Information Retrieval

WS 2017 / 2018

Lecture 4, Tuesday November 14th, 2017
(Compression, Codes, Entropy)

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Overview of this lecture

- Organizational
  - Your experiences with ES3 Efficient List Intersection
- Compression
  - Motivation saves space **and** query time
  - Codes Elias, Golomb, Variable-Byte
  - Entropy Shannon's famous theorem
  - **Exercise Sheet 4:** three nice proofs → part of Shannon's theorem + optimality of Golomb + size of inverted index

    We take a break from implementation work this week

Experiences with ES3 1/2

- Summary / excerpts
  - Very interesting topic and exercise (performance tuning)
  - First-time contact with Java or C++ for some
  - Large variations in runtimes between runs, especially in Java

    Reasons: JIT Compiler, Garbage Collection, Caching effects
  - It was surprisingly (for many) hard to beat the baseline

    "Whenever I tried an improvement I got a worse result"

    That is, in fact, an important lesson: it can be very hard to beat a well-implemented baseline, even with an algorithm that is asymptotically much better (but also more complex)
  - "The dude never actually bowls in the whole movie"

Experiences with ES3 2/2

Query film+bowling+rug:

The Big Lebowski

Fact "that rug really tied the room together"

mentioned four times

- Results
  - Three inverted lists of different lengths

film	1,983,510 postings
bowling	131,068 postings
rug	5,132 postings

  - Query **film+rug**, list length ratio **386**

    Any of galloping, skip ptrs, bin. search give large speedup
  - Query **bowling+rug**, list length ratio **25**

    Skipping helps, but not too much
  - Query **film+bowling**, list length ratio **15**

    Skipping costs more than it helps, switch to tuned baseline

Compression 1/6

- Motivation
  - Inverted lists can become very large

    Recall: the length of an inverted list of a word = total number of occurrences of that word in the collection

  - In a web-scale collection, the size can be millions, tens of millions, or hundreds of millions

    Just try a few single-keyword queries on Google and note the (estimated) number of results

algorithm	about 161,000,000 results
retrieval	about 661,000,000 results

  - Compression potentially saves space **and** time … the next slides will explain why

Compression 2/6

- Index in **memory**
  - Then compression saves memory (obviously)
  - Also: the index might be too large to fit into memory without compression, and with compression it does
  - Fitting in memory is good because reading from memory is much much **much** faster than reading from disk

    Transfer rate from memory ≈ 2 – 6 GB / sec

    Transfer rate from disk ≈ 50 – 500 MB / sec

    On a single disk, more like 50 MB / sec, for much higher bandwidths parallel disks are needed

Compression 3/6

- Index on **disk**:
  - Then compression saves disk space (obviously)
  - But it also saves query time, here is a realistic example:

Disk transfer time:	50 MB / second	
Compression rate:	Factor 5	
Decompression time:	30 MB / second	
Inverted list of size:	50 MB	
Reading uncompressed:	1.0 seconds	→ 50 MB
Reading compressed:	0.2 seconds	→ 10 MB
Decompressing:	0.3 seconds	→ 50 MB

Reading compressed + decompression **twice faster** compared to reading uncompressed

Compression 4/6

ids $> 2^{32} \approx 4B$
documents

- Gap encoding
  - Example inverted list (doc ids only):

    3, 17, 21, 24, 34, 38, 45, ..., 11876, 11899, 11913, ...
  - Numbers small in the beginning and large in the end, using an **int** requires **4 bytes** or even **8 bytes** per id
  - Alternative: store differences from one item to next:

    +3, +14, +4, +3, +10, +4, +7, ..., +12, +23, +14, ...
  - This is called **gap encoding**
  - Works as long as we process the lists from left to right
  - Now we have a sequence of mostly (but not always) small numbers ... how do we store these in little space?

