21-651: General Topology

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Contents

Chapter 1

A Recap of Undergraduate Topology

We begin by making a few slightly non-standard notational choices.

Notation. Given a function $f : X \to Y$ and a subset $A \subseteq X$, we write

$$f[A] := \{f(x) \in Y \mid x \in A\}$$

In similar fashion, given $B \subseteq Y$, we write

$$f^{-1}[B] := \{x \in X \mid f(x) \in B\}$$

The reason it is important to distinguish between the notations $f(A)$ and $f[A]$ is that there might be a situation in which $A \in X$ *and* $A \subseteq X$. We will not make any more notational choices at this stage that differ from standard mathematical conventions. As and when we do, we will introduce them.

We now recall basic facts about metric spaces.

1.1 Metric Spaces

Recall the definition of a metric space.

Definition 1.1.1 (Metric Spaces). A **metric space** is a pair (X, d) consisting of a set X and a function $d : X \times X \to \mathbb{R}$ such that

1. for all $x, y \in X$, $d(x, y) \geq 0$ for all $x, y \in X$
2. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. for all $x, y \in X$, $d(x, y) = d(y, x)$
4. for all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

We call the function d a **metric on** X.

We give several familiar examples.

Example 1.1.2 (Some Familiar Metric Spaces).

1. $\mathbb{R}^n$ under the Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

2. Any set X under the equality metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$

3. Any subset $Y \subseteq X$ of a metric space (X, d) under the restriction of d to $Y \times Y \subseteq X \times X$.
4. Given two metric spaces (X_1, d_1) and (X_2, d_2), there are numerous viable metrics we can define on $X_1 \times X_2$. One of them would be taking the *maximum* of d_1 and d_2; another would be the *sum*; a third would be

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}$$

for all $(x_1, y_1) \in X_1$ and $(x_2, y_2) \in Y_2$. We define this third metric space to be the

product metric, and it is easily seen that the product of Euclidean spaces (under the Euclidean metric) is indeed a Euclidean space (under the Euclidean metric).

5. The set $C^0([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$ under the supremum metric

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x)-g(x)|$$

for all $f,g \in C^0([0,1])$. More generally, any compact set works (not just $[0,1]$).

6. The set $C^0([0,1])$ under the metric

$$d(f,g) = \sqrt{\int_0^1 (f(x)-g(x))^2 \, \mathrm{d}x}$$

for all $f,g \in C^0([0,1])$, which we know is positive-definite because continuous functions that are zero almost everywhere are zero (and nonnegative functions whose integral is zero are zero almost everywhere).

7. Consider the set

$$l^2(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; x_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} x_n^2 < \infty \right\}$$

We can define the l^2 metric on this set by

$$d(x,y) := \sqrt{\sum_{n=0}^{\infty} (x_i - y_i)^2}$$

for all $x,y \in l^2(\mathbb{R})$. More than showing that this satisfies the properties of a metric, what is tricky here is showing that this metric is well-defined. But this is doable, and we will end the discussion of this example on that note.

After this barrage of examples of metric spaces, we are finally ready to move onto more interesting definitions.

1.1.1 Continuity of Functions

We begin by discussing the notion of continuity of functions between metric spaces.

Definition 1.1.3 (Continuity). Let (X, d) and (X', d') be metric spaces. We say that a function $f : X \to X'$ is **continuous at a point** $x_0 \in X$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$, if $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \varepsilon$. We say that f is **continuous** if f is continuous at every point $x_0 \in X$.

We mention two interesting facts that we do not bother to prove.

Exercise 1.1.4 (Argument-Wise Continuity of Metrics). If (X, d) is a metric space, for all $a \in X$, the function

$$x \mapsto d(a, x) : X \to \mathbb{R}$$

is continuous.

Exercise 1.1.5 (Composition of Continuous Functions). A composition of continuous functions is continuous.

1.1.2 Sequences, Convergence and Uniqueness of Limits

We recall what it means for a sequence to converge in a metric space.

Definition 1.1.6 (Convergence of a Sequence in a Metric Space). Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. Given some $x \in X$, we say that x_n **converges to** x, denoted $x_n \to x$, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x) < \varepsilon$. We say x **is the limit of** x_n **as** $n \to \infty$.

We can show that limits in a metric space are unique.

Proposition 1.1.7. *Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. There is at most one x such that $x_n \to x$.*

Proof. Suppose, for contradiction, that there exist distinct points $x, x' \in X$ such that $x_n \to x$ and $x_n \to x'$. Pick $\varepsilon := d(x, x')/2$. For this ε, we know there is some $N \in \mathbb{N}$ such that for $n \geq N$, $d(x_n, x) < \varepsilon$. Similarly, we know there is some $N' \in \mathbb{N}$ such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. Pick

$M := \max(N, N') + 1$. Then, applying the fact that $d(x_M, x) = d(x, x_M)$,

$$d(x, x') \leq d(x, x_M) + d(x_M, x') < \varepsilon + \varepsilon = d(x, x')$$

which clearly is a contradiction. So, x and x' cannot be distinct. $\square$

Warning. In this course, we will encounter spaces where a sequence can have more than one limit. Proceed with caution!

Finally, we discuss topological properties of subsets of topological spaces.

1.1.3 Open and Closed Sets

We recall the definition of an open ball.

Definition 1.1.8 (Open Ball). Let (X, d) be a metric space. Fix $a \in X$ and $\varepsilon > 0$. We define the **open ball of radius ε centred at a** to be

$$B_\varepsilon(a) := \{b \in X \mid d(a, b) < \varepsilon\}$$

The reason for the term 'open' in the above definition is the following definition.

Definition 1.1.9 (Open Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **open** if for all $a \in U$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$.

It is easy to see that an open ball is indeed an open set.

We can also define a dual notion.

Definition 1.1.10 (Closed Sets). Let (X, d) be a metric space. We say that $U \subseteq X$ is **closed** if its complement $X \setminus U$ is open.

We recall basic properties of open and closed sets.

Proposition 1.1.11. *Let (X, d) be a metric space.*

1. *Both $\emptyset$ and X are both open and closed.*
2. *An arbitrary union of open sets is open.*
3. *A finite intersection of open sets is open.*

We omit the proof, because it is easy and basic.

We are now ready to venture into more general waters.

1.2 Introduction to Topological Spaces

A topological space, broadly speaking, is one in which we wish to be able to discuss the notion of convergence without depending on the notion of distance. The definition of convergence in metric spaces really only relies on the openness of open balls. We can generalise it merely by generalising the definition of open sets.

We begin by defining the notion of a **topological space**.

Definition 1.2.1 (Topological Space). Let X be a set. A **topology** on X is a family τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

We say a subset of X is **open** with respect to a topology τ if it lies in τ and **closed** if its complement lies in τ. We call the pair (X, τ) a **topological space**.

The terminology for open sets is visibly consistent with the terminology used in metric spaces. In fact, this is exactly what Proposition 1.1.11 demonstrates: that the open sets of a metric space, defined as in Definition 1.1.9, do indeed define a topology on it. Every metric space is thus also a topological space. Note that sets can be both open and closed, such as $\emptyset$ and the entire set, just as with metric spaces. We sometimes refer to such sets as being "clopen".

1.2.1 Sequences, Convergence and Limits

The reason why we introduced topological spaces to begin with is because we wanted a more general setting in which to talk about convergence. The question is, what is the best way of talking about convergence in arbitrary topological spaces?

Since every metric space is a topological space, and seeing as we already have a notion of convergence in metric spaces, we would want to define the notion of convergence in a topological space to be a *generalisation*. To that end, we restate the definition of convergence in metric spaces using only the language of open sets.

Proposition 1.2.2. *Let (X, d) be a metric space. For every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, the following are equivalent:*

1. *x_n converges to x as in Definition 1.1.6*
2. *For every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.*

We do not prove this proposition here, but note that it is not difficult to prove.

Since the second statement in Proposition 1.2.2 does not actually mention any *metric space* properties of X, it is a viable definition for convergence in topological spaces.

Definition 1.2.3 (Convergence of a Sequence in a Topological Space)**.** Let (X, τ) be a topological space, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence in X. Given some $x \in X$, we say that x_n **converges to** x, denoted $x_n \to x$, if for every open set $U \subseteq X$ such that $x \in U$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

Proposition 1.2.2 essentially tells us that if (X, d) is a metric space, then Definitions 1.1.6 and 1.2.3 are equivalent, where we apply Definition 1.2.3 to the topological space structure induced by the metric space structure.

Example 1.2.4 (Convergence in the Indiscrete Topology)**.** Consider a set X along with the indiscrete topology (ie, view it as the topological space $(X, \{\emptyset, X\})$). Then, *every sequence converges to every point.*

Example 1.2.4 demonstrates that the uniqueness of limits seen in Proposition 1.1.7 does not always hold in topological spaces. See Definition 1.2.5 for more.

Note that going forward, we often omit the topology when talking about topological spaces, unless it has specific properties that are essential for our purposes.

Recall that limits in a metric space are unique (Proposition 1.1.7). We noted that this is not always true, but we are interested enough in distinguishing spaces where this is true from spaces where this is not true that we have a name for such spaces.

Definition 1.2.5 (Hausdorff/T_2 Spaces). Let (X, τ) be a topological space. We say (X, τ) is **Hausdorff**, or T_2, if for all $a, b \in X$ with $a \neq b$, there exist $U, V \in \tau$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

The Hausdorff/T_2 property is an instance of a *separation property*. There are many more such properties, including T_1, T_3, $T_{3\frac{1}{2}}$, and T_4, and it is highly non-trivial to prove facts like $T_4 \implies T_{3\frac{1}{2}}$. We will see such things as the course progresses.

Note that Proposition 1.1.7 tells us precisely that Metric Spaces are Hausdorff/T_2. Indeed, this tells us that Hausdorffitude is a *necessary* condition for a topology on a space to be **metrisable**—that is, for it to be induced by a metric. In particular, Example 1.2.4 tells us that the indiscrete topology is non-metrisable. There is the more interesting question of what conditions might be *sufficient* for a topology to be metrisable. Metrisability is, indeed, highly non-trivial property, and we will see some metrisation theorems in this course.

1.2.2 Continuity

Just as we generalised the notion of convergence from metric spaces to topological spaces, so too can we generalise the notion of continuity.

Definition 1.2.6 (Continuity). Let X and Y be topological spaces, and let $f : X \to Y$ be a function. We say f is **continuous** if for every $U \subseteq Y$ open in Y, the pre-image $f^{-1}[U]$ is open in X.

It is easy to show that Definition 1.2.6 is equivalent to Definition 1.1.3. We do not do this here.

There are many examples of continuous functions, some familiar and some slightly pathological.

Example 1.2.7 (Weird Topologies lead to Weird Notions of Continuity)**.** Let X and Y be topological spaces.

1. If X and Y both have the *discrete* topology on them, then *any* function $f : X \to Y$ is continuous, because all subsets of X and Y are open.
2. If X and Y both have the *indiscrete* topology on them, then `sorry`

1.2.3 Interiors and Closures

We briefly discuss how to "make subsets of a topological space open or closed". We do this by defining interiors and closures.

Definition 1.2.8 (The Interior of a Set)**.** Let X be a topological space and let $A \subseteq X$. We define the **interior** of A, denoted $\text{int}(A)$ or A°, to be the union of all open subsets of A.

Note that $\text{int}(A)$ is always open. Moreover, it is the largest (with respect to inclusion) open subset of A. Finally, note that the sets that are equal to their interiors are precisely the open sets.

We have a 'dual' notion of interiors that capture closure properties.

Definition 1.2.9 (Closure)**.** Let X be a topological space and let $A \subseteq X$. We define the **closure** of A, denoted $\text{cl}(A)$ or $\overline{A}$, to be the intersection of all closed subsets of X containing A.

Dually to the interior, $\text{cl}(A)$ is always closed. Moreover, it is the smallest (with respect to inclusion) closed subset of X containing A. Finally, note that the sets that are equal to their closures are precisely the closed sets.

1.3 A Closer Look at Topologies

1.3.1 Bases and Sub-Bases

It is interesting to talk about whether a topology can be *generated* by a family of subsets by taking unions. In the case of metric spaces, it is not difficult to show that this is true.

Proposition 1.3.1 (Metric Space Topologies are Generated by Open Balls). *Let (X, d) be a metric space. Any $U \subseteq X$ is open if and only if U is an open of open balls.*

Proof. Proposition 1.1.11 tells us that if U is a union of open sets, then it is open. Conversely, for all $x \in U$, there is some ε_x such that $B_{\varepsilon_x}(x) \subseteq U$. Then, it is easily seen that

$$U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$$

showing that U is a union of open sets. □

We have a special term for families of open sets that 'generate' a topology.

Definition 1.3.2 (Basis of a Topology). Let (X, τ) be a topological space. A **basis for τ** is a subset $B \subseteq \tau$ such that every $U \in \tau$ is a union of sets in B.

Proposition 1.3.1 tells us precisely that the open balls in a metric space form a *basis* for the topology induced by the metric.

We have a nice criterion to check if a subset of a topology is a basis.

Proposition 1.3.3. *Let (X, τ) be a topological space. Let $B \subseteq \tau$. TFAE:*

1. *B is a basis for τ.*
2. *X is a union of elements of B. Moreover, for all $U, V \in B$, $U \cap V$ is a union of elements of B.*

It is immediate that the first statement implies the second. The converse is an exercise in set algebra, which we do not do here.

It is also possible to define bases using 'sub-bases'. To that end, we have a preliminary fact.

Lemma 1.3.4. *Let X be a set and let Π be a non-empty set of topologies on X. Then,*

$$\bigcap \Pi = \{A \subseteq X \mid \forall \tau \in \Pi,\, A \in \tau\}$$

is a topology on X.

We do not prove this here. We merely mention that the proof has the same flavour as such statements in algebra as "an arbitrary intersection of subgroups/ideals/subfields is a subgroup/ideal/subfield".

A particular consequence of the above is that we can make the following definition.

Definition 1.3.5 (Topology Generated by a Set)**.** Let (X) be a set and let C be any collection of subsets of X. We can define a topology on X by

$$\Pi(C) := \bigcap \{\tau \text{ a topology on } X \mid C \subseteq \tau\}$$

We call this the **topology on X generated by C**.

Note that it is *always* possible to define a topology generated by a subset, since $\Pi(C)$ is always non-empty: it always contains the discrete topology.

We can now define a sub-basis of a topology.

Definition 1.3.6 (Sub-Basis of a Topology)**.** Let X be a set and let $C \subseteq X$. If τ is the least topology of X that contains C, then we say that C **is a sub-basis of τ**.

Indeed, any subset is a sub-basis of *some* topology, namely, the topology it generates.

1.3.2 New Topologies from Old Ones

There are numerous techniques to define new topologies from old ones. We begin with the most obvious definition imaginable.

Definition 1.3.7 (The Subspace Topology). Let X be a topological space. Any subset $Y \subseteq X$ inherits a topology from X, known as the **subspace topology**, the open sets of which are precisely those sets of the form $U \cap Y$ for open sets $U \subseteq X$.

With that out of the way, we move onto more interesting examples.

We begin with a topology that is defined using continuity of functions.

Definition 1.3.8 (Initial Topology). Let X be a set. Given an index set $\mathcal{I}$, topological spaces X_i and functions $f_i : X \to X_i$ for $i \in \mathcal{I}$, we define the **initial topology on X with respect to X_i and f_i** to be the least topology on X (with respect to inclusion) such that for all $i \in \mathcal{I}$, f_i is continuous.

Note that it is always possible to do this, because at worst, we the discrete topology on X renders continuous every function from X to any topological space.

Given the definition of continuity (Definition 1.2.6), it is possible to be explicit about the initial topology.

Lemma 1.3.9. *Let $X, \mathcal{I}, X_i, f_i$ be as in Definition 1.3.8. The corresponding initial topology τ is given by*

$$\tau = \left\{ f_i^{-1}[U] \subseteq X \;\middle|\; i \in \mathcal{I} \text{ and } U \subseteq X_i \text{ is open} \right\}$$

We give an important example illustrating the notion of an initial topology.

Example 1.3.10 (Products). Let $\mathcal{I}$ be an index set and let X_i be topologies. Consider the Cartesian product X of these X_i, defined as follows:

$$X = \prod_{i \in \mathcal{I}} X_i = \left\{ (x_i)_{i \in \mathcal{I}} \;\middle|\; \forall i \in \mathcal{I}\, x_i \in X_i \right\}$$

We know (either by the category theoretic definition of a product, of which the Cartesian product is the instance in the category of sets, or by sheer common sense) that there are projections $\pi_i : X \to X_i$ that map any $(x_j)_{j \in \mathcal{I}}$ to x_i, indexed by $i \in \mathcal{I}$. We can compute the initial topologies of these projections to obtain a topology on X.

We can give a special name to the topology defined above, which is a key way of constructing new topologies from old ones.

Definition 1.3.11 (The Product Topology). Let $\mathcal{I}, X_i, X, \pi_i$ be as in Example 1.3.10. We call the initial topology of the π_i the **product topology** on X.

We can give a more explicit description of the product topology.

Proposition 1.3.12. *Let $\mathcal{I}, X_i, X, \pi_i$ be as in Example 1.3.10 and Definition 1.3.11. Then,*

1. *A sub-basis of the product topology is given by all sets of the form*

$$\left\{(x_j)_{j\in\mathcal{I}} \;\middle|\; x_i \in U\right\}$$

for $i \in \mathcal{I}$ and $U \subseteq X_i$ open.

2. *A basis of the product topology is given by all sets of the form*

$$\left\{(x_j)_{j\in\mathcal{I}} \;\middle|\; x_{i_1} \in U_1, \dots, x_{i_n} \in U_n\right\}$$

for $n \in \mathbb{N}$, $i_1, \dots, i_n \in \mathcal{I}$, and $U_k \subseteq X_{i_k}$ open for $1 \leq k \leq n$.

We do not prove this here.

Dually to how we defined a topology on the Cartesian product of spaces as the initial topology of the projections, we can define a topology on any subset of a space as the initial topology of the inclusion.

Write this out either as an example followed by a defi-

Finally[1], we define the *final* topology.

