Sphere Packing in Lean

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Abstract

This blueprint consists of an adaptation of Maryna Viazovska's Fields Medal-winning paper proving that no packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing. This blueprint is a work in progress, and will be frequently updated and restructured as the formalisation effort progresses. We recommend that you look at this webpage for the latest version.

Contents

1 Sphere packings

The Sphere Packing problem is a classic optimisation problem with widespread applications that go well beyond mathematics. The task is to determine the "densest" possible arrangement of spheres in a given space. It remains unsolved in all but finitely many dimensions.

It was famously determined, in [12], that the optimal arrangement in $\mathbb{R}^8$ is given by the E_8 lattice. The result is strongly dependent on the Cohn-Elkies linear programming bound (Theorem 3.1 in [3]), which, if a $\mathbb{R}^d \to \mathbb{R}$ function satisfying certain conditions exists, bounds the optimal density of sphere packings in $\mathbb{R}^d$ in terms of it. The proof in [12] uses the theory of modular forms to construct a function that can be used to bound the density of all sphere packings in $\mathbb{R}^8$ above by the density of the E_8 lattice packing. This then allows us to conclude that no packing in $\mathbb{R}^8$ can be denser than the E_8 lattice packing.

1.1 The Setup

This subsection gives an overview for the setup of the problem, both informally and in Lean. Throughout this blueprint, $\mathbb{R}^d$ will denote the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\mathrm{Vol}(\cdot)$. For any $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$, we denote by $B_d(x, r)$ the open ball in $\mathbb{R}^d$ with center x and radius r. While we will give a more formal definition of a sphere packing below (and in Lean), the underlying idea is that it is a union of balls of equal radius centred at points that are far enough from each other that the balls do not overlap.

Arguably the most important definition in this subsection is that of *packing density*, which measures which portion of d-dimensional Euclidean space is covered by a given sphere packing. Taking the supremum over all packings gives what we refer to as the *sphere packing constant*, which is the quantity we are interested in optimising.

Definition 1.1. *Given a set $X \subset \mathbb{R}^d$ and a real number $r > 0$ (known as the* separation radius*) such that $\|x - y\| \geq r$ for all distinct $x, y \in X$, we define the* sphere packing *$\mathcal{P}(X)$ with centres at X to be the union of all open balls of radius r centred at points in X:*

$$\mathcal{P}(X) := \bigcup_{x \in X} B_d(x, r)$$

Remark 1.2. *Note that a sphere packing is uniquely defined from a given set of centres (which, in order to be a valid set of centres, must admit a corresponding separation radius). Therefore, as a conscious choice during the formalisation process, we will define everything that depends on sphere packings in terms of* `SpherePacking`*, a* `structure` *that bundles all the identifying information of a packing, but not the actual balls themselves. For the purposes of this blueprint, however, we will* *refrain from making this distinction.*

We now define a notion of density for bounded regions of space by considering the density inside balls of finite radius.

Definition 1.3. *The* finite density *of a packing $\mathcal{P}$ is defined as*

$$\Delta_{\mathcal{P}}(R) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0,R))}{\mathrm{Vol}(B_d(0,R))}, \quad R > 0.$$

As intuitive as it seems to take the density of a packing to be the limit of the finite densities as the radius of the ball goes to infinity, it is not immediately clear that this limit exists. Therefore, we define the density of a sphere packing as a limit superior instead.

Definition 1.4. *We define the* density *of a packing $\mathcal{P}$ as the limit superior*

$$\Delta_{\mathcal{P}} := \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R).$$

We may now define the sphere packing constant, the quantity that the sphere packing problem requires us to compute.

Definition 1.5. *The* sphere packing constant *is defined as supremum of packing densities over all possible packings:*

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

1.2 Scaling Sphere Packings

Given that the problem involves the *arrangement* of balls in space, it is intuitive not to worry about the radius of the balls (so long as they are all equal to each other). However, Definition 1.1 involves a choice of separation radius. In principle, we would want two sphere packing configurations that differ only in separation radii to 'encode the same information'. In this brief subsection, we will describe how to change the separation radius of a sphere packing by *scaling* the packing by a positive real number and prove that this does not affect its density. This will give us the freedom to choose any separation radius we like when attempting to define the optimal sphere packing in $\mathbb{R}^d$.

Definition 1.6. *Given a sphere packing $\mathcal{P}(X)$ with separation radius r, we defined the* scaled packing *with respect to a real number $c > 0$ to be the packing $\mathcal{P}(cX)$, where $cX = \{cx \in V \mid x \in X\}$ has separation radius cr.*

Lemma 1.7. *Let $\mathcal{P}(X)$ be a sphere packing and c a positive real number. Then, for all $R > 0$,*

$$\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R).$$

Proof. The proof follows by direct computation:

$$\Delta_{\mathcal{P}(cX)}(cR) = \frac{\mathrm{Vol}(\mathcal{P}(cX) \cap B_d(0, cR))}{\mathrm{Vol}(B_d(0, cR))} = \frac{c^d \cdot \mathrm{Vol}(\mathcal{P}(X) \cap B_d(0, R))}{c^d \cdot \mathrm{Vol}(B_d(0, R))} = \Delta_{\mathcal{P}(X)}(R)$$

where the second equality follows from applying the fact that scaling a (measurable) set by a factor of c scales its volume by a factor of c^d to the fact that $\mathcal{P}(cX) \cap B_d(0, cR) = c \cdot (\mathcal{P}(X) \cap B_d(0, cR))$. □

Lemma 1.8. *Let $\mathcal{P}(X)$ be a sphere packing and c a positive real number. Then, the density of the scaled packing $\mathcal{P}(cX)$ is equal to the density of the original packing $\mathcal{P}(X)$.*

Proof. One can show, using relatively unsophisticated real analysis, that

$$\limsup_{R \to \infty} \Delta_{\mathcal{P}(cX)}(R) = \limsup_{cR \to \infty} \Delta_{\mathcal{P}(cX)}(cR)$$

Lemma 1.7 tells us that $\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R)$ for every $R > 0$. Therefore,

$$\limsup_{cR \to \infty} \Delta_{\mathcal{P}(cX)}(cR) = \limsup_{cR \to \infty} \Delta_{\mathcal{P}(X)}(R) = \limsup_{R \to \infty} \Delta_{\mathcal{P}(X)}(R)$$

where the second equality is the result of a similar change of variables to the one done above. □

Therefore, as expected, we do not need to worry about the separation radius when constructing sphere packings. This will be useful when we attempt to construct the optimal sphere packing in $\mathbb{R}^8$—and even more so when attempting to *formalise* this construction—because the underlying structure of the packing is given by a set known as the E_8 lattice, which has separation radius $\sqrt{2}$.

We can also use Lemma 1.8 to simplify the computation of the sphere packing constant by taking the supremum not over *all* sphere packings but only over those with density 1.

Lemma 1.9.

$$\Delta_d = \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \textit{sphere packing} \\ \textit{sep. rad.}=1}} \Delta_{\mathcal{P}}$$

Proof. That the supremum over packings of unit density is at most the sphere packing constant is obvious. For the reverse inequality, let $\mathcal{P}(X)$ be any sphere packing with separation radius r. We know, from Lemma 1.8, that the density of $\mathcal{P}(X)$ is equal to that of the scaled packing $\mathcal{P}\left(\frac{X}{r}\right)$. Since the scaled packing has separation radius 1, its density is naturally at most the supremum over all packings of unit density, meaning that the same is true of $\mathcal{P}(X)$. $\square$

1.3 Lattices and Periodic packings

We begin by defining what a lattice is in Euclidean space.

Definition 1.10. *We say that an additive subgroup $\Lambda \leq \mathbb{R}^d$ is a* lattice *if it is discrete and its $\mathbb{R}$-span contains all the elements of $\mathbb{R}^d$.*

There is also a corresponding dual notion, which will become relevant when we start doing Fourier analysis on functions over lattices.

Definition 1.11. *The* dual lattice *of a lattice Λ is the set*

$$\Lambda^* := \left\{ v \in \mathbb{R}^d \;\middle|\; \forall l \in \Lambda, \langle v, l \rangle \in \mathbb{Z} \right\}$$

As one might expect,

Theorem 1.12. *The dual of a lattice is also a lattice.*

Proof. Let Λ be a lattice and Λ^* its dual. We need to show three things: that Λ^* is an additive subgroup of $\mathbb{R}^d$; that Λ^* is discrete; and that the $\mathbb{R}$-span of Λ^* contains all of $\mathbb{R}^d$.

It is easy enough to see that Λ^* is an additive subgroup of $\mathbb{R}^d$: it clearly contains the zero vector (whose inner-product with any vector is zero), and is closed under addition and negation because the inner-product is bilinear and $\mathbb{Z}$ is closed under addition and negation. $\square$

Definition 1.13. *We say that a sphere packing $\mathcal{P}(X)$ is (Λ-)*periodic *if there exists a lattice $\Lambda \subset \mathbb{R}^d$ such that for all $x \in X$ and $y \in \Lambda$, $x + y \in X$ (ie, X is Λ-periodic).*

There is a natural definition of density for periodic sphere packings, namely the "local" density of balls in a fundamental domain. However, *a priori* the density of sphere packing above need not to coincide with this alternative definition. In Theorem 2.5, we will prove that this is the case.

Now that we have simplified the process of computing the packing densities of specific packings, we can simplify that of computing the sphere packing constant. It turns out that once again, periodicity is key.

Definition 1.14. *The periodic sphere packing constant is defined to be*

$$\Delta_d^{periodic} := \sup_{\substack{P \subset \mathbb{R}^d \\ periodic\ packing}} \Delta_P$$

Theorem 1.15. *For all d, the periodic sphere packing constant in $\mathbb{R}^d$ is equal to the sphere packing constant in $\mathbb{R}^d$.*

Proof. **State this in Lean (ready). Fill in proof here (see [3, Appendix A])** ☐

Thus, one can show a sphere packing to be optimal by showing its density to be equal to the *periodic* sphere packing constant instead of the regular sphere packing constant. The determination of the periodic constant is easier than that of the general constant, as we shall see when investigating the Linear Programming bounds derived by Cohn and Elkies in [3].

1.4 Main Result

With the terminologies above, we can state the main theorem of this project.

Theorem 1.16. *All* periodic *packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8} = \frac{\pi^4}{384}$, the density of the E_8 sphere packing (see Definition 3.9).*

Proof. Directly follows from Theorem 5.1 applied to the function $f(x) = g(x/\sqrt{2})$ of Theorem 5.2. ☐

Corollary 1.17. *All packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$.*

Proof. This is a direct consequence of Theorem Theorem 1.15 and Theorem 1.16. ☐

Corollary 1.18. $\Delta_8 = \Delta_{E_8}$.

Proof. By definition, $\Delta_{E_8} \leq \Delta_8$, while Corollary 1.17 shows $\Delta_8 = \sup_{\text{packing } \mathcal{P}} \leq \Delta_{E_8}$, and the result follows. ☐

2 Density of packings

The definition of density given in Section 1 is inconvenient to work with, especially when our packing is a structured one, such as a periodic/lattice packing. This section fixes this problem.

Note that some of the proofs in this section have only been sketched. The finer details are formalised in Lean.

Observe that the finite density evaluated at some $R > 0$ measures the density of sphere packings within a bounded, open ball of radius R. As one might expect, there is a relationship between the finite density and the number of centers in the ball of radius R.

Lemma 2.1. *For any $R > 0$,*

$$\left|X \cap \mathcal{B}_d\left(R - \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))} \leq \Delta_{\mathcal{P}}(R) \leq \left|X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))}$$

Proof. The high level idea is to prove that $\mathcal{P} \cap \mathcal{B}_d(R) = \left(\bigcup_{x \in X} \mathcal{B}_d\left(x, \frac{r}{2}\right)\right) \subseteq \bigcup_{x \in X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)} \mathcal{B}_d\left(x, \frac{r}{2}\right)$, and a similar inequality for the upper bound. The rest follows by rearranging and using the fact that the balls are pairwise disjoint. □

Suppose further that X is a periodic packing w.r.t. the lattice $\Lambda \subseteq \mathbb{R}^d$. Let $\mathcal{D}$ be a (bounded) fundamental region of Λ, say the parallelopiped $[0,1]^n\Lambda$, and let L be the bound on the norm of vectors in $\mathcal{D}$, i.e. a number satisfying $\forall x \in \mathcal{D}, \|x\| \leq L$.

