21-603: Model Theory I

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Chapter 1

An Introduction to Model Theory

Model Theory is one of the two main branches of mathematical logic, alongside Set Theory. We can view Model Theory as a translation of algebra to the world of set theory, or as a *generalisation* of algebra, with many applications to algebra. Indeed, it is possible to view model theory as a more specialised version of category theory, at least in its power to deal with generality. We can prove tremendously deep theorems about algebra, or other fields of maths, purely using model theory.

Before we can talk about model theory in any detail, we need to recall a few notions from the world of logic.

1.1 A Crash Course on First-Order Logic

1.1.1 Languages and Structures

We begin by recalling the notion of a first-order language.

Definition 1.1.1 (Language). A **language** is a disjoint union

$$\mathbf{L} = \mathbf{F} \cup \mathbf{R} \cup \mathbf{C}$$

of countable sets of symbols, where

- $\mathbf{F}$ is a set of function symbols

- **R** is a set of relation symbols
- **C** is a set of constant symbols

Next, we recall the notion of a structure in a language.

Definition 1.1.2 (Structure). Let **L** be a structure. An **L-structure** is a tuple

$$M = \langle U; F, R, C \rangle$$

consisting of a non-empty set U and functions, relations, and constants that live in **F**, **R** and **C** respectively. U known as the **universe** of M, and is denoted by $|M|$. The function, relation and constant symbols of M are denoted F^M, R^M and C^M respectively.

Any function or relation in a structure has an **arity**, which is informally the number of arguments it takes. An important fact to note is that arities are not a feature of functions and relations themselves, but of their corresponding *symbols*. In other words, **arity is a syntactic notion**. Semantically speaking, when we seek an interpretation of a function symbol of some arity n, we are forced to limit our search to the set of functions from U^n to U.

We now describe the notion of structure-preserving bijections, known as isomorphisms.

MAKE MORE PRECISE - FOLLOW DEF 2.9 IN BOOK

Definition 1.1.3 (Isomorphism). Let **L** be a language and let M, N be **L**-structures. We say that a function $g : |M| \to |N|$ is an **isomorphism** if

1. g is a bijection
2. "g commutes with functions"
3. "g commutes with relations"
4. "g agrees on constants"

where the double-quotes for the second and third point above refer to the fact that we implicitly require an equality of arities condition before we can talk about composing isomorphisms with multi-ary functions.

We are now ready to define submodels.

Definition 1.1.4 (Submodel). Let M, N be **L**-structures. We say that M is a submodel of N, denoted $M \subseteq N$, if

1. $|M| \subseteq |N|$.
2. For all function symbols $F(x_1, \ldots, x_n)$, the interpretation in M agrees with the interpretation in N.
3. For all relation symbols $R(a_1, \ldots, a_n)$, the interpretation in M agrees with the interpretation in N.
4. For all constant symbols C, the interpretation in M agrees with the interpretation in N.

In particular, a submodel of a model is also a model. For instance, if G is a group, ie, a model of the group axioms, then any submodel of G is, in fact, a subgroup of G, and a group in its own right (in that it again models the group axioms).

We now talk about more syntactic elements of a language.

1.1.2 Syntax: Terms, Formulae, Sentences and Theories

Definition 1.1.5 (Terms). Let **L** be a language. **Term**(**L**) is defined to be the minimal set of finite sequences of symbols[a] from

$$\{(,),[,]\} \cup \mathbf{C} \cup \mathbf{F} \cup \{x_1, x_2, x_3, \ldots\}$$

satisfying the following rules:

1. Every constant symbol is a term.
2. Every variable is a term.
3. For all n-ary functions f and n terms $t_1, \ldots, t_n$, $f(t_1, \ldots, t_n)$ is a term.
4. Every term arises in this way.

[a]Here, the set of variable symbols is countable, but we might, on occasion, need uncountably many variable symbols

Remark. In other words, the elements of **Term**(**L**) are exactly the constants, variables, and functions of such (constants and variables).

