21-602: Introduction to Set Theory I

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Contents

Chapter 1

A Recap of Undergraduate Set Theory

Set Theory can be viewed as a field of mathematics that encompasses all other mathematics. We can view set theory as beginning at the empty set $\emptyset$, also denoted V_0. We can construct subsequent sets $V_1, V_2, V_3, \ldots$ by taking power sets, ie, $V_1 = \mathcal{P}(V_0)$, $V_2 = \mathcal{P}(V_1)$, and so on. We can then define V_ω to be the union of all V_n, where $n < \omega$. We can then define $V_{\omega+1} = \mathcal{P}(\omega)$, and so on. All of analysis, the field of mathematics Davoud Cheraghi once described as the "rigorous study of infinite constructions", is essentially about $V_{\omega+1}$. In this way, set theory encompasses analysis, for example.

We call the hierarchy

$$V_0 \subset V_1 \subset \cdots \subset V_\omega \subset V_{\omega+1} \subset \cdots$$

the **universe of sets**. Indeed, this is often represented diagrammatically as a V that contains each of these Vs, with V_0 at the bottom, V_1 above V_0, and so on.

There is a problem with the universe of sets: *the power set operation is too strong*. That is, when we take the power set of a set, it "throws too many elements in". This leads to questions like the *continuum hypothesis*, which asks the question of whether $\aleph_0 = |\mathbb{R}| = |V_{\omega+1}|$ is equal to $\aleph_1$.

Something we will study quite closely in this course is a slightly different version of this universe, known as **Gödel's Constructible Universe**. This is a universe of sets that is not built up from $\emptyset$ using the power set, but using a *different* operation $\mathcal{D}$, defined as follows: for a set A, viewing it as a first-order structure $(A; \in)$,

$$\mathcal{D}(A) := \{x \mid x \text{ is a 1st-order definable structure over } (A; \in) \text{ using parameters from } A\}$$

We can show that for finite sets, $\mathcal{D}$ agrees with $\mathcal{P}$. Moreover, we can show that for a countable set A, $\mathcal{D}(A)$ is countable as well. The point is that if we then define a universe

$$(L_0 = \emptyset) \subset (L_1 = \mathcal{D}(L_0)) \subset \cdots \subset \left(L_\omega = \bigcup_{n<\omega} L_n \right) \subset \cdots$$

we not only have agreement between L_n and V_n for all $n \leq \omega$, but we can build a model of ZFC in which the continuum hypothesis is true!

This is something we will explore in great detail in this course.

We will begin by reviewing the ZFC axioms.

1.1 The Zermelo-Fraenkel Axioms and the Axiom of Choice

We begin by reviewing the Zermelo-Fraenkel Axioms of Set Theory. We work in a first-order language with only one relation symbol, denoted $\in$. In principle, we would want to distinguish between the symbol in the language and its interpretation in any model, but we will not do this in practice.

We begin with the Axiom of Extensionality.

ZFC Axiom 1 (The Axiom of Extensionality).

$$\forall A \forall B \; [(\forall x \, (x \in A \leftrightarrow x \in B)) \rightarrow (A = B)]$$

We define the $\subseteq$ symbol, used infix as $A \subseteq B$, to be shorthand for the formula $\forall x \; (x \in A \rightarrow x \in B)$. ZFC Axiom 1 tells us that

$$\forall A \forall B \, ((A \subseteq B \wedge B \subseteq A) \rightarrow A = B)$$

We now define the Axiom Scheme of Comprehension.

ZFC Axiom 2 (The Axiom (Scheme) of Comprehension). Let x and y be free variables. For all formulae $\varphi(x, y)$, the following is an axiom:

$$\forall A \forall y \exists B \forall x \, [x \in B \leftrightarrow (x \in A \wedge \varphi(x, y))]$$

Intuitively, this means we can define

$$B = \{x \in A \mid \varphi(x, y)\}$$

This is an axiom scheme because we can increase the arity of φ and have more free variables $y_1, y_2, y_3, \ldots, y_n$ for any n.

