An Introduction to Partial Differential Equations

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Chapter 1

Model Equations

A standard philosophy in analyzing problems in applied mathematics involves three major steps:

1. Formulation
2. Solution
3. Interpretation

We begin with a standard formulation of the three basic models which lie at the cornerstone of applied mathematics − the diffusion equation, Laplace's equation and the wave equation.

1.1 Diffusion Equation

The diffusion equation is used to describe the flow of heat and is often called the heat equation. The flow of heat is due to a transfer of thermal energy caused by an agitation of molecular matter. The two basic processes that take place in order for thermal energy to move is 1) conduction − collisions of neighboring molecules not moving appreciably and 2) convection − vibrating molecules moving locations.

1.1.1 Conduction in a One-Dimensional Rod

Consider a rod of length L, cross section area A and density $\rho(x)$.

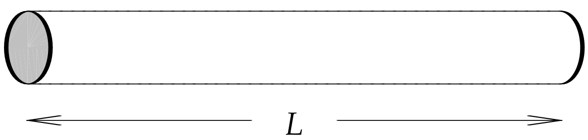


Figure 1. The typical cross-section.

We introduce the thermal energy density function $e(x,t)$ defined by

$$e(x,t) = c(x)\rho(x)u(x,t)A, \tag{1.1}$$

where $c(x)$ is the specific heat, $\rho(x)$ is the density along the rod and $u(x,t)$ is the temperature in the rod at the location x and time t and A the cross sectional area of the rod. We define the total heat $H(x,t)$ as

$$H(x,t) = \int_0^L c(x)\rho(x)u(x,t)A\,dx. \tag{1.2}$$

The amount of thermal energy per unit time flowing to the right per unit surface area is called "heat flux" defined by $\phi(x,t)$.

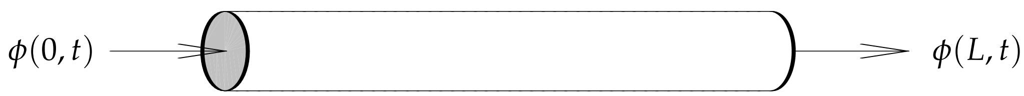


Figure 2. Change in the flux in a rod of length L.

If heat is generated within the rod and its heat density $Q(x,t)$ is given, then the total heat generated is

$$\int_0^L Q(x,t)A\,dx.$$

The conservation of heat energy states:

rate of change of total heat	=	heat flowing across boundaries	+	heat energy generated inside

or, mathematically

$$\frac{dH}{dt} = \left(\phi(0,t) - \phi(L,t)\right)A + \int_0^L Q(x,t)A\,dt,$$

so

$$\frac{d}{dt}\int_0^L c(x)\rho(x)u(x,t)A\,dx = \left(\phi(0,t) - \phi(L,t)\right)A + \int_0^L Q(x,t)A\,dt. \tag{1.3}$$

Cancelling out the $A's$ and using Leibniz' rule and the fundamental theorem of calculus, eqn. (1.3) becomes

$$\int_0^L c(x)\rho(x)\frac{\partial u(x,t)}{\partial t}dx = -\int_0^L \frac{\partial \phi}{\partial x}dx + \int_0^L Q(x,t)dt,$$

which gives

$$\int_0^L \left(c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t) \right) dx = 0.$$

Since this applies to any length L, then

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t) = 0,$$

or

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t). \tag{1.4}$$

Flux and Fourier's Law

Fourier's law is a description on how heat flows in a temperature field and is based on the following:

1. If the temperature is constant, then there is no heat flow,
2. If there is a temperature difference, then there will be flow and it will flow hot to cold,
3. The greater the temperature difference, the greater the heat flow,
4. Heat flows differently for different materials.

With these in mind, Fourier suggested the following form for heat flux:

$$\phi = -k\frac{\partial u}{\partial x}, \tag{1.5}$$

where k is the thermal conductivity. This then give eqn. (1.4) as

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t,u)\,\frac{\partial u}{\partial x} \right) + Q(x,t). \tag{1.6}$$

In the case where the density, the specific heat and thermal conductivity are all constant, the eqn. (1.6) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{1.7}$$

where $D = k/\rho c$, is the coefficient of diffusion, and $f = Q/\rho c$, is the source term. Equation (1.7) is often referred to as the $1-D$ diffusion or heat equation with constant diffusion D with source f.

Boundary and Initial Conditions

In solving eqn. (1.7) for the temperature $u(x,t)$, it is necessary to prescribe information at the endpoints of the rod as well as a initial temperature distribution. These are referred to boundary conditions (BCs) and initial conditions (ICs).

Boundary Conditions

These are conditions at the end of the rod. There are several types of which the following are the most common.

(i) *Prescribed Temperature*

Usually given as

$$u(0,t) = u_l, \qquad u(L,t) = u_r.$$

where u_l and u_r are constant, or

$$u(0,t) = u_l(t), \qquad u(L,t) = u_r(t).$$

with u_l and u_r varying with respect to time.

(ii) *Prescribed Temperature Flux*

If we were to prescribe the heat flow at the boundary, we would impose

$$-k_0 \frac{\partial u}{\partial x} = \phi(x,t),$$

so

$$-k_0 \frac{\partial u}{\partial x}(0,t) = \phi(0,t), \qquad -k_0 \frac{\partial u}{\partial x}(L,t) = \phi(L,t)$$

where $\phi(0,t)$ and $\phi(L,t)$ are given functions of t. In the case of insulated boundaries *i.e.* zero flux, then these would be

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

(iii) *Radiating Boundary Conditions*

In the case where one or both ends are such that the temperature change is proportional to the temperature itself, we would use

$$\frac{\partial u}{\partial x}(0,t) = k_1 u(0,t), \qquad \frac{\partial u}{\partial x}(L,t) = k_2 u(L,t),$$

where $k_i > 0$ is gain where $k_i < 0$ is loss.

Initial Conditions

In addition to describing the temperature at the boundary, it is also necessary to describe the temperature initially (at $t = 0$). This is typically given as

$$u(x,0) = f(x).$$

1.1.2 Conduction in three-dimensions

In analogy to the heat flow in one-dimension, we define the heat flow in three dimensions. If we define the heat energy density $e(x, y, z, t)$ as

$$e(x,y,z,t) = c(x,y,z)\rho(x,y,z)u(x,y,z,t) \tag{1.8}$$

where as before c is the specific heat of the material, ρ the density of the material and u is the temperature within the material, all of which are dependent on the position (x, y, z), and for temperature, time t.
The total heat H is therefore given by

$$\begin{aligned} H(x,y,z,t) &= \iiint_V e(x,y,z,t)dV \\ &= \iiint_V c(x,y,z)\rho(x,y,z)u(x,y,z)dV, \end{aligned} \tag{1.9}$$

We also define the heat flux $\phi(x, y, z, t)$ as the amount of thermal energy per unit time flowing out from the surface S, the boundary of the volume V. The rate at which the heat flows across a surface element dS is given by

$$\bar{\phi} \cdot \hat{n}\ dS.$$

Thus, the net rate of heat flow across the entire surface is given by

$$\iint_S \bar{\phi} \cdot \hat{n}\ dS. \tag{1.10}$$

If heat is generated (or lost) within the material and is given by $Q(x, y, z, t)$, then the total heat generated (or lost) in the volume is given by

$$\iiint_V Q(x,y,z,t)\ dV. \tag{1.11}$$

Applying the conservation of energy where the change in total heat equals the heat flux through the surface plus the heat generated (or lost) in the volume itself is represented mathematically by

$$\frac{d}{dt}\iiint_V c\ \rho\ u\ dV = -\iint_S \bar{\phi} \cdot \hat{n}\ dS + \iiint_V Q\ dV \tag{1.12}$$

From the divergence theorem from Calculus, *i.e.*

$$\iint_S \bar{\phi} \cdot \hat{n}\ dS = \iiint_V \nabla \cdot \phi\ dV,$$

equation (1.12) maybe re-written as

$$\frac{d}{dt}\iiint_V c\,\rho\,u\,dV = -\iiint_V \nabla\cdot\phi\,dV + \iiint_V Q\,dV,$$

or

$$\iiint_V \left(c\,\rho\,\frac{\partial u}{\partial t} + \nabla\cdot\phi - Q\right) dV = 0. \tag{1.13}$$

Since the volume V is arbitrary, this implies that

$$c\,\rho\,\frac{\partial u}{\partial t} + \nabla\cdot\phi - Q = 0.$$

or

$$c\,\rho\,\frac{\partial u}{\partial t} = -\nabla\cdot\phi + Q. \tag{1.14}$$

Fourier's Law

We now use Fourier's law recalling from the previous section that the heat energy flux ϕ in one dimension was given by

$$\phi = -k\frac{\partial u}{\partial x}. \tag{1.15}$$

For 3-D, the heat energy flux ϕ is given

$$\begin{aligned}\vec{\phi} &= -k\nabla u \\ &= -k\left\langle u_x, u_y, u_z\right\rangle.\end{aligned} \tag{1.16}$$

With the conservation of energy as given in equation (1.14) and the heat energy flux given above, we now obtain the three dimensional heat equation, namely,

$$c\,\rho\,\frac{\partial u}{\partial t} = \nabla\cdot(k\nabla u) + Q,$$

or, in expanded form

$$c\,\rho\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right) + Q. \tag{1.17}$$

If we assume that the material is homogeneous, that the specific heat is constant throughout the material and there is no heat generation or loss throughout the material, then we can assume that ρ, c and k are all constant and $Q(x, y, z, t) = 0$. If we let $D = k/\rho\ c$ then the 3D heat equation becomes

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{1.18}$$

Often, equation (1.18) is written as

$$u_t = D\ \nabla^2 u, \tag{1.19}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.20}$$

which is commonly referred to as the Laplacian.

Boundary Conditions

In the one-dimensional case we required boundary condition, *i.e.* conditions at the end points of the rod. For example, if the temperature was fixed at zero at each end, then the BCs

$$u(0,t) = 0, \quad u(L,t) = 0. \tag{1.21}$$

If the ends were insulated, (*i.e.* no heat flow from the ends), then we would impose the following BCs

$$u_x(0,t) = 0, \quad u_x(L,t) = 0. \tag{1.22}$$

Similarly, we must also impose boundary condition for the $3-D$ volume. In analogy to fixed end point temperatures in one dimension, in three dimensions, we would prescribed a surface temperature. For example, we might impose

$$u\,(x,y,z,t)\,\big|_S = F(x,y,t),$$

if the surface was denoted by $z = f(x,y)$. For the insulated boundary conditions in $3D$, we would impose

$$\frac{\partial u}{\partial n} = 0,$$

or

$$\nabla u\ \cdot \hat{n} = 0. \tag{1.23}$$

where $\nabla u\ \cdot \hat{n}$ is the directional derivative of u in the outward normal direction.

Steady State Temperature

If the body was in equilibrium then there would be no change in temperature with respect to time. This is referred to as "steady state" and is such that

$$\frac{\partial u}{\partial t} = 0. \tag{1.24}$$

In one-dimensional, this would given

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.25}$$

In two and three dimensions, we obtain respectively

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (2D), \tag{1.26}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad (3D), \tag{1.27}$$

and can conveniently written as

$$\nabla^2 u = 0. \tag{1.28}$$

These equations are commonly known as "Laplace's equation." It is one of the primary equations used in applied mathematics.

1.2 Wave equation

Consider the motion of a perfectly elastic string in which the horizontal motion is negligible and that vertical motion is small. Let us represent this vertical displacement by $y = u(x,t)$. The string will move according to a change in tension throughout the string and, in particular, on $[x, x + \Delta x]$ where the tension at the endpoints is $T(x,t)$ and $T(x + \Delta x, t)$, respectively (see fig. 3 and 4).

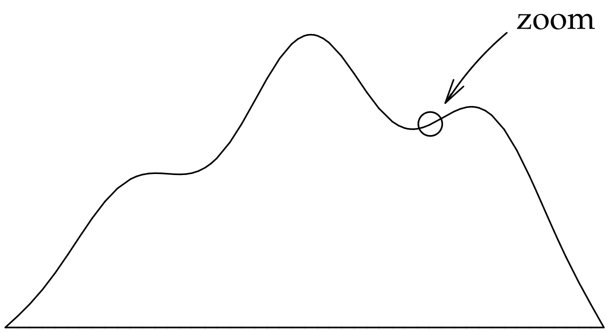


Figure 3. A plucked string.

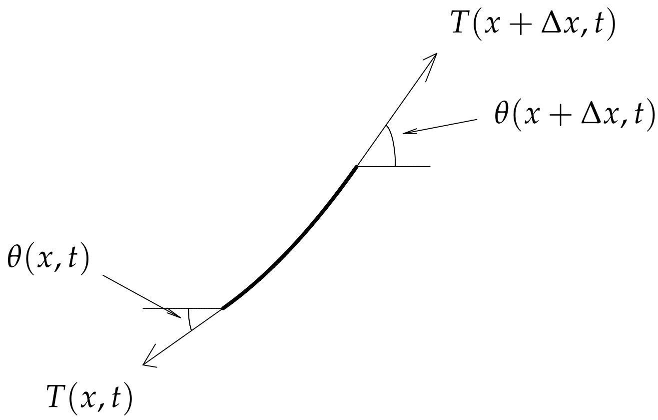


Figure 4. Tension in a small length of string.

If the mass throughout the string is denoted by $\rho = \rho(x)$ using Newton's second law $F = ma$ gives

$$F = ma = \rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}. \tag{1.29}$$

This must be balanced by the resultant tension, namely, $T \uparrow - T \downarrow$. At the end-points we have

$$T \downarrow = T(x,t) \ \sin\ \theta(x,t), \tag{1.30a}$$

$$T \uparrow = T(x+\Delta x,t) \ \sin\ \theta(x+\Delta x,t). \tag{1.30b}$$

and thus the difference gives

$$F = T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t). \tag{1.31}$$

The balance of forces, namely equations (1.29) and (1.31), gives rise to the following

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t),$$

and in the limit as $\Delta x \to 0$ then

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x \to 0}\left(\frac{T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t)}{\Delta x}\right),$$

from which we obtain

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\ \sin\ \theta(x,t)\Big). \tag{1.32}$$

Since we are assuming that displacements are small, then $\theta(x,t) \simeq 0$, so that

$$\sin\ \theta(x,t) \simeq\ \tan\ \theta(x,t)$$

leading us to replace equation (1.32) with

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\ \tan\ \theta(x,t)\Big). \tag{1.33}$$

Since

$$\tan\ \theta(x,t) = \frac{\partial u}{\partial x} \tag{1.34}$$

this gives equation (1.33) as

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\ \frac{\partial u}{\partial x}\right). \tag{1.35}$$

a partial differential equation commonly know as the "wave equation".
If we assume that the string is homogeneous and perfectly elastic, then $\rho(x) = \rho_0$ and $T(x,t) = T_0$, both constant. This, in turn, gives the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\,\frac{\partial^2 u}{\partial x^2}. \tag{1.36}$$

We note that the units of T_0 is "$kg\ m/s^2$" and the units of ρ_0 is "kg/m" giving the units of T_0/ρ_0 as "m^2/s^2" - the units of speed. Thus, we introduce the term "wave speed", the variable c, where $c^2 = T_0/\rho_0$, this gives the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}. \tag{1.37}$$

Boundary Conditions
As in the case of the heat equation, it is necessary that boundary conditions be prescribed, *i.e.* conditions at the end points of the string. For example, if the endpoints are fixed at zero, then the following BCs would be used

$$u(0,t) = 0, \quad u(L,t) = 0. \tag{1.38}$$

If the ends were forced to move, then we would impose the following BCs

$$u(0,t) = f(t), \quad u(L,t) = g(t), \tag{1.39}$$

where f and g would be specified. If the ends were free to move, then we would impose the following BCs

$$u_x(0,t) = 0, \quad u_x(L,t) = 0. \tag{1.40}$$

Similarly, as with the heat equation, we must also impose initial conditions, however, unlike the case of the heat equation, we must prescribe two initial conditions − both position and speed. For example, we might impose

$$u\,(x,0) = f(x), \quad \text{position}, \tag{1.41a}$$
$$\frac{\partial u}{\partial t}\,(x,0) = g(x), \quad \text{velocity}. \tag{1.41b}$$

Higher Dimensional Wave Equations

As with the heat equation, it is also possible to have higher dimensional wave equations. For example, if we assume that a surface is moving in the vertical direction *i.e.* the surface of the ocean or a drum head, then the appropriate wave equations would be

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{1.42}$$

in two dimensions. In three dimensions

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{1.43}$$

If we where to consider the vibrations of a circular drum, the PDE that we would consider is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right). \tag{1.44}$$

If we were to assume that the drumhead preserved its symmetry, the equation (1.44) would reduce to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right). \tag{1.45}$$

The ultimate goal would then be to solve the wave equation subject to suitable boundary conditions and initial conditions. For example, we would be interested in solving (1.37) with BCs (1.38) and ICs (1.41).

Chapter 2

First Order PDEs

Partial differential equations of the form

$$\begin{aligned} u_x - 2u_y = u, \quad xu_x + yu_u = 0, \\ u_x + uu_y = 1, \quad u_x^2 + u_y^2 = 1, \end{aligned} \tag{2.1}$$

and all examples of first order partial differential equations. The first equation is constant coefficient, the second equation is linear, the third equation quasilinear and the last equation nonlinear. In general, equations of the form

$$F(x, y, u, u_x, u_y) = 0, \tag{2.2}$$

are first order partial differential equations. This chapter deals with techniques to construct general solutions of (2.2) when they are constant coefficient, linear, quasilinear and fully nonlinear equations.