Lemma 2.2. *For all R, we have the following inequality relating the number of lattice points from Λ in a ball with the volume of the ball and the fundamental region $\mathcal{D}$:*

$$\frac{\mathrm{Vol}(\mathcal{B}_d(R - L))}{\mathrm{Vol}(\mathcal{D})} \leq |\Lambda \cap \mathcal{B}_d(R)| \leq \frac{\mathrm{Vol}(\mathcal{B}_d(R + L))}{\mathrm{Vol}(\mathcal{D})}$$

Proof. For the first inequality, it suffices to prove that $\mathcal{B}_d(R - L) \subseteq \bigcup_{x \in \Lambda \cap \mathcal{B}_d(R)}(x + \mathcal{D})$, since the cosets on the right are disjoint. For a vector $v \in \mathcal{B}_d(R - L)$, we have $\|v\| < R - L$ by definition. Since $\mathcal{D}$ is a fundamental domain, there exists a lattice point $x \in \Lambda$ such that $v \in x + \mathcal{D}$. Rearranging gives $v - x \in \mathcal{D}$, which means $\|v - x\| \leq L$. By the triangle inequality, $\|x\| < R$, i.e. $x \in \mathcal{B}_d(L)$, concluding the proof.

For the second inequality, we prove that $\bigcup_{x \in \Lambda \cap \mathcal{B}_d(R)}(x + \mathcal{D}) \subseteq \mathcal{B}_d(R + L)$. Consider a vector $x \in \Lambda \cap \mathcal{B}_d(R)$ and a vector $y \in x + \mathcal{D}$. From above, we know $\|x\| < R$ and $\|y - x\| \leq L$, so $\|y\| < R + L$, concluding the proof. □

To link Lemma 2.1 and Lemma 2.2, we need a lemma relating $|\Lambda \cap \mathcal{B}|$ with $|X \cap \mathcal{B}|$, which is what the following lemma does:

Lemma 2.3. *For all R, we have the following inequality relating the number of points from X (periodic w.r.t. Λ) in a ball with the number of points from Λ:*

$$|\Lambda \cap \mathcal{B}_d(R-L)|\,|X/\Lambda| \leq |X \cap \mathcal{B}_d(R)| \leq |\Lambda \cap \mathcal{B}_d(R+L)|\,|X/\Lambda|$$

Proof. For the first inequality, we notice that $\bigcup_{x \in \Lambda \cap \mathcal{B}_d(R-L)} (x+\mathcal{D}) \subseteq \mathcal{B}_d(R)$, because for $x \in \Lambda \cap \mathcal{B}_d(R-L)$ and $y \in x+\mathcal{D}$, we have $\|x\| < R-L$ and $\|y-x\| \leq L$, so $\|y\| < R$ by triangle inequailty. Intersecting both sides with X and simplifying, we have

$$\left(\bigcup_{x \in \Lambda \cap \mathcal{B}_d(R-L)} (x+\mathcal{D})\right) \cap X = \bigcup_{x \in \Lambda \cap \mathcal{B}_d(R-L)} ((x+\mathcal{D}) \cap X) \subseteq \mathcal{B}_d(R) \cap X$$

Consider the (finite) cardinality on both sides and noting that $|(x+\mathcal{D}) \cap X| = |X/\Lambda|$ for all x, we see that $|\Lambda \cap \mathcal{B}_d(R-L)||X/\Lambda| \leq |X \cap \mathcal{B}_d(R)|$, as desired.

The proof of the second inequality is similar. We again observe that $\mathcal{B}_d(R) \subseteq \bigcup_{x \in \Lambda \cap \mathcal{B}_d(R+L)} (x+\mathcal{D})$, which follows from the tiling property of fundamental domain (i.e. every point can be translated by a Λ lattice point into $\mathcal{D}$). Intersecting both sides with X and considering cardinality of both sides concludes the proof.

There are several technicalities when formalising in Lean, such as having to prove $|\Lambda \cap \mathcal{B}_d(R)|$ is countable and finite. Those are handled at `aux3`. □

When we combine the inequalities above, we need one additional computational lemma.

Lemma 2.4. *For any constant $C > 0$, we have*

$$\lim_{R \to \infty} \frac{\mathrm{Vol}(\mathcal{B}_d(R))}{\mathrm{Vol}(\mathcal{B}_d(R+C))} = 1$$

Proof. Write out the formula for volume of a ball and simplify. More specifically, we have $\mathrm{Vol}(\mathcal{B}_d(R)) = R^d \pi^{d/2} / \Gamma\left(\frac{d}{2}+1\right)$, so $\mathrm{Vol}(\mathcal{B}_d(R))/\mathrm{Vol}(\mathcal{B}_d(R+C)) = R^d/(R+C)^d = \left(1 - \frac{1}{R+C}\right)^d = 1$. □

Finally, we can relate the density of a periodic sphere packing to the natural definition of density given by any of its fundamental domain:

Theorem 2.5. *For a periodic sphere packing $\mathcal{P} = \mathcal{P}(X)$ with centers X periodic to the lattice Λ and separation r,*

$$\Delta_{\mathcal{P}} = |X/\Lambda| \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(r/2))}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}$$

Proof. Fix any fundamental domain $\mathcal{D}$ (induced by any basis) of the lattice Λ. Combining Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get the following inequality for the *finite* density:

$$|X/\Lambda| \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(r/2))}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(R - r/2 - 2L))}{\mathrm{Vol}(\mathcal{B}_d(R))} \leq \Delta_{\mathcal{P}}(R) \leq |X/\Lambda| \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(r/2))}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(R + r/2 + 2L))}{\mathrm{Vol}(\mathcal{B}_d(R))}$$

Taking limit on both sides as $R \to \infty$ and apply the Sandwich theorem and Lemma 2.4, we get

$$\Delta_{\mathcal{P}} = \limsup_{R \to \infty} \Delta_{\mathcal{P}}(R) = \lim_{R \to \infty} \Delta_{\mathcal{P}}(R) = |X/\Lambda| \cdot \frac{\mathrm{Vol}(\mathcal{B}_d(r/2))}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}$$

□

3 The E_8 lattice

3.1 Definitions of E_8 lattice

There are several equivalent definitions of the E_8 lattice. Below, we formalise two of them, and prove they are equivalent.

Definition 3.1. *(E_8-lattice, Definition 1) We define the* E_8-lattice *(as a subset of* $\mathbb{R}^8$*) to be*

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 | \sum_{i=1}^{8} x_i \equiv 0 \pmod 2\}.$$

Definition 3.2. *(E_8-lattice, Definition 2) We define the* E_8 basis vectors *to be the set of vectors*

$$\mathcal{B}_8 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Theorem 3.3. *The two definitions above coincide, i.e.* $\Lambda_8 = \mathrm{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

Proof. We prove each side contains the other side.

For a vector $\vec{v} \in \Lambda_8 \subseteq \mathbb{R}^8$, we have $\sum_i \vec{v}_i \equiv 0 \pmod 2$ and $\vec{v}_i$ being either all integers or all half-integers. After some modulo arithmetic, it can be seen that $\mathcal{B}_8^{-1}\vec{v}$ as integer coordinates (i.e. it is congruent to 0 modulo 1). Hence, $\vec{v} \in \mathrm{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

For the opposite direction, we write the vector as $\vec{v} = \sum_i c_i \mathcal{B}_8^i \in \mathrm{span}_{\mathbb{Z}}(\mathcal{B}_8)$ with c_i being integers and $\mathcal{B}_8^i$ being the i-th basis vector. Expanding the definition then gives $\vec{v} = \left(c_1 - \frac{1}{2}c_7, -c_1 + c_2 - \frac{1}{2}c_7, \cdots, -\frac{1}{2}c_7\right)$. Again, after some modulo arithmetic, it can be seen that $\sum_i \vec{v}_i$ is indeed 0 modulo 2 and are all either integers and half-integers, concluding the proof.

(Note: this proof doesn't depend on that $\mathcal{B}_8$ is linearly independent.) $\square$

3.2 Basic Properties of E_8 lattice

In this section, we establish basic properties of the E_8 lattice and the $\mathcal{B}_8$ vectors.

Lemma 3.4. *B_8 is a $\mathbb{R}$-basis of $\mathbb{R}^8$.*

Proof. It suffices to prove that $\mathcal{B}_8 \in \mathrm{GL}_8(\mathbb{R})$. We prove this by explicitly defining the inverse matrix $\mathcal{B}_8^{-1}$ and proving $\mathcal{B}_8\mathcal{B}_8^{-1} = I_8$, which implies that $\det(\mathcal{B}_8)$ is nonzero. See the Lean code for more details., □

Lemma 3.5. *Λ_8 is an additive subgroup of $\mathbb{R}^8$.*

Proof. Trivially follows from that $\Lambda_8 \subseteq \mathbb{R}^8$ is the $\mathbb{Z}$-span of $\mathcal{B}_8$ and hence an additive group. □

Lemma 3.6. *All vectors in Λ_8 have norm of the form $\sqrt{2n}$, where n is a nonnegative inteeger.*

Proof. Writing $\vec{v} = \sum_i c_i \mathcal{B}_8^i$, we have $\|v\|^2 = \sum_i \sum_j c_i c_j (\mathcal{B}_8^i \cdot \mathcal{B}_8^j)$. Computing all 64 pairs of dot products and simplifying, we get a massive term that is a quadratic form in c_i with even integer coefficients, concluding the proof. □

Lemma 3.7. *$c\Lambda_8$ is discrete, i.e. that the subspace topology induced by its inclusion into $\mathbb{R}^8$ is the discrete topology.*

Proof. Since Λ_8 is a topological group and $+$ is continuous, it suffices to prove that $\{0\}$ is open in Λ_8. This follows from the fact that there is an open ball $\mathcal{B}(\sqrt{2}) \subseteq \mathbb{R}^8$ around it containing no other lattice points, since the shortest nonzero vector has norm $\sqrt{2}$. □

Lemma 3.8. *$c\Lambda_8$ is a $\mathbb{Z}$-lattice, i.e. it is discrete and spans $\mathbb{R}^8$ over $\mathbb{R}$.*

Proof. The first part is by Lemma 3.7, and the second part follows from that $\mathcal{B}_8$ is a basis (Lemma 3.4) and hence linearly independent over $\mathbb{R}$. □

Prove E_8 is unimodular. Prove E_8 is positive-definite.

3.3 The E_8 sphere packing

Definition 3.9. *The E_8* sphere packing *is the (periodic) sphere packing with separation $\sqrt{2}$, whose set of centres is Λ_8.*

Lemma 3.10. $\mathrm{Vol}(\Lambda_8) = \mathrm{Covol}(\mathbb{R}^8/\Lambda_8) = 1$.

Proof. **In theory this should follow directly from $\det(\Lambda_8) = 1$, but Lean hates me and `EuclideanSpace` is being annoying.** $\square$

Theorem 3.11. *We have* $\Delta_{\mathcal{P}(E_8)} = \frac{\pi^4}{384}$.

Proof. By Theorem 2.5, we have $\Delta_{\mathcal{P}(E_8)} = |E_8/E_8| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_8(\sqrt{2}/2)\right)}{\mathrm{Covol}(E_8)} = \frac{\pi^4}{384}$, where the last equality follows from Lemma 3.10 and the formula for volume of a ball: $\mathrm{Vol}(\mathcal{B}_d(R)) = R^d \pi^{d/2} / \Gamma\left(\frac{d}{2} + 1\right)$. $\square$

4 Facts from Fourier analysis

Recall the definition of a Fourier transform.

Definition 4.1. *The Fourier transform of an L^1-function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as*

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,y \rangle} \, \mathrm{d}x, \quad y \in \mathbb{R}^d$$

where $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$ is the standard scalar product in $\mathbb{R}^d$.

The following computational result will be of use later on.

Lemma 4.2.

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} \, e^{\pi i \|y\|^2 \, (\frac{-1}{z})}.$$

Proof. **Fill in proof.** □

Of great interest to us will be a specific family of functions, known as Schwartz Functions. The Fourier transform behaves particularly well when acting on Schwartz functions. We elaborate in the following subsections.

4.1 On Schwartz Functions

Definition 4.3. *A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a* Schwartz function *if it decays to zero as $\|x\| \to \infty$ faster then any inverse power of $\|x\|$, and the same holds for all partial derivatives of f, ie, if for all $k, n \in \mathbb{N}$, there exists a constant $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $\|x\|^k \cdot \left\|f^{(n)}(x)\right\| \leq C$, where $f^{(n)}$ denotes the n-th derivative of f considered along with the appropriate operator norm. The set of all Schwartz functions from $\mathbb{R}^d$ to $\mathbb{C}$ is called the* Schwartz space. *It is an $\mathbb{R}$-vector space.*

Lemma 4.4. *The Fourier transform is a continuous, linear automorphism of the space of Schwartz functions.*

Proof. We do not elaborate here as the result already exists in Mathlib. We do, however, mention that the Lean implementation *defines* a continuous linear equivalence on the Schwartz space *using* the Fourier transform (see `SchwartzMap.fourierTransformCLM`). The 'proof' that for any Schwartz function f, its Fourier transform and its image under this continuous linear equivalence are, indeed, the same $\mathbb{R}^d \to \mathbb{R}$ function, is stated in Mathlib solely for the purpose of `rw` and `simp` tactics, and is proven simply by `rfl`. □

Another reason we are interested in Schwartz Functions is that they behave well under infinite sums. This will be useful to us when proving the Cohn-Elkies linear programming bound.

