

21-651: General Topology

Lecturer: James Cummings

Scribes: Sidharth Hariharan and Teresa Pollard

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Chapter 1

A Recap of Undergraduate Topology

We begin by making a few slightly non-standard notational choices.

Notation. Given a function $f : X \rightarrow Y$ and a subset $A \subseteq X$, we write

$$f[A] := \{f(x) \in Y \mid x \in A\}$$

In similar fashion, given $B \subseteq Y$, we write

$$f^{-1}[B] := \{x \in X \mid f(x) \in B\}$$

The reason it is important to distinguish between the notations $f(A)$ and $f[A]$ is that there might be a situation in which $A \in X$ and $A \subseteq X$. We will not make any more notational choices at this stage that differ from standard mathematical conventions. As and when we do, we will introduce them.

We now recall basic facts about metric spaces.

1.1 Metric Spaces

Recall the definition of a metric space.

Definition 1.1.1 (Metric Spaces). A **metric space** is a pair (X, d) consisting of a set X and a function $d : X \times X \rightarrow \mathbb{R}$ such that

1. for all $x, y \in X$, $d(x, y) \geq 0$ for all $x, y \in X$
2. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. for all $x, y \in X$, $d(x, y) = d(y, x)$
4. for all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

We call the function d a **metric on X** .

We give several familiar examples.

Example 1.1.2 (Some Familiar Metric Spaces).

1. \mathbb{R}^n under the Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$

2. Any set X under the equality metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$

3. Any subset $Y \subseteq X$ of a metric space (X, d) under the restriction of d to $Y \times Y \subseteq X \times X$.
4. Given two metric spaces (X_1, d_1) and (X_2, d_2) , there are numerous viable metrics we can define on $X_1 \times X_2$. One of them would be taking the *maximum* of d_1 and d_2 ; another would be the *sum*; a third would be

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$

for all $(x_1, y_1) \in X_1$ and $(x_2, y_2) \in Y_2$. We define this third metric space to be the

product metric, and it is easily seen that the product of Euclidean spaces (under the Euclidean metric) is indeed a Euclidean space (under the Euclidean metric).

5. The set $C^0([0, 1])$ of continuous functions from $[0, 1]$ to \mathbb{R} under the supremum metric

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

for all $f, g \in C^0([0, 1])$. More generally, any compact set works (not just $[0, 1]$).

6. The set $C^0([0, 1])$ under the metric

$$d(f, g) = \sqrt{\int_0^1 (f(x) - g(x))^2 dx}$$

for all $f, g \in C^0([0, 1])$, which we know is positive-definite because continuous functions that are zero almost everywhere are zero (and nonnegative functions whose integral is zero are zero almost everywhere).

7. Consider the set

$$l^2(\mathbb{R}) = \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} x_n^2 < \infty \right\}$$

We can define the l^2 metric on this set by

$$d(x, y) := \sqrt{\sum_{n=0}^{\infty} (x_i - y_i)^2}$$

for all $x, y \in l^2(\mathbb{R})$. More than showing that this satisfies the properties of a metric, what is tricky here is showing that this metric is well-defined. But this is doable, and we will end the discussion of this example on that note.

After this barrage of examples of metric spaces, we are finally ready to move onto more interesting definitions.

1.1.1 Continuity of Functions

We begin by discussing the notion of continuity of functions between metric spaces.

Definition 1.1.3 (Continuity). Let (X, d) and (X', d') be metric spaces. We say that a function $f : X \rightarrow X'$ is **continuous at a point** $x_0 \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \varepsilon$. We say that f is **continuous** if f is continuous at every point $x_0 \in X$.

We mention two interesting facts that we do not bother to prove.

Exercise 1.1.4 (Argument-Wise Continuity of Metrics). If (X, d) is a metric space, for all $a \in X$, the function

$$x \mapsto d(a, x) : X \rightarrow \mathbb{R}$$

is continuous.

Exercise 1.1.5 (Composition of Continuous Functions). A composition of continuous functions is continuous.

1.1.2 Sequences, Convergence and Uniqueness of Limits

We recall what it means for a sequence to converge in a metric space.

Definition 1.1.6 (Convergence of a Sequence in a Metric Space). Let (X, d) be a metric space, and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence in X . Given some $x \in X$, we say that x_n **converges to** x , denoted $x_n \rightarrow x$, if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, d(x_n, x) < \varepsilon$. We say x **is the limit of** x_n **as** $n \rightarrow \infty$.

We can show that limits in a metric space are unique.

Proposition 1.1.7. *Let (X, d) be a metric space and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence in X . There is at most one x such that $x_n \rightarrow x$.*

Proof. Suppose, for contradiction, that there exist distinct points $x, x' \in X$ such that $x_n \rightarrow x$ and $x_n \rightarrow x'$. Pick $\varepsilon := d(x, x')/2$. For this ε , we know there is some $N \in \mathbb{N}$ such that for $n \geq N$, $d(x_n, x) < \varepsilon$. Similarly, we know there is some $N' \in \mathbb{N}$ such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. Pick

$M := \max(N, N') + 1$. Then, applying the fact that $d(x_M, x) = d(x, x_M)$,

$$d(x, x') \leq d(x, x_M) + d(x_M, x') < \varepsilon + \varepsilon = d(x, x')$$

which clearly is a contradiction. So, x and x' cannot be distinct. \square

Warning. In this course, we will encounter spaces where a sequence can have more than one limit. Proceed with caution!

Finally, we discuss topological properties of subsets of topological spaces.

1.1.3 Open and Closed Sets

We recall the definition of an open ball.

Definition 1.1.8 (Open Ball). Let (X, d) be a metric space. Fix $a \in X$ and $\varepsilon > 0$. We define the **open ball of radius ε centred at a** to be

$$B_\varepsilon(a) := \{b \in X \mid d(a, b) < \varepsilon\}$$

The reason for the term ‘open’ in the above definition is the following definition.

Definition 1.1.9 (Open Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **open** if for all $a \in U$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$.

It is easy to see that an open ball is indeed an open set.

We can also define a dual notion.

Definition 1.1.10 (Closed Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **closed** if its complement $X \setminus U$ is open.

We recall basic properties of open and closed sets.

Proposition 1.1.11. *Let (X, d) be a metric space.*

1. *Both \emptyset and X are both open and closed.*
2. *An arbitrary union of open sets is open.*
3. *A finite intersection of open sets is open.*

We omit the proof, because it is easy and basic.

We are now ready to venture into more general waters.