2.1 Constant coefficient equations

Partial differential equation (PDEs) of the form

$$a\,u_x + b\,u_y = c\,u. \tag{2.3}$$

where a, b and c are all constant are called constant coefficient PDEs. For example, consider

$$u_x - u_y = 0. \tag{2.4}$$

Our goal is to find a solution $u = u(x, y)$ of this equation. If we introduce the change of variables

$$r = x + y, \quad s = x - y, \tag{2.5}$$

then the derivatives transform as

$$u_x = u_r + u_s, \quad u_y = u_r - u_s,$$

and equation (2.4) becomes

$$u_r + u_s - u_r + u_s = 0 \quad \Rightarrow \quad u_s = 0.$$

Integration of this gives

$$u = f(r),$$

or

$$u = f(x+y). \tag{2.6}$$

Substituting (2.6) into equation (2.4) shows that it is satisfied and thus is the solution. Let us introduce a new set of coordinates, say

$$r = x + y, \quad s = x^2 - y^2. \tag{2.7}$$

The derivatives transform as

$$u_x = u_r + \left(\frac{s}{r} + r\right) u_s, \quad u_y = u_r + \left(\frac{s}{r} - r\right) u_s,$$

and equation (2.4) becomes

$$2ru_s = 0 \quad \Rightarrow \quad u_s = 0.$$

Upon integration yields

$$u = f(r),$$

or

$$u = f(x+y),$$

the same solution as given in (2.6). Finally, consider a third change of variables

$$r = x - y, \quad s = x^2 - y^2. \tag{2.8}$$

The derivatives transform as

$$u_x = u_r + \left(r + \frac{s}{r}\right) u_s, \quad u_y = -u_r + \left(r - \frac{s}{r}\right) u_s,$$

and equation (2.4) becomes, after simplification

$$ru_r + su_s = 0,$$

a new first order PDE, however this equation is actually more complicated than the one we start with! As the changes of variables (2.5) and (2.7) transforms the original PDE to equations that are simple and the change of variables (2.8) to a more complicated PDE, the natural question is: "What is common in the change of variables (2.5) and (2.7) that is not in (2.8)"? The answer is that one of the variables is $r = x + y$. In factor, if we choose $r = x + y$ and $s = s(x, y)$, arbitrary, then

$$u_x = u_r + s_x u_s, \quad u_y = u_r + s_y u_s,$$

and equation (2.4) becomes, after simplification

$$\left(s_x - s_y\right) u_s = 0, \tag{2.9}$$

from we deduce that $u_s = 0$, recovering the solution found in (2.6).

The question is, how did we know how to choose $r = x + y$ as one of the new variables? For example, suppose we consider

$$2u_x - u_y = 0, \tag{2.10}$$

or

$$u_x + 5u_y = 0, \tag{2.11}$$

what would be the right choice of $r(x, y)$ that leads to a simple equation like $u_s = 0$? In an attempt to try and answer this, let us introduce the change of variables $r = r(x, y)$ and $s = s(x, y)$ and try to find r so that the original PDE becomes $u_s = 0$. Under a general changes of variables

$$u_x = r_x u_r + s_x u_s, \quad u_y = r_y u_r + s_y u_s,$$

equation (3.19) becomes

$$\left(r_x - r_y\right) u_r + \left(s_x - s_y\right) u_s = 0 \tag{2.12}$$

and in order to obtain our target PDE, it is necessary to choose

$$r_x - r_y = 0. \tag{2.13}$$

However, to solve (2.13) is to solve (2.4)! In fact to solve any PDE in the form of (2.3), using a general change of variables, it would be necessary to have one solution to

$$a\, r_x - b\, r_y = 0. \tag{2.14}$$

So without knowing one of the variables is $r = x + y$, our first attempt at trying to solve (2.4) would have failed!

For our second attempt, we will try and work backwards. We will start with the answer, $u_s = 0$ and try and target our original PDE, equation (2.4). Therefore, if we start with

$$u_s = 0, \tag{2.15}$$

and use a general chain rule

$$u_s = u_x x_s + u_y y_s,$$

then (2.15) becomes

$$u_x x_s + u_y y_s = 0. \tag{2.16}$$

Choosing

$$x_s = 1, \quad y_s = -1, \tag{2.17}$$

gives the original equation (2.4). Integrating (2.17) gives

$$x = s + a(r), \quad y = -s + b(r), \tag{2.18}$$

where $a(r)$ and $b(r)$ are arbitrary functions of r. From (2.15), we obtain $u = A(r)$, and eliminating s from (2.18) gives

$$x + y = a(r) + b(r) = c(r) \quad \Rightarrow \quad r = c^{-1}(x + y) \tag{2.19}$$

so

$$u = A\left(c^{-1}(x + y)\right) \quad \Rightarrow \quad u = f(x + y) \tag{2.20}$$

since the addition, the inverse and composition of arbitrary functions is arbitrary. The next two examples illustrate further.

Example 1
Consider

$$2u_x + u_y = 0. \tag{2.21}$$

If

$$u_s = u_x x_s + u_y y_s,$$

and

$$u_s = 0, \tag{2.22}$$

we obtain (3.21) by choosing

$$x_s = 2, \quad y_s = 1. \tag{2.23}$$

Integrating (2.23) gives

$$x = 2s + a(r), \quad y = s + b(r). \tag{2.24}$$

The solution of (2.22) is $u = c(r)$, and eliminating s from (2.24) $x - 2y = a(r) - 2b(r) = d(r)$ (some arbitrary function) and solving for r gives $r = d^{-1}(x - 2y)$. Using this, we obtain

$$u = f(x - 2y) \tag{2.25}$$

as the solution of (3.21) noting that when we compose, we identify that $c(d^{-1}) = f$.

Example 2
Consider

$$u_x + 4u_y = u. \tag{2.26}$$

In this case our target is

$$u_s = u. \tag{2.27}$$

If

$$u_s = u_x x_s + u_y y_s,$$

then (2.27) becomes

$$u_x x_s + u_y y_s = u,$$

and choosing

$$x_s = 1, \quad y_s = 4, \tag{2.28}$$

gives the original equation (3.32). Integrating (2.28) gives

$$x = s + a(r), \quad y = 4s + b(r), \tag{2.29}$$

while the solution of (2.27) is $u = f(r)\mathrm{e}^s$. Eliminating s from (2.29) gives $r = c(4x - y)$, and further, from (2.29) we obtain $s = x - a(r)$. This then leads to

$$\begin{aligned}
u &= f_1(r)\mathrm{e}^{x-a(r)}, \\
&= \underbrace{f_1(r)\mathrm{e}^{-a(r)}}_{f_2(r)}\,\mathrm{e}^x, \\
&= \underbrace{f_2(c}_{f}(4x - y))\,\mathrm{e}^x \\
&= f(4x - y)\,\mathrm{e}^x,
\end{aligned}$$

the solution of the PDE (3.32).

2.2 Linear equations

We now turn our attention to first order PDEs of the form

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u, \tag{2.30}$$

where a, b and c are all function of the variables x and y. These types of equations are called linear PDEs. For example,

$$xu_x - yu_y = 0. \tag{2.31}$$

So now the question is, does the method from the preceding section work here? The answer $-$ yes! If we let

$$u_s = u_x x_s + u_y y_s,$$

and choose

$$x_s = x, \quad y_s = -y, \tag{2.32}$$

then (2.31) becomes

$$u_s = 0, \tag{2.33}$$

and we are required to solve (2.32) and (2.33). The solution of these are

$$x = a(r)\mathrm{e}^s, \quad y = b(r)\mathrm{e}^{-s}, \quad u = c(r). \tag{2.34}$$

Eliminating s from the first two gives

$$xy = a(r)b(r) \quad \Rightarrow \quad r = A(xy), \tag{2.35}$$

and from the third of (2.34) we obtain

$$u = c\left(A(xy)\right) \quad \Rightarrow \quad u = f(xy). \tag{2.36}$$

In general, for linear equations in the form (2.30), if we introduce a general chain rule

$$u_s = u_x x_s + u_y y_s,$$

then we can target this PDE by choosing

$$x_s = a(x,y), \quad y_s = b(x,y), \quad u_s = c(x,y)u. \tag{2.37}$$

Example 3
Consider

$$xu_x + yu_y = u. \tag{2.38}$$

Here, we must solve

$$x_s = x, \quad y_s = y, \quad u_s = u, \tag{2.39}$$

The solution of these are

$$x = a(r)\mathrm{e}^s, \quad y = b(r)\mathrm{e}^s, \quad u = c(r)\mathrm{e}^s. \tag{2.40}$$

Eliminating s from the first and second and first and third gives

$$\frac{y}{x} = \frac{b(r)}{a(r)}, \quad \text{and} \quad \frac{u}{x} = \frac{c(r)}{a(r)}, \tag{2.41}$$

and further elimination of r gives

$$\frac{u}{x} = f\left(\frac{y}{x}\right) \quad \text{or} \quad u = xf\left(\frac{y}{x}\right). \tag{2.42}$$

Example 4
Consider

$$u_t + xu_x = 1, \quad u(x,0) = -x^2. \tag{2.43}$$

Here we must solve

$$t_s = 1, \quad x_s = x, \quad u_s = 1. \tag{2.44}$$

But as this is an initial value problem, so it is necessary to find appropriate initial conditions for the system of equations (2.44). In the (x,t) plane, the line $t = 0$ is where the initial condition u is defined. So we must associate a curve in the (r,s) plane. As any choice will work, we choose $s = 0$ and identify that $r = x$ so that $u = -x^2$ when $t = 0$ becomes

$$t(r,0) = 0, \quad x(r,0) = r, \quad u(r,0) = -r^2. \tag{2.45}$$

Solving (2.44) gives

$$t = s + a(r), \quad x = b(r)\mathrm{e}^s, \quad u = s + c(r), \tag{2.46}$$

which, subject to the initial conditions, (2.45), gives

$$t = s, \quad x = r\mathrm{e}^s, \quad u = s - r^2, \tag{2.47}$$

Elimination of r and s in (2.47) gives

$$u = t - x^2\mathrm{e}^{-2t}. \tag{2.48}$$

Example 5
Consider

$$yu_x + xu_y = u. \tag{2.49}$$

Here, we must solve

$$x_s = y, \quad y_s = x, \quad u_s = u, \tag{2.50}$$

As the first two of (2.50) is a system, their solution will be coupled. The solution of these are

$$x = a(r)\mathrm{e}^s + b(r)\mathrm{e}^{-s}, \quad y = a(r)\mathrm{e}^s - b(r)\mathrm{e}^{-s}, \quad u = c(r)\mathrm{e}^s. \tag{2.51}$$

In previous example, eliminating s was easy. Here this is not the case. Noting that

$$x + y = 2a(r)\mathrm{e}^s, \quad \text{and} \quad x - y = 2b(r)\mathrm{e}^{-s} \tag{2.52}$$

and multiplying these gives

$$(x+y)(x-y) = 4a(r)b(r) \quad \Rightarrow \quad r = g(x^2 - y^2) \tag{2.53}$$

and further, using (2.53) in conjunction with (2.51) leads to finally the solution

$$u = (x+y)f\left(x^2 - y^2\right), \tag{2.54}$$

but this example clearly shows that even though this technique works, trying to eliminate the variables r and s can be quite tricky. In the next section we will bypass the introduction of the variables r and s.

2.3 Method of Characteristics

In solving first order PDEs of the form

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)u, \tag{2.55}$$

it was necessary to solve the system of ODEs

$$x_s = a(x,y), \quad y_s = b(x,y), \quad u_s = c(x,y)u. \tag{2.56}$$

As seen in example (3.32), in solving

$$u_x + 4u_y = u. \tag{2.57}$$

we associate the system

$$x_s = 1, \quad y_s = 4, \quad u_s = u, \tag{2.58}$$

or

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = 4, \quad \frac{\partial u}{\partial s} = u. \tag{2.59}$$

If we can rewrite this system as

$$4\,\partial x - \partial y = 0, \quad \frac{\partial u}{u} = \partial x, \tag{2.60}$$

then we have just eliminated the introduction of the variable s. Further, integrating leads to

$$4\,x - y = A(r), \quad \ln|u| - x = B(r) \tag{2.61}$$

and eliminate of r leads to

$$\ln|u| - x = f(4\,x - y) \tag{2.62}$$

or

$$u = \mathrm{e}^x f(4\,x - y) \tag{2.63}$$

after exponentiation noting that we have replaced $e^f = f$. If we essentially treat r in (2.61) as constant, then we can treat the partial derivatives as ordinary derivatives in (2.60) then

$$4\,dx - dy = 0, \quad \frac{du}{u} = dx. \tag{2.64}$$

If we we to integrate (2.64) we would obtain

$$4\,x - y = c_1, \quad \ln|u| - x = c_2. \tag{2.65}$$

Comparing (2.65) and (2.61) shows that the constants c_1 and c_2 play the role of $A(r)$ and $B(r)$ and since we wish to eliminate r this is equivalent to having $c_2 = f(c_1)$. This would be the solution of the PDE. We typically write equation (2.64) as

$$\frac{dx}{1} = \frac{dy}{4} = \frac{du}{u}. \tag{2.66}$$

These are called characteristics equations, and the method − the method of characteristics. In general, for linear PDEs of the form (2.55) the method of characteristics requires us to solve

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y)u}. \tag{2.67}$$

Example 6
Here, we revisit example 5, equation (2.49), considered earlier. The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}. \tag{2.68}$$

Solving the first pair we obtain

$$x^2 - y^2 = c_1. \tag{2.69}$$

For the second pair, we use a result derived in #5 in the exercises that

$$\frac{d(x+y)}{x+y} = \frac{du}{u}, \tag{2.70}$$

which leads to

$$\frac{u}{x+y} = c_2, \tag{2.71}$$

and the general solution is $c_2 = f(c_1)$ which leads to

$$u = (x+y)\, f(x^2 - y^2). \tag{2.72}$$

Example 7
Consider

$$2x\, u_x + y\, u_y = 2x, \quad u(x,x) = x^2 + x. \tag{2.73}$$

The characteristic equations are

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{du}{2x}. \tag{2.74}$$

Solving the first and second and first and third gives

$$\frac{y^2}{x} = c_1, \quad u - x = c_2 \tag{2.75}$$

giving the general solution as

$$u = x + f\left(\frac{y^2}{x}\right). \tag{2.76}$$

Using the initial condition gives

$$x + f(x) = x + x^2 \quad \Rightarrow \quad f(x) = x^2 \tag{2.77}$$

thus, giving the solution

$$u = x + \left(\frac{y^2}{x}\right)^2. \tag{2.78}$$

2.4 Quasilinear equations

First order PDEs of the form

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u), \tag{2.79}$$

where a, b and c are all functions of the variables x, y and u are called quasilinear PDEs. For example,

$$yu_x + (x-u)u_y = y. \tag{2.80}$$

The method of characteristics can also be used here giving

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}, \tag{2.81}$$

noting the difference in this and the linear case is the resulting ODEs are fully coupled. For the example given in (2.80), the characteristic equations become

$$\frac{dx}{y} = \frac{dy}{x-u} = \frac{du}{y}. \tag{2.82}$$

From the first and third we obtain $u - x = c_1$ and then using this in the first and second, we obtain

$$\frac{dx}{y} = -\frac{dy}{c_1} \tag{2.83}$$

which yields

$$c_1 x + \frac{1}{2}y^2 = c_2,$$

or

$$(u-x)x + \frac{1}{2}y^2 = c_2.$$

Eliminating the constants gives rise to the solution

$$(u-x)x + \frac{1}{2}y^2 = f\left(u-x\right).$$

Example 9
Consider

$$(x+u)\,u_x + (y+u)\,u_y = x-y. \tag{2.84}$$

The characteristic equations are

$$\frac{dx}{x+u} = \frac{dy}{y+u} = \frac{du}{x-y}. \tag{2.85}$$

To solve these, we re-write them as

$$\frac{dx}{du} = \frac{x+u}{x-y}, \quad \frac{dy}{du} = \frac{y+u}{x-y}. \tag{2.86}$$

Subtracting gives

$$\frac{d(x-y)}{du} = \frac{x-y}{x-y} = 1, \tag{2.87}$$

which integrates giving

$$x - y = u + c_1. \tag{2.88}$$

Using this in the first and third of (2.85) gives

$$\frac{dx}{x+u} = \frac{du}{u+c_1}, \tag{2.89}$$

which is a linear ODE. It has as its solution

$$\frac{x}{u+c_1} - \ln|u+c_1| - \frac{c_1}{u+c_1} = c_2, \tag{2.90}$$

and using c_1 above gives

$$\frac{y+u}{x-y} - \ln|x-y| = c_2. \tag{2.91}$$

Therefore the solution of the original PDE is given by

$$\frac{y+u}{x-y} - \ln|x-y| = f(x-y-u). \tag{2.92}$$

As the final example in this section, we consider the PDE

$$u_t + c(x,t,u)u_x = 0. \tag{2.93}$$

Equation (2.93) is commonly referred to the first order wave equation where c is usually referred to as the "wave speed." In particular, we will consider the following three equations

$$u_t + 2u_x = 0, \quad u_t + 2xu_x = 0, \quad u_t + 2uu_x = 0, \tag{2.94}$$

all subject to the initial condition $u(x,0) = \operatorname{sech} x$. Each can be solved using the method of characteristics giving

$$u = f(x-2t), \quad u = f\left(x\mathrm{e}^{-2t}\right), \quad u = f(x-2tu), \tag{2.95}$$

respectively, and imposing the initial conditions gives the solutions

$$u = \operatorname{sech}(x-2t), \quad u = \operatorname{sech}\left(x\mathrm{e}^{-2t}\right), \quad u = \operatorname{sech}(x-tu). \tag{2.96}$$

Figures 1, 2 and 3 show their respective solutions.

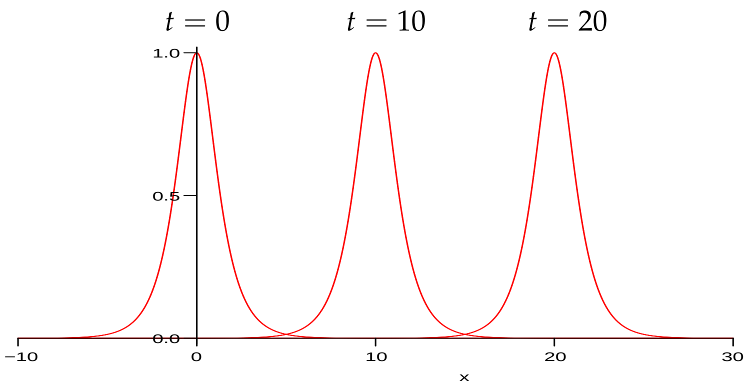


Figure 1. The solution (2.96)(i) at times t = 0, 10 and 20.