Note that ZFC Axiom 1 tells us that we can replace the existential quantifier for B in ZFC Axiom 2 with an existence and uniqueness quantifier without changing its meaning.

Before proceeding further, we define the empty set. First, observe that in first order logic, structures are required to have non-empty universes. Thus, for every structure (A, E) in the language of set theory, with $E \subseteq (A \times A)$ representing equality, we must have

$$(A, E) \models \exists x \, (x = x)$$

The Completeness Theorem then tells us that

$$\vdash \exists x \, (x = x)$$

ie, that the formula $\exists x \, (x = x)$ *must be a theorem* in the language of sets. Hence, the axioms **??** 1**??** 2 tell us that

$$\vdash \exists ! A \forall x \; (x \notin A)$$

In other words, it is a theorem in the language of sets that there is a unique set that contains no members whatsoever. This unique set is denoted $\emptyset$, and is called the **empty set**. Its existence (and uniqueness) is not a distinct axiom, but a direct consequences of **??** 1**??** 2.

Next, we give the axiom of pairing.

ZFC Axiom 3 (The Axiom of Pairing).

$$\forall x\, \forall y\, \exists A\, (x \in A \wedge y \in A)$$

ZFC Axiom 3 essentially gives us a way of constructing, for any x and y, the unique set $A = \{x, y\}$ with the property that

$$\forall z\, (z \in A \leftrightarrow (z = x) \vee (z = y))$$

In particular, it allows us to construct, for any x, the set $\{x, x\}$, which, by ZFC Axiom 1, is exactly $\{x\}$. In other words, it tells us that we can stick any x into a set that only contains x.

In a similar flavour, we can construct a *union* of any family of sets. Note that this is distinct from pairing, because to construct $\{x, y\}$ as a union of $\{x\}$ and $\{y\}$, one needs to know that one can construct $\{x\}$ and $\{y\}$ in the first place. This requires pairing (or some version of it).

Now, we starte the axiom of unions.

ZFC Axiom 4 (The Axiom of Unions). If we denote by $\mathcal{F}$ a family of sets,

$$\forall \mathcal{F}\, \exists B\, \forall A\, [A \in \mathcal{F} \rightarrow A \subseteq B]$$

As usual, we can use **??** 1**??** 2 to show that

$$\bigcup \mathcal{F} = \{x \mid \exists A \subseteq \mathcal{F}\ (x \in A)\}$$

is well (and uniquely) defined.

Next comes the power set axiom.

ZFC Axiom 5 (The Axiom of Power Sets).

$$\forall A\, \exists \mathcal{F}\, \forall X\, [X \subseteq A \rightarrow X \in \mathcal{F}]$$

Again, **??** 1**??** 2 tell us that this is equivalent to

$$\forall A\, \exists! \mathcal{F}\, \forall X\, (X \in \mathcal{F} \leftrightarrow X \subseteq A)$$

This justifies the definition

$$\mathcal{P}(A) := \{X \mid X \subseteq A\}$$

Next comes the second (and last) axiom scheme in ZFC, the axiom scheme of replacement.

ZFC Axiom 6 (The Axiom (Scheme) of Replacement)**.** For each formula $\varphi(x, y, z)$ that does not contain B as a free variable, the following is an axiom:

$$\forall A \, \forall z \, [[\forall x \in A \, \exists! y \, \varphi(x, y, z)] \to [[\exists B \, \forall x \in A \, \exists y \in B \, \varphi(x, y, z)]]]$$

As in ZFC Axiom 2, we can introduce more free variables in lieu of z.

We can unpack this slightly complicated formula using **??** 1**??** 2: it essentially tells us that we can define

$$\{y \mid \exists x \in A \, \varphi(x, y, z)\}$$

The quantifiers $\forall A$ and $\forall z$ merely introduce variables. What the rest of the axiom (scheme) tells us is that **if** for every x, there is a unique *witness* y for which the formula $\varphi(x, y, z)$ holds, **then** we can collect all these witnesses into a set that is contained B, ranging over $x \in A$. Intuitively, "the number of such witnesses is *small enough* that it *can be collected into a set*."