1.2 Introduction to Topological Spaces

A topological space, broadly speaking, is one in which we wish to be able to discuss the notion of convergence without depending on the notion of distance. The definition of convergence in metric spaces really only relies on the openness of open balls. We can generalise it merely by generalising the definition of open sets.

We begin by defining the notion of a **topological space**.

Definition 1.2.1 (Topological Space). Let X be a set. A **topology** on X is a family τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

We say a subset of X is **open** with respect to a topology τ if it lies in τ and **closed** if its complement lies in τ . We call the pair (X, τ) a **topological space**.

The terminology for open sets is visibly consistent with the terminology used in metric spaces. In fact, this is exactly what Proposition 1.1.11 demonstrates: that the open sets of a metric space, defined as in Definition 1.1.9, do indeed define a topology on it. Every metric space is thus also a topological space. Note that sets can be both open and closed, such as \emptyset and the entire set, just as with metric spaces. We sometimes refer to such sets as being “clopen”.

1.2.1 Sequences, Convergence and Limits

The reason why we introduced topological spaces to begin with is because we wanted a more general setting in which to talk about convergence. The question is, what is the best way of talking about convergence in arbitrary topological spaces?

Since every metric space is a topological space, and seeing as we already have a notion of convergence in metric spaces, we would want to define the notion of convergence in a topological space to be a *generalisation*. To that end, we restate the definition of convergence in metric spaces using only the language of open sets.

Proposition 1.2.2. *Let (X, d) be a metric space. For every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$, the following are equivalent:*

1. x_n converges to x as in Definition 1.1.6
2. For every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

We do not prove this proposition here, but note that it is not difficult to prove.

Since the second statement in Proposition 1.2.2 does not actually mention any *metric space* properties of X , it is a viable definition for convergence in topological spaces.

Definition 1.2.3 (Convergence of a Sequence in a Topological Space). Let (X, τ) be a topological space, and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence in X . Given some $x \in X$, we say that x_n **converges to** x , denoted $x_n \rightarrow x$, if for every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Proposition 1.2.2 essentially tells us that if (X, d) is a metric space, then Definitions 1.1.6 and 1.2.3 are equivalent, where we apply Definition 1.2.3 to the topological space structure induced by the metric space structure.

Example 1.2.4 (Convergence in the Indiscrete Topology). Consider a set X along with the indiscrete topology (ie, view it as the topological space $(X, \{\emptyset, X\})$). Then, every sequence converges to every point.

Example 1.2.4 demonstrates that the uniqueness of limits seen in Proposition 1.1.7 does not always hold in topological spaces. See Definition 1.2.5 for more.

Note that going forward, we often omit the topology when talking about topological spaces, unless it has specific properties that are essential for our purposes.

Recall that limits in a metric space are unique (Proposition 1.1.7). We noted that this is not always true, but we are interested enough in distinguishing spaces where this is true from spaces where this is not true that we have a name for such spaces.

Definition 1.2.5 (Hausdorff/ T_2 Spaces). Let (X, τ) be a topological space. We say (X, τ) is **Hausdorff**, or T_2 , if for all $a, b \in X$ with $a \neq b$, there exist $U, V \in \tau$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

The Hausdorff/ T_2 property is an instance of a *separation property*. There are many more such properties, including T_1 , T_3 , $T_{3\frac{1}{2}}$, and T_4 , and it is highly non-trivial to prove facts like $T_4 \implies T_{3\frac{1}{2}}$. We will see such things as the course progresses.

Note that Proposition 1.1.7 tells us precisely that Metric Spaces are Hausdorff/ T_2 . Indeed, this tells us that Hausdorffitude is a *necessary* condition for a topology on a space to be **metrisable**—that is, for it to be induced by a metric. In particular, Example 1.2.4 tells us that the indiscrete topology is non-metrisable. There is the more interesting question of what conditions might be *sufficient* for a topology to be metrisable. Metrisability is, indeed, highly non-trivial property, and we will see some metrisation theorems in this course.

1.2.2 Continuity

Just as we generalised the notion of convergence from metric spaces to topological spaces, so too can we generalise the notion of continuity.

Definition 1.2.6 (Continuity). Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a function. We say f is **continuous** if for every $U \subseteq Y$ open in Y , the pre-image $f^{-1}[U]$ is open in X .

It is easy to show that Definition 1.2.6 is equivalent to Definition 1.1.3. We do not do this here.

There are many examples of continuous functions, some familiar and some slightly pathological.

Example 1.2.7 (Weird Topologies lead to Weird Notions of Continuity). Let X and Y be topological spaces.

1. If X and Y both have the *discrete* topology on them, then *any* function $f : X \rightarrow Y$ is continuous, because all subsets of X and Y are open.
2. If X and Y both have the *indiscrete* topology on them, then **sorry**

1.2.3 Interiors and Closures

We briefly discuss how to “make subsets of a topological space open or closed”. We do this by defining interiors and closures.

Definition 1.2.8 (The Interior of a Set). Let X be a topological space and let $A \subseteq X$. We define the **interior** of A , denoted $\text{int}(A)$ or A° , to be the union of all open subsets of A .

Note that $\text{int}(A)$ is always open. Moreover, it is the largest (with respect to inclusion) open subset of A . Finally, note that the sets that are equal to their interiors are precisely the open sets.

We have a ‘dual’ notion of interiors that capture closure properties.

Definition 1.2.9 (Closure). Let X be a topological space and let $A \subseteq X$. We define the **closure** of A , denoted $\text{cl}(A)$ or \overline{A} , to be the intersection of all closed subsets of X containing A .

Dually to the interior, $\text{cl}(A)$ is always closed. Moreover, it is the smallest (with respect to inclusion) closed subset of X containing A . Finally, note that the sets that are equal to their closures are precisely the closed sets.

1.3 A Closer Look at Topologies

1.3.1 Bases and Sub-Bases

It is interesting to talk about whether a topology can be *generated* by a family of subsets by taking unions. In the case of metric spaces, it is not difficult to show that this is true.

Proposition 1.3.1 (Metric Space Topologies are Generated by Open Balls). *Let (X, d) be a metric space. Any $U \subseteq X$ is open if and only if U is a union of open balls.*

Proof. Proposition 1.1.11 tells us that if U is a union of open sets, then it is open. Conversely, for all $x \in U$, there is some ε_x such that $B_{\varepsilon_x}(x) \subseteq U$. Then, it is easily seen that

$$U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$$

showing that U is a union of open sets. □

We have a special term for families of open sets that ‘generate’ a topology.

Definition 1.3.2 (Basis of a Topology). Let (X, τ) be a topological space. A **basis for τ** is a subset $B \subseteq \tau$ such that every $U \in \tau$ is a union of sets in B .