Compression 5/6

- Binary representation
  - We can write number x in binary using $\lfloor \log_2 x \rfloor + 1$ bits

x	binary	number of bits	
1	1	1	$\lfloor \log_2 1 \rfloor + 1 = 1$
2	10	2	$\lfloor \log_2 2 \rfloor + 1 = 2$
3	11	2	$\lfloor \log_2 3 \rfloor + 1 = 2$
4	100	3	$\lfloor \log_2 4 \rfloor + 1 = 3$
5	101	3	

  - This encoding is optimal in a sense … see later slides
  - So why not just (gap-)encode like this and concatenate:

    +3, +14, +4, … → 11, 1110, 100, … → 111110100…

Compression 6/6

- Prefix-free codes, definition
  - Decode bit sequence from the last slide: 111110100

    This could be: +3, +14, +4 → 11, 1110, 100

    Could also be: +7, +6, +4 → 111, 110, 100

    Or: +3, +3, +2, + 4 → 11, 11, 10, 100
  - Problem: we have no way to tell where one code ends and the next code begins

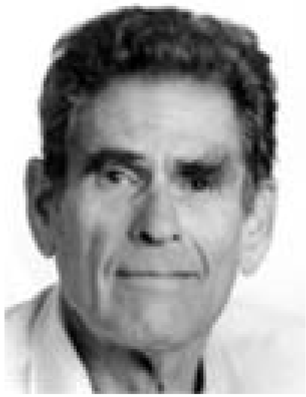
    Equivalently: some codes are prefixes of other codes
  - In a **prefix-free code**, no code is a prefix of another

    Then decoding from left to right is unambiguous !

Codes 1/5

- Elias-Gamma ... from 1975
  - Write $\lfloor \log_2 x \rfloor$ zeros, then x in binary like on slide 9
  - Prefix-free, because the number of initial zeros tells us exactly how many bits of the code come afterwards
  - Code for x has a length of exactly $2 \cdot \lfloor \log_2 x \rfloor + 1$ bits

1	1
2	010
3	011
4	00100
⋮	
10	0001010



Peter Elias
1923 – 2001

Codes 2/5

+1 because there is
no code for 0 in Elias-Gamma

- Elias-Delta ... also from 1975
  - Write $\lfloor \log_2 x \rfloor + 1$ in Elias-Gamma, followed by x in binary (like on slide 9) but **without** the leading 1
  - Prefix-free because the (prefix-free) Elias-Gamma code tells us exactly how many bits of the code come afterwards
  - Code for x has $\lfloor \log_2 x \rfloor + 2 \log_2 \log_2 x + O(1)$ bits

1	1
2	010 0
3	010 1
4	011 00
⋮	
10	00100 010

ELIAS-GAMMA

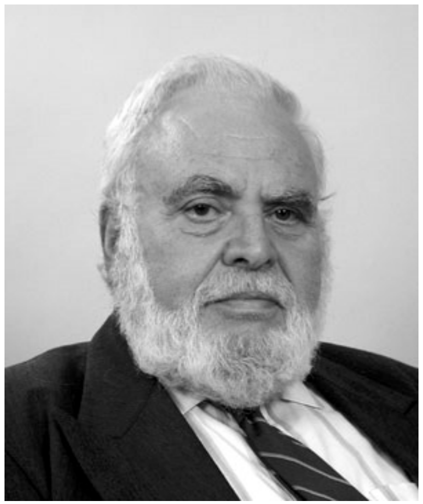
1	1
2	010
3	011
4	00100
⋮	⋮

Codes 3/5

q = quotient r = remainder

- Golomb (not Gollum) ... from 1966
  - Comes with an integer parameter M, called **modulus**
  - Write x as q · M + r, where q = x div M and r = x mod M
  - The code for x is then the concatenation of:
    - q written in unary with 0s $\lfloor \frac{x}{M} \rfloor$ bits
    - a single 1 (as a delimiter) 1 bit
    - r written in binary $\lceil \log_2 M \rceil$ bits

Solomon Golomb
1932 – 2016



$M = 16$, $x = 42$

$q = 42$ div $16 = 2 = 00$ in unary

$r = 42$ mod $16 = 10 = 1010$ in binary

Golomb code is 0 0 1 1 0 1 0

Codes 4/5

- Variable-Byte (VB)
  - Idea: use **whole bytes**, in order to avoid the (expensive) bit fiddling needed for the previous schemes