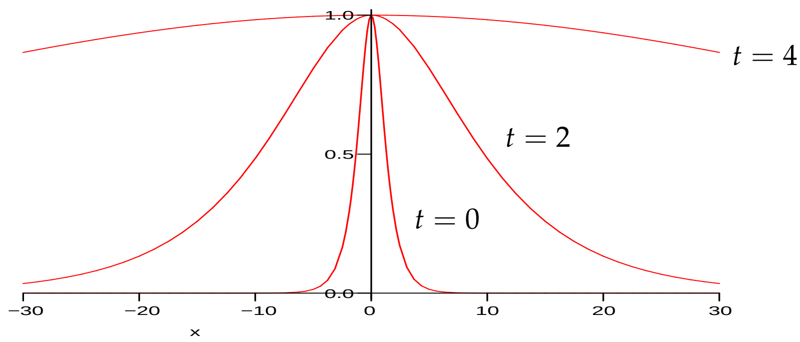


Figure 2. The solution (2.96)(ii) at times t = 0, 2 and 4.

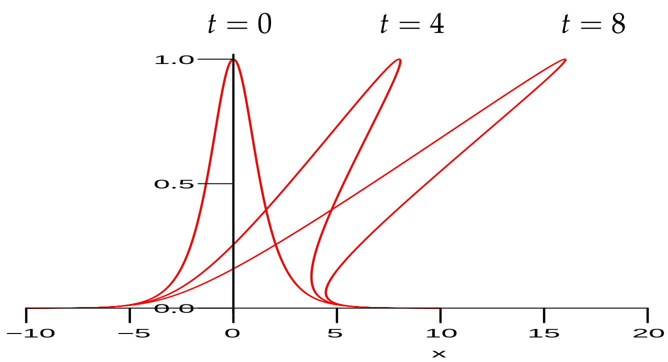


Figure 3. The solution (2.96)(iii) at times t = 0, 4 and 8.

It is interesting to note that in Fig. 1, the wave moves to the right without changing its shape, in Fig. 2, the wave spread out and, in Fig. 3, the wave moves to the right with its speed changing according to height.

2.5 Higher dimensional equations

The method of characteristics easily extends to PDEs with more than two independent variables. For example

$$x\,u_x + y\,u_y - z\,u_z = u \tag{2.97}$$

The characteristic equations for this would be

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-z} = \frac{du}{u} \tag{2.98}$$

We now pick in pairs

$$\begin{aligned} (i) \quad & \frac{dx}{x} = \frac{dy}{y}, \\ (ii) \quad & \frac{dx}{x} = \frac{dz}{-z}, \\ (iii) \quad & \frac{dx}{x} = \frac{du}{u}. \end{aligned}$$

Integrating each, we obtain,

$$\frac{y}{x} = c_1, \quad xz = c_2, \quad \frac{u}{x} = c_3.$$

Here, in analogy to the case we have two independent variables, the general solution is $c_3 = f(c_1, c_2)$. For this problem, it would be

$$u = xf\left(\frac{y}{x}, xz\right),$$

Example 10
Consider

$$u_t + y\,u_x - xu_y = -u, \quad u(x,y,0) = \frac{y}{\sqrt{x^2+y^2}}\,\mathrm{e}^{-\mathrm{x}^2-\mathrm{y}^2}. \tag{2.100}$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{y} = \frac{dy}{-x} = \frac{du}{-u}. \tag{2.101}$$

Solving the second and third and first and last gives

$$x^2 + y^2 = c_1, \quad u\mathrm{e}^t = c_2. \tag{2.102}$$

In order to integrate a third pair, it is necessary to use one of (2.102). Isolating y gives

$$y = \pm\sqrt{c_1 - x^2},$$

so

$$\frac{dt}{1} = \frac{dx}{\pm\sqrt{c_1 - x^2}} \quad \Rightarrow t = \tan^{-1}\frac{x}{y} + c_3,$$

and the general solution is

$$u = \mathrm{e}^{-t} f\left(x^2 + y^2,\, t - \tan^{-1}\frac{y}{x}\right).$$

Using the initial condition gives

$$f\left(x^2 + y^2,\, -\tan^{-1}\frac{y}{x}\right) = \frac{y}{\sqrt{x^2+y^2}}\,\mathrm{e}^{-x^2-y^2} \quad \Rightarrow \quad f(a,b) = \sin b\,\mathrm{e}^{-a},$$

thus giving the solution

$$u = \frac{x\sin t + y\cos t}{\sqrt{x^2+y^2}}\,\mathrm{e}^{-t-x^2-y^2}.$$

2.6 Fully nonlinear first order equations

In this section we introduce two methods for obtain exact solutions to first order PDEs $-$ the method of characteristics and Charpit's method.

2.6.1 Method of Characteristics

In order to develop the method of characteristics for fully nonlinear first order equation

$$F(x,y,u,u_x,u_y) = 0 \tag{2.103}$$

we return to the case of quasilinear equations

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{2.104}$$

and define F as

$$F(x,y,u,u_x,u_y) = a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \tag{2.105}$$

Recall that the characteristic equations for equation (2.104) are

$$\frac{dx}{ds} = a(x,y,u), \quad \frac{dy}{ds} = b(x,y,u), \quad \frac{du}{ds} = c(x,y,u). \tag{2.106}$$

We see that these can also be obtained from (2.105) be computing

$$\frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{du}{ds} = pF_p + qF_q, \tag{2.107}$$

where, as usual, we have used the notation $p = u_x$ and $q = u_y$. As (2.107) are independent of the nature of F (*i.e* quasilinear or nonlinear), we will use these for the more general equation (2.103). As (2.107) may contain the variable p and q, it is necessary to add to (2.107) two more equations, one for $\frac{dp}{ds}$ and one for $\frac{dq}{ds}$. Differentiating (2.103) with respect to x and y gives

$$F_x + pF_u + p_xF_p + q_xF_q = 0, \quad F_y + qF_u + p_yF_p + q_yF_q = 0. \tag{2.108}$$

Using the fact that

$$q_x = p_y, \qquad p_y = q_x, \tag{2.109}$$

gives

$$F_x + pF_u + p_xF_p + p_yF_q = 0, \quad F_y + qF_u + q_xF_p + q_yF_q = 0. \tag{2.110}$$

If we consider $\frac{dp}{ds}$, then from the chain rule (and (2.110))

$$\begin{aligned} \frac{dp}{ds} &= \frac{dp}{dx}\frac{dx}{ds} + \frac{dp}{dy}\frac{dy}{ds} \\ &= p_xF_p + p_yF_q \\ &= -F_x - pF_u. \end{aligned} \tag{2.111}$$

Similarly, if we consider $\frac{dp}{ds}$, then from the chain rule (and (2.110))

$$\begin{aligned} \frac{dq}{ds} &= \frac{dq}{dx}\frac{dx}{ds} + \frac{dq}{dy}\frac{dy}{ds}, \\ &= q_xF_p + q_yF_q \\ &= -F_y - qF_u. \end{aligned} \tag{2.112}$$

Thus we have the following characteristic equations

$$\begin{aligned} &\frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{du}{ds} = pF_p + qF_q, \\ &\frac{dp}{ds} = -F_x - pF_u, \quad \frac{dq}{ds} = -F_y - qF_u. \end{aligned} \tag{2.113}$$

We now consider two examples.

Example 11
Solve

$$u_x = u_y^2, \quad u(0,y) = -\frac{y^2}{2}. \tag{2.114}$$

Here, we identify that

$$F = p - q^2, \tag{2.115}$$

so that the characteristic equations (2.113) become

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = -2q, \quad \frac{du}{ds} = p - 2q^2, \quad \frac{dp}{ds} = 0 \quad \frac{dq}{ds} = 0. \tag{2.116}$$

As this is an initial value problem, it is necessary to find appropriate initial conditions for the equations (2.116). In the (x,y) plane, the line $x = 0$ is where u is defined. To this, we must associate a curve in the (r,s) plane. Given the flexibility, we can choose $s = 0$ and $r = y$. Therefore, we have

$$x(r,0) = 0, \quad y(r,0) = r, \quad u(r,0) = -\frac{r^2}{2}. \tag{2.117}$$

To determine p and q on $s = 0$, it is necessary to consider the initial condition $u(0,y) = -\frac{y^2}{2}$. Differentiating with respect to y gives $u_y(0,y) = -y$, and from the original equation $u_x(0,y) = u_y^2(0,y) = y^2$. This, then gives

$$p(r,0) = r^2, \quad q(r,0) = -r. \tag{2.118}$$

From the last two equations of (2.116) we obtain

$$p = a(r), \quad q = b(r) \tag{2.119}$$

where a and b are arbitrary functions. From the initial condition (2.118), we find that

$$p = r^2, \quad q = -r, \tag{2.120}$$

for all s. Further, using these we have from (2.116)

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 2r, \quad \frac{du}{ds} = -r^2. \tag{2.121}$$

These integrate to give

$$x = s + f(r), \quad y = 2rs + g(r), \quad u = -r^2 s + h(r). \tag{2.122}$$

Using the initial conditions (2.117) gives

$$x = s, \quad y = 2rs + r, \quad u = -r^2 s - \frac{r^2}{2}. \tag{2.123}$$

On elimination of r and s in u gives

$$u = -\frac{y^2}{2(2x+1)}. \tag{2.124}$$

Example 12
Solve

$$u_x u_y = 4y, \qquad u(x,x) = \frac{8}{3}x^{3/2}. \tag{2.125}$$

Here, we use the method of characteristics. If we set

$$F = pq - 4y, \tag{2.126}$$

then the characteristic equations (2.113) become

$$\frac{dx}{ds} = q, \quad \frac{dy}{ds} = p, \quad \frac{du}{ds} = 2pq, \quad \frac{dp}{ds} = 0 \quad \frac{dq}{ds} = 4. \tag{2.127}$$

As this is an initial value problem, it is again necessary to find appropriate initial conditions for the characteristic equations. In the (x, y) plane, the line $y = x$ is where u is defined. To this, we must associate a curve in the (r, s). As we have freedom here, we will choose $s = 0$ and identify that $r = x$. Therefore, we have

$$x(r,0) = r, \quad y(r,0) = r, \quad u(r,0) = \frac{8}{3}r^{3/2}. \tag{2.128}$$

To determine p and q on $s = 0$, it is necessary to consider the initial condition $u(x,x) = \frac{8}{3}x^{3/2}$. Differentiating this with respect to x gives

$$u_x(x,x) + u_y(x,x) = 4x^{1/2},$$

and from the original equation we find that

$$u_x(x,x)u_y(x,x) = 4x.$$

This, corresponds to

$$p(r,0) + q(r,0) = 4r^{1/2}, \quad p(r,0)q(r,0) = 4r, \tag{2.129}$$

from which we find that

$$p(r,0) = 2r^{1/2}, \quad q(r,0) = 2r^{1/2}, \tag{2.130}$$

Thus, we must solve (2.127) subject to (2.128) and (2.130). Solving the last two of (2.127) gives

$$p = f(r), \quad q = 4s + g(r),$$

and using (2.130) gives

$$p = 2r^{1/2}, \quad q = 4s + 2r^{1/2}. \tag{2.131}$$

From (2.127), using (2.131) we then have

$$\frac{dx}{ds} = 4s + 2r^{1/2}, \quad \frac{dy}{ds} = 2r^{1/2}, \quad \frac{du}{ds} = 2(2r^{1/2})(4s + 2r^{1/2}).$$

Integrating and imposing the initial conditions (2.128) gives

$$x = 2s^2 + 2\sqrt{r}\,s + r, \quad y = 2\sqrt{r}\,s + r, \quad u = 8\sqrt{r}\,s^2 + 8rs + \frac{8}{3}r^{3/2}. \tag{2.132}$$

From (2.132), we find that

$$r = \left(\sqrt{\frac{x+y}{2}} - \sqrt{\frac{x-y}{2}}\right)^2, \quad s = \sqrt{\frac{x-y}{2}}, \tag{2.133}$$

(using only the positive case) and further from (2.132), we find the exact solution as

$$u = \frac{2\sqrt{2}}{3}\left((x+y)^{3/2} - (x-y)^{3/2}\right). \tag{2.134}$$

2.6.2 Charpit's Method

An alternate method for deriving exact solutions of first order nonlinear partial differential equations is known as *Charpit's Method*. Consider the nonlinear PDE

$$u_t = u_x^2, \tag{2.135}$$

and the linear PDE

$$2t\,u_t + x\,u_x = 0. \tag{2.136}$$

The solution of (2.136) is found to be

$$u = f\left(\frac{x^2}{t}\right). \tag{2.137}$$

Substitution of the solution (2.137) into the nonlinear PDE (2.135) gives

$$-\frac{x^2}{t^2}f' = 4\frac{x^2}{t^2}f'^2,$$

which simplifies to

$$f'\left(4f'+1\right) = 0.$$

From this we can deduce $f = c$, a constant or $f(\lambda) = -\frac{\lambda}{4}$ which leads to the exact solution $u = -\frac{x^2}{4t}$.

Consider the first order PDE

$$t\,u_t - x\,u_x = -3u,$$

whose solution is found to be

$$u = \frac{1}{t^3}f\left(xt\right). \tag{2.138}$$

Substitution of the solution (2.138) into the nonlinear PDE (2.135) gives

$$f'^2 - \lambda f' + 3f = 0 \tag{2.139}$$

where $\lambda = xt$. The general solution of (2.139) is

$$f = \frac{2c}{3}\left(\frac{c+\sqrt{c^2-2x}}{2}\right)^3 - \left(\frac{c+\sqrt{c^2-2x}}{2}\right)^4,$$

thus giving an exact solution solution to the PDE (2.135) as

$$u = \frac{2c}{3t^3}\left(\frac{c+\sqrt{c^2-2xt}}{2}\right)^3 - \frac{1}{t^3}\left(\frac{c+\sqrt{c^2-2xt}}{2}\right)^4,$$

noting that upon setting $c = 0$ we recover the previous solution.

The question we now pose is: “How did we know how to pick a second PDE whose solution would lead to a solution of the original nonlinear PDE?” Before we answer this question it is interesting to consider the following pairs of equations

$$(i)\quad u_t = u_x^2, \quad 2t\,u_t + x\,u_x = 0, \tag{2.140a}$$

$$(ii)\quad u_t = u_x^2, \quad t\,u_t - x\,u_x = -3u. \tag{2.140b}$$

In the first pair, we augmented the nonlinear PDE with one that is linear (and hence solvable). We substituted the solution of the linear equation into the nonlinear equations and obtained an ODE which we solved. This gave rise to an exact solution to the original equation. Thus, we were able to show that they share a common solution. If two PDEs share a common solution, they are said to be *compatible*. We also did this for the second pair. However, it should be noted that not all first order PDEs will be compatible. Consider, for example,

$$u_x = 2x,$$

Integrating gives

$$u = x^2 + f(t), \tag{2.141}$$

and substituting (2.141) into (2.135) gives

$$f'(t) = 4x^2 \tag{2.142}$$

and clearly no function $f(t)$ will work.

So we ask, is it possible to determine whether two PDEs are compatible before trying to solve one and see if a solution can be obtained to the other? We consider the first pair in (2.140a) and construct higher order PDEs by differentiating with respect to t and x. This leads to the following

$$u_{tt} = 2u_x u_{tx}, \quad u_{tx} = 2u_x\, u_{xx} \tag{2.143a}$$

$$2tu_{tt} + xu_{tx} + 2u_t = 0, \quad 2tu_{tx} + xu_{xx} + u_x = 0 \tag{2.143b}$$

Solving the first three of (2.143) gives

$$u_{tt} = -4\,\frac{u_t u_x}{4\,t\,u_x + x},\ u_{tx} = -2\,\frac{u_t}{4\,t\,u_x + x},\ u_{xx} = -\,\frac{u_t}{u_x(4\,t\,u_x + x)},$$

and substitution in the last of (2.143) gives

$$\frac{u_x^2 - u_t}{u_x} = 0,$$

which is identically true by virtue of the original equation. Thus, we have a way to check whether two equations are compatible and the method is known a Charpit's method.

Consider the compatibility of the following first order PDEs

$$F(x, y, u, p, q) = 0, \tag{2.144a}$$

$$G(x, y, u, p, q) = 0. \tag{2.144b}$$

where $p = u_x$ and $q = u_y$. Calculating second order derivatives of (2.144a) gives

$$\begin{aligned}
F_x + pF_u + u_{xx}F_p + u_{xy}F_q &= 0,\\
F_y + qF_u + u_{xy}F_p + u_{yy}F_q &= 0,\\
G_x + pG_u + u_{xx}G_p + u_{xy}G_q &= 0,\\
G_y + qG_u + u_{xy}G_p + u_{yy}G_q &= 0.
\end{aligned}$$

Solving the first three (2.145) for u_{xx}, u_{xy} and u_{yy} gives

$$\begin{aligned}
u_{xx} &= \frac{-F_x\,G_q - p\,F_u\,G_q + F_q\,G_x + p\,F_q\,G_u}{F_p\,G_q - F_q\,G_p},\\
u_{xy} &= \frac{-F_p\,G_x - p\,F_p\,G_u + F_x\,G_p + p\,F_u\,G_p}{F_p\,G_q - F_q\,G_p},\\
u_{yy} &= \frac{\begin{array}{c}F_p^2\,G_x + p\,F_p^2\,G_u - F_y\,F_p\,G_q - q\,F_u\,F_p\,G_q\\ +q\,F_u\,F_q\,G_p - F_x\,F_p\,G_p - p\,F_u\,F_p\,G_p + F_y\,F_q\,G_p\end{array}}{(F_p\,G_q - F_q\,G_p)F_q}.
\end{aligned}$$

Substitution into the last of (2.145) gives

$$F_p\,G_x + F_q\,G_y + (p\,F_p + q\,F_q)G_u - (F_x + p\,F_u)G_p - (F_y + q\,F_u)G_q = 0,$$

or conveniently written as

$$\begin{vmatrix} D_xF & F_p \\ D_xG & G_p \end{vmatrix} + \begin{vmatrix} D_yF & F_q \\ D_yG & G_q \end{vmatrix} = 0, \tag{2.147}$$

where $D_x\,F = F_x + p\,F_u$, $D_y\,F = F_y + q\,F_u$ and $|\cdot|$ the usual determinant.