Definition 1.3.13 (Final Topology). Let X be a set. Given an index set $\mathcal{I}$, topological spaces X_i and functions $f_i : X_i \to X$ for $i \in \mathcal{I}$, we define the **final topology on X with respect to X_i and f_i** to be the largest topology on X (with respect to inclusion) such that

[1] Pun intended

for all $i \in \mathcal{I}$, f_i is continuous.

Remark. To state it explicitly, we observe that if X has more open sets (i.e. the topology on X is "larger"), then it is "harder" for a function $f_i : X_i \to X$ to be continuous (as $f_i^{-1}(U)$ must be open in X_i for all open $U \subseteq X$, which forces the topology on X_i to be larger as well).

Note that it is always possible to construct a final topology, because at worst, we the indiscrete topology on X renders continuous every function into X from any topological space.

Given the definition of continuity (Definition 1.2.6), it is possible to be explicit about the initial topology.

Lemma 1.3.14. *Let $X, \mathcal{I}, X_i, f_i$ be as in Definition 1.3.13. The corresponding initial topology τ is given by*

$$\tau = \{U \subseteq X \mid i \in \mathcal{I} \text{ and } f^{-1}(U) \subseteq X_i \text{ is open}\}$$

There are many good examples of final topologies, some of which are likely to be familiar to the reader.

Example 1.3.15 (Topologies on Quotients)**.** Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $q : X \twoheadrightarrow X/_\sim$ be the canonical surjection from X to the quotient set. The final topology of q is a topology on $X/_\sim$.

We give a special name to such a topology on a quotient.

Definition 1.3.16 (Quotient Topology)**.** Let X, $\sim$, and q be as in (1.3.15). We call the final topology on $X/_\sim$ with respect to X and q the **quotient topology**.

Note that the above definition really applies to *all* situations where X is a topological space and Y is a set such that there is a surjection $q : X \twoheadrightarrow Y$: in this case, Y is essentially the quotient of X by the relation $\sim$ where $x \sim y$ iff $f(x) = f(y)$.

We can express the quotient topology in a more familiar way.

Proposition 1.3.17. *sorry*

1.3.3 Separation Properties

Throughout this subsection, let X denote a topological space.

We recall the definition of T_2 spaces from Definition 1.2.5. Definition 1.2.5 already exists btw - just link to it or smth if you want

Definition 1.3.18 (T_2 (i.e. the Hausdorff condition)). We say that a topological space is T_2, or **Hausdorff**, if $\forall x \neq y$ there exist open sets U, V such that $x \in U, y \in V$ that we have $U \cap V = \emptyset$

There is also a (weaker) notion of separation called the T_1 property.

Definition 1.3.19 (T_1 property). We say that X is a T_1 **space** if for all *distinct* $x, y \in X$, there is an open set $U \subseteq X$ with $x \in U$ and $y \notin U$.

We can give an equivalent characterisation of the T_1 property.

Proposition 1.3.20. *X is T_1 if and only if every singleton in X is closed.*

Proof. First, notice that in the trivial cases where $|X| = 0$ or $|X| = 1$, we are done. So, going forward, assume that X contains at least two distinct elements.

($\implies$) Assume that X is T_1. If $|X| = 1$, then we are done. Fix $x \in X$ and consider the singleton $\{x\}$. We know that for any $y \in X \setminus \{x\}$, there exists some open $U_y \subseteq X$ such that $y \in U_y$ and $x \notin U_y$.[2] Then, it is easy to see that

$$X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$$

Indeed, the $\subseteq$ inclusion is obvious, since $y \in U_y$ for all $y \in X \setminus \{x\}$, and the $\supseteq$ inclusion is also clear because x does not lie in any of the U_y. Since each U_y is open, so is the union of

[2]Define a pairing (y, U_y) from the definition of X being T_1 using the Axiom of Choice.

all of them, making $X \setminus \{x\}$ a union of open sets, hence open. Thus, $\{x\}$ is closed.

($\Longleftarrow$) Assume that every singleton $\{x\} \subseteq X$ is closed. Then, fix $x, y \in X$ and assume $x \neq y$. Since $\{x\}$ is closed, $X \setminus \{x\}$ is open, and since x and y are distinct, $y \in X \setminus \{x\}$. Thus, $X \setminus \{x\}$ is an open subset of X containing y but not x.

Therefore, the T_1 condition is equivalent to the condition that every singleton is closed. □

There is a notion that is weaker still.

Definition 1.3.21 (T_0 property). We say X is a T_0 **space** if for all distinct $x, y \in X$,

$$\{U \subseteq X \mid U \text{ is open and } x \in U\} \neq \{V \subseteq X \mid V \text{ is open and } y \in U\}$$

That is, either there is an open set U with $x \in U$ and $y \notin U$ or there is an open set V with $y \in V$ and $x \notin V$.

Remark. We emphasise that here, we are not able to choose which of x and y is contained in the open set U witnessing the T_0 property.

Before we give a classic example of a T_0 topology, we recall the definition of a partially ordered set (also abbreviated 'poset').

Definition 1.3.22 (Partial Order). Let P be a set. We say that a binary relation $\leq$ on P is a **partial order** if it is reflexive, antisymmetric and transitive. We call the pair $(P, \leq)$ a **partially ordered set**, often abbreviated **poset**.

There are many familiar examples of posets in mathematics.

Example 1.3.23 (A Collection of Sets as a Poset). Given any collection of sets, the collection can always be ordered by inclusion. So any time you take a family of sets and order them by inclusion, you will wind up with a poset structure.

Given that topologies are sets of sets, in particular, topologies are *also* partially ordered by inclusion. Thus, we will find the theory of posets to be quite useful throughout this course.

There is also a “converse” relationship between posets and topology: we can define a topology on any poset by taking advantage of the properties of the partial order.

Example 1.3.24 (A T_0 Topology on a Poset)**.** Let $(P, \leq)$ be a poset. We introduce a topology τ on P as follows: we deem a set $U \subseteq P$ to be open iff for all $p \in U$,

$$\{q \in P \mid q \leq P\} \subseteq U$$

That is, we define the open sets to be precisely those U that contain all downward cones of elements in U. We can show that this is, indeed, a topology on P.

The topology described in Example 1.3.24 is the ‘natural’ topology for *forcing posets* in set theory.

1.4 A Closer Look at Continuity

1.4.1 Neighbourhoods and Neighbourhood Bases

Next, we introduce the notion of a neighbourhood.

Definition 1.4.1 (Nbhd)**.** We say that a point $x \in X$ has nbhd $A \subseteq X$ if there exists some open set $O \subseteq X$ such that $x \in O \subseteq A$.

Abbreviation. Because nobody has the time to write “neighbourhood” more than a few times in their life, Professor Cummings will abbreviate it by “Nbhd” in the future. We will not, by default, assume that a neighbourhood (or nbhd) of a point is open.

We can talk about neighbourhoods in a collective sense, reminiscent of `Filter.nhds` in Lean...

Definition 1.4.2 (Nbhd Basis)**.** Given $x \in X$, a **nbhd basis** for x is a set $\mathcal{N}$ of nbhds of x such that for every nbhd A of x there exists some $B \in \mathcal{N}$ such that $B \subseteq A$ - equivalently, for every open nbhd U of x there exists $B \in \mathcal{N}$ such that $B \subseteq U$.

Example 1.4.3 (The Closed Ball Nbhd Basis). Let (X, d) be a metric space. Fix $x \in X$. Let $\mathcal{N}$ be a set of nbhds of the form

$$\{y \in X \mid d(x, y) \leq \varepsilon\}$$

for all $\varepsilon > 0$. Then, this set all of closed balls at x is a nbhd basis.

Remark. It is a "trivial fact" that if we have a neighbourhood basis at every point $x \in X$, then we can recover the topology of X. (See this as an exercise.)

Finally, we note that nbhds give us a definition of continuity that more closely resembles the definition to which we are accustomed in metric spaces.

Proposition 1.4.4. *Given topological spaces X and Y and a function $f : X \to Y$, TFAE:*

1. *f is continuous*
2. *for all $x \in X$ and N a nbhd of $f(x)$, there is a nbhd M of x such that $f[M] \subseteq N$.*

In particular, specialising second characterisation of continuity in the above result to a single point gives us a way of talking about the continuity of a function *at a single point*.

1.4.2 Homeomorphisms

Throughout this subsection, fix topological spaces X and Y.

We first define the notion of open and closed functions.

Definition 1.4.5 (Open and Closed Functions). For any $f : X \to Y$, we say:

- f is **open** iff for all open $U \subseteq X$ we have $f(U)$ open in Y.
- f is **closed** iff for all closed $C \subseteq X$ we have $f(C)$ closed in Y.

Next, we define the notion of an isomorphism in the category of topological spaces.

Definition 1.4.6 (Homeomorphism). We say $f : X \to Y$ is a **homeomorphism** if f satisfies the following conditions:

1. f is a bijection

2. For all $U \subseteq X$, U is open in X iff $f[U]$ is open in Y

We now mention an easy result (that effectively shows that homeomorphisms are isomorphisms in the category of topological spaces).

Lemma 1.4.7. *For any function $f : X \to Y$, the following are equivalent:*

1. *f is a homeomorphism*
2. *There is $g : Y \to X$ such that*
   (a) *g is continuous*
   (b) *$f \circ g = \mathrm{id}_Y$*
   (c) *$g \circ f = \mathrm{id}_X$*

We do not prove this result here.

We note that the continuity assumption on the inverse is strictly necessary:

Warning. A bijective continuous map need not be a homeomorphism!

This is because continuity is in some sense "one sided" (to take care of the other side, maybe we will need to also require our map be open...).

We illustrate this using a simple counterexample.

Counterexample 1.4.8 (A Continuous Bijection that is NOT a Homeomorphism)**.** Let X be a set of cardinality at least 2. Consider the spaces (X, τ_1) and (X, τ_2), with τ_1 being the discrete topology and τ_2 being the indiscrete topology. Then, the identity function id_X is a bijection from (X, τ_1) to (X, τ_2); moreover, it is continuous because every function out of (X, τ_1) is continuous. However, its inverse (which is also the identity) is *not* continuous.

1.5 Connectedness

Again, fix a topological space X. In this section, we discuss different notions of connectedness of a toplogical space.

Definition 1.5.1 (Disconnectedness)**.** We say X is **disconnected** if there exist disjoint open sets $U, V \subsetneq X$ with $U \cup V = X$. If U and V are disjoint ant cover X, we say that they **disconnect** X.

Remark. One must be very careful when disconnecting sets: when you have a disconnected set X, it is not at all obvious that the sets you use to disconnect X (namely, U and V) are actually disjoint in the rest of X. (Remember this for your General Topology basic exam!)

We can use this definition to define connectedness in the obvious way.

Definition 1.5.2 (Connectedness)**.** We say X is **connected** if X is not disconnected, ie, if there do not exist disjoint open sets U and V that disconnect X.

As one would expect, we define connectedness for subsets to be connectedness with respect to the subspace topology.

It turns out we can use the definition of connectedness to say something about the clopen sets of a connected space.

Lemma 1.5.3. *Let $(A_i)_{i \in \mathcal{I}}$ be some family of subsets of X indexed by some non-empty set $\mathcal{I}$ such that*

1. *For all $i \in \mathcal{I}$, A_i is a connected subset of X*
2. *For all $i, j \in \mathcal{I}$, $A_i \cap A_j \neq 0$*

That is, (A_i) is a family of connected, non-disjoint sets. Then,

$$A := \bigcup_{i \in \mathcal{I}} A_i$$

is a connected subset of X.

Proof. Suppose that A is not connected. Then, there exist open subsets $U, V \subsetneq X$ such that

$$A = (A \cap U) \cup (A \cap V)$$

with $(A \cap U)$ and $(A \cap V)$ being disjoint, non-empty, and proper subsets of A. Then, for any $i \in \mathcal{I}$, we have that

$$A_i \subseteq A = (A \cap U) \cup (A \cap V)$$

Since the A_i are all connected, we know that for all $i \in \mathcal{I}$, either $A_i \subseteq U$ or $A_i \subseteq V$. It turns out that we can do better: either every A_i is contained in U or every A_i is contained in V. Indeed, if this were not the case, then there would be $i, j \in \mathcal{I}$ with $A_i \in U$ and $A_j \in V$. Then, we would have $A_i \cap A_j \subseteq U \cap V$. Moreover, $A_i \cap A_j \subseteq A$, since $A_i \subseteq A$ and $A_j \subseteq A$. Thus,

$$A_i \cap V_j \subseteq A \cap \cap U \cap V = (A \cap U) \cap (A \cap V)$$

Finally, we assumed, in our setup, that $A_i \cap A_j \neq \emptyset$, meaning that there is an element living in the disjoint set $A \cap U$ and $A \cap V$, which is obviously a contradiction. Hence, either every A_i is contained in U or every A_i is contained in V.

Without loss of generality, say that every A_i is contained in U. Then, $A \subseteq U$ also, meaning that $A \cap U = A$. Hence, $A \cap V = \emptyset$, which contradicts the assumption that A is disconnected, because disconnecting subsets must be proper. □

We next use the definition of connectedness to establish a "canonical" decomposition of X.

1.5.1 Connected Components

Define the binary relation $\sim$ on X such that $a \sim b \iff \exists A \subseteq X$ with A connected and $a, b \in A$. We then claim that $\sim$ is an equivalence relation, and (having shown this) we let the equivalence classes of $\sim$ partition X into sets X_i.

Definition 1.5.4 (Connected Components)**.** The **connected components** of X are the equivalence classes in ${X}/{\sim}$, with $\sim$ being defined as above.

Let us describe these equivalence classes. Fix $x \in X$. Denote its equivalence class in ${X}/{\sim}$ by $[x]$. Then, it is possible to show that

$$[x] = \{y \in X \mid y \sim x\} = \bigcup \{A \subseteq X \mid x \in A \text{ and } A \text{ is connected}\}$$

The $\subseteq$ inclusion is clear, because $[x]$ is itself connected, meaning that any $y \in [x]$ clearly lies in some connected subset of X containing x. Conversely, any element of any connected subset of X containing x must be connected to x, putting it in $[x]$.

Lemma 1.5.5. *If $A \subseteq X$ and A is connected, then $\mathrm{cl}(A)$ is also connected.*

Proof. Suppose that $\mathrm{cl}(A)$ is not connected. then, there are open sets $u, V \subseteq X$ such that $\mathrm{cl}(A) \subseteq U \cup V$, $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \emptyset$, and $\mathrm{cl}(A) \cap U, \mathrm{cl}(A) \cap V \neq \emptyset$. We can see, since $A \subseteq \mathrm{cl}(A)$, that $A = (A \cap U) \cup (A \cap V)$, with $A \cap U \cap V = \emptyset$. Since A is connected, we need either $A \cap U = \emptyset$ or $A \cap V = \emptyset$. Assume it is the former. sorry □

Corollary 1.5.6. *The connected components of X are closed subsets of X.*

Proof. Fix $x \in X$. Then, by Lemma 1.5.5, the closure $\mathrm{cl}([x])$ of the connected component $[x]$ containing X is a connected subset of X. Since all such connected sets are contained in $[x]$, we have that $\mathrm{cl}([x]) \subseteq [x]$, meaning $[x]$ is equal to its closure, making it closed. □

1.5.2 Preserving Connectedness

Surprisingly—or unsurprisingly—connectedness is not preserved by taking subsets.

Counterexample 1.5.7 (A Disconnected Subset of a Connected Space)**.** We know that $\mathbb{R}$ is connected under the Euclidean topology. Consider $\mathbb{Q} \subset \mathbb{R}$. We can clearly see that $\left(-\infty, \sqrt{2}\right) \cap \mathbb{Q}$ and $\left(\sqrt{2}, \infty\right) \cap \mathbb{Q}$ are disjoint, open, proper subsets of $\mathbb{Q}$ that disconnect it.

However, it turns out the *image* of a connected set in a continuous function *is* connected.

Lemma 1.5.8. *Let X and Y be topological spaces. Let $A \subseteq X$ be connected and let $f : X \to Y$ be continuous. Then, $f[A]$ is connected.*

Proof. If not, let $U, V \subseteq Y$ be open with $f[A]$ sorry □

This allows us to construct many examples of connected spaces.

Example 1.5.9 (Connectedness of the Unit Interval)**.** Consider $[0, 1]$ along with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. We show that $[0, 1]$ is connected.

Indeed, suppose $[0, 1]$ is *not* connected. Then, there exist open sets $U, V \subseteq [0, 1]$ such that $U \cup V = [0, 1]$ and $U \cap V = \emptyset$. Thus, $0 \in U$ or $U \in V$. WLOG, say $0 \in U$. Define

$$S = \{a \in [0, 1] \mid [0, a] \subseteq U\}$$

Observe that S has the following properties.

1. $0 \in S$, so S is non-empty.
2. S is bounded above (by 1, for instance)

Therefore, S has a finite supremum (indeed, a supremum in $[0, 1]$). Denote this b.

Now, observe that b has the following properties.

1. $b > 0$. The reason for this is that since $0 \in U$ and U is open, there is some $\varepsilon > 0$ such that $[0, \varepsilon) \subseteq S$. Thus, $b \geq \varepsilon > 0$.
2. $[0, b) \subseteq U$. This is true by definition of b and S.
3. $b \in U$. Otherwise, certainly $b \in V$, and therefore, by openness, there is some $\varepsilon > 0$ such that $(b - \varepsilon, b] \subseteq V$, but for all $\varepsilon > 0$, we must have that $b - \varepsilon \in U$ by definition of b (and S), and U and V are disjoint.

The idea is to show that $b \in U \cap V$ or $b = 1$, a contradiction. `sorry`

1.5.3 Path Connectedness

Definition 1.5.10 (Path)**.** Given X a space, let $a, b \in X$. We say that a **path** in X is a continuous function $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = a, \gamma(1) = b$.

Remark. You may have many paths, be unable to use sub or super-scripts, but still need to name them all. Professor Cummings is here to help: $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi, \pi, \phi, \rho$, etc.

As a preliminary, we note that we can compose paths to obtain new paths. (`sorry`)

Definition 1.5.11 (Path-Connected Space). We say that X is **path-connected** iff for all $a, b \in X$ there is a path from a to b.

