# 21-651: General Topology

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# Contents

# Chapter 1

# An Introduction to Topology

We begin by recalling basic notions about metric spaces.

## 1.1 A Word on Metric Spaces

Recall the definition of a metric space.

**Definition 1.1.1** (Metric Spaces). A **metric space** is a pair $(X, d)$ consisting of a set $X$ and a function $d : X \times X \to \mathbb{R}$ such that

1. for all $x, y \in X$, $d(x, y) \geq 0$ for all $x, y \in X$
2. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. for all $x, y \in X$, $d(x, y) = d(y, x)$
4. for all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

We call the function $d$ a **metric on** $X$.

We give several familiar examples.

**Example 1.1.2** (Some Familiar Metric Spaces).

1. $\mathbb{R}^n$ under the Euclidean metric
$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$
2. Any set $X$ under the equality metric
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$
for all $x, y \in X$
3. Any subset $Y \subseteq X$ of a metric space $(X, d)$ under the restriction of $d$ to $Y \times Y \subseteq X \times X$.
4. Given two metric spaces $(X_1, d_1)$ and $(X_2, d_2)$, there are numerous viable metrics we can define on $X_1 \times X_2$. One of them would be taking the *maximum* of $d_1$ and $d_2$; another would be the *sum*; a third would be
$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$
for all $(x_1, y_1) \in X_1$ and $(x_2, y_2) \in Y_2$. We define this third metric space to be the **product metric**, and it is easily seen that the product of Euclidean spaces (under the Euclidean metric) is indeed a Euclidean space (under the Euclidean metric).
5. The set $C^0([0, 1])$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ under the supremum metric
$$d(f, g) = \|f - g\|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|$$
for all $f, g \in C^0([0, 1])$. More generally, any compact set works (not just $[0, 1]$).
6. The set $C^0([0, 1])$ under the metric
$$d(f, g) = \sqrt{\int_0^1 (f(x) - g(x))^2 \, dx}$$
for all $f, g \in C^0([0, 1])$, which we know is positive-definite because continuous functions that are zero almost everywhere are zero (and nonnegative functions whose

integral is zero are zero almost everywhere).

7. Consider the set

$$l^2(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, x_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} x_n^2 < \infty \right\}$$

We can define the $l^2$ metric on this set by

$$d(x, y) := \sqrt{\sum_{n=0}^{\infty} (x_i - y_i)^2}$$

for all $x, y \in l^2(\mathbb{R})$. More than showing that this satisfies the properties of a metric, what is tricky here is showing that this metric is well-defined. But this is doable, and we will end the discussion of this example on that note.

After this barrage of examples of metric spaces, we are finally ready to move onto more interesting definitions. We begin by discussing the notion of continuity of functions between metric spaces.

**Definition 1.1.3** (Continuity of Functions). Let $(X, d)$ and $(X', d')$ be metric spaces. We say that a function $f : X \to X'$ is **continuous at a point** $x_0 \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \varepsilon$. We say that $f$ is **continuous** if $f$ is continuous at every point $x_0 \in X$.

We mention two interesting facts that we do not bother to prove.

**Exercise 1.1.4** (Argument-Wise Continuity of Metrics). If $(X, d)$ is a metric space, for all $a \in X$, the function

$$x \mapsto d(a, x) : X \to \mathbb{R}$$

is continuous.

**Exercise 1.1.5** (Composition of Continuous Functions). A composition of continuous functions is continuous.

## 1.2 Another Section

# Chapter 2

# Another Chapter

You get the idea.

## 2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

## 2.2 Another Section

Boy do I love LaTeX!

Visit https://thefundamentaltheor3m.github.io/TopologyNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to my repository.