21-603: Model Theory I

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Chapter 1

An Introduction to Model Theory

Model Theory is one of the two main branches of mathematical logic, alongside Set Theory. We can view Model Theory as a translation of algebra to the world of set theory, or as a *generalisation* of algebra, with many applications to algebra. Indeed, it is possible to view model theory as a more specialised version of category theory, at least in its power to deal with generality. We can prove tremendously deep theorems about algebra, or other fields of maths, purely using model theory.

Before we can talk about model theory in any detail, we need to recall a few notions from the world of logic.

1.1 A Crash Course on First-Order Logic

1.1.1 Languages and Structures

We begin by recalling the notion of a first-order language.

Definition 1.1.1 (Language). A **language** is a disjoint union

$$\mathbf{L} = \mathbf{F} \cup \mathbf{R} \cup \mathbf{C}$$

of countable sets of symbols, where

- $\mathbf{F}$ is a set of function symbols

- **R** is a set of relation symbols
- **C** is a set of constant symbols

Next, we recall the notion of a structure in a language.

Definition 1.1.2 (Structure). Let **L** be a structure. An **L-structure** is a tuple

$$M = \langle U; F, R, C \rangle$$

consisting of a non-empty set U and functions, relations, and constants that live in **F**, **R** and **C** respectively. U known as the **universe** of M, and is denoted by $|M|$. The function, relation and constant symbols of M are denoted F^M, R^M and C^M respectively.

Any function or relation in a structure has an **arity**, which is informally the number of arguments it takes. An important fact to note is that arities are not a feature of functions and relations themselves, but of their corresponding *symbols*. In other words, **arity is a syntactic notion**. Semantically speaking, when we seek an interpretation of a function symbol of some arity n, we are forced to limit our search to the set of functions from U^n to U.

We now describe the notion of structure-preserving bijections, known as isomorphisms.

MAKE MORE PRECISE - FOLLOW DEF 2.9 IN BOOK

Definition 1.1.3 (Isomorphism). Let **L** be a language and let M, N be **L**-structures. We say that a function $g : |M| \to |N|$ is an **isomorphism** if

1. g is a bijection
2. "g commutes with functions"
3. "g commutes with relations"
4. "g agrees on constants"

where the double-quotes for the second and third point above refer to the fact that we implicitly require an equality of arities condition before we can talk about composing isomorphisms with multi-ary functions.

We are now ready to define submodels.

Definition 1.1.4 (Submodel). Let M, N be $\mathbf{L}$-structures. We say that M is a submodel of N, denoted $M \subseteq N$, if

1. $|M| \subseteq |N|$.
2. For all function symbols $F(x_1, \ldots, x_n)$, the interpretation in M agrees with the interpretation in N.
3. For all relation symbols $R(a_1, \ldots, a_n)$, the interpretation in M agrees with the interpretation in N.
4. For all constant symbols C, the interpretation in M agrees with the interpretation in N.

In particular, a submodel of a model is also a model. For instance, if G is a group, ie, a model of the group axioms, then any submodel of G is, in fact, a subgroup of G, and a group in its own right (in that it again models the group axioms).

We now talk about more syntactic elements of a language.

1.1.2 Syntax: Terms, Formulae, Sentences and Theories

Definition 1.1.5 (Terms). Let $\mathbf{L}$ be a language. $\mathbf{Term}(\mathbf{L})$ is defined to be the minimal set of finite sequences of symbols[a] from

$$\{(,),[,]\} \cup \mathbf{C} \cup \mathbf{F} \cup \{x_1, x_2, x_3, \ldots\}$$

satisfying the following rules:

1. Every constant symbol is a term.
2. Every variable is a term.
3. For all n-ary functions f and n terms $t_1, \ldots, t_n$, $f(t_1, \ldots, t_n)$ is a term.
4. Every term arises in this way.

[a] Here, the set of variable symbols is countable, but we might, on occasion, need uncountably many variable symbols

Remark. In other words, the elements of $\mathbf{Term}(\mathbf{L})$ are exactly the constants, variables, and functions of such (constants and variables).

Recall, from Definition 1.1.3, that isomorphisms are, in particular, bijections that agree on constants. It is possible to show that they also agree on the interpretations of terms in models. We will show this later.

In similar fashion, we can define the formulae in a language.

Definition 1.1.6 (Formulae). Let $\mathbf{L}$ be a language. $\mathbf{Fml}(\mathbf{L})$ is defined to be the minimal set of finite sequences of symbols from

$$\mathbf{Term}(L) \cup \{\wedge, \vee, \rightarrow, \neg, \ldots\} \cup \{=\} \cup \{\forall, \exists\}$$

satisfying the following rules:

1. For all $\tau_1, \tau_2 \in \mathbf{Term}(\mathbf{L})$, $\tau_1 = \tau_2$ is a formula.
2. For all n-ary relations R and n terms $t_1, \ldots, t_n$, $R(t_1, \ldots, t_n)$ is a formula.
3. For all connectives $\star$ and formulae Φ and Ψ, $\Phi \star \Psi$ is a formula.
4. For a quantifier Q, variable x and formula $\varphi(x)$, $Qx\,(\varphi(x))$ is a formula.
5. Every formula arises in this way.

Formulae consisting only of a single relation symbol (including formulae that only consist of an equality) are called **atomic formulae**. The atomic formulae of $\mathbf{L}$ are denoted $\mathbf{AFml}(\mathbf{L})$.

For a formula φ, denote by $\mathbf{FV}(\varphi)$ the set of free variables of φ. It is sometimes useful to distinguish those formulae in a language that contain no free variables.

Definition 1.1.7. Define the set of **sentences** of a language $\mathbf{L}$ to be

$$\mathbf{Sent}(\mathbf{L}) := \{\varphi \in \mathbf{Fml}(\mathbf{L}) \mid \mathbf{FV}(\varphi) = \emptyset\}$$

Essentially, a formula with no free variables is called a sentence. A theory is simply a set of sentences.

Definition 1.1.8 (Theory). Let $\mathbf{L}$ be a language. An $\mathbf{L}$-theory is any subset $T \subseteq \mathbf{Sent}(\mathbf{L})$.

There are many well-known theories in mathematics. The most familiar examples come from algebra.

Example 1.1.9 (The Theory of Fields). The theory of fields is a first-order theory in the language of fields. This is a language with function symbols $+, \times, ^{-1}$, relation symbol $=$, and constant symbols 0 and 1. It also has other first-order symbols, such as quantifiers, connectives and punctuation, but we ignore these (indeed, we will always take for granted that these exist). The theory of fields consists of the following sentences in this language:

1. $\forall x \forall y \forall z \, [(x+y)+z = x+(y+z)]$
2. $\forall x \forall y \, [x+y = y+x]$
3. $\forall x \exists y \, [x+y=0] \wedge \forall x \, [x+0=x]$
4. $\forall x \, [x \neq 0 \rightarrow \exists y \, [xy=1]]$, where $\neq$ is the obvious shorthand
5. $\forall x \, [x \times 1 = x]$
6. $\forall x \forall y \forall z \, [x \times (y+z) = x \times y + x \times z]$

Collectively, these sentences are known as the **theory of fields**.

There are numerous well-known examples of fields. There is the question of how we can formally describe what it means for some structure, such as the rational numbers, to *satisfy* the above sentences. To that end, we introduce the semantics of interpretation, namely, the notion of **satisfaction**.

1.1.3 Semantics: Satisfaction

We begin with the most important definition of this entire section.

Definition 1.1.10 (Satisfaction - Sentences). Let $\mathbf{L}$ be a language and M an $\mathbf{L}$-structure. For any $\varphi \in \mathbf{Fml}(\mathbf{L})$, we say that M **models** φ, denoted $M \models \varphi$, if `sorry`

Remark. We note that, (a) $M \vDash \forall x \varphi(x) \iff N \vDash \neg \exists x \neg \varphi(x)$ and (b) $M \vDash \exists x \varphi(x) \iff M \vDash \neg \forall x \neg \varphi(x)$. This is worthy of note as for proofs, we will often need to induct on (the number of symbols in) formulas - just as we only need the logical connectives $\neg, \vee$ to get the rest, we only need to check satisfcation for a single quantifier and $\neg$.

We can define satisfaction for theories in the obvious way.

Definition 1.1.11 (Satisfaction - Theories). Let $\mathbf{L}$ be a language and M an $\mathbf{L}$-structure. Given an $\mathbf{L}$-theory T, we say that $M \models T$ if $M \models \psi$ for all $\psi \in T$.

We are now ready to state a simple-sounding but rather non-trivial result.

Lemma 1.1.12. *Suppose M and N are both $\mathbf{L}$-structures. If $M \subseteq N$, then for all $\tau \in \mathbf{Term}(\mathbf{L})$, $\tau^M[a] = \tau^N[a]$, where $a \in |M| \times \cdots \times |M|$ and $\tau^M[a]$ and $\tau^N[a]$ denote interpretations of τ in M and N with the variables all being interpreted as the components of a.*

We do not prove this result. It is not difficult.