It is possible to show that every path-connected space is connected (`sorry`), but not the other way round (`sorry`).

1.6 Compactness and Completeness

1.6.1 Compactness

Recall that last time (when we were learning about Monsters) we showed the following facts:

- $[a, b]$ is compact.
- A closed subset of a compact space is compact.
- If $C \subseteq \mathbb{R}$ (in the usual topology is closed and bounded, then C is compact.

Generally speaking, when we consider some new property of a topological space, it is worthwhile to think about whether it passes to subspaces.

Lemma 1.6.1. *Any continuous image of a compact set is compact.*

Proof. Let $C \subseteq X$ be compact with $f : X \to Y$ a continuous function. Letting F denote an open cover of $f[C]$ (i.e. a collection of open sets covering $f[C]$) , the collection of sets $\{f^{-1}[U] : U \in F\}$ will give an open covering of C, and now the finite subcover of $f[C]$ can be seen to follow from the existence of our finite subcover of C (select finitely many U such that $f^{-1}[U]$ cover C). $\square$

Lemma 1.6.2. *Given X a Hausdorff space with $C \subseteq X$ a compact subset of X, necessarily C is closed in X.*

Proof. Given $y \notin C$, for each $x \in C$ we let $N(x)$ denote an open neighborhood of x disjoint from some open neighborhood $N_x(y)$ of y. Noting that $\{N(x) : x \in C\}$ forms an open cover of C, as

C is a compact set it follows that there exist finitely many $x_i \in C$ such that $\{N(x_i) : i < n\}$ forms an open cover of C.

But then we observe that $\cap_{i<n} N_{x_i}(y) = \mathcal{N}_C(y)$ is a finite intersection of open sets containing y which is disjoint from C. In particular, $\mathcal{N}(y) \subseteq X \setminus C$. It follows that such an open neighborhood must exist for every $y \in X \setminus C \implies X \setminus C$ is open $\implies C$ must be closed. □

Remark. "This is just logic, in its most pejorative sense."

Before moving on, we pause here for a brief interlude which will allow us to proceed in more generality (oh yay!).

Definition 1.6.3 (Bounded Subset). Let Y be a metric space, $B \subseteq Y$. Then we say that B is "bounded" if and only if there exists some $y \in Y, C > 0$ such that $d(y, b) < C$ holds for all $b \in B$.

Let's get on the straight and narrow:

Lemma 1.6.4. *If B is a compact subset of a nonempty metric space Y, then B is bounded.*

Proof. We can choose $y \in Y$ and let $U_n = B_n(y) = \{x \in Y : d(x, y) < n\}$. Then $B \subseteq X = \bigcup_{n\in\mathbb{N}} U_n$, and as B is compact and $U_n \subseteq U_{n+1}$, it follows that there must exist some $N \in \mathbb{N}$ such that $B \subseteq U_N$ (so by definition of a bounded subset, B is bounded). □

And now with all these lemmas at our disposal, it's time for another lemma (a "fact"):

Lemma 1.6.5. *Let X be a compact space, and let $f : X \to \mathbb{R}$ denote a continuous function. Then if f is bounded, it must attain its bounds.*

Proof. Let $C = f[X]$. As continuous images of compact sets are compact, $f[X]$ must be compact, and further as a compact subset of $\mathbb{R}$ it must be closed and bounded. If we now let $M = \sup(C)$ and $m = \inf(C)$ then $m \leq f(x) \leq M$ for all $x \in X$. As C is closed, we must have $m, M \in C = f[X]$ (and hence f must attain its bounds). □

Remark. $C \subseteq \mathbb{R}$ is in fact compact *if and only if* C is closed and bounded.

Another brief digression - there are several "phrase-ologies" in use to compare various topologies:

Remark. Given topologies σ, τ both topologies on some space Z, we say that σ is smaller (i.e. weaker, coarser) than τ is $\sigma \subseteq \tau$, and correspondingly τ is larger (i.e. stronger, finer).

Remark. X is compact if and only if, given any open cover $\mathcal{O}$ of X by basis elements, there exists a finite subcover $\mathcal{O}' \subseteq \mathcal{O}$ which covers X.

1.6.2 Completions in a Metric Space

For context, we have been speaking about completeness, and we are now going to build a new complete metric space out of the Cauchy sequences in our metric space (X, d):

- Let (X, d) be a metric space, and let $Z = \{(x_n)_{n \in \mathbb{N}} : (x_n)_n \text{ is a Cauchy sequence in } X\}$.
- We first claim (1) that If $(x_n)_n, (y_n)_n \in Z$, then $(d(x_n, y_n))_n$ is a Cauchy sequence in $\mathbb{R}$ - so in particular, it has a limit $d^*((x_n)_n, (y_n)_n) = d^*(x, y)$.
- We next claim (2) that the set of $[x] = \{y \in Z : d^*(x, y) = 0\}$ for $x \in Z$ will partition Z, or in other words, that R defined such that $xRy \leftrightarrow d^*(x, y)$ forms an equivalence relation on Z.
- Now we claim (3) that $d^*(x, y)$ depends only on the classes of $x = (x_n)_n, y = (y_n)_n \in Z$, and we then define $\overline{d}([x], [y]) = d^*(x, y)$.
- And then we claim (4) that $\overline{d}$ is a metric on the set of equivalence classes $\overline{X}$.
- Jumping off from here, we let $i : X \to \overline{X}$ such that $i(x) = [(x_n)_n]$, where $(x_n)_n = (x, x, x, \dots)$.
- We would claim (5) that $\overline{d}(i(x), i(y)) = d(x, y)$ (i.e. that i is an "isometric embedding").
- Then we clam that $i[X]$ is dense in $\overline{X}$, and that every point in $\overline{X}$ is the limit of a sequence of points in $i[X]$.
- (7) * We claim that $(\overline{X}, \overline{d})$ is complete. *
- (Then to finish, once can identify $x \in X$ with $i(x) \in \overline{X}$ and viola.)

So now that we have a way to construct a completion of X, then we have the following fact:

Lemma 1.6.6. *If X is a metric space and Y is a complete metric space with $f : X \to Y$ an isometric embedding, then there is a unique $\overline{f} : \overline{X} \to \overline{Y}$ which is an isometric embedding with $\overline{f} \upharpoonright X = f$.*

Proof. Exercise! Or rather, "just do what comes naturally." "I'm obviously being very glib. But this is not the main topic of this course." □

Remark. As an exercise, use the above to show that any 2 completions of X must be isomorphic.

Now is where the instructor exercises his freedom in determining the order of topics to cover...

Definition 1.6.7 (Subsequence)**.** An infinite subsequence of an infinite sequence $(x_n)_{n\in\mathbb{N}}$ is a sequence $(x_{i_n})_{n\in\mathbb{N}}$ where $i_k < i_{k+1}$ holds for all $k \in \mathbb{N}$.

Definition 1.6.8 (Sequentially Compact)**.** We say that X is "sequentially compact" if and only if every infinite sequence has a convergent subsequence.

"I keep on saying this, then it keeps on being true."

We now digress briefly to recall that convergence can be defined topologically (without reference to a metric). We also have this fact that in a metric space X, for any $A \subseteq X$ we will have

$$\mathrm{cl}(A) = \{x \in X : \text{ there is some } (x)n)_{n\in\mathbb{N}} \text{ such that } x_n \in A \text{ and } x_n \to x\}$$

and the proof of this **fact** is "a proof by extreme obviousness" that involves "ridiculously clear contradictions" - in other words, the proof is an exercise. However

Warning. This **fact** about metric spaces is very not true in general spaces! In a general space, all convergent sequences consisting of points in A converge to a point of $\mathrm{cl}(A)$, but this may not give all points of $\mathrm{cl}(A)$.

Next week, we will introduce the concept of "nets" to deal with this.

Definition 1.6.9 (Limit Point)**.** Given a space X and $A \subseteq X$ with $x \in X$, we say that x is a "limit point for A" to mean that, for every open neighborhood U of x, there is some $z \in A \cap U$ such that $z \neq x$.

The following is "a kind of logical truism" (which maybe seems less obvious than some of our other obvious facts):

Lemma 1.6.10. *$cl(A) = A \cup \{x \in X : x$ is a limit point of $A\}$.*

In general topological spaces, sequential compactness and compactness are **not** the same (as perhaps is to be expected) - however, in a metric space, the two notions do coincide.

1.6.3 The Relationship between Compactness and Completeness

Definition 1.6.11 (Totally Bounded)**.** A metric space X is **totally bounded** iff for all $\varepsilon > 0$, X is a finite union of open ε-balls.

The reason we define total boundedness is that it connects compactnss and completeness. Before we can establish that connection, we need an intermediate lemma.

Lemma 1.6.12. *Let X be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with no repetitions. For all $x \in X$, there is some $\varepsilon > 0$ such that there is no $n \in \mathbb{N}$ with $x_n = x$ and $x_n \in B(x, \varepsilon)$.*

Proof. We argue by contradiction. Assume it does not hold. That is, assume there is some $x \in X$ such that for all $\varepsilon > 0$, there is $n \in \mathbb{N}$ with $x_n \neq x$ and $x_n \in B(x, \varepsilon)$. Now, perform the following steps.

- Let $\varepsilon_0 = 1$. There is some $x_{n_0} \neq x$ with $x_{n_0} \in B(x, \varepsilon_0)$.
- Let $\varepsilon_1 := \min(2^{-1}), d(x, x_{n_0})$. There is some $x_{n_1} \neq x$ with $x_{n_1} \in B(x, \varepsilon_1)$.
- Let $\varepsilon_2 := \min(2^{-2}), d(x, x_{n_0})$. There is some $x_{n_1} \neq x$ with $x_{n_1} \in B(x, \varepsilon_2)$.
- ... and so on.

Then, one can show that each x_{n_k} satisfies $d(x_{n_k}, x) < 2^{-k}$. One can then take an increasing subsequence of these x_{n_k} and this gives us what we need. □

We prove another intermediate result, sometimes referred to as the 'existence of a Lebesgue number for open covers' when viewed in the context of metric spaces.

Lemma 1.6.13. *Let (X, d) be a sequentially compact metric space, and let $(U_i)_{i\in\mathcal{I}}$ be an open cover of X. There is some $\varepsilon > 0$ such that for all $x \in X$, there is $i \in \mathcal{I}$ such that $B(x, \varepsilon) \subseteq U_i$.*

Proof. Suppose that this is not true. That is, suppose that $\forall\varepsilon > 0$, $\exists x \in X$ such that $\forall i \in \mathcal{I}$, $B(x, \varepsilon) \not\subseteq U_i$.

We consider a discrete sequence of εs tending to 0, such as the sequence $(2^{-n})_{n\in\mathbb{N}}$. That is, for all $n \in \mathbb{N}$, we know that there exists some $x_n \in X$ such that $\forall i \in \mathcal{I}$, $B(x, 2^{-n}) \not\subseteq U_i$.

Since X is sequentially compact, we can find a subsequence $\left(x_{n_j}\right)_{j\in\mathbb{N}}$ such that $x_{n_j} \to x$ for some $x \in X$. Since $(U_i)_{i\in\mathcal{I}}$ is an open covering, we know that there is some $i_0 \in \mathcal{I}$ such that $x \in U_{i_0}$. Since U_{i_0} is open, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_i$.

We choose $k \in \mathbb{N}$ so large that both the following inequalities hold:

$$d(x_{n_k}, x) < \frac{\varepsilon}{2} \qquad\qquad 2^{-n_k} < \frac{\varepsilon}{2}$$

Then,

$$B\left(x_{n_k}, 2^{-n_k}\right) \subseteq B(x, \varepsilon) \subseteq U_i$$

which contradicts the choice of subsequence x_{n_k}. □

Theorem 1.6.14. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is compact*

(2) *X is sequentially compact*

(3) *X is complete and totally bounded*

Proof. (1) $\implies$ (2). Say X is compact, $(x_n)_n$ with $x \in X$. If $\{x_n \mid n \in \mathbb{N}\}$ is finite then, by the infinite pigeonhole principle there must exist some infinite constant subsequence (and then we're done). Else, if \$ $\{x_n \mid n \in \mathbb{N}\}$ is infinite, we must be able to replace it with a subsequence with no repetitions. Abuse notation and call this $(x_n)_{n\in\mathbb{N}}$ as well.

Towards contradiction, assume that (x_n) has no convergent subsequences. We claim that for all $x \in X$, there is some $\varepsilon > 0$ such that there is no $n \in \mathbb{N}$ with $x_n = x$ and $x_n \in B(x, \varepsilon)$.

Lemma 1.6.12 effectively tells us that there are balls $B(x, \varepsilon_x)$ for every $x \in X$, with $\varepsilon_x > 0$, such that for no $n \in \mathbb{N}$ is $x_n = x$ and $x_n \in B(x, \varepsilon_x)$. In particular, these balls form an open cover of X, and the assumption that x is compact tells us that this open cover has a finite subcover. Enumerate this subcover

$$B(y_1, \varepsilon_{y_1}), \ldots, B(y_s, \varepsilon_{y_s})$$

Since $(x_n)_{n\in\mathbb{N}}$ has been trimmed down to have no repetitions, find now some $n \in \mathbb{N}$ such that $x_n \neq x$ and $x_n \neq y_1, \ldots, y_s$. But there is some $1 \leq i \leq s$ such that $x_n \in B(y_i, \varepsilon_{y_i})$. Thus, we have a contradiciton, meaning that (x_n) must have a convergent subsequence.

(2) $\implies$ (3). Assume X is sequentially compact.

- X is totally bounded. Fix $\varepsilon > 0$. Assume, for contradiction, that X is not a union of finitely many open ε-balls. Choose, by induction, some $(x_n)_{n\in\mathbb{N}}$ such that

  $$x_n \notin \bigcup_{i<n} B(x_i, \varepsilon)$$

  Then, $(x_n)_{n\in\mathbb{N}}$ couldn't possibly have a convergent subsequence, a contradiction. Thus, X must be totally bounded.

- X is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. As X is sequentially compact, there is some subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_i} \to x$ for some $x \in X$. We show that $x_n \to x$.

  Fix $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ so large that for all $i, j \geq N_1$, we have $d(x_i, x_j) < \frac{\varepsilon}{2}$. Choose N_2 so large that for all $k \geq N_2$, we have $d(x_{n_k}) < \frac{\varepsilon}{2}$. Then, choose $N = \max(N_1, N_2)$. By the triangle inequality, for all $n \geq N$, we have that $d(x_n, x) < \varepsilon$.

(3) $\implies$ (2). Suppose that X is totally bounded and complete. Fix a sequence $(x_n)_{n\in\mathbb{N}}$. To show that (x_n) has a convergent subsequence, it is enough to show that $(x_n)_{n\in\mathbb{N}}$ has a *Cauchy* subsequence.

Letting $A_0 \subseteq X$ be a finite set such that we can cover X with balls of radius $2^{-0}(= 1)$ such that

$$X = \bigcup_{y\in A_0} B(y, 2^{-0})$$

We can do this because we know, by assumption, that X is totally bounded. Then it follows by the infinite pigeonhole principle that there must exist some $y \in A_0$ such that $B_1(y)$ contains infinitely many points in our sequence - we now choose $y = y_0$.

Next, we consider a covering of X by balls of radius 2^{-1}. As there are infinitely many points in $B_1(y_0)$, it follows that there must exist some y_1 such that $B_{2^{-1}}(y_1) \subseteq B_1(y_0)$ with infinitely many points contained in $B_{2^{-1}}(y_1)$. It follows by similar argument that we can choose some y_2 with $B_{2^{-2}}(y_2) \subseteq B_{2^{-1}}(y_1) \subseteq B_1(y_0)$ with infinitely many points contained in $B_{2^{-2}}(y_2)$, and similarly for $y_3, y_4, \ldots$.

After continuing in this manner, we claim that we have defined a Cauchy subsequence of (x_n): define $x_{n_i} = y_i$. To show it is Cauchy, fix $\varepsilon > 0$. Observe that given $k < i$, with $n_k < n_i$, we know that $n_k \in Y_k$ and $n_i \in Y_i \subseteq Y_k$, giving us that $n_i \in Y_k$. sorry

(2) $\implies$ (1). Assume that X is sequentially compact. Since (2) $\implies$ (3), we know that X is complete and totally bounded. Now, fix an open cover $(U_i)_{i\in\mathcal{I}}$ of X. Since X is totally bounded, X is a finite union of ε-balls; that is, we can write

$$X = \bigcup_{s=1}^{t} B(y_s, \varepsilon)$$

By the choice of ε, for all s, there is $s \in I$ such that $B(y_s, \varepsilon) \subseteq U_{i_s}$. So

$$X = \bigcup_{s=1}^{t} U_{i_s}$$

proving that $(U_i)_{i\in\mathcal{I}}$ has a finite subcover. Thus, X is compact.

□

Chapter 2

Nets and Filters

In this chapter, we talk about nets and filters, both of which are very Bourbarki-esque. A net is a generalisation of the notion of a sequence, which will allow us to do very interesting things in the future.

2.1 Some "Completely Trivial Combinatorics"

Fix a topological space X.

2.1.1 Directed Sets and Nets

We recall the definition of a poset from earlier in the course (Definition 1.3.22). We begin with a generalisation of this concept.

Definition 2.1.1 (Directed Set). A **directed set** is a poset $(\mathbb{D}, \leq)$ such that for all $x, y \in \mathbb{D}$, there is some $z \in \mathbb{D}$ such that $x, y \leq z$.

It is easy to extend the definition, by induction, to show that for any directed set $(\mathbb{D}, \leq)$ and $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{D}$, there is some $z \in \mathbb{D}$ such that $x_1, \ldots, x_n \leq z$.

Example 2.1.2 (Familiar Examples of Directed Sets).

1. $(\mathbb{N}, \leq)$ is a directed set.
2. If X is any infinite set, then the set
$$\mathbb{D} = \{a \subseteq X \mid |a| < \omega\}$$
ordered by inclusion is a directed set.
3. Let X be a topological space and $x \in X$ a point. Define
$$\mathbb{D}\{U \subseteq X \mid U \text{ is open and } x \in U\}$$
$\mathbb{D}$ is ordered by the relation $U \leq V \iff V \subseteq U$, and with this ordering, forms a directed set.

