21-651: General Topology

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Contents

Chapter 1

A Recap of Undergraduate Topology

We begin by making a few slightly non-standard notational choices.

Notation. Given a function $f : X \to Y$ and a subset $A \subseteq X$, we write

$$f[A] := \{f(x) \in Y \mid x \in A\}$$

In similar fashion, given $B \subseteq Y$, we write

$$f^{-1}[B] := \{x \in X \mid f(x) \in B\}$$

The reason it is important to distinguish between the notations $f(A)$ and $f[A]$ is that there might be a situation in which $A \in X$ *and* $A \subseteq X$. We will not make any more notational choices at this stage that differ from standard mathematical conventions. As and when we do, we will introduce them.

We now recall basic facts about metric spaces.

1.1 Metric Spaces

Recall the definition of a metric space.

Definition 1.1.1 (Metric Spaces). A **metric space** is a pair (X, d) consisting of a set X and a function $d : X \times X \to \mathbb{R}$ such that

1. for all $x, y \in X$, $d(x, y) \geq 0$ for all $x, y \in X$
2. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. for all $x, y \in X$, $d(x, y) = d(y, x)$
4. for all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

We call the function d a **metric on** X.

We give several familiar examples.

Example 1.1.2 (Some Familiar Metric Spaces).

1. $\mathbb{R}^n$ under the Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

2. Any set X under the equality metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$

3. Any subset $Y \subseteq X$ of a metric space (X, d) under the restriction of d to $Y \times Y \subseteq X \times X$.
4. Given two metric spaces (X_1, d_1) and (X_2, d_2), there are numerous viable metrics we can define on $X_1 \times X_2$. One of them would be taking the *maximum* of d_1 and d_2; another would be the *sum*; a third would be

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$

for all $(x_1, y_1) \in X_1$ and $(x_2, y_2) \in Y_2$. We define this third metric space to be the

product metric, and it is easily seen that the product of Euclidean spaces (under the Euclidean metric) is indeed a Euclidean space (under the Euclidean metric).

5. The set $C^0([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$ under the supremum metric

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x)-g(x)|$$

for all $f,g \in C^0([0,1])$. More generally, any compact set works (not just $[0,1]$).

6. The set $C^0([0,1])$ under the metric

$$d(f,g) = \sqrt{\int_0^1 (f(x)-g(x))^2 \, \mathrm{d}x}$$

for all $f,g \in C^0([0,1])$, which we know is positive-definite because continuous functions that are zero almost everywhere are zero (and nonnegative functions whose integral is zero are zero almost everywhere).

7. Consider the set

$$l^2(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; x_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} x_n^2 < \infty \right\}$$

We can define the l^2 metric on this set by

$$d(x,y) := \sqrt{\sum_{n=0}^{\infty} (x_i - y_i)^2}$$

for all $x,y \in l^2(\mathbb{R})$. More than showing that this satisfies the properties of a metric, what is tricky here is showing that this metric is well-defined. But this is doable, and we will end the discussion of this example on that note.

After this barrage of examples of metric spaces, we are finally ready to move onto more interesting definitions. We begin by discussing the notion of continuity of functions between metric spaces.

Definition 1.1.3 (Continuity). Let (X,d) and (X',d') be metric spaces. We say that a function $f: X \to X'$ is **continuous at a point** $x_0 \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $d(x,x_0) < \delta$, then $d'(f(x), f(x_0)) < \varepsilon$. We say that f is **continuous** if f is continuous at every point $x_0 \in X$.

We mention two interesting facts that we do not bother to prove.

Exercise 1.1.4 (Argument-Wise Continuity of Metrics)**.** If (X, d) is a metric space, for all $a \in X$, the function

$$x \mapsto d(a, x) : X \to \mathbb{R}$$

is continuous.

Exercise 1.1.5 (Composition of Continuous Functions)**.** A composition of continuous functions is continuous.

We recall what it means for a sequence to converge in a metric space.

Definition 1.1.6 (Convergence of a Sequence in a Metric Space)**.** Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. Given some $x \in X$, we say that x_n **converges to** x, denoted $x_n \to x$, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x) < \varepsilon$. We say x **is the limit of** x_n **as** $n \to \infty$.

