MATH-314: Representation Theory of Finite Groups

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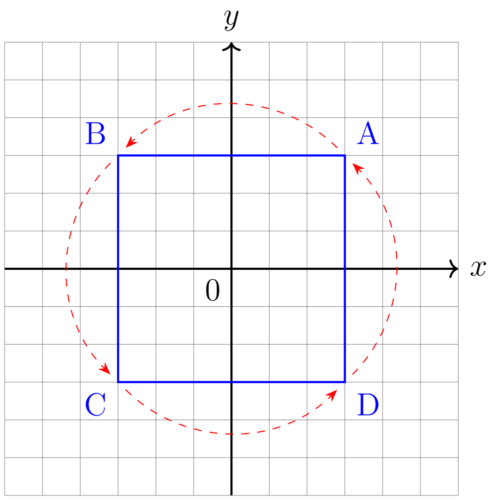
Chapter 1

An Introduction to the Theory of Representations of Groups

As I understand it, the fundamental idea behind Representation Theory is to study the actions of groups on vector spaces. While arbitrary vector spaces over arbitrary fields might not have naturally visualisable geometric properties, representations of groups in the ones that do can greatly illustrate the nature of these groups, especially to individuals like myself who delight in (somewhat literally) *seeing* mathematics come alive.

A key motivating example in the study of representation theory would be the representations of Dihedral groups over $\mathbb{R}^2$. It is very natural to (at least informally) view the Dihedral group D_n of order $2n$ as the group of symmetries of the regular n-gon; in other words, elements of D_n have natural actions on a regular n-gon that preserve its structure. For instance, D_4 contains an element that rotates a square clockwise by $90°$, an action under which the square is, of course, invariant.



If one were to now plot this square in $\mathbb{R}^2$, then action of the same element on the square can be extended to an orthogonal transformation of $\mathbb{R}^2$ that maps the x-axis to the y-axis and vice-versa, but in a manner preserving orientation (ie, that *rotates the plane clockwise by* $90°$). In a similar fashion, one can extend the actions of all dihedral groups D_n to actions on the entirety of $\mathbb{R}^2$. More precisely, to every element of a dihedral group, one can ascribe a specific *matrix* that transforms $\mathbb{R}^2$ in a manner preserving the regular n-gon.

This motivates the formal definition of a representation.

1.1 Important Definitions

1.1.1 What is a Representation?

It turns out that representations can be defined quite broadly, sidestepping the geometric niceties (or are they constraints?) of Euclidean spaces.

Definition 1.1.1 (Group Representation)**.** Let G be a group. A representation of G is a pair (V, ρ) of a vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Here, $\mathrm{GL}(V)$ refers to the **G**eneral **L**inear group over V, consisting of all vector space automorphisms of V equipped with the binary operation of composition.

Definition 1.1.2 (Degree of a Representation)**.** Let G be a group and let (V, ρ) be a representation of G. We define the degree of V to be the dimension of V over its base field.

There exist innumerable examples of representations throughout mathematics. Below, we give some important ones.

Example 1.1.3 (Important Classes of Representations)**.**

1. The trivial representation. Let G be a group and V be any vector space. The map $\rho : G \to \mathrm{GL}(V) : g \mapsto \mathrm{id}_V$ is a representation.
2. The zero representation. Let G be a group and let $V = \{0\}$ be the zero vector space over an arbitrary field K. The trivial representation over V is known as the zero representation.

3. The sign representation. Let $G = S_n$, the symmetric group on n elements, and let $V = K$, a field. Then, $\mathrm{GL}(V) = K^\times$, the multiplicative group of K. Denoting by ξ the canonical map from $\mathbb{Z}$ to K, the map

$$\rho : G \to \mathrm{GL}(V) : \sigma \mapsto \xi(\mathrm{sgn}(\sigma))$$

is a representation, where $\mathrm{sgn} : G \to \{-1, 1\}$ denotes the sign homomorphism.

4. Permutation representations. Let G be a group acting on a finite set X, and let $V = K[X]$, the free vector space (over some field K) generated by X. Consider a K-basis $\{e_x \in V : x \in X\}$ of V. Then, the map $\rho : G \to \mathrm{GL}(V)$ given by

$$\rho(g)(e_x) = e_{g(x)}$$

is a representation.

5. The regular representation. Let G be a *finite* group. The permutation representation corresponding to the canonical action of G on itself by left-multiplication gives a representation of G over $K[G]$, the free vector space generated by G (as a set) over any field K.

Non-Example 1.1.4. Let G be a group and let V be a nonzero vector space over an arbitrary field. The map $g \mapsto 0 : G \to (V \to V)$ is not a representation because the zero map $0 : V \to V$ is not invertible.

As it turns out, we also have a notion of morphisms of representations.