    VB is often used in practice, for exactly that reason

    In particular, for UTF-8 encoding … see Lecture 7
  - Use one bit of each byte to indicate whether this is the last byte in the current code or not

$x = 542$; x div $128 = 4 = 100$ in binary

x mod $128 = 30 = 11110$ in binary

1	0000100

Byte 1

says that there are more bytes to come

0	0011110

Byte 2

says that this is the last byte of the code

The "code point" is 0000100 : 0011110 which is 542 in binary

Codes 5/5

- Other codes
  - There is a huge variety of codes with different trade-offs wrt space reduction, compression speed, decompression speed
  - One particularly interesting recent coding scheme is called

    Asymmetric numeral systems (ANS) ... Duda et al, PCS'14
  - Close to optimal space reduction **and** fast (de)compression
  - It is actively used at Facebook, Apple, Google, ...
  - The basic idea is to encode a **whole message** (not just individual symbols) into a single (then fairly big) **integer**
  - It is an intricate scheme which, interestingly, is much easier to implement than to understand mathematically

    Check out the Wikipedia article if you are interested

Entropy 1/12

- Motivation
  - Which code compresses the best ?

    It depends !

    But on what ?
  - Roughly: it depends, on the relative frequency on the numbers / symbols we want to encode

    For example, in natural language, an "e" is much more frequent than a "z"

    So we should encode "e" with less bits than "z"
  - The next slides will make this more precise

Entropy 2/12

$H(X) = -\sum_{i=1}^{m} \frac{1}{m} \log_2 \frac{1}{m}$

$= -m \cdot \frac{1}{m} \cdot \log_2 \frac{1}{m} = \log_2 m$

- Definition
  - Entropy of a discrete random variable **X**

    Without loss of generality range of X = {1, ..., m}

    Think of X as generating the symbols of a message
  - Then the **entropy of X** is written and defined as

    $H(X) = - \Sigma_i \, p_i \log_2 p_i$ where $p_i = \mathrm{Prob}(X = i)$
  - We will see: H(X) is the optimal number of bits to encode a random symbol generated according to X
  - Example: all $p_i = 1/m$, then $H(X) = \log_2 m$

    If all m symbols are equally likely, the best encoding is the standard encoding with $\log_2$ m bits / symbol

Entropy 3/12

- Shannon's source coding theorem ... from 1948
  - Let X be a random variable with finite range
  - For an arbitrary prefix-free (**PF**) encoding, let L_i be the length of the code for $i \in \text{range}(X)$

    (1) For any PF encoding it holds: $\mathbf{E}\, L_X \geq H(X)$

    (2) There is a PF encoding with: $\mathbf{E}\, L_X \leq H(X) + 1$

    where $\mathbf{E}$ denotes the expectation
  - In words:

    No code can be better than the entropy, and there is always a code that is (almost) as good

Claude Shannon
1916 – 2001

Entropy 4/12

- Central Lemma ... to prove the source coding theorem
  - Denote by L_i the length of the code for the i-th symbol, then

    (1) Given a PF code with lengths L_i ⇨ $\Sigma_i\, 2^{-L_i} \leq 1$

    (2) Given L_i with $\Sigma_i\, 2^{-L_i} \leq 1$ ⇨ exists PF code with length L_i
  - Note: $\Sigma_i\, 2^{-L_i} \leq 1$ is known as "Kraft's inequality"
  - Intuitively: not all L_i can be small ... small L_i → large 2^{-L_i}

    For example, the lemma says that a prefix-free code where three $L_i = 1$ is not possible, because $2^{-1} + 2^{-1} + 2^{-1} > 1$

Entropy 5/12

- Proof of central lemma, part (1)
  - Show: given prefix-free code with lengths L_i then $\Sigma_i\ 2^{-L_i} \leq 1$
  - Generate a random binary sequence as follows:

