Partial Fraction Decomposition

and its Formalisation in Lean

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Imperial College London Motivations

How would you simplify the following?

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Imperial College London Motivations

How would you simplify the following?

$$\sum_{x=2}^{n} \frac{1}{x - x^2} = \sum_{x=2}^{n} \left(\frac{1}{x} - \frac{1}{x - 1} \right)$$

Motivations

How would you simplify the following?

Imperial College London Discussion

At this point, we ask a very natural question:

Imperial College London Discussion

At this point, we may make the following interesting observation:

Under what conditions can we thus decompose fractions of polynomials?

The short answer is:

It depends on the properties of the polynomials and our base structure.

Our goal today is:

- To understand what properties are sufficient for a decomposition to exist.
- To prove that a decomposition exists when said properties are satisfied.
- To explore the formalisation of this result in Lean.

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But first: a bit of background on Lean.

Lean: Some Background

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mathlib conventions include

- Formalising to maximum generality
- Inheriting rather than duplicating code
- Optimising code to minimise compile time

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The Lean compiler checks proofs down to the axiomatic level!

So, if we want to "teach" it a proof, we need to lay out our argument in a manner it can understand.

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Then, we'll look at the proof in Lean.

On Domains

Polynomial coefficients generally come from Rings, from which are inherited polynomial addition and multiplication.

Typically, they come from Integral Domains, which are a very broad class of rings that tend to be "well-behaved"—whatever that means.

Definition (Domain)

A Ring R is called a Domain if $\forall a, b \in R \setminus \{0\}$, we have $ab \neq 0$.

Definition (Integral Domain)

An Integral Domain is a Commutative Ring that's also a Domain.

Polynomial Rings

We're now ready for the following:

Definition (Polynomial Ring over an Integral Domain)

If R is an Integral Domain, the set of polynomials with coefficients in R under addition (p+q)(X)=p(X)+q(X) and multiplication (pq)(X)=p(X)q(X) is called the Polynomial Ring R[X].

^aIn fact, R[X] is also an Integral Domain.

Note that X does **not** have to belong to R. It is called an "indeterminate" or a "variable" and is treated somewhat like a constant when one manipulates polynomials.

Can you think of an example where one can apply a polynomial to an object not in the base ring?

More Definitions

Definition (Monic)

Let R be an ID. $f \in R[X]$ is monic if its leading coefficient is 1, ie, if f is of the form

$$f(X) = a_0 + a_1 X + a_2 X^2 + \dots + a_{n-1} X^{n-1} + X^n$$

where $n = \deg(f)$ and $a_i \in R$ for all i.

Definition (Coprime)

Let R be an ID. Then, $f, g \in R[X]$ are coprime if $\exists a, b \in R[X]$ s.t.

$$af + bg = 1$$

Imperial College London Useful Results

Theorem

Let R be an ID. For $f, g \in R[X]$ with g monic, $\exists Q, R \in R[X]$ s.t. R + Qg = f.

This is given by polynomial.mod_by_monic_add_div in Lean's math library mathlib. Note also that Q and R are respectively denoted f $/_m$ g and f $%_m$ g in Lean. Note $\deg(R) < \deg(g)$, given by polynomial.degree_mod_by_monic_lt.

Lemma

Let R be an ID. Fix $f, g, h \in R[X]$ and let f, g and f, h be coprime. Then, f and gh are coprime.

In mathlib, this exists (more generally) as is_coprime.mul_right.

Imperial College London Main Result

Theorem (Partial Fraction Decomposition)

Let R be an Integral Domain. Fix $f, g_1, g_2, \dots, g_n \in R[X]$ and let the g_i s be *monic* and *pairwise coprime*. Then, $\exists q, r_1, r_2, \dots, r_n \in R[X]$ such that $\deg(r_i) < \deg(g_i)$ for all i, and

$$\frac{f}{\prod_{i=1}^n g_i} = q + \sum_{i=1}^n \frac{r_i}{g_i}$$

Imperial College London Proof Sketch

We start by proving the n=2 case and then proceed by induction.

Proof Sketch: n = 2

By coprimality of g_1 and g_2 , we know $\exists c, d \in R[X]$ s.t. $cg_1 + dg_2 = 1$. Then, we write $f = f \cdot 1$, and then get

$$\frac{f}{g_1g_2} = \frac{f(cg_1 + dg_2)}{g_1g_2} = \frac{cf}{g_2} + \frac{df}{g_1}$$

We know that $\exists q_1, q_2, r_1, r_2 \in R[X]$ s.t.

$$\frac{cf}{g_2} + \frac{df}{g_1} = \left(q_2 + \frac{r_2}{g_2}\right) + \left(q_1 + \frac{r_1}{g_1}\right)$$
$$= \left(q_1 + q_2\right) + \frac{r_1}{g_1} + \frac{r_2}{g_2}$$

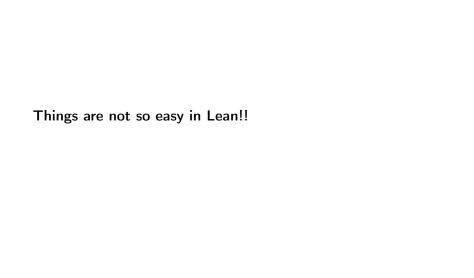
Proof Sketch: General *n*

We proceed by induction on n. The n = 1 base case follows directly from the quotient-remainder result¹.

