimal sphere packing in $n+1$ dimensions. [4]. In fact, this appraoch is known to fail in dimensions as low as 10 [7]. This is not obvious, not least because the approach does, in fact, succeed in the visualisable dimensions of 1, 2 and 3.

The 1-dimensional case is uninteresting. Visually, one can easily see that the densest possible arrangement of disjoint intervals of the form $(-r, r)$ on the real line consists of intervals centred at all points $2rm$ for $m \in \mathbb{Z}$. Indeed, one can fix r to be $\frac{1}{2}$ by rescaling the real line. The optimal packing therefore consists of open intervals of unit length centred at points on the lattice $\mathbb{Z} \subset \mathbb{R}$.

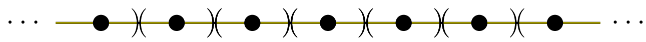

Figure 1.1: The $\mathbb{Z}$ lattice packing in dimension 1.

Rescaling gives us a powerful—if straightforward—simplification of the sphere packing problem where we can fix the radius of the spheres to a convenient value. Indeed, we only mention rescaling explicitly because it needs to be explicitly dealt with when formalising the problem. We will resume this discussion in . For now, we will take for granted the fact that rescaling does not affect the density of a sphere packing, meaning that we can talk about optimal sphere packings without worrying about the radius of the spheres in question. Bearing in mind that the spheres must all have the same radius, as per the statement of Problem 1.1.1, we will henceforth describe sphere packings simply by describing the points at which the spheres are centred.

ADD REFERENCE

The sphere packing problem in dimension 2 turns out to be more interesting. A reasonable strategy for finding the densest packing is to 'stack' the $\mathbb{Z}$ lattice packing from dimension 1 onto itself in some manner, but the question remains as to exactly how this should be done. 'Stacking' it onto itself would involve extending the lattice $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{R}^2$ into a lattice in $\mathbb{R}^2$ by extending the $\mathbb{R}$-basis $\{(1,0)\}$ of $\mathbb{R}$ (viewed as a subspace of $\mathbb{R}^2$) to an $\mathbb{R}$-basis of $\mathbb{R}^2$, and taking its $\mathbb{Z}$-span.

One natural way of doing this is to extend the lattice $\mathbb{Z} \subset \mathbb{R}$ to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ consisting of points with integer coordinates. This corresponds to the natural extension of $\{(1,0)\}$ to the standard $\mathbb{R}$-basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$. Unfortunately, this packing turns out to be sub-optimal,

1.2 The Work of Maryna Viazovska

1.3 The Formalisation Movement

1.4 Progress in Formalising Viazovska's Solution in Dimension 8

1.5 The Scope of this Project

Chapter 2

A Roadmap to Constructing the Magic Function

We mentioned, in the introduction, that the scope of this project is to construct Viazovska's Magic Function in Lean and prove that it satisfies certain specific properties, such as satisfying the hypotheses of the Cohn-Elkies Linear Programming Bound. In this chapter, we will outline the steps we will take to achieve this goal. In particular, we will list all the conditions we need to prove that the Magic Function satisfies. Our approach will be to construct the Magic Function in terms of two intermediary functions. Proving it satisfies the necessary conditions will then be a matter of proving that these intermediary functions satisfy certain properties. We will list these properties as well.

2.1 On Schwartz Functions

2.2 The Cohn-Elkies Conditions

2.3 The Desired Properties of the Magic Function

2.4 The Magician's Assistants: a and b

Bibliography

[1] M. S. Viazovska. The sphere packing problem in dimension 8. *Annals of Mathematics*, 185 (3):991–1015, 2017.

[2] H. Cohn and N. Elkies. New Upper Bounds on Sphere Packings I. *Annals of Mathematics*, 157(2):689–714, 2003.

[3] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska. The sphere packing problem in dimension 24. *Annals of Mathematics*, 185(3):1017–1033, 2017.

[4] H. Cohn. The work of Maryna Viazovska. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 82–105. EMS Press, 2023. Presented at ICM, July 6–14, 2022.

[5] T. C. Hales. A Proof of the Kepler Conjecture. *Annals of Mathematics*, 162(3):1065–1185, 2005.

[6] E. Klarreich. Sphere Packing Solved in Higher Dimensions. *Quanta Magazine*, March 2016.

[7] H. Cohn. A conceptual breakthrough in sphere packing. *Notices of the American Mathematical Society*, 64(02):102–115, Feb. 2017.