We can show that limits in a metric space are unique.

Proposition 1.1.7. *Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. There is at most one x such that $x_n \to x$.*

Proof. Suppose, for contradiction, that there exist distinct points $x, x' \in X$ such that $x_n \to x$ and $x_n \to x'$. Pick $\varepsilon := d(x, x')/2$. For this ε, we know there is some $N \in \mathbb{N}$ such that for $n \geq N$, $d(x_n, x) < \varepsilon$. Similarly, we know there is some $N' \in \mathbb{N}$ such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. Pick $M := \max(N, N') + 1$. Then, applying the fact that $d(x_M, x) = d(x, x_M)$,

$$d(x, x') \leq d(x, x_M) + d(x_M, x') < \varepsilon + \varepsilon = d(x, x')$$

which clearly is a contradiction. So, x and x' cannot be distinct. □

Warning. In this course, we will encounter spaces where a sequence can have more than one limit. Proceed with caution!

We recall the definition of an open ball.

Definition 1.1.8 (Open Ball). Let (X, d) be a metric space. Fix $a \in X$ and $\varepsilon > 0$. We define the **open ball of radius ε centred at a** to be

$$B_\varepsilon(a) := \{b \in X \mid d(a, b) < \varepsilon\}$$

The reason for the term 'open' in the above definition is the following definition.

Definition 1.1.9 (Open Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **open** if for all $a \in U$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$.

It is easy to see that an open ball is indeed an open set.

We can also define a dual notion.

Definition 1.1.10 (Closed Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **closed** if its complement $X \setminus U$ is open.

We recall basic properties of open and closed sets.

Proposition 1.1.11. *Let (X, d) be a metric space.*

1. *Both $\emptyset$ and X are both open and closed.*
2. *An arbitrary union of open sets is open.*
3. *A finite intersection of open sets is open.*

We omit the proof, because it is easy and basic.

We are now ready to venture into more general waters.

1.2 Topological Spaces

A topological space, broadly speaking, is one in which we wish to be able to discuss the notion of convergence without depending on the notion of distance. The definition of convergence in metric spaces really only relies on the openness of open balls. We can generalise it merely by generalising the definition of open sets.

We begin by defining the notion of a **topological space**.

Definition 1.2.1 (Topological Space). Let X be a set. A **topology** on X is a family τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

We say a subset of X is **open** with respect to a topology τ if it lies in τ. We call the pair (X, τ) a **topological space**.

The terminology for open sets is visibly consistent with the terminology used in metric spaces. In fact, this is exactly what Proposition 1.1.11 demonstrates: that the open sets of a metric space, defined as in Definition 1.1.9, do indeed define a topology on it. Every metric space is thus also a topological space.

The reason why we introduced topological spaces to begin with is because we wanted a more general setting in which to talk about convergence. The question is, what is the best way of talking about convergence in arbitrary topological spaces?

Since every metric space is a topological space, and seeing as we already have a notion of convergence in metric spaces, we would want to define the notion of convergence in a topological space to be a *generalisation*. To that end, we restate the definition of convergence in metric spaces using only the language of open sets.

Proposition 1.2.2. *Let (X, d) be a metric space. For every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, the following are equivalent:*

1. *x_n converges to x as in Definition 1.1.6*
2. *For every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.*

We do not prove this proposition here, but note that it is not difficult to prove.

Since the second statement in Proposition 1.2.2 does not actually mention any *metric space* properties of X, it is a viable definition for convergence in topological spaces.

Definition 1.2.3 (Convergence of a Sequence in a Topological Space)**.** Let (X, τ) be a topological space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. Given some $x \in X$, we say that x_n **converges to** x, denoted $x_n \to x$, if for every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Proposition 1.2.2 essentially tells us that if (X, d) is a metric space, then Definitions 1.1.6 and 1.2.3 are equivalent, where we apply Definition 1.2.3 to the topological space structure induced by the metric space structure.

Example 1.2.4 (Convergence in the Indiscrete Topology)**.** Consider a set X along with the indiscrete topology (ie, view it as the topological space $(X, \{\emptyset, X\})$). Then, *every sequence converges to every point*.