    1. Pick one bit after the other, and independently

    2. Stop when you have a valid code, or when no more code is possible (= no code starts with that bit sequence)
  - Let C_i = the event that code i is generated
  - Observation: for prefix-free code, the C_i are **independent**
  - Thus $\underbrace{\Pr(C_1)}^{2^{-L_1}} + \ldots + \underbrace{\Pr(C_m)}^{2^{-L_m}} = \Pr(C_1 \cup \ldots \cup C_m) \leq 1$

    holds because $C_1, \ldots, C_m$ are independent
  - We also have $\Pr(C_i) = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2}}_{L_i \text{ times}} = 2^{-L_i}$
  - This proves $\Sigma_i\ 2^{-L_i} \leq 1$

Entropy 6/12

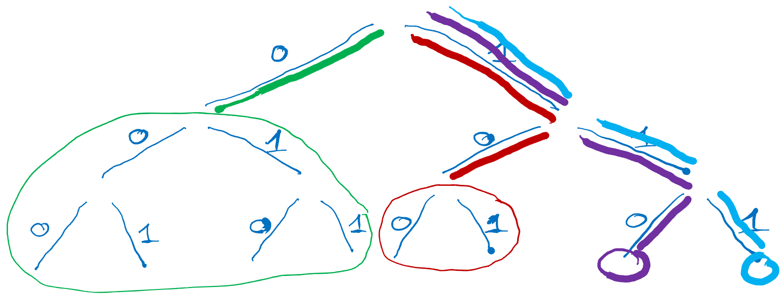
$L_1 = 1, \; L_2 = 2, \; L_3 = 3, \; L_4 = 3$

$\sum_{i=1}^{4} 2^{-L_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$ ✓

THIS IS CALLED HUFFMAN ENCODING

- Proof of central lemma, part (2)
  - To show: L_i with $\Sigma_i \, 2^{-L_i} \leq 1$ ⇨ exists PF code with lengths L_i
  - Complete binary tree of depth $M = \max L_i$ ... has 2^M leaves

    in the example $M = 3$, $2^M = 8$
  - Mark all left edges 0, and all right edges 1
  - Consider the code lengths L_i in sorted order, smallest first
  - Then iterate: pick subtree with $2^{M - L_i}$ leaves that does not overlap with already picked subtrees ... path to that subtree gives code for symbol i and $\Sigma_i \, 2^{M - L_i} = 2^M \cdot \Sigma_i \, 2^{-L_i} \leq 2^M$



$L_1 = 1$, $2^{M-L_1} = 4$, Code $= 0$

$L_2 = 2$, $2^{M-L_2} = 2$, Code $= 10$

$L_3 = 3$, $2^{M-L_3} = 1$, Code $= 110$

$L_4 = 3$, $2^{M-L_4} = 1$, Code $= 111$

Entropy 7/12

- Proof of source coding theorem, part (1)
  - To show: for any PF encoding $\mathbf{E}\, L_X \geq H(X)$
  - By definition of expectation: $\mathbf{E}\, L_X = \Sigma_i\, p_i \cdot L_i$ (1)
  - By Kraft's inequality: $\Sigma_i\, 2^{-L_i} \leq 1$ (2)
  - Using Lagrange, it can be shown that, under the constraint (2), (1) is **min**imized for $L_i = \log_2 1/p_i$

    This is Exercise 1 from ES4 … it is perfect to practice Lagrangian optimization and to deepen understanding of the source coding theorem

  - Then $E\, L_X = \Sigma_i\, p_i \cdot L_i \geq \Sigma_i\, p_i \cdot \overbrace{\log_2 1/p_i}^{= -\log_2 p_i} = H(X)$

Entropy 8/12

- Proof of source coding theorem, part (2)
  - Show: there is a PF encoding with $\mathbf{E}\ L_X \le H(X) + 1$ $\quad \geq \log_2 \frac{1}{p_i}$ $\quad = \log_2 p_i$ $\quad = 1$
  - Let $L_i = \lceil \log_2 1/p_i \rceil$, then $\Sigma_i\ 2^{-L_i} = \sum_i 2^{-\lceil \log_2 \frac{1}{p_i} \rceil} \le \sum_i 2^{-\log_2 \frac{1}{p_i}} = \sum_i p_i$