4.2 On the Summability of Schwartz Functions

We begin by stating a general summability result over specific subsets of $\mathbb{R}^d$.

Lemma 4.5. *Let $X \subset \mathbb{R}^d$ be a set of sphere packing centres of separation* 1 *that is periodic with some lattice $\Lambda \subset \mathbb{R}^d$. Then, there exists $k \in \mathbb{N}$ sufficiently large such that*

$$\sum_{x \in X} \frac{1}{\|x\|^k} < \infty.$$

Proof. First, note that it does not matter how we number the (countably many) elements of the discrete set X: if we prove absolute convergence for one numbering, we prove absolute convergence for *any* numbering. The idea will be to bound the sequence of partial sums by considering the volumes of concentric d-spheres of the appropriate radii (or scaled versions of a 0-centred fundamental domain).

Finish!

$\square$

The decaying property of Schwartz functions means that they can be compared to the absolutely convergent power series above.

Lemma 4.6. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function and let $X \subset \mathbb{R}^d$ be periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$. Then,*

$$\sum_{x \in X} |f(x)| < \infty.$$

Proof. Without loss of generality, assume that $0 \notin X$: if $0 \in X$, then we can add the $f(0)$ term to the sum over nonzero elements of X, which, if the sum over the nonzero elements converges absolutely, will be equal to the sum over all of X. Now, we know that for all $k \in \mathbb{N}$, there exists some constant C such that $|f(x)| \leq C \|x\|^{-k}$ for all $x \in \mathbb{R}^d$. Choosing k to be sufficiently large, we see that

$$\sum_{x \in X} |f(x)| \leq \sum_{x \in X} \frac{C}{\|x\|^k} = C \sum_{x \in X} \frac{1}{\|x\|^k} < \infty.$$

$\square$

We end with a crucial result on Schwartz functions, the statement of which only makes sense because of the above result.

Theorem 4.7 (Poisson summation formula)**.** *Let Λ be a lattice in $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function. Then, for all $v \in \mathbb{R}^d$,*

$$\sum_{\ell \in \Lambda} f(\ell + v) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) e^{-2\pi i \langle v, m \rangle}.$$

Proof. **Fill in proof.** □

While the Poisson Summation Formula over lattices can be stated in greater generality (and probably should be formalised as such in Mathlib due to the many applications it admits), we stick with Schwartz functions because that should be sufficient for our purposes.

5 Cohn-Elkies linear programming bounds

In 2003 Cohn and Elkies [3] developed linear programming bounds that apply directly to sphere packings. The goal of this section is to formalize the Cohn–Elkies linear programming bound.

The following theorem is the key result of [3]. (The original theorem is stated for a class of functions more general then Schwartz functions)

Theorem 5.1. *(Cohn–Elkies [3]) Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function that is not identically zero and satisfies the following conditions:*

$$f(x) \leq 0 \text{ for } \|x\| \geq 1 \tag{1}$$

and

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d. \tag{2}$$

Then the density of d-dimensional sphere packings is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \mathrm{vol}(B_d(0, 1/2)).$$

Proof. Here we reproduce the proof given in [3]. We will first prove the theorem for periodic packings.

Let $X \subset \mathbb{R}^d$ be a discrete subset such that $\|x - y\| \geq 1$ for any distinct $x, y \in X$. Suppose that X is Λ-periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$.

The inequality

$$\sharp(X/\Lambda) \cdot f(0) \geq \sum_{x \in X} \sum_{y \in X/\Lambda} f(x-y) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l) \tag{3}$$

follows from the condition (1) of the theorem and the assumption on the distances between points in X. The equality

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x-y+l) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)}.$$

follows from the Poisson summation formula. The right hand side of the above equation can be written as

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)} = \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \big| \sum_{x \in X/\Lambda} e^{2\pi i m x} \big|^2.$$

Note that $\big|\sum_{x \in X/\Lambda} e^{2\pi i m x}\big|^2 \geq 0$ for all $m \in \Lambda^*$. Moreover, the term corresponding to $m = 0$ satisfies

$\left|\sum_{x\in X/\Lambda} e^{2\pi i 0x}\right|^2 = \sharp(X/\Lambda)^2$. Now we use the condition (2) and estimate

$$\frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{m\in\Lambda^*} \widehat{f}(m)\cdot \Big|\sum_{x\in X/\Lambda} e^{2\pi i m(x-y)}\Big|^2 \geq \frac{\sharp(X/\Lambda)^2}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}\cdot \widehat{f}(0). \tag{4}$$

Comparing inequalities (3) and (4) we arrive at

$$\frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \leq \frac{f(0)}{\widehat{f}(0)}.$$

Now we see that the density of the periodic packing $\mathcal{P}_X$ with balls of radius $1/2$ is bounded by

$$\Delta(\mathcal{P}_X) = \frac{\sharp(X/\Lambda)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}\cdot \mathrm{vol}(B_d(0,1/2)) \leq \frac{f(0)}{\widehat{f}(0)}\cdot \mathrm{vol}(B_d(0,1/2)).$$

This finishes the proof of the theorem for periodic packings. Theorem 1.15 implies the desired result for arbitrary packings. □

The main step in our proof of Theorem 1.16 is the explicit construction of an optimal function. It will be convenient for us to scale this function by $\sqrt{2}$.

Theorem 5.2. *There exists a radial Schwartz function $g:\mathbb{R}^8\to\mathbb{R}$ which satisfies:*

$$g(x)\leq 0 \text{ for } \|x\|\geq\sqrt{2} \tag{5}$$

$$\widehat{g}(x)\geq 0 \text{ for all } x\in\mathbb{R}^8 \tag{6}$$

$$g(0)=\widehat{g}(0)=1. \tag{7}$$

Theorem 5.1 applied to the optimal function $f(x)=g(x/\sqrt{2})$ immediately implies Theorem 1.16.

6 Modular forms

In this section, we recall and develop some theory of (quasi)modular forms.

6.1 Modular forms and examples

Let $\mathfrak{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Lemma 6.1. *The modular group $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by linear fractional transformations*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}.$$

Let N be a positive integer.

Definition 6.2. *The* level N principal congruence subgroup *of Γ_1 is*

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N \right\}.$$

Definition 6.3. *A subgroup $\Gamma \subset \Gamma_1$ is called a* congruence subgroup *if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{N}$.*

Definition 6.4. *Define the matrices*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma_1, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1, \alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma_2 \subset \Gamma_1, \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \Gamma_2 \subset \Gamma_1.$$

It is easily verifiable that $\alpha = T^2$ and $\beta = -S\alpha^{-1}S = -ST^{-2}S$.

The following two lemmas tell us the group structure of $\Gamma(1) = \Gamma_1$ and $\Gamma(2)$, which we will use later on to define the theta forms.

Lemma 6.5. *We have $\Gamma(1) = \langle S, T, -I \rangle$.*

Proof. See [4, Exercise 1.1.1]. $\square$

Lemma 6.6. *We have $\Gamma(2) = \langle \alpha, \beta, -I \rangle$.*

Proof. See [4, Exercise 1.2.4]. $\square$

Let $z \in \mathfrak{H}$, $k \in \mathbb{Z}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We omit many of the proofs below when they exist in Mathlib already.

Definition 6.7. *The* automorphy factor *of weight k is defined as*

$$j_k(z, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := (cz+d)^{-k}.$$

Lemma 6.8. *The automorphy factor satisfies the* chain rule

$$j_k(z, \gamma_1\gamma_2) = j_k(z, \gamma_1)\, j_k(\gamma_2 z, \gamma_1).$$

Definition 6.9. *Let F be a function on $\mathfrak{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then the* slash operator *acts on F by*

$$(F|_k\gamma)(z) := j_k(z,\gamma)\, F(\gamma z).$$

Lemma 6.10. *The chain rule implies*

$$F|_k\gamma_1\gamma_2 = (F|_k\gamma_1)|_k\gamma_2.$$

In particular, this lemma implies that if $\Gamma = \langle M_i \rangle_{i\in\mathcal{I}}$, then the slash action $F|\gamma$ is uniquely determined by the action of generators, i.e. $F|M_i$ and $F|M_i^{-1}$.

Lemma 6.11. *For even k, $F|_k(-I) = F$.*

Proof. Follows from the definition of the slash operator: $(F|_k(-I))(z) = (-1)^{-k}F((-I)z) = F(z)$. □

Definition 6.12. *A* (holomorphic) modular form *of integer weight k and congruence subgroup Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that:*

1. *(Slash invariant) $f|_k\gamma = f$ for all $\gamma \in \Gamma$*
2. *(Holomorphic at $i\infty$) for each $\alpha \in \Gamma_1$ $f|_k\alpha$ has the Fourier expansion $f|_k\alpha(z) = \sum_{n=0}^{\infty} c_f(\alpha, \frac{n}{n_\alpha})\, e^{2\pi i \frac{n}{n_\alpha} z}$ for some $n_\alpha \in \mathbb{N}$ and Fourier coefficients $c_f(\alpha, m) \in \mathbb{C}$.*

Definition 6.13. *Let $M_k(\Gamma)$ be the space of modular forms of weight k and congruence subgroup Γ.*

Let us consider several examples of modular forms.

Definition 6.14. *For an even integer $k \geq 4$ we define the* weight k Eisenstein series *as*

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} (cz+d)^{-k}. \tag{8}$$

Lemma 6.15. *For all k, $E_k \in M_k(\Gamma_1)$. Especially, we have*

$$E_k\left(-\frac{1}{z}\right) = z^k E_k(z). \tag{9}$$

Proof. This follows from the fact that the sum converges absolutely. Now apply slash operator with $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ gives (9). $\square$

Lemma 6.16. *The Eisenstein series possesses the Fourier expansion*

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i z}, \tag{10}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. In particular, we have

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z} \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}. \end{aligned}$$

The infinite sum (8) does not converge absolutely for $k = 2$. On the other hand, the expression (10) converges to a holomorphic function on the upper half-plane and we will take it as a definition of E_2 (See Definition 6.18 below).

The discriminant form is a unique normalized cusp form of weight 12, which can be defined as:

Definition 6.17. *The* discriminant form $\Delta(z)$ *is given by*

$$\Delta(z) = e^{2\pi i z} \prod_{n\geq 1} (1 - e^{2\pi i n z})^{24}.$$

This product formula allows us to prove positivity of $\Delta(it)$ for $t > 0$ later. But we need to first check its a modular form. For this we first need some definitions/ results.

We define it as a q-series, which gives a holomorphic function on $\mathfrak{H}$.

Definition 6.18. *We set*

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i n z}. \tag{11}$$

Lemma 6.19. *This function is not modular, however it satisfies*

$$z^{-2}\, E_2\left(-\frac{1}{z}\right) = E_2(z) - \frac{6i}{\pi}\,\frac{1}{z}. \tag{12}$$

Proof. This is excercise 1.2.8 of [4].

∎

Definition 6.20. *The Dedekind eta function is defined as*

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

where $q = e^{2\pi i z}$.

Lemma 6.21. *The Dedekind eta function transforms as*

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\eta(z).$$

Proof. Cosider the logarithmic derivative of η, which one can easily see is equal to $\frac{\pi i}{12} E_2$. The result then follows from the transformation of E_2.

See [4, proposition 1.2.5]. ∎

Lemma 6.22. $\Delta(z) \in M_{12}(\Gamma_1)$. *Especially, we have*

$$\Delta\left(-\frac{1}{z}\right) = z^{12}\Delta(z).$$

Also, it vanishes at the unique cusp, i.e. it is a cusp form of level Γ_1 *and weight* 12.

Proof. The fact that it is invariant under translation is clear from the definition, so we only need to check transformation under S. Now, note that $\eta^2 4 = \Delta$, and from 6.21 we have $\eta(-1/z) = \sqrt{-iz}\eta(z)$, so $\Delta(-1/z) = z^{12}\Delta(z)$ as required.

∎

Using this one can now easily check that we have

Lemma 6.23.

$$(cz+d)^{-2}E_2\left(\frac{az+b}{cx+d}\right) = E_2(z) - \frac{6ic}{\pi(cz+d)}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{13}$$

Proof. Modularity of $\Delta(z)$ gives $(cz+d)^{-12}\Delta(\frac{az+b}{cz+d}) = \Delta(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$, and by differentiating it we get

$$(cz+d)^{-14}\Delta'\left(\frac{az+b}{cz+d}\right) = \Delta'(z) - \frac{6ic}{\pi(cz+d)}\Delta(z).$$

Now, divide both sides with $\Delta(z)$ proves (13). ☐

Lemma 6.24. *We have*

$$\Delta(z) = (E_4^3 - E_6^2)/1728.$$

Proof. We only need to show its a cuspform, since once we have this, dividing the rhs by Δ would give a modular form of weight 0 which is a constant, and so we can determine the constant easily.