Recall, from Definition 1.1.3, that isomorphisms are, in particular, bijections that agree on constants. It is possible to show that they also agree on the interpretations of terms in models. We will show this later.

In similar fashion, we can define the formulae in a language.

Definition 1.1.6 (Formulae). Let $\mathbf{L}$ be a language. $\mathbf{Fml}(\mathbf{L})$ is defined to be the minimal set of finite sequences of symbols from

$$\mathbf{Term}(L) \cup \{\wedge, \vee, \rightarrow, \neg, \ldots\} \cup \{=\} \cup \{\forall, \exists\}$$

satisfying the following rules:

1. For all $\tau_1, \tau_2 \in \mathbf{Term}(\mathbf{L})$, $\tau_1 = \tau_2$ is a formula.
2. For all n-ary relations R and n terms $t_1, \ldots, t_n$, $R(t_1, \ldots, t_n)$ is a formula.
3. For all connectives $\star$ and formulae Φ and Ψ, $\Phi \star \Psi$ is a formula.
4. For a quantifier Q, variable x and formula $\varphi(x)$, $Qx\,(\varphi(x))$ is a formula.
5. Every formula arises in this way.

Formulae consisting only of a single relation symbol (including formulae that only consist of an equality) are called **atomic formulae**. The atomic formulae of $\mathbf{L}$ are denoted $\mathbf{AFml}(\mathbf{L})$.

For a formula φ, denote by $\mathbf{FV}(\varphi)$ the set of free variables of φ. It is sometimes useful to distinguish those formulae in a language that contain no free variables.

Definition 1.1.7. Define the set of **sentences** of a language $\mathbf{L}$ to be

$$\mathbf{Sent}(\mathbf{L}) := \{\varphi \in \mathbf{Fml}(\mathbf{L}) \mid \mathbf{FV}(\varphi) = \emptyset\}$$

Essentially, a formula with no free variables is called a sentence. A theory is simply a set of sentences.

Definition 1.1.8 (Theory). Let $\mathbf{L}$ be a language. An $\mathbf{L}$-theory is any subset $T \subseteq \mathbf{Sent}(\mathbf{L})$.

There are many well-known theories in mathematics. The most familiar examples come from algebra.

Example 1.1.9 (The Theory of Fields). The theory of fields is a first-order theory in the language of fields. This is a language with function symbols $+, \times, ^{-1}$, relation symbol $=$, and constant symbols 0 and 1. It also has other first-order symbols, such as quantifiers, connectives and punctuation, but we ignore these (indeed, we will always take for granted that these exist). The theory of fields consists of the following sentences in this language:

1. $\forall x \forall y \forall z\, [(x+y)+z = x+(y+z)]$
2. $\forall x \forall y\, [x+y = y+x]$
3. $\forall x \exists y\, [x+y=0] \wedge \forall x\, [x+0=x]$
4. $\forall x\, [x \neq 0 \to \exists y\, [xy=1]]$, where $\neq$ is the obvious shorthand
5. $\forall x\, [x \times 1 = x]$
6. $\forall x \forall y \forall z\, [x \times (y+z) = x \times y + x \times z]$

Collectively, these sentences are known as the **theory of fields**.

There are numerous well-known examples of fields. There is the question of how we can formally describe what it means for some structure, such as the rational numbers, to *satisfy* the above sentences. To that end, we introduce the semantics of interpretation, namely, the notion of **satisfaction**.

1.1.3 Semantics: Satisfaction

We begin with the most important definition of this entire section.

Definition 1.1.10 (Satisfaction - Sentences). Let $\mathbf{L}$ be a language and M an $\mathbf{L}$-structure. For any $\varphi \in \mathbf{Fml}(\mathbf{L})$, we say that M **models** φ, denoted $M \models \varphi$, if `sorry`

Remark. We note that, (a) $M \vDash \forall x \varphi(x) \iff N \vDash \neg\exists x \neg\varphi(x)$ and (b) $M \vDash \exists x \varphi(x) \iff M \vDash \neg\forall x \neg\varphi(x)$. This is worthy of note as for proofs, we will often need to induct on (the number of symbols in) formulas - just as we only need the logical connectives $\neg, \vee$ to get the rest, we only need to check satisfcation for a single quantifier and $\neg$.

