21-651: General Topology

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Contents

Chapter 1

An Introduction to Topology

We begin by making a few slightly non-standard notational choices.

Notation. Given a function $f : X \to Y$ and a subset $A \subseteq X$, we write

$$f[A] := \{f(x) \in Y \mid x \in A\}$$

In similar fashion, given $B \subseteq Y$, we write

$$f^{-1}[B] := \{x \in X \mid f(x) \in B\}$$

The reason it is important to distinguish between the notations $f(A)$ and $f[A]$ is that there might be a situation in which $A \in X$ *and* $A \subseteq X$. We will not make any more notational choices at this stage that differ from standard mathematical conventions. As and when we do, we will introduce them.

We now recall basic facts about metric spaces.

1.1 A Word on Metric Spaces

Recall the definition of a metric space.

Definition 1.1.1 (Metric Spaces). A **metric space** is a pair (X, d) consisting of a set X and a function $d : X \times X \to \mathbb{R}$ such that

1. for all $x, y \in X$, $d(x, y) \geq 0$ for all $x, y \in X$
2. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. for all $x, y \in X$, $d(x, y) = d(y, x)$
4. for all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

We call the function d a **metric on** X.

We give several familiar examples.

Example 1.1.2 (Some Familiar Metric Spaces).

1. $\mathbb{R}^n$ under the Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

   for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$
2. Any set X under the equality metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

   for all $x, y \in X$
3. Any subset $Y \subseteq X$ of a metric space (X, d) under the restriction of d to $Y \times Y \subseteq X \times X$.
4. Given two metric spaces (X_1, d_1) and (X_2, d_2), there are numerous viable metrics we can define on $X_1 \times X_2$. One of them would be taking the *maximum* of d_1 and d_2; another would be the *sum*; a third would be

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$

   for all $(x_1, y_1) \in X_1$ and $(x_2, y_2) \in Y_2$. We define this third metric space to be the

product metric, and it is easily seen that the product of Euclidean spaces (under the Euclidean metric) is indeed a Euclidean space (under the Euclidean metric).

5. The set $C^0([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$ under the supremum metric

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|$$

for all $f, g \in C^0([0,1])$. More generally, any compact set works (not just $[0,1]$).

6. The set $C^0([0,1])$ under the metric

$$d(f,g) = \sqrt{\int_0^1 (f(x) - g(x))^2 \, \mathrm{d}x}$$

for all $f, g \in C^0([0,1])$, which we know is positive-definite because continuous functions that are zero almost everywhere are zero (and nonnegative functions whose integral is zero are zero almost everywhere).

7. Consider the set

$$l^2(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; x_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} x_n^2 < \infty \right\}$$

We can define the l^2 metric on this set by

$$d(x,y) := \sqrt{\sum_{n=0}^{\infty} (x_i - y_i)^2}$$

for all $x, y \in l^2(\mathbb{R})$. More than showing that this satisfies the properties of a metric, what is tricky here is showing that this metric is well-defined. But this is doable, and we will end the discussion of this example on that note.

After this barrage of examples of metric spaces, we are finally ready to move onto more interesting definitions. We begin by discussing the notion of continuity of functions between metric spaces.

Definition 1.1.3 (Continuity of Functions). Let (X,d) and (X',d') be metric spaces. We say that a function $f : X \to X'$ is **continuous at a point** $x_0 \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \varepsilon$. We say that f is **continuous** if f is continuous at every point $x_0 \in X$.

We mention two interesting facts that we do not bother to prove.

Exercise 1.1.4 (Argument-Wise Continuity of Metrics)**.** If (X, d) is a metric space, for all $a \in X$, the function

$$x \mapsto d(a, x) : X \to \mathbb{R}$$

is continuous.

Exercise 1.1.5 (Composition of Continuous Functions)**.** A composition of continuous functions is continuous.

We recall what it means for a sequence to converge in a metric space.

Definition 1.1.6 (Convergence of a Sequence)**.** Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. Given some $x \in X$, we say that x_n **converges to** x, denoted $x_n \to x$, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x) < \varepsilon$. We say x **is the limit of** x_n **as** $n \to \infty$.

We can show that limits in a metric space are unique.

Proposition 1.1.7. *Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. There is at most one x such that $x_n \to x$.*

Proof. Suppose, for contradiction, that there exist distinct points $x, x' \in X$ such that $x_n \to x$ and $x_n \to x'$. Pick $\varepsilon := d(x, x')/2$. For this ε, we know there is some $N \in \mathbb{N}$ such that for $n \geq N$, $d(x_n, x) < \varepsilon$. Similarly, we know there is some $N' \in \mathbb{N}$ such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. Pick $M := \max(N, N') + 1$. Then, applying the fact that $d(x_M, x) = d(x, x_M)$,

$$d(x, x') \leq d(x, x_M) + d(x_M, x') < \varepsilon + \varepsilon = d(x, x')$$

which clearly is a contradiction. So, x and x' cannot be distinct. $\square$

Warning. In this course, we will encounter spaces where a sequence can have more than one limit. Proceed with caution!

We recall the definition of an open ball.

Definition 1.1.8 (Open Ball). Let (X, d) be a metric space. Fix $a \in X$ and $\varepsilon > 0$. We define the **open ball of radius ε centred at a** to be

$$B_\varepsilon(a) := \{b \in X \mid d(a, b) < \varepsilon\}$$

The reason for the term 'open' in the above definition is the following definition.

Definition 1.1.9 (Open Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **open** if for all $a \in U$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$.

It is easy to see that an open ball is indeed an open set.

We can also define a dual notion.

Definition 1.1.10 (Closed Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **closed** if its complement $X \setminus U$ is open.

We recall basic properties of open and closed sets.

Proposition 1.1.11. *Let (X, d) be a metric space.*

1. *Both $\emptyset$ and X are both open and closed.*
2. *An arbitrary union of open sets is open.*
3. *A finite intersection of open sets is open.*

We omit the proof, because it is easy and basic.

We are now ready to venture into more general waters.

1.2 Topological Spaces

A topological space, broadly speaking, is one in which we wish to be able to discuss the notion of convergence without depending on the notion of distance. The definition of convergence in metric spaces really only relies on the openness of open balls. We can generalise it merely by generalising the definition of open sets.

We begin by defining the notion of a **topological space**.

Definition 1.2.1 (Topological Space). Let X be a set. A **topology** on X is a family τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

We say a subset of X is **open** with respect to a topology τ if it lies in τ. We call the pair (X, τ) a **topological space**.

The terminology for open sets is visibly consistent with the terminology used in metric spaces. In fact, this is exactly what Proposition 1.1.11 demonstrates: that the open sets of a metric space, defined as in Definition 1.1.9, do indeed define a topology on it. Every metric space is thus also a topological space.

Chapter 2

Another Chapter

You get the idea.

2.1 Introducing the Main Object of Study in this Chapter

Woah. Very cool.

2.2 Another Section

Boy do I love LaTeX!

Visit https://thefundamentaltheor3m.github.io/TopologyNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to my repository.