Unit-2 (Part-3)

Predicate Logic

Predicate Logic

- Limitations of Prepositional logic
  - It is difficult to draw conclusions from the given facts.

    Ex:- John is a boy => A

    Paul is a boy => B

    Peter is a boy => C
- Predicate logic can represent the fact using general statements and variables can have any possible values.

  In general we can write as boy(X).

  boy- predicate Symbol

  X- argument / variable and has {John, Paul, Peter}

❖ When X is given its actual value then the predicate struct boy(X) becomes either true or false.

  Ex:- boy(Peter)=true

  boy(Mary)=false.

Predicate Logic

- Predicate logic is an extension of propositional logic which deals with validity ,Satisfiability & Unsatisfiability of a formula along with inference rules to derive new formula.
- When inference rules are added to predicate calculus ,it becomes predicate logic.
- Example: “All birds fly” cannot be represented in propositional logic but can be represented in predicate logic.

Predicate Calculus

- Term: A term is defined as either a variable, or constant or n-place function. A function is defined as a mapping that maps n terms to a single term. An n-place function is written as $f(t_1, t_2....t_n)$ where $t_1, t_2....t_n$ are terms.
- Predicate: A predicate is defined as a relation that maps n terms to a truth value {true, false}.
- Quantifiers: Quantifiers are used with variables, there are two types of quantifiers, namely, universal quantifiers, $\forall$ (for all), and existential quantifiers, $\exists$ (there exists).

Predicate Calculus

- A well-formed formula is defined as follows in predicate calculus

✔ Atomic formula $p(t_1, t_2 \ldots . t_n)$ (also called an atom) is a WFF where p is predicate symbol and $t_1, t_2 \ldots . t_n$ are terms.

✔ If α and β are WFF, then $\sim\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$ are also WFF.

✔ If α is a WFF and X is a free variable in α, then $(\forall X)\alpha$, $(\exists X)\alpha$ are also WFF. These quantifiers define the scope of the variable X.

✔ Well formed formulae may be generated by applying the rules, a finite number of times.

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Predicate Logic

Examples of Predicates

- ✔ French(x) → x is French → one place predicate
- ✔ Speaks French(x) → x speaks French
- ✔ P(x,y)='3x+5y=25' → two place predicate
- ✔ R(x,y,z)='$6x^2+3y^2+z=21$' → two place predicate
- ✔ Rides(ali,bicycle) → Ali rides a bicycle.
- ✔ Two wheeled(bicycle) → A bicycle is two-wheeled.
- ✔ Book(the-stranger) → The stranger is a book.
- ✔ Owns(Mona, Book(The-stranger)) → Mona owns a book called The-stranger.
- ✔ Teacher saw that the children were reading a book. → See(teacher, Read(children, book))

Predicate Logic

Examples of Predicates

- All girls love music → □X(Girl(X) □ love(X,music))
- All linguists are bald → □X(linguist(X) □ bald(X))
- Some linguists are bald → □X(linguist(X) ∧ bald(X))
- Some girls love music → □X(Girl(X) ∧ love(X,music))
- No girls love music → ~X(Girl(X) ∧ love(X,music))
- Every body is happy → □X(person(X) □ happy(X))
- John likes some animals → □X(Animal(X) ∧ likes(John,X))

Predicate Logic

Examples of Predicates

- ✔ Nobody likes Mary
  - → ~X(Person(X) □ like(X, Mary))
  - → ~□X(Person(X) ∧ like(X, Mary))
- ✔ Some problems are difficult
  - → □X(Problems(X) ∧ difficult(X))
- ✔ All students that study AI are good at logic
  - → □X(student(X) ∧ study AI(X) □ good at logic(X))
- ✔ No student is allowed to carry mobiles
  - → □X(student(X) □ ~carry-mobiles(X))

First order predicate calculus

- If the quantification in predicate formula is on simple variables and not on predicates or functions, then it is called first order predicate calculus.

$$\square X,\ \square Y(\ p(X) \leftrightarrow p(Y))$$

- If the quantification is over predicates or functions, then it becomes second order predicate calculus.

$$\square p(\ p(X) \leftrightarrow p(Y))$$

- When inference rules are added to first order predicate calculus, it becomes First Order Predicate Logic.