To checke its a cuspform, we just look at the q-expansions of E_4 and E_6 and prove directly that the first term vanishes.

☐

Corollary 6.25. $\Delta(it) > 0$ *for all* $t > 0$.

Proof. By 6.17, we have

$$\Delta(it) = e^{-2\pi t}\prod_{n\geq 1}(1 - e^{-2\pi nt})^{24} > 0.$$

☐

The following nonvanishing result, which directly follows from Definition 6.17, will be used in the construction of the magic function.

Corollary 6.26. $\Delta(z) \neq 0$ *for all* $z \in \mathfrak{H}$.

Proof. This follows from the product formula. ☐

A key fact in the theory of modular forms is that the spaces $M_k(\Gamma)$ are finite-dimensional. To prove this we will do use the following non-standard proof. First we have the following result.

Theorem 6.27. *Let* $k \in \mathbb{Z}$ *with* $k < 0$. *Then* $M_k(\Gamma_1) = \{0\}$ *and moreover* $\dim M_0(\Gamma(1)) = 1$.

Proof. The proof makes use of the maximum modulus principle, as its already been formalised we skip the details here but see the lean proof for details. $\square$

Theorem 6.28. *Let $k \in \mathbb{Z}$ with $k \geq 0$ and even. Then* $\dim M_k(\Gamma_1) = \lfloor k/12 \rfloor$ *if* $k \equiv 2 \mod 12$ *and* $\dim M_k(\Gamma_1) = \lfloor k/12 \rfloor + 1$ *if* $k \not\equiv 2 \mod 12$.

Proof. First we note that for $2 < k$ we have $\dim(M_k(\Gamma_1)) = 1 + \dim S_k(\Gamma_1)$. This follows since we know the E_k are in M_k so by scalling appropriately, any non-cuspform $f \in M_k$ we would have $f - aE_k \in S_k$ for some a.

Next, note that $S_k(\Gamma_1)$ is isomorphic to $M_{k-12}(\Gamma_1)$, since if $f \in S_k$ then f/Δ is now a modular form (using the product expansion of Δ and its non-vanishing on $\mathfrak{H}$) of weight $k - 12$. Note its important that f is a cuspform so that the quotient by Δ is a modular form.

So we only need to know the dimensions of $M_k(\Gamma_1)$ for $0 \leq k \leq 12$. For $k = 0$ we have $\dim M_0(\Gamma_1) = 1$ by Theorem 6.27. For $k = 4$ we have $\dim M_4(\Gamma_1) = 1$ since if there was a cuspform f of weight 4 then f/Δ would be a modular form of negative weight, i.e. zero, so $f = 0$. Similarly for $k = 6, 8, 10$. For $k = 12$ we have $\dim S_{12}(\Gamma_1) = 2$ since the discriminant form is a cusp form of weight 12 and any other cusp form of weight 12 would be a scalar multiple of Δ (since their ratio would be a modular form of weight 0). So we have $\dim M_{12}(\Gamma_1) = 2$.

Finally we need to check that $\dim M_2(\Gamma_1) = 0$. Firstly, there can't be any cuspforms here by the same argument as above. So we need to check that there are no modular forms of weight 2. If we did have one, call it f then f^2 would be a non-cuspform of weight 4 and so $f^2 = aE_4$, where in fact $a = a_0(f)^2$ (since $(f^2 - a_0(f)E_4)$ is now a cuspform of weight 4 which means its zero). Similarly, $f^3 = a_0(f)^3 E_6$. But now taking powers to make them weight 12 forms we see that $a_0(f)^6(E_4^3 - E_6^2) = 0 = 1728 a_0(f)^6 \Delta$ but $a_0(f) \neq 0$ (since its assumed to not be a cuspform), this would mean $\Delta = 0$ which we know can't happen.

$\square$

Theorem 6.29. *Let Γ be a congruence subgroup. Then $M_k(\Gamma)$ is finite-dimensional.*

Proof. We know that $\dim(M_k(\Gamma_1))$ is finite dimensional from the above, now this means that there is some r_k such that any element of $M_k(\Gamma_1)$ vanishing at infinity to degree $> r_k$ must be zero. Now take $f \in M_k(\Gamma)$ and vanishes to degree n at infinity, then consider $F = \prod_\gamma f \mid_k \gamma$ where the product is over a set of representatives of $\Gamma_1 \backslash \Gamma$. Then F is a modular form of weight kd where $d = [\Gamma_1 : \Gamma]$ and vanishes at infinity to degree at least n. So if $n > r_{kd}$ then $F = 0$ meaning the $f = 0$. $\square$

Corollary 6.30. *We have*

$$\begin{aligned}\dim M_4(\mathrm{SL}_2(\mathbb{Z})) &= 1,\\ \dim M_6(\mathrm{SL}_2(\mathbb{Z})) &= 1,\\ \dim M_8(\mathrm{SL}_2(\mathbb{Z})) &= 1,\\ \dim M_2(\Gamma(2)) &= 2,\end{aligned} \tag{14}$$

Another examples of modular forms we would like to consider are *theta functions* [13, Section 3.1].

Definition 6.31. *We define three different theta functions (so called "Thetanullwerte") as*

$$\begin{aligned}\Theta_2(z) = \theta_{10}(z) \ &= \sum_{n\in\mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 z}.\\ \Theta_3(z) = \theta_{00}(z) \ &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}\\ \Theta_4(z) = \theta_{01}(z) \ &= \sum_{n\in\mathbb{Z}} (-1)^n \, e^{\pi i n^2 z}\end{aligned}$$

For convenience, we use the following notations for the fourth powers of the theta functions.

Definition 6.32. *Define*

$$H_2 = \Theta_2^4, \quad H_3 = \Theta_3^4, \quad H_4 = \Theta_4^4.$$

Note that we only need these fourth powers to define (7.18).

The group Γ_1 is generated by the elements $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and $-I = \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$ (Lemma 6.5), and the transformation of functions under $\Gamma(2)$ is determined by that under Γ_1 (by Lemma 6.10). The following lemma shows how the theta functions (and their powers) transform under the slash action of these matrices.

Lemma 6.33. *These elements act on the theta functions in the following way*

$$H_2|S = -H_4 \tag{15}$$

$$H_3|S = -H_3 \tag{16}$$

$$H_4|S = -H_2 \tag{17}$$

and

$$\begin{aligned} H_2|T &= -H_2 \\ H_3|T &= H_4 \\ H_4|T &= H_3 \end{aligned} \tag{18}$$

Proof. The last three identities easily follow from the definition. For example, (18) follows from

$$\begin{aligned} \Theta_2(z+1) &= \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2(z+1)} = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2} e^{\pi i(n+\frac{1}{2})^2 z} \\ &= \sum_{n\in\mathbb{Z}} e^{\pi i(n^2+n+\frac{1}{4})} e^{\pi i(n+\frac{1}{2})^2 z} = \sum_{n\in\mathbb{Z}} (-1)^{n^2+n} e^{\pi i/4} e^{\pi i(n+\frac{1}{2})^2 z} \\ &= e^{\pi i/4}\Theta_2(z) \end{aligned}$$

and taking 4th power. (15) and (17) are equivalent under $z \leftrightarrow -1/z$, so it is enough to show (15) and (16). These identities follow from the identities of the *two-variable* Jacobi theta function, which is defined as (be careful for the variables, where we use τ instead of z)

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}$$

and already formalized by David Loeffler. This function specialize to the theta functions as

$$\begin{aligned} \Theta_2(\tau) &= e^{\pi i \tau/4}\theta(-\tau/2,\tau) \\ \Theta_3(\tau) &= \theta(0,\tau) \\ \Theta_4(\tau) &= \theta(1/2,\tau) \end{aligned}$$

Possion summation formula gives

$$\theta(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i z^2}{\tau}} \theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

and applying the specializations above yield the identities. For example, (17) follows from

$$\Theta_4(\tau) = \theta\left(\frac{1}{2},\tau\right) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}} \theta\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}} \Theta_2\left(-\frac{1}{\tau}\right)$$

and taking 4th power. □

Using the above identities, we can prove that these are modular forms.

Lemma 6.34. *H_2, H_3, and H_4 are slash invariant under $\Gamma(2)$, i.e. for all $\gamma \in \Gamma(2)$ and $i \in \{2,3,4\}$, we have $H_i|\gamma = H_i|\gamma^{-1} = H_i$.*

Proof. By Lemma 6.6 and Lemma 6.10, it suffices to show that the H_i are invariant under slash actions with respect to α, β, and $-I$. Invariance under $-I$ follows from Lemma 6.11. The rest follows from Lemma 6.10, 6.33, and the matrix identities

$$\alpha = T^2, \quad \beta = -S\alpha^{-1}S = -ST^{-2}S.$$

For example, invariance for H_2 can be proved by

$$\begin{aligned} H_2|\alpha &= H_2|T^2 = -H_2|T = H_2 \\ H_2|\beta &= H_2|(-S\alpha^{-1}S) = H_2|(S\alpha^{-1}S) = -H_4|(\alpha^{-1}S) = -H_4|S = H_2. \end{aligned}$$

□

Lemma 6.35. *For all $\gamma \in \Gamma_1$, $H_2|_2\gamma$, $H_3|_2\gamma$, and $H_4|_2\gamma$ are holomorphic at $i\infty$.*

Proof. We want to show that for $\gamma \in \Gamma_1$, $\|H_2|_2\gamma(z)\|$ is bounded as $z \in \mathbb{H} \to i\infty$. Firstly, by Lemma 6.33, Lemma 6.6 and induction on group elements, we notice that $\{\pm H_2, \pm H_3, \pm H_4\}$ is closed under action by Γ_1. Hence, it suffices to prove that H_2, H_3 and H_4 are bounded at $i\infty$. Consider $z \in \mathbb{H}$ with $\Im(z) \geq A$. We proceed by direct algebraic manipulation:

$$\begin{aligned} \|H_2(z)\| &= \left\| \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(n + \frac{1}{2}\right)^2 z\right) \right\|^4 \leq \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(\pi i \left(n + \frac{1}{2}\right)^2 z\right)\right\| \right)^4 \\ &= \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(-\pi \left(n + \frac{1}{2}\right)^2 \Im(z)\right)\right\| \right)^4 \leq \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(-\pi \left(n + \frac{1}{2}\right)^2 A\right)\right\| \right)^4 \end{aligned}$$

Where we prove the final term is convergent by noticing that it equals $\exp(-\pi A/4)\theta(iA/2, iA)$, which has been shown to converge in `Mathlib`. The proofs for H_3 and H_4 are similar (actually easier) and have been omitted.

It seems the `MDifferentiable` requirement is missing. □

Lemma 6.36. *H_2, H_3, and H_4 belong to $M_2(\Gamma(2))$.*

Proof. From Lemma 6.34 and Lemma 6.35, it remains ot prove that H_2, H_3 and H_4 are holomorphic on $\mathbb{H}$. **fill in proof.** ☐

They have Fourier expansions as follows.

Proposition 6.37. *H_2 admits a Fourier series of the form*

$$H_2(z) = \sum_{n \geq 1} c_{H_2}(n) e^{\pi i n z}$$

for some $c_{H_2}(n) \in \mathbb{R}_{\geq 0}$, with $c_{H_2}(1) = 16$ and $c_{H_2}(n) = O(n^k)$ for some $k \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}
H_2(z) &= \Theta_2(z)^4 \\
&= \left(\sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 z} \right)^4 \\
&= \left(2 \sum_{n \geq 0} e^{\pi i (n+\frac{1}{2})^2 z} \right)^4 \\
&= \left(2 e^{\pi i z/4} + 2 \sum_{n \geq 1} e^{\pi i (n^2+n+\frac{1}{4}) z} \right)^4 \\
&= 16 e^{\pi i z} \left(1 + \sum_{n \geq 1} e^{\pi i (n^2+n) z} \right)^4 \\
&= 16 e^{\pi i z} + \sum_{n \geq 2} c_{H_2}(n) e^{\pi i n z} \\
&= \sum_{n \geq 1} c_{H_2}(n) e^{\pi i n z}.
\end{aligned}$$

☐

Proposition 6.38. *H_3 admits a Fourier series of the form*

$$H_3(z) = \sum_{n \geq 0} c_{H_3}(n) e^{\pi i n z}$$

for some $c_{H_3}(n) \in \mathbb{R}_{\geq 0}$ with $c_{H_3}(0) = 1$ and $c_{H_3}(n) = O(n^k)$ for some $k \in \mathbb{N}$. Especially, H_3 is not cuspidal.