Proposition 1.3.1 tells us precisely that the open balls in a metric space form a *basis* for the topology induced by the metric.

We have a nice criterion to check if a subset of a topology is a basis.

Proposition 1.3.3. *Let (X, τ) be a topological space. Let $B \subseteq \tau$. TFAE:*

1. B is a basis for τ .
2. X is a union of elements of B . Moreover, for all $U, V \in B$, $U \cap V$ is a union of elements of B .

It is immediate that the first statement implies the second. The converse is an exercise in set algebra, which we do not do here.

It is also possible to define bases using ‘sub-bases’. To that end, we have a preliminary fact.

Lemma 1.3.4. *Let X be a set and let Π be a non-empty set of topologies on X . Then,*

$$\bigcap \Pi = \{A \subseteq X \mid \forall \tau \in \Pi, A \in \tau\}$$

We do not prove this here. We merely mention that the proof has the same flavour as such statements in algebra as “an arbitrary intersection of subgroups/ideals/subfields is a subgroup/ideal/subfield”.

A particular consequence of the above is that we can make the following definition.

Definition 1.3.5 (Topology Generated by a Set). Let (X) be a set and let C be any collection of subsets of X . We can define a topology on X by

$$\Pi(C) := \bigcap \{\tau \text{ a topology on } X \mid C \subseteq \tau\}$$

We call this the **topology on X generated by C** .

Note that it is *always* possible to define a topology generated by a subset, since $\Pi(C)$ is always non-empty: it always contains the discrete topology.

We can now define a sub-basis of a topology.

Definition 1.3.6 (Sub-Basis of a Topology). Let X be a set and let $C \subseteq X$. If τ is the least topology of X that contains C , then we say that C is a **sub-basis of τ** .

Indeed, any subset is a sub-basis of *some* topology, namely, the topology it generates.

1.3.2 New Topologies from Old Ones

There are numerous techniques to define new topologies from old ones. We begin with the most obvious definition imaginable.

Definition 1.3.7 (The Subspace Topology). Let X be a topological space. Any subset $Y \subseteq X$ inherits a topology from X , known as the **subspace topology**, the open sets of which are precisely those sets of the form $U \cap Y$ for open sets $U \subseteq X$.

With that out of the way, we move onto more interesting examples.

We begin with a topology that is defined using continuity of functions.

Definition 1.3.8 (Initial Topology). Let X be a set. Given an index set \mathcal{I} , topological spaces X_i and functions $f_i : X \rightarrow X_i$ for $i \in \mathcal{I}$, we define the **initial topology on X with respect to X_i and f_i** to be the least topology on X (with respect to inclusion) such that for all $i \in \mathcal{I}$, f_i is continuous.

Note that it is always possible to do this, because at worst, we the discrete topology on X renders continuous every function from X to any topological space.

Given the definition of continuity (Definition 1.2.6), it is possible to be explicit about the initial topology.

Lemma 1.3.9. *Let X, \mathcal{I}, X_i, f_i be as in Definition 1.3.8. The corresponding initial topology τ is given by*

$$\tau = \{f_i^{-1}[U] \subseteq X \mid i \in \mathcal{I} \text{ and } U \subseteq X_i \text{ is open}\}$$

We give an important example illustrating the notion of an initial topology.

Example 1.3.10 (Products). Let \mathcal{I} be an index set and let X_i be topologies. Consider the Cartesian product X of these X_i , defined as follows:

$$X = \prod_{i \in \mathcal{I}} X_i = \{(x_i)_{i \in \mathcal{I}} \mid \forall i \in \mathcal{I} \ x_i \in X_i\}$$

We know (either by the category theoretic definition of a product, of which the Cartesian product is the instance in the category of sets, or by sheer common sense) that there are projections $\pi_i : X \rightarrow X_i$ that map any $(x_j)_{j \in \mathcal{I}}$ to x_i , indexed by $i \in \mathcal{I}$. We can compute the

initial topologies of these projections to obtain a topology on X .

We can give a special name to the topology defined above, which is a key way of constructing new topologies from old ones.

Definition 1.3.11 (The Product Topology). Let $\mathcal{I}, X_i, X, \pi_i$ be as in Example 1.3.10. We call the initial topology of the π_i the **product topology** on X .

We can give a more explicit description of the product topology.

Proposition 1.3.12. Let $\mathcal{I}, X_i, X, \pi_i$ be as in Example 1.3.10 and Definition 1.3.11. Then,

1. A sub-basis of the product topology is given by all sets of the form

$$\{(x_j)_{j \in \mathcal{I}} \mid x_i \in U\}$$

for $i \in \mathcal{I}$ and $U \subseteq X_i$ open.

2. A basis of the product topology is given by all sets of the form

$$\{(x_j)_{j \in \mathcal{I}} \mid x_{i_1} \in U_1, \dots, x_{i_n} \in U_n\}$$

for $n \in \mathbb{N}$, $i_1, \dots, i_n \in \mathcal{I}$, and $U_k \subseteq X_{i_k}$ open for $1 \leq k \leq n$.

We do not prove this here.

Dually to how we defined a topology on the Cartesian product of spaces as the initial topology of the projections, we can define a topology on any subset of a space as the initial topology of the inclusion.

Finally¹, we define the *final* topology.

Definition 1.3.13 (Final Topology). Let X be a set. Given an index set \mathcal{I} , topological spaces X_i and functions $f_i : X_i \rightarrow X$ for $i \in \mathcal{I}$, we define the **final topology on X with respect to X_i and f_i** to be the largest topology on X (with respect to inclusion) such that

¹Pun intended

Write
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for all $i \in \mathcal{I}$, f_i is continuous.

Remark. To state it explicitly, we observe that if X has more open sets (i.e. the topology on X is “larger”), then it is “harder” for a function $f_i : X_i \rightarrow X$ to be continuous (as $f_i^{-1}(U)$ must be open in X_i for all open $U \subseteq X$, which forces the topology on X_i to be larger as well).

Note that it is always possible to construct a final topology, because at worst, we the indiscrete topology on X renders continuous every function into X from any topological space.

Given the definition of continuity (Definition 1.2.6), it is possible to be explicit about the initial topology.

Lemma 1.3.14. *Let X, \mathcal{I}, X_i, f_i be as in Definition 1.3.13. The corresponding initial topology τ is given by*

$$\tau = \{U \subseteq X \mid i \in \mathcal{I} \text{ and } f_i^{-1}(U) \subseteq X_i \text{ is open}\}$$

There are many good examples of final topologies, some of which are likely to be familiar to the reader.