We can define satisfaction for theories in the obvious way.

Definition 1.1.11 (Satisfaction - Theories). Let $\mathbf{L}$ be a language and M an $\mathbf{L}$-structure. Given an $\mathbf{L}$-theory T, we say that $M \models T$ if $M \models \psi$ for all $\psi \in T$.

We are now ready to state a simple-sounding but rather non-trivial result.

Lemma 1.1.12. *Suppose M and N are both $\mathbf{L}$-structures. If $M \subseteq N$, then for all $\tau \in \mathbf{Term}(\mathbf{L})$, $\tau^M[a] = \tau^N[a]$, where $a \in |M| \times \cdots \times |M|$ and $\tau^M[a]$ and $\tau^N[a]$ denote interpretations of τ in M and N with the variables all being interpreted as the components of a.*

We do not prove this result. It is not difficult.

Next is a less trivial result.

Lemma 1.1.13. *If $M \subseteq N$, then $M \models \varphi$ if and only if $N \models \varphi$ for all quantifier-free formulae φ.*

Going back to Example 1.1.9, we can now say the following.

Example 1.1.14 (Models of the Theory of Fields). Recall the theory of fields, seen in Example 1.1.9. It can be shown that the following structures in the language of fields satisfy the theory of fields:

$$\mathbb{Q}, \mathbb{R}, \mathbb{C}, {}^{\mathbb{Z}}\!/_{7\mathbb{Z}}$$

A model of the theory of fields is known simply as a **field**. We all know that this is an incredibly rich theory. It might have been even richer if Évariste Galois had had better control over his faculties...

1.1.4 Elementary Equivalence

Recall the definition of an isomorphism of structures (Definition 1.1.3). We can regard isomorphism as a *syntactic* notion of equivalence of structures. In this subsection, we explore a *semantic* notion of equivalence of models.

Definition 1.1.15 (Elementary Equivalence). Let $\mathbf{L}$ be a langauge and let M and N be $\mathbf{L}$-structures. We say M **is elementarily equivalent to** N if for every sentence $\varphi \in \mathbf{Sent}(\mathbf{L})$, we have that M satisfies φ if and only if N satisfies φ. We denote this

$$M \equiv N$$

We can now relate isomorphisms to equivalence in the following manner.

Theorem 1.1.16. *Let $\mathbf{L}$ be a language and let M and N be $\mathbf{L}$-structures. If $M \cong N$, then for every $\varphi(x_1, \ldots, x_n) \in \mathbf{Fml}(\mathbf{L})$ and every $a_1, \ldots, a_n \in |M|$, we have that*

$$M \models \varphi[a_1, \ldots, a_n] \iff N \models \varphi[f(a_1), \ldots, f(a_n)]$$

Proof. Fix a formula $\varphi(x_1, \ldots, x_n) \in \mathbf{Fml}(\mathbf{L})$ with n free variables. We prove the result by performing induction[1] on φ.

Suppose φ is atomic. Then, there are two cases.

- $\varphi(x_1, \ldots, x_n)$ is of the form $\tau_1(x_1, \ldots, x_n) = \tau_2(x_1, \ldots, x_n)$ for terms $\tau_, \tau_2 \in \mathbf{Term}(\mathbf{L})$. In this case, it is possible to show, by cases on what τ_1 and τ_2 can be (see Definition 1.1.5) that f is compatible with φ.

- φ is of the form $R(x_1, \ldots, x_n)$ for some $R \in \mathbf{R}(\mathbf{L})$. This is true by definition of an isomorphism (see Definition 1.1.3).