Next is a less trivial result.

Lemma 1.1.13. *If $M \subseteq N$, then $M \models \varphi$ if and only if $N \models \varphi$ for all quantifier-free formulae φ.*

Going back to Example 1.1.9, we can now say the following.

Example 1.1.14 (Models of the Theory of Fields). Recall the theory of fields, seen in Example 1.1.9. It can be shown that the following structures in the language of fields satisfy the theory of fields:

$$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/7\mathbb{Z}$$

A model of the theory of fields is known simply as a **field**. We all know that this is an incredibly rich theory. It might have been even richer if Évariste Galois had had better control over his faculties...

More generally, we can define the theory *of a model* in the natural way.

Definition 1.1.15 (Theory of a Model). Let M be an $\mathbf{L}$-structure. We define the **theory**

of M, denoted $\mathbf{Th}(M)$, to be the sentences satisfied by M. That is,

$$\mathbf{Th}(M) := \{\varphi \in \mathbf{Sent}(\mathbf{L}) \mid M \models \varphi\}$$

This simple but powerful definition allows us to express many ideas model-theoretically. We will see this when we discuss models of Peano arithmetic.

1.1.4 Elementary Equivalence

Recall the definition of an isomorphism of structures (Definition 1.1.3). We can regard isomorphism as a *syntactic* notion of equivalence of structures. In this subsection, we explore a *semantic* notion of equivalence of models.

Definition 1.1.16 (Elementary Equivalence)**.** Let $\mathbf{L}$ be a langauge and let M and N be $\mathbf{L}$-structures. We say M **is elementarily equivalent to** N if for every sentence $\varphi \in \mathbf{Sent}(\mathbf{L})$, we have that M satisfies φ if and only if N satisfies φ. We denote this

$$M \equiv N$$

We can now relate isomorphisms to equivalence in the following manner.

Theorem 1.1.17. *Let* $\mathbf{L}$ *be a language and let* M *and* N *be* $\mathbf{L}$*-structures. If* $f : M \cong N$*, then for every* $\varphi(x_1, \ldots, x_n) \in \mathbf{Fml}(\mathbf{L})$ *and every* $a_1, \ldots, a_n \in |M|$*, we have that*

$$M \models \varphi[a_1, \ldots, a_n] \iff N \models \varphi[f(a_1), \ldots, f(a_n)]$$

Proof. Fix a formula $\varphi(x_1, \ldots, x_n) \in \mathbf{Fml}(\mathbf{L})$ with n free variables. We prove the result by performing induction[1] on φ.

Suppose φ is atomic. Then, there are two cases.

- $\varphi(x_1, \ldots, x_n)$ is of the form $\tau_1(x_1, \ldots, x_n) = \tau_2(x_1, \ldots, x_n)$ for terms $\tau_, \tau_2 \in \mathbf{Term}(\mathbf{L})$. In this case, it is possible to show, by cases on what τ_1 and τ_2 can be (see Definition 1.1.5) that f

[1] That is, cases

is compatible with φ.

- φ is of the form $R(x_1, \ldots, x_n)$ for some $R \in \mathbf{R}(\mathbf{L})$. This is true by definition of an isomorphism (see Definition 1.1.3).

Suppose, now, that φ is not atomic. It is enough to show that the result shows if φ is of the form $\psi_1(x_1, \ldots, x_n) \wedge \psi_2(x_1, \ldots, x_n)$, $\psi_1(x_1, \ldots, x_n) \vee \psi_2(x_1, \ldots, x_n)$, $\neg\psi(x_1, \ldots, x_n)$, and $\forall x_1 \forall x_2 \ldots \forall x_n \psi(x_1, \ldots, x_n)$, as these would be adequate.

- φ is of the form $\psi_1(x_1, \ldots, x_n) \wedge \psi_2(x_1, \ldots, x_n)$. This is immediate from the definition of satisfaction: for all $a_1, \ldots, a_n \in |M|$, we have that

$$\begin{aligned} M \models \varphi[a_1, \ldots, a_n] &\iff M \models \psi_1[a_1, \ldots, a_n] \text{ and } M \models \psi_2[a_1, \ldots, a_n] \\ &\iff N \models \psi_1[f(a_1), \ldots, f(a_n)] \text{ and } N \models \psi_2[f(a_1), \ldots, f(a_n)] \\ &\iff N \models \varphi[f(a_1), \ldots, f(a_n)] \end{aligned}$$

  as required.

- φ is of the form $\psi_1(x_1, \ldots, x_n) \vee \psi_2(x_1, \ldots, x_n)$. Similar.

- φ is of the form $\neg\psi(x_1, \ldots, x_n)$. sorry

- φ is of the form $\exists x_1, \psi(x_1, \ldots, x_n)$.[2] Fix $a_1, \ldots, a_n \in |M|$. Then,

$$\begin{aligned} M \models \varphi[a_1, \ldots, a_n] &\iff M \models \exists x \varphi[x, a_2, \ldots, a_n] \\ &\iff \text{There is some } a \in |M| \text{ such that } M \models \varphi[b, a_2, \ldots, a_n] \\ &\iff \text{There is some } a \in |M| \text{ such that } N \models \varphi[f(b), f(a_2), \ldots, f(a_n)] \\ &\iff \text{There is some } b \in |N| \text{ such that } N \models \varphi[b, f(a_2), \ldots, f(a_n)] \\ &\iff N \models \exists y \varphi[y, f(a_2), \ldots, f(a_n)] \end{aligned}$$

  where we note that the '$\implies$' direction of the fourth $\iff$ comes from taking $b = f(a)$ and the '$\impliedby$' direction comes from the fact that f is surjective, meaning that we can take a to be any element of $|M|$ such that $f(a) = b$.

[2] This is enough because you can induct on the number of free variables, with exactly this being the inductive step.

We can conclude by noting that the above cases are adequate. See sorry. □

Corollary 1.1.18. *Let* $\mathbf{L}$ *be a language and let* M *and* N *be* $\mathbf{L}$*-structures. If* $M \cong N$*, then* $M \equiv N$*.*

Proof. Let $f : M \xrightarrow{\sim} N$ be an isomorphism from M to N. sorry □

We warn the reader that this implication is strict.

Warning. The notion of isomorphism is finer than elementary equivalence!

The reason for this is that in elementary equivalence, we do not insist that the interpretations in M and N are the same! That is, M could be a sub-structure of N in which the interpretations used for elementary equivalence *differ*. In the case of isomorphism, we insist that the interpretations are the same (or rather, bijective with f).

We end our discourse on first-order logic by briefly discussing the theory of deduction and proof.

1.1.5 Deduction and Proof

Let $\mathbf{L}$ be a language. Recall that $\mathbf{Sent}(\mathbf{L})$ is the set of *sentences* in $\mathbf{L}$. Throughout this subsection, fix a theory $T \subseteq \mathbf{Sent}(\mathbf{L})$.

Definition 1.1.19 (Provability)**.** We say a sentence $\varphi \in \mathbf{Sent}(\mathbf{L})$ is **provable from** T, denoted $T \vdash \varphi$, if there exists a sequence $\langle \varphi_1, \dots, \varphi_n \rangle$ of $\mathbf{L}$-sentences such that $\varphi_n = \varphi$ and for all $i < n$, either $\varphi_i \in T$ or φ_i is obtained from $\langle \varphi_1, \dots, \varphi_{i-1} \rangle$ via the standard deduction rules of first-order logic, namely, Modus Ponens and Generalisation.

We can say something about what makes T a "sensible" set from which to deduce things.

Definition 1.1.20 (Consistency)**.** We say that T is **consistent** if there is no $\varphi \in \mathbf{Sent}(\mathbf{L})$ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Consistency is equivalent to model existence.

Theorem 1.1.21 (Gödel-Henkin). *T is consistent if and only if there is an* **L**-*structure M such that $M \models T$.*

We do not prove this theorem here, but we will make extensive use of it.

We end by recalling the compactness theorem for first-order logic.

Theorem 1.1.22 (Compactness, Gödel-Malcev). *If for any finite $T_0 \subseteq T$, there is a*

We now discuss the *size* of a model and a theory.

1.2 Cardinality and Categoricity

Throughout this section, let **L** be a language.

Definition 1.2.1 (Cardinality of a Structure). Let M be an **L**-structure. The **cardinality of M**, denoted $\|M\|$, is the cardinality of its universe $|M|$.

We can also talk about the size of a theory.

Definition 1.2.2 (Categoricity of a Theory). Let T be an **L**-theory. Suppose $\lambda \geq |L|$ is a cardinal. We say that T **is λ-Categorical**, or that T **is categorical in** λ, if for all **L**-structures M and N such that $M, N \models T$ and $\|M\| = \|N\| = \lambda$, we have that $M \cong N$.

Categoricity brings up interesting questions, such as the so-called *spectrum problem*.