Interpretations of Formulae in FOL

- Interpretation of a formulae α in FOL refers to an assignment of truth values to atoms from domain D.

- Each formula α is evaluated to be true or false under a given interpretation *I* over a given domain *D* .

— (□X) p(X) = true if an only if p(X) = true, ∀X ∈ D otherwise it is false.

— (∃X) p(X) = true if an only if ∃ c ∈ D otherwise such that p(X) = true, otherwise it is false.

Example : Evaluate the truth value of an FOL formula α : (∀X)(∃Y) p(X,Y) under the following interpretation I: D={1,2}, p(1,1)=F, p(1,2)=T, p(2,1)=T, p(2,2)=F

Solution: 1. For X=1, then ∃ 2 ∈D such that p(1,2)=T

2. For X=2, then ∃ 1 ∈D such that p(2,1)=T

Hence α is true under interpretation I.

Interpretations of Formulae in FOL

Example : Evaluate the truth value of an FOL formula

α : $(\forall X)[\ p(X) \rightarrow, q(f(X),c)$ under the following interpretation I:

D={1,2}, c=1 (c is a constant from 7the domain D)

f(1) = 2, f(2) = 1, p(1) = F, p(2) = T

q(1,1) = T, q(1,2) = F, q(2,1) = T, q(2,2) = T

Solution: 1. For X=1,

$p(1) \rightarrow q(f(1),1) \equiv p(1) \rightarrow q(2,1) = F \rightarrow q(2,1) \equiv T$

2. For X=2,

$p(2) \rightarrow q(f(2),1) \equiv p(2) \rightarrow q(1,1) = T \rightarrow q(1,1) \equiv T \rightarrow T \equiv T$

Hence α is true for all values of X $\in$ D under interpretation I.

Satisfiability and Unsatisfiability

- A formula α is said to be satisfiable if and only if there exists an interpretation *I* such that α is evaluated to be true under *I*. *I* satisfies α
- A formula α is said to be unsatisfiable if and only if ∃ no interpretation that satisfies α or ∃ no model for α.
- A formula α is said to be valid if and only if for every interpretation *I*, α is true.
- A formula α is called a logical consequence of a set of formulae $\{\alpha_1, \alpha_2 \ldots. \alpha_n\}$ if and only if for every interpretation *I* if $\alpha_1 \wedge \alpha_2 \ldots. \wedge \alpha_n$ is evaluated to be true, then α is also evaluated to be true.

Transformation of a formula into PNF

- In propositional logic the clauses are represented in CNF and DNF.
- Prenex Normal Form is a notation used for obtaining the clauses from a First order Logic formula.
- A formula is said to be in closed form of all the variables appearing in it are quantified and there are no free variables.
- Definition: A closed formula α in FOL is said to be in PNF if and only if α is represented as $(Q_1X_1)\ (Q_2X_2)\ldots\ldots\ (Q_nX_n)\ M$ where Q_k are quantifiers ($\forall$ or $\exists$) X_k are variables, for $1 \leq k \leq n$, while M is a formula free from quantifiers.
- The list of quantifiers $[(Q_1X_1)\ (Q_2X_2)\ldots\ldots\ (Q_nX_n)]$ is called prefix and M is called the matrix of a formula α.