Suppose, now, that φ is not atomic. It is enough to show that the result shows if φ is of the form $\psi_1(x_1, \ldots, x_n) \wedge \psi_2(x_1, \ldots, x_n)$, $\psi_1(x_1, \ldots, x_n) \vee \psi_2(x_1, \ldots, x_n)$, $\neg\psi(x_1, \ldots, x_n)$, and $\forall x_1 \forall x_2 \ldots \forall x_n \psi(x_1, \ldots, x_n)$, as these would be adequate.

- φ is of the form $\psi_1(x_1, \ldots, x_n) \wedge \psi_2(x_1, \ldots, x_n)$. This is immediate from the definition of satisfaction: for all $a_1, \ldots, a_n \in |M|$, we have that

$$M \models \varphi[a_1, \ldots, a_n] \iff M \models \psi_1[a_1, \ldots, a_n] \text{ and } M \models \psi_2[a_1, \ldots, a_n]$$

[1] That is, cases

$$\begin{aligned}
&\iff N \models \psi_1[f(a_1), \ldots, f(a_n)] \text{ and } N \models \psi_2[f(a_1), \ldots, f(a_n)] \\
&\iff N \models \varphi[f(a_1), \ldots, f(a_n)]
\end{aligned}$$

as required.

- φ is of the form $\psi_1(x_1, \ldots, x_n) \vee \psi_2(x_1, \ldots, x_n)$. Similar.

- φ is of the form $\neg\psi(x_1, \ldots, x_n)$. sorry

- φ is of the form $\exists x_1, \psi(x_1, \ldots, x_n)$.[2] Fix $a_1, \ldots, a_n \in |M|$. Then,

$$\begin{aligned}
M \models \varphi[a_1, \ldots, a_n] &\iff M \models \exists x \varphi[x, a_2, \ldots, a_n] \\
&\iff \text{There is some } a \in |M| \text{ such that } M \models \varphi[b, a_2, \ldots, a_n] \\
&\iff \text{There is some } a \in |M| \text{ such that } N \models \varphi[f(b), f(a_2), \ldots, f(a_n)] \\
&\iff \text{There is some } b \in |N| \text{ such that } N \models \varphi[b, f(a_2), \ldots, f(a_n)] \\
&\iff N \models \exists y \varphi[y, f(a_2), \ldots, f(a_n)]
\end{aligned}$$

where we note that the '$\implies$' direction of the fourth $\iff$ comes from taking $b = f(a)$ and the '$\impliedby$' direction comes from the fact that f is surjective, meaning that we can take a to be any element of $|M|$ such that $f(a) = b$.

We can conclude by noting that the above cases are adequate. See sorry. □

Corollary 1.1.17. *Let* $\mathbf{L}$ *be a language and let* M *and* N *be* $\mathbf{L}$*-structures. If* $M \cong N$*, then* $M \equiv N$*.*

Proof. Let $f : M \xrightarrow{\sim} N$ be an isomorphism from M to N. sorry □

Warning. The notion of isomorphism is (much?) finer than elementary equivalence.

We end our discourse on first-order logic by briefly discussing the theory of deduction and proof.

[2]This is enough because you can induct on the number of free variables, with exactly this being the inductive step.

1.1.5 Deduction and Proof

Let $\mathbf{L}$ be a language. Recall that $\mathbf{Sent}(\mathbf{L})$ is the set of *sentences* in $\mathbf{L}$. Throughout this subsection, fix a theory $T \subseteq \mathbf{Sent}(\mathbf{L})$.

Definition 1.1.18 (Provability). We say a sentence $\varphi \in \mathbf{Sent}(\mathbf{L})$ is **provable from** T, denoted $T \vdash \varphi$, if there exists a sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $\mathbf{L}$-sentences such that $\varphi_n = \varphi$ and for all $i < n$, either $\varphi_i \in T$ or φ_i is obtained from $\langle \varphi_1, \ldots, \varphi_{i-1} \rangle$ via the standard deduction rules of first-order logic, namely, Modus Ponens and Generalisation.

We can say something about what makes T a "sensible" set from which to deduce things.