1.1.2 Morphisms of Representations

Definition 1.1.5 (Homomorphism of Representations)**.** Let G be a group and let (V, ρ) and (V', ρ') be two representations of G. A homomorphism of representations $T : V \to V$ is a linear map $T : V \to V'$ such that $\forall g \in G$,

$$T \circ \rho(g) = \rho'(g) \circ T$$

or equivalently, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd} \tag{1.1.1}$$

Such a map T is said to be *G-linear*.

A natural way to define two representations to be equal, or 'isomorphic,' is as follows.

Definition 1.1.6 (Equivalence of Representations)**.** Let G be a group and let (V,ρ) and (V',ρ') be two representations of G. We say that (V,ρ) and (V',ρ') are equivalent, denoted $(V,\rho) \sim (V',\rho')$, if there exists a homomorphism $T : (V,\rho) \to (V',\rho')$ that is invertible as a linear map—ie, that gives a linear isomorphism between V and V'.

The point of morphisms of representations is to be able to move from one vector space to another without losing the structural information captured by the representation. This is precisely illustrated in (1.1.1).

Example 1.1.7 (Representations of Cyclic Groups over $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** Consider the cyclic group $C_n = \langle g \rangle$ of order n. Let $V = \mathbb{R}^2, V' = \mathbb{R}^3$. Together with the respective maps

$$\rho : G \to \mathrm{GL}\big(\mathbb{R}^2\big) : g^m \mapsto \begin{bmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{bmatrix}$$

$$\rho' : G \to \mathrm{GL}\big(\mathbb{R}^3\big) : g^m \mapsto \begin{bmatrix} \cos(2\pi/m) & -\sin(2\pi/m) & 0 \\ \sin(2\pi/m) & \cos(2\pi/m) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

they give representations of C_n. Consider now the inclusion $T : \mathbb{R}^2 \to \mathbb{R}^3$ whose matrix with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. One can see that T gives a map from (V,ρ) to (V',ρ'). Indeed, the corestriction of T to its image is a linear isomorphism, which gives an equivalence between (V,ρ) and $(T(V),\rho)$, where we restrict the domains of each $\rho(g^m)$ to $T(V)$.

The above example leads to an interesting question. Can we think of one representation as being "contained" in another?

It turns out that we can.

1.1.3 Subrepresentations

We have the objects; we have the morphisms. It is only natural to think about what the subobjects would be in the context of group representations. And if Example 1.1.7 is any indication, they involve something more than just an inclusion. There is some structural property of a sub-vector space of a representation that makes it *compatible* with the representation structure. In the case of Example 1.1.7, for instance, this is the fact that the representation ρ' acted only "horizontally"–ie, "parallel" to the subspace $T(V)$.

More generally, it turns out that the property we really require a subspace to have in order to be 'compatible' with the representation structure is the following.

Definition 1.1.8 (*G*-Invariance)**.** Let G be a group and let (V, ρ) be a representation of G. We say that a sub-vector space $W \leq V$ is G-invariant if for all $w \in W$ and $g \in G$,

$$\rho(g)(w) \in W$$

In other words, W is G-invariant if W is $\rho(g)$-invariant for every $g \in G$.

One can make the following observation. Let G be a group, (V, ρ) a representation of G, and $W \leq V$ a G-invariant subspace. Then, $\forall g \in G$, $\rho(g) \in \mathrm{GL}(W)$. That is, $\rho(g)$ is a linear automorphism of W whose inverse, $\rho(g^{-1})$, is *also* a linear automorphism of W. This then leads to the following definition of a subrepresentation.

Definition 1.1.9 (Subrepresentation)**.** Let G be a group and let (V, ρ) be a representation of G. A subrepresentation of V is a pair $(W, \rho|_W)$ consisting of a G-invariant subspace $W \leq V$ and the map

$$\rho|_W : G \to \mathrm{GL}(W) : g \mapsto \rho(g)|_W$$

It is very important to note that the map $\rho|_W$ is *not actually a restriction of ρ to a specific domain*. Rather, it is a map that restricts the domain of $\rho(g)$ for every $g \in G$.

One can also observe easily that a subrepresentation is given uniquely by a G-invariant subspace. Hence, we will often abuse notation and not distinguish between the pair $(W, \rho|_W)$ (which is actually a representation) and simply W (which is merely a subspace).

Example 1.1.10. Let G be a finite group and K a field. Consider the regular representation $\rho : G \to K[G]$. Let $\{e_g : g \in G\}$ denote a basis of $K[G]$. Then, the subspace $W := \mathrm{Span}\left(\sum_{g \in G} e_g\right)$ is G-invariant.

1.1.4 Irreducibility

Having discussed the subobjects of representations (namely, subrepresentation), it is only natural to wish to describe whether a representation ever contains a nontrivial subrepresentation. I say "nontrivial" because any representation naturally admits two (uninteresting) subrepresentations: the trivial representation and itself.

Akin to the definition of simple groups, where we answer a similar question, we have the following definition that captures this idea.

Definition 1.1.11 (Irreducibility)**.** Let G be a group and (V, ρ) a nonzero representation of G. We say (V, ρ) is irreducible if V contains no proper, nonzero G-invariant subspaces.