Proof. We have

$$H_3(z) = \Theta_3(z)^4 = \left(\sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}\right)^4 = \left(1 + 2\sum_{n\geq 1} e^{\pi i n^2 z}\right)^4 = 1 + O(e^{\pi i z}).$$

□

Proposition 6.39. *H_4 admits a Fourier series of the form*

$$H_4(z) = \sum_{n\geq 0} c_{H_4}(n) e^{\pi i n z}$$

for some $c_{H_4}(n) \in \mathbb{R}$ with $c_{H_4}(0) = 1$ and $c_{H_4}(n) = O(n^k)$ for some $k \in \mathbb{N}$. Especially, H_4 is not cuspidal.

We also have a nontrivial relation between these theta functions.

Lemma 6.40. *These three theta functions satisfy the* Jacobi identity

$$H_2 + H_4 = H_3 \Leftrightarrow \Theta_2^4 + \Theta_4^4 = \Theta_3^4. \tag{19}$$

Proof. By (14), H_2, H_3, and H_4 are linearly dependent. The relation can be identified by comparing the first two q-coefficients of the Fourier expansions. □

These are also related to E_4, E_6, and Δ as follows.

Lemma 6.41. *We have*

$$E_4 = \frac{1}{2}(H_2^2 + H_3^2 + H_4^2) = H_2^2 + H_2H_4 + H_4^2 \tag{20}$$

$$E_6 = \frac{1}{2}(H_2 + H_3)(H_3 + H_4)(H_4 - H_2) = \frac{1}{2}(H_2 + 2H_4)(2H_2 + H_4)(H_4 - H_2) \tag{21}$$

$$\Delta = \frac{1}{256}(H_2H_3H_4)^2. \tag{22}$$

Proof. We can prove these similarly as Lemma 6.40. Right hand sides of (20), (21), and (22) are all modular forms of level Γ_1 and desired weights, where (22) is a cusp form since H_2 is. Now the identities follow from the dimension calculations $\dim M_4(\Gamma_1) = \dim M_6(\Gamma_1) = \dim S_{12}(\Gamma_1) = 1$ and comparing the first nonzero q-coefficients. □

The *strict* positivity of Jacobi theta functions might needed later.

Corollary 6.42. *All three functions $t \mapsto H_2(it), H_3(it), H_4(it)$ are positive for $t > 0$.*

Proof. By Lemma 6.40 and the transformation law (15), it is enough to prove the positivity for $\Theta_2(it)$, which is clear from its definition:

$$\Theta_2(it) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+\frac{1}{2})^2 t} > 0.$$

$\square$

6.2 Quasimodular forms and derivatives

Morally, quasimodular forms can be thought as *modular forms with differentiations.* It can be defined formally as follows: Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function, and let k and $s \geq 0$ be integers. The function f is a *quasimodular form of weight* k*, level* Γ*, and depth* s if there exist holomorphic functions $f_0, \dots, f_s : \mathfrak{H} \to \mathbb{C}$ such that

$$(f|_k\gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{s} f_j(z) \left(\frac{c}{cz+d}\right)^j$$

for all $z \in \mathfrak{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

By taking $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, one can check that we should have $f_0 = f$. Thus, a quasimodular form of depth 0 is just a modular form of same weight and level. Also, it is easy to see that the space of quasimodular forms is closed under the normalized derivative.

Definition 6.43. *Let* F *be a quasimodular form. We define the (normalized) derivative of* F *as*

$$F' = DF := \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}F. \tag{23}$$

D is normalized as in (23) because of the following lemma.

Lemma 6.44. *We have an equality of operators* $D = q\frac{\mathrm{d}}{\mathrm{d}q}$*. In particular, the* q*-series of the derivative of a quasimodular form* $F(z) = \sum_{n\geq n_0} a_n q^n$ *is* $F'(z) = \sum_{n\geq n_0} n a_n q^n$*.*

Proof. Directly follows from the definition (6.43), where $\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}e^{2\pi i n z} = n e^{2\pi i n z}$. $\square$

The most important quasimodular form is the weight 2 Eisenstein series E_2.

Definition 6.45. *For* $k \in \mathbb{R}$*, define the weight* k *Serre derivative* ∂_k *of a modular form* F *as*

$$\partial_k F := F' - \frac{k}{12}E_2 F.$$

Theorem 6.46. *Let F be a modular form of weight k and level Γ. Then, $\partial_k F$ is a modular form of weight $k+2$ of the same level.*

Proof. Let $G = \partial_k F = F' - \frac{k}{12} E_2 F$. It is enough to show that G is invariant under $|_{k+2}\gamma$ for $\gamma \in \Gamma$. From $F \in M_k(\Gamma)$, we have

$$(F|_k\gamma)(z) := (cz+d)^{-k} F\left(\frac{az+b}{cz+d}\right) = F(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

By taking the derivative of the above equation, we get

$$\begin{aligned} &-kc(cz+d)^{-k-1} F\left(\frac{az+b}{cz+d}\right) + (cz+d)^{-k}(cz+d)^{-2}\frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{d}F}{\mathrm{d}z}(z) \\ &\Leftrightarrow (cz+d)^{-k-2} F'\left(\frac{az+b}{cz+d}\right) = F'(z) - \frac{ikc}{2\pi(cz+d)} F(z). \end{aligned}$$

Combined with (13), we get

$$\begin{aligned} ((\partial_k F)|_{k+2}\gamma)(z) &= (cz+d)^{-k-2}\left(F'\left(\frac{az+b}{cz+d}\right) - \frac{k}{12} E_2\left(\frac{az+b}{cz+d}\right) F\left(\frac{az+b}{cz+d}\right)\right) \\ &= F'(z) - \frac{ikc}{2\pi(cz+d)} F(z) - \frac{k}{12}\left(E_2 - \frac{6c}{\pi(cz+d)}\right) F(z) \\ &= F'(z) - \frac{k}{12} E_2(z) F(z) = (\partial_k F)(z) \end{aligned}$$

so $\partial_k F \in M_{k+2}(\Gamma)$. $\square$

Remark 6.47. *More generally, the following theorem holds: if F is a quasimodular form of weight k and depth s, then $\partial_{k-s} F$ is a quasimodular form of weight $k+2$* and depth $\leq s$ *of the same level. We will not prove this here.*

Theorem 6.48. *We have*

$$E_2' = \frac{E_2^2 - E_4}{12} \tag{24}$$

$$E_4' = \frac{E_2 E_4 - E_6}{3} \tag{25}$$

$$E_6' = \frac{E_2 E_6 - E_4^2}{2} \tag{26}$$

Proof. In terms of Serre derivatives, these are equivalent to

$$\begin{aligned} \partial_1 E_2 &= -\frac{1}{12} E_4 \\ \partial_4 E_4 &= -\frac{1}{3} E_6 \end{aligned}$$

$$\partial_6 E_6 = -\frac{1}{2}E_4^2$$

By Theorem 6.46, all the serre derivatives are, in fact, modular. To be precise, the modularity of $\partial_4 E_4$ and $\partial_6 E_6$ directly follows from Theorem 6.46, and that of $\partial_1 E_2$ follows from (13). Differentiating and squaring then gives us the following:

$$\begin{aligned} E_2'|_4\gamma &= E_2' - \frac{ic}{\pi(cz+d)}E_2 - \frac{3c^2}{\pi^2(cz+d)^2} \\ E_2^2|_4\gamma &= E_2^2 - \frac{12ic}{\pi(cz+d)}E_2 - \frac{36c^2}{\pi^2(cz+d)^2} \end{aligned} \tag{27}$$

Hence, (24)$-\frac{1}{12}$(27) is a modular form of weight 4. By Corollary 6.30, they should be multiples of E_4, E_6, E_4^2, and the proportionality constants can be determined by observing the constant terms of q-expansions. □

Corollary 6.49.

$$\Delta' = E_2\Delta. \tag{28}$$

Proof. By Ramanujan's formula (25) and (26),

$$\Delta' = \frac{3E_4^2E_4' - 2E_6E_6'}{1728} = \frac{1}{1728}\left(3E_4^2 \cdot \frac{E_2E_4 - E_6}{3} - 2E_6 \cdot \frac{E_2E_6 - E_4^2}{2}\right) = \frac{E_2(E_4^3 - E_6^2)}{1728} = E_2\Delta.$$

□

Similar argument allow us to compute (Serre) derivatives of H_2, H_3, H_4.

Proposition 6.50. *We have*

$$\begin{aligned} H_2' &= \frac{1}{6}(H_2^2 + 2H_2H_4 + E_2H_2) \\ H_3' &= \frac{1}{6}(H_2^2 - H_4^2 + E_2H_3) \\ H_4' &= -\frac{1}{6}(2H_2H_4 + H_4^2 - E_2H_4) \end{aligned}$$

or equivalently,

$$\partial_2 H_2 = \frac{1}{6}(H_2^2 + 2H_2H_4) \tag{29}$$

$$\partial_2 H_3 = \frac{1}{6}(H_2^2 - H_4^2) \tag{30}$$

$$\partial_2 H_4 = -\frac{1}{6}(2H_2H_4 + H_4^2) \tag{31}$$

Proof. Equivalences are obvious from the definition of the Serre derivative. By Theorem 6.46, all the

Serre derivatives are modular forms of weight 4 and level $\Gamma(2)$. We have $\dim M_4(\Gamma(2)) = 3$ with basis H_2^2, H_2H_4, H_4^2, and comparing the first three q-coefficients give (29), (30), and (31). $\square$

Theorem 6.51. *The Serre derivative satisfies the following product rule: for any quasimodular forms F and G,*

$$\partial_{w_1+w_2}(FG) = (\partial_{w_1}F)G + F(\partial_{w_2}G).$$

Proof. It follows from the definition:

$$\begin{aligned}
\partial_{w_1+w_2}(FG) &= (FG)' - \frac{w_1+w_2}{12}E_2(FG) \\
&= F'G + FG' - \frac{w_1+w_2}{12}E_2(FG) \\
&= \left(F' - \frac{w_1}{12}E_2F\right)G + F\left(G' - \frac{w_2}{12}E_2G\right) \\
&= (\partial_{w_1}F)G + F(\partial_{w_2}G).
\end{aligned}$$

$\square$

We also have the following useful theorem for proving positivity of quasimodular forms on the imaginary axis, which is [7, Proposition 3.5, Corollary 3.6].

Theorem 6.52. *Let F be a holomorphic quasimodular cusp form with real Fourier coefficients. Assume that there exists k such that $(\partial_k F)(it) > 0$ for all $t > 0$. If the first Fourier coefficient of F is positive, then $F(it) > 0$ for all $t > 0$.*

Proof. By (28), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{\Delta(it)^{\frac{k}{12}}}\right) &= (-2\pi)\frac{F'(it)\Delta(it)^{\frac{k}{12}} - F(it)\frac{k}{12}E_2(it)\Delta(it)^{\frac{k}{12}}}{\Delta(it)^{\frac{k}{6}}} \\
&= (-2\pi)\frac{(\partial_k F)(it)}{\Delta(it)^{\frac{k}{12}}} < 0,
\end{aligned}$$

hence

$$t \mapsto \frac{F(it)}{\Delta(it)^{\frac{k}{12}}}$$

is monotone decreasing. Because of the assumption on the positivity of the first nonzero Fourier coefficient of F, $F(it) > 0$ for sufficiently large t since

$$F = \sum_{n \geq n_0} a_n q^n \Rightarrow e^{2\pi n_0 t}F(it) = a_{n_0} + e^{-2\pi t}\sum_{n \geq n_0+1} a_n e^{-2\pi(n-n_0-1)t}$$

and $\lim_{t\to\infty} e^{2\pi n_0 t}F(it) = a_{n_0} > 0$, hence the result follows. $\square$

7 Fourier eigenfunctions with double zeroes at lattice points

In this section we construct two radial Schwartz functions $a, b : \mathbb{R}^8 \to i\mathbb{R}$ such that

$$\mathcal{F}(a) = a \tag{32}$$

$$\mathcal{F}(b) = -b \tag{33}$$

which double zeroes at all Λ_8-vectors of length greater than $\sqrt{2}$. Recall that each vector of Λ_8 has length $\sqrt{2n}$ for some $n \in \mathbb{N}_{\geq 0}$. We define a and b so that their values are purely imaginary because this simplifies some of our computations. We will show in Section 8 that an appropriate linear combination of functions a and b satisfies conditions (5)–(7).