Recall that at the start of the section, we fixed a topological space X. We define a **net in** X to be a *sequence in* X *indexed by a directed set*.

Definition 2.1.3 (Net). A **net in** X is a function $\mathbb{D} \to X$, where $\mathbb{D}$ is a nonempty directed set. We denote a net by $(x_a)_{a\in\mathbb{D}}$, just as we would any sequence.

It is clear that every sequence $\mathbb{N} \to X$ is a net $(\mathbb{N}, \leq) \to X$.

2.1.2 Convergence of Nets

What do we mean when we use words like "eventually" and "frequently"? Let's find out.

Definition 2.1.4 (Eventuality of Occurrence). Let $P(x)$ be a property of points $x \in X$. Let $(x_a)_{a\in\mathbb{D}}$ be a net. We say P **occurs eventually** if there is some $a \in \mathbb{D}$ such that for all $b \in \mathbb{D}$, if $b \geq a$ then $P(x_b)$ holds in X.

Definition 2.1.5 (Frequency of Occurrence). `sorry`

Next, we state what it means for a net to converge.

Definition 2.1.6 (Convergence). `sorry`

We can say something about sorry.

Proposition 2.1.7. *Consider a subspace $A \subseteq X$ and a point $x \in X$. TFAE:*

(1) *There is some directed set $\mathbb{D}$ and some net $(x_a)_{a \in \mathbb{D}}$ such that the following both hold:*

- $x_a \in A$ *for all* $a \in D$
- $(x_a)_{a \in \mathbb{D}}$ *converges to* x

(2) $x \in \mathrm{cl}(A)$

Proof.

(1) $\implies$ (2). sorry

(2) $\implies$ (1). Consider the directed set

$$\mathbb{D}\{U \subseteq X \mid U \text{ is open and } x \in U\}$$

ordered by the relation $U \leq V \iff V \subseteq U$ (the third example in Example 2.1.2). We claim we can find a net $(x_U)_{U \in \mathbb{D}}$ such that $x_U \in U \cap A$. sorry

□

It is an "amusing (if not very useful) fact" that a space is Hausdorff if and only if nets have limits.

2.1.3 Subnets

Before we define what a subnet is, we define a property of maps between directed posets.

Definition 2.1.8 (Cofinality). Let $\mathbb{E}, \mathbb{D}$ be directed posets. We say a function $\phi : \mathbb{E} \to \mathbb{D}$ is **cofinal** if for all $d \in \mathbb{D}$, there is some $e \in \mathbb{E}$ such that $d \leq \phi(e)$.

For the Topology Basic Exam, it is "good to know what the actual definition of a subnet is." Sounds like sage advice!(!!)

Definition 2.1.9 (Subnet). Let $(x_d)_{d \in \mathbb{D}}$ be a net in X. A **subnet** of this is some net $(y_e)_{e \in \mathbb{E}}$ together with a map $\phi : \mathbb{E} \to \mathbb{D}$ such that

1. ϕ is order-preserving.
2. ϕ is cofinal.
3. $y_e = x_{\phi(e)}$ for all $e \in \mathbb{E}$.

Now that we have the right definition of a subnet, we define a very tempting mistake to make. It is tempting to define the property of being a subnet to be a specialisation of the definintion given in Definition 2.1.9 to the case where $\mathbb{E} \subseteq \mathbb{D}$ and ϕ is the inclusion. While there do exist subnets of this form, not every subnet takes this form.

2.1.4 Ultranets

This closely resembles the theory of filters and ultrafilters from set theory (or from Bourbaki, or from the topological corners of `mathlib`).

Definition 2.1.10 (Ultranet). Let X be a topological space and let $(x_d)_{d \in \mathbb{D}}$ be a net. We say that $(x_d)_{d \in \mathbb{D}}$ is an **ultranet** if for all $Y \subseteq X$, *at least one* of the following holds:

1. There is some $a \in \mathbb{D}$ such that for all $b \geq a$, we have $x_b \in Y$
2. There is some $a \in \mathbb{D}$ such that for all $b \geq a$, we have $x_b \notin Y$

The reason we said *at least one* of the conditions should hold is that the two are not mutually exclusive. `sorry`

Give example where both hold

This is related to the concept of locales in Grothendieck topology.

Professor Cummings leaves us with the following "utterly trivial remark".

Exercise 2.1.11. Let X and Y be topological spaces. If $(x_a)_{a \in \mathbb{D}}$ is an ultranet in X and $f : X \to Y$ is any function, then $(f(x_d))_{d \in \mathbb{D}}$ is an ultranet in Y.

Remark. "Inverse images play very nicely with complements in set theory...and I'm done, with a little bit of fast talking."

2.2 Generalising Properties of Metric Spaces

To set the stage, recall that in a metric space X with $A \subseteq X$, we have

$$\mathrm{cl}(A) = \left\{x \in X \;\middle|\; \exists\, (a_n)_{n\in\mathbb{N}} \text{ s.t. } a_n \in A \text{ and } a_n \to x \text{ as } n \to \infty\right\}$$

Moreover, recall that a metric space X is compact if and only if X is sequentially compact, in the sense that every sequence has a convergent subsequence.

We will show that in an arbitrary topological space, we can replace sequences with nets and subnets and show that they still hold.

2.2.1 Compactness and Sequential Compactness

While we know that compactness and sequential compactness are not equal in general topological spaces, we show that they *are* if we use nets instead.

Theorem 2.2.1. *Let X be a topological space. The following are equivalent.*

(1) *X is compact.*

(2) *Ever net has a convergent subnet.*

Proof.

(2) $\implies$ (1). Suppose that every net has a convergent subnet. Define an open cover $\{U_i \mid i \in I\}$ and define `sorry`

Let $(x_a)_{a\in\mathbb{D}}$ be such that

$$x_a \in \bigcap_{i\in a} F_i$$

We know a convergent subnet exists. Let $(y_b)_{b\in\mathbb{E}}$ be such a subnet, with associated map $\phi : \mathbb{E} \to \mathbb{D}$, and say that $(y_b)_{b\in\mathbb{E}}$ converges to $y \in X$. We claim that

$$y \in \bigcap_{i\in I} F_i$$

Indeed, if this is not true, then there is some $i \in I$ such that $y \notin F_i$. That is, $y \in X \setminus F_i$, which is open because F_i is closed. Moreover, we know that $y_b \to y$. By the definition of convergence, there is some $c \in \mathbb{E}$ such that for all $c' \in \mathbb{E}$, if $c' \geq c$ then $y_{c'} \in X \setminus F_i$.

Our setup essentially tells us that for all $i \in I$ and $a \in \mathbb{D}$,

$$a \geq \{i\} \iff i \in a \iff x_a \in F_i \tag{2.2.1}$$

Since ϕ is cofinal, there is some index $\overline{c} \in \mathbb{E}$ such that $\phi(\overline{c}) \geq \{i\}$ (that is, $i \in \phi(\overline{c})$). Let $e \in \mathbb{E}$ be such that $e \geq c, \overline{c}$ (such an e exists by directedness). Then,

$$x_{\phi(e)} = y_e \in X \setminus F_i$$

Indeed, $\phi(e) \geq \phi(\overline{c}) \geq \{i\}$, with the ordering being inclusion. Therefore, $i \in \phi(e)$. But (2.2.1) tells us that this is equivalent to $x_{\phi(e)} \in F_i$, which is a contradiction.

<u>$(1) \implies (2)$.</u> Let X be compact. Let $(x_a)_{a\in\mathbb{D}}$ be a net. For each $a \in \mathbb{D}$, define

$$F_a := \mathrm{cl}(\{x_b \mid b \geq a\})$$

We claim that this family $\{F_a \mid a \in \mathbb{D}\}$ of closed sets has the finite intersection property.

Indeed, fix $a_1, \ldots, a_n \in \mathbb{D}$. Find, by directedness, some $a \in A$ such that

$$a \geq a_1, \ldots, a_n$$

Then, $x_a \in F_{a_i}$ for all $1 \leq i \leq n$, just by unfolding definitions. Thus,

$$x_a \in \bigcap_{i=1}^{n} F_{a_i}$$

Since X is compact, we can produce, by "dubious magic", some

$$x \in \bigcap_{a\in\mathbb{D}} F_a$$

We construct a subnet of $(x_a)_{a\in\mathbb{D}}$ which converges to x.

Define the following set:

$$\mathbb{E} = \{(a, U) \mid a \in \mathbb{D},\ U \text{ is an open nbhd of } x, \text{ and } x_a \in U\}$$

We order $\mathbb{E}$ by the ordering $(a, U) \leq (b, V)$ iff $a \leq_{\mathbb{D}} b$ and $V \subseteq U$. We show that under this ordering, $\mathbb{E}$ is directed. That is, we show that for all $(a_1, U_1), (a_2, U_2) \in \mathbb{E}$, there exists some (c, V) such that $(a_1, U_1), (a_2, U_2) \leq_{\mathbb{E}} (c, V)$. Indeed, for all $(a_1, U_1), (a_2, U_2) \in \mathbb{E}$, we can find some $b \in \mathbb{D}$ such that $b \geq a_1, a_2$, because $\mathbb{D}$ is directed. Moreover, $U_1 \cap U_2$ is an open neighbourhood of x. Indeed, if we take

$$F_b := \mathrm{cl}(\{x_c \in X \mid c \geq b\})$$

then we have $x \in F_b$, so the triple intersection

$$\{x_c \in c \mid c \geq b\} \cap U_1 \cap U_2$$

is nonempty. Find $c \geq b$ such that $x_c \in U_1 \cap U)2$. Then, we have $(c, U_1 \cap U_2) \in \mathbb{E}$. Moreover, $c \geq b \geq a_1, a_2$ and $U_1 \cap U_2 \subseteq U_1, U_2$. Thus, $(a_1, U_1), (a_2, U_2) \leq_{\mathbb{E}} (c, U_1 \cap U_2)$.

To define a subnet, we need first a map $\phi : \mathbb{E} \to \mathbb{D}$ satisfying the desired conditions. Define

$$\phi : \mathbb{E} \to \mathbb{D} : (a, U) \mapsto a$$

Define the net $\left(y_{(a,U)}\right)_{(a,U) \in \mathbb{E}}$ by $y_{(a,U)} = x_a$. Then,

1. ϕ is order-preserving because for all $(a, U), (b, V) \in \mathbb{D}$, if $(a, U) \leq (b, V)$ then $a \leq b$.

2. ϕ is cofinal because for all $a \in \mathbb{D}$, we can see that $(a, X) \in \mathbb{E}$ has the property that $a \leq \phi((a, X)) = a$.

3. For all $(a, U) \in \mathbb{E}$, clearly $y_{(a,U)} = x_a$ by definition.

Lastly, we show that $\left(y_{(a,U)}\right)_{(a,U) \in \mathbb{E}}$ converges to x. `sorry`

□

Warning. When defining a subnet, always first make sure the index set is in fact directed. Then one needs to specify the values of the function ϕ, and then make sure that ϕ is a valid map ("as advertised").

2.2.2 Cluster Points

Before we describe how the concept of closures being defined by sequences is generalised to arbitrary topological spaces using nets, we will find it useful to ask (and answer) the following question:

A subset of a subset is a subset. Is a subnet of a subnet also a subnet?

Remark. The answer is "yes it is", and you get no prizes for showing so (let $\psi : \mathbb{F} \to \mathbb{E}$ and $\phi : \mathbb{E} \to \mathbb{D}$ define a subnet and a subnet of that subnet respectively - then consider the map $\phi \circ \psi$, and show that this defines a subnet of $\mathbb{D}$).

For the remainder of this subsection,

- A topological space X
- Directed posets $\mathbb{D}, \mathbb{E}, \mathcal{F}$
- Nets $(x_d)_{d\in\mathbb{D}}$, $(y_e)_{e\in\mathbb{E}}$, $(z_f)_{f\in\mathcal{F}}$ in X
- Cofinal and order-preserving maps $\phi : \mathbb{E} \to \mathbb{D}$ and $\psi : \mathcal{F} \to \mathbb{E}$

We define the notion of a cluster point.

Definition 2.2.2 (Cluster Points). Fix $x \in X$. We say that x is a **cluster point of** $(x_d)_{d\in\mathbb{D}}$ if and only if for all open sets $U \ni x$ and $a \in \mathbb{D}$, there is some $b \geq a$ such that $x_b \in U$.

There is an equivalent characterisation of cluster points in terms of subnets. The proof is "one of those proofs where you follow your nose, and just do some stenography with the definitions."

Proposition 2.2.3 (An Equivalent Characterisation of Cluster Points). *The following are equivalent.*

(1) $x \in X$ *is a cluster point of* $(x_d)_{d\in\mathbb{D}}$

(2) $(x_a)_{a\in\mathbb{D}}$ *has a subnet which converges to* x

Proof. We begin the process of nose-following and definitional stenography. Yay - fun!

(2) $\implies$ (1). Let $(y_e)_{e\in\mathbb{E}}$ and $\phi : \mathbb{E} \to \mathbb{D}$ together form a subnet of $(x_d)_{d\in\mathbb{D}}$. We show that

$y_e \to x$.

Let $U \ni x$ be open. Fix $d \in \mathbb{D}$. Let $e \in \mathbb{E}$ be such that for all $e' \geq e$, $y_{e'} \in U$. Let $e_1 \in \mathbb{E}$ be such that $\phi(e_1) \geq_{\mathbb{D}} d$. Since $\mathbb{E}$ is directed, we can find some $e' \in \mathbb{E}$ that simultaneously satisfies $e' \geq_{\mathbb{D}} e, e_1$.

Since ϕ is order-preserving, we have that

$$\phi(e') \geq \phi(e_1) \geq \phi(d)$$

Since $e' \geq e$, we know that

$$x_{\phi(e')} = y_{e'} \in U$$

Thus, x is a cluster point. As Professor Cummings said, "At this point, e have "won the game." Yippee!!

(2) $\implies$ (1). "As you may have guessed, that was the less painful part of the proof."

Let x be a cluster point of $(x_d)_{d \in \mathbb{D}}$. We are going to cook [up a convergent subnet of x]. Define

$$\mathbb{E} := \{(a, U) \mid a \in \mathbb{D},\ U \ni x,\ U \text{ is open},\ x_a \in U\}$$

Define the ordering

$$(a, U) \leq_{\mathbb{E}} (b, V) \iff a \leq_{\mathbb{D}} b \text{ and } V \subseteq U$$

Define the map

$$\phi : \mathbb{E} \to \mathbb{D} : \phi((a, U)) = a$$

We now need to verify that ϕ is order-preserving and cofinal. The former is "just a joke" (i.e. follows by definition). The latter is true for the "dumbest possible reason": for all $d \in \mathbb{D}$, $(d, X) \in \mathbb{E}$ satisfies the property that $\phi(d, X) = d$.

Next, we need to show that $\mathbb{E}$ is directed - which is pretty much immediate from the observation that x is a cluster point. In particular, given $(a_1, U_1), (a_2, U_2) \in \mathbb{E}$ with

$b \geq a_1 \cap a_2, U_1 \cap U_2$ open, as x is a cluster point of the net we can find $c \geq b$ such that $x_c \in U_1 \cup U_2$, and then $(c, U_1 \cap U_2) \geq (a_1, U_1), (a_2, U_2)$.

We can now define a subnet $y_{(a,U)} := x_a$. All that remains is to show that $(y_e)_{e \in \mathbb{E}}$ actually *converges* to x. Fix $V \ni x$ an open neighbourhood of x. Find any old $b \in \mathbb{D}$ such that $x_b \in V$. Take $e = (b, V)$. Then, for all $e' = (b', V') \geq_{\mathbb{E}} (b, V)$, we have

$$y_{e'} = x_{b'} \in V' \subseteq V$$

"I've told this joke before in a slightly different context." Also, verifying the subnet criteria is "as fun as watching paint dry - but it's good to be thorough." □

"We are kind of compelled to turn things upside down."

We now state an important fact that relates cluster points to ultranets

Proposition 2.2.4. *Let X be a topological space, x a point in X, and $(x_d)_{d \in \mathbb{D}}$ an ultranet in X (cf. Definition 2.1.10). The following are equivalent.*

(1) *x is a cluster point of $(x_d)_{d \in \mathbb{D}}$*

(2) *$x_d \to x$*

Proof. sorry □

Corollary 2.2.5. *If X is compact and $(x_a)_{a \in \mathbb{D}}$ is an ultranet, then $(x_a)_{a \in \mathbb{D}}$ converges to some point.*

Proof. Since X is compact and $(x_a)_{a \in \mathbb{D}}$ is a net, we know that $(x_a)_{a \in \mathbb{D}}$ has a convergent subnet converging to some x. This makes x a cluster point of $(x_a)_{a \in \mathbb{D}}$. Thus, by (2) $\implies$ (1) in Proposition 2.2.4, $x_a \to x$. □

Next, we state "Fact 4 in [Professor Cummings's] litany of facts" (the first two being the points in Proposition 2.2.4 and the third being Corollary 2.2.5).

Corollary 2.2.6. *If X and Y are topological spaces and $(x_a)_{a\in\mathbb{D}}$ is an ultranet in X and $f : X \to Y$ is any function, then $(f(x_a))_{a\in\mathbb{D}}$ is an ultranet in Y.*

Proof sketch. Let $B \subseteq Y$ be arbitrary. Define $A := f^{-1}[B] \subseteq X$. Use ultranet properties. □

Finally, we state a powerful theorem about ultranet (ultrasubnet? subultranet?) existence.

Proposition 2.2.7. *Every net has a subnet which is an ultranet.*

Proof. sorry □

Professor Cummings shall now cheat us in a somewhat innocuous way by using ultranets to prove Tychonoff's Theorem.

2.2.3 Tychonoff's Theorem

We are now ready to prove the main result of this chapter (indeed, the result that motivated our development of the theory of nets, subnets and ultranets).