Indeed, if it looks like $\varphi(x, y, z)$ defines a function with domain A, then it really does! If we can define $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$ (which we have yet to show we can do, but let's say we can), then we can put

$$f = \{(x, y) \in A \times B \mid \varphi(x, y, z)\}$$

which is a set by ZFC Axiom 2. Effectively, ZFC Axiom 6 tells us that *if the domain of a function is a set, then so is its range*: everything in its "image" (a *witness*) can be collected into some set. Of course, we only include this comment for the sake of intuition; as far as the formalism goes, we cannot talk about domains and ranges of functions yet, because functions are not yet defined. That being said, ZFC Axiom 6 is an essential step in the process of actually *defining* functions (or showing that this is something we can *do*).

One situation in which ZFC Axiom 6 is used (and will be used once sufficiently many axioms are stated) is in the situation we talked about at the start of Chapter 1. We'll see that there is a formula $\varphi(n, S)$ so that $\varphi(n, S) \leftrightarrow (n \in \omega \wedge S = V_n)$. Then, ZFC Axiom 6 would imply that

$$\langle V_n \mid n < \omega \rangle$$

really is a sequence. This allows us to define $V_\omega = \bigcup_{n<\omega} V_n$ and continue.

We now state the most interesting of the ZFC axioms: the Axiom of Choice.

ZFC Axiom 7 (The Axiom of Choice - (AC)).

$$\forall \mathcal{F} \, \exists c \; [\forall A \in \mathcal{F} \; (A \neq \emptyset \rightarrow A \in \mathrm{dom}(c) \wedge c(A) \in A)]$$

where $\mathcal{F}$ is a family of sets, c is a function, and $\mathrm{dom}(c)$ is the domain of c. We call c a **Choice Function on** $\mathcal{F}$.

The reason we state the Axiom of Choice at this point in our discussion is that modulo the ZFC Axioms that we have seen so far, the Axiom of Choice is equivalent to each of the following.

1. $\forall A \, \exists R \subseteq A \times A \; (A, R)$ is a well-ordering.

2. $\forall A \, \exists \alpha \, \exists f : A \xrightarrow{\sim} \alpha$, where α is an ordinal.

Note that none of the above axioms imply the existence of the natural numbers, without which virtually none of the mathematics we know and love would be possible. We therefore come to the axiom of infinity, which posits the existence of infinite sets, the 'smallest' of which is the natural numbers.

ZFC Axiom 8 (The Axiom of Infinity).

$$\exists I \; (\emptyset \in I \wedge \forall x \in I \; (x \cup \{x\} \in I))$$

A set I as above is called an **inductive set**, and it is an axiom of ZFC that such a set exists.

It is possible to prove, with relative ease, that given the existence of inductive sets (as posited above), there is a unique inductive set ω such that for all inductive sets J, $\omega \subseteq J$. Indeed, for any inductive set K, ω can be defined as the intersection of all inductive subsets $J \subseteq K$. It can then be shown that this intersection ω lives in all inductive sets. This minimal inductive set ω is known as the **set of natural numbers**. It is easy to see that the famed axioms of Peano are satisfied by ω.

Recall the sets $V_0, V_1, \ldots, V_\omega, \ldots$ defined at the start of Chapter 1. It is worth asking ourselves the following question: *Why does V_ω exist?* Or, more precisely, *why is V_ω a set?*

By induction, we know that each V_n is a set. We want to essentially see that the map $n \mapsto V_n$ is a *set* (ie, a *function*). Then, if we take the union of the range of this function, whose domain is the set ω whose existence is given by ZFC Axiom 8, we can define V_ω to be that union.

We invoke ZFC Axiom 6, the Axiom of Replacement. Let $\varphi(x, y)$ be the formula capturing the following facts:

1. y is a function with domain $\mathrm{dom}(y) = x \cup \{x\}$.
2. If $\emptyset \in \mathrm{dom}(y)$, then $y(\emptyset) = \emptyset$.
3. If $z \cup \{z\} \in \mathrm{dom}(y)$, then $z \in \mathrm{dom}(y)$ and $y(z \cup \{z\}) = \mathcal{P}(z)$.