We assume the result for general n. Then, we write

$$\frac{f}{\prod_{i=1}^{n+1} g_i} = \frac{f}{(\prod_{i=1}^n g_i) \cdot g_{n+1}}
= q' + \frac{r_{n+1}}{g_{n+1}} + \frac{f}{\prod_{i=1}^n g_i}
= (q' + Q) + \frac{r_{n+1}}{g_{n+1}} + \sum_{i=1}^n \frac{r_i}{g_i} \qquad \Box$$

¹In informal mathematics, yes; in Lean, not quite!



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There's (mainly) two things that are different in Lean:

- We use a different style of induction
- The distinction between an Integral Domain and its Field of Fractions is significantly more pronounced

The Use of Finsets

In Lean, rather than indexing over \mathbb{N} , we index over an arbitrary type ι and look at an arbitrary finite subset s of ι . This allows for more generality.

Induction therefore looks a bit different:

- Base Case: True over $(\emptyset : finset \ \iota)$
- Inductive Case: True over (b : finset ι) \Longrightarrow true over (insert a b : finset ι), where (a : ι) and (a \notin b).

Proving this base case and inductive case amounts to proving the result for any finite subset of ι .

The Distinction between an ID and its Field of Fractions

Roughly speaking, the Field of Fractions K over an Integral Domain R is

"
$$\{p/q : p \in R, q \in R \setminus \{0\}\}$$
"

But, this is a pretty terrible definition... can anyone see why?

Imperial College London Fields of Fractions

To fix the problem, we make use of the following two observations:

- Each element "p/q" consists of two ring elements: p and q
- Each "p/q" would also be expected to be the same as "px/qx" for some $x \in R \setminus 0$. So, each element is not necessarily represented by a *unique* pair (p,q).

Field of Fractions

Let R be an ID. Consider the set $J := R \times (R \setminus \{0\})$. Define an equivalence relation \sim on J by $(a,b) \sim (c,d)$ iff ad = bc. This loosely models what we would intuitively expect:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

Definition (Field of Fractions)

Given an ID R, we construct a set J as above. Then, the Field of Fractions K of R is the set of Equivalence Classes on J given by \sim , which forms a field under the operations

$$[(a,b)] + [(c,d)] = [(ad + bc,bd)]$$
$$[(a,b)] \cdot [(c,d)] = [(ac,bd)]$$

Field of Fractions

At this point, we note the following things:

- The construction of the Field of Fractions mirrors identically the construction of \mathbb{Q} from \mathbb{Z} .
- The reason we need R to be an ID is so that we don't get 0 in the denominator when we multiply two "fractions".
- It is **improper**² to say that $R \subseteq K$ (even though we were taught in school that $\mathbb{Z} \subseteq \mathbb{Q}$). What is true, however, is that there is an injective "inclusion map" going from R to K. In fact, one can even show it to be a ring homomorphism. [1]

(Note: we usually drop the [(p,q)] notation and just use normal fraction notation p/q.)

²At least, from a *strictly formal perspective*

Imperial College London Why does this matter?

In Lean, all the lemmas we have about polynomials in the ID R[X] (where R is an ID) is

Imperial College London The Lean Proof

Let's see the proof in VS Code!

Imperial College London Some Discussion

One question that you might have is, "Why did we need all that theory at the beginning?"

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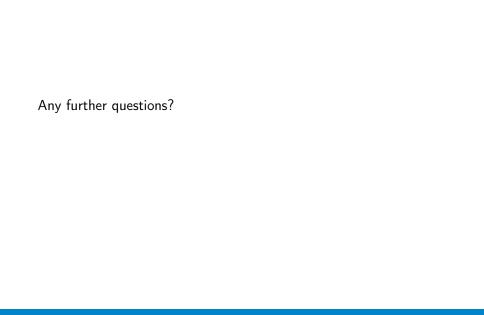
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$$\frac{1}{(x+1)(x-1)}$$
 example.

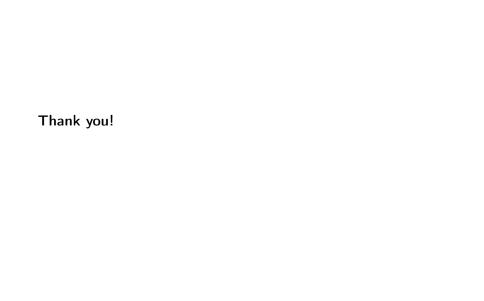
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The reason we could decompose in the FoF of $\mathbb{R}[x]$ but not $\mathbb{Z}[x]$ is that x+1 and x-1 are coprime in $\mathbb{R}[x]$ but not in $\mathbb{Z}[x]$.





Imperial College London References

[1] James McKernan. MATH 103B. Field of Fractions. University of California San Diego, 2016. URL: https://mathweb.ucsd.edu/~jmckerna/Teaching/15-16/Spring/103B/l_14.pdf.

Imperial College London Further Resources

Click on the item to visit the linked page.

- Mathlib documentation
- Lean code for this project, written in collaboration with Dr.
 Kevin Buzzard and merged into mathlib on Jan 17th, 2023
- Imperial College MathWiki: Links to notes and resources from our second-year course MATH50005 Groups and Rings