Example 13
Consider

$$u_t = u_x^2. \tag{2.148}$$

This is the example we considered already, however now we will determine all classes of equation that are compatible with this one. Denoting

$$G = u_t - u_x^2 = p - q^2,$$

where $p = u_t$ and $q = u_x$, then

$$G_t = 0,\;\; G_x = 0,\;\; G_u = 0,\;\; G_p = 1,\;\; G_q = -2q,$$

and the Charpit equations are

$$\begin{vmatrix} D_t F & F_p \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} D_x F & F_q \\ 0 & -2q \end{vmatrix} = 0,$$

or, after expansion

$$F_t - 2qF_x + \left(p - 2q^2\right) F_u = 0,$$

noting that the third term can be replaced by $-pF_u$ due to the original equation. Solving this linear PDE by the method of characteristics gives the solution as

$$F = F(x + 2tu_x,\ u + tu_t,\ u_t,\ u_x). \tag{2.149}$$

If we set F in (2.149)

$$F(a,b,c,d) = ad, \text{ or } F(a,b,c,d) = ad - 3b,$$

and consider

$$F + 2t\left(u_t - u_x^2\right) = 0,$$

we obtain the first and second compatible equations found in (2.140), whereas, if we choose

$$F(a,b,c,d) = ad - b,$$

and consider

$$F + 2t\left(u_t - u_x^2\right) = 0,$$

we obtain a new compatible equation

$$t\,u_t + x\,u_x = u.$$

Example 14
Consider

$$u_x^2 + u_y^2 = u^2. \tag{2.150}$$

Denoting $p = u_x$ and $q = u_y$, then

$$G = u_x^2 + u_y^2 - u^2 = p^2 + q^2 - u^2.$$

Thus

$$G_x = 0,\ G_y = 0,\ G_u = -2u,\ G_p = 2p,\ G_q = 2q,$$

and the Charpit equation's are

$$\begin{vmatrix} D_x F & F_p \\ -2pu & 2p \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ -2qu & 2q \end{vmatrix} = 0,$$

or, after expansion

$$pF_x + qF_y + \left(p^2 + q^2\right) F_u + puF_p + quF_q = 0, \tag{2.151}$$

noting that the third term can be replaced by $u^2 F_u$ due to the original equation. Solving (2.151), a linear PDE, by the method of characteristics gives the solution as

$$F = F\left(x - \frac{p}{u}\ln u, y - \frac{q}{u}\ln u, \frac{p}{u}, \frac{q}{u}\right).$$

Consider the following particular example

$$x - \frac{p}{u}\ln u + y - \frac{q}{u}\ln u = 0,$$

or

$$u_x + u_y = (x+y)\frac{u}{\ln u}.$$

If we let $u = \mathrm{e}^{\sqrt{v}}$ then this becomes

$$v_x + v_y = 2(x+y),$$

which, by the method of characteristics, has the solution

$$v = 2xy + f(x-y).$$

This, in turn, gives the solution for u as

$$u = \mathrm{e}^{\sqrt{2xy + f(x-y)}}. \tag{2.152}$$

Substitution into the original equation (2.150) gives the following ODE

$$f'(r)^2 - 2rf'(r) - 2f(r) + 2r^2 = 0, \quad r = x - y.$$

If we let $f(r) = g(r) + \frac{1}{2}r^2$ then we obtain

$$g'^2 - 2g = 0 \tag{2.153}$$

whose solution is given by

$$g = \frac{(r+c)^2}{2} \tag{2.154}$$

where c is an arbitrary constant of integration. This, in turn, gives

$$f = r^2 + cr + \frac{1}{2}c^2 \tag{2.155}$$

and substitution into (2.152) gives

$$u = e^{\sqrt{x^2+y^2+c(x-y)+\frac{1}{2}c^2}},$$

as an exact solution to the original PDE.

It is interesting to note that when we substitute the solution of the compatible equation into the original it reduces to an ODE. A natural question is, does this always happen? This was recently proven to be true in two independent variables by D. J. Arrigo, *J. Non Math Phys.* **12**(3), 321-329 (2005).

Exercises

1. Solve the following first order PDE using a change of coordinates $(x, y) \to (r, s)$

(i) $u_x - 2u_y = -u$,
(ii) $2xu_x + 3y\, u_y = x, \quad u(x, x) = 1$,
(iii) $2u_t - u_x = 4$,
(iv) $u_x + u_y = 6y$,
(v) $xu_x - 2uu_y = x$,
(vi) $u_t - xu_y = t, \quad u(x, 0) = \sin x$.

2. Solve the following first order PDEs using the method of characteristics

(i) $xyu_x + (x^2 + y^2)u_y = yu$,
(ii) $u_x + (y + 1)u_y = u + x$,
(ii) $x^2u_x - y^2u_y = u^2$,
(iv) $xu_x + (x + y)u_y = x$,
(v) $yu_x + xu_y = xy, \quad u(x, 0) = x^2$.

3. Solve the following first order PDEs subject to the initial condition $u(x, 0) = \text{sech}(x)$. Explain the behavior of each solution for $t > 0$

(i) $u_t + (x + 1)u_x = 0$,
(ii) $u_t + (u - 2)u_x = 0$,
(iii) $u_t + (u - 3x)u_x = 0$,
(iv) $u_t + (1 + 2x + 3u)u_x = 0$.

4. Use the method characteristics to solve the following.

(i) $u_x^2 + u_y^2 = x^2$
(ii) $u_t + uu_x^2 = 0$
(iii) $u_xu_y = 1$

5. Use Charpit's method to find compatible first order equation to the one given. Use any one of your equation to find an exact solution.

(i) $u_x^2 + u_y^2 = x^2$

(ii) $u_t + uu_x^2 = 0$

(iii) $u_x u_y = 1$

6. If the characteristic equations are

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)},$$

show for some constant α, β and γ that

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = \frac{d(\alpha a + \beta b + \gamma c)}{\alpha a + \beta b + \gamma c}.$$

6. Generalize Charpit's method for nonlinear first order PDEs of the form

$$F\left(x_1, x_2, \cdots x_n, u_{x_1}, u_{x_2}, \cdots u_{x_n}\right) = 0.$$

Chapter 3

Second Order Linear PDEs

3.1 Introduction

The general class of second order linear PDEs are of the form:

$$\begin{aligned} a(x,y)u_{xx} + b(x,y)u_{xy} &+ c(x,y)u_{yy} \\ + d(x,y)u_x &+ e(x,y)u_y + f(x,y)u = g(x,y). \end{aligned} \tag{3.1}$$

The three PDEs that lie at the cornerstone of applied mathematics are: the heat equation, the wave equation and Laplace's equation,*i.e.*

$$\begin{aligned} (i) \quad u_t &= u_{xx}, && \text{the heat equation} \\ (ii) \quad u_{tt} &= u_{xx}, && \text{the wave equation} \\ (iii) \quad u_{xx} + u_{yy} &= 0, && \text{Laplace's equation} \end{aligned}$$

or, using the same independent variables, x and y

$$(i) \quad u_{xx} - u_y = 0, \qquad \text{the heat equation} \tag{3.3a}$$

$$(ii) \quad u_{xx} - u_{yy} = 0, \qquad \text{the wave equation} \tag{3.3b}$$

$$(iii) \quad u_{xx} + u_{yy} = 0. \qquad \text{Laplace's equation} \tag{3.3c}$$

Analogous to characterizing quadratic equations

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

as either hyperbolic, parabolic or elliptic determined by

$$\begin{aligned} b^2 - 4ac &> 0, && \text{hyperbolic,} \\ b^2 - 4ac &= 0, && \text{parabolic,} \\ b^2 - 4ac &< 0, && \text{elliptic,} \end{aligned}$$

we do the same for PDEs. So, for the heat equation $a = 1, \ b = 0, \ c = 0$ so $b^2 - 4ac = 0$ and so the heat equation is parabolic. Similarly, the wave equation is hyperbolic and Laplace's equation is elliptic. This leads to a natural question. Is it possible to transform one PDE to another where the new PDE is simpler? Namely, under a change of variable

$$r = r(x, y), \qquad s = s(x, y),$$

can we transform to one of the following *canonical* forms:

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0, \qquad \text{hyperbolic,} \tag{3.5a}$$

$$u_{ss} + \text{l.o.t.s.} = 0, \qquad \text{parabolic,} \tag{3.5b}$$

$$u_{rr} + u_{ss} + \text{l.o.t.s.} = 0, \qquad \text{elliptic,} \tag{3.5c}$$

where the term "l.o.t.s" stands for lower order terms. For example, consider the PDE

$$2u_{xx} - 2u_{xy} + 5u_{yy} = 0. \tag{3.6}$$

This equation is elliptic since the elliptic $b^2 - 4ac = 4 - 40 = -36 < 0$. If we introduce new coordinates,

$$r = 2x + y, \qquad s = x - y, \tag{3.7}$$

then by a change of variable using the chain rule

$$\begin{aligned}
u_{xx} &= u_{rr}r_x^2 + 2u_{rs}r_xs_x + u_{ss}s_x^2 + u_rr_{xx} + u_ss_{xx}, \\
u_{xy} &= u_{rr}r_xr_y + u_{rs}(r_xs_y + r_ys_x) + u_{ss}s_xs_y + u_rr_{xy} + u_ss_{xy}, \\
u_{yy} &= u_{rr}r_y^2 + 2u_{rs}r_ys_y + u_{ss}s_y^2 + u_rr_{yy} + u_ss_{yy},
\end{aligned} \tag{3.8}$$

gives

$$\begin{aligned}
u_{xx} &= 4u_{rr} + 4u_{rs} + u_{ss}, \\
u_{xy} &= 2u_{rr} - u_{rs} - u_{ss}, \\
u_{yy} &= u_{rr} - 2u_{rs} + u_{ss}.
\end{aligned}$$

Under (3.7), equation (3.6) becomes

$$u_{rr} + u_{ss} = 0,$$

which is Laplace's equation (also elliptic). Before we consider transformations for PDEs in general, it is important to determine whether the equation type could change under transformation. Consider the general class of PDEs

$$au_{xx} + bu_{xy} + cu_{yy} = 0 \tag{3.9}$$

where a, b, and c are functions of x and y and noting that we have suppressed the lower order terms as they will not affect the type. Under a change of variable $(x, y) \to (r, s)$ with the change of variable formulas (3.8) gives

$$\begin{aligned} & a\left(u_{rr}r_x^2 + 2u_{rs}r_xs_x + u_{ss}s_x^2 + u_rr_{xx} + u_ss_{xx}\right) \\ + \; & b\left(u_{rr}r_xr_y + u_{rs}(r_xs_y + r_ys_x) + u_{ss}s_xs_y + u_rr_{xy} + u_ss_{xy}\right) \\ + \; & c\left(u_{yy} + u_{rr}r_y^2 + 2u_{rs}r_ys_y + u_{ss}s_y^2 + u_rr_{yy} + u_ss_{yy}\right) = 0 \end{aligned} \tag{3.10}$$

Rearranging (3.10), and again neglecting lower order terms, gives

$$\begin{aligned} (ar_x^2 + br_xr_y + cr_y^2)u_{rr} \; & + \; (2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y)u_{rs} \\ & + \; (as_x^2 + bs_xs_y + cs_y^2)u_{ss} = 0. \end{aligned} \tag{3.11}$$

Setting

$$\begin{aligned} A &= ar_x^2 + br_xr_y + cr_y^2, \\ B &= 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y, \\ C &= as_x^2 + bs_xs_y + cs_y^2, \end{aligned} \tag{3.12}$$

gives (again suppressing lower order terms)

$$Au_{rr} + Bu_{rs} + Cu_{ss} = 0,$$

whose type is given by

$$B^2 - 4AC = (b^2 - 4ac)\left(r_xs_y - r_ys_x\right)^2,$$

from which we deduce that

$$\begin{aligned} b^2 - 4ac &> 0, \quad \Rightarrow \quad B^2 - 4AC > 0, \\ b^2 - 4ac &= 0, \quad \Rightarrow \quad B^2 - 4AC = 0, \\ b^2 - 4ac &< 0, \quad \Rightarrow \quad B^2 - 4AC < 0, \end{aligned}$$

giving that the equation type is unchanged under transformation. We now consider transformations to canonical form. As there are three standard forms (hyperbolic, parabolic and elliptic) we will deal with each type separately.

3.2 Standard Forms

If we introduce the change of coordinates

$$r = r(x, y), \qquad s = s(x, y), \tag{3.13}$$

the derivatives change according to:
First Order

$$u_x = u_r r_x + u_s s_x, \qquad u_y = u_r r_y + u_s s_y, \tag{3.14}$$

Second Order

$$\begin{aligned} u_{xx} &= u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx}, \\ u_{xy} &= u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy}, \\ u_{yy} &= u_{rr} r_y^2 + 2u_{rs} r_y s_y + u_{ss} s_y^2 + u_r r_{yy} + u_s s_{yy}, \end{aligned} \tag{3.15}$$

If we substitute (3.14) and (3.15) into the general linear equation (3.1) and re-arrange we obtain

$$\begin{aligned} (ar_x^2 &+ br_x r_y + cr_y^2) u_{rr} + \left(2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y\right) u_{rs} \\ &+ (as_x^2 + bs_x s_y + cs_y^2) u_{ss} + \text{l.o.t.s.} = 0. \end{aligned} \tag{3.16}$$

Our goal now is to target a given standard form and solve a set of equations for the new coordinates r and s.

3.2.1 Parabolic Standard Form

Comparing (3.16) with the parabolic standard form (3.5b) leads to choosing

$$ar_x^2 + br_x r_y + cr_y^2 = 0, \tag{3.17a}$$

$$2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y = 0, \tag{3.17b}$$

Since in the parabolic case $b^2 - 4ac = 0$, then substituting $c = \frac{b^2}{4a}$ we find both equations of (3.17) are satisfied if

$$2ar_x + br_y = 0. \tag{3.18}$$

with the choice of $s(x, y)$ arbitrary. The following examples demonstrate.

Example 1.
Consider

$$u_{xx} + 6u_{xy} + 9u_{yy} = 0. \tag{3.19}$$

Here, $a = 1$, $b = 6$ and $c = 9$ showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$r_x + 3r_y = 0,$$

gives

$$r = f(3x - y).$$

As we wish to find new coordinates as to transform the original equation to standard form, we choose

$$r = 3x - y, \quad s = y.$$

Calculating second derivatives

$$u_{xx} = 9u_{rr}, \quad u_{xy} = -3u_{rr} + 3u_{rs}, \quad u_{yy} = u_{rr} - 2u_{rs} + u_{ss}. \tag{3.20}$$

Substituting (3.20) into (3.19) gives

$$u_{ss} = 0!^{\dagger}$$

Integrating twice gives

$$u = f(r)s + g(r).$$

where f and g are arbitrary functions of r. In terms of the original variables, we obtain the solution

$$u = yf(3x - y) + g(3x - y).$$

Example 2.
Consider

$$x^2 u_{xx} - 4xyu_{xy} + 4y^2 u_{yy} + xu_x = 0. \tag{3.21}$$

Here, $a = x^2$, $b = -4xy$ and $c = 4y^2$ showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$x^2 r_x - 2xyr_y = 0,$$

or

$$xr_x - 2yr_y = 0,$$

† Not to be confused with factorial (!).

gives

$$r = f(x^2 y).$$

As we wish to find new coordinates, *i.e.* r and s, we choose simple

$$r = x^2 y, \qquad s = y.$$

Calculating first derivatives gives

$$u_x = 2xyu_r. \tag{3.22}$$

Calculating second derivatives

$$\begin{aligned}
u_{xx} &= 4x^2y^2u_{rr} + 2yu_r, &(3.23a)\\
u_{xy} &= 2x^3yu_{rr} + 2xyu_{rs} + 2xu_r, &(3.23b)\\
u_{yy} &= x^4u_{rr} + 2x^2u_{rs} + u_{ss}. &(3.23c)
\end{aligned}$$

Substituting (3.22) and (3.23) into (3.21) gives

$$4y^2u_{ss} - 4x^2yu_r = 0.$$

or, in terms of the new variables, r and s,

$$u_{ss} - \frac{r}{s^2}u_r = 0. \tag{3.24}$$

An interesting question is whether different choices of the arbitrary function f and the variable s would lead to a different standard forms. For example, suppose we chose

$$r = 2\ln x + \ln y, \qquad s = \ln y,$$

we would obtain

$$u_{ss} - u_r - u_s = 0, \tag{3.25}$$

a constant coefficient parabolic equation, whereas, choosing

$$r = 2\ln x + \ln y, \qquad s = 2\ln x,$$

we would obtain

$$u_{ss} - u_r = 0, \tag{3.26}$$

the heat equation.

3.2.2 Hyperbolic Standard Form

In order to obtain the standard form for the hyperbolic type, *i.e.*

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0, \tag{3.27}$$

from (3.16), we find it is necessary to choose

$$\begin{aligned} ar_x^2 + br_xr_y + cr_y^2 &= -\left(as_x^2 + bs_xs_y + cs_y^2\right), \\ 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y &= 0. \end{aligned} \tag{3.28}$$

The problem is that this system is still a very hard problem to solve (both PDEs are nonlinear and coupled!). Therefore, we introduce a modified hyperbolic form that is much easier to work with.