The idea is that $y = \langle V_0, V_1, V_2, \ldots, V_{n+1} \rangle$[1] for some $n \in \omega$. Then,

$$\mathbf{ZFC} \vdash \forall n \in \omega \ \exists! y \, \varphi(n, y)$$

Applying the Axiom of Replacement (ZFC Axiom 6), we know that the map that takes any $n \in \omega$ to its unique witness y for n *is, indeed, a function*. In other words,

$$\langle \langle V_m \mid m \leq n \rangle \mid n < \omega \rangle$$

is a set. From this, we can read off

$$\langle V_n \mid n < \omega \rangle$$

by taking a union.

Remark. Recall that $(V_\omega, \in)$ is shorthand for (V_ω, R), where $R = \{(x, y) \in V_\omega \times V_\omega \mid x \in y\}$. It is

[1] By this, we mean it is a *sequence*, ie, a function from ω to something that takes 0 to V_0, 1 to V_1, and so on.

possible to show not only that

$$(V_\omega, \in) \models \text{ZFC without the Axiom of Infinity}$$

but the stronger fact that

$$(V_\omega, \in) \models \text{ZFC without the Axiom of Infinity but with its } \textit{negation}$$

Thus,

$$\mathbf{ZFC} \vdash \text{ZFC without Infinity but with } \neg\text{Infinity is consistent}$$

Ie, from ZFC, we can deduce that the entirety of ZFC, except with the negation of the axiom of infinity instead of the axiom itself, is consistent.

We note that the typical background setting for this course is the entirety of ZFC. However, as and when we need to either get rid of an axiom, or introduce a new one, we will mention it explicitly. We also use the following notational conventions.

Notation. We use **ZFC** to mean we have all the ZFC axioms. As and when we need variations, we do the following.

$$\mathbf{ZF} := \text{“ZFC without the Axiom of Choice”}$$

$$\mathbf{ZFC} - \mathbf{F} := \text{“ZFC without the Axiom of Foundation”}$$

$$\mathbf{ZF} - \mathbf{F} := \text{“ZF without the Axiom of Foundation”}$$

Here, we end our discussion on the axioms of set theory, and proceed to consider a broader type of object than sets, known as classes.

1.2 Classes

Recall Russell's Paradox: there is no set of all sets. One reason for this (which relies on the Axiom of Foundation) is that if V is the set of all sets, then $V \in V$, so V has no $\in$-minimal member, which contradicts the Axiom of Foundation. Without the Axiom of Foundation, if V is the set of all sets, then if we set

$$W = \{x \in V \mid x \notin x\}$$

(which is a set by the Axiom of Comprehension), then if $W \in W$, then $W \notin W$, and if $W \notin W$, then $W \in W$, which is a contradiction, meaning there can be no set of all sets.

Nevertheless, we really do want to write things like

$$V = \{x \mid x \text{ is a set}\} = \{x \mid x = x\}$$

We would also like to write things like

$$\mathbf{OR} = \{\alpha \mid \alpha \text{ is an ordinal}\} \{V_\alpha \mid \alpha \in \mathbf{OR}\}$$

The only problem is, none of these are sets. They are, however, **classes**. In fact, these are examples of **proper classes**.

Note that we would want sets to be classes. Indeed, anything of the form $\{x \mid \varphi(x)\}$, with $\varphi(x)$ a formula in the language of set theory, should be a class. Unfortunately, this is not generous enough: we want, for each set A, that

$$A = \{x \mid x \in A\} = \{x \mid \varphi(x, A)\}$$

to be included as well. In other words, we want to have formulae φ that not only allow us to substitute free variables with input sets x but also allow us to substitute more free variables with parameter sets A.

Formulae and parameters determine the class, but the converse is not true. In other words, if

$$\forall x \, (\varphi(x) \leftrightarrow \varphi'(x))$$

then

$$\{x \mid \varphi(x)\} = \{x \mid \varphi'(x)\}$$

For example, we know that

$$\varphi(x) \leftrightarrow (\varphi(x) \wedge x = x)$$

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Yup, `\lipsum` time. Boy do I love LaTeX!

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