Theorem 2.2.8 (Tychonoff's Theorem). *An arbitrary product of compact spaces is compact.*

Proof. Let $(X_i)_{i\in\mathcal{I}}$ be a family of compact spaces. Let $X := \prod_{i\in\mathcal{I}} X_i$. We will show that X is compact by showing that every net in X has a convergent subnet. Then, sorry will tell us that X is compact. Add reference

Let $(x_a)_{a\in\mathbb{D}}$ be a net in X. We doubly index $(x_a)_{a\in\mathbb{D}}$ in the following manner: for each $a \in \mathbb{D}$, we write

$$x_a = (x_{a,i})_{i\in\mathcal{I}}$$

That is, we denote by $x_{a,i}$ the ith component of x_a.

Let $(y_b)_{b\in\mathbb{E}}$ be a subnet of $(x_a)_{a\in\mathbb{D}}$ such that $(y_b)_{b\in\mathbb{E}}$ is an ultranet in X (as given by Proposition 2.2.7). We again doubly index $(y_b)_{b\in\mathbb{E}}$ by writing the ith component of every y_b as $y_{b,i}$. We will show that $(y_b)_{b\in\mathbb{E}}$ converges.

For each $i \in \mathcal{I}$, let $\pi_i : X \to X_i$ be the canonical projection map sending $z \in X$ to its ith component, denoted z_i. By Fact 4 in the "litany of facts" (Corollary 2.2.6), since $(y_b)_{b\in\mathbb{E}}$ is an ultranet, so is $(\pi_i(y_b))_{b\in\mathbb{E}}$. Indeed, observe that for all $i \in \mathcal{I}$ and $b \in \mathbb{E}$, $\pi_i(y_b) = y_{b,i}$. Thus, for all $i \in \mathcal{I}$, the ultranet $(\pi_i(y_b))_{b\in\mathbb{E}}$ is exactly equal to $(y_{b,i})_{b\in\mathbb{E}}$.

Fix $i \in \mathcal{I}$. Since X_i is compact, Fact 3 in the "litany" (Corollary 2.2.5), there is some $y_i \in X_i$ such that $(y_{b,i})_{b\in\mathbb{E}}$ converges to y_i. This gives us an element $y \in X$ defined to be the tuple $(y_i)_{i\in\mathcal{I}}$.

We now claim that $(y_b)_{b\in\mathbb{E}}$ converges to y. If we can prove this claim, we will be done with the proof, as it will establish that $(y_b)_{b\in\mathbb{E}}$ is indeed a convergent subnet of $(x_a)_{a\in\mathbb{D}}$.

Recall that there is an inherent *finiteness* built into the product topology. In particular, the product topology is **strictly contained** in the box topology, because we require the box topology to be a product object in the category of topological spaces (see Section A.1 for more). This will be absolutely crucial in establishing the claim that $(y_b)_{b\in\mathbb{E}}$ converges to y.

Fix a neighbourhood $U \ni y$ that is open with respect to the product topology on X. Fix a basic open set $\prod_{i\in\mathcal{I}} U_i$ containing y such that $\prod_{i\in\mathcal{I}} U_i \subseteq U$. We know there are finitely many indices $\{i_1, \ldots, i_n\} \subseteq \mathcal{I}$ such that for $1 \leq s \leq n$, $U_{i_s} \subsetneq X_{i_s}$.

For each $1 \leq s \leq n$, the net $(y_{b,i_s})_{b\in\mathbb{E}}$ converges to y_{i_s} in X_{i_s}. Indeed, $y_{i_s} \in U_{i_s}$, and U_{i_s} is open in X_{i_s}, so there is some index $b_s \in \mathbb{E}$ such that for all $c \in \mathbb{E}$, if $c \geq_{\mathbb{E}} b_s$, then $y_{c,i_s} \in U_{i_s}$ Thus, we get a finite set of indices $b_1, \ldots, b_n \in \mathbb{E}$ such that each b_s gives the threshold in *that* X_{i_s} beyond which elements of $(y_{b,i_s})_{b\in\mathbb{E}}$ all lie in U_{i_s}.

Since $\mathbb{E}$ is directed, we know there exists some $b \in \mathbb{E}$ such that $b \geq b_1, \ldots, b_n$. We claim that for all $c \geq b$, $y_c \in U$.

Fix some $c \geq b$. We show that for all $i \in \mathcal{I}$, $y_{c,i} \in U_i$ for all $i \in \mathcal{I}$. This would then imply that $y_c \in \prod_{i\in\mathcal{I}} U_i$, and we know that $\prod_{i\in\mathcal{I}} U_i \subseteq U$, so we would be done.

So, fix $i \in \mathcal{I}$. Either $i = i_s$ for some $1 \le s \le n$ or not.

Case 1: $i = i_s$ for some $1 \le s \le n$. In this case, $y_{c,i_s} \in U_{i_s}$ because $c \ge b \ge b_s$.

Case 2: $i \ne i_s$ for any $1 \le s \le n$. In this case, $U_i = X_i$. Thus, $y_{c,i} \in X_i = U_i$.

Either way, each component of y_c is contained in each U_i. Therefore, $y_c \in \prod_{i \in \mathcal{I}} U_i$, and since $\prod_{i \in \mathcal{I}} U_i \subseteq U$, we can conclude that $y_c \in U$, and we are done. $\square$

Warning. The distinction between the product and box topologies is **absolutely critical** for Tychonoff's Theorem! In fact, the conclusion typically **fails** when we take a box product.

A detailed discussion on the difference between

2.3 Filters and Ultrafilters

Yayy, now we learn informally what we learnt formally to do Lean parce que Patrick Massot adore Bourbaki! (Actually, was that the reason? I'm not even sure... mais probablement...)

2.3.1 First Definitions

According to Professor Cummings, this is "just pure set theory"—nothing to do with topology. At least... *not yet...*

Throughout this subsection, fix a **nonempty** set X.

Definition 2.3.1 (Filter). A **filter on** X is a family Fs of X satisfying the properties

F1. $\emptyset \notin F$ but we always have $X \in F$

F2. For all $A \in F$ and all $B \subseteq X$ which satisfy $A \subseteq B \subseteq X$ we have $B \in F$

F3. For all $A, B \in F$ we will have $A \cap B \in F$

We can think of a filter as a "notion of largeness" we can think of a filter as specifying exactly which subsets of a set have this property. Indeed, the empty set shouldn't be large, but we would expect the ambient set to be considered large. We would also like to think about being able to *enlarge* large things, which is why we want upwards closure. Finally, we can understand the intersection property by thinking complementarily: if we have two small sets, we'd want their union to be small (bigger than them, yes, but not so big as to actually be *big*). Thus, if we have two large sets, it's reasonable to think we'd want their intersection to be small.

Definition 2.3.2 (Ultrafilter). An **ultrafilter on** X is a filter U satisfying the additional property

F4. For all $A \subseteq X$, either $A \in U$ or $X \setminus A \in U$

Note

Not every filter is an ultrafilter.

Counterexample 2.3.3 (A Filter that is <u>not</u> an Ultrafilter). We define the following filter on $\mathbb{N}$:

$$F = \{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$$

We call this the **cofinite filter** or the **Fréchet filter**. While F is a filter, it is certainly **not** an ultrafilter. For instance, neither the evens nor their complement (the odds) live in F because they are both infinite.

We can show, however, that every filter can be *extended* to an ultrafilter. This is similar in spirit to how we extend ideals to maximal ideals in commutative rings. We will do it using Zorn's Lemma.

(Incidentally, there is also a notion of an *ideal* of sets, which is essentially complementary to the notion of a filter. Lots of similarities!)

2.3.2 Zorn's Lemma: A Quick Recap

Throughout this subsection, let $(\mathbb{P}, \leq)$ be a poset.

We begin by defining notions of maximality in a poset.

Definition 2.3.4 (Maximum and Maximal Elements). We say a point $p \in \mathbb{P}$ is

- **maxim*um*** iff for all $q \in \mathbb{P}$, $q \leq \mathbb{P}$.

- **maxim*al*** iff there is no $q \in \mathbb{P}$ such that $p < q$

Next, we define the notion of a chain.

Definition 2.3.5 (Chain in a Poset). A **chain in** $\mathbb{P}$ is some subset $C \subseteq \mathbb{P}$ which is linearly ordered, ie, where for all $c, d \in C$, either $c \leq d$ or $d \leq c$.

Indeed, a chain is bounded (above) by $q \in \mathbb{P}$ if and only if for all $p \in C$, $p \leq q$.

We are now ready to state the famous Zorn's Lemma.

Theorem 2.3.6 (Zorn's Lemma). *If every chain in $\mathbb{P}$ is bounded, then for all $p \in \mathbb{P}$, there is some $q \geq p$ such that q is maximal.*

As the saying goes,

> "The Axiom of Choice is obviously true; the Well-Ordering Principle is obviously false; and as for Zorn's Lemma, who can say?"

More seriously, though, it is possible to show that the Zermelo-Fraenkel axioms imply an equivalence between Zorn's Lemma and the Axiom of Choice. The rough idea of this proof is that if you allow larger and larger things, you keep taking greater and greater ordinals, until you need to take a limit ordinal, and then you take greater and greater ordinals, and then you take a limit ordinal, and you just keep going, until you hit a limit ordinal beyond which you will not need to go.

2.3.3 Extending Filters to Ultrafilters

The reason why we took that Zorny detour is because we need to apply Theorem 2.3.6 to extend filters to ultrafilters.

Throughout this subsection, fix a set $X \neq \emptyset$. We can show that the set

$$\mathbb{P} = \{F \subseteq \mathcal{P}(X) \mid F \text{ is a filter on } X\}$$

is partially ordered by inclusions of sets. This allows us to make the following characterisation.

Lemma 2.3.7. *A filter F on X is an ultrafilter if and only if F is maximal in $\mathbb{P}$.*

Proof.

($\implies$) Let F be an ultrafilter. Towards contradiction, suppose that F is not maximal in $\mathbb{P}$. Then there must be some filter $F' \in \mathbb{P}$ such that $F \subsetneq F'$. So, there exists some $A \in F' \setminus F$.

Since F is an ultrafilter, either $A \in F$ or $X \setminus A \in F$. Since $A \notin F$, $X \setminus A \in F$. Since $F \subseteq F'$, $X \setminus A \in F'$. But then, since filters are closed under intersections, we get

$$\emptyset = A \cap (X \setminus A) \in F'$$

which is impossible, because F' is a filter and filters cannot contain the empty set.

($\impliedby$) Let X be maximal. Towards contradiction, suppose that F is not an ultrafilter. Then, there must be some $A \subseteq X$ such that $A \notin F$ and $X \setminus A \notin F$. Define

$$F' := \{X \subseteq A \mid \exists C \in F \text{ s.t. } A \cap C \subseteq B\}$$

One can show that F' is a filter that strictly contains F, which would contradict the maximality of F. Professor Cummings leaves this as an exercise.

Do exercise.

$\square$

This allows us to prove the following important result, which has the flavour of extending ideals to maximal ideals in commutative algebra.

Proposition 2.3.8. *Any filter on X extends to a filter which is maximal in $\mathbb{P}$.*

Proof. By Theorem 2.3.6, it is enough to show that every chain in $\mathbb{P}$ has an upper-bound in $\mathbb{P}$. So, fix a chain C in $\mathbb{P}$. If $C = \emptyset$, then the trivial filter $\{X\}$ is an upper-bound of C (thus, the "empty set brigade", led by the almighty Lean kernel, can be satisfied that this proof is valid). We now treat the case where C is nonempty.

Define the subset $S \subseteq \mathcal{P}(X)$ by

$$S := \bigcup C = \{A \subseteq X \mid \exists F \in C \text{ s.t. } A \in F\}$$

S clearly contains every element of C, making it an upper-bound for C. All we need to show is that S is a filter. This will show that $S \in \mathbb{P}$, meaning that C has an upper-bound in $\mathbb{P}$.

First, we need to show that S is nonempty. But this is clear, because $X \in S$: X is contained in every filter in C.

sorry □

Combining Lemma 2.3.7 and Proposition 2.3.8 gives us the following.

Corollary 2.3.9. *Any filter on F extends to an ultrafilter on X.*

Youpie!

2.4 Relating Filters to Nets

We begin with an "irritating and bafflingly slick proof" that essentially boils down to a clever choice of index set. Let X be a topological space.

Theorem 2.4.1. *Let $(x_a)_{a\in\mathbb{D}}$ be a net in X. There exists an ultranet in X that is a subnet of $(x_a)_{a\in\mathbb{D}}$.*

Proof. Our argument will essentially be an application of Corollary 2.3.9.

Define the following collection of subsets of $\mathbb{D}$:

$$F := \{A \subseteq \mathbb{D} \mid \exists a \in \mathbb{D} \text{ s.t. } \{b \in \mathbb{D} \mid a \leq b\} \subseteq A\}$$

That is, we define F to be the set of all subsets of $\mathbb{D}$ containing some nonempty cone. We claim that F is a filter on $\mathbb{D}$. According to Professor Cummings, the only "tinily nontrivial" thing to show there is showing that F is closed under intersections. The point is that by directedness, the intersection of two cones contains a another cone.

Apply Corollary 2.3.9 to obtain an ultrafilter $\mathcal{U} \supseteq F$ on $\mathbb{D}$. We are now ready to define the index set and maps giving a subnet of $(x_a)_{a\in\mathbb{D}}$. Define

$$E := \{(a, A) \mid a \in \mathbb{D},\ a \in A,\ A \in \mathcal{U}\}$$

Define the following ordering on $\mathbb{E}$:

$$(a_1, A_1) \leq_{\mathbb{E}} (a_2, A_2) \iff a_1 \leq_{\mathbb{D}} a_2 \text{ and } A_2 \subseteq A_1$$

It is not difficult to show that $\mathbb{E}$ is a poset (Professor Cummings would give "no points" for that). "The point really is to show it is directed." To that end, fix $(a_1, A_1), (a_2, A_2) \in \mathbb{E}$. Since $\mathbb{D}$ is directed, we can find $b \in \mathbb{D}$ such that $b \geq_{\mathbb{D}} a_1, a_2$. Now define $B = A_1 \cap A_2 \cap \{b \in \mathbb{D} \mid b \geq_{\mathbb{D}} a\}$. It is not difficult to see that $(a_1, A_1), (a_2, A_2) \leq_{\mathbb{E}} (b, B)$.

We now find an order-preserving and cofinal map from $\mathbb{E}$ to $\mathbb{D}$. Define $\phi : \mathbb{E} \to \mathbb{D}$ by $\phi((a, A)) = a$. It is immediate that ϕ is order-preserving. Moreover, since $\phi((a, \mathbb{D})) = a$, we can see that ϕ is cofinal.

Thus, for all $(a, A) \in \mathbb{E}$, if we define

$$y_{(a,E)} = x_{\phi((a,A))} = x_a$$

then $(y_e)_{e\in\mathbb{E}}$ is a subnet of $(x_a)_{a\in\mathbb{D}}$.

All that remains is to show that $(y_e)_{e\in\mathbb{E}}$ is an ultranet.

One must remember what is the decisive property of ultranets - we either want to find a point where all the indices above that point hit $Y \subseteq X$, or all indices above this point hit $X \setminus Y$.

Let $A = \{a \in \mathbb{D} : x_a \in Y\}$, as $\mathcal{U}$ is an ultrafilter, either $A \in \mathcal{U}$ or $\mathbb{D} \setminus A \in \mathcal{U}$. Now supposing that $A \in \mathcal{U}$, we can choose $a \in A$ and let $e = (a, A)$. Then for all $f \geq e$, $f = (b, B)$ where $a \leq b, B \subseteq A$ we have $b \in B \subseteq A \implies x_b \in Y \implies y_d = y_{(b,B)} = x_b$.

Hopefully you can see that "rest, similar." `sorry` $\square$

And now we've reached a bit of a high-point in the course - we have proved Tychnoff's Theorem!

You knew we were not done with the definitions - in fact we have only just begun.

Chapter 3

More Topological Properties

In this chapter, we dive deeper into properties of topological spaces. Specifically, we will focus on how properties are propagated by subspaces and constructions, such as products and quotients.

3.1 (More) Separation Properties

Now where were we...

- T_0: Given any $x, y \in X$, either there exists some open U with $x \in U$, $y \notin U$.
- T_1: Singleton sets (not "points" "because this is a picky and pedantic discipline") are closed.
- T_2: Hausdorff!

It is now time to prepare ourselves for the metrization theorems to come. One may notice that the following has a slightly different flavour than our previous separation properties:

Definition 3.1.1 (Regular Space). X a space is "regular" if for all $x \in X$, $C \subseteq X$ with C closed and $x \notin C$, there are open U with $x \in U$ and $V \supseteq C$ such that $U \cap V = \emptyset$.

Definition 3.1.2 (Normal Space). We say that our space X is "normal" if for all closed $C, D \subseteq X$ with $C \cap D = \emptyset$, then there are $U \supseteq C, V \supseteq D$ with U, V open subsets of X such that $U \cap V = \emptyset$.

Definition 3.1.3 (Completely Regular Space). Our favorite and least favorite space X is "completely regular" if for all $x \in X, C \subseteq X$ closed with $x \notin C$, there is a continuous $f : X \to [0,1]$ such that $f(x) = 0, f[c] = [1]$.

Remark. Getting to the idea of separating sets with continuous functions - will be discussed much more after fall break...

Remark. Note also that we do NOT require that "regular", "normal", or "completely regular" spaces are Hausdorff.

[also note that we had lots of T but I spent too long typing and did not have the time to grab any before it was removed from the board :(`sorry`]

Definition 3.1.4 (T_4 Space). A topological space is T_4 if it is Hausdorff and normal.

Definition 3.1.5 ($T_{3.5}$/Tychonoff Space). A topological space is $T_{3.5}$ or **Tychonoff** if it is Hausdorff and completely regular.

We can offer another characterisation of Tychonoff spaces.

3.2 Countability Properties

Definition 3.2.1 (1st, 2nd Countability). Let X be a space, then

1. we say that X is 1st countable to say that every point in X has a countable neighbourhood basis (and...)
2. we say that X is second countable to say that X has a countable basis.

"one whose relationship with the first two might remain a little questionable for now..."

"those of you who took field theory might hate the name of this next one..."

Definition 3.2.2 (Separable ("ohhhhhhhhhh..." insert disappointment)). We say that X is "separable" to say that X has a countable dense subset.