Definition 1.1.19 (Consistency). We say that T is **consistent** if there is no $\varphi \in \mathbf{Sent}(\mathbf{L})$ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Consistency is equivalent to model existence.

Theorem 1.1.20 (Gödel-Henkin). *T is consistent if and only if there is an $\mathbf{L}$-structure M such that $M \models T$.*

We do not prove this theorem here, but we will make extensive use of it.

We end by recalling the compactness theorem for first-order logic.

Theorem 1.1.21 (Compactness, Gödel-Malcev). *If for any finite $T_0 \subseteq T$, there is a*

We now discuss the *size* of a model and a theory.

1.2 Cardinality and Categoricity

Throughout this section, let $\mathbf{L}$ be a language.

Definition 1.2.1 (Cardinality of a Structure). Let M be an $\mathbf{L}$-structure. The **cardinality of** M, denoted $\|M\|$, is the cardinality of its universe $|M|$.

We can also talk about the size of a theory.

Definition 1.2.2 (Categoricity of a Theory). Let T be an **L**-theory. Suppose $\lambda \geq |L|$ is a cardinal. We say that T **is λ-Categorical**, or that T **is categorical in** λ, if for all **L**-structures M and N such that $M, N \models T$ and $\|M\| = \|N\| = \lambda$, we have that $M \cong N$.

Categoricity brings up interesting questions, such as the so-called *spectrum problem*.

1.2.1 The Spectrum Problem

The spectrum of a theory with respect to a cardinal is defined as follows.

Definition 1.2.3 (Spectrum). Let T be an **L**-theory and let λ be a cardinal. We define the **spectrum of T with respect to λ** to be

$$I(\lambda, T) := \left|\left\{ {}^{M}/_{\cong} \;\middle|\; M \models T \text{ and } \|M\| = \lambda \right\}\right|$$

ie, $I(\lambda, T)$ denotes the number of isomorphism classes of models of T of cardinality λ.

It is obvious, from Definition 1.2.2, that a theory T is λ-categorical if and only if $I(\lambda, T) = 1$. However, if T is not λ-categorical, then it is, in general, quite difficult to compute $I(\lambda, T)$. In fact, for most theories and cardinals, computing the spectrum is an *open problem*, referred to as the **spectrum problem**.

There has been some progress on this problem. Steinitz made the following determinations.

Theorem 1.2.4 (Steinitz). *Let* **L** *be the language of fields, and let T be the theory of algebraically closed fields of characteristic p (obtained by adding the appropriate sentences to the Theory of Fields encountered in Example 1.1.9). Then,*

1. $I(\aleph_0, T) = \aleph_0$.
2. *For all* $\lambda > \aleph_0$, $I(\lambda, T) = 1$.

The spectrum problem has been worked on by some of the most eminent logicians of our time, including Rami's advisor, Saharon Shelah, who proved a famous conjecture by Morley (1965). More on the Spectrum Problem can be found on the associated Wikipedia page, and while this is not the most authoritative source, its contents are nonetheless interesting.

Morley also proved a famous conjecture by Łos from the 1950s, which since became known as Morley's Categoricity Theorem.

Theorem 1.2.5 (Morley's Categoricity Theorem, Morley 1965). *Let T be a theory in a language $\mathbf{L}$. Assume that $|\mathbf{L}| \leq \aleph_0$. If $\exists\lambda > \aleph_0$ such that T is λ-categorical, them $\forall\lambda > \aleph_0$, T is λ-categorical.*

One of our objectives in this course is to prove Morley's Categoricity Theorem.

As a side note, Morley was initially a PhD student of Saunders MacLane's at the University of Chicago. Morley didn't initially finish his PhD, to the point of losing his stipend at Chicago, but somehow landed a job at Berkeley, where he proved this famous theorem. MacLane, a staunch category theorist, didn't believe Morley's work was quite enough to merit a PhD; nevertheless, after being persuaded by the then-nascent (and very excited) model theory community, he eventually relented and awarded Morley his degree.