3.2.3 Modified Hyperbolic Form

The modified hyperbolic standard form is defined as

$$u_{rs} + \text{l.o.t.s.} = 0, \tag{3.29}$$

noting that $a = 0$, $b = 1$ and $c = 0$ and that $b^2 - 4ac > 0$ still! In order to target the modified hyperbolic form, from (3.16) it is now necessary to choose

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \tag{3.30a}$$
$$as_x^2 + bs_xs_y + cs_y^2 = 0. \tag{3.30b}$$

If we re-write (3.30a) and (3.30b) as

$$a\left(\frac{r_x}{r_y}\right)^2 + 2b\,\frac{r_x}{r_y} + c = 0, \tag{3.31a}$$
$$a\left(\frac{s_x}{s_y}\right)^2 + 2b\,\frac{s_x}{s_y} + c = 0, \tag{3.31b}$$

we can solve equations (3.31a) and (3.31b) separately for $\frac{r_x}{r_y}$ and $\frac{s_x}{s_y}$. This leads to two first order linear PDEs for r and s. The solutions of these then gives rise to the correct standard variables. The following examples demonstrate.

Example 3.
Consider

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0 \tag{3.32}$$

Here, $a = 1$, $b = -5$ and $c = 6$ showing that $b^2 - 4ac = 1 > 0$, so the PDE is hyperbolic. Thus, (3.30) becomes

$$r_x^2 - 5r_xr_y + 6r_y^2 = 0, \quad s_x^2 - 5s_xr_y + 6s_y^2 = 0,$$

and factoring gives

$$\left(r_x - 2r_y\right)\left(r_x - 3r_y\right) = 0, \quad \left(s_x - 2s_y\right)\left(s_x - 3s_y\right) = 0,$$

from which we choose

$$r_x - 2r_y = 0, \quad s_x - 3s_y = 0,$$

giving rise to solutions

$$r = R(2x + y), \quad s = S(3x + y)$$

where R and S are arbitrary functions of their arguments. As we wish to find new coordinates as to transform the original equation to standard form, we choose

$$r = 2x + y, \quad s = 3x + y.$$

Calculating second derivatives

$$\begin{aligned} u_{xx} &= 4u_{rr} + 12u_{rs} + 9u_{ss}, \\ u_{xy} &= 2u_{rr} + 5u_{rs} + 3u_{ss}, \\ u_{yy} &= u_{rr} + 2u_{rs} + u_{ss}. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.32) gives

$$u_{rs} = 0.$$

Solving gives

$$u = f(r) + g(s).$$

where f and g are arbitrary functions. In terms of the original variables, we obtain the solution

$$u = f(2x + y) + g(3x + y).$$

Example 4.
Consider

$$xu_{xx} - (x+y)u_{xy} + yu_{yy} = 0. \tag{3.34}$$

Here, $a = x$, $b = -(x+y)$ and $c = y$ showing that $b^2 - 4ac = (x-y)^2 > 0$, so the PDE is hyperbolic. We thus need to solve

$$xr_x^2 - (x+y)r_x r_y + yr_y^2 = 0,$$

or, upon factoring

$$\left(xr_x - yr_y\right)\left(r_x - r_y\right) = 0.$$

As s satisfies the same equation, we choose the first factor for r and the second for s, *i.e.*

$$xr_x - yr_y = 0, \quad s_x - s_y = 0. \tag{3.35}$$

Upon solving (3.35), we obtain

$$r = f(xy), \qquad s = g(x+y).$$

The simplest choice for the new coordinates r and s are

$$r = xy, \qquad s = x+y.$$

Calculating first derivatives gives

$$u_x = yu_r + u_s, \qquad u_y = xu_r + u_s. \tag{3.36}$$

Calculating second derivatives

$$\begin{aligned} u_{xx} &= y^2 u_{rr} + 2yu_{rs} + u_{ss}, \\ u_{xy} &= xyu_{rr} + (x+y)u_{rs} + u_{ss} + u_r, \\ u_{yy} &= x^2 u_{rr} + 2xu_{rs} + u_{ss}. \end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.34) gives

$$\left(4xy - (x+y)^2\right) u_{rs} - (x+y)u_r = 0,$$

or, in terms of the new variables, r and s,

$$u_{rs} + \frac{s}{s^2 - 4r} u_r = 0.$$

3.2.4 Regular Hyperbolic Form

We now wish to transform a given hyperbolic PDE to its regular standard form

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0. \tag{3.38}$$

First, let us consider the following example.

$$x^2 u_{xx} - y^2 u_{yy} = 0. \tag{3.39}$$

If we were to transform to modified standard form, we would solve

$$x r_x - y r_y = 0, \quad x s_x + y s_y = 0,$$

which gives

$$r = f(xy), \quad s = g(x/y).$$

To find new coordinates r and s we again choose simple

$$r = xy, \quad s = x/y.$$

In doing so, the original PDE then becomes

$$u_{rs} - \frac{1}{2r} u_s = 0. \tag{3.40}$$

However, if we choose

$$r = \ln x + \ln y, \quad s = \ln x - \ln y,$$

then the original PDE becomes

$$u_{rs} - u_s = 0, \tag{3.41}$$

which is clearly an easier PDE. However, if we introduce new coordinates α and β such that

$$\alpha = \frac{r+s}{2}, \quad \beta = \frac{r-s}{2},$$

noting that derivatives transform

$$u_r = \frac{1}{2} u_\alpha + \frac{1}{2} u_\beta, \quad u_s = \frac{1}{2} u_\alpha - \frac{1}{2} u_\beta \quad u_{rs} = \frac{1}{4} u_{\alpha\alpha} - \frac{1}{4} u_{\beta\beta}, \tag{3.42}$$

and the PDE (4.3) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} - 2u_\alpha + 2u_\beta = 0,$$

a PDE in regular hyperbolic form. Thus, combining the variables r and s and α and β gives directly

$$\alpha = \ln x, \qquad \beta = \ln y.$$

In fact, one can show that if

$$\alpha = \frac{r+s}{2}, \qquad \beta = \frac{r-s}{2},$$

where r and s satisfies (3.30a) and (3.30b) then α and β satisfies

$$a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2 = -\left(a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2\right), \tag{3.43a}$$

$$2a\alpha_x\beta_x + b(\alpha_x\beta_y + \alpha_y\beta_x) + 2c\alpha_y\beta_y = 0. \tag{3.43b}$$

which is (3.76a) with r and s replaces with α and β. This give a convenient way to go directly to the coordinates that lead to the regular hyperbolic form. We note that

$$\alpha, \beta = \frac{r \pm s}{2}, \tag{3.44}$$

so we can essentially consider

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \tag{3.45a}$$

$$as_x^2 + bs_xs_y + cs_y^2 = 0. \tag{3.45b}$$

but instead of factoring, treat each as a quadratic equation in r_x/r_y or s_x/s_y and solve according. We demonstrate with an example.

Example 5.
Consider

$$8u_{xx} - 6u_{xy} + u_{yy} = 0. \tag{3.46}$$

The corresponding equations for r and s are

$$8r_x^2 - 6r_xr_y + r_y^2 = 0, \tag{3.47a}$$

$$8s_x^2 - 6s_xs_y + s_y^2 = 0, \tag{3.47b}$$

but as they are identical it suffices to only consider one. Dividing (3.75a) by r_y^2 gives

$$8\left(\frac{r_x}{r_y}\right)^2 - 6\frac{r_x}{r_y} + 1 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{6 \pm 2}{16},$$

or

$$8r_x - (3 \pm 1)r_y = 0.$$

The method of characteristics gives

$$\frac{dx}{8} = -\frac{dy}{3 \pm 1}; \quad dr = 0.$$

which gives

$$r = f\left((3 \pm 1)x + 8y\right),$$

which we choose

$$r = 3x + 8y \pm x,$$

which leads to the chose

$$r = 3x + 8y, \qquad s = x,$$

Under this transformation, the original equation (3.46) becomes

$$u_{rr} - u_{ss} = 0,$$

the desired standard form.

Example 6.
Consider

$$xy^3 u_{xx} - x^2y^2 u_{xy} - 2x^3 y u_{yy} - y^2 u_x + 2x^2 u_y = 0. \tag{3.48}$$

The corresponding equations for r and s are

$$xy^3 r_x^2 - x^2y^2 r_x r_y - 2x^3 y r_y^2 = 0, \tag{3.49a}$$
$$xy^3 s_x^2 - x^2y^2 s_x r_y - 2x^3 y s_y^2 = 0, \tag{3.49b}$$

and choosing the first gives

$$y^2 \left(\frac{r_x}{r_y}\right)^2 - xy\frac{r_x}{r_y} - 2x^2 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(1 \pm 3)x}{2y},$$

or

$$2yr_x - (1 \pm 3)\, xr_y = 0.$$

Solving gives

$$r = f\left(x^2 + 2y^2 \pm 3x^2\right).$$

If we choose f to be simple and split according to the $\pm$ gives

$$r = x^2 + 2y^2, \quad s = 3x^2,$$

Under this transformation, the original equation (5.92) becomes

$$u_{rr} - u_{ss} = 0,$$

the desired standard form.

3.2.5 Elliptic Standard Form

In order to obtain the standard form for the elliptic type, *i.e.*

$$u_{rr} + u_{ss} + \text{l.o.t.s.} = 0,$$

then from (3.16), it is necessary to choose

$$\begin{aligned} ar_x^2 + br_xr_y + cr_y^2 &= \left(as_x^2 + bs_xs_y + cs_y^2\right), \\ 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y &= 0. \end{aligned} \tag{3.50}$$

The problem, like the regular hyperbolic type, is that it is still difficult to solve. However, we find that if we let [†]

$$r = \frac{\alpha + \beta}{2}, \quad s = \frac{\alpha - \beta}{2i} \tag{3.51}$$

where α and β satisfy

$$a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2 = 0, \tag{3.52a}$$

$$a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2 = 0, \tag{3.52b}$$

then (3.50) is satisfied. This is much like the connection between modified and regular hyperbolic standard form. As solving (3.52a) gives rise to complex roots, the

[†] Please note the switch in the variables r and s and α and β.

formulas (3.51) will take real and complex parts of the solved α and β equations as new variables. The next few examples will illustrate.

Example 7.
Consider

$$u_{xx} - 4u_{xy} + 5u_{yy} = 0. \tag{3.53}$$

The corresponding equations for r and s are

$$r_x^2 - 4r_x r_y + 5r_y^2 = 0, \tag{3.54a}$$

$$s_x^2 - 4s_x s_y + 5s_y^2 = 0, \tag{3.54b}$$

but as they are identical it suffices to only consider one. Dividing (3.54a) by r_y^2 gives

$$\left(\frac{r_x}{r_y}\right)^2 - 4\frac{r_x}{r_y} + 5 = 0.$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = 2 \pm i,$$

or

$$r_x - (2 \pm i)r_y = 0.$$

The method of characteristics gives

$$\frac{dx}{1} = -\frac{dy}{2 \pm i}, \quad dr = 0.$$

which gives

$$r = f\left(2x + y \pm ix\right),$$

which we choose

$$r = 2x + y \pm ix,$$

and taking the real and imaginary parts leads to the choice

$$r = 2x + y, \qquad s = x,$$

Under this transformation, the original equation (3.53) becomes

$$u_{rr} + u_{ss} = 0,$$

the desired standard form.

Example 8.
Consider

$$2\left(1+x^2\right)^2 u_{xx} - 2\left(1+x^2\right)\left(1+y^2\right)u_{xy} + \left(1+y^2\right)^2 u_{yy} + 4x\left(1+x^2\right)u_x = 0. \tag{3.55}$$

The corresponding equations for r and s are

$$2\left(1+x^2\right)^2 r_x^2 - 2\left(1+x^2\right)\left(1+y^2\right)r_x r_y + \left(1+y^2\right)^2 r_y^2 = 0, \tag{3.56a}$$

$$2\left(1+x^2\right)^2 s_x^2 - 2\left(1+x^2\right)\left(1+y^2\right)s_x s_y + \left(1+y^2\right)^2 s_y^2 = 0, \tag{3.56b}$$

but as they are identical it suffices to only consider one. Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(2\pm i)\left(1+y^2\right)}{1+x^2},$$

or

$$2\left(1+x^2\right)r_x - (1\pm i)\left(1+y^2\right)r_y = 0.$$

The method of characteristics gives the solution as

$$r = f\left(\tan^{-1}x + 2\tan^{-1}y \pm i\tan^{-1}x\right),$$

which we choose

$$r = \tan^{-1}x + 2\tan^{-1}y \pm i\tan^{-1}x,$$

which leads to the choice

$$r = \tan^{-1}x + 2\tan^{-1}y, \qquad s = \tan^{-1}x,$$

Under this transformation, the original equation (3.55) becomes

$$u_{rr} + u_{ss} - 2yu_r = 0,$$

and upon using the original transformation gives

$$u_{rr} + u_{ss} - 2\tan\frac{r-s}{2}u_r = 0,$$

the desired standard form.

3.3 The wave equation - Revisited

We now re-visit the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty \tag{3.57}$$

subject to the initial conditions

$$u(x,0) = f(x), \ \ -\infty < x < \infty, \tag{3.58}$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty. \tag{3.59}$$

As seen previously, we can target the modified hyperbolic equation by making the change of variables: $r = x - ct$ and $s = x + ct$. Equation (3.57) becomes

$$u_{rs} = 0, \tag{3.60}$$

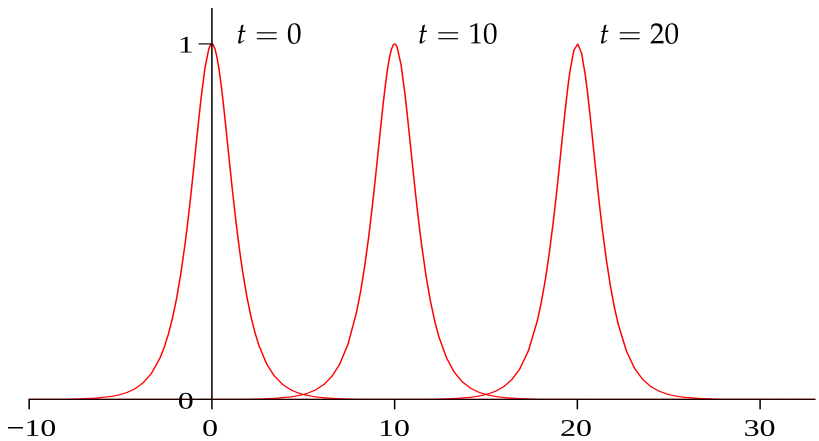
which we previously solved (see example 3). There, we obtained the solution

$$u = F(r) + G(s) \tag{3.61}$$

or in terms of x and t

$$u = F(x - ct) + G(x + ct). \tag{3.62}$$

To understand what this solution means, we look at the following example. Let $u = F(x - ct)$ (that is, $G = 0$) and let $F(x - ct) = \text{sech}^2(x - ct)$. When $c = 1$ this simplifies to $u = \text{sech}^2(x - t)$. This is just a wave traveling to the right. Similarly if we look at $u = \text{sech}^2(x + ct)$ this is a wave traveling to the left. Graphs for this at various times for t are given below



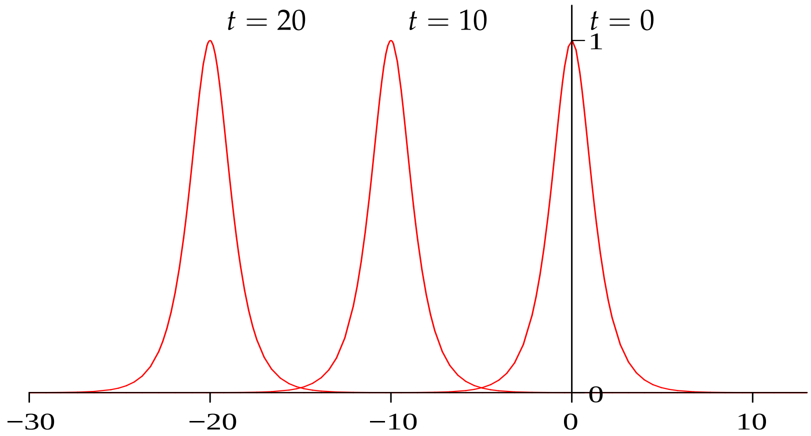


Figure 1. Traveling wave solutions $u = \operatorname{sech}(x \pm t)$ at times t = 0, 10 and 20.

Thus in general

$$u = F(x - ct) + G(x + ct), \tag{3.63}$$

consists of two waves, one traveling right and one traveling left.
We now incorporate general initial conditions, *i.e.*

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{3.64}$$

Imposing these initial conditions on the solution (5.25) we obtain

$$F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x). \tag{3.65}$$

From these we obtain

$$F'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x), \quad G'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x), \tag{3.66}$$

and upon integration

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds, \tag{3.67a}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds. \tag{3.67b}$$

Thus, we obtain

$$u(x,t) = \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c}\left\{\int_0^{x+ct} g(s)ds - \int_0^{x-ct} g(s)ds\right\},$$

which simplifies to

$$u(x,t) = \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{3.68}$$

This is know as the *d'Alembert* solution.

Example:

$$u_{tt} = u_{xx}$$

subject to the initial conditions:

$$u(x,0) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{if otherwise} \end{cases}, \quad u_t(x,0) = 0 \tag{3.69}$$

If we let

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x, \end{cases} \tag{3.70}$$

then (3.69) can be rewritten as

$$u(x,0) = 2\left(H(x+1) - H(x-1)\right) \tag{3.71}$$

so the solution for this problem is given by:

$$u(x,t) = H(x+1+t) - H(x-1+t) + H(x+1-t) - H(x-1-t) \tag{3.72}$$

The following figure illustrates the solution at times $t = 0, .5, 1, 1.5$ and 2.

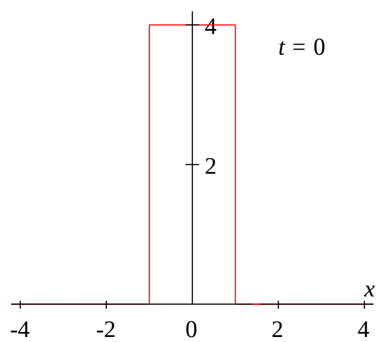


Figure 2. The initial condition at $t = 0$.