    Note that the code length must be an integer, and we must round upwards, so that Kraft's inequality holds
  - By the central lemma, part (2), there then exists a PF code with code lengths L_i
  - By definition of expectation: $\mathbf{E}\ L_X = \Sigma_i\ p_i \cdot L_i$
  - Hence $\mathbf{E}\ L_X = \sum_i p_i \cdot \lceil \log_2 \frac{1}{p_i} \rceil \le \sum_i p_i \left(\log_2 \frac{1}{p_i} + 1 \right) = \underbrace{\sum_i p_i \log_2 \frac{1}{p_i}}_{= H(X)} + \underbrace{\sum_i p_i}_{= 1}$

Entropy 9/12

- Entropy-optimal codes
  - Consider a PF code with L_i = code length for symbol i and p_i = probability for symbol i
  - We say that the code is **optimal for distribution p_i** if

    $L_i \leq \log_2 1/p_i + 1$

  - This implies

    $\mathbf{E}\, L_X = \Sigma_i\, p_i \cdot L_i \leq \Sigma_i\, p_i \cdot (\log_2 1/p_i + 1) = H(X) + 1$

  - By Shannon's theorem this is the best we can hope for

    For the optimality proof from Exercise 2 from ES4, it suffices that you show $L_i \leq \log_2 1/p_i + \mathbf{O(1)}$

Entropy 10/12

- Universal codes
  - A prefix-free code is called **universal** if for **every** probability distribution over the symbols to be encoded

    $\mathbf{E}\, L_X = O(\, H(X)\,)$
  - That is, the expected code length is within a constant factor of the optimum for **any** distribution
  - Elias-Gamma, Elias-Delta, Golomb, and Variable-Byte are all universal in this sense
  - Note the difference to the stronger inequality from the previous slide for optimality for a particular distribution

    $\mathbf{E}\, L_X \le H(X) + 1$ versus $\mathbf{E}\, L_X = O(\, H(X)\,)$

Entropy 11/12

- Entropy-optimality of Elias-Gamma
  - Recall: code length for Elias-Gamma is $L_i = 2 \lfloor \log_2 i \rfloor + 1$
  - For which probability distribution is this entropy-optimal?
  - We need $L_i = 2 \lfloor \log_2 i \rfloor + 1 \leq \log_2 1/p_i + 1$
  - This suggests $p_i = 1 / i^2$ because:

    $p_i = 1 / i^2 \rightarrow \log_2 1/p_i = \log_2 i^2 = 2 \cdot \log_2 i$
  - We have to take care that the p_i sum to 1, however:

    $\Sigma_{i=1..\infty} 1 / i^2 = \pi^2 / 6 = 1.6449\ldots$
  - Hence let $p_i = 1 / i^2$ for $i \geq 2$, and p_1 such that $\Sigma_i\, p_i = 1$

    We have thus established a distribution, for which Elias-Gamma is entropy-optimal in the sense of slide 24

Entropy 12/12

- Entropy-optimality of Golomb
  - Consider the following random experiment for the generation of an inverted list L of length m :

    Include each document in L with probability p = m/n, independently of each other, where n = #documents

  - Let X be a fixed **gap** in this inverted list, then

    $\Pr(X = i) = (1 - p)^{i-1} \cdot p =: p_i$ for $i = 1, 2, 3, \ldots$

    Exercise 2 from ES4: Golomb is optimal for this distrib.

  - Bottom line: Golomb is optimal for gap-encoded lists

    But not too practical, because of the bit fiddling, see the definition of Golomb encoding on slide 13

References

- Textbook

  Section 5: Index compression

  Section 5.3: Postings file compression (some codes only)

- Wikipedia

  http://en.wikipedia.org/wiki/Elias_gamma_coding

  http://en.wikipedia.org/wiki/Elias_delta_coding

  http://en.wikipedia.org/wiki/Golomb_coding

  http://en.wikipedia.org/wiki/Variable-width_encoding

  http://en.wikipedia.org/wiki/Source_coding_theorem

  http://en.wikipedia.org/wiki/Kraft_inequality