Example 1.3.15 (Topologies on Quotients). Let X be a topological space and let \sim be an equivalence relation on X . Let $q : X \rightarrow X/\sim$ be the canonical surjection from X to the quotient set. The final topology of q is a topology on X/\sim .

We give a special name to such a topology on a quotient.

Definition 1.3.16 (Quotient Topology). Let X, \sim , and q be as in (1.3.15). We call the final topology on X/\sim with respect to X and q the **quotient topology**.

Note that the above definition really applies to *all* situations where X is a topological space and Y is a set such that there is a surjection $q : X \rightarrow Y$: in this case, Y is essentially the quotient of X by the relation \sim where $x \sim y$ iff $f(x) = f(y)$.

We can express the quotient topology in a more familiar way.

Proposition 1.3.17. *sorry*

1.3.3 Separation Properties

Throughout this subsection, let X denote a topological space.

We recall the definition of T_2 spaces from Definition 1.2.5. Definition 1.2.5 already exists btw - just link to it or smth if you want

Definition 1.3.18 (T_2 (i.e. the Hausdorff condition)). We say that a topological space is T_2 , or **Hausdorff**, if $\forall x \neq y$ there exist open sets U, V such that $x \in U, y \in V$ that we have $U \cap V = \emptyset$

There is also a (weaker) notion of separation called the T_1 property.

Definition 1.3.19 (T_1 property). We say that X is a T_1 **space** if for all *distinct* $x, y \in X$, there is an open set $U \subseteq X$ with $x \in U$ and $y \notin U$.

We can give an equivalent characterisation of the T_1 property.

Proposition 1.3.20. X is T_1 if and only if every singleton in X is closed.

Proof. First, notice that in the trivial cases where $|X| = 0$ or $|X| = 1$, we are done. So, going forward, assume that X contains at least two distinct elements.

(\implies) Assume that X is T_1 . If $|X| = 1$, then we are done. Fix $x \in X$ and consider the singleton $\{x\}$. We know that for any $y \in X \setminus \{x\}$, there exists some open $U_y \subseteq X$ such that $y \in U_y$ and $x \notin U_y$.² Then, it is easy to see that

$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$$

Indeed, the \subseteq inclusion is obvious, since $y \in U_y$ for all $y \in X \setminus \{x\}$, and the \supseteq inclusion is also clear because x does not lie in any of the U_y . Since each U_y is open, so is the union of

²Define a pairing (y, U_y) from the definition of X being T_1 using the Axiom of Choice.

all of them, making $X \setminus \{x\}$ a union of open sets, hence open. Thus, $\{x\}$ is closed.

(\Leftarrow) Assume that every singleton $\{x\} \subseteq X$ is closed. Then, fix $x, y \in X$ and assume $x \neq y$. Since $\{x\}$ is closed, $X \setminus \{x\}$ is open, and since x and y are distinct, $y \in X \setminus \{x\}$. Thus, $X \setminus \{x\}$ is an open subset of X containing y but not x .

Therefore, the condition that X is T_1 is equivalent to the condition that every singleton is closed. □

There is a notion that is weaker still.

Definition 1.3.21 (T_0 property). We say X is a T_0 space if for all distinct $x, y \in X$,

$$\{U \subseteq X \mid U \text{ is open and } x \in U\} \neq \{V \subseteq X \mid V \text{ is open and } y \in V\}$$

That is, either there is an open set U with $x \in U$ and $y \notin U$ or there is an open set V with $y \in V$ and $x \notin V$.

Remark. We emphasise that here, we are not able to choose which of x and y is contained in the open set U witnessing the T_0 property.

Before we give a classic example of a T_0 topology, we recall the definition of a partially ordered set (also abbreviated ‘poset’).

Definition 1.3.22 (Partial Order). Let P be a set. We say that a binary relation \leq on P is a **partial order** if it is reflexive, antisymmetric and transitive. We call the pair (P, \leq) a **partially ordered set**, often abbreviated **poset**.

There are many familiar examples of posets in mathematics.

Example 1.3.23 (A Collection of Sets as a Poset). Given any collection of sets, the collection can always be ordered by inclusion. So any time you take a family of sets and order them by inclusion, you will wind up with a poset structure.

Given that topologies are sets of sets, in particular, topologies are *also* partially ordered by inclusion. Thus, we will find the theory of posets to be quite useful throughout this course.

There is also a “converse” relationship between posets and topology: we can define a topology on any poset by taking advantage of the properties of the partial order.

Example 1.3.24 (A T_0 Topology on a Poset). Let (P, \leq) be a poset. We introduce a topology τ on P as follows: we deem a set $U \subseteq P$ to be open iff for all $p \in U$,

$$\{q \in P \mid q \leq p\} \subseteq U$$

That is, we define the open sets to be precisely those U that contain all downward cones of elements in U . We can show that this is, indeed, a topology on P .

The topology described in Example 1.3.24 is the ‘natural’ topology for *forcing posets* in set theory.

1.4 A Closer Look at Continuity

1.4.1 Neighbourhoods and Neighbourhood Bases

Next, we introduce the notion of a neighbourhood.

Definition 1.4.1 (Nbhd). We say that a point $x \in X$ has nbhd $A \subseteq X$ if there exists some open set $O \subseteq X$ such that $x \in O \subseteq A$.

Abbreviation. Because nobody has the time to write “neighbourhood” more than a few times in their life, Professor Cummings will abbreviate it by “Nbhd” in the future. We will not, by default, assume that a neighbourhood (or nbhd) of a point is open.

We can talk about neighbourhoods in a collective sense, reminiscent of `Filter.nhds` in Lean...

Definition 1.4.2 (Nbhd Basis). Given $x \in X$, a **nbhd basis** for x is a set \mathcal{N} of nbhds of x such that for every nbhd A of x there exists some $B \in \mathcal{N}$ such that $B \subseteq A$ - equivalently, for every open nbhd U of x there exists $B \in \mathcal{N}$ such that $B \subseteq U$.

Example 1.4.3 (The Closed Ball Nbhd Basis). Let (X, d) be a metric space. Fix $x \in X$.

Let \mathcal{N} be a set of nbhds of the form

$$\{y \in X \mid d(x, y) \leq \varepsilon\}$$

for all $\varepsilon > 0$. Then, this set all of closed balls at x is a nbhd basis.

Remark. It is a “trivial fact” that if we have a neighbourhood basis at every point $x \in X$, then we can recover the topology of X . (See this as an exercise.)