Here, we end our discussion on the spectrum problem. Before proceeding further, we recall the basics of cardinal arithmetic.

1.2.2 Cardinal Arithmetic

We begin by introducing notation.

Notation. We denote by

- **ZF** the Zermelo-Fraenkel Axioms of Set Theory
- **AC** the Axiom of Choice
- **ZFC** the Zermelo-Fraenkel Axioms with the Axiom of Choice

We denote cardinality of a set A by $|A|$ or $\mathrm{card}(A)$ and write $|A| = |B|$ if and only if there is a bijection from A to B. Informally, a **cardinal** is a measure of cardinality. That is, a set λ is a cardinal if $\lambda = |A|$ for some set A. We denote by $\aleph_0$ the cardinal of the natural numbers, which we will denote ω in any cardinal- or ordinal-theoretic context.

There are more precise ways in which we can define the notions of ordinals and cardinals. We do not do this here, but we mention that there is an appendix in Rami's book and several sections in my undergrad logic lecture notes that discuss this.

Add references

Definition 1.2.6 (Cardinal Arithmetic). Let λ, μ be cardinals, with $\lambda = |A|$ and $\mu = |B|$. We denote

$$\lambda + \mu := \texttt{sorry}$$
$$\lambda \cdot \mu := |A \times B|$$

The following is a famed theorem of Tarski, a direct consequence of which is precisely the fundamental theorem of cardinal arithmetic.

Theorem 1.2.7 (Tarski). *We can make the following deduction:*

$$\mathbf{ZF} \vdash (\mathbf{AC} \leftrightarrow \forall A, |A| \geq \aleph_0 \to |A \times A| = |A|)$$

Equivalently,

$$\mathbf{ZF} \vdash (\mathbf{AC} \leftrightarrow \forall A, \lambda \geq \aleph_0 \to \lambda \cdot \lambda = \lambda)$$

The fundamental theorem of cardinal arithmetic, which states that $|\omega \times \omega| = |\omega|$, is clearly just the specialisation of the above result to the case where $\lambda = \aleph_0$.

There is another fact that will be important for our purposes.

Theorem 1.2.8. *For infinite cardinals $\lambda, \mu \geq \aleph_0$, we have*

$$\lambda \cdot \mu = \max(\lambda, \mu) = \lambda + \mu$$

The reason for discussing cardinal arithmetic is that we can exploit it to prove the existence of submodels of specific cardinalities.

1.2.3 Submodel Existence

We begin by defining the cardinality of a strucutre..

Definition 1.2.9 (Cardinality of a Structure). Let N be a **L**-structure. We define card(N) to be the cardinality of the union of

We begin with the famed submodel theorem.

Theorem 1.2.10 (The Submodel Theorem [**MOAB**]). *Let M be a* **L***-structure. Define* $\lambda := |\mathbf{L}| + \aleph_0$. *If $A \leq |M|$, then there exists a substructure $N \leq M$ such that*

(a) $\mathrm{card}(N) \geq A$

(b) $\mathrm{card}(|N|) \leq |A| + \lambda$

Proof. By recursion on $n < \omega$, define sets $\{B_n subseteq |M| \mid n < \omega\}$ such that

1. $B_0 = \left\{c \in \mathbf{C}^M \mid c \text{ is a constant symbol of } c\right\} \cup A$
2. If $n < \omega$, then $|B_n| \leq |A| + \lambda$
3. For all $n < \omega$, define $B_{n+1} := \left\{F^M(\bar{a}) \mid \bar{a} \in B_n\right\} \cup B_n$

This is enough: if we have such a sequence of B_n, then we could take $B := \bigcup_{n<\omega} B_n$ and define $N := \langle B, F^M, R^M, C^M \rangle$. We can show that this satisfies the desired conditions.

(a) `sorry`

(b) `sorry`

Given these, all that remains now is to show that this is possible. `sorry` □

Finish using text-book proof

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, `\lipsum` time. Boy do I love LaTeX!

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