Definition 3.2.3 (Lindelof ("Poor Man's Subcover")). We say that X is Lindelof to say that every open covering of X has a countable subcover.

Now another definition, to get some feel for the "geometry" of a metric space

Definition 3.2.4 (distance from a point to a set in a metric space). Let (X, d) be a metric space, $A \subseteq X, A \neq \emptyset$ (to pre-emptively take care of the empty set police) and $x \in X$. Then $d(x, A) = \inf\{d(x, a) : a \in A\}$.

Theorem 3.2.5. *$d(x, A)$ is a continuous function of x.*

Proof. Let $x, y \in X$ and $a \in A$, then $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \implies d(y, a) \geq d(x, A) - d(x, y)$ for all $a \in A \implies d(y, A) \geq d(x, A) - d(x, y)$. Now one can fill in the details at their leisure. □

With this, we can show that X a metric space implies X is T_4:

Proof. Let $C, D \subseteq X$ with (X, d) is a metric space with C, D closed and $C \cap D = \emptyset$. If we then let $g(x) = d(x, C) - d(x, D)$ [we NEED that C, D are closed and disjoint!], then g is continuous as a difference of two continuous functions. Let $X = g^{-1}[(-\infty, 0)]$ and $V = g^{-1}[(0, \infty)]$ - then maybe we're basically done `sorry`. □

Definition 3.2.6 (Embedding). We say $f : X \to Y$ is an "embedding" of X into Y if f is an injective map with the additional constraint that f is a homeomorphism from X to $f[X]$ (where $f[X] \subseteq Y$ is endowed with the subspace topology).

Warning. Crucially, note $f[U]$ need not be open in Y for U open in X - we only require $f[U]$ open in subspace topology of $f[X] \subseteq Y$.

Recall that for spaces X, Y_i $(i \in I)$ that $F : X \to \prod_{i \in I} Y_i$ is continuous iff for all i, $\pi_i \circ F$ is continuous. In what follows, we let $f_i := \pi_i \circ F$ given some $F(x) = (f_i(x))_{i \in I}$.

Our goal now is to find conditions on $f_i : X \to Y_i$ to ensure that F is an embedding of X in $\prod_{i \in I} Y_i$. To ensure that F is continuous, each f_i needs to be continuous. To be sure F is injective, we need that for all $x, x' \in X$ that $x \neq x' \implies \exists i \in I$ such that $f_i(x) \neq f_i(x')$.

"and you're going to groan when I give you the next definition..."

Definition 3.2.7 (Separating Points from Closed Sets)**.** Given $X, (Y_i)_{i \in I}$ and continuous functions $f_i : X \to Y_i$ we say that $(f_i)_{i \in I}$ "separates points from closed sets" iff for all closed $C \subseteq X$ with $x \in X, x \notin C$ here is some $i \in I$ such that $f_i(x) \notin \text{cl}(f_i[C])$.

Theorem 3.2.8. *If $f_i : X \to Y_i$ with $i \in I$ are such that each f_i separates points, separates points on closed sets with $i \in I$ and is continuous, then F is an embedding.*

Here is the proof, but "you're not going to like it" - is there set-theoretic nonsense??

Proof. Given $U \subseteq X$ open, we show $F[U]$ is relatively open in $F[X]$. In fact, we will show that for all $x \in U$ that there is $V \subseteq \prod_i Y_i$, $F(x) \in V$, with V open and $V \cap F[X] \subseteq F[U]$. However observing that $x \notin X \setminus U$ a closed set, as $(f_i)_{i \in I}$ separates points from closed sets, there is i such that $f_i(x) \notin \text{cl}(f_i[X \setminus U])$.

We now let $E = \text{cl}(f_i[X \setminus U]) \subseteq Y_i$ with E closed. But then $f_i(x) \in Y_i \setminus E$ (an open set in Y_i). If we now let $V = \pi^{-1}[Y_i \setminus E]$, we can find that V is open in $\prod_i Y_i$, and $F(x) \in V$ - we are left only to substantiate this claim.

Finally then, we let $F(z) = V \cap F[X]$. As $F(z) \in V$, we have that $f_i(z) = \pi_i \circ F(z) \in Y_i \setminus E$. Now "chasing through definitions" we claim $z \in U$ and then proceed to observe that $z \notin X \setminus U \implies f_i(z) \in f_i[X \setminus U]$ and $f_i(z) \in \text{cl}(f_i[X \setminus U]) = E$. □

So to ensure you get a topological embedding, ensure you separate points from points, then separate points from closed sets.

3.3 Regularity

Throughout this subsection, fix a topological space X.

We recall the definition of a regular space.

Definition 3.3.1 (Regular Space)**.** We say that X is **regular** if for all closed $F \subseteq X$ and $x \in X \setminus F$, there are $U, V \subseteq X$ that are open and contain x such that $F \subseteq V$ and $U \cap V = \emptyset$.

There is a strictly stronger notion as well.

Definition 3.3.2 (Complete Regularity)**.** We say X is **completely regular** if for all $F \subseteq X$ closed and $x \in X \setminus F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f[F] =\subseteq \{1\}$.

It makes complete sense for complete regularity to strictly generalise regularity. (See what I did there?) We do not show this here, but we reassure the reader that it is true.

In this section, we will explore how regularity and complete regularity interact with topological constructions.

3.3.1 Regularity and Subspaces

Fix a subset A of X, and consider it as a topological space, endowed with the subspace topology inherited from X. Recall that $B \subseteq A$ is closed with respect to the subspace topology if and only if $B = A \cap F$ for some $F \subseteq X$ closed.

It turns out that subspaces of regular spaces are themselves regular.

Lemma 3.3.3. *If X is regular, then so is A.*

Proof. Fix $B \subseteq A$ closed and fix $a \in A \setminus B$. We know that $B = A \cap F$ for some closed $F \subseteq X$. Since $a \notin B$, we must also have $a \notin F$. Then, since X is regular, we can find $U, V \subseteq X$ that are open and contain a such that $F \subseteq V$ and $U \cap V = \emptyset$. Then, $U \cap A$ and $V \cap A$ are relatively open subsets of A that also contain a and satisfy the properties that $B \subseteq V \cap A$ and $(U \cap A) \cap (V \cap A) = \emptyset$. $\square$

We can say something analogous about complete regularity.

Lemma 3.3.4. *If X is completely regular then so is A.*

Proof. Fix $B \subseteq A$ relatively closed and $a \in A \setminus B$. We know $B = A \cap F$ for some $F \subseteq X$ closed. We know there is some continuous $f : X \to [0,1]$ such that $f(a) = 0$ and $f[F] = \{1\}$. Consider its restriction $g = f \restriction A$. Obviously $g(a) = 0$ because $a \in A$. Moreover, since g agrees with f on A, it is also clear that $g[A] = f[A] \subseteq f[F] = \{1\}$.[1] □

Proposition 3.3.5. *The following are equivalent.*

(1) *X is regular.*

(2) *for all $x \in X$ and all open $U \ni x$, there exists an open $V \ni x$ such that $\operatorname{cl} V \subseteq U$.*

Proof. The picture we want to have in mind is given in Figure 3.1. `sorry` Finish!

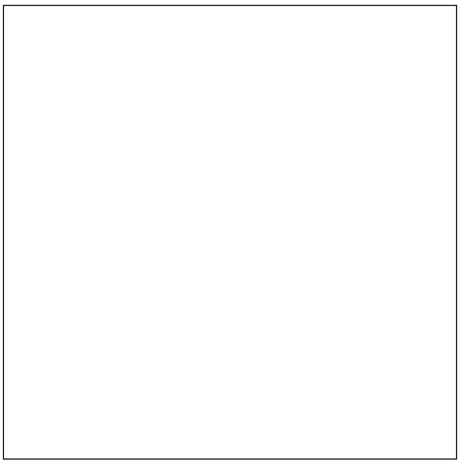

Figure 3.1: The picture we want to have in mind for Proposition 3.3.5.

(1) $\implies$ (2). Fix $x \in X$ and some open $U \ni x$. Evidently, $x \notin X \setminus U$, and since $X \setminus U$ is closed and X is regular, we can find some open $V, W \subseteq X$ such that $x \in W$, $X \setminus U \subseteq W$, and $W \cap V = \emptyset$. Then, the closure $\operatorname{cl} V \subseteq X \setminus W$, and this is contained in U.

(2) $\implies$ (1). Exercise. `sorry` Finish!

[1] I suppose the 'empty set brigade' might have some objections but honestly who even cares about such people... (I kinda do but I am choosing to ignore my feelings in the interest of convenience!)

□

3.3.2 Regularity and Products

Our eventual goal in this subsection will be to show that products of regular spaces are regular.

Throughout this subsection, fix topological spaces $(X_i)_{i\in I}$. Denote by X the product $\prod_{i\in I} X_i$.

Lemma 3.3.6. *If $(B_i)_{i\in I}$ are basic open sets of the X_i, then*

$$\prod_{i\in I} \operatorname{cl} B_i = \operatorname{cl} \prod_{i\in I} B_i$$

Proof. sorry □

We can now show that a product of regular spaces is regular.

Theorem 3.3.7. *If each X_i is regular, then so is X.*

Proof. Fix $x = (x_i)_{i\in I}$ lie in some open set $U \subseteq X$. We can write U_i as a product of open sets U_i, with only finitely many U_i being *properly* contained in X_i. sorry □

We can say something analogous for completely regular spaces.

Theorem 3.3.8. *If each X_i is completely regular, then so is X.*

Proof. Fix $x \in X$ and $F \subseteq X$ closed, and assume $x \notin F$. Let U be a basic open set containing x and disjoint from F. We know that there are open subsets $U_i \subseteq X_i$ for every $i \in I$ such that

$$U = \prod_{i\in I} U_i \qquad \text{and} \qquad A := \{i \in I \mid U_i \subsetneq X_i\} \text{ is finite}$$

We know that $F \subseteq X \setminus U$. Moreover, since $x \in U$, $x \notin X \setminus U$. Finally, note that

$$\begin{aligned} y \in U &\iff \text{For all } i \in I,\ y_i \in U_i \\ &\iff \text{For all } i \in A,\ y_i \in U_i \end{aligned}$$

Since X_i is completely regular, for each $i \in A$, we can find continuous functions $f_i : X_i \to [0,1]$ such that $f_i(x_i) = 0$ and $f_i(z) = 1$ for all $z \in X \setminus U_i$. We can then define

$$f : X \to [0,1] : y \mapsto \max_{i \in A} f_i(y_i)$$

Observe that $f(x) = 0$. For all $y \in F$, $y \in X \setminus U$, so there is some $i \in A$ such that $y_i \in X_i \setminus U_i$ for all i. We then see that $f_i(y_i) = 1$, so $f(y) = 1$. □

Definition 3.3.9 (Hilbert Cube)**.** Let I be a nonempty set. The "Hilbert cube" associated with I (i.e. "from I") is the space $[0,1]^I$, the product of copies of $[0,1]$ indexed by elements of I endowed with the product topology.

This is 'a potentially very large cube-like thingy'.

What properties of the unit interval would we expect the Hilbert Cube to have? Compactness? Hausdorffness? Thus, T_4? Thus $T_{3.5}$? All of the above, actually - and purely by results we have seen so far! Cool, eh?

We can ask ourselves what spaces can be embedded in Hilbert cubes. Indeed, we will be asking a lot of questions of this type over the course of this course.

Certainly we would want such a space to be $T_{3.5}$. Indeed, we know that subspaces of $T_{3.5}$ spaces are $T_{3.5}$. So being $T_{3.5}$ is necessary for being embeddable in the Hilbert Cube.

But astonishingly, it turns out that the converse is also true!

Theorem 3.3.10. *If X is a $T_{3.5}$ space, then X is embeddable into some Hilbert cube.*

Proof. Define $I := \{f : X \to [0,1] \mid f \text{ is continuous}\}$. Since X is $T_{3.5}$, I separates points and I separates points from closed sets—that is, for $x \in X$ and $C \subseteq X$ with $x \notin C$, there is some $f \in I$ such that $f(x) \notin \operatorname{cl} f[C]$.

An 'old theorem' says that if we define $H : X \to [0,1]^I : x \mapsto (f(x))_{f \in I}$, then H is the desired embedding. □

Add ref from earlier in course

3.4 Urysohn's Lemma

In this section, we develop the tools needed to prove Urysohn's Lemma, which (according to Professor Cummings) should really be a *theorem*.

It goes as follows.

Theorem 3.4.1 (Urysohn's Lemma). *If X is a normal topological space, for all closed $C, D \subseteq X$, if $C \cap D = \emptyset$ then there is a continuous $f : X \to [0, 1]$ such that $f[C] \subseteq \{0\}$ and $f[D] \subseteq \{1\}$.*

In some sense, normality tells us we can 'separate closed sets using open sets', and Urysohn's Lemma says that if we can do that, then we can also 'separate closed sets using continuous functions'.

For the remainder of this section, fix a normal topological space X. Our proof will be *constructive*: we will give an explicit function.

Lemma 3.4.2. *Let F be a closed subset of X and O an open subset of X. If $F \subseteq O$, then there exist $O_1, F_1 \subseteq X$ such that O_1 is open, F_1 is closed, and*

$$F \subseteq O_1 \subseteq F_1 \subseteq O$$

Proof. The proof is just an "irritating game involving naïve set theory".

If $F \subseteq O$, then $F \cap (X \setminus O) = \emptyset$. Normality then tells us that we have disjoint open sets $U, V \subseteq X$ such that $F \subseteq U$ and $X \setminus O \subseteq V$. Simply take $O_1 = U$ and $F_1 = X \setminus V$. $\square$

We will also briefly explain what the dyadic rationals are.

Definition 3.4.3 (Dyadic Rational). We say $q \in \mathbb{Q}$ is **dyadic** if there is some $c \in \mathbb{Z}$ and $i \in \mathbb{N}$ such that $q = \frac{c}{2^i}$.

The dyadic rationals are dense in $\mathbb{R}$: indeed, this is why real numbers admit binary expansions.

Local Notation. Denote by D the set of all dyadic rationals contained in $(0, 1)$.

We are now ready to prove Urysohn's Lemma.

Proof of Urysohn's Lemma (Theorem 3.4.1). We will proceed by constructing sets $\{U_q \mid q \in D\}$ such that

1. $C \subseteq U_q \subseteq X \setminus D$ for all $q \in D$.
2. If $p, q \in D$ and $p < q$ then $\operatorname{cl} U_p \subseteq U_q$.

To do this, we will actually build $\{U_q, V_q \mid q \in D\}$ such that for all $q \in D$,

1. U_q is open
2. V_q is closed
3. $V_q \subseteq U_r$ for some $r > q$

At this stage, it will be useful to draw a picture.

Here's a more formal way of describing the construction.

1. By Lemma 3.4.2, we can sorry
2. Then, given $C \subseteq U_{\frac{1}{2}}$, we can apply Lemma 3.4.2 to get $U_{\frac{1}{4}}$ open and $V_{\frac{1}{4}}$ closed such that
$$C \subseteq U_{\frac{1}{4}} \subseteq V_{\frac{1}{4}} \subseteq U_{\frac{1}{2}}$$
Similarly, since $V_{\frac{1}{2}} \subseteq X \setminus D$, we can again apply Lemma 3.4.2 to get $U_{\frac{3}{4}}$ open and $V_{\frac{3}{4}}$ such that
$$V_{\frac{1}{2}} \subseteq U_{\frac{3}{4}} \subseteq V_{\frac{3}{4}} \subseteq X \setminus D$$

It will be incredibly useful to bear in mind that for all dyadic rationals $p < q$, we get

$$C \subseteq U_p \subseteq X \setminus D \qquad \text{and} \qquad \operatorname{cl} U_p \subseteq U_{q^2}$$

We will now get on to the construction of our continuous function. To do so, we define $f : X \to [0, 1]$ such that, if there us no $r \in D$ with $x \in U_r$, then we let $f(x) := 1$ - otherwise, we define

$f(x) := \inf\{r \in D \mid x \in U_r\}$.

It is "easy" to see that if $x \in D$, then $f(x) = 1$, as $x \in C \implies f(x) = \inf(D) = 0$. However we must now get on to the trickier part...we claim that for all $x \in X$ and $p \in D$,

(a) If $x \in \operatorname{cl} U_p$, $f(x) \leq p$
(b) If $x \notin \operatorname{cl} U_p$, $f(x) \geq p$

Indeed, if $x \in \operatorname{cl} U_p$, then $x \in U_q$ for all $q \in D$ with $q > p$, so $f(x) \leq p$. On the other hand, if $x \notin \operatorname{cl} U_p$, then assume, for contradiction, that $f(x) < p$. Then we can easily see that there must be some $r < p$ and $x \in U_r$ such that $f(x) \leq r < p$. sorry

Taking the contrapositives of the above claims, we see that for all $x \in X$ and $p \in D$,

(a) If $f(x) > p$ then $x \notin \operatorname{cl} U_p$ (and thus $x \notin U_p$, so $f(x) \geq p$).
(b) If $f(x) < p$ then $x \in U_p$ (and thus $f(x) \leq p$).

Consider the following basis for the Euclidean (subspace) topology on $[0, 1]$:

$$\{[0, a)] \mid 0 < a\} \cup \{(a, b) \mid 0 \leq a < b \leq 1\} \cup \{(b, 1] \mid b < 1\}$$

We show f is continuous by showing that the pre-image in f of every basic open subset of $[0, 1]$ is open.

We begin by showing that $f^{-1}[(a, b)]$ is open. We do this by showing every point has an open neighbourhood. Fix $x \in f^{-1}[(a, b)]$. We know that $a < f(x) < b$. Find $p, q \in D$ such that $a < p < f(x) < q < b$. Then $f(x) > p$, so $x \notin \operatorname{cl} U_p$. Similarly, $f(x) < q$, so $x \in U_q$. Now, let $V = U_q \setminus \operatorname{cl} U_p$. Observe that $x \in V$. Moreover, for all $y \in V$, $y \in U_q$, so $f(y) \leq q$, and $y \notin U_p$, so $f(y) \geq p$. Then, we can see that $f[V] \subseteq [p, q] \subseteq (a, b)$.