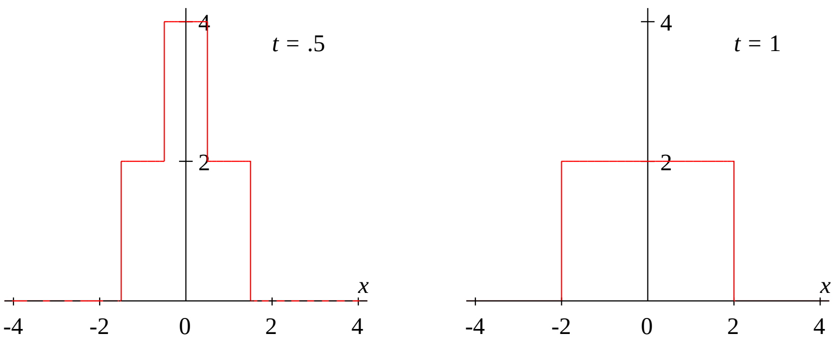


Figure 3. Traveling wave solutions at times t = 0.5 and 1.

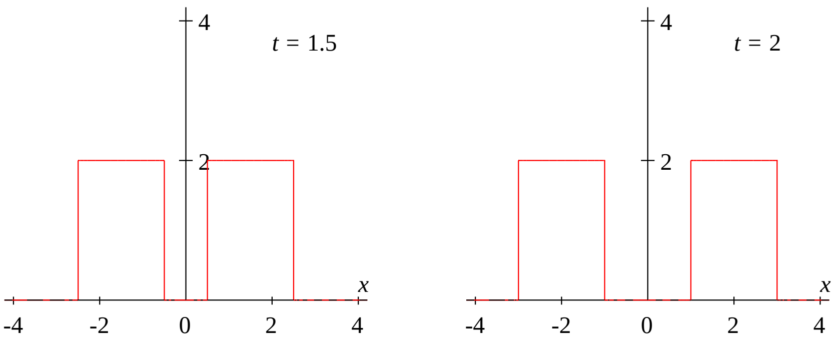


Figure 4. Traveling wave solutions at times t = 1.5 and 2.

Exercises

1. Determine the type of the following second order PDEs

(i) $x^2u_{xx} - y^2u_{yy} = u_x + u_y$
(ii) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(iii) $y^2u_{xx} + 2yu_{xy} - u_{yy} = 0$
(iv) $u_{xy} + u = u_x + u_y$

2. Transform the following parabolic PDEs to standard form. Find the general solution if possible.

(i) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(ii) $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = 0.$
(iii) $y^2 u_{xx} + 2xy u_{xy} + x^2 u_{yy} - 2x u_x = 0.$

3. Reduce the following second order PDEs to modified hyperbolic standard form

(i) $2u_{xx} - 3u_{xy} + u_{yy} = u_x + u_y,$
(ii) $x^2 u_{xx} - 3xy u_{xy} + 2y^2 u_{yy} = 0.$

4. Reduce the following second order PDEs to standard form

(i) $4u_{xx} - 8u_{xy} + 3u_{yy} = 0,$
(ii) $4u_{xx} + 4u_{xy} + 5u_{yy} = 1,$
(iii) $x^2 u_{xx} + y^2 u_{yy} = 1,$
(iv) $u_{xx} - (1 + y^2)^2 u_{yy} = 0.$

5. If a linear second order PDE

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{lots} = 0.$$

is hyperbolic, then it is possible to transform to a modified hyperbolic standard form

$$u_{rs} + \text{l.o.t.s.} = 0,$$

by choosing r and s such that they satisfy

$$\begin{aligned} ar_x^2 + 2br_x r_y + cr_y^2 &= 0, \\ as_x^2 + 2bs_x s_y + cs_y^2 &= 0. \end{aligned}$$

5(i) Show that by introducing new variables α and β such that

$$\alpha = r + s, \quad \beta = r - s,$$

then the following equations are satisfied

$$\begin{aligned} a\alpha_x^2 + 2b\alpha_x\alpha_y + c\alpha_y^2 &= -\left(a\beta_x^2 + 2b\beta_x\beta_y + c\beta_y^2\right), \\ a\alpha_x\beta_x + b(\alpha_x\beta_y + r_y\beta_x) + c\alpha_y\beta_y &= 0. \end{aligned}$$

This then leads to the hyperbolic standard form

$$u_{\alpha\alpha} - u_{\beta\beta} + \text{l.o.t.s.} = 0.$$

5(ii). If, instead of the α and β introduced in 5(i), we introduce α and β such that

$$\alpha = c_{11}r + c_{12}s, \quad \beta = c_{21}r + c_{22}s,$$

where c_{11}, c_{12}, c_{21} and c_{22} are constant and r and s satisfy the above first order PDEs, namely

$$\begin{aligned} ar_x^2 + 2br_x r_y + cr_y^2 &= 0, \\ as_x^2 + 2bs_x s_y + cs_y^2 &= 0, \end{aligned}$$

find conditions on c_{11}, c_{12}, c_{21} and c_{22} such that

$$\begin{aligned} a\alpha_x^2 + 2b\alpha_x\alpha_y + c\alpha_y^2 &= -\left(a\beta_x^2 + 2b\beta_x\beta_y + c\beta_y^2\right), \\ a\alpha_x\beta_x + b(\alpha_x\beta_y + r_y\beta_x) + c\alpha_y\beta_y &= 0. \end{aligned}$$

are still satisfied.

Chapter 4

Fourier Series

At this point we now consider the major technique for solving standard boundary value equations. Consider, for example the heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0 \tag{4.1}$$

subject to

$$u(x,0) = 2\sin x, \quad u(0,t) = u(\pi,t) = 0. \tag{4.2}$$

Here, we will assume that the solutions are separable and are of the form

$$u(x,t) = X(x)T(t). \tag{4.3}$$

Substituting into the heat equation (5.1) gives

$$XT' = X''T,$$

or, after dividing by TX

$$\frac{T'}{T} = \frac{X''}{X}. \tag{4.4}$$

Since each side is a function of a different variable, they therefore must be independent of each other and thus

$$\frac{T'}{T} = \lambda = \frac{X''}{X}, \tag{4.5}$$

or

$$T' = \lambda T, \quad X'' = \lambda X, \tag{4.6}$$

where λ is a constant. The boundary conditions in (5.2) become, accordingly

$$X(0) = X(\pi) = 0. \tag{4.7}$$

Integrating the X equation in (5.4) gives rise to three cases depending on the sign of λ. These are

$$X(x) = \begin{cases} c_1 e^{nx} + c_2 e^{-nx} & \text{if } \lambda = n^2, \\ c_1 x + c_2 & \text{if } \lambda = 0, \\ c_1 \sin nx + c_2 \cos nx & \text{if } \lambda = -n^2, \end{cases}$$

where n is a constant. We consider each case separately. In the first case where $\lambda = n^2$ imposing the boundary conditions (5.5) gives that

$$c_1 e^0 + c_2 e^0 = 0, \quad c_1 e^{n\pi} + c_2 e^{-n\pi} = 0, \tag{4.8}$$

and solving for c_1 and c_2 shows that both are identically zero. In the case where $\lambda = 0$, imposing the boundary conditions (5.5) gives that

$$c_1 0 + c_2 = 0, \quad c_1 \pi + c_2 = 0, \tag{4.9}$$

and solving for c_1 and c_2 shows again that both are identically zero. In the remaining case, $\lambda = -n^2$, using the boundary conditions (5.5) gives

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin n\pi + c_2 \cos n\pi = 0, \tag{4.10}$$

which leads to

$$c_2 = 0, \quad \sin n\pi = 0 \quad \Rightarrow \quad n = 0, 1, 2, \ldots. \tag{4.11}$$

From (5.4) with $\lambda = -n^2$ gives

$$T' = -n^2 T, \tag{4.12}$$

from which we find that

$$T(t) = c_3 e^{-n^2 t} \tag{4.13}$$

where c_3 is a constant of integration and thus, giving a solution to the original PDE as

$$u = XT = c e^{-n^2 t} \sin nx \tag{4.14}$$

where we have set $c = c_1 c_3$. Finally, imposing the initial condition $u(x, 0) = 2 \sin x$ gives

$$u(x, 0) = c e^0 \sin nx = 2 \sin x \tag{4.15}$$

from which we obtain $c = 2$ and $n = 1$. Therefore the solution to the PDE subject to the initial and boundary conditions is

$$u(x, t) = 2 e^{-t} \sin x. \tag{4.16}$$

If the initial condition was different, say $u(x,0) = 4\sin 3x$, then $c = 4$ and $n = 3$ and the solution would be

$$u(x,t) = 4e^{-9t}\sin 3x. \tag{4.17}$$

However, if the initial condition was $u(x,0) = 2\sin x + 4\sin 3x$ it would be impossible to choose n and c to satisfy both. However, if we were to solve the heat equation with each initial condition separately, we can simply just add the solutions together. This is called the **principle of superposition**.

Theorem Principle of Superposition
If u_1 and u_2 are two solution to the heat equation, then $u = c_1u_1 + c_2u_2$ is also a solution.

Proof. Calculating the derivatives $u_t = c_1u_{1t} + c_2u_{2t}$ and $u_{xx} = c_1u_{1xx} + c_2u_{2xx}$ and substituting into the heat equation show it is identically satisfied if each of u_1 and u_2 satisfy the heat equation. □

Therefore to solve the heat equation subject to $u(x,0) = 2\sin x + 4\sin 3x$ we would obtain

$$u(x,t) = 2e^{-t}\sin x + 4e^{-9t}\sin 3x.$$

The principle of superposition easily extends to more than 2 solutions. Thus, if

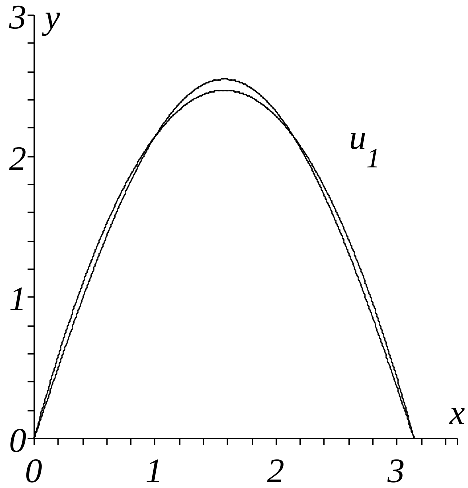
$$u(x,t) = (a_n\cos nx + b_n\sin nx)\,e^{-n^2t}, \quad n = 1,2,3,4,\ldots$$

are solutions to the heat equation, then so is

$$u(x,t) = \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-n^2t}. \tag{4.18}$$

If the initial conditions were such that they only involved sine and cosine functions, we could choose the integers n and constants a_n and b_n according to match the terms in the initial condition. However, if the initial condition was, for example, $u(x,0) = \pi x - x^2$, then it is not obvious how to proceed as (5.25) contains no x terms. However, consider the following

$$\begin{aligned}
u_1 &= \frac{8}{\pi}e^{-t}\sin x,\\
u_2 &= \frac{8}{\pi}\left(e^{-t}\sin x + \frac{1}{27}e^{-9t}\sin 3x\right),\\
u_3 &= \frac{8}{\pi}\left(e^{-t}\sin x + \frac{1}{27}e^{-9t}\sin 3x + \frac{1}{125}e^{-25t}\sin 5x\right).
\end{aligned} \tag{4.19}$$



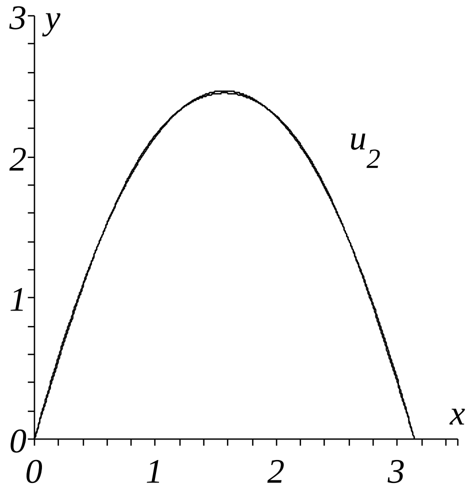


Figure 1. The solutions (4.19) with one and two terms.

From figure 1, one will notice that with each additional term added in (4.19), the solution is a better match to the initial condition. In fact, if we consider

$$u = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)^2 t} \sin(2n-1)x \tag{4.20}$$

we get a perfect match to the initial condition. Thus, we are lead to ask "How are the integers n and constants a_n and b_n chosen as to match the initial condition?"

4.1 Fourier Series

It is well known that infinitely many functions can be represented by a power series

$$f(x) = \sum_{i=0}^{\infty} a_n \left(x - x_0\right)^n ,$$

where x_0 is the center of the series and a_n, constants determined by

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \quad i = 0, 1, 2, 3, \dots .$$

For functions that require different properties, say for example, fixed points at the endpoints of an interval, a different type of series is required. An example of such a series is called Fourier series. For example, suppose that $f(x) = \pi x - x^2$ has a Fourier series

$$\pi x - x^2 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x \dots . \tag{4.21}$$

How do we choose b_1, b_2, b_3 *etc.* such that the Fourier series looks like the function? Notice that if multiply (4.21) by $\sin x$ and integrate from 0 to π

$$\int_0^{\pi} (\pi x - x^2) \sin x \, dx = b_1 \int_0^{\pi} \sin^2 x \, dx + b_2 \int_0^{\pi} \sin x \sin 2x \, dx + \dots .$$

then we obtain

$$4 = b_1 \frac{\pi}{2} \quad \Rightarrow \quad b_1 = 8/\pi, \tag{4.22}$$

since

$$\int_0^{\pi} (\pi x - x^2) \sin x \, dx = 4, \quad \int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2},$$
$$\text{and} \quad \int_0^{\pi} \sin x \sin nx \, dx = 0, \;\; n = 2, 3, 4, \dots$$

Similarly, if multiply (4.21) by $\sin 2x$ and integrate from 0 to π

$$\int_0^{\pi} (\pi x - x^2) \sin 2x \, dx = b_1 \int_0^{\pi} \sin x \sin 2x \, dx + b_2 \int_0^{\pi} \sin^2 2x \, dx + \dots .$$

then we obtain

$$0 = b_2 \frac{\pi}{2} \quad \Rightarrow \quad b_2 = 0.$$

Multiply (4.21) by $\sin 3x$ and integrate from 0 to π

$$\int_0^{\pi} (\pi x - x^2) \sin 3x \, dx = b_1 \int_0^{\pi} \sin x \sin 3x \, dx + b_2 \int_0^{\pi} \sin 2x \sin 3x \, dx + \dots ,$$

then we obtain

$$\frac{4}{27} = b_3\,\frac{\pi}{2} \quad \Rightarrow \quad b_3 = \frac{8}{27\pi}. \tag{4.23}$$

Continuing in this fashion, we would obtain

$$b_4 = 0,\;\; b_5 = \frac{8}{125\pi},\;\; b_6 = 0,\;\; b_7 = \frac{8}{343\pi}. \tag{4.24}$$

Substitution of (4.22), (4.23) and (4.24) into (4.21) gives (4.19).

4.2 Fourier Series on $[-\pi,\pi]$

Consider the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \tag{4.25}$$

where a_0, a_n and b_n are constant coefficients. The question is: "How do we choose the coefficients as to give an accurate representation of $f(x)$?" Well, we use the following properties of $\cos n\pi x$ and $\sin n\pi x$

$$\int_{-\pi}^{\pi} \cos nx\,dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx\,dx = 0, \tag{4.26}$$

$$\int_{-\pi}^{\pi} \cos nx\;\cos mx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{4.27}$$

$$\int_{-\pi}^{\pi} \sin nx\;\sin mx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{4.28}$$

$$\int_{-\pi}^{\pi} \sin nx\;\cos mx\,dx = 0. \tag{4.29}$$

First, if we integrate (4.25) from $-\pi$ to π, then by the properties in (4.26), we are left with

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0,$$

from which we deduce

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Next we multiply the series (4.25) by $\cos mx$ giving

$$f(x)\cos mx = \frac{1}{2}a_0 \cos mx + \sum_{n=1}^{\infty}\left(a_n \cos nx\, \cos mx + b_n \sin nx\, \cos mx\right).$$

Again, integrate from $-\pi$ to π. From (4.26), the integration of $a_0 \cos mx$ is zero, from (4.27), the integration of $\cos nx\, \cos mx$ is zero except when $n = m$ and further from (4.29) the integrations of $\sin nx\, \cos mx$ is zero for all m and n. This leaves

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi a_n,$$

or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

Similarly, if we multiply the series (4.25) by $\sin mx$ then we obtain

$$f(x)\sin mx = \frac{1}{2}a_0 \sin mx + \sum_{n=1}^{\infty}\left(a_n \cos nx\, \sin mx + b_n \sin nx\, \sin mx\right),$$

which we integrate from $-\pi$ to π. From (4.26), the integration of $a_0 \sin m\pi x$ is zero, from (4.29) the integration of $\sin nx\, \cos mx$ is zero for all m and n and further from (4.28) the integration of $\sin nx\, \sin mx$ is zero except when $n = m$. This leaves

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi b_n,$$

or

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where the coefficients a_0, a_n and b_n are chosen such that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \quad \text{for} \;\; n = 1, 2, \ldots. \end{aligned} \tag{4.30}$$

As the integral formula's for a_0 and a_n are so similar, it is more convenient for us to shift the factor of $1/2$ in a_0 into the Fourier series, thus

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where all of the a_n's are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{4.31}$$

Example 1
Consider

$$f(x) = x^2, \quad [-\pi,\pi]. \tag{4.32}$$

From (4.31) we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\,\frac{x^3}{3}\bigg|_{-\pi}^{\pi} = \frac{2\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} + 2\,\frac{x\cos nx}{n^2} - 2\,\frac{\sin nx}{n^3}\right]\bigg|_{-\pi}^{\pi} \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

and from (4.30)

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx \\
&= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + 2\,\frac{x\sin nx}{n^2} + 2\,\frac{\cos nx}{n^3}\right]\bigg|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

Thus, the Fourier series for $f(x) = x^2$ on $[-\pi,\pi]$ is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}. \tag{4.33}$$

Figures 2 and 3 show consecutive plots of the Fourier series (4.33) with 5 and 10 terms on the interval [$-\pi$,π] and [-3π,3π].