Finally, we note that nbhds give us a definition of continuity that more closely resembles the definition to which we are accustomed in metric spaces.

Proposition 1.4.4. *Given topological spaces X and Y and a function $f : X \rightarrow Y$, TFAE:*

1. f is continuous
2. for all $x \in X$ and N a nbhd of $f(x)$, there is a nbhd M of x such that $f[M] \subseteq N$.

In particular, specialising second characterisation of continuity in the above result to a single point gives us a way of talking about the continuity of a function *at a single point*.

1.4.2 Homeomorphisms

Throughout this subsection, fix topological spaces X and Y .

We first define the notion of open and closed functions.

Definition 1.4.5 (Open and Closed Functions). For any $f : X \rightarrow Y$, we say:

- f is **open** iff for all open $U \subseteq X$ we have $f(U)$ open in Y .
- f is **closed** iff for all closed $C \subseteq X$ we have $f(C)$ closed in Y .

Next, we define the notion of an isomorphism in the category of topological spaces.

Definition 1.4.6 (Homeomorphism). We say $f : X \rightarrow Y$ is a **homeomorphism** if f satisfies the following conditions:

1. f is a bijection

2. For all $U \subseteq X$, U is open in X iff $f[U]$ is open in Y

We now mention an easy result (that effectively shows that homeomorphisms are isomorphisms in the category of topological spaces).

Lemma 1.4.7. *For any function $f : X \rightarrow Y$, the following are equivalent:*

1. f is a homeomorphism
2. There is $g : Y \rightarrow X$ such that
 - (a) g is continuous
 - (b) $f \circ g = \text{id}_Y$
 - (c) $g \circ f = \text{id}_X$

We do not prove this result here.

We note that the continuity assumption on the inverse is strictly necessary:

Warning. A bijective continuous map need not be a homeomorphism!

This is because continuity is in some sense “one sided” (to take care of the other side, maybe we will need to also require our map be open...).

We illustrate this using a simple counterexample.

Counterexample 1.4.8 (A Continuous Bijection that is NOT a Homeomorphism). Let X be a set of cardinality at least 2. Consider the spaces (X, τ_1) and (X, τ_2) , with τ_1 being the discrete topology and τ_2 being the indiscrete topology. Then, the identity function id_X is a bijection from (X, τ_1) to (X, τ_2) ; moreover, it is continuous because every function out of (X, τ_1) is continuous. However, its inverse (which is also the identity) is *not* continuous.

1.5 Connectedness

Again, fix a topological space X . In this section, we discuss different notions of connectedness of a topological space.

Definition 1.5.1 (Disconnectedness). We say X is **disconnected** if there exist disjoint open sets $U, V \subsetneq X$ with $U \cup V = X$. If U and V are disjoint and cover X , we say that they **disconnect** X .

Remark. One must be very careful when disconnecting sets: when you have a disconnected set X , it is not at all obvious that the sets you use to disconnect X (namely, U and V) are actually disjoint in the rest of X . (Remember this for your General Topology basic exam!)

We can use this definition to define connectedness in the obvious way.

Definition 1.5.2 (Connectedness). We say X is **connected** if X is not disconnected, ie, if there do not exist disjoint open sets U and V that disconnect X .

As one would expect, we define connectedness for subsets to be connectedness with respect to the subspace topology.

It turns out we can use the definition of connectedness to say something about the clopen sets of a connected space.

Lemma 1.5.3. Let $(A_i)_{i \in \mathcal{I}}$ be some family of subsets of X indexed by some non-empty set \mathcal{I} such that

1. For all $i \in \mathcal{I}$, A_i is a connected subset of X
2. For all $i, j \in \mathcal{I}$, $A_i \cap A_j \neq \emptyset$

That is, (A_i) is a family of connected, non-disjoint sets. Then,

$$A := \bigcup_{i \in \mathcal{I}} A_i$$

is a connected subset of X .

Proof. Suppose that A is not connected. Then, there exist open subsets $U, V \subsetneq X$ such that

$$A = (A \cap U) \cup (A \cap V)$$

with $(A \cap U)$ and $(A \cap V)$ being disjoint, non-empty, and proper subsets of A . Then, for any $i \in \mathcal{I}$,

we have that

$$A_i \subseteq A = (A \cap U) \cup (A \cap V)$$

Since the A_i are all connected, we know that for all $i \in \mathcal{I}$, either $A_i \subseteq U$ or $A_i \subseteq V$. It turns out that we can do better: either every A_i is contained in U or every A_i is contained in V . Indeed, if this were not the case, then there would be $i, j \in \mathcal{I}$ with $A_i \subseteq U$ and $A_j \subseteq V$. Then, we would have $A_i \cap A_j \subseteq U \cap V$. Moreover, $A_i \cap A_j \subseteq A$, since $A_i \subseteq A$ and $A_j \subseteq A$. Thus,

$$A_i \cap A_j \subseteq A \cap U \cap V = (A \cap U) \cap (A \cap V)$$

Finally, we assumed, in our setup, that $A_i \cap A_j \neq \emptyset$, meaning that there is an element living in the disjoint set $A \cap U$ and $A \cap V$, which is obviously a contradiction. Hence, either every A_i is contained in U or every A_i is contained in V .

Without loss of generality, say that every A_i is contained in U . Then, $A \subseteq U$ also, meaning that $A \cap U = A$. Hence, $A \cap V = \emptyset$, which contradicts the assumption that A is disconnected, because disconnecting subsets must be proper. \square

We next use the definition of connectedness to establish a “canonical” decomposition of X .

1.5.1 Connected Components

Define the binary relation \sim on X such that $a \sim b \iff \exists A \subseteq X$ with A connected and $a, b \in A$. We then claim that \sim is an equivalence relation, and (having shown this) we let the equivalence classes of \sim partition X into sets X_i .

Definition 1.5.4 (Connected Components). The **connected components** of X are the equivalence classes in X/\sim , with \sim being defined as above.

Let us describe these equivalence classes. Fix $x \in X$. Denote its equivalence class in X/\sim by $[x]$. Then, it is possible to show that

$$[x] = \{y \in X \mid y \sim x\} = \bigcup \{A \subseteq X \mid x \in A \text{ and } A \text{ is connected}\}$$

The \subseteq inclusion is clear, because $[x]$ is itself connected, meaning that any $y \in [x]$ clearly lies in

some connected subset of X containing x . Conversely, any element of any connected subset of X containing x must be connected to x , putting it in $[x]$.

