# MATH-314: Representation Theory of Finite Groups

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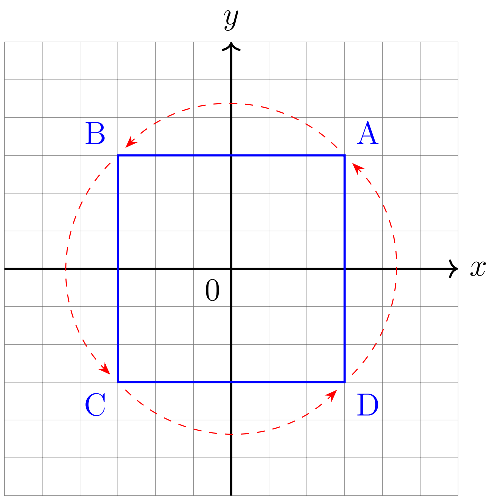
# Chapter 1

# An Introduction to the Theory of Representations of Groups

As I understand it, the fundamental idea behind Representation Theory is to study the actions of groups on vector spaces. While arbitrary vector spaces over arbitrary fields might not have naturally visualisable geometric properties, representations of groups in the ones that do can greatly illustrate the nature of these groups, especially to individuals like myself who delight in (somewhat literally) *seeing* mathematics come alive.

A key motivating example in the study of representation theory would be the representations of Dihedral groups over $\mathbb{R}^2$. It is very natural to (at least informally) view the Dihedral group $D_n$ of order $2n$ as the group of symmetries of the regular $n$-gon; in other words, elements of $D_n$ have natural actions on a regular $n$-gon that preserve its structure. For instance, $D_4$ contains an element that rotates a square clockwise by $90°$, an action under which the square is, of course, invariant.




If one were to now plot this square in $\mathbb{R}^2$, then action of the same element on the square can be extended to an orthogonal transformation of $\mathbb{R}^2$ that maps the $x$-axis to the $y$-axis and vice-versa, but in a manner preserving orientation (ie, that *rotates the plane clockwise by* $90°$). In a similar fashion, one can extend the actions of all dihedral groups $D_n$ to actions on the entirety of $\mathbb{R}^2$. More precisely, to every element of a dihedral group, one can ascribe a specific *matrix* that transforms $\mathbb{R}^2$ in a manner preserving the regular $n$-gon.

This motivates the formal definition of a representation.

## 1.1 Important Definitions

### 1.1.1 What is a Representation?

It turns out that representations can be defined quite broadly, sidestepping the geometric niceties (or are they constraints?) of Euclidean spaces.

**Definition 1.1.1** (Group Representation)**.** Let $G$ be a group. A representation of $G$ is a pair $(V, \rho)$ of a vector space $V$ and a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Here, $\mathrm{GL}(V)$ refers to the **G**eneral **L**inear group over $V$, consisting of all vector space automorphisms of $V$ equipped with the binary operation of composition.

**Definition 1.1.2** (Degree of a Representation)**.** Let $G$ be a group and let $(V, \rho)$ be a representation of $G$. We define the degree of $V$ to be the dimension of $V$ over its base field.

There exist innumerable examples of representations throughout mathematics. Below, we give some important ones.

**Example 1.1.3** (Important Classes of Representations)**.**

1. The zero representation. Let $G$ be a group and $V$ be any vector space. The map $\rho : G \to \mathrm{GL}(V)$ that maps any $g \in G$ to the zero map in $\mathrm{GL}(V)$ is a representation.

2. The trivial representation. Let $G$ be a group and $V$ be any vector space. The map $\rho : G \to \mathrm{GL}(V) : g \mapsto \mathrm{id}_V$ is a representation.

3. The sign representation. Let $G = S_n$, the symmetric group on $n$ elements, and let $V = K$, a field. Then, $\mathrm{GL}(V) = K^\times$, the multiplicative group of $K$. Denoting by $\xi$ the canonical map from $\mathbb{Z}$ to $K$, the map

$$\rho : G \to \mathrm{GL}(V) : \sigma \mapsto \xi(\mathrm{sgn}(\sigma))$$

is a representation, where $\mathrm{sgn} : G \to \{-1, 1\}$ denotes the sign homomorphism.

4. Permutation representations. Let $G$ be a group acting on a finite set $X$, and let $V = K[X]$, the free vector space (over some field $K$) generated by $X$. Consider a $K$-basis $\{e_x \in V : x \in X\}$ of $V$. Then, the map $\rho : G \to \mathrm{GL}(V)$ given by

$$\rho(g)(e_x) = e_{g(x)}$$

is a representation.

5. The regular representation. Let $G$ be a *finite* group. The permutation representation corresponding to the canonical action of $G$ on itself by left-multiplication gives a representation of $G$ over $K[G]$, the free $K$-vector space generated by the set $G$.

As it turns out, we also have a notion of morphisms of representations.

### 1.1.2 Morphisms of Representations

**Definition 1.1.4** (Homomorphism of Representations)**.** Let $G$ be a group and let $(V, \rho)$ and $(V', \rho')$ be two representations of $G$. A homomorphism of representations $T : V \to V$ is a linear map $T : V \to V'$ such that $\forall g \in G$,

$$T \circ \rho(g) = \rho'(g) \circ T \tag{1.1.1}$$

or equivalently, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd} \tag{1.1.2}$$

A natural way to define two representations to be equal, or 'isomorphic,' is as follows.

**Definition 1.1.5** (Equivalence of Representations)**.** Let $G$ be a group and let $(V, \rho)$ and $(V', \rho')$ be two representations of $G$. We say that $(V, \rho)$ and $(V', \rho')$ are equivalent, denoted $(V, \rho) \sim (V', \rho')$, if there exists a homomorphism $T : (V, \rho) \to (V', \rho')$ that is invertible as a linear map—ie, that gives a linear isomorphism between $V$ and $V'$.

The point of morphisms of representations is to be able to move from one vector space to another without losing the structural information captured by the representation. This is precisely illustrated in (1.1.2).

### 1.1.3 Subrepresentations