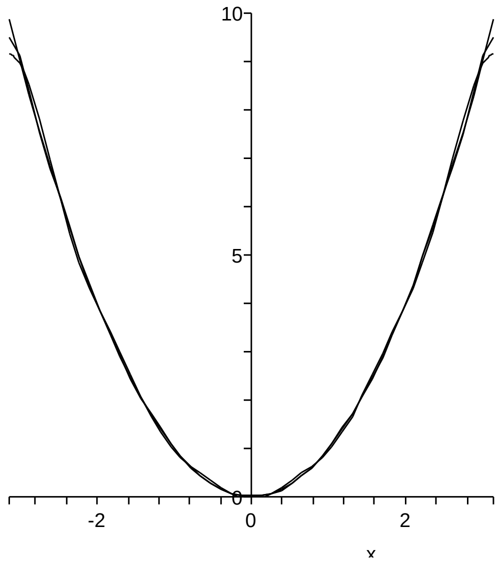


Figure 2. The solutions (4.33) with five and ten terms on $[-\pi,\pi]$.

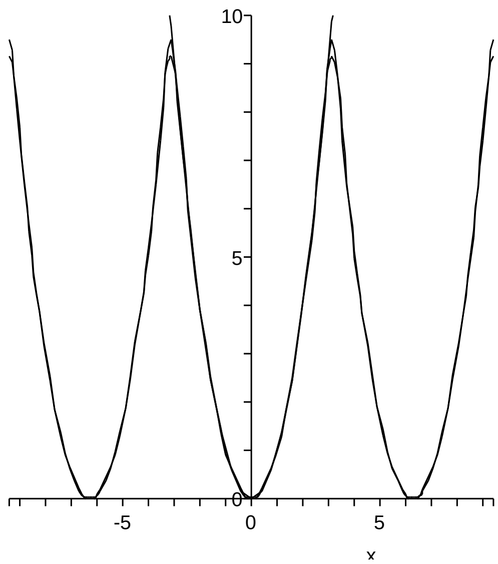


Figure 3. The solutions (4.33) with five and ten terms on $[-3\pi,3\pi]$.

Example 2
Consider

$$f(x) = x, \quad [-\pi, \pi] \tag{4.34}$$

From (4.31) we have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} = 0, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \\ &= \frac{1}{\pi}\left.\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\right|_{-\pi}^{\pi} \\ &= 0, \end{aligned}$$

and from (4.30)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \frac{1}{\pi}\left.\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]\right|_{-\pi}^{\pi} \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}. \tag{4.35}$$

Figures 4 and 5 show consecutive plots of the Fourier series (4.35) with 5 and 50 terms on the interval [$-\pi$,π] and [-3π,3π].

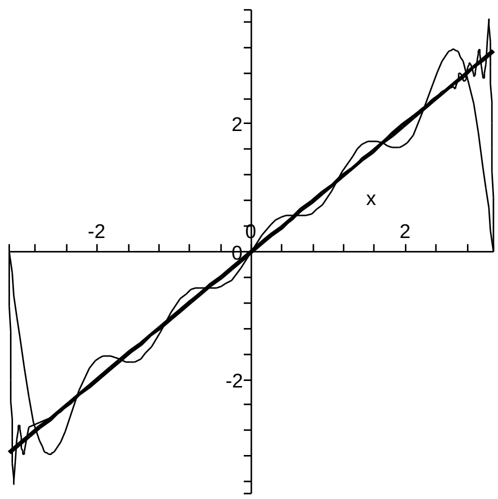


Figure 4. The solutions (4.35) with five and fifty terms on [$-\pi$,π].

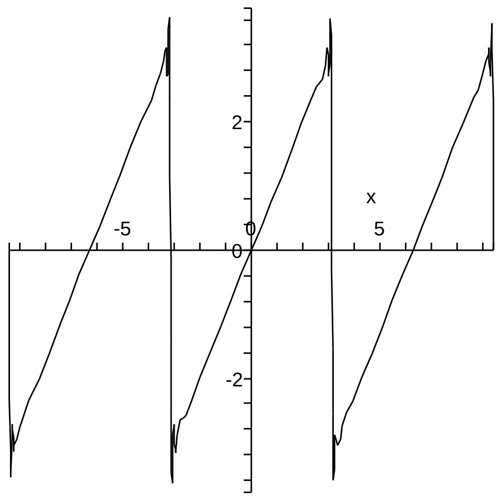


Figure 5. The solutions (4.35) with five and fifty terms on $[-3\pi,3\pi]$.

Example 3
Consider

$$f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0, \\ \mathrm{x}+1 & \text{if } 0 < x < \pi, \end{cases}$$

From (4.31) we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} x+1\,dx = \frac{1}{\pi}\,x|_{-\pi}^{0} + \frac{1}{\pi}\,\frac{x^2}{2}+x\Big|_0^{\pi} = \frac{\pi}{2}+2, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+1)\cos nx\,dx \\
&= \frac{1}{\pi}\,\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[(x+1)\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]\Big|_0^{\pi} \\
&= \frac{1}{\pi}\frac{(-1)^n-1}{n^2},
\end{aligned}$$

and from (4.30)

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+1)\sin nx\,dx \\
&= \frac{1}{\pi}\,-\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[-(x+1)\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]\Big|_0^{\pi} \\
&= \frac{1}{\pi}\frac{(-1)^n-1}{n} + \frac{1}{\pi}\frac{(\pi+1)(-1)^{n+1}+1}{n} \\
&= \frac{(-1)^{n+1}}{n}.
\end{aligned}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + 1 + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \frac{(-1)^n - 1}{n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right). \tag{4.36}$$

Figure 6 shows a plot of the Fourier series (4.36) with 10 terms on the interval $[-\pi,\pi]$.

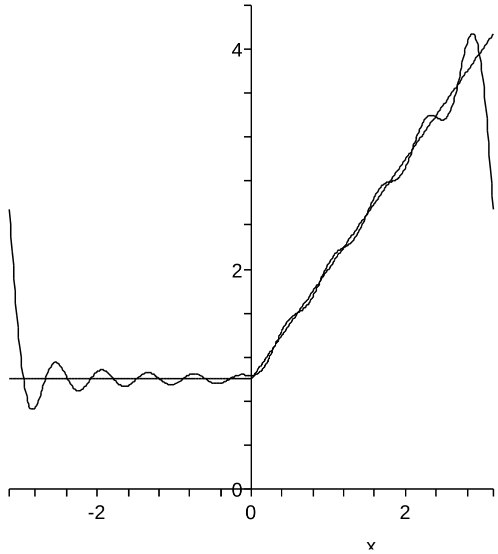


Figure 6. The solutions (4.36) with 10 terms.

As many problems are on intervals more general than $[-\pi, \pi]$, it is natural for us to extend to more general intervals.

4.3 Fourier Series on [$-L$,L]

Consider the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{4.37}$$

where L is a positive number and a_0, a_n and b_n constant coefficients. The question is: "How are the coefficients chosen as to give an accurate representation of $f(x)$?" Here, we use the following properties of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} dx = 0, \qquad \int_{-L}^{L} \sin \frac{n\pi x}{L} dx = 0, \tag{4.38}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4.39}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4.40}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L} dx = 0. \tag{4.41}$$

If we integrate (4.37) from $-L$ to L, then by the properties in (4.38), we are left with

$$\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-L}^{L} a_0 dx = La_0,$$

from which we deduce

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

Next we multiply the series (4.37) by $\cos\frac{m\pi x}{L}$ giving

$$f(x)\cos\frac{m\pi x}{L} = \frac{1}{2}a_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right).$$

Again, integrate from $-L$ to L. From (4.38), the integration of $a_0\cos\frac{m\pi x}{L}$ is zero, from (4.39), the integration of $\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero except when $n = m$ and further from (4.41) the integrations of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and n. This leaves

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}dx = La_n,$$

or

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx.$$

Similarly, if we multiply the series (4.37) by $\sin\frac{m\pi x}{L}$ then we obtain

$$f(x)\sin\frac{m\pi x}{L} = \frac{1}{2}a_0\sin\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right),$$

which we integrate from $-L$ to L. From (4.38), the integration of $a_0\sin\frac{m\pi x}{L}$ is zero, from (4.41) the integration of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and n and further from (4.40) the integration of $\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$ is zero except when $n = m$. This leaves

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}dx = La_n,$$

or

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where the coefficients a_n and b_n are chosen such that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{4.42}$$

for $n = 0, 1, 2, \ldots.$

Example 4
Consider

$$f(x) = 9 - x^2, \quad [-3, 3] \tag{4.43}$$

In this case $L = 3$ so from (4.42) we have

$$\begin{aligned}
a_0 &= \frac{1}{3}\int_{-3}^{3} 9 - x^2\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]\Bigg|_{-3}^{3} = 12,\\
a_n &= \frac{1}{3}\int_{-3}^{3}(9 - x^2)\cos\frac{n\pi x}{3}dx\\
&= \frac{1}{3}\left[\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\sin\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\cos\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3}\\
&= \frac{36(-1)^{n+1}}{n^2\pi^2},
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3}(9 - x^2)\cos\frac{n\pi x}{3}dx\\
&= \frac{1}{3}\left[-\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\cos\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\sin\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3}\\
&= 0.
\end{aligned}$$

Thus, the Fourier series for (4.43) on $[-3, 3]$ is

$$f(x) = 6 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{3}. \tag{4.44}$$

Figure 6 shows the graph of this Fourier series (4.44) with the first 20 terms.

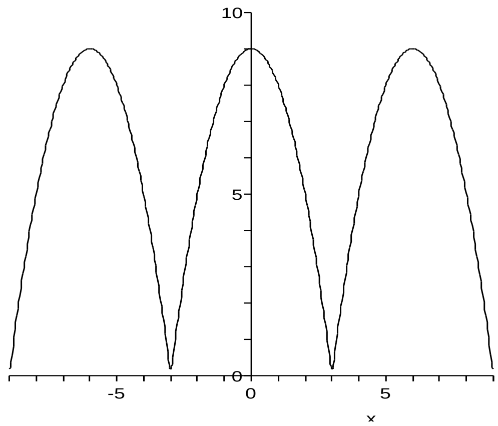


Figure 6. The solutions (4.44) using the first 20 terms.

Example 5
Consider

$$f(x) = \begin{cases} -2-x & \text{if } -2 < x < -1, \\ x & \text{if } -1 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \end{cases} \tag{4.45}$$

In this case $L = 2$ so from (4.42) we have

$$\begin{aligned} a_0 &= \frac{1}{2}\left\{ \int_{-2}^{-1} (-2-x)\,dx + \int_{-1}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx \right\} \\ &= \frac{1}{2}\left\{ \left[-2x - \frac{x^2}{2} \right]\Bigg|_{-2}^{-1} + \frac{x^2}{2}\Bigg|_{-1}^{1} + \left[2x - \frac{x^2}{2} \right]\Bigg|_{1}^{2} \right\} = 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2}\left\{ \int_{-2}^{-1} (-2-x)\cos\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2} (2-x)\cos\frac{n\pi x}{2}\,dx \right\} \\ &= \left[-\frac{(x+2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2} \right]\Bigg|_{-2}^{-1} \\ &+ \left[\frac{x}{n\pi}\sin\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2} \right]\Bigg|_{-1}^{1} \\ &+ \left[-\frac{(x-2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2} \right]\Bigg|_{1}^{2} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\sin\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\sin\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}\,dx\right\} \\
&= \left[\frac{(x+2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-2}^{-1} \\
&+ \left[-\frac{x}{n\pi}\cos\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-1}^{1} \\
&+ \left[\frac{(x-2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{1}^{2} \\
&= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} \frac{16}{n^2\pi^2} & \text{if } n = 1,5,9\ldots, \\ -\frac{16}{n^2\pi^2} & \text{if } n = 3,7,11\ldots, \end{cases}
\end{aligned}$$

Thus, the Fourier series for (4.45) on $[-2,2]$ is

$$\begin{aligned}
f_k(x) &= \frac{16}{\pi^2}\left\{\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - + \ldots\right\} \\
&= \frac{16}{\pi^2}\sum_{n=1}^{k}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2} \qquad (4.46)
\end{aligned}$$

Figure 7 show the graph of this Fourier series (4.46) with the first 5 terms.

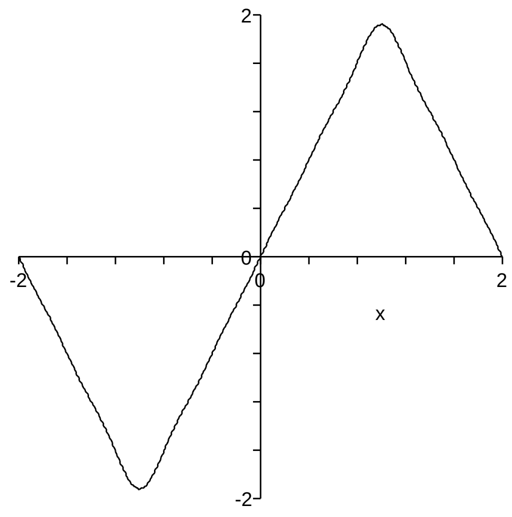


Figure 7. The solutions (4.46) with 5 terms.

Example 6

Consider

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1, \end{cases}$$

In this case $L = 1$ so from (4.42) we have

$$\begin{aligned} a_0 &= \int_0^1 dx = 1 \\ a_n &= \int_0^1 \cos n\pi x \, dx = \frac{1}{n\pi} \sin n\pi x \Big|_0^1 = 0 \end{aligned}$$

and

$$b_n = \int_0^1 \sin n\pi x \, dx - \frac{1}{n\pi} \cos\{n\pi x \Big|_0^1 = \frac{1 - (-1)^n}{n\pi}$$

Thus, the Fourier series for (4.3) on $[-1, 1]$ is

$$\begin{aligned} f_k(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{k} \frac{1 - (-1)^n}{n} \sin n\pi x \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{k} \frac{\sin(2n-1)\pi x}{2n-1} \end{aligned} \tag{4.47}$$

Figure 8 and 9 show the graph of this Fourier series (4.47) with thre first 5 and 50 terms.

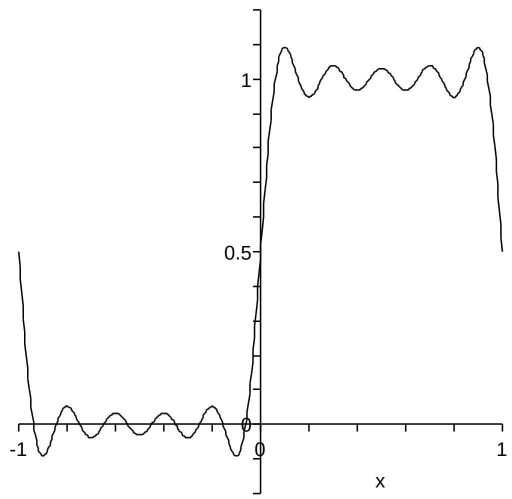


Figure 8. The solutions (4.47) with 5 terms.

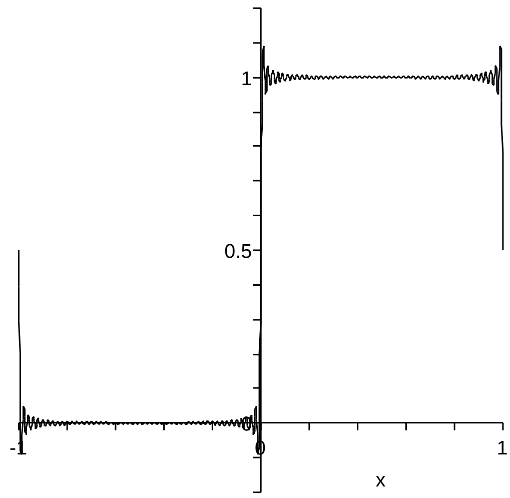


Figure 9. The solutions (4.47) with 50 terms.

It is interesting to note that regardless of the number of terms we have in the Fourier series, we cannot eliminate the spikes at $x = -1, 0, 1$ etc. This phenomena is know as **Gibb's phenomena**.

4.4 Odd and Even Extensions

Consider $f(x) = x$ on $[0,\pi]$. Here the interval is half the interval $[-\pi,\pi]$. Can we still construct a Fourier series for this? Well, it really depends on what $f(x)$ looks like on the interval $[-\pi,0]$. For example, if $f(x) = x$ on $[-\pi,0]$, then yes. If $f(x) = -x$ on $[-\pi,0]$, then also yes. In either case, as long a we are given $f(x)$ on $[-\pi,0]$, then the answer is yes. If we are just given $f(x)$ on $[0,\pi]$, then it is natural to extend $f(x)$ to $[-\pi,0]$ as either an odd extension or even extension. Recall that a function is even it $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. For example, if

$$f(x) = x, \text{then} f(-x) = -x = -f(x)$$

so $f(x) = x$ is odd. Similarly, if

$$f(x) = x^2, \text{then} f(-x) = (-x)^2 = x^2 = f(x)$$

so $f(x) = x^2$ is even. For each extension, the Fourier series constructed will contain only sine terms or cosine terms. These series respectively are called *Sine series* and *Cosine series*. Before we consider each series separately, it is necessary to establish the following lemma's.

LEMMA 1 *If $f(x)$ is an odd function then*

$$\int_{-l}^{l} f(x)\,dx = 0$$

and if $f(x)$ is an even function, then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

Proof
Consider

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx.$$

Under a change of variables $x = -y$, the second integral changes and we obtain

$$\int_{-l}^{l} f(x)\,dx = -\int_{l}^{0} f(-y)\,dy + \int_{0}^{l} f(x)\,dx.$$

If $f(x)$ is odd, then $f(-y) = -f(y)$ then

$$\begin{aligned}\int_{-l}^{l} f(x)\,dx &= \int_{l}^{0} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= -\int_{0}^{l} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= 0.\end{aligned}$$

If $f(y)$ is even, then $f(-y) = f(y)$ then

$$\begin{aligned}\int_{-l}^{l} f(x)\,dx &= -\int_{l}^{0} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= \int_{0}^{l} f(y)\,dy + \int_{0}^{l} f(x)\,dx \\ &= 2\int_{0}^{l} f(x)\,dx,\end{aligned}$$

establishing the result. At this point we a ready to consider each series separately.