1.2.1 The Spectrum Problem

The spectrum of a theory with respect to a cardinal is defined as follows.

Definition 1.2.3 (Spectrum). Let T be an **L**-theory and let λ be a cardinal. We define the **spectrum of T with respect to λ** to be

$$I(\lambda, T) := \left|\left\{ {}^{M}/_{\cong} \mid M \models T \text{ and } \|M\| = \lambda \right\}\right|$$

ie, $I(\lambda, T)$ denotes the number of isomorphism classes of models of T of cardinality λ.

It is obvious, from Definition 1.2.2, that a theory T is λ-categorical if and only if $I(\lambda, T) = 1$. However, if T is not λ-categorical, then it is, in general, quite difficult to compute $I(\lambda, T)$. In fact, for most theories and cardinals, computing the spectrum is an *open problem*, referred to as the **spectrum problem**.

There has been some progress on this problem. Steinitz made the following determinations.

Theorem 1.2.4 (Steinitz). *Let* **L** *be the language of fields, and let T be the theory of algebraically closed fields of characteristic p (obtained by adding the appropriate sentences to the Theory of Fields encountered in Example 1.1.9). Then,*

1. $I(\aleph_0, T) = \aleph_0$.
2. *For all* $\lambda > \aleph_0$, $I(\lambda, T) = 1$.

The spectrum problem has been worked on by some of the most eminent logicians of our time, including Rami's advisor, Saharon Shelah, who proved a famous conjecture by Morley (1965). More on the Spectrum Problem can be found on the associated Wikipedia page, and while this is not the most authoritative source, its contents are nonetheless interesting.

Morley also proved a famous conjecture by Łos from the 1950s, which since became known as Morley's Categoricity Theorem.

Theorem 1.2.5 (Morley's Categoricity Theorem, Morley 1965). *Let T be a theory in a language* **L**. *Assume that* $|\mathbf{L}| \leq \aleph_0$. *If* $\exists\lambda > \aleph_0$ *such that T is λ-categorical, them* $\forall\lambda > \aleph_0$, *$T$ is λ-categorical.*

One of our objectives in this course is to prove Morley's Categoricity Theorem.

As a side note, Morley was initially a PhD student of Saunders MacLane's at the University of Chicago. Morley didn't initially finish his PhD, to the point of losing his stipend at Chicago, but somehow landed a job at Berkeley, where he proved this famous theorem. MacLane, a staunch category theorist, didn't believe Morley's work was quite enough to merit a PhD; nevertheless, after being persuaded by the then-nascent (and very excited) model theory community, he eventually relented and awarded Morley his degree.

Here, we end our discussion on the spectrum problem. Before proceeding further, we recall the basics of cardinal arithmetic.

1.2.2 Cardinal Arithmetic

We begin by introducing notation.

Notation. We denote by

- **ZF** the Zermelo-Fraenkel Axioms of Set Theory
- **AC** the Axiom of Choice
- **ZFC** the Zermelo-Fraenkel Axioms with the Axiom of Choice

We denote cardinality of a set A by $|A|$ or $\text{card}(A)$ and write $|A| = |B|$ if and only if there is a bijection from A to B. Informally, a **cardinal** is a measure of cardinality. That is, a set λ is a cardinal if $\lambda = |A|$ for some set A. We denote by $\aleph_0$ the cardinal of the natural numbers, which we will denote ω in any cardinal- or ordinal-theoretic context.

There are more precise ways in which we can define the notions of ordinals and cardinals. We do not do this here, but we mention that there is an appendix in Rami's book and several sections in my undergrad logic lecture notes that discuss this.

Add references

Definition 1.2.6 (Cardinal Arithmetic)**.** Let λ, μ be cardinals, with $\lambda = |A|$ and $\mu = |B|$. We denote

$$\lambda + \mu := \texttt{sorry}$$

$$\lambda \cdot \mu := |A \times B|$$

The following is a famed theorem of Tarski, a direct consequence of which is precisely the fundamental theorem of cardinal arithmetic.

Theorem 1.2.7 (Tarski). *We can make the following deduction:*

$$\mathbf{ZF} \vdash (\mathbf{AC} \leftrightarrow \forall A,\ |A| \geq \aleph_0 \to |A \times A| = |A|)$$

Equivalently,

$$\mathbf{ZF} \vdash (\mathbf{AC} \leftrightarrow \forall A,\ \lambda \geq \aleph_0 \to \lambda \cdot \lambda = \lambda)$$

The fundamental theorem of cardinal arithmetic, which states that $|\omega \times \omega| = |\omega|$, is clearly just the specialisation of the above result to the case where $\lambda = \aleph_0$.

There is another fact that will be important for our purposes.

Theorem 1.2.8. *For infinite cardinals $\lambda, \mu \geq \aleph_0$, we have*

$$\lambda \cdot \mu = \max(\lambda, \mu) = \lambda + \mu$$

The reason for discussing cardinal arithmetic is that we can exploit it to prove the existence of submodels of specific cardinalities.

1.3 A Word on Submodels

Fix a language $\mathbf{L}$.

1.3.1 Submodel Existence

We begin by defining the cardinality of a structure.

Definition 1.3.1 (Cardinality of a Structure). Let N be a $\mathbf{L}$-structure. We define $\text{card}(N)$ to be the cardinality of the union of $\mathbf{F}^N(\mathbf{L}) \cup \mathbf{C}^N(\mathbf{L}) \cup \mathbf{R}^N(\mathbf{L}) \cup |N|$.

We begin with the famed submodel theorem.

Theorem 1.3.2 (The Submodel Theorem [**MOAB**]). *Let M be a $\mathbf{L}$-structure. Define $\lambda := |\mathbf{L}| + \aleph_0$. If $A \leq |M|$, then there exists a substructure $N \leq M$ such that*

(a) $\mathrm{card}(N) \geq A$

(b) $\mathrm{card}(|N|) \leq |A| + \lambda$

Proof. By recursion on $n < \omega$, define sets $\{B_n \subseteq |M| \mid n < \omega\}$ such that

1. $B_0 = \left\{c \in \mathbf{C}^M \mid c \text{ is a constant symbol of } c\right\} \cup A$

2. If $n < \omega$, then $|B_n| \leq |A| + \lambda$

3. For all $n < \omega$, define $B_{n+1} := \left\{F^M(\bar{a}) \mid \bar{a} \in B_n\right\} \cup B_n$

This is enough: if we have such a sequence of B_n, then we could take $B := \bigcup_{n<\omega} B_n$ and define $N := \langle B, F^M, R^M, C^M \rangle$. We can show that this satisfies the desired conditions.

(a) sorry

(b) sorry

Given these, all that remains now is to show that this is possible. sorry □

Finish using text-book proof

1.3.2 Elementary Submodels

Recall the definition of elementary substructures (sorry). In this subsection, we define an analogous notion for models.

Definition 1.3.3 (Elementary Submodels). Let M, N be $\mathbf{L}$-structures. We say that M **is an elementary submodel of** N, denoted $M \preceq N$, if

1. $M \subseteq N$
2. $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[a_1, \ldots, a_n]$ for every $\varphi \in \mathbf{Fml}(\mathbf{L})$ and $a_1, \ldots, a_n \in |M|$.

We can relate this to the notion of elementary substructures in the following manner.

Theorem 1.3.4 (Tarski-Vaught 1956). *Let M, N be* **L***-structures with $M \subseteq N$. If $M \preceq N$, then $M \equiv N$.*

The converse is not true.

Counterexample 1.3.5. `sorry`

1.3.3 The Tarski-Vaught Test

In this subsection, we explore a monumental result by Tarski and Vaught that gives a sufficient and necessary condition for a substructure to be elementary.

We begin by introducing some notation.

Local Notation. Denote by $\star_\psi$ the statement

$$\text{For all } a_1, \ldots, a_n \in |M|, \qquad M \models \psi[a_1, \ldots, a_n] \iff N \models \psi[a_1, \ldots, a_n] \tag{1.3.1}$$

for some $\psi \in \mathbf{Fml}(L)$.

Next, we note a fact about substructures.

Lemma 1.3.6. *Let N be an L-structure and let $M \subseteq N$ be a substructure of N. Then, $\star_\psi$ holds for all quantifier-free formulae $\psi \in \mathbf{Fml}(L)$.*

Proof. `sorry` □

Theorem 1.3.7 (The Tarski-Vaught Test). *Let M, N be L-structures with $M \subseteq N$. Then, the following are equivalent.*

1. $M \preceq N$
2. *If, for every $\varphi(y, x_1, \ldots, x_n) \in \mathbf{Fml}(L)$ and $a_1, \ldots, a_n \in |M|$,*

   $$N \models \exists y\, \varphi(y, a_1, \ldots, a_n)$$

   then there is some $b \in |M|$ such that $N \models \varphi[b, a_1, \ldots, a_n]$

Remark. We can see this as 'a more "algebraic" notion of being a submodel.' What is a formula? A list of quantifiers, connectives, etc. - for example, we can think of polynomials in several variables, which we wish to solve. If $\varphi(y, x)$ is a set of finitely many equations (which we wish to solve), we can see this result as telling us that if there exists a solution to the system $y \in N$, there is also a b in the substructure M which also solves the same system.