In similar fashion, we say a representation is reducible if it is not irreducible.

Given that MATH-314 focuses on *finite* groups, the following result is quite useful.

Proposition 1.1.12. *Let G be a group and let (V, ρ) be a representation of G. If G is finite* *and (V, ρ) is irreducible, then V is finite-dimensional.*

Proof. Since (V, ρ) is irreducible, in particular, $V \supsetneq \{0\}$—ie, $\exists v \in V$ such that $v \neq 0$. Let $W := \operatorname{Span}(\{\rho(g)(v) : g \in G\})$. Since $0 \neq v \in W$, W is a nonzero subspace of V. Furthermore, since G is finite, W is finite-dimensional. We show that W is, in fact, G-invariant. Then, since V is irreducible, W could not possibly be a proper subspace of V, meaning that $W = V$, making V finite-dimensional as well.

Fix $h \in G$, and consider an arbitrary element $w = \sum_{g \in G} \lambda_g \rho(g)(v) \in W$. Then,

$$\begin{aligned} \rho(h)(w) &= \sum_{g \in G} \lambda_g \rho(h)(\rho(g)(v)) \\ &= \sum_{g \in G} \lambda_g \left(\rho(h) \circ \rho(g)\right)(v) \\ &= \sum_{g \in G} \lambda_g \rho(hg)(v) \in W \end{aligned}$$

proving that W is $\rho(h)$-invariant for every $h \in G$, making it a G-invariant subspace of V. Therefore, as argued above, $W = V$, proving that V is finite-dimensional. □

Example 1.1.13 (Simple Examples of Irreducible Representations)**.**

1. Any representation of degree 1 is irreducible.
2. Let K be a field. The trivial embedding $\mathrm{SL}(n, K) \hookrightarrow \mathrm{GL}(n, K)$ gives an irreducible representation of $\mathrm{SL}(n, K)$ over K^n.

   Proof. Assume $n > 1$ (else, the result follows from the previous point). For the sake of contradiction, suppose there exists a nonzero, $\mathrm{SL}(n, K)$-invariant subspace W of K^n having dimension $m < n$. Let $\mathcal{B} = \{e_1, \ldots, e_m\}$ be a basis of W, extending to a basis $\bar{\mathcal{B}} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of V. Consider the linear map $T \in \mathrm{SL}(n, K)$ having matrix

$$[T]_{\bar{\mathcal{B}}} = \begin{bmatrix} & & & (-1)^{n+1} \\ & & \iddots & \\ & -1 & & \\ 1 & & & \end{bmatrix}$$

with respect to $\bar{B}$. Clearly, $T(e_1) = e_n$, even though $e_1 \in W$ and $e_n \notin W$, contradicting the $\mathrm{SL}(n, K)$-invariance of W. $\square$

Non-Example 1.1.14. Let G be a finite group and K a field. Consider the regular representation $(K[G], \rho)$. In the notation of Example 1.1.10, we know that $W := \mathrm{Span}\left(\sum_{g \in G} e_g\right)$ is G-invariant. If $|G| > 1$, then W is a proper subspace of $K[G]$, as it has dimension 1 (whereas $K[G]$ has dimension $|G|$). Furthermore, W is nonzero. Hence, $(K[G], \rho)$ is not irreducible (unless $|G| = 1$, in which case it follows from the first point of Example 1.1.13 that $(K[G], \rho)$ is irreducible).

1.2 Invariant Constructions

In this section, we briefly examine how ordinary linear algebraic constructions can interact with representations. We are particularly interested in the notion of *invariance*, wherein a construction respects the structure of the representation(s) involved.

1.2.1 Direct Sums of Representations

The most elementary operation we can think about when we have two objects is *putting them together*. One of the most meaningful ways of doing so in the context of linear algebra is the direct sum of two vector spaces. It turns out that this extends rather naturally to representations.

Definition 1.2.1. Let G be a group and let (V, ρ) and (V', ρ') be representations of G. We define the direct sum of (V, ρ) and (V', ρ') to be the pair $(V \oplus V', \rho \oplus \rho')$, where $V \oplus V'$ is the direct sum of V and V' as vector spaces and $\rho \oplus \rho' : G \to \mathrm{GL}(V \oplus V')$ is the map given by

$$(\rho \oplus \rho')(g)(v \oplus v') = \rho(g)(v) \oplus \rho'(g)(v')$$

for all $g \in G$.

Proposition 1.2.2. *Let G be a group and let (V, ρ) and (V', ρ') be representations of G.*

1. *The direct sum $(V \oplus V', \rho \oplus \rho')$ of (V, ρ) and (V', ρ') is, indeed, a representation of G.*
2. *V and V' are G-invariant subspaces*[1] *of $V \oplus V'$.*

1.2.2 The G-Invariant Inner-Product

[1]Technically, isomorphic to the subspaces $V \oplus \{0\}$ and $\{0\} \oplus V'$, but we overlook such distinctions.