So for $f^{-1}[[0, a)]$ we let $f(x) < a$ and find $p \in D$ with $f(x) < p < a$, $x \in U_p$ and $f[U_p] \subseteq [0, p] \subseteq [0, a)$. Meanwhile, for $f^{-1}[(b, 1]]$ we let $f(x) > b$ we find $q \in D$ such that $b < q < f(x)$ and the $x \notin \operatorname{cl} U_q$ and $y \in X \setminus \operatorname{cl} U_q$ then $y \notin U_q \implies f(y) \geq q > b$. $\square$

The good news is that we have a very rich supply of normal spaces. Indeed, we know that Metric spaces are Hausdorff and normal (T_4).

3.5 The Stone-Čech Compactification

Definition 3.5.1 (Compactification)**.** Given a space X, we say that X has the **compactification** Y if there is a space Y such that the following hold:

1. X is a subspace of Y.
2. X is dense in Y.
3. Y is compact.

3.5.1 One-Point Compactification

Definition 3.5.2 (One-Point Compactification)**.** Let X be a locally compact, non-compact Hausdorff space. Let $Y = X \cup \{\infty\}$, where ∞ is a formal symbol that does not lie in X, endowed with the topology τ consisting of

1. Open subsets of X.
2. Sets of the form $\infty \cup (X \setminus K)$, where $K \subseteq X$ is compact.

We call Y the **one-point compactification of** X. We sometimes denote Y by αX.

Think of projective space.

Suppose, as before, that X is a $T_{3.6}$ topological space. Define

$$I = \{f : X \to [0, 1] \mid f \text{ is continuous}\}$$

3.5.2 Constructing the Stone-Čech Compactification

Throughout this subsection, let X be a $T_{3.5}$/Tychonoff space. That is, X is Hausdorff and completely regular (cf. Definition 3.1.5). Write

$$\mathcal{F} := \{f : X \to [0, 1] \mid f \text{ is continuous}\}$$

Observe that $\mathcal{F}$ separates points (because X is Hausdorff) and also separates points from closed sets (because X is completely regular).

Let $Y = [0, 1]^{\mathcal{F}}$, endowed with the product topology. We know, by Tychonoff's Theorem (`sorry`), that Y is compact. We also know Y is Hausdorff. So Y is $T_{3.5}$ as well.

Consider $H : X \to Y : x \mapsto (f(x))_{f\in\mathcal{F}}$. We can show that H is an embedding of topological spaces, ie, that H is continuous and injective.

Definition 3.5.3 (Stone-Čech Compactification). The **Stone-Čech Compactification of** X, denoted βX, is the closure in Y of the image of X under H, where H is the embedding described above. Ie,

$$\beta X = \operatorname{cl} H[X]$$

From a categorical standpoint, the data of the Stone-Čech Compactification consists of both the space βX and the embedding $H : X \to \beta X$.

As one would hope from the name, βX is indeed compact. Moreover, βX contains $H[X]$, which is homeomorphic to X. So it does indeed make sense to call βX a compactification of X. As an added bonus, βX is Hausdorff.

It will also be sensible to show that the above compactification process behaves exactly as one would hope on spaces that are already compact: it "does nothing".

Let K be compact and Hausdorff. K is T_4, so K is $T_{3.5}$, and so one can define its Stone-Čech Compactification βK. Define

$$G := \{f : K \to [0,1] \mid g \text{ is continuous}\}$$

Consider the embedding $\overline{H} : K \to [0,1]^G : k \mapsto (g(k))_{g\in G}$. Then $\beta G = \operatorname{cl} \overline{H}[K]$.

Since K is compact and $\overline{H}$ is continuous, $\overline{H}[K]$ is compact. Moreover, $\overline{H}$ is a bijection from K to $\overline{H}[K]$, and $\overline{H}[K]$ is Hausdorff. Thus, the corestriction of $\overline{H}$ to its image is a continuous bijection from a compact space to a Hausdorff space, and thus a homeomorphism. So indeed, βK is homeomorphic to K.

3.5.3 The Universal Property of the Stone-Čech Compactification

It turns out that we can characterise the Stone-Čech Compactification using a universal property. Indeed, as one might guess, β is functorial, so it makes sense that we can do something categorical here...

The desired universal property turns out to be an **extension property**. What it says is that the Stone-Čech Compactification is universal in the sense of every compactification of X into a compact, Hausdorff space factors uniquely through βX.

Theorem 3.5.4 (Universal Property of the Stone-Čech Compactification)**.** *Let X be $T_{3.5}$ and let $H : X \to \beta X$ be the embedding described above. Let K be any compact, Hausdorff space and let $f : X \to K$ be any continuous function. Then there is a unique continuous $g : \beta X \to K$ such that $f = g \circ H$. That is, the following diagram commutes (in the category of topological spaces):*

$$\begin{tikzcd} \beta X \arrow[dr, dashed, "\exists! g"] & \\ X \arrow[u, hook, "H"] \arrow[r, "f"'] & K \end{tikzcd}$$

Proof. "I'm not doing this in the *most* bafflingly slick way, but I'm still doing it in a somewhat bafflingly slick way." "I have made everything maximally confusing—it's a gift."

As above, denote

$$\mathcal{F} := \{f : X \to [0,1] \mid f \text{ is continuous}\}$$

Then, $\beta X = \operatorname{cl} H[X]$ and $H(x) = (f(x))_{f \in \mathcal{F}}$.

1. Special Case: $K = [0,1]$. Fix a continuous map $f : X \to [0,1]$. We need to show there is a unique $g : \beta X \to [0,1]$ such that $g \circ H = f$. Since $f \in \mathcal{F}$ and $\beta X \subseteq [0,1]^{\mathcal{F}}$, we can define $g := \pi_f \restriction \beta X$, where π_f is the projection from $[0,1]^{\mathcal{F}}$ to the f coordinate. Obviously π_f is continuous and makes the triangle commute.

   It remains to show uniqueness. Indeed, the key is that $H[X]$ is dense in $\operatorname{cl} H[X] = \beta X$. So if we had $g' : \beta X \to K$ such that $g' \circ H = g \circ H$, then we would require g' and g to agree on the dense subset $H[X]$ of βX, so they must agree on all of βX.

2. General case: K is an arbitrary (Hausdorff) compactification of X. Fix a continuous $f : X \to K$. Denote by $\mathcal{G}$ the set of all continuous functions from K to $[0,1]$. For all $t \in \mathcal{G}$, we know $t \circ f \in \mathcal{F}$.

Define $G : \beta X \to [0,1]^{\mathcal{G}} : (u_f)_{f\in\mathcal{F}} \mapsto (u_{t\circ f})_{t\in\mathcal{G}}$.

First, we show that G is continuous. Denote by π_t the projection from $[0,1]^{\mathcal{G}}$ to the tth coordinate. One can show that $\pi_t \circ G = \pi_{t\circ f}$, and it follows readily from this that G is continuous.

Next, we show that $G[\beta X] \subseteq \operatorname{cl} \overline{H}[K]$. Fix $u = (u_s)_{s\in\mathcal{F}} \in \beta X$. We know that $G(u)$ is contained in some basic open subset of $[0,1]^{\mathcal{G}}$. Such sets are finite intersections of images of projections: that is, there exist $t_1, \ldots, t_n \in \mathcal{G}$ and open sets $V_1, \ldots, V_n \subseteq [0,1]$ such that

$$G(u) \in \bigcap_{i=1}^{n} \pi_{t_i^{-1}}[V_i]$$

Then, $u_{t_i\circ f} = \pi_{t_i}(G(u)) \in V_i$ for all $1 \leq i \leq n$. Thus,

$$u \in \bigcap_{i=1}^{n} \pi_{t_i\circ f}^{-1}[V_i]$$

Since $u \in \beta X = \operatorname{cl} H[X]$, we can find a point $x \in X$ such that $H(x) \in \bigcap_{i=1}^{n} \pi_{t_i\circ f}^{-1}[V_i]$. So for $1 \leq i \leq n$, $H(x)_{t_i\circ f} = (t_i \circ f)(x) \in V_i$.

"Our suffering is almost over." Let $k = f(x) \in K$. Then, $t_i(k) \in V_i$ for $1 \leq i \leq n$, so $\overline{H}(k) \in \bigcap_{i=1}^{n} \pi_{t_i^{-1}}[V_i]$. So $G(u) \in \operatorname{cl} \overline{H}[K]$, so $G[\beta X] \subseteq \operatorname{cl} \overline{H}[X] = \overline{H}[X] = \beta K$. We then find ourselves in the following situation:

$$\begin{tikzcd}
 & \beta X \arrow[d, dashed, "g"] \arrow[dr, "G"] & \\
X \arrow[ur, "H"] \arrow[r, "f"] & K \arrow[r, "\sim", "\overline{H}"'] & \overline{H}[K]
\end{tikzcd}$$

where for all $u \in \beta X$, we define $g(u)$ to be the unique $k \in H$ such that $\overline{H}(k) = G(k)$.

"Ok, I'm tired of this proof, you're probably also tired of this proof, so let's just check a few more details and move on."

We finish by verifying that g is continuous. sorry

∎

3.5.4 Applications of Stone-Čech Compactification

We already saw how the Stone-Čech Compactification behaves with compact spaces. It turns out to be interesting to explore how it behaves with locally compact spaces.

We begin with a fact (the proof of which is left as an exercise).

Exercise 3.5.5. If Y is compact and Hausdorff and X is an open subspace of Y, then X is locally compact.

Remark. Note that when X is an open subspace of Y, then the subspace topology on X consists exactly of those open sets (in the topology on Y) which are contained in X (trivially so, but still this may be a helpful fact)...

Theorem 3.5.6. *Let X be a $T_{3.5}$ space with $H : X \to \beta X$ the Stone-Čech Compactification, then TFAE:*

(1) *X is locally compact.*

(2) *$H[X]$ is open in βX.*

Proof.

(2) $\implies$ (1) Follows from Exercise 3.5.5.

(1) $\implies$ (2) If X is compact, then $H[X] = \beta X$. If X is not compact, let αX denote its one-point compactification, with inclusion $\iota : X \hookrightarrow \alpha X$. Recall that αX is compact and Hausdorff. The Universal Property (Theorem 3.5.4) then gives us some (uniquely defined) $g : \beta X \to X$ such that $g \circ H = \iota$.

$$\begin{tikzcd} \beta X \arrow[dr, dashed, "\exists! g"] & \\ X \arrow[u, hook, "H"] \arrow[r, "\iota"'] & K \end{tikzcd}$$

One can show that $\beta X \setminus H[X] = g^{-1}[\{\infty\}]$[2]. Since αX is Hausdorff, $\{\infty\}$ is closed, so $\beta X \setminus H[X]$ is closed, thus $H[X]$ is open.

[2] "I'd rather put that on the homework than show it in class" - Professor Cummings

□

3.6 Urysohn's Metrisation Theorem

Urysohn's Metrisation Theorem gives conditions under which we can construct a metric on a topological space that induces the same topology on it.

Theorem 3.6.1 (Urysohn's Metrisation Theorem). *Let X be a topological space. If X is both T_3 and second-countable, then X is metrisable.*

The way we will prove this theorem is by embedding X into a metrisable space, namely, $[0,1]^{\mathbb{N}}$. This is the same sort of object that comes up in the construction of the Stone-Čech Compactification.

It's not obvious that $[0,1]^{\mathbb{N}}$ really is metrisable - we will 'brush it under the rug' because it is 'a consequence of some nonsense on this week's homework'. The metric we define is

$$d(x,y) = \sum_{n\in\mathbb{N}} 2^{-n} |x_n - y_n|$$

and it isn't hard to show that this sum converges absolutely for all $x, y \in [0,1]^{\mathbb{N}}$.

We begin by proving some more general facts, which we will later relate to the proof of Urysohn's Metrisation Theorem (Theorem 3.6.1).

Recall the definition of a Lindelöf space.

Definition 3.6.2 (Lindelöf Space). A space X is **Lindelöf** if every open cover has a countable subcover.

It is not hard to show that any closed subset of a Lindelöf space is Lindelöf (when endowed with the subspace topology).

Lemma 3.6.3. *Any 2nd countable space is Lindelöf.*

Proof. `sorry` □

Lemma 3.6.4. *Any space that is both Lindelöf and T_3 is T_4.*

Proof. Let $A, B \subseteq X$ be closed and disjoint. Notice that both A and B (as subspaces) are Lindelöf themselves. We will use the hypothesis that X is T_3 to cook up open covers of A and B.

For all $a \in A$, we know that $a \notin B$, so $a \in X \setminus B$, which is an open set. Then there is some open $U_a \ni a$ such that $U \cap B = \emptyset$. This gives us an open cover of A, which we know admits a countable subcover $(U_{a_n})_{n \in \mathbb{N}}$. In similar fashion, we can construct an open cover $(V_{b_n})_{n \in \mathbb{N}}$.

Observe that for all $n \in \mathbb{N}$,

$$\mathrm{cl}(U_{a_n}) \cap B = \emptyset = \mathrm{cl}(V_{b_n}) \cap A$$

Now, define

$$Y_n := U_n \cap \left(\bigcap_{k=1}^{n} (X \setminus \mathrm{cl}(V_{b_n})) \right) \qquad Y := \bigcup_{n \in \mathbb{N}} Y_n$$

$$Z_n := V_n \cap \left(\bigcap_{k=1}^{n} (X \setminus \mathrm{cl}(U_{a_n})) \right) \qquad Z := \bigcup_{n \in \mathbb{N}} Z_n$$

The point is that for each $n \in \mathbb{N}$, since $A \cap \mathrm{cl}(V_{b_n})$, we know $A \subseteq X \setminus \mathrm{cl}(V_{b_n})$. Similarly,for each $n \in \mathbb{N}$, since $B \cap \mathrm{cl}(U_{a_n})$, we know $B \subseteq X \setminus \mathrm{cl}(U_{a_n})$. Thus,

$$A \subseteq \bigcap_{n \in \mathbb{N}} (X \setminus \mathrm{cl}(V_{b_n})) \qquad B \subseteq \bigcap_{n \in \mathbb{N}} (X \setminus \mathrm{cl}(U_{a_n}))$$

Moreover, it turns out that for all $m, n \in \mathbb{Z}$, $Y_m \cap Z_n = \emptyset$. Indeed, suppose this is not true. Then there are some $m, n \in \mathbb{N}$ (with $m \leq n$ WLoG) such that there is some $x \in Y_m \cap Z_n$. Since $x \in Y_m$, $x \in U_m$, and since $x \in Z_n$ and $m \leq n$, $x \in X \setminus \mathrm{cl}(U_m)$. But this is impossible. Thus, for all $m, n \in \mathbb{Z}$, $Y_m \cap Z_n = \emptyset$.

This implies that $Y \cap Z = \emptyset$. So X is T_4. □

We are now ready to prove Urysohn's Metrisation Theorem.

Proof of Theorem 3.6.1. Let X be a 2nd countable T_3 space. We need to show that X is metrisable.

By Lemma 3.6.3, if X is 2nd countable, then X is Lindelöf. To embed X into $[0,1]^{\mathbb{N}}$, it will suffice to find a countable set $\mathcal{F}$ of continuous functions from X to $[0,1]$ that separate points from closed sets.

Why? Well, first enumerate such a set $\mathcal{F}$ as functions $\{f_1, f_2, \ldots\}$ and define $F : X \to [0,1]^{\mathbb{N}} : x \mapsto (f_n(x))_{n\in\mathbb{N}}$. As $\mathcal{F}$ separates points from closed sets, sorry. Note that it is crucial that $\mathcal{F}$ is countable for this to work.

What?

So now let's get to work and actually find such a family!

We will make use of the fact that X is second countable. This means precisely that the topology of X has a countable basis B. For all pairs (U,V) with $U,V \in B$ and $\mathrm{cl}(U) \subseteq V$, we observe that $\mathrm{cl}(U) \cap (X \setminus V) = \emptyset$ So define $f_{U,V} : X \to [0,1]$ by sorry.

Is this right??

It just remains to show that $\mathcal{F}$ separates points from closed sets. Let $F \subseteq X$ be closed. Fix some $x \in X \setminus F$. Clearly $X \setminus F$ is open, so there is some $V \in B$ such that $x \in V$ and $V \subseteq X \setminus F$ (meaning $V \cap F = \emptyset$). By regularity, we know that there is an open set U' that is a neighbourhood of x whose closure is contained in V. Ie, we have $x \in U'$ and $\mathrm{cl}(U') \subseteq V$. Indeed, we can find a *basic* $U \in B$ such that $x \in U \subseteq U'$ and $\mathrm{cl}(U) \subseteq \mathrm{cl}(U') \subseteq V$.

which follows from...?

sorry □

This was a bit of a slog, so let's break it down into its component parts and understand the weird hypotheses.

First, you have a modest separation property, T_3, and a strong countability property, 2nd countability. And really the weaker consequence of being Lindelöf is what makes the argument work, because that gives us T_4, which is comforting because metric spaces are T_4. We then ned to use all our ingredients to construct a clever family of functions that separate points from closed sets, and this allows us to embed our space into $[0,1]^{\mathbb{N}}$, showing metrisability.

There are even sharper metrisability theorems. This theorem works great, as far as it goes, but the hypotheses are overkill. In particular, they're not *equivalent* to the property of metrisability. A much better theorem is the Nagata-Smirnov theorem, which gives sufficient *and* necessary conditions for metrisability. (Of course, an obvious equivalent condition to metrisability *is just metrisability itself.*

We're talking about equivalent conditions that are actually *helpful*/that occur not uncommonly.)

Chapter 4

Topologies on Sets of Functions

In this chapter of the course, we will venture into the wonderfully weird realm of function spaces and functional analysis. We will talk about Hilbert spaces, operators on Hilbert spaces, and topologies on spaces of operators on Hilbert spaces.

If you look at the Wikipedia page on topologies on spaces of operators on Hilbert spaces, there are about 10 different topologies mentioned. So this is quite a rich theory, and we will explore it in detail.

We begin with set-theoretic notation (which we really should've introduced earlier—unless we have—because we've kinda been using it all along...).

Notation. For sets X and Y, denote $Y^X := X \to Y$, the set of functions from X to Y.