Proof of Theorem 1.3.7. $\underline{1 \implies 2.}$ Fix $\varphi(y, x_1, \dots, x_n) \in \mathbf{Fml}(L)$ and $a_1, \dots, a_n \in |M|$. Suppose

$$N \models \exists y\, \varphi(y, a_1, \dots, a_n)$$

Since $M \preceq N$, by definition of satisfaction, we know that

$$M \models \exists y\, \varphi(y, a_1, \dots, a_n)$$

This tells us that there is $b \in |M|$ witnessing φ, meaning that

$$M \models \exists y\, \varphi(b, a_1, \dots, a_n)$$

Then, since $M \preceq N$, we have that

$$N \models \exists y\, \varphi(b, a_1, \dots, a_n)$$

as required.

$\underline{2 \implies 1.}$ We show, by induction on $\varphi(y, x_1, \dots, x_n) \in \mathbf{Fml}(L)$, that $\star_\psi$ holds for all $\psi \in \mathbf{Fml}(L)$. Recall, from Lemma 1.3.6, that $\star_\psi$ does hold for quantifier-free formulae ψ. In particular, it holds for atomic formulae. We can now consider the different possible cases on ψ.

1. $\underline{\psi \text{ is of the form } \psi_1 \wedge \psi_2.}$

   `sorry`

2. $\underline{\psi(x_1, \dots, x_n) \text{ is of the form } \exists y\, \varphi(y, x_1, \dots, x_n).}$

   Assume that $M \models \psi[a_1, \dots, a_n]$. Then, by assumption, there is some $b \in |M|$ such that $M \models \varphi[b, a_1, \dots, a_n]$. Then, $N \models \varphi[b, a_1, \dots, a_n]$ because $b \in |M| \subseteq |N|$. Thus,

$$N \models \exists y\, \varphi(y, a_1, \dots, a_n)$$

ie,

$$N \models \psi[a_1, \ldots, a_n]$$

By $\star_\psi$, $b \in |M|$ □

1.3.4 Chains of Substructures

Throughout this subsection, fix a linear order $(\mathcal{I}, \leq)$.

Definition 1.3.8 (Chain). Let $\{M_i \mid i \in \mathcal{I}\}$ be **L**-structures. We say they form a **chain** if for all $i_1, i_2 \in \mathcal{I}$, if $i_1 < i_2$ then $M_{i_1} \subseteq M_{i_2}$.

Local Notation. For the remainder of this subsection, fix a chain $\{M_i \mid i \in \mathcal{I}\}$. Define

$$N := \bigcup_{i \in \mathcal{I}} M_i$$

It is easy to show that N is an **L**-structure as well. Moreover, $M_i \subseteq N$ for all $i \in \mathcal{I}$.

We can ask ourselves a natural question: suppose T is a theory in **L** and that $\forall i \in \mathcal{I}$, $M_i \models T$. Is it necessarily true that $N \models T$ as well?

The answer turns out to be no when $|I| \geq \aleph_0$.

Counterexample 1.3.9. Take $I = \omega$ and `sorry`

We can instead define elementary chains, which are the analogues of chains for elementary substructures.

Definition 1.3.10 (Elementary Chain). We say that $\{M_i \mid i \in \mathcal{I}\}$ form an **elementary chain** if for all $i_1, i_2 \in \mathcal{I}$, if $i_1 < i_2$ then $M_{i_1} \preceq M_{i_2}$.

We can apply Theorem 1.3.7 to prove an important result on elementary chains.

Theorem 1.3.11 (Tarski-Vaught Chain Theorem). *Assume $\{M_i \mid i \in I\}$ is an elementary chain. Let N be an L-structure. Then, the L-structure*

$$N = \bigcup_{i \in I} M_i$$

satisfies the property that for all $i \in I$, $M_i \preceq N$.

Proof. Since we already know that $M_i \subseteq N$, it is enough to show that for every $\psi(x_1, \ldots, x_n) \in$ **Fml**(L) and every $i \in I$, $\star_\psi$ holds (where $\star_\psi$ is the formula defined as local notation in the previous subsection, considered along with the substructure M_i of N).

We know that for all $i \in I$, since $M_i \subseteq N$, $\star_\psi$ holds for atomic formula. We induct on logical connectives and quantifiers to exhaustively prove that the statement $\forall i \in I, \star_\psi$ is true. We only do a few cases explicitly.

1. $\psi(x_1, \ldots, x_n)$ is of the form $\neg\varphi(x_1, \ldots, x_n)$.

   Then, for all $i \in I$, $M_i \models \psi[a_1, \ldots, a_n]$ iff $M_i \not\models \varphi[a_1, \ldots, a_n]$. By the induction hypothesis that $\forall i \in I, \star_\phi$ holds for all formulae ϕ with fewer quantifiers than ψ, we can conclude that

   $$M_i \not\models \varphi[a_1, \ldots, a_n] \iff N \not\models \varphi[a_1, \ldots, a_n]$$

   This tells us that $N \models \psi[a_1, \ldots, a_n]$ for every interpretation a_i of x_i.

2. $\psi(x_1, \ldots, x_n)$ is of the form $\exists y\, \varphi(y, x_1, \ldots, x_n)$.

   Fix $i \in I$. Then, $M_i \models \psi[a_1, \ldots, a_n] \implies M_i \models \exists y\, \varphi(y, a_1, \ldots, a_n)$. Then, there is some $b \in |M_i|$ such that $M_i \models \varphi[b, a_1, \ldots, a_n]$. This tells us, by the induction hypothesis, that $N \models \varphi[b, a_1, \ldots, a_n]$ for some $b \in |M_i| \subseteq |N|$. Thus, $N \models \exists y\, \varphi(y, x_1, \ldots, x_n)$, meaning $N \models \psi[a_1, \ldots, a_n]$ as required.

We can argue similarly for other quantifiers and connectives. □

Corollary 1.3.12. *If T is an L-theory, then if $M_i \models T$ for all $i \in I$ then $N \models T$ as well.*

We don't prove this corollary here.

We finally mention an additional nuance. Suppose $i \in I$ satisfies $a_1, \ldots, a_n \in |M_i|$ and $N \models \psi[a_1, \ldots, a_n]$. Say that ψ is of the form $\exists y\, \varphi(y, x_1, \ldots, x_n)$. Then, by the definition of satisfaction, we know that there is some $b \in |N|$

Definition 1.3.13 (directed poset). Let $(I, <)$ be a poset, we say that I is "directed" if $\forall i, j \in I$ there exists $b \in I$ such that $i \leq k, j \leq k$.

Remark. This theorem can be extended - we don't need a linearly ordered set, possibly a directed poset would suffice? (We won't see this in this course.)

1.3.5 Restrictions and Expansions

Before going any further, we will need to define a central tool: restrictions and expansions. Throughout this subsection, fix a language L and an L-structure M.

Definition 1.3.14 (Restriction/Expansion). Let $L_1 \subseteq L$, so that L contains relations, functions, and constants $\mathbf{R}(L_1)$, $\mathbf{F}(L_1)$, and $\mathbf{C}(L_1)$. The **restriction of** M **to** L is the L_1-structure

$$M|_{L_1} := \langle |M| \,, \mathbf{R}^M(L_1) \,, \mathbf{F}^M(L_1) \,, \mathbf{C}^M(L_1) \rangle$$

Dually, we say that M **is the expansion of** $M|_{L_1}$ **to** L.

A good example of this is to model-theoretically encode the fact that every field is also an abelian group (additively).

Example 1.3.15 (Restriction: Fields to Abelian Groups). Let L be the language of fields and let L_1 be the language of (additively expressed) abelian groups. Then, if

$$M = \langle \mathbb{Q}, +, \cdot, 0, 1 \rangle$$

then its restriction to L_1 is

$$M|_{L_1} = \langle \mathbb{Q}, +, 0 \rangle$$

1.3.6 The Löwenheim-Skolem Theorems

Throughout, let L be a language.

We begin with the downwards theorem, which gives us substructures with control over cardinality. It looks similar to the Submodel Theorem (Theorem 1.3.2).

Theorem 1.3.16 (Downwards Löwenheim-Skolem-Tarski Theorem). *Let M be an L-structure. Define $\lambda := |L| + \aleph_0$. For all $A \subseteq |M|$, there is some $N \preceq M$ with $|N| \supseteq A$ and $||N|| \leq |A| + \lambda$, where $||N||$ refers to the cardinality of the universe of N.*

Recall that Theorem 1.3.2 gives us the existence of $N \leq M$ with the desired properties. The difference is that in Theorem 1.3.16, we have elementarity.