Lemma 1.5.5. *If $A \subseteq X$ and A is connected, then $\text{cl}(A)$ is also connected.*

Proof. Suppose that $\text{cl}(A)$ is not connected. then, there are open sets $U, V \subseteq X$ such that $\text{cl}(A) \subseteq U \cup V$, $\text{cl}(A) \cap \text{cl}(B) = \emptyset$, and $\text{cl}(A) \cap U, \text{cl}(A) \cap V \neq \emptyset$. We can see, since $A \subseteq \text{cl}(A)$, that $A = (A \cap U) \cup (A \cap V)$, with $A \cap U \cap V = \emptyset$. Since A is connected, we need either $A \cap U = \emptyset$ or $A \cap V = \emptyset$. Assume it is the former. sorry □

Corollary 1.5.6. *The connected components of X are closed subsets of X .*

Proof. Fix $x \in X$. Then, by Lemma 1.5.5, the closure $\text{cl}([x])$ of the connected component $[x]$ containing x is a connected subset of X . Since all such connected sets are contained in $[x]$, we have that $\text{cl}([x]) \subseteq [x]$, meaning $[x]$ is equal to its closure, making it closed. □

1.5.2 Preserving Connectedness

Surprisingly—or unsurprisingly—connectedness is not preserved by taking subsets.

Counterexample 1.5.7 (A Disconnected Subset of a Connected Space). We know that \mathbb{R} is connected under the Euclidean topology. Consider $\mathbb{Q} \subset \mathbb{R}$. We can clearly see that $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $(\sqrt{2}, \infty) \cap \mathbb{Q}$ are disjoint, open, proper subsets of \mathbb{Q} that disconnect it.

However, it turns out the *image* of a connected set in a continuous function *is* connected.

Lemma 1.5.8. *Let X and Y be topological spaces. Let $A \subseteq X$ be connected and let $f : X \rightarrow Y$ be continuous. Then, $f[A]$ is connected.*

Proof. If not, let $U, V \subseteq Y$ be open with $f[A]$ sorry □

This allows us to construct many examples of connected spaces.

Example 1.5.9 (Connectedness of the Unit Interval). Consider $[0, 1]$ along with the subspace topology inherited from the Euclidean topology on \mathbb{R} . We show that $[0, 1]$ is connected.

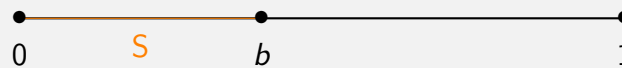
Indeed, suppose $[0, 1]$ is *not* connected. Then, there exist open sets $U, V \subseteq [0, 1]$ such that $U \cup V = [0, 1]$ and $U \cap V = \emptyset$. Thus, $0 \in U$ or $0 \in V$. WLOG, say $0 \in U$. Define

$$S = \{a \in [0, 1] \mid [0, a] \subseteq U\}$$

Observe that S has the following properties.

1. $0 \in S$, so S is non-empty.
2. S is bounded above (by 1, for instance)

Therefore, S has a finite supremum (indeed, a supremum in $[0, 1]$). Denote this b .



Now, observe that b has the following properties.

1. $b > 0$. The reason for this is that since $0 \in U$ and U is open, there is some $\varepsilon > 0$ such that $[0, \varepsilon] \subseteq U$. Thus, $b \geq \varepsilon > 0$.
2. $[0, b) \subseteq U$. This is true by definition of b and S .
3. $b \in U$. Otherwise, certainly $b \in V$, and therefore, by openness, there is some $\varepsilon > 0$ such that $(b - \varepsilon, b] \subseteq V$, but for all $\varepsilon > 0$, we must have that $b - \varepsilon \in U$ by definition of b (and S), and U and V are disjoint.

The idea is to show that $b \in U \cap V$ or $b = 1$, a contradiction. sorry

1.5.3 Path Connectedness

Definition 1.5.10 (Path). Given X a space, let $a, b \in X$. We say that a **path** in X is a continuous function $\gamma : [0, 1] \rightarrow X$ satisfying $\gamma(0) = a, \gamma(1) = b$.

Remark. You may have many paths, be unable to use sub or super-scripts, but still need to name them all. Professor Cummings is here to help: $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, \pi, \phi, \rho$, etc.

As a preliminary, we note that we can compose paths to obtain new paths. (sorry)

Definition 1.5.11 (Path-Connected Space). We say that X is **path-connected** iff for all $a, b \in X$ there is a path from a to b .

It is possible to show that every path-connected space is connected (**sorry**), but not the other way round (**sorry**).

1.6 Compactness and Completeness

1.6.1 Compactness

Recall that last time (when we were learning about Monsters) we showed the following facts:

- $[a, b]$ is compact.
- A closed subset of a compact space is compact.
- If $C \subseteq \mathbb{R}$ (in the usual topology is closed and bounded, then C is compact.

Generally speaking, when we consider some new property of a topological space, it is worthwhile to think about whether it passes to subspaces.

Lemma 1.6.1. *Any continuous image of a compact set is compact.*

Proof. Let $C \subseteq X$ be compact with $f : X \rightarrow Y$ a continuous function. Letting F denote an open cover of $f[C]$ (i.e. a collection of open sets covering $f[C]$), the collection of sets $\{f^{-1}[U] : U \in F\}$ will give an open covering of C , and now the finite subcover of $f[C]$ can be seen to follow from the existence of our finite subcover of C (select finitely many U such that $f^{-1}[U]$ cover C). \square

Lemma 1.6.2. *Given X a Hausdorff space with $C \subseteq X$ a compact subset of X , necessarily C is closed in X .*

Proof. Given $y \notin C$, for each $x \in C$ we let $N(x)$ denote an open neighborhood of x disjoint from some open neighborhood $N_x(y)$ of y . Noting that $\{N(x) : x \in C\}$ forms an open cover of C , as

C is a compact set it follows that there exist finitely many $x_i \in C$ such that $\{N(x_i) : i < n\}$ forms an open cover of C .

But then we observe that $\bigcap_{i < n} N_{x_i}(y) = \mathcal{N}_C(y)$ is a finite intersection of open sets containing y which is disjoint from C . In particular, $\mathcal{N}(y) \subseteq X \setminus C$. It follows that such an open neighborhood must exist for every $y \in X \setminus C \implies X \setminus C$ is open $\implies C$ must be closed. \square

Remark. "This is just logic, in its most pejorative sense."

Before moving on, we pause here for a brief interlude which will allow us to proceed in more generality (oh yay!).

Definition 1.6.3 (Bounded Subset). Let Y be a metric space, $B \subseteq Y$. Then we say that B is "bounded" if and only if there exists some $y \in Y, C > 0$ such that $d(y, b) < C$ holds for all $b \in B$.