We begin with some motivations and first examples.

4.1 Motivation and First Examples

We begin by asking ourselves what kinds of topologies we can endow sets of functions with. We will be particularly interested in metrisable topologies and even more so in completely metrisable topologies. We have already seen one very natural topology in cases where X is an arbitrary set and Y has a topology. We will investigate this further and study completeness properties in the process.

4.1.1 The Topology of Pointwise Convergence

Let Y be a topological space and X an arbitrary set.

Definition 4.1.1 (The Topology of Pointwise Convergence)**.** The **topology of pointwise convergence** on Y^X is defined to be the product topology on Y^X.

We can give a characterisation of this topology in terms of nets.

Exercise 4.1.2. Let $(f_a)_{a\in\mathbb{D}}$ be a net in Y^X. For all $f \in Y^X$, TFAE:

(1) $f_a \to f$

(2) For all $x \in X$, $f_a(x) \to f(x)$

Note that (2) is a sensible statement to make because for all $x \in X$, $(f_a(x))_{a\in\mathbb{D}}$ is a net in Y.

The big advantage of this topology is that it is very general: it does not require X to have any topological structure whatsoever.

But pointwise convergence is not the strongest notion of convergence out there. We can define uniform convergence of nets, and then define a topology of uniform convergence.

4.1.2 The Topology of Uniform Convergence

Consider the following setup.

Let X be a set and let (Y, d) be a metric space. Define a new metric $\overline{d}$ on Y by

$$\overline{d}(y_1, y_2) := \begin{cases} d(y_1, y_2) & \text{if } d(y_1, y_2) < 1 \\ 1 & \text{if } d(y_1, y_2) \geq 1 \end{cases} = \min(d(y_1, y_2), 1)$$

One way to view this new metric space is as the original metric space ‘squashed’ to a ball of radius 1. It is a ‘trivial exercise’ to show that this is a metric.

Note that (Y, d) and $(Y, \overline{d})$ have the same open sets. As a result, one can show they also have the same Cauchy sequences and the same convergent sequences.

Along these lines, we can define a 'uniform metric' and 'uniform topology' on Y^X.

Definition 4.1.3 (The Topology of Uniform Convergence)**.** Let X be a set and (Y, d) be a metric space. Denote by $\overline{d}$ the metric described above. The **uniform metric** is defined for all $f_1, f_2 \in Y^X$ by

$$d_{\text{uniform}}(f_1, f_2) := \sup \left\{ \overline{d}(f_1(x), f_2(x)) \,\middle|\, x \in X \right\}$$

We call the induced topology the **uniform topology** or the **topology of uniform convergence**.

The reason for this terminology is that a sequence of functions converges with respect to the uniform metric obtained from d if and only if it converges uniformly with respect to d.

It turns out that the uniform metric construction preserves completeness.

Proposition 4.1.4. *If (Y, d) is a complete metric space and X is an arbitrary set, then* $\left(Y^X, d_{\text{uniform}}\right)$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be Cauchy in $\left(Y^X, d_{\text{uniform}}\right)$. It is easy to see that for all $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent in (Y, d). Since this metric space is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges to some limit. We can therefore define

$$f : X \to Y : x \mapsto \lim_{n\to\infty} f_n(x)$$

ie, we take f to be the pointwise limit of the $(f_n)_{n\in\mathbb{N}}$. We will show that in (Y, d_{uniform}), the (Cauchy) sequence $(f_n)_{n\in\mathbb{N}}$ converges to f.

Fix $\varepsilon > 0$ and assume that $\varepsilon < 1$[1]. Since $(f_n)_{n\in\mathbb{N}}$ is Cauchy, we can find some $N \in \mathbb{N}$ such that for all $n_1, n_2 \geq N$, $d_{\text{uniform}}(f_{n_1}, f_{n_2}) < \infty$. Then, by definition of the uniform metric, for all $x \in X$, $d(f_{n_1}(x), f_{n_2}(x)) < \varepsilon$.

[1]We don't need to do this right away, but it makes the rest of the argument significantly simpler because d_{uniform} is defined in terms of $\overline{d}$, and $\overline{d}$ always takes values ≤ 1.

Send $n_2 \to \infty$. Then, $f_{n_2}(x) \to f(x)$. Since d is continuous, we can see that $d(f_{n_1}(x), f(x)) \leq \varepsilon$. Since this is true for all $x \in X$, it follows that $d_{\text{uniform}}(f_{n_1}, f) \leq \varepsilon$ for all $n_1 \geq N$. This is enough, because the real numbers "have lots of room" - so even if we "replace the OG epsilon by a smaller one" we are fine! Yay :) □

We now move to a slightly different context.

Notation. For all topological (and metric) spaces X and Y, denote by $C(X, Y)$ the set of all continuous functions from X to Y.

For the remainder of this subsection, let X be a topological space and let (Y, d) be a metric space. Since $C(X, Y)$ is a subset of Y^X and d_{uniform} is a metric on Y^X, we can restrict d_{uniform} to $C(X, Y)$ and thus view $(C(X, Y), d_{\text{uniform}})$ as a metric space in its own right.

The following "is like a really important fact".

Theorem 4.1.5. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C(X, Y)$, and assume that there is some $f \in Y^X$ such that $f_n \to f$ with respect to d_{uniform}. Then, f is continuous, that is, we can view f as an element of $C(X, Y)$.*

Proof. To show that f is continuous at all $x \in X$, we let $\varepsilon > 0, \varepsilon < 1$, and n such that $d_{unif}(f_n, f) < \varepsilon/3$ - note the $\varepsilon/3$ providing evidence that we are thinking ahead! We now know that f_n is continuous at x, so we find some open set U containing x for which we have $d(f_n(x), f_n(x')) < \varepsilon/3$, and also $d(f_n(x), f_n(x')) < \varepsilon/3$ for all $x, x' \in U$. Noting now that $d(f(x), f(x')) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x')) + d(f(x'), f_n(x')) \leq 3 \cdot \varepsilon/3 = \varepsilon$. □

The above is best summarised by saying that **a uniform limit of continuous functions is continuous**.

Corollary 4.1.6. *Let X be an arbitrary topological space and let (Y, d) be a complete metric space. The metric space $(C(X, Y), d_{\text{uniform}})$ is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(X, Y)$. As before, if we define

$$f : X \to Y : x \mapsto \lim_{n\to\infty} f(x)$$

then we know that $f_n \to f$ in $\left(Y^X, d_{\text{uniform}}\right)$. Theorem 4.1.5 then tells us that f is continuous, so $f_n \to f$ in $(C(X, Y), d_{\text{uniform}})$. □

4.1.3 An Extension Property for Normal Spaces

The uniform metric has many applications. In this subsection, we will investigate one of them: an extension property for normal spaces.

Theorem 4.1.7 (Tietze Extension Theorem). *Let X be a normal space and let $A \subseteq X$ be closed. Let $f : A \to \mathbb{R}$ be continuous. There exists some $g : X \to \mathbb{R}$ continuous such that $g \restriction A = f$.*

> *"The zen of the proof is that I'm going to reduce the complexity of the situation just a little bit, and then build, pretty much by hand, a Cauchy sequence of continuous functions that converge with respect to the uniform metric. I'm then going to take a limit and get the function that I want."*

Proof. Let $h : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Replacing f by $h \circ f$, we may assume that f is bounded. For any real-valued function g, define the following notation:

$$\|g\|_A := \sup_{a\in A} |g(a)|$$
$$\|g\|_X := \sup_{x\in X} |g(x)|$$

We construct a sequence of functions $(f_n)_{n\in\mathbb{N}}$, the limit of which will be our candidate.

Define $f_0 : X \to \mathbb{R}$ by $f_0 = 0$. Find $c_0 > 0$ such that $\|f - f_0\|_A = \|f\|_A \leq c_0$. This is something we can do because f is bounded. We will proceed by recursion.

Suppose we have found a continuous function $f_n : X \to \mathbb{R}$ such that $\|f - f_n\|_A \leq c_n$. Define the following disjoint subsets of A:

$$A_n^- := \left\{ a \in A \;\middle|\; f(a) - f_n(a) \in \left[-c_n, -\frac{c_n}{3} \right] \right\}$$
$$A_n^+ := \left\{ a \in A \;\middle|\; f(a) - f_n(a) \in \left[\frac{c_n}{3} \right], c_n \right\}$$

Since A is closed and $f - f_n$ is continuous, A_n^- and A_n^+ are both disjoint, closed subsets of X. Since X is normal, we can apply Urysohn's Lemma (Theorem 3.4.1) to find some continuous $\phi_n : X \to [0,1]$ with $\phi_n \restriction A_n^- = 0$ and $\phi_n \restriction A_n^+ = 1$.

Define[2]

$$g_n := \frac{2c_n}{3}\phi_n - \frac{c_n}{3}$$

Observe that on A_n^-, g_n is constant with value $-\frac{c_n}{3}$, and on A_n^+, g_n is constant with value $\frac{c_n}{3}$. Indeed, $g_n(x) \in \left[-\frac{c_n}{3}, \frac{c_n}{3}\right]$ for all $x \in X$. That is, $\|g_n\| \leq \frac{c_n}{3}$.

Let $f_{n+1} := f_n + g_n$. Let's examine how $f - f_{n+1}$ behaves on A. Indeed, $f - f_{n+1} = (f - f_n) - g_n$. For all $a \in A$, we can see that $|f(a) - f_{n+1}(a)| \leq \frac{2c_n}{3}$, so indeed $\|f - f_{n+1}\|_A \leq \frac{2c_n}{3}$.

We can then perform the recursive construction choosing $c_{n+1} = \frac{2c_n}{3}$, so we can see that for all $n \in \mathbb{N}$, $c_n = \left(\frac{2}{3}\right)^n c_0$.

By an 'easy calculation' one can show that $(f_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the uniform metric. Theorem 4.1.5 then tells us that this limit, denoted $g : X \to \mathbb{R}$, is indeed continuous. Since $\|f - f_n\|_A \leq c_n$ for all $n \in \mathbb{N}$ and $c_n \to 0$ as $n \to \infty$, we can conclude that $g \restriction A = f$. □

Note that it is **vital** that A be closed in the above theorem. If this is not true, the conclusion is no longer true.

Counterexample 4.1.8. Let $X = S^1$, the circle. Consider a point $p \in X$ and define $A := X \setminus \{p\}$.

[2]This is the sort of thing that would *not* feature in Anand/Tim's database of motivated proofs...

4.2 Families of Functions

Professor Cummings, in his own words, is about to bore us with a long list of topologies. But I highly doubt they'll be boring... something tells me they'll be quite the opposite!

4.2.1 Equicontinuity

Throughout this subsection, fix a topological space X and a metric space (Y, d).

Definition 4.2.1 (Equicontinuity)**.** Fix a family $\mathcal{F} \subseteq Y^X$.

1. For some point $x_0 \in X$, the family $\mathcal{F}$ is **equicontinuous at** x_0 iff for all $\varepsilon > 0$, there is an open neighbourhood $U \ni x_0$ such that for all $f \in \mathcal{F}$ and $x \in U$, $d(f(x_0), f(x)) < \varepsilon$.
2. $\mathcal{F}$ is **equicontinuous** if $\mathcal{F}$ is equicontinuous at all points in X.

Note that equicontinuity is **not the same notion as uniform continuity**: we are describing a family of functions *collectively* when we talk about equicontinuity. Moreover, they're continuous "in the same way" (ie, for the same ε, the same conclusions hold *simultaneously* for *all of them*).

Here is a motivating theorem.

Theorem 4.2.2. *Fix a family $\mathcal{F} \subseteq C(X, Y)$ of continuous functions from X to Y. If $\mathcal{F}$, viewed as a sub-metric space of the space $(Y^X, d_{\text{uniform}})$, is totally bounded, then $\mathcal{F}$ is equicontinuous.*

One may recall we had "a rather arduous" discussion of compactness in metric spaces, which involved discussion of "total-boundedness" (which stated that for any fixed $\varepsilon > 0$, we could cover our space with finitely many ε-balls - this condition can be seen to capture the idea of finite volume). Now we can view $F \subseteq C(X, Y)$ with $C(X, Y)$ considered as a metric space with the uniform metric, and then if F is totally bounded in the uniform metric, our F must be equicontinuous.

Proof of Theorem 4.2.2. "This is one of those proofs that writes itself as long as you're a bit careful and do things in the right order."

Fix $0 < \varepsilon < 1$ (we are working in the uniform metric, so we can take $\varepsilon < 1$). Fix an arbitrary point $x_0 \in X$. We will show that $\mathcal{F}$ is equicontinuous at x_0.

Let $\delta = \frac{\varepsilon}{3}$. Note that $0 < \delta < 1$ also. Since $\mathcal{F}$ is totally bounded, $\mathcal{F}$ is expressible as a finite union of δ-balls: there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{n} B_{d_{\text{uniform}}}(f_i, \delta)$$

The key to uniform boundedness is that we get a finite family that covers $\mathcal{F}$. We can now do the following.

For each i, f_i is continuous at x_0, so we can find open neighbourhoods $U_i \ni x_0$ such that for all $x \in U_i$, $d(f_i(x_0), f_i(x)) < \delta$, with d being the metric on Y.

Let $U = \bigcap_{i=1}^{n} U_i$. We know that U is open and that $x_0 \in U$ because $x_0 \in U_i$ for every $i \in \{1, \ldots, n\}$. We're now ready to show that $\mathcal{F}$ is equicontinuous at x_0.

We now consider an arbitrary function $f \in \mathcal{F}$ and an arbitrary $x \in U$, and find $1 \leq i \leq n$ such that $d_{\text{uniform}}(f, f_i) < \delta$. For our fixed $x \in U$, we know that

$$\begin{aligned} d(f_i(x), f_i(x_0)) &< \delta \\ d(f(x), f_i(x)) &< \delta \\ d(f(x_0), f_i(x_0)) &< \delta \end{aligned}$$

Applying the triangle inequality, we can see that

$$d(f(x), f(x_0)) \leq d(f_i(x), f_i(x_0)) + d(f(x), f_i(x)) + d(f(x_0), f_i(x_0)) < 3\delta = \varepsilon$$

Since $\varepsilon < 1$, $d(f(x), f(x_0)) = \overline{d}(f(x), f(x_0))$, and we're done. $\square$

Recall that the *product topology* on Y^X (where Y is a space and X is a set) is called the "**topology of pointwise convergence**" (cf. Definition 4.1.1). We also recall that the *metric topology* on Y^X (where Y a metric space and X a set) which is induced by the uniform metric is called the "**topology of uniform convergence**" (cf. Definition 4.1.1).

4.2.2 A Study in Compactness

It's time for another topology! After all, there can never be enough topologies...

As usual, let X be a topological space and let (Y, d) be a metric space. A word of warning: the following definition is one that'll "leave a bit of work to you."

Definition 4.2.3 (The Topology of Compact Convergence). The **topology of compact convergence** on Y^X is the topology generated by the basic open sets

$$B_C(f, \varepsilon) := \left\{ g \in Y^X \;\middle|\; \sup_{x \in C}(d(f(x), g(x))) < \varepsilon \right\}$$

indexed by

- Compact sets $C \subseteq X$
- Functions $f \in Y^X$
- $\varepsilon > 0$

The reason this definition requires some work is that we need to show that the above 'basic open sets' do, indeed, form a basis for some topology on Y^X, by showing the appropriate covering property and the appropriate intersection property. This is an "amusing little exercise."

Exercise 4.2.4 (An Amusing Little Exercise using Properties of Compact Sets). Show that the sets $B_C(f, \varepsilon)$ defined in Definition 4.2.3 do, indeed, form the basis of a topology.

This is useful because in complex analysis, we know that if sequences of holomorphic functions converge with respect to this topology of compact convergence, their limit is holomorphic too.

Definition 4.2.5 (Compactly Generated Space). We say that the space X is "compactly generated" to mean that for all $A \subseteq X$, A is open in X exactly when, for all compact $C \subseteq X$ we have $A \cap C$ relatively open in C.

One can think of a compactly generated space as a space in which the topological properties of our space are captured by its compact subspaces.

Proposition 4.2.6. *If X is locally compact or first-countable, then X is compactly generated.*

Proof. We show each case separately.

Case 1: X is locally compact.

Let $A \subseteq X$ be such that $A \cap C$ is relatively open for all compact $C \subseteq X$. Fix $x \in A$. Since X is locally compact, there is some compact set $C \subseteq X$ and some open set $U \subseteq C$ such that $x \in U$. $A \cap C$ is relatively open in C, so $A \cap U$ is open. Thus, A is open.

Case 2: X is first-countable.

Let $B \subseteq X$ be such that $B \cap C$ is relatively closed for all compact $C \subseteq X$. It will now suffice to show that B is closed. To do so, we let $x \in \mathrm{cl}(B)$, and let $(U_n)_{n \in \mathbb{N}}$ be a countable neighbourhood basis of open sets - by a little topological joke, we observe that $V_n = \cap_{i \leq n} U_i$ will also form a (nested) countable neighbourhood basis for x. We now let $x_n \in B \cap V_n$ for each $n \in \mathbb{N}$, and note that the sequence of (x_n) converges to x - finish the proof as an exercise? `sorry`

$\square$

"by an ancient and seemingly pointless homework exercise" "I'm getting close to the danger zone here [running out of time] - EASY!"

Appendices

In this chapter, we add a few appendices that are relevant to the main text.

A A Categorical Perspective

A.1 Product Spaces

B Professor Cummings's Top(ological) Tips

1. Follow your nose!

2. Proof by picture always works!

3. The surface of this blackboard is a *super* present topological space - it is a closed subset of $\mathbb{R}^2$!

4. Sometimes, pictures can be helpful; sometimes, they can be totally misleading. The more sets there are, the worse the pictures become...

5. You should always check what you want to do in your heart, but you should also check that what you want to do in your heart works.

6. Just figure it out.

7. Pictures can be seductively useful...

8. In a topological setting, you often don't really care too much about boundedness.

9. You really want to read the fine print when I'm defining these metrics and topologies...

10.

Visit https://thefundamentaltheor3m.github.io/TopologyNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to the associated GitHub repository.