Proof of Theorem 1.3.16. Fix $A \subseteq |M|$. Fix $\varphi(y, x_1, \ldots, x_n) \in \mathbf{Fml}(L)$. Fix a well-ordering $\leq$ of $|M|$, which we pick using the Axiom of Choice.

We define the function $G_\varphi : |M| \times \cdots \times |M| \to |M|$ as follows: for all $b_1, \ldots, b_n$, define

$$G_\varphi(b_1, \ldots, b_n) = \begin{cases} \min_{\leq} |M| & \text{if } M \not\models \exists y\, \varphi(y, x_1, \ldots, x_n) \\ \min_{\leq} \{a \in |M| \mid M \models \varphi[a, b_1, \ldots, b_n]\} & \text{if } M \models \exists y\, \varphi(y, x_1, \ldots, x_n) \end{cases} \tag{1.3.2}$$

Thus, for every $b_1, \ldots, b_n \in |M|$, G_φ gives the least element $a \in |M|$ such that $\varphi[a, b_1, \ldots, b_n]$ is satisfied (and returns a junk value of there is no such element).

We now augment our language L in the following manner. Define

$$L_1 := L \cup \{G_\varphi(x_1, \ldots, x_n) \mid \varphi(y, x_1, \ldots, x_n) \in \mathbf{Fml}(L)\} \tag{1.3.3}$$

That is, we create L_1 by adding to L all the constant symbols that correspond to `sorry`.

Let M_1 be the expansion of M to L (cf. Definition 1.3.14). Observe that

$$|L_1| \leq |L| + |\mathbf{Fml}(L)| \leq \lambda + \aleph_0 \cdot \lambda = \lambda$$

Now, we can apply the Submodel Theorem (Theorem 1.3.2) to M_1 and A to obtain some $N_1 \subseteq M_1$

(as L_1-structures) such that $|N_1| \supseteq A$ and $||N|| \leq |A| + \lambda$.

Define $N := N_1|_L$, the restriction of N_1 to L. To show that N has the properties desire, we really only need to show that $N|_L \preceq M|_L$. We do this by using the Tarski-Vaught test (Theorem 1.3.7) to prove that $M_1|_L \preceq N_1|_L$.

Fix $\psi(y, x_1, \ldots, x_n) \in \mathbf{Fml}(L)$. Suppose that $M \models \exists y\, \psi(y, a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in |N_1|$. Then, by definition of G for formulae, we have that $G_\psi(a_1, \ldots, a_n) \in |M_1|$ is the smallest $b \in |M_1|$ such that $M_1 \models \psi[b, a_1, \ldots, a_n]$. Since $a_1, \ldots, a_n \in |N_1|$, and since N is closed under taking G_ψ, we must have $b \in N_1$.

□

We will find that the techniques of defining functions like G_φ in (1.3.2) and augmenting languages as in (1.3.3) will come up time and time again in model theory.

Theorem 1.3.17 (Upwards Löwenheim-Skolem Theorem). *Suppose T has an infinite model. Then given any cardinal $\lambda \geq \aleph_0 + |\mathbf{L}(T)|$ there exists some L-structure M of cardinality $||M|| = \lambda$ and $M \models T$.*

Proof. We will just add λ many constants to a model of T. In particular, we let $T^k := T \cup \{c_i \neq c_j | i \neq j < \lambda\}$. By the compactness theorem T^k has a model N, and if we now let $A = \{c_i^M | i < \lambda\}$ then applying Downward Lowenheim-Skolem Tarski we can obtain some $M < N \upharpoonright \mathbf{L}(T)$ of cardinality λ (completing the proof). □

1.3.7 Complete and Elementary Diagrams

Definition 1.3.18 (Complete Diagram on M). Let M be an L-structure and $L_M := \mathbf{L}(M) \cup \{c_a | a \in M\}$. Letting $M' := \{M, c_a\}_{a \in |M|}$ with the constants interpreted such that $c_a^{M'} = a$ for all $a \in |M|$, we define $\mathbf{CD}(M) = Th(M')$ denote the "complete diagram of M."

Definition 1.3.19 (Elementary Diagram of M). We now define

$$\mathbf{ED}(M) = \{\varphi \in \mathbf{CD}(M) | \varphi \text{ is quantifier free}\}$$

Lemma 1.3.20 (Lemma 1). *Suppose $N \models \mathbf{ED}(M)$ and let $N^* := N \restriction L$. Then there exists $\varphi : |M| \to |N^*|$ an injective homomorphism (not necessarily injective).*

For a proof, we can just take $\varphi(a) = c_a^{N^*}$.

Lemma 1.3.21 (Lemma 2). *Suppose $N \models \mathbf{CD}(M)$ with $N^* := N \restriction L$. The there exists $\varphi : |M| \to |N|$ an elementary embedding [i.e. $\varphi[M] \prec N$].*

As an application, given M and some Γ a set of sentences, if $\mathbf{CD}(M) \cup \Gamma$ which is consistent, it follows that there must exist some N such that M is an elementary submodel of N and $N \models \Gamma$.

Theorem 1.3.22 (Upward Lowenheim-Skolem-Tarski). *Given an infinite L-structure M and any cardinal $\lambda \geq ||M|| + |\mathbf{L}(M)|$ there exists some $N > M$ of cardinality λ.*

For this one-line proof, just apply sorryto $\mathbf{CD}(M)$.

What are the basic theorems of model theory? ULS, DLST, and Compactness Theorem - and it turn out that these are equivalent to AC in ZFC!

Remark. A fact proved by Lauchli and Levi is that $ZF \vdash [CT \leftrightarrow BPI]$ where BPI deals with "Boolean Prime Ideals" sorry(see book), and further in ZF all of this is equivalent to Tychonoff's Theorem. The effect of this is that it's very difficult to do interesting model theory without choice.

Remark. "I feel it in my bones, it is true." - Shelah's response to (Grossberg's) the question of how one can feel comfortable using Axiom of Choice in Model Theory. Made quite an impression on Professor Grossberg.

1.4 Models of Peano Arithmetic

In this section, we build many models of Peano arithmetic, the most standard and universal of which is ω, the natural numbers.

1.4.1 The Language and Theory of Peano Arithmetic

In this subsection, we set up our study of Peano arithmetic by defining the language in which we will work and the theory we will seek to model. We begin with some notation.

Notation. Let **PA** denote the theory of Peano arithmetic expressed in a language L, the 'language of Peano arithmetic'.

sorry

1.4.2 Existence of Non-Standard Models of Peano Arithmetic

We know that there is a standard model of Peano arithmetic, denoted N. This consists intuitively of the natural numbers—that is, ordinals less than ω—with the usual addition, multiplication, additive and multiplicative identities. It turns out there are also other models, whose existence we will see in this subsection.

Before that, we mention that any model of Peano arithmetic admits a linear order.

Theorem 1.4.1. *The following is a valid deduction.*

$$\mathbf{PA} \vdash \forall x \forall y \exists z \, (y = x + z)$$

Theorem 1.4.2. *There exists some $M \models$ **PA** such that M is not isomorphic to the standard model N.*

Proof. Augment the language L to the $L_1 = L_{\mathbf{PA}} \cap \{c\}$ by adding a new constant symbol c. Let

$$T_1 := \mathbf{PA} \cup \{c \neq 0\} \cup \left\{ c \neq \underbrace{1 + \cdots + 1}_{n \text{ times}} \;\middle|\; n < \omega \right\}$$

We show, using the Compactness Theorem (sorry), that there is a model M of T_1. Given $T_0 \subseteq T_1$ finite, let $n_0 < \omega$ be the largest natural number such that

$$c \neq \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

lies in T_0. Define the L_1-structure

$$M_0 := \langle \omega, +, \cdot, 0, 1, a \rangle$$

be the expansion of N to L_1., with $c^{M_0} = a$. Then, $M_0 \models T_0$. Hence, since every finite subset of T_1 has a model, so does T_1. Call this model M_1. Denote by M the restriction of M_1 to $L_{\mathbf{PA}}$.

Now that we have established the existence of M, all that remains is to show that M is not isomorphic to N. Let $b = c^M$. Suppose $M \cong N$ via an isomorphism f. Then, $f(b) \in \omega$, so there is some $n < \omega$ such that

$$f(b) = \underbrace{1 + \cdots + 1}_{n \text{ times}}$$

□

1.4.3 Famous Results on Peano Arithmetic

Recall the definition of the *spectrum of a theory* (Definition 1.2.3). It turns out, we can use the idea of a spectrum to say something rather sophisticated about countable models of Peano Arithmetic.

Theorem 1.4.3. $I(\aleph_0, \mathbf{PA}) = 2^{\aleph_0}$.

We know that **PA** admits a model (indeed, a standard model). Therefore, we know that **PA** is a consistent theory. However, Gödel famously showed that it is not possible to prove consistency of **PA** using **PA** alone. That is, he proved the following.