Example 1.2.4 demonstrates that the uniqueness of limits seen in Proposition 1.1.7 does not always hold in topological spaces. See Definition 1.2.7 for more.

Note that going forward, we often omit the topology when talking about topological spaces, unless it has specific properties that are essential for our purposes.

Just as we generalised the notion of convergence from metric spaces to topological spaces, so too can we generalise the notion of continuity.

Definition 1.2.5 (Continuity)**.** Let X and Y be topological spaces, and let $f : X \to Y$ be a function. We say f is **continuous** if for every $U \subseteq Y$ open in Y, the pre-image $f^{-1}[U]$ is open in X.

It is easy to show that Definition 1.2.5 is equivalent to Definition 1.1.3. We do not do this here.

There are many examples of continuous functions, some familiar and some slightly pathological.

Example 1.2.6 (Weird Topologies lead to Weird Notions of Continuity)**.** Let X and Y be topological spaces.

1. If X and Y both have the *discrete* topology on them, then *any* function $f : X \to Y$ is continuous, because all subsets of X and Y are open.
2. If X and Y both have the *indiscrete* topology on them, then `sorry`

Recall that limits in a metric space are unique (Proposition 1.1.7). We noted that this is not always true, but we are interested enough in distinguishing spaces where this is true from spaces where this is not true that we have a name for such spaces.

Definition 1.2.7 (Hausdorff/T_2 Spaces)**.** Let (X, τ) be a topological space. We say (X, τ) is **Hausdorff**, or T_2, if for all $a, b \in X$ with $a \neq b$, there exist $U, V \in \tau$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

The Hausdorff/T_2 property is an instance of a *separation property*. There are many more such properties, including T_1, T_3, $T_{3\frac{1}{2}}$, and T_4, and it is highly non-trivial to prove facts like $T_4 \implies T_{3\frac{1}{2}}$. We will see such things as the course progresses.

Note that Proposition 1.1.7 tells us precisely that Metric Spaces are Hausdorff/T_2. Indeed, this tells us that Hausdorffitude is a *necessary* condition for a topology on a space to be **metrisable**—that is, for it to be induced by a metric. In particular, Example 1.2.4 tells us that the indiscrete topology is non-metrisable. There is the more interesting question of what conditions might be *sufficient* for a topology to be metrisable. Metrisability is, indeed, highly non-trivial property, and we will see some metrisation theorems in this course.

It is interesting to talk about whether a topology can be *generated* by a family of subsets by taking unions. In the case of metric spaces, it is not difficult to show that this is true.

Proposition 1.2.8 (Metric Space Topologies are Generated by Open Balls)**.** *Let (X, d) be a metric space. Any $U \subseteq X$ is open if and only if U is an open of open balls.*

Proof. Proposition 1.1.11 tells us that if U is a union of open sets, then it is open. Conversely, for all $x \in U$, there is some ε_x such that $B_{\varepsilon_x}(x) \subseteq U$. Then, it is easily seen that

$$U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$$

showing that U is a union of open sets. □

We have a special term for families of open sets that 'generate' a topology.

Definition 1.2.9 (Basis of a Topology). Let (X, τ) be a topological space. A **basis for τ** is a subset $B \subseteq \tau$ such that every $U \in \tau$ is a union of sets in B.

Proposition 1.2.8 tells us precisely that the open balls in a metric space form a *basis* for the topology induced by the metric.

We now discuss how to "make subsets of a topological space open or closed". We do this by defining interiors and closures.

Definition 1.2.10 (The Interior of a Set). Let X be a topological space and let $A \subseteq X$. We define the **interior** of A, denoted $\text{int}(A)$ or A°, to be the union of all open subsets of A.

Note that $\text{int}(A)$ is open. Moreover, it is the largest (with respect to inclusion) open subset of A. Finally, note that the sets that are equal to their interiors are precisely the open sets.

We have a 'dual' notion of interiors that capture closure properties.

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Boy do I love $\LaTeX$!

Visit https://thefundamentaltheor3m.github.io/TopologyNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to my repository.