Let's get on the straight and narrow:

Lemma 1.6.4. *If B is a compact subset of a nonempty metric space Y , then B is bounded.*

Proof. We can choose $y \in Y$ and let $U_n = B_n(y) = \{x \in Y : d(x, y) < n\}$. Then $B \subseteq X = \bigcup_{n \in \mathbb{N}} U_n$, and as B is compact and $U_n \subseteq U_{n+1}$, it follows that there must exist some $N \in \mathbb{N}$ such that $B \subseteq U_N$ (so by definition of a bounded subset, B is bounded). \square

And now with all these lemmas at our disposal, it's time for another lemma (a "fact"):

Lemma 1.6.5. *Let X be a compact space, and let $f : X \rightarrow \mathbb{R}$ denote a continuous function. Then if f is bounded, it must attain its bounds.*

Proof. Let $C = f[X]$. As continuous images of compact sets are compact, $f[X]$ must be compact, and further as a compact subset of \mathbb{R} it must be closed and bounded. If we now let $M = \sup(C)$ and $m = \inf(C)$ then $m \leq f(x) \leq M$ for all $x \in X$. As C is closed, we must have $m, M \in C = f[X]$ (and hence f must attain its bounds). \square

Remark. $C \subseteq \mathbb{R}$ is in fact compact *if and only if* C is closed and bounded.

Another brief digression - there are several “phrase-ologies” in use to compare various topologies:

Remark. Given topologies σ, τ both topologies on some space Z , we say that σ is smaller (i.e. weaker, coarser) than τ is $\sigma \subseteq \tau$, and correspondingly τ is larger (i.e. stronger, finer).

Remark. X is compact if and only if, given any open cover \mathcal{O} of X by basis elements, there exists a finite subcover $\mathcal{O}' \subseteq \mathcal{O}$ which covers X .

1.6.2 Completions in a Metric Space

For context, we have been speaking about completeness, and we are now going to build a new complete metric space out of the Cauchy sequences in our metric space (X, d) :

- Let (X, d) be a metric space, and let $Z = \{(x_n)_{n \in \mathbb{N}} : (x_n)_n \text{ is a Cauchy sequence in } X\}$.
- We first claim (1) that If $(x_n)_n, (y_n)_n \in Z$, then $(d(x_n, y_n))_n$ is a Cauchy sequence in \mathbb{R} - so in particular, it has a limit $d^*((x_n)_n, (y_n)_n) = d^*(x, y)$.
- We next claim (2) that the set of $[x] = \{y \in Z : d^*(x, y) = 0\}$ for $x \in Z$ will partition Z , or in other words, that R defined such that $xRy \leftrightarrow d^*(x, y) = 0$ forms an equivalence relation on Z .
- Now we claim (3) that $d^*(x, y)$ depends only on the classes of $x = (x_n)_n, y = (y_n)_n \in Z$, and we then define $\bar{d}([x], [y]) = d^*(x, y)$.
- And then we claim (4) that \bar{d} is a metric on the set of equivalence classes \bar{X} .
- Jumping off from here, we let $i : X \rightarrow \bar{X}$ such that $i(x) = [(x_n)_n]$, where $(x_n)_n = (x, x, x, \dots)$.
- We would claim (5) that $\bar{d}(i(x), i(y)) = d(x, y)$ (i.e. that i is an “isometric embedding”).
- Then we claim that $i[X]$ is dense in \bar{X} , and that every point in \bar{X} is the limit of a sequence of points in $i[X]$.
- (7) * We claim that (\bar{X}, \bar{d}) is complete. *
- (Then to finish, once can identify $x \in X$ with $i(x) \in \bar{X}$ and viola.)

So now that we have a way to construct a completion of X , then we have the following fact:

Lemma 1.6.6. *If X is a metric space and Y is a complete metric space with $f : X \rightarrow Y$ an isometric embedding, then there is a unique $\bar{f} : \bar{X} \rightarrow \bar{Y}$ which is an isometric embedding with $\bar{f} \upharpoonright X = f$.*

Proof. Exercise! Or rather, “just do what comes naturally.” “I’m obviously being very glib. But this is not the main topic of this course.” □

Remark. As an exercise, use the above to show that any 2 completions of X must be isomorphic.

Now is where the instructor exercises his freedom in determining the order of topics to cover...

Definition 1.6.7 (Subsequence). An infinite subsequence of an infinite sequence $(x_n)_{n \in \mathbb{N}}$ is a sequence $(x_{i_n})_{n \in \mathbb{N}}$ where $i_k < i_{k+1}$ holds for all $k \in \mathbb{N}$.

Definition 1.6.8 (Sequentially Compact). We say that X is “sequentially compact” if and only if every infinite sequence has a convergent subsequence.

“I keep on saying this, then it keeps on being true.”

We now digress briefly to recall that convergence can be defined topologically (without reference to a metric). We also have this fact that in a metric space X , for any $A \subseteq X$ we will have

$$\text{cl}(A) = \{x \in X : \text{there is some } (x_n)_{n \in \mathbb{N}} \text{ such that } x_n \in A \text{ and } x_n \rightarrow x\}$$

and the proof of this **fact** is “a proof by extreme obviousness” that involves “ridiculously clear contradictions” - in other words, the proof is an exercise. However

Warning. This **fact** about metric spaces is very not true in general spaces! In a general space, all convergent sequences consisting of points in A converge to a point of $\text{cl}(A)$, but this may not give all points of $\text{cl}(A)$.

Next week, we will introduce the concept of “nets” to deal with this.

Definition 1.6.9 (Limit Point). Given a space X and $A \subseteq X$ with $x \in X$, we say that x is a “limit point for A ” to mean that, for every open neighborhood U of x , there is some $z \in A \cap U$ such that $z \neq x$.

The following is “a kind of logical truism” (which maybe seems less obvious than some of our other obvious facts):

Lemma 1.6.10. $cl(A) = A \cup \{x \in X : x \text{ is a limit point of } A\}$.

In general topological spaces, sequential compactness and compactness are **not** the same (as perhaps is to be expected) - however, in a metric space, the two notions do coincide.

1.6.3 The Relationship between Compactness and Completeness

Definition 1.6.11 (Totally Bounded). A metric space X is **totally bounded** iff for all $\varepsilon > 0$, X is a finite union of open ε -balls.

The reason we define total boundedness is that it connects compactness and completeness. Before we can establish that connection, we need an intermediate lemma.