Theorem 1.4.4 (Gödel's Second Incompleteness Theorem)**.** **PA** $\nvdash$ **PA** *is consistent*

Remark. Note that no number theorist would accept this as an example of a statement of interest in number theory which is not provable in **PA** (so says the model theorist).

Another interesting impossibility result involves Ramsey theorem.

Theorem 1.4.5 (Paris-Harrington 1976)**.** **PA** $\nvdash$ *A special case of Ramsey's Theorem.*

1.4.4 True Arithmetic and the Twin Prime Conjecture

Recall the definition of the theory of a model (Definition 1.1.15). Since $\langle \omega, +, \cdot, 0, 1\rangle$ is obviously a model of Peano Arithmetic, its *theory* strictly contains the theory of Peano Arithmetic. We call this new theory the theory of true arithmetic.

Definition 1.4.6 (True Arithmetic)**.** Define the **theory of true arithmetic**, denoted **TA**, to be

$$\mathbf{TA} := \mathbf{Th}(\langle \omega, +, \cdot, 0, 1\rangle) \supsetneq \mathbf{PA}$$

The following is known about the spectrum of true arithmetic.

Theorem 1.4.7. *For all cardinals $\lambda \geq \aleph_0$, $I(\lambda, \mathbf{TA}) = 2^\lambda$.*

Indeed, taking $\lambda = |\omega| = \aleph_0$ gives us something reminiscent of Theorem 1.4.3.

For the remainder of this subsection, we will talk about how we can use the theory of true arithmetic to study the twin prime conjecture. We begin with notation.

Local Notation. Denote by ψ the formula in the language of true arithmetic expressing that there are infinitely many twin primes. Let $\mathcal{P}$ denote the set of prime numbers, a subset of ω. We will also use the symbol $|$ in an in-fix manner to denote

$$x \mid y \iff \exists x \, [y = z \cdot x]$$

Lemma 1.4.8. *For every $S \subseteq \mathcal{P}$, there is a countable model $M_S \models$ **TA**. Moreover, $\exists a_S \in |M_S|$ such that for all $p \in \mathcal{P}$, $M_S \models p \mid a_S$ if and only if $p \in S$.*

Proof. $I(\aleph_0, TA) = 2^{\aleph_0}$ there exists $\mu < 2^{\aleph_0}$ infinite such that $\exists \{M_i | i < \mu\}$ a sequence of countable structures a;; models of TA such that **if** $M \models TA$ countable **then** $\exists n < \mu$ such that $M \cong M_i$. Let $L' = L_{PA} \cup \{c\}$ where c is a constant. Given the assumption of our lemma, there exists $\{N_\alpha | \alpha < 2^{\aleph_0}\}$ all countable L'-structures such that $N_\alpha \models TA$ and $\alpha \neq \beta \implies N_\alpha \not\cong N_\beta$.

[We can let $M_s' := \langle M_s, a_s \rangle$ denote the expansion of M_s to L'. For $S_1 \neq S_2$ we note there exists

$q \in S_1, q \notin S_2$. For such q we will thus have $M'_{s_1} \models q \mid c$ but $M'_{s_2} \models\neq (g \mid c)$, from which we conclude that M'_{s_1}, M'_{s_2} cannot be elementarily equivalent.]

Since $N_2 \models TA$ we will have $\left\{ N_\alpha|_{L_{\mathsf{PA}}}/\cong \;\middle|\; \alpha < 2^{\aleph_0} \right\} \subseteq \{M_i \mid i < \mu\}$ but then we have $\{N_\alpha/\cong |\alpha < 2^{\aleph_0}\} \subseteq \{(M_i, a)/\cong |a \in |M_i|, i < \mu\}$. Considering the cardinalities of these sets we will have $\sum_{i<\mu} ||M_i|| \leq \mu\aleph_0 = \mu$ but $2^{\aleph_0} \leq \mu$ contradicts $\mu < 2^{\aleph_0}$. sorry

(if we don't have continuum many we can write down the countable subsets indexed by μ, adding a countable single we will get continuum many isomorphism types of expanded language, but how many interpretations of constant? countable. This is a kind of approximation

(take a_s interpreting the constant)

(taking 2 diff sets, prime in one will not be in other) $\square$

class 9-17 (move where you please)

Definition 1.4.9 (Ordered Field). We define an ordered field $(F, +, \cdot, 0, 1, \leq)$ as satisfying the sentence $\forall x\forall y\forall z[x \leq y \to x+z < y+z]$ and $\forall x\forall y[x < y \to \forall z[z > 0 \to x\cdot z < y\cdot z]]]$

Definition 1.4.10 (Archimedean Ordered Field). An ordered field is "Archimedean" if it satisfies $\phi = \begin{cases} \forall a \in F \text{ if } a > 0 \text{ then there is integer } n \text{such that } a_0 + \ldots + a_n \geq 1 \end{cases}$

For example, $\mathbb{R}$ is an Archimedean ordered field.

One may ask if there is some finite set of sentences T in the language $\langle +, \cdot, 0, 1, \leq\rangle$ equivalent to ϕ - the answer is no:

Theorem 1.4.11. *There exists an extension $M > (\mathbb{R}, +\cdot, 0, 1, \leq)$ which is not Archimedean.*

Proof. We let c denote a new constant and then define $T^* := CD(\mathbb{R}) \cup \{c > 0\} \cup \{n\cdot c < 1 | n < \omega\}$. Observing that for any finite subset of these sentences in $\{n \cdot c < 1 | n < \omega\}$ we can find some interpretation of c which works, it follows that each finite subset of T^* has a model, and thus compactness lets us conclude that T^* is consistent. $\square$

Remark. This brings us to nonstandard analysis - c is an infinitesimal! In particular, because ϕ cannot be axiomatized, we get this interesting model which does not satisfy the Archimedean property.

We now recall the definition of a well-ordered set.

Definition 1.4.12 (Well-Ordered Set). We say that $(P, \leq)$ is a "well-order"(ed set) if we have $<$ a linear order such that $\forall S \subseteq P$ with $S \neq \emptyset$...`sorry`

`sorrysorrysorry`(!) (showed there's an elementary extension of $CD(\omega, <)$ with $(|M|, <^M)$ is not well-ordered. then talked about periodic and locally finite groups)

Chapter 2

Abstract Elementary Classes

I guess the time has come to start a new chapter. (?)

2.1 A Word on Infinitary Logic

We begin by discussing a fundamental type of logic, the development of which involved the likes of Erdos, Tarski, Henken, Chang, Keisler and Morley.

2.1.1 The Syntax and Semantics of $L_{\omega_1,\omega}$

Let's start with a language L. We define a new language out of L called $L_{\omega_1,\omega}$, whose syntax and semantics are as follows.

We define $\mathbf{Fml}(L_{\omega_1,\omega})$ to be a superset of the set of first-order formulae of L, closed under first-order operations, containing also all countable conjunctions and disjunctions of formulae in L.

As for the semantics of $L_{\omega_1,\omega}$, for some $n < \omega$, given formulae

$$\{\varphi_i(x_1, \ldots, x_n) \mid i < \omega\}$$

we say that the $L_{\omega_1,\omega}$-formulae formed by conjunction and disjunction are satisfied by a structure

M and $a_1, \ldots, a_n \in M$ as follows:

$$M \models \bigwedge_{i<\omega} \varphi[a_1, \ldots, a_n] \iff \text{For every } i < \omega,\ M \models \varphi_i[a_1, \ldots, a_n]$$

$$M \models \bigvee_{i<\omega} \varphi[a_1, \ldots, a_n] \iff \text{There exists } i < \omega \text{ such that } M \models \varphi_i[a_1, \ldots, a_n]$$

Example 2.1.1. Say we let $L = L_{PA} = \langle +, \cdot, 0, 1 \rangle$ denote the language of Peano Arithmetic. While the compactness theorem gives us nonstandard models of Peano Arithmetic, in infinitary logic we have the sentence $\psi = \wedge PA \wedge [\forall x[x = 0) \vee (\vee_{n<\omega}(x = \sum_{i=1}^{n} 1)]]$ - in particular we have that $M \models \psi \Leftrightarrow M \cong (\omega, +, \cdot, 0, 1)$.
Similarly, with the use of similar infinite sentences (using only finitely many variables) kwe can axiomatize Archimedean fields and periodic groups.

2.1.2 New Languages using Infinite Cardinals

We can actually talk about infinitary logic more generally, where given a language L, we construct languages $L_{\lambda^+,\omega}$, with the specialisation $\lambda = \aleph_0$ giving us $L_{\omega_1,\omega}$. The semantics of a language $L_{\lambda^+,\omega}$ allow formulae of the form

$$\bigwedge_{\alpha<\lambda} \varphi_\alpha(x_1, \ldots, a_n) \qquad \text{and} \qquad \bigvee_{\alpha<\lambda} \varphi_\alpha(x_1, \ldots, a_n)$$

That is, $L_{\lambda^+,\omega}$ allows quantification of κ-many elements for any cardinal $\kappa < \lambda$.