Transformation of a formula into PNF

Conversion of Formula into PNF Notation

- α - FOL formula α without a variable X.
- $\alpha[X]$ – FOL formula α which contains a variable X
- Q – Quantifier ($\forall$ or $\exists$)

Equivalence Laws

Law 1 : $(QX)\ \alpha[X] * \beta \equiv (QX)\ (\alpha[X] * \beta)$

Law 2 : $\alpha * (QX)\ \beta[X] \equiv (QX)\ (\alpha * \beta[X])$

Law 3 : $\sim(\forall X)\ \alpha[X] \equiv (\exists X)\ (\sim\alpha[X])$

Law 4 : $\sim(\exists X)\ \alpha[X] \equiv (\forall X)\ (\sim\alpha[X])$

Law 5 : $(\forall X)\ \alpha[X] \wedge (\forall X)\ \beta[X] \equiv (\forall X)\ (\alpha[X] \wedge \beta[X])$

Law 6 : $(\exists X)\ \alpha[X] \vee (\exists X)\ \beta[X] \equiv (\exists X)\ (\alpha[X] \vee \beta[X])$

Conversion of PNF to its standard form (Skolemization)

- Prenex Normal Form of a given formula can be further transformed into a special form called Skolemization or standard form.
- The process of eliminating existential quantifiers from the prefix of a PNF notation and replacing the corresponding variable by a constant or a function is called Skolemization.
- The constant or function is called as Skolem constant or Skolem function.

Conversion of PNF to its standard form (Skolemization)

Skolemization Procedure:

Let $(Q_1X_1)\ (Q_2X_2)\ldots\ldots\ (Q_nX_n)$ M be in PNF notation where $(Q_1X_1)\ (Q_2X_2)\ldots\ldots\ (Q_nX_n)$ is prefix and M is matrix

1. Scan prefix from left to right till we obtain the first existential quantifier.
   - If Q_1 is the first existential quantifier then choose a new constant $c \notin$ {set of constants in M}. Replace all occurrence of X_1 appearing in matrix M by c and delete (Q_1X_1) from prefix to obtain new prefix and matrix.
   - If Q_r is the first existential quantifier & $Q_1, Q_2\ldots\ Q_{r-1}$ are universal quantifiers appearing before Q_r then choose a new (r-1) place function symbol $c \notin$ {set of functions appearing in M}. Replace all occurrences of X_r in M by $f(X_1, \ldots.\ X_{r-1})$ and remove $(Q_r\ X_r)$ from prefix.
2. Repeat the process till all existential quantifiers are removed from M.

Clauses in FOL

- A clause is defined as a closed formula written in the form $(L_1 \vee \ldots\ldots \vee L_m)$ where L_i is a literal and the variables occurring in literals are universal quantifiers.
- A formula α_n is said to be unsatisfiable if and only if its corresponding set S is unsatisfiable.
- S is said to be unsatisfiable if and only if there ∃ no interpretation that satisfies all the clauses of S simultaneously.
- S is said to be satisfiable if and only if each clause is satisfiable. i.e ∃ an interpretation that satisfies all the clauses of S simultaneously.
- Alternatively, an interpretation I is said to model S if and only if I models each clause of S.

Resolution Refutation Method in FOL

- Resolution refutation is used to test unsatisfiability of a set of clauses corresponding to the predicate formula.
- L is a logical consequence of S if and only if $\{S \cup \sim L\} = \{C_1, \ldots C_m, \sim L\}$ is unsatisfiable. We can say that goal L is deduced from S. Here ~L may give rise to more than one clause.
- Soundness and complete of resolution refutation theorem states that there is a resolution refutation of S if and only if S is unsatisfiable. A deduction of a contradiction from a set S of clauses is called resolution refutation of S.
- L is said to be a logical consequence of S if and only if there is a resolution of $S \cup \{\sim L\}$.

Example 3:

Example 4.20 Show that the formula $\alpha : (\forall X)\ (p(X) \land \sim[q(X) \rightarrow p(X)])$ is unsatisfiable.

Solution To prove the above statement, we need to convert α into a set of clauses with the help of equivalence laws.

$$p(X) \land \sim[q(X) \rightarrow p(X)] \equiv p(X) \land \sim[\sim q(X) \lor p(X)]$$
$$\equiv p(X) \land \sim\sim q(X) \land \sim p(X)$$
$$\equiv p(X) \land q(X) \land \sim p(X)$$

The set of clauses is written as $S = \{p(X), q(X), \sim p(X)\}$. Since there is a contradiction in S itself [because of $p(X)$ and $\sim p(X)$], S is unsatisfiable and consequently α is unsatisfiable.