4.4.1 Sine Series

If $f(x)$ is given on [0,L] we assume that $f(x)$ is an odd function which gives us $f(x)$ on the interval [$-L$,0]. We now consider the Fourier coefficients a_n and b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.48}$$

Since $f(x)$ is odd and $\cos\frac{n\pi x}{L}$ is even, then their product is odd and by lemma 1

$$a_n = 0, \quad \forall n.$$

Similarly, since $f(x)$ is odd and $\sin\frac{n\pi x}{L}$ is odd, then their product is even and by lemma 1

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.49}$$

The Fourier series is therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{4.50}$$

where b_n is given in (4.49).

Example 1
Find a Fourier sine series for

$$f(x) = x^2, \quad [0,1]. \tag{4.51}$$

The coefficient b_n is given by

$$\begin{aligned} b_n &= 2\int_0^1 x\sin n\pi x\,dx \\ &= \left[-\frac{x^2\cos n\pi\, x}{n\pi} + 2\,\frac{x\sin n\pi\, x}{n^2\pi^2} + 2\,\frac{\cos n\pi\, x}{n^3\pi^3}\right]\Bigg|_0^1 \\ &= 2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi} \end{aligned}$$

giving the Fourier Sine series as

$$f = 2\sum_{n=1}^{\infty}\left(2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right)\sin n\pi x. \tag{4.52}$$

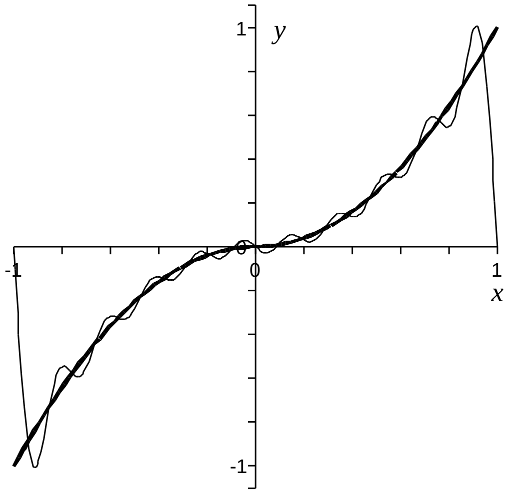


Figure 10. The function (4.51) with its odd extension and its Fourier Sine series (4.52) with 10 terms.

Example 2

Find a Fourier Sine series for

$$f(x) = \cos x, \quad [0, \pi] \tag{4.53}$$

Two cases need to be considered here. The case where $n = 1$ and the case where $n \neq 1$. The coefficient b_1 is given by

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin \pi x \, dx = 0$$

and the coefficient b_n is given by

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \cos x \sin n\pi x \, dx \\ &= \left[-\frac{1}{2} \frac{\cos(n-1)x}{n-1} - \frac{1}{2} \frac{\cos(n+1)x}{n+1} \right] \Bigg|_0^{\pi} \\ &= \frac{n\,(1 + (-1)^n)}{n^2 - 1}. \end{aligned}$$

The Fourier Sine series is then given by

$$\begin{aligned} f &= \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{n\,(1 + (-1)^n)}{n^2 - 1} \sin nx \\ &= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin 2nx. \end{aligned} \tag{4.54}$$

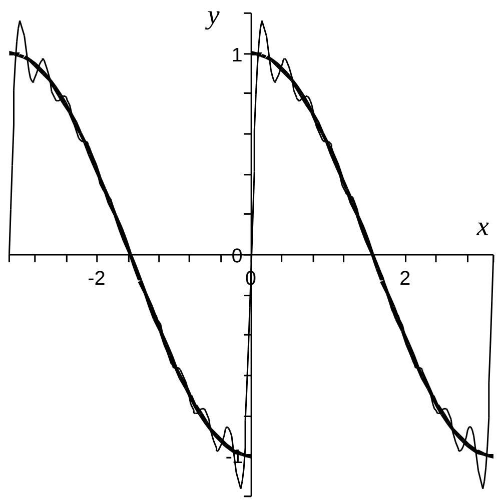


Figure 11. The function (4.53) with its odd extension and its Fourier Sine series (4.54) with 20 terms.

4.4.2 Cosine Series

If $f(x)$ is given on $[0,L]$ we assume that $f(x)$ is an even function which gives us $f(x)$ on the interval $[-L,L]$. We now consider the Fourier coefficients a_n and b_n.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \tag{4.55}$$

Since $f(x)$ is even and $\cos\frac{n\pi x}{L}$ is even, then their product is even and by lemma 1

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \tag{4.56}$$

Similarly

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{4.57}$$

Since $f(x)$ is even and $\sin\frac{n\pi x}{L}$ is odd, then their product is odd and by lemma 1

$$b_n = 0, \quad \forall\, n. \tag{4.58}$$

The Fourier series is therefore

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} \tag{4.59}$$

where a_n is given in (4.58).

Example 3
Find a Fourier cosine series for

$$f(x) = x, \quad [0,2]. \tag{4.60}$$

The coefficient a_0 is given by

$$a_0 = \frac{2}{2}\int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2.$$

The coefficient n_n is given by

$$\begin{aligned} a_n &= \int_0^2 x\cos\frac{n\pi}{2}x\,dx \\ &= \left.\left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\right|_0^2 \\ &= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \end{aligned}$$

giving the Fourier Cosine series as

$$f = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{2}. \tag{4.61}$$

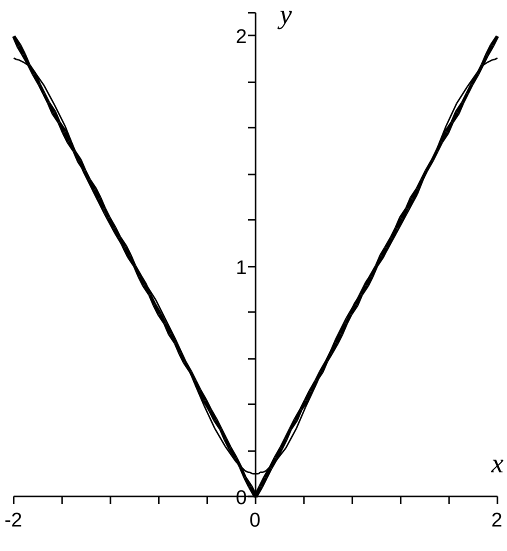


Figure 12. The function (4.60) with its even extension and its Fourier Sine series (4.61) with 3 terms.

Example 4
Find a Fourier cosine series for

$$f(x) = \sin x, \quad [0, \pi]. \tag{4.62}$$

The coefficient a_0 is given by

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi \sin x \, dx \\ &= \frac{2}{\pi} \left[-\cos\ x \right] \Big|_0^\pi \\ &= \frac{4}{\pi}. \end{aligned} \tag{4.63}$$

For the remaining coefficients a_n, the case a_1 again needs to be considered separately. For a_1

$$a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x \, dx = 0,$$

and the coefficient $a_n, \;\; n \geq 2$ is given by

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \sin x \cos n\pi x \, dx \\ &= \left[\frac{1}{2} \frac{\cos(n-1)x}{n-1} - \frac{1}{2} \frac{\cos(n+1)x}{n+1} \right] \Bigg|_0^\pi \\ &= -\frac{n\,(1+(-1)^n)}{n^2-1}. \end{aligned}$$

Thus, the Fourier series is

$$\begin{aligned} f &= -\frac{2}{\pi} \sum_{n=2}^{k} \frac{n\,(1+(-1)^n)}{n^2-1} \sin nx \\ &= -\frac{8}{\pi} \sum_{n=1}^{k} \frac{n}{4n^2-1} \sin 2nx. \end{aligned} \tag{4.64}$$

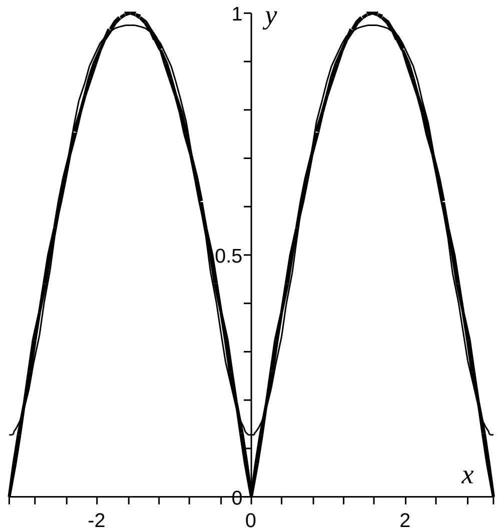


Figure 13. The function (4.62) with its even extension and its Fourier sine series (4.64) with 5 terms.

Example 5

Find a Fourier Sine and Cosine series for

$$f(x) = \begin{cases} 4x - x^2 & \text{for } 0 \le x \le 2 \\ 8 - 2x & \text{for } 2 < x < 4. \end{cases} \tag{4.65}$$

For the Fourier Sine series $a_n = 0$ and b_n are obtained by

$$\begin{aligned} b_n &= \frac{2}{4}\int_0^2 (4x - x^2)\sin\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8-2x)\sin\frac{n\pi x}{4}\,dx \\ &= \left[\left(\frac{32-16x}{n^2\pi^2}\right)\sin\frac{n\pi x}{4} + \left(\frac{2x^2-8x}{n\pi} - \frac{64}{n^3\pi^3}\right)\cos\frac{n\pi x}{4}\right]\Bigg|_0^2 \\ &+ \left[\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4} - \frac{4x-16}{n\pi}\cos\frac{n\pi x}{4}\right]\Bigg|_2^4 \\ &= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{64}{n^3\pi^3}\left(1 - \cos\frac{n\pi}{2}\right). \end{aligned}$$

Thus, the Fourier Sine series is given by

$$f = \frac{16}{\pi^2}\sum_{n=1}^{k}\left(\frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{4}{n^3\pi}\left(1 - \cos\frac{n\pi}{2}\right)\right)\sin\frac{n\pi x}{4}. \tag{4.66}$$

For the Fourier cosine series $b_n = 0$ and a_0 and a_n are given by

$$\begin{aligned} a_0 &= \frac{2}{4}\int_0^2 4x - x^2\,dx + \frac{2}{4}\int_2^4 8 - 2x\,dx \\ &= \frac{1}{2}\left[2x^2 - \frac{x^3}{3}\right]\Big|_0^2 + \frac{1}{2}\left[8x - x^2\right]\Big|_2^4 \\ &= \frac{8}{3} + 2 = \frac{14}{3}, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^2 (4x - x^2)\cos\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8 - 2x)\cos\frac{n\pi x}{4}\,dx \\ &= \left[\left(\frac{32 - 16x}{n^2\pi^2}\right)\cos\frac{n\pi x}{4} - \left(\frac{2x^2 - 8x}{n\pi} - \frac{64}{n^3\pi^3}\right)\sin\frac{n\pi x}{4}\right]\Big|_0^2 \\ &+ \left[-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4} + \frac{4x - 16}{n\pi}\sin\frac{n\pi x}{4}\right]\Big|_2^4 \\ &= \frac{16}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{64}{n^3\pi^3}\sin\frac{n\pi}{2}. \end{aligned}$$

Thus, the Fourier Cosine series is given by

$$f = \frac{14}{3} + \frac{16}{\pi^2}\sum_{n=1}^{k}\left(\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{4}{n^3\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{4}, \tag{4.67}$$

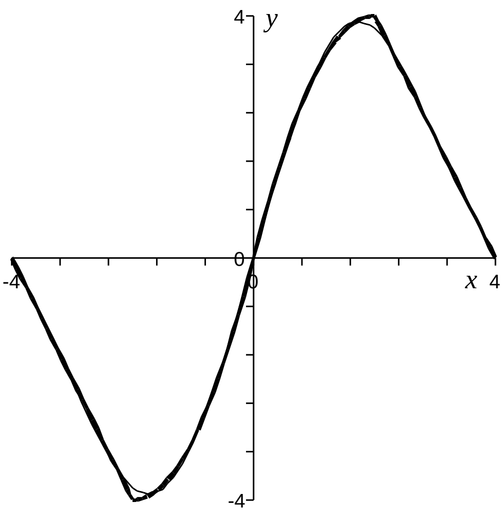


Figure 14. The function (4.65) with its odd extension and its Fourier Sine series (4.66) with 3 terms.

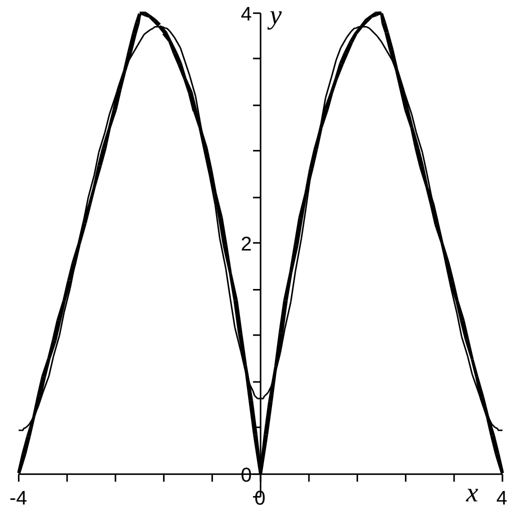


Figure 15. The function (4.65) with its even extension and its Fourier Cosine series (4.67) with 3 terms.

As shown in the examples in this chapter, often only a few terms are needed to obtain a fairly good representation of the function. It is interesting to note that if discontinuity is encountered on the extension, Gibb's phenomena occurs. In the next chapter, we return to solving the heat equation, Laplace's equation and the wave equation using separation of variables as introduced at the beginning of this chapter.

Exercises

1. Find Fourier series for the following

(i) $f(x) = \mathrm{e}^{-x}$ on $[-1, 1]$

(ii) $f(x) = |\,x\,|$ on $[-2, 2]$

(iii) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ x - 1 & \text{if } \;\; 0 < x < 1, \end{cases}$

(iv) $f(x) = \mathrm{e}^{-x^2}$ on $[-5, 5]$.

2. Find Fourier sine and cosine series for the following and illustrate the function and its corresponding series on $[-L, L]$ and $[-2L, 2L]$

(i) $f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 < x < 2, \\ 3 - x & \text{if } 2 < x < 3, \end{cases}$

(ii) $f(x) = x - x^2$ on $[0, 2]$

(iii) $f(x) = \begin{cases} x + 1 & \text{if } 0 < x < 1, \\ 4 - 2x & \text{if } 1 < x < 2, \end{cases}$,

(iv) $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{cases}$,

3. Find the first 10 terms numerically of the Fourier series of the following

(i) $f(x) = e^{-x^2}$, on $[-5, 5]$

(ii) $f(x) = \sqrt{x}$, on $[0, 4]$

(iii) $f(x) = -x \ln x$ on $[0, 1]$.

Chapter 5

Separation of Variables

We are ready to now resume our work on solving the three main equations: the heat equation, Laplace's equation and the wave equation using the method of separation of variables.

5.1 The heat equation

Consider, the heat equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{5.1}$$

subject to the initial and boundary conditions

$$u(x,0) = x - x^2, \quad u(0,t) = u(1,t) = 0. \tag{5.2}$$

Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.3}$$

the heat equation (5.1) becomes

$$XT' = X''T,$$

which, after dividing by XT and expanding gives

$$\frac{T'}{T} = \frac{X''}{X}.$$

As T is a function of t only and X a function of x only, this implies that

$$\frac{T'}{T} = \frac{X''}{X} = \lambda,$$

where λ is a constant. This gives that

$$T' = \lambda T, \quad X'' = \lambda X. \tag{5.4}$$

From (5.2) and (5.3), the boundary conditions become

$$X(0) = X(1) = 0. \tag{5.5}$$

Integrating the X equation in (5.4) gives rise to three cases depending on the sign of λ but as seen in the last chapter, only the case where $\lambda = -k^2$ for some constant k is applicable. This has the solution

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.6}$$

Imposing the boundary conditions (5.5) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin k + c_2 \cos k = 0, \tag{5.7}$$

which leads to

$$c_2 = 0, \quad c_1 \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots n\pi, \ldots \tag{5.8}$$

where n is an integer. From (5.4), we further deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where we have set $c_1 c_3 = b_n$. Using the initial condition gives

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

From the last chapter, we recognize this as a Fourier sine series and know that the coefficients b_n are chosen such that

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2) \sin n\pi x \, dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2}\cos n\pi x + \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right)\cos n\pi x\right]\Bigg|_0^1 \\ &= \frac{4}{n^3\pi^3}(1 - (-1)^n). \end{aligned}$$

Thus, the solution of the PDE as

$$u(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \tag{5.9}$$

Figure 1 shows the solution at times $t = 0, 0.1$ and 0.2.

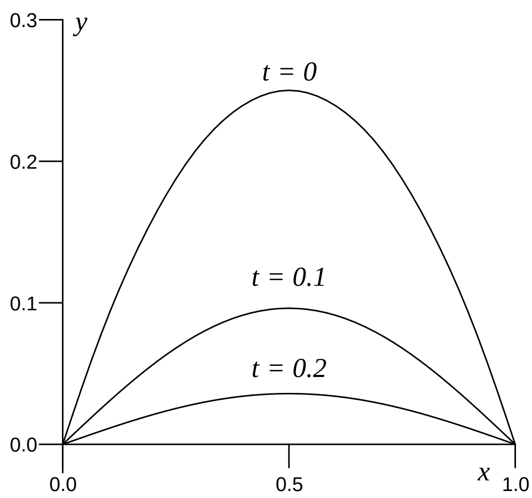


Figure 1. The solution of the heat equation with fixed boundary conditions.

Example 1
Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{5.10}$$

subject to

$$u(x,0) = x - x^2, \quad u_x(0,t) = u_x(1,t) = 0. \tag{5.11}$$

This problem is similar to the proceeding problem except the boundary conditions are different. The last problem had the boundaries fixed at zero whereas in this problem, the boundaries are insulated (*i.e.* no flux across the boundary condition). Again, assuming that the solutions are separable

$