Interestingly, if we consider cardinals $\aleph_0 \leq \lambda < \mu$, we can show that $L_{\lambda^+,\omega} \subsetneq L_{\mu^+,\omega,\omega}$.

It was believed by the great Shelah, when he first came to the US in the 1950s, that the future of logic was not model theory but infinitary logic. He defined the notion of an Abstract Elementary Class (AEC), the title of this chapter, which is also the sort of thing Professor Grossberg and his PhD students do research on. The point of AEC is that it defines a concrete category, consisting of objects that are sets with structure and morphisms that are structure-preserving injections, and admitting projective limits and certain other category theoretic properties. But we will not discuss these here.

2.2 The Basics of Abstract Elementary Classes

2.2.1 The Intuition of Abstract Elementary Classes

Fix a language L. Let T be a consistent first-order theory in L.

First, observe that if $K = \mathbf{Mod}(T)$ is the class of models of this theory, then for any $M, N \in \mathbf{Mod}(T)$, the relation $M \leq_{\mathbf{Mod}(T)} N$, defined to hold if and only if M is an elementary submodel of N, defines a partial order on $\mathbf{Mod}(T)$. Moreover, any subclass $K \subseteq \mathbf{Mod}(T)$ is also partially ordered by the restriction of this relation, which we denote $\leq_K$.

We begin by defining the concept of a Löwenheim-Skolem Cardinal.

Definition 2.2.1 (The Löwenheim-Skolem Cardinal). For any subclass $K \subseteq \mathbf{Mod}(T)$, we define its **Löwenheim-Skolem Cardinal LS**(K) to be the smallest cardinal $\lambda \geq \aleph_0 + |L|$ such that for all $M \in K$ and $A \subseteq |M|$, there is $N \in K$ such that $N \leq_K M$, $|N| \geq A$ and $||N|| \leq \lambda + |A|$

In essence, the Löwenheim-Skolem Cardinal is the first cardinal at which the conclusion of the Downwards Löwenheim-Skolem Theorem (Theorem 1.3.16) holds.

We can use this to define an AEC in the following manner.

Definition 2.2.2 (Abstract Elementary Class). We define an **Abstract Elementary Class** to be *any* class K of L-structures modelling some consistent theory T, partially ordered by $\leq_K$ as discussed above, satisfying

1. Coherence: if $M_1, M_2, M_3 \in K$, and

$$M_1 \leq_K M_3 \qquad \text{and} \qquad M_2 \leq_K M_3 \qquad \text{and} \qquad M_1 \subseteq M_2$$

then $M_1 \leq_K M_2$.

2. Closure under Isomorphisms: for all $M \in K$, if N is any L-structure such that $M \cong N$, then $N \in K$.
3. The Tarski-Vaught Chain Axioms:
   (a) For all ordinals α and for all sequences $\{M_j \mid j < \alpha\} \subseteq K$, if $i < j$ then $M_i \leq_K$

M_j. Moreover,

$$M^* = \bigcup_{i<\alpha} M_i$$

also lies in K, and for all $i < \alpha$, $M_i \leq M^*$.

(b) If, in addition, we have $N \in K$ such that $\forall i < \alpha$, $M_i \leq_K N$, then $M^* \leq_K N$.

4. The Löwenheim-Skolem Axiom: sorry

We also define what it means for a poset to be directed.

Definition 2.2.3 (Directed Poset). Let $(\mathcal{I}, \leq_{\mathcal{I}})$ be a poset. We say it is **directed** if for all $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$.

Lemma 2.2.4. *Suppose L is a countable language. Let M be an uncountable structure with cardinality λ. Then, there exists some chain $\{M_i \mid i < \lambda\}$ such that*

1. *For all $i_1, i_2 < \lambda$, $i_1 < i_2 \implies M_{i_1} \leq_K M_{i_2}$*
2. *For all $i < \lambda$, $||M_i|| < \lambda$*
3. $M = \bigcup_{i<\lambda} M_i$

Proof. sorry □

We can now state an important theorem about AECs. Note that the first point, while similar, is subtly different (if not very significantly so) from the Tarski-Vaught Axioms because it involves posets rather than ordinals.

Theorem 2.2.5. *Let K be an AEC and let $(\mathcal{I}, \leq)$ a directed poset. Then,*

(A) *For all families $\{M_i \mid i \in \mathcal{I}\} \subseteq K$ such that if $i \leq j \implies M_i \leq_K M_j$, if we define*

$$M^* = \bigcup_{i \in \mathcal{I}} M_i$$

then $M^* \in K$ and $\forall j \in \mathcal{I}$, $M_j \leq_K M^*$.

(B) *If, in addition, there is some $N \in K$ such that $\forall i \in \mathcal{I}$, $M_i \leq_K N$, then $M^* \leq_K N$.*

Categorically speaking, the first point says something about the existence of limits.

Proof. We argue by induction on $|I| = \aleph_\alpha$.

1. $\underline{\alpha = 0.}$ In this case, enumerate $\{a_n \mid n < \omega\} = I$. By induction on n, using the directedness property of the poset I, define $\{b_n \mid n < \omega\} \subseteq I$ such that $b_{n+1} \geq b_n$ and $b_{n+1} \geq a_n$. Notice that

$$M^* = \bigcup_{i \in I} M_i = \bigcup_{n < \omega} M_{b_n}$$

a fact that is true because of condition (A). Apply the Tarski-Vaught Chain Axioms to $J = \{b_n \mid n < \omega\}$ to conclude.

2. $\underline{\alpha > 0.}$ Let λ be the Löwenheim-Skolem Cardinal. By the lemma (sorry- see HW 2), there is an elementary chain $\{I_j \leq I \mid j < \lambda\}$ such that $|I_j| < \lambda$ for all j and $\bigcup_j I_j = I$.

Since I is directed,

$$(I, \leq) \models \forall x \forall y \exists z \, [z \geq y \wedge z \geq x]$$

This implies that $(I_j, \leq)$ is also directed. Apply the induction hypothesis to $\{M_i \mid i \in I_j\}$ for all j. Let

$$M_j^* = \bigcup_{i \in I_j}$$

Then, since $j_1 < j_2 \implies I_{j_1} \subseteq I_{j_2}$, we have that $M_{j_1}^* = M_{j_2}^*$. By (A), for all $i \in I_{j_l}$, $M_i \leq_K M_{j_l}$. Applying (B) to $\{M_i \mid i \in I_1\}$ to obtain sorry

□

2.2.2 Decomposing Uncountable Models into Countable Elementary Submodels

We have an interesting consequence, which effectively says "an uncountable model can be broken down elementarily into countable pieces".

Theorem 2.2.6. *Let K be an AEC. For al $\lambda > \mathbf{LS}(K)$ and for all $M \in K_\lambda$, there exists a directed poset $(I, \leq)$ and a chain $\{M_i \mid i \in I\} \subseteq K_{\mathbf{LS}(K)}$ such that whenever $i \leq_I j$, we have $M_i \leq_K M_j$ and $\bigcup_{i \in I} M_i = M$.*

Proof. Define I to be the set of all countable submodels of M:

$$I := \{A \subseteq |M| \mid |A| < \aleph_0\}$$

This a directed poset under inclusion.

By induction on $n < \omega$, define M_A for all $A \subseteq |M|$ with $|A| = n$ in the following manner:

$\underline{n = 0.}$ By the Löwenheim-Skolem Axiom, we can define some $M_\emptyset \leq_K M$ of cardinality $\mathbf{LS}(K)$.

$\underline{n + 1.}$ Given $A \subseteq |M|$ of cardinality $n + 1$, we can again apply the Löwenheim-Skolem Axiom to the set

$$A^* := A \cup \bigcup_{B \subsetneq A} |M_B|$$

to obtain $M_A \leq_K M$ of cardinality $\mathbf{LS}(K)$ containing A^*.

We can show, from our construction, that

$$|M| = \bigcup_{\substack{A \subseteq |M| \\ |A| = \aleph_0}} M_A$$

All that remains now is to show that the set

$$\{M_A \mid A \in I\}$$

is indeed directed. We prove this using the Coherence Axiom.

Fix $A_1, A_2 \in I$ and assume that $A_1 \subseteq A_2$. We can observe, from the inductive step of our construction, that $M_{A_1} \subseteq M_{A_2}$. Then, since both M_1 and M_2 are both elementary submodels of M, we can see that in fact, M_{A_1} is an *elementary* submodel of M_{A_2}. Therefore, we have that whenever $A_1 \subseteq A_2$, $M_{A_1} \leq_K M_{A_2}$, which tells us that $\{M_A \mid A \in I\}$ is directed. $\square$

Corollary 2.2.7. *Let T be a first-order countable theory in a countable language. For all models M of T, there exists a countable chain $\{M_i \mid i \in I\}$ of models of T, with $|I| \leq \aleph_0$ being the distinguished set such that*

$$\bigcup_{i \in I} M_i = M$$

and $\forall i, j \in I,\ i < j \implies M_i \leq M_j$.