Example 4:

Example 4.21 Show that $q(a)$ is a logical consequence of formulae α and β where

$\alpha : (\forall X)\ [p(X) \rightarrow q(X)]$

$\beta : p(a)$

Solution To prove the statement given in Example 4.21, we need to convert α and β into a set of clauses by using equivalence laws. The set of clauses, S, may be written as

$S = \{\sim p(X) \vee q(X), p(a)\}$

When we add the negation of goal, that is, $\sim q(a)$ to S, we get the new set, S', where $S' = \{\sim p(X) \vee q(X), p(a), \sim q(a)\}$. The resolution tree is given in Fig. 4.3.

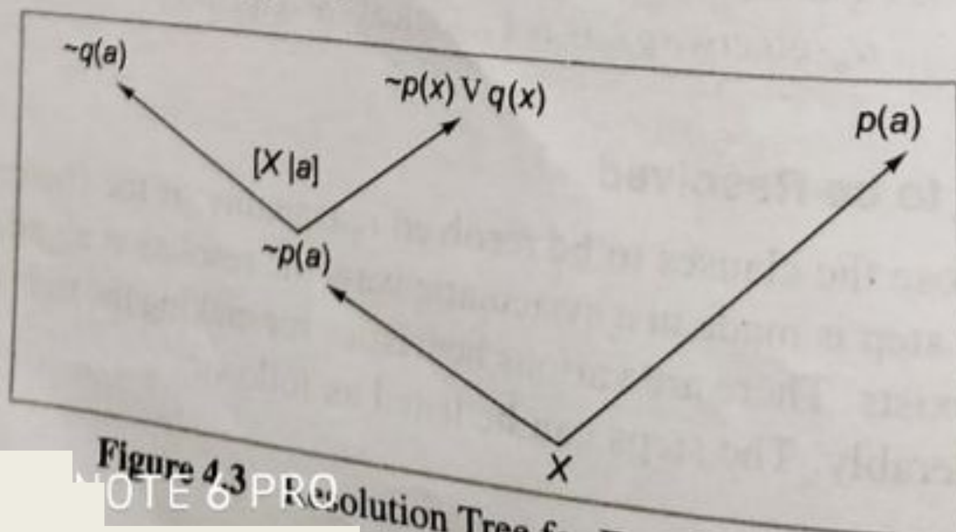

Figure 4.3 Resolution Tree for Example 4.21

General Procedure to convert predicate logic to CNF

1. Eliminate all implications
2. Reduce the scope of all ~ to single term.
3. Make all variables names unique
4. Move Quantifiers left(PNF)
5. Eliminate existential Quantifiers
6. Eliminate Universal Quantifiers
7. Connect to conjunctions of disjunctions
8. Create separate clause for each conjunct.

Example 8:

25

1. $\sim[\text{pass}(x, \text{history}) \wedge \text{win}(x, \text{lottery})] \vee \text{happy}(x)$

   $\sim\text{pass}(x, \text{history}) \vee \sim\text{win}(x, \text{lottery}) \vee \text{happy}(x)$ — C1

2. $\sim[\text{study}(x) \vee \text{lucky}(x)] \vee \text{pass}(x, y)$

   $[\sim\text{study}(x) \wedge \sim\text{lucky}(x)] \vee \text{pass}(x, y)$

   $[\sim\text{study}(x) \vee \text{pass}(x, y)] \wedge [\sim\text{lucky}(x) \vee \text{pass}(x, y)]$