First, we will define function a. To this end we consider the following functions:

Definition 7.1.

$$\begin{aligned}
\phi_{-4} &:= \frac{E_4^2}{\Delta} \\
\phi_{-2} &:= \frac{E_4(E_2E_4 - E_6)}{\Delta} \\
\phi_0 &:= \frac{(E_2E_4 - E_6)^2}{\Delta}
\end{aligned}$$

The function $\phi_0(z)$ is not modular; however, it satisfies the following transformation rules:

Lemma 7.2. *We have*

$$\phi_0(z+1) = \phi_0(z) \tag{34}$$

$$\phi_0\left(-\frac{1}{z}\right) = \phi_0(z) - \frac{12i}{\pi}\,\frac{1}{z}\,\phi_{-2}(z) - \frac{36}{\pi^2}\,\frac{1}{z^2}\,\phi_{-4}(z). \tag{35}$$

Proof. (34) easily follows from periodicity of Eisenstein series and $\Delta(z)$. For (35),

$$\begin{aligned}
\phi_0\left(-\frac{1}{z}\right) &= \frac{(E_2(-1/z)E_4(-1/z) - E_6(-1/z))^2}{\Delta(-1/z)} \\
&= \frac{((z^2E_2(z) - 6iz/\pi)\cdot z^4E_4(z) - z^6E_6(z))^2}{z^{12}\Delta(z)} \\
&= \frac{\left(E_2(z)E_4(z) - E_6(z) - \frac{6i}{\pi z}E_4(z)\right)^2}{\Delta(z)} \\
&= \frac{(E_2(z)E_4(z) - E_6(z))^2 - \frac{12i}{\pi z}(E_2(z)E_4(z) - E_6(z))E_4(z) - \frac{36}{\pi^2z^2}E_4(z)^2}{\Delta(z)} \\
&= \phi_0(z) - \frac{12i}{\pi z}\phi_{-2}(z) - \frac{36}{\pi^2z^2}\phi_{-4}(z).
\end{aligned}$$

□

Definition 7.3. *For $x \in \mathbb{R}^8$ we define*

$$a(x) := \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i \|x\|^2 z}\,dz + \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i \|x\|^2 z}\,dz \tag{36}$$
$$-2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i \|x\|^2 z}\,dz + 2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i \|x\|^2 z}\,dz.$$

We observe that the contour integrals in (36) converge absolutely and uniformly for $x \in \mathbb{R}^8$. Indeed, $\phi_0(z) = O(e^{-2\pi i z})$ as $\Im(z) \to \infty$. Therefore, $a(x)$ is well defined. Now we prove that a satisfies condition (32). The following lemma will be used to prove Schwartzness of a and b.

Lemma 7.4. *Let $f(z)$ be a holomorphic function with a Fourier expansion*

$$f(z) = \sum_{n \geq n_0} c_f(n) e^{\pi i n z}$$

with $c_f(n_0) \neq 0$. Assume that $c_f(n)$ has a polynomial growth, i.e. $|c_f(n)| = O(n^k)$ for some $k \in \mathbb{N}$. Then there exists a constant $C_f > 0$ such that

$$\left|\frac{f(z)}{\Delta(z)}\right| \leq C_f e^{-\pi(n_0-2)\Im z}$$

for all z with $\Im z > 1/2$.

Note that the assumption on the polynomial growth holds when f is a holomorphic modular form, where the proof can be found in [11, p. 94] for the case of level 1 modular forms. But we just add this for simplicity, and we can prove it for "specific" f such as Eisenstein series, theta functions, and their combinations.

Proof. By the product formula (**??**),

$$\begin{aligned}
\left|\frac{f(z)}{\Delta(z)}\right| &= \left|\frac{\sum_{n\geq n_0} c_f(n) e^{\pi i n z}}{e^{2\pi i z}\prod_{n\geq 1}(1-e^{2\pi i n z})^{24}}\right| \\
&= |e^{\pi i (n_0-2)z}| \cdot \frac{|\sum_{n\geq n_0} c_f(n) e^{\pi i (n-n_0)z}|}{\prod_{n\geq 1}|1-e^{2\pi i n z}|^{24}} \\
&\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n\geq n_0} |c_f(n)| e^{-\pi(n-n_0)\Im z}}{\prod_{n\geq 1}(1-e^{-2\pi n \Im z})^{24}} \\
&\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n\geq n_0} |c_f(n)| e^{-\pi(n-n_0)/2}}{\prod_{n\geq 1}(1-e^{-\pi n})^{24}} \\
&= C_f \cdot e^{-\pi(n_0-2)\Im z}
\end{aligned}$$

where

$$C_f = \frac{\sum_{n\geq n_0} |c_f(n)| e^{-\pi(n-n_0)/2}}{\prod_{n\geq 1}(1-e^{-\pi n})^{24}}.$$

Note that the summation in the numerator converges absolutely because of polynomial growth. The denominator also converges, which is simiply $e^{\pi} \cdot \Delta(i/2)$. ☐

As corollaries, we have the following bound for ϕ_0, ϕ_{-2}, and ϕ_{-4}.

Corollary 7.5. *There exists a constant $C_0 > 0$ such that*

$$|\phi_0(z)| \leq C_0 e^{-2\pi \Im z} \tag{37}$$

for all z with $\Im z > 1/2$.

Proof. By Ramanujan's formula, $E_2 E_4 - E_6 = 3E_4' = 720 \sum_{n\geq 1} n\sigma_3(n) e^{2\pi i n z}$ and

$$(E_2(z)E_4(z) - E_6(z))^2 = 720^2 e^{4\pi i z} + O(e^{5\pi i z}).$$

Then the result follows from Lemma 7.4 with $f(z) = (E_2 E_4 - E_6)^2$ and $n_0 = 4$. ☐

Corollary 7.6. *There exists a constant $C_{-2} > 0$ such that*

$$|\phi_{-2}(z)| \leq C_{-2} \tag{38}$$

for all z with $\Im z > 1/2$.

Corollary 7.7. *There exists a constant $C_{-4} > 0$ such that*

$$|\phi_{-4}(z)| \leq C_{-4} e^{2\pi \Im z} \tag{39}$$

for all z with $\Im z > 1/2$.

Note that we can take the constants C_0, C_{-2}, and C_{-4} as

$$\begin{aligned} C_0 &= 9 \cdot 240^2 \cdot e^{\pi} \cdot \frac{E_4'(i/2)^2}{\Delta(i/2)} \\ C_{-2} &= 3 \cdot \frac{E_4(i/2)E_4'(i/2)}{\Delta(i/2)} \\ C_{-4} &= e^{-\pi} \cdot \frac{E_4(i/2)^2}{\Delta(i/2)}. \end{aligned}$$

Proposition 7.8. *$a(x)$ is a Schwartz function.*

Proof. We estimate the first summand in the right-hand side of (36). By (37), we have

$$\left|\int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2 z}\,dz\right| = \left|\int\limits_{i\infty}^{-1/(i+1)} \phi_0(z)\,z^{-4}\,e^{\pi i r^2(-1/z-1)}\,dz\right| \leq$$

$$C_1\int\limits_{1/2}^{\infty} e^{-2\pi t}\,e^{-\pi r^2/t}\,dt \leq C_1\int\limits_{0}^{\infty} e^{-2\pi t}\,e^{-\pi r^2/t}\,dt = C_2\,r\,K_1(2\sqrt{2}\,\pi\,r)$$

where C_1 and C_2 are some positive constants and $K_\alpha(x)$ is the modified Bessel function of the second kind defined as in [1, Section 9.6]. This estimate also holds for the second and third summand in (36). For the last summand we have

$$\left|\int\limits_{i}^{i\infty} \phi_0(z)\,e^{\pi i r^2 z}\,dz\right| \leq C\int\limits_{1}^{\infty} e^{-2\pi t}\,e^{-\pi r^2 t}\,dt = C_3\,\frac{e^{\pi(r^2+2)}}{r^2+2}.$$

Therefore, we arrive at

$$|a(r)| \leq 4C_2\,r\,K_1(2\sqrt{2}\pi r) + 2C_3\,\frac{e^{-\pi(r^2+2)}}{r^2+2}.$$

It is easy to see that the left hand side of this inequality decays faster then any inverse power of r. Analogous estimates can be obtained for all derivatives $\frac{\mathrm{d}^k}{\mathrm{d}r^k}a(r)$. ☐

Proposition 7.9. *$a(x)$ satisfies (32).*

Proof. We recall that the Fourier transform of a Gaussian function is

$$\mathcal{F}(e^{\pi i\|x\|^2 z})(y) = z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}. \tag{40}$$

Next, we exchange the contour integration with respect to z variable and Fourier transform with respect to x variable in (36). This can be done, since the corresponding double integral converges absolutely. In this way we obtain

$$\widehat{a}(y) = \int\limits_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz + \int\limits_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz$$

$$-2\int\limits_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz + 2\int\limits_{i}^{i\infty} \phi_0(z)\,z^{-4}\,e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz.$$

Now we make a change of variables $w=\frac{-1}{z}$. We obtain

$$\widehat{a}(y) = \int\limits_{1}^{i} \phi_0\Big(1-\frac{1}{w-1}\Big)\,(\frac{-1}{w}+1)^2\,w^2\,e^{\pi i\|y\|^2\,w}\,dw$$

$$+\int_{-1}^{i} \phi_0\Big(1-\frac{1}{w+1}\Big)\,(\frac{-1}{w}-1)^2\, w^2\, e^{\pi i \|y\|^2\, w}\, dw$$
$$-2\int_{i\infty}^{i} \phi_0(w)\, e^{\pi i \|y\|^2\, w}\, dw + 2\int_{i}^{0} \phi_0\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|y\|^2\, w}\, dw.$$

Since ϕ_0 is 1-periodic we have

$$\begin{aligned}\widehat{a}(y) \;=\; & \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i \|y\|^2\, z}\, dz + \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i \|y\|^2\, z}\, dz \\ & +2\int_{i}^{i\infty} \phi_0(z)\, e^{\pi i \|y\|^2\, z}\, dz - 2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i \|y\|^2\, z}\, dz \\ & = a(y).\end{aligned}$$

This finishes the proof of the proposition. □

Next, we check that a has double zeroes at all Λ_8-lattice points of length greater then $\sqrt{2}$. Using (37), (38), and (39), we can control the behavior of ϕ_0 near 0 and $i\infty$.

Corollary 7.10. *We have*

$$\phi_0\left(\frac{i}{t}\right) = O(e^{-2\pi/t}) \quad \text{as } t\to 0$$
$$\phi_0\left(\frac{i}{t}\right) = O(t^{-2}e^{2\pi t}) \quad \text{as } t\to \infty.$$

Proof. The first estimate follows from (37) with $z=i/t$. For the second estimate, by (35), (38), and (39), we have

$$\left|\phi_0\left(\frac{i}{t}\right)\right| \le |\phi_0(it)| + \frac{12}{\pi t}|\phi_{-2}(it)| + \frac{36}{\pi^2 t^2}|\phi_{-4}(it)| \le C_0 e^{-2\pi t} + \frac{12}{\pi t}\cdot C_{-2} + \frac{36}{\pi^2 t^2}\cdot C_{-4}e^{2\pi t} = O(t^{-2}e^{2\pi t}).$$

□

Proposition 7.11. *For $r>\sqrt{2}$ we can express $a(r)$ in the following form*

$$a(r) = -4\sin(\pi r^2/2)^2 \int_{0}^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2\, z}\, dz. \tag{41}$$

Proof. We denote the right hand side of (41) by $d(r)$. Convergence of the integral for $r>\sqrt{2}$ follows from Corollary 7.10. We can write

$$d(r) = \int_{-1}^{i\infty-1} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz - 2\int_0^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz$$
$$+\int_1^{i\infty+1} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz.$$

From (35) we deduce that if $r > \sqrt{2}$ then $\phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z} \to 0$ as $\Im(z) \to \infty$. Therefore, we can deform the paths of integration and rewrite

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz + \int_{i}^{i\infty} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz$$
$$-2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz - 2\int_{i}^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz$$
$$+\int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz + \int_{i}^{i\infty} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz.$$

Now from (35) we find

$$\phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2 - 2\phi_0\Big(\frac{-1}{z}\Big)\,z^2 + \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2 =$$
$$\phi_0(z+1)\,(z+1)^2 - 2\phi_0(z)\,z^2 + \phi_0(z-1)\,(z-1)^2$$
$$-\frac{12i}{\pi}\Big(\phi_{-2}(z+1)\,(z+1) - 2\phi_{-2}(z)\,z + \phi_{-2}(z-1)\,(z-1)\Big)$$
$$-\frac{36}{\pi^2}\Big(\phi_{-4}(z+1) - 2\phi_{-4}(z) + \phi_{-4}(z-1)\Big) =$$
$$2\phi_0(z).$$

Thus, we obtain

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\,e^{\pi i r^2\,z}\,dz - 2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\,z^2\,e^{\pi i r^2\,z}\,dz$$
$$+\int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2\,z}\,dz + 2\int_{i}^{i\infty} \phi_0(z)\,e^{\pi i r^2\,z}\,dz = a(r).$$

This finishes the proof. $\square$

Finally, we find another convenient integral representation for a and compute values of $a(r)$ at $r = 0$ and $r = \sqrt{2}$.