Countable models are an important object of study, though they are not yet completely understood. The following is a(n as yet unproved) conjecture of Vaught's.

Conjecture 2.2.8 (Vaught 1962). *Let T be a first-order consistent countable theory with $|L(T)| \leq \aleph_0$, and let $I(\alpha, T)$ denote the number of models of T with cardinality α up to isomorphism. Then either $I(\aleph_0, T) \leq \aleph_0$ or $I(\aleph_0, T) = 2^{\aleph_0} = |\mathbb{R}|$. (Want to do this without assuming Continuum Hypothesis, of course...)*

The shortest way to get a PhD from Rami Grossberg is to solve Vaught's Conjecture. The closest anybody every came was Leo Harrington, whose supposed proof so offended the great Shelah that he trashed it almost immediately after reading the first page. A simple way to get a PhD is then to dig through some sort of landfill somewhere in California to find and rehash Harrington's solution, though it would need to better impress Professor Grossberg than it did his advisor.

2.3 The Erdős-Rado Theorem

As usual (comme d'habitude), fix a language L.

2.3.1 Une Perspective Galoisienne

We begin by defining the notion of a type[1].

[1] No, this is NOT a computer sciency thing!

Definition 2.3.1 (Type). Let M be an L-structure. Fix $A \subseteq |M|$. Let $\overline{b} = (b_1, \ldots, b_n) \in |M|^n$. We define

$$\mathbf{tp}\left({}^{\overline{b}}\!/_{A}\, , M \right) = \left\{ \varphi(\overline{x}; \overline{a}) \;\middle|\; \varphi(\overline{x}; \overline{y}) \in \mathbf{Fml}(L) \text{ and } \overline{a} \in A \text{ with } M \models \varphi\left[\overline{b}, \overline{a}\right] \right\}$$

That is, we define $\mathbf{tp}\left({}^{\overline{b}}\!/_{A}\, , M \right)$ to be the set of all L-formulae with parameters $\overline{a} \in A$ and free variables $\overline{x}$ such that when the $\overline{x}$ are interpreted as $\overline{b}$ in M, φ is satisfied.

We state a preliminary result about types.

Lemma 2.3.2. *Suppose $M_1 \leq M_2$ are L-structures and $A \subseteq |M_1|$, $\overline{b} \in |M_1|$. Then*

$$\mathbf{tp}\left({}^{\overline{b}}\!/_{A}\, , M_1 \right) = \mathbf{tp}\left({}^{\overline{b}}\!/_{A}\, , M_2 \right)$$

Proof. The idea is to exploit the elementarity of $M_1 \leq M_2$. sorry □

We will use the theory of types to take an approach reminiscent of Galois Theory to prove the famous **Erdős-Rado Theorem**, which can be thought of as a version of the pigeonhole principle for uncountable cardinals.

2.3.2 Pigeons and Holes - A Study in Regularity

We begin by recalling the pigeonhole principle (for finite sets).

Theorem 2.3.3 (Finite Pigeonhole Principle). *Let A, B be sets with $|A| = n$ and $|B| > n$. For all functions $f : B \to A$, $\exists x, y \in B$ with $x \neq y$ and $f(x) = f(y)$.*

Informally, we think of it as stuffing a set B of pigeons into a set A of holes.

There is an infinite analogue of this principle that we will seek to establish.

Theorem 2.3.4 (Finite to Infinite Pigeonhole Principle). *Let A and B be sets with $|A| < \aleph_0$ and $|B| \geq \aleph_0$. That is, A is finite and B is infinite. Then, there is a set $S \subseteq B$ and an element $a \in A$ such that $|S| \geq \aleph_0$ and for all $x \in S$, we have $f(x) = a$.*

That is, there are infinitely many (distinct) elements of A that must be fixed by f.

In order to establish an even more general pigeonhole principle, we introduce the concept of regularity.

Definition 2.3.5 (Regularity of Cardinals). Let $\lambda \geq \aleph_0$ be an infinite cardinal. We say λ is **regular** if for all cardinals $\mu < \lambda$ and for all $f : \lambda \to \mu$, there is some $S \subseteq \lambda$ with $|S| = \lambda$ and some $a < \mu$ such that for all $x \in S$, we have $f(x) = a$.

We also recall the concept of the successor of a cardinal.

Definition 2.3.6 (Successor of a Cardinal). Let λ be a cardinal. We define its successor to be the cardinal

$$\lambda^{\dagger} := \min \{\mu \text{ a cardinal} \mid \mu > \lambda\}$$

Definition 2.3.7 (α-limit). Given α an ordinal, we define the "α-limit" to be the cardinal (sorrycheck?) given by $\aleph_\alpha := \sum_{\beta<\alpha} \aleph_\beta = \sup_{\beta<\alpha} \aleph_\beta$

We can see that consecutive elements in the sequence of alephs are, indeed, successors. The same is true of the sequence of beths.

We will more formally define the sequences of alephs and beths for *all* cardinals (rather than just finite cardinals) below.

Definition 2.3.8 (The Sequence of Alephs). Let $\lambda \geq \aleph_0$ be a cardinal and let α be an ordinal. We define

$$\aleph_\alpha(\lambda) = \begin{cases} \lambda & \text{if } \alpha = 0 \\ [\aleph_\beta(\lambda)]^{\dagger} & \text{if } \alpha = \beta + 1 \\ \sum_{\beta<\alpha} \aleph_\beta(\lambda) & \text{if } \alpha \text{ is a limit (cf. Definition 2.3.7)} \end{cases}$$

We denote by $\aleph_\alpha$ the cardinal $\aleph_\alpha(\aleph_0)$.

We can show that the following is a theorem of **ZFC**. Indeed, it is a theorem of **all** of **ZFC**: we use the Axiom of Choice in the proof.

Proposition 2.3.9. *For all cardinals $\lambda \geq \aleph_0$, $\lambda^\dagger$ is regular.*

Proof. Fix a cardinal $\mu < \lambda^\dagger$ and a function $f : \lambda^\dagger \to \mu$. If $\lambda^\dagger$ is not regular, then for all $\alpha < \mu$, we have

$$\left|f^{-1}(\alpha)\right| \leq \lambda$$

We can then show that

$$\lambda^\dagger = \bigcup_{\alpha<\mu} f^{-1}(\alpha) \leq \sum_{\alpha<\mu} \left|f^{-1}(\alpha)\right| \leq \mu \cdot \lambda \leq \lambda \cdot \lambda = \lambda$$

□

The reason why we need the Axiom of Choice here is that we apply the Fundamental Theorem of Cardinal Arithmetic towards the end of the proof.

2.3.3 Ramsey and Sierpiński Join the Fray

We begin by associating, to any set, a set of ordinals.

Definition 2.3.10. Let A be a set and $n < \omega$. We define $[A]^n$ to be

$$[A]^n := \{(i_1, \ldots, i_n) \mid \forall l,\ \ i_l \in A \text{ and } i_l < i_{l+1}\}$$

Definition 2.3.11 (Extension of Cardinals)**.** Let $n < \omega$ be a natural number and let λ, μ, κ be cardinals. We say that $(\mu)^1_\kappa$ **extends** λ if

$$\lambda \to (\mu)^n_\kappa \text{ is true}$$

if and only if (check this!) for all $f : [\lambda]^n \to \kappa$, there exists some $S \subseteq \lambda$ with $|S| = \mu$ such that $\exists j < k$ with $\forall i_1 < \ldots < i_n \in S$ we have $f(i_1, \ldots, i_n) = j$ and a set of ordinals $[A]^n = \{(i_1, \ldots, i_n)|i_\ell \in A_\ell, i_\ell < i_{\ell+1}$ holding $\forall \ell < n\}$

Remark. We can think of the function f as a colouring, and the set S as a monochromatic set when coloured by the given f sorry.

Recall the statement of Ramsey's Theorem. We express it in terms of the definition above.

Theorem 2.3.12 (Ramsey's Theorem - Finite Version). *For all $0 < n, \mu, \kappa < \omega$, there exists some λ such that*

$$\lambda \to (\mu)^n_\kappa$$

Imagine you're playing the following game against the devil. You have a map, with cities in a grid, indexed by ω. So it's quite a big map. Any two cities in the map are connected by a line, corresponding to a road. The devil asks you to go to sleep, and while you're sleeping, he's painting these roads either black or white. When you wake up, your goal is to find an infinite subset of cities all connected by a line of the same colour.

Remark. Saying an uncountable cardinal λ is regular is equivalent to saying that $\forall \kappa < \lambda$, $\lambda \to (\lambda)^1_\kappa$.

Sierpiński proved that the following extension is impossible.

Theorem 2.3.13 (Sierpiński). *The following extensions are impossible:*

$$\aleph_1 \not\to (\aleph_1)^2_2 \qquad 2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Visit https://thefundamentaltheor3m.github.io/ModelTheoryNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to the associated GitHub repository.