   $\therefore$ $\sim\text{study}(y) \vee \text{pass}(y, z)$ — C2

   $\sim\text{lucky}(w) \vee \text{pass}(w, v)$ — C3

3. $\sim\text{study}(\text{John})$ — C4

   $\text{lucky}(\text{John})$ — C5

4. $\sim\text{lucky}(x) \vee \text{win}(x, \text{lottery})$

   $\sim\text{lucky}(u) \vee \text{win}(u, \text{lottery})$ — C6

5. Conclusion $\text{happy}(\text{John})$

   Negation $\sim\text{happy}(\text{John})$ — C7

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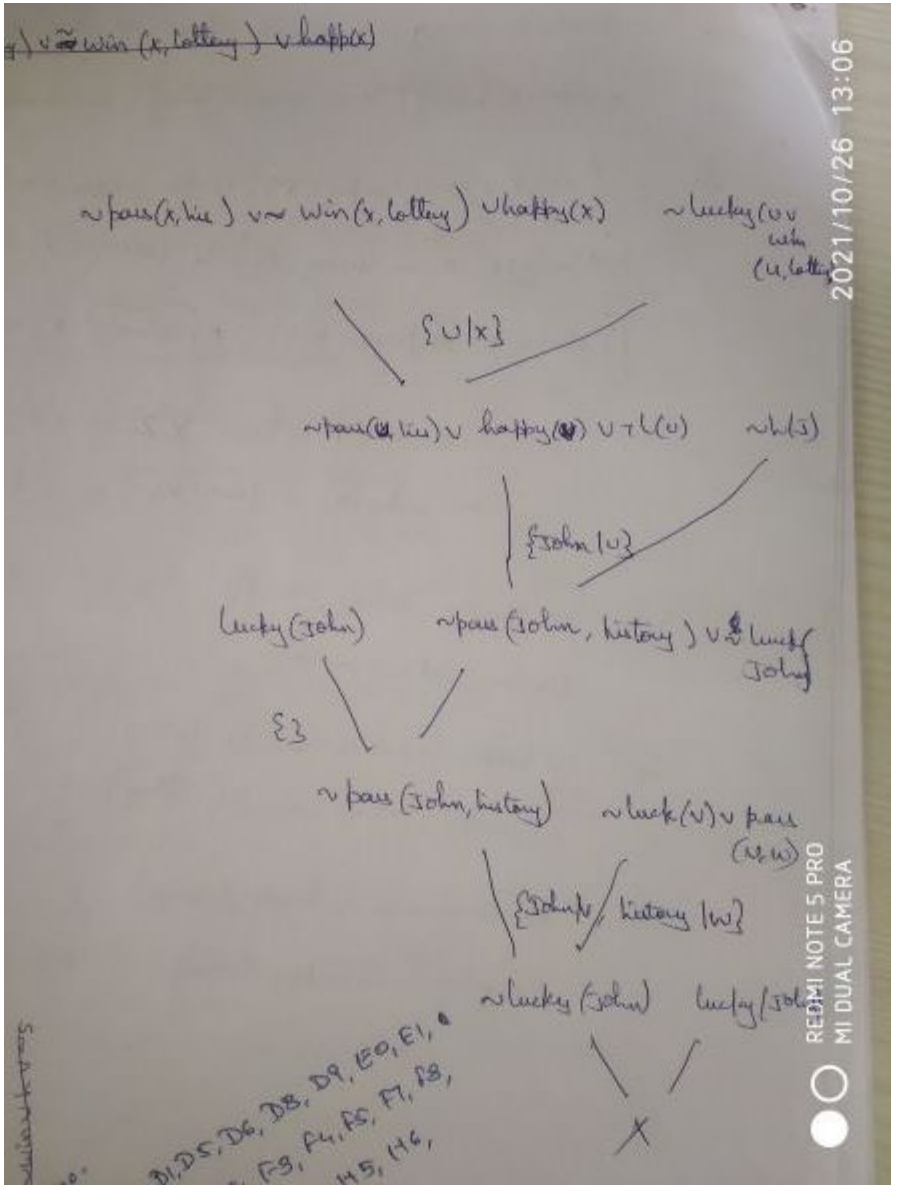
Solution to Example 8:



G. NARAYANAMMA INSTITUTE OF TECHNOLOGY & SCIENCE
HYDERABAD