Lemma 1.6.12. Let X be a metric space, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence with no repetitions. For all $x \in X$, there is some $\varepsilon > 0$ such that there is no $n \in \mathbb{N}$ with $x_n = x$ and $x_n \in B(x, \varepsilon)$.

Proof. We argue by contradiction. Assume it does not hold. That is, assume there is some $x \in X$ such that for all $\varepsilon > 0$, there is $n \in \mathbb{N}$ with $x_n \neq x$ and $x_n \in B(x, \varepsilon)$. Now, perform the following steps.

- Let $\varepsilon_0 = 1$. There is some $x_{n_0} \neq x$ with $x_{n_0} \in B(x, \varepsilon_0)$.
- Let $\varepsilon_1 := \min(2^{-1}, d(x, x_{n_0}))$. There is some $x_{n_1} \neq x$ with $x_{n_1} \in B(x, \varepsilon_1)$.
- Let $\varepsilon_2 := \min(2^{-2}, d(x, x_{n_0}))$. There is some $x_{n_1} \neq x$ with $x_{n_1} \in B(x, \varepsilon_2)$.
- ... and so on.

Then, one can show that each x_{n_k} satisfies $d(x_{n_k}, x) < 2^{-k}$. One can then take an increasing subsequence of these x_{n_k} and this gives us what we need. \square

We prove another intermediate result, sometimes referred to as the ‘existence of a Lebesgue number for open covers’ when viewed in the context of metric spaces.

Lemma 1.6.13. *Let X be a sequentially compact space, and let $(U_i)_{i \in \mathcal{I}}$ be an open cover of X . There is some $\varepsilon > 0$ such that for all $x \in X$, there is $i \in \mathcal{I}$ such that $B(x, \varepsilon) \subseteq U_i$.*

Proof. sorry \square

Theorem 1.6.14. *Let X be a metric space. The following are equivalent:*

- (1) X is compact
- (2) X is sequentially compact
- (3) X is complete and totally bounded

Proof. (1) \implies (2). Say X is compact, $(x_n)_{n \in \mathbb{N}}$ with $x \in X$. If $\{x_n \mid n \in \mathbb{N}\}$ is finite then, by the infinite pigeonhole principle there must exist some infinite constant subsequence (and then we’re done). Else, if $\{x_n \mid n \in \mathbb{N}\}$ is infinite, we must be able to replace it with a subsequence with no repetitions. Abuse notation and call this $(x_n)_{n \in \mathbb{N}}$ as well.

Towards contradiction, assume that (x_n) has no convergent subsequences. We claim that for all $x \in X$, there is some $\varepsilon > 0$ such that there is no $n \in \mathbb{N}$ with $x_n = x$ and $x_n \in B(x, \varepsilon)$.

Lemma 1.6.12 effectively tells us that there are balls $B(x, \varepsilon_x)$ for every $x \in X$, with $\varepsilon_x > 0$, such that for no $n \in \mathbb{N}$ is $x_n = x$ and $x_n \in B(x, \varepsilon_x)$. In particular, these balls form an open cover of X , and the assumption that x is compact tells us that this open cover has a finite subcover. Enumerate this subcover

$$B(y_1, \varepsilon_{y_1}), \dots, B(y_s, \varepsilon_{y_s})$$

Since $(x_n)_{n \in \mathbb{N}}$ has been trimmed down to have no repetitions, find now some $n \in \mathbb{N}$ such that $x_n \neq x$ and $x_n \neq y_1, \dots, y_s$. But there is some $1 \leq i \leq s$ such that $x_n \in B(y_i, \varepsilon_{y_i})$. Thus, we have a contradiction, meaning that (x_n) must have a convergent subsequence.

(2) \implies (3). Assume X is sequentially compact.

- X is totally bounded. Fix $\varepsilon > 0$. Assume, for contradiction, that X is not a union of finitely many open ε -balls. Choose, by induction, some $(x_n)_{n \in \mathbb{N}}$ such that

$$x_n \notin \bigcup_{i < n} B(x_i, \varepsilon)$$

Then, $(x_n)_{n \in \mathbb{N}}$ couldn't possibly have a convergent subsequence, a contradiction. Thus, X must be totally bounded.

- X is complete. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. As X is sequentially compact, there is some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $x_{n_k} \rightarrow x$ for some $x \in X$. We show that $x_n \rightarrow x$.

Fix $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ so large that for all $i, j \geq N_1$, we have $d(x_i, x_j) < \frac{\varepsilon}{2}$. Choose N_2 so large that for all $k \geq N_2$, we have $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Then, choose $N = \max(N_1, N_2)$. By the triangle inequality, for all $n \geq N$, we have that $d(x_n, x) < \varepsilon$.

(3) \implies (2). Suppose that X is totally bounded and complete. Fix a sequence $(x_n)_{n \in \mathbb{N}}$. To show that (x_n) has a convergent subsequence, it is enough to show that $(x_n)_{n \in \mathbb{N}}$ has a *Cauchy* subsequence.

Letting $A_0 \subseteq X$ be a finite set such that we can cover X with balls of radius $2^{-0}(= 1)$ such that

$$X = \bigcup_{y \in A_0} B(y, 2^{-0})$$

We can do this because we know, by assumption, that X is totally bounded. Then it follows by the infinite pigeonhole principle that there must exist some $y \in A_0$ such that $B_1(y)$ contains infinitely many points in our sequence - we now choose $y = y_0$.

Next, we consider a covering of X by balls of radius 2^{-1} . As there are infinitely many points in $B_1(y_0)$, it follows that there must exist some y_1 such that $B_{2^{-1}}(y_1) \subseteq B_1(y_0)$ with infinitely many points contained in $B_{2^{-1}}(y_1)$. It follows by similar argument that we can choose some y_2 with $B_{2^{-2}}(y_2) \subseteq B_{2^{-1}}(y_1) \subseteq B_1(y_0)$ with infinitely many points contained in $B_{2^{-2}}(y_2)$,

and similarly for y_3, y_4, \dots

After continuing in this manner, we claim that we have defined a Cauchy subsequence of (x_n) : define $x_{n_i} = y_i$. To show it is Cauchy, fix $\varepsilon > 0$. Observe that given $k < i$, with $n_k < n_i$, we know that $n_k \in Y_k$ and $n_i \in Y_i \subseteq Y_k$, giving us that $n_i \in Y_k$. sorry

(2) \implies (1). Assume that X is sequentially compact. Let $(U_i)_{i \in \mathcal{I}}$ be an open cover.

□

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Boy do I love L^AT_EX!

Visit <https://thefundamentaltheor3m.github.io/TopologyNotes/main.pdf> for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to the associated [GitHub repository](#).