Proposition 7.12. *For $r \geq 0$ we have*

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \Bigg(\frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2} + \int_0^\infty \left(t^2\,\phi_0\!\left(\frac{i}{t}\right) - \frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t - \frac{18144}{\pi^2} \right) e^{-\pi r^2 t}\,dt \Bigg). \tag{42}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. Suppose that $r > \sqrt{2}$. Then by Proposition 7.11

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \int_0^\infty \phi_0(i/t)\,t^2\,e^{-\pi r^2 t}\,dt.$$

From (**??**)–(35) we obtain

$$\phi_0(i/t)\,t^2 = \frac{36}{\pi^2}\,e^{2\pi t} - \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} + O(t^2\,e^{-2\pi t}) \quad \text{as } t \to \infty. \tag{43}$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \left(\frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} \right) e^{-\pi r^2 t}\,dt = \frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2}.$$

Therefore, the identity (42) holds for $r > \sqrt{2}$.

On the other hand, from the definition (36) we see that $a(r)$ is analytic in some neighborhood of $[0,\infty)$. The asymptotic expansion (43) implies that the right hand side of (42) is also analytic in some neighborhood of $[0,\infty)$. Hence, the identity (42) holds on the whole interval $[0,\infty)$. This finishes the proof of the proposition. □

From the identity (42) we see that the values $a(r)$ are in $i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$.

Proposition 7.13. *We have $a(0) = -\frac{i}{8640}$.*

Proof. These identities follow immediately from the previous proposition. □

Now we construct function b. To this end we consider the function

Definition 7.14.

$$h(z) := 128\frac{H_3(z) + H_4(z)}{H_2(z)^2}. \tag{44}$$

It is easy to see that $h \in M^!_{-2}(\Gamma_0(2))$. Indeed, first we check that $h|_{-2}\gamma = h$ for all $\gamma \in \Gamma_0(2)$. Since the group $\Gamma_0(2)$ is generated by elements $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ it suffices to check that h is invariant under their action. This follows immediately from (15)–(17) and (44). Next we analyze the poles of h. It is known [8, Chapter I Lemma 4.1] that θ_{10} has no zeros in the upper-half plane and hence h has poles only at the cusps. At the cusp $i\infty$ this modular form has the Fourier expansion

$$h(z) \;=\; q^{-1} + 16 - 132q + 640q^2 - 2550q^3 + O(q^4).$$

Let $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ be elements of Γ_1.

Definition 7.15. *We define the following three functions*

$$\begin{aligned} \psi_I &:= h - h|_{-2}ST && (45)\\ \psi_T &:= \psi_I|_{-2}T \\ \psi_S &:= \psi_I|_{-2}S. && (46) \end{aligned}$$

Lemma 7.16. *More explicitly, we have*

$$\begin{aligned} \psi_I(z) &= 128\frac{H_3(z)+H_4(z)}{H_2(z)^2} + 128\frac{H_4(z)-H_2(z)}{H_3(z)^2} \\ \psi_T(z) &= 128\frac{H_3(z)+H_4(z)}{H_2(z)^2} + 128\frac{H_2(z)+H_3(z)}{H_4(z)^2} \\ \psi_S(z) &= 128\frac{H_2(z)+H_3(z)}{H_4(z)^2} - 128\frac{H_2(z)-H_4(z)}{H_3(z)^2} && (47) \end{aligned}$$

Lemma 7.17. *The Fourier expansions of these functions are*

$$\begin{aligned} \psi_I(z) &= q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O(q^{5/2}) && (48)\\ \psi_T(z) &= q^{-1} + 144 + 5120q^{1/2} + 70524q + 626688q^{3/2} + 4265600q^2 + O(q^{5/2}) \\ \psi_S(z) &= -10240q^{1/2} - 1253376q^{3/2} - 48328704q^{5/2} - 1059078144q^{7/2} + O(q^{9/2}). \end{aligned}$$

Definition 7.18. *For $x \in \mathbb{R}^8$ define*

$$\begin{aligned} b(x) := &\int_{-1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz && (49)\\ &-2\int_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz. \end{aligned}$$

Now we prove that b is a Schwartz function and satisfies condition (33).

Lemma 7.19. *$\psi_S(z)$ can be written as*

$$\psi_S(z) = -\frac{H_2^3(2H_2^3 + 5H_2H_4 + 5H_4^2)}{2\Delta}. \tag{50}$$

Proof. Using (47) and (19) gives

$$\begin{aligned}
\psi_S &= -128\frac{H_3 + H_2}{H_4^2} - 128\frac{H_2 - H_4}{H_3^2} \\
&= -128\frac{H_3^2(H_2 - H_4) + H_4^2(H_2 - H_4)}{H_3^2H_4^2} \\
&= -128\frac{(H_2 + H_4)^2(2H_2 + H_4) + H_4^2(H_2 + H_4)}{H_3^2H_4^2} \\
&= -128\frac{H_2(2H_2^2 + 5H_2H_4 + 5H_4^2)}{H_3^2H_4^2} \\
&= -128\frac{H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2)}{H_2^2H_3^2H_4^2} \\
&= -\frac{1}{2}\frac{H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2)}{\Delta}.
\end{aligned}$$

□

Lemma 7.20. *There exists a constant $C_S > 0$ such that*

$$|\psi_S(z)| \le C_S e^{-\pi \Im z} \tag{51}$$

for all z with $\Im z > 1/2$.

Proof. Proof is similar to that of Lemma 7.5. By Proposition 6.37, 6.38 and 6.39, we can write Fourier expansion of the numerator of ψ_S as

$$H_2(z)^3(2H_2(z)^2 + 5H_2(z)H_4(z) + 5H_4(z)^2) = \sum_{n\ge 3} a_n e^{\pi i n z}$$

with $a_3 = 16^3 \cdot 5 = 20480$ and $a_n = O(n^k)$ for some $k > 0$. Now the result follows from Lemma 7.4. □

Proposition 7.21. *$b(x)$ is a Schwartz function.*

Proof. We have

$$\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz = \int_{0}^{i+1} \psi_I(z)\, e^{\pi i r^2 (z-1)}\, dz =$$

$$\int_{i\infty}^{-1/(i+1)} \psi_I\Big(\frac{-1}{z}\Big)\, e^{\pi i r^2(-1/z-1)}\, z^{-2}\, dz = \int_{i\infty}^{-1/(i+1)} \psi_S(z)\, z^{-4}\, e^{\pi i r^2(-1/z-1)}\, dz.$$

Using (51), we can estimate the first summand in the left-hand side of (49)

$$\left|\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz\right| \le C_1\, r\, K_1(2\pi r).$$

We combine this inequality with analogous estimates for the other three summands and obtain

$$|b(r)| \le C_2\, r\, K_1(2\pi r) + C_3\, \frac{e^{-\pi(r^2+1)}}{r^2+1}.$$

Here C_1, C_2, and C_3 are some positive constants. Similar estimates hold for all derivatives $\frac{\mathrm{d}^k}{\mathrm{d}^k r} b(r)$. □

Proposition 7.22. *$b(x)$ satisfies (33).*

Proof. Here, we repeat the arguments used in the proof of Proposition 7.9. We use identity (40) and change contour integration in z and Fourier transform in x. Thus we obtain

$$\mathcal{F}(b)(x) = \int_{-1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz + \int_{1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz$$
$$-2\int_{0}^{i} \psi_I(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz.$$

We make the change of variables $w = \frac{-1}{z}$ and arrive at

$$\mathcal{F}(b)(x) = \int_{1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw + \int_{-1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw$$
$$-2\int_{i\infty}^{i} \psi_I\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw - 2\int_{i}^{0} \psi_S\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw.$$

Now we observe that the definitions (45)–(46) imply

$$\psi_T|_{-2}S = -\,\psi_T$$
$$\psi_I|_{-2}S = \psi_S$$

$$\psi_S|_{-2}S = \psi_I.$$

Therefore, we arrive at

$$\begin{aligned}\mathcal{F}(b)(x) = &\int_1^i -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{-1}^i -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz \\ &+2\int_i^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz + 2\int_0^i \psi_I(z)\, e^{\pi i \|x\|^2 w}\, dw.\end{aligned}$$

Now from (49) we see that

$$\mathcal{F}(b)(x) = -b(x).$$

□

Now we regard the radial function b as a function on $\mathbb{R}_{\geq 0}$. We check that b has double roots at Λ_8-points.

Lemma 7.23. *There exists a constant $C_I > 0$ such that*

$$|\psi_I(z)| \leq C_I e^{2\pi \Im z}$$

for all z with $\Im z > 1/2$.

Proof. By (50), (46), (15), and (17),

$$\psi_I(z) = \frac{H_4^3(2H_4^2 + 5H_4H_2 + 5H_2^2)}{2\Delta}.$$

The denominator is not a cusp form (i.e. has a nonzero constant term), hence Lemma 7.4 concludes the proof with $n_0 = 0$. □

Corollary 7.24. *We have*

$$\psi_I(it) = O(t^2 e^{\pi/t}) \quad \text{as } t \to 0 \tag{52}$$

$$\psi_I(it) = O(e^{2\pi t}) \quad \text{as } t \to \infty. \tag{53}$$

Proof. By (46), we have

$$\psi_I(it) = (it)^{-2}\psi_S\left(\frac{-1}{it}\right) = -t^{-2}\psi_S\left(\frac{i}{t}\right).$$

and combined with (51) we get (52). (53) follows from Lemma 7.23. □

Proposition 7.25. *For $r > \sqrt{2}$ function $b(r)$ can be expressed as*

$$b(r) = -4\sin(\pi r^2/2)^2 \int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz. \tag{54}$$

Proof. We denote the right hand side of (54) by $c(r)$. By Corollary 7.24, the integral in (54) converges for $r > \sqrt{2}$. Then we rewrite it in the following way:

$$c(r) = \int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz.$$

From the Fourier expansion (48) we know that $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\Im(z) \to \infty$. By assumption $r^2 > 2$, hence we can deform the path of integration and write

$$\int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz$$
$$\int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz.$$

We have

$$\begin{aligned} c(r) = &\int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz \\ &+ 2\int_i^{i\infty} (\psi_T(z) - \psi_I(z))\, e^{\pi i r^2 z}\, dz. \end{aligned} \tag{55}$$

Next, we check that the functions ψ_I, ψ_T, and ψ_S satisfy the following identity:

$$\psi_T + \psi_S = \psi_I. \tag{56}$$

Indeed, from definitions (45)-(46) we get

$$\begin{aligned} \psi_T + \psi_S =&(h - h|_{-2}ST)|_{-2}T + (h - h|_{-2}ST)|_{-2}S \\ =&h|_{-2}T - h|_{-2}ST^2 + h|_{-2}S - h|_{-2}STS. \end{aligned}$$

Note that ST^2S belongs to $\Gamma_0(2)$. Thus, since $h \in M^!_{-2}\Gamma_0(2)$ we get

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS.$$

Now we observe that T and $STS(ST)^{-1}$ are also in $\Gamma_0(2)$. Therefore,

$$\psi_T + \psi_S = h|_{-2}T - h|_{-2}STS = h|_{-2} - h|ST = \psi_I.$$

Combining (55) and (56) we find

$$\begin{aligned} c(r) =& \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_{0}^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz \\ & - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i r^2 z}\, dz \\ =& b(r). \end{aligned}$$

□

At the end of this section we find another integral representation of $b(r)$ for $r \in \mathbb{R}_{\geq 0}$ and compute special values of b.

Proposition 7.26. *For $r \geq 0$ we have*

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \left(\frac{144}{\pi\, r^2} + \frac{1}{\pi\,(r^2-2)} + \int_0^\infty \left(\psi_I(it) - 144 - e^{2\pi t}\right)\, e^{-\pi r^2 t}\, dt \right). \tag{57}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 7.12. First, suppose that $r > \sqrt{2}$. Then by Proposition 7.25

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \int_0^\infty \psi_I(it)\, e^{-\pi r^2 t}\, dt.$$

From (48) we obtain

$$\psi_I(it) = e^{2\pi t} + 144 + O(e^{-\pi t}) \quad \text{as } t \to \infty. \tag{58}$$

For $r > \sqrt{2}$ we have

$$\int_0^\infty \left(e^{2\pi t} + 144\right)\, e^{-\pi r^2 t}\, dt = \frac{1}{\pi\,(r^2-2)} + \frac{144}{\pi\, r^2}.$$

Therefore, the identity (57) holds for $r > \sqrt{2}$.

On the other hand, from the definition (49) we see that $b(r)$ is analytic in some neighborhood of $[0,\infty)$. The asymptotic expansion (58) implies that the right hand side of (57) is also analytic in some neighborhood of $[0,\infty)$. Hence, the identity (57) holds on the whole interval $[0,\infty)$. This finishes the proof of the proposition. $\square$

We see from (57) that $b(r) \in i\mathbb{R}$ far all $r \in \mathbb{R}_{\geq}0$. Another immediate corollary of this proposition is

Proposition 7.27. *We have* $b(0) = 0$.

8 Proof of Theorem 5.2

Our proof of the Theorem 5.2 relies on the following two inequalities for modular objects.

Proposition 8.1. *Consider the function* $A : (0, \infty) \to \mathbb{C}$ *defined as*

$$A(t) := -t^2 \phi_0(i/t) - \frac{36}{\pi^2}\, \psi_I(it).$$

Then

$$A(t) < 0 \tag{59}$$

for all $t > 0$.

Proposition 8.2. *Consider the function* $B : (0, \infty) \to \mathbb{C}$ *defined as*

$$B(t) := -t^2 \phi_0(i/t) + \frac{36}{\pi^2}\, \psi_I(it)$$

Then

$$B(t) > 0 \tag{60}$$

for all $t > 0$.

Here we formalize the proof of the inequalities by Lee [7]. First, we can rewrite the inequality in 8.1 as follows.

Definition 8.3. *Define two (quasi) modular forms as*

$$\begin{aligned} F(z) &= (E_2(z)E_4(z) - E_6(z))^2 \\ G(z) &= H_2(z)^3(2H_2(z)^2 + 5H_2(z)H_4(z) + 5H_4(z)^2). \end{aligned}$$

Lemma 8.4. *We have*

$$\phi_0 = \frac{F}{\Delta} \tag{61}$$

$$\psi_S = -\frac{1}{2}\frac{G}{\Delta} \tag{62}$$

Proof. (61) is clear. (62) is already shown in Lemma 7.19. □

Lemma 8.5. *Inequality (59) and (60) are equivalent to*

$$F(it) + \frac{18}{\pi^2} G(it) > 0 \tag{63}$$

$$F(it) - \frac{18}{\pi^2} G(it) > 0 \tag{64}$$

respectively.

Proof. By (46),

$$\psi_I(it) = (\psi_S|_{-2}S)(it) = (it)^2 \psi_S\left(-\frac{1}{it}\right) = -t^2 \psi_S\left(\frac{i}{t}\right).$$

Combined with Lemma 8.4 we can rewrite (59) as

$$A(t) = -t^2 \phi_0\left(\frac{i}{t}\right) + \frac{36}{\pi^2}\psi_S\left(\frac{i}{t}\right) < 0 \Leftrightarrow \frac{F(it)}{\Delta(it)} + \frac{18}{\pi^2}\frac{G(it)}{\Delta(it)} > 0$$

for $t > 0$, which is equivalent to (63) by Corollary 6.25. Equivalences of (60) and (64) follows similarly; just change the sign. □

Now, the first inequality (63) follows from the positivity of each $F(it)$ and $G(it)$.

Lemma 8.6. *For all $t > 0$, we have $F(it) > 0$ and $G(it) > 0$.*

Proof. By Ramanujan's identity (25), we have $F(z) = 9E_4'(z)^2$ and

$$F(it) = 9E_4'(it)^2 = 9\left(240 \sum_{n \geq 1} n\sigma_3(n) e^{-2\pi n t}\right)^2 > 0.$$

$G(it) > 0$ follows from positivity of $H_2(it)$ and $H_4(it)$ (Lemma 6.42). □

Corollary 8.7. *(63) holds.*

Proof. This directly follows from Lemma 8.6. □

To prove the second inequality (64), we need some identities satisfied by F and G.

Lemma 8.8. *F and G satisfy the following differential equations:*

$$\partial_{12}\partial_{10}F - \frac{5}{6}E_4 F = 7200\Delta(-E_2') \tag{65}$$

$$\partial_{12}\partial_{10}G - \frac{5}{6}E_4 G = -640\Delta H_2 \tag{66}$$

Proof. Both can be shown by direct computations. By Ramanujan's identities (Theorem 6.48) and the product rule of Serre derivatives (Theorem 6.51), we have

$$\begin{aligned}
\partial_5(E_2E_4 - E_6) &= (E_2E_4 - E_6)' - \frac{5}{12}E_2(E_2E_4 - E_6) \\
&= \frac{E_2^2 - E_4}{12} \cdot E_4 + E_2 \cdot \frac{E_2E_4 - E_6}{3} - \frac{E_2E_6 - E_4^2}{2} - \frac{5}{12}E_2(E_2E_4 - E_6) \\
&= -\frac{5}{12}(E_2E_6 - E_4^2) \\
\partial_7(E_2E_6 - E_4^2) &= (E_2E_6 - E_4^2)' - \frac{7}{12}E_2(E_2E_6 - E_4^2) \\
&= \frac{E_2^2 - E_4}{12} \cdot E_6 + E_2 \cdot \frac{E_2E_6 - E_4^2}{2} - 2E_4 \cdot \frac{E_2E_4 - E_6}{3} - \frac{7}{12}E_2(E_2E_6 - E_4^2) \\
&= -\frac{7}{12}E_4(E_2E_4 - E_6)
\end{aligned}$$

and using these we can compute

$$\begin{aligned}
\partial_{10}F &= \partial_{10}(E_2E_4 - E_6)^2 \\
&= 2(E_2E_4 - E_6)\partial_5(E_2E_4 - E_6) \\
&= -\frac{6}{5}(E_2E_4 - E_6)(E_2E_6 - E_4^2), \\
\partial_{12}\partial_{10}F &= -\frac{5}{6}\partial_{12}((E_2E_4 - E_6)(E_2E_6 - E_4)) \\
&= -\frac{5}{6}(\partial_5(E_2E_4 - E_6))(E_2E_6 - E_4^2) - \frac{5}{6}(E_2E_4 - E_6)(\partial_7(E_2E_6 - E_4)) \\
&= \frac{25}{72}(E_2E_6 - E_4^2)^2 + \frac{35}{72}E_4(E_2E_4 - E_6)^2, \\
\partial_{12}\partial_{10}F - \frac{5}{6}E_4F &= \frac{25}{72}(E_2E_6 - E_4^2)^2 + \frac{35}{72}E_4(E_2E_4 - E_6)^2 - \frac{5}{6}E_4(E_2E_4 - E_6)^2 \\
&= \frac{25}{72}((E_2E_6 - E_4^2)^2 - E_4(E_2E_4 - E_6)^2) \\
&= \frac{25}{72}(-E_2^2E_4^3 + E_2^2E_6^2 + E_4^4 - E_4E_6^3) \\
&= -\frac{25}{72}(E_4^3 - E_6^2)(E_2^2 - E_4) \\
&= 7200 \cdot \frac{E_4^3 - E_6^2}{1728} \cdot \frac{-E_2^2 + E_4}{12} \\
&= 7200\Delta(-E_2')
\end{aligned}$$

which proves (65). Similarly, (66) can be proved using Proposition 6.50 and Lemma 6.41. $\square$

Corollary 8.9. (65) *(resp.* (66)*) is positive (resp. negative) on the (positive) imaginary axis.*

Proof. From (11) and Lemma 6.25,

$$7200(-E_2'(it))\Delta(it) = 7200 \cdot 24 \left(\sum_{n \geq 1} n\sigma_1(n) e^{-2\pi n t} \right) \cdot \Delta(it) > 0.$$

Negativity of (66), i.e. $-640\Delta(it)H_2(it) < 0$ follows from Corollary 6.42 and 6.25. $\square$

The second inequality (64) follows from the following two observations. Since $G(it) > 0$ for all $t > 0$, we can define the quotient

$$Q(t) := \frac{F(it)}{G(it)}$$

as a function on $(0, \infty)$.

Lemma 8.10. *We have*

$$\lim_{t \to 0^+} Q(t) = \frac{18}{\pi^2}.$$

Proof. We have

$$\lim_{t \to 0^+} Q(t) = \lim_{t \to 0^+} \frac{F(it)}{G(it)} = \lim_{t \to \infty} \frac{F(i/t)}{G(i/t)}.$$

By using the transformation laws of Eisenstein series (12), (9) (for $k = 4, 6$) and the thetanull functions, (15), (17), we get

$$\begin{aligned} F\left(\frac{i}{t}\right) &= t^{12} F(it) - \frac{12 t^{11}}{\pi} (E_2(it) E_4(it) - E_6(it)) E_4(it) + \frac{36 t^{10}}{\pi^2} E_4(it)^2, \\ G\left(\frac{i}{t}\right) &= t^{10} H_4(it)^3 (2 H_4(it)^2 + 5 H_4(it) H_2(it) + 5 H_2(it)^2). \end{aligned}$$

Since F, $E_2 E_4 - E_6$ and H_2 are cusp forms, we have $\lim_{t \to \infty} t^k A(it) = 0$ when $A(z)$ is one of these forms and $k \geq 0$. From $\lim_{t \to \infty} E_4(it) = 1 = \lim_{t \to \infty} H_4(it)$, we get

$$\begin{aligned} \lim_{t \to \infty} \frac{F(i/t)}{G(i/t)} &= \lim_{t \to \infty} \frac{t^{12} F(it) - \frac{12 t^{11}}{\pi} (E_2(it) E_4(it) - E_6(it)) E_4(it) + \frac{36 t^{10}}{\pi^2} E_4(it)^2}{t^{10} H_4(it)^3 (2 H_4(it)^2 + 5 H_4(it) H_2(it) + 5 H_2(it)^2)} \\ &= \lim_{t \to \infty} \frac{t^2 F(it) - \frac{12 t}{\pi} (E_2(it) E_4(it) - E_6(it)) E_4(it) + \frac{36}{\pi^2} E_4(it)^2}{H_4(it)^3 (2 H_4(it)^2 + 5 H_4(it) H_2(it) + 5 H_2(it)^2)} \\ &= \frac{18}{\pi^2}. \end{aligned}$$

$\square$

Proposition 8.11. *The function $t \mapsto Q(t)$ is monotone decreasing.*

Proof. It is enough to show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{G(it)}\right) < 0 &\Leftrightarrow (-2\pi)\frac{F'(it)G(it)-F(it)G'(it)}{G(it)^2} < 0\\ &\Leftrightarrow F'(it)G(it)-F(it)G'(it) > 0\\ &\Leftrightarrow (\partial_{10}F)(it)G(it)-F(it)(\partial_{10}G)(it) > 0.\end{aligned}$$

Let $\mathcal{L}_{1,0} := (\partial_{10}F)G - F(\partial_{10}G)$. Then its Fourier expansion starts with

$$\mathcal{L}_{1,0} = 5308416000q^{\frac{7}{2}} + O(q^{\frac{9}{2}})$$

and its Serre derivative $\partial_{22}\mathcal{L}_{1,0}$ is positive by Corollary 8.9:

$$\partial_{22}\mathcal{L}_{1,0} = (\partial_{12}\partial_{10}F)G - F(\partial_{12}\partial_{10}G) = \Delta(7200(-E_2')G + 640H_2F) > 0.$$

Hence $\mathcal{L}_{1,0}(it) > 0$ by Theorem 6.52, and the monotonicity follows. □

Corollary 8.12. (64) *holds.*

Proof.

$$\frac{F(it)}{G(it)} = Q(t) < \lim_{u\to 0^+} Q(u) = \frac{18}{\pi^2}$$

and by Lemma 8.6, (64) follows. □

Finally, we are ready to prove Theorem 5.2.

Theorem 8.13. *The function*

$$g(x) := \frac{\pi\, i}{8640}a(x) + \frac{i}{240\pi}\, b(x)$$

satisfies conditions (5)–(7).

Proof. First, we prove that (5) holds. By Propositions 7.11 and 7.25 we know that for $r > \sqrt{2}$

$$g(r) = \frac{\pi}{2160}\,\sin(\pi r^2/2)^2 \int_0^\infty A(t)\, e^{-\pi r^2 t}\, dt \tag{67}$$

where

$$A(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2}\,\psi_I(it).$$

from the Proposition 8.1 we know that $A(t) < 0 \quad$ for $t \in (0,\infty)$. Therefore identity (67) implies (5).

Next, we prove (6). By Propositions 7.12 and 7.26 we know that for $r > 0$

$$\widehat{g}(r) = \frac{\pi}{2160}\,\sin(\pi r^2/2)^2 \int_0^\infty B(t)\,e^{-\pi r^2 t}\,dt$$

where

$$B(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

Finally, the property (7) readily follows from Proposition 7.13 and Proposition 7.27. This finishes the proof of Theorems 8.13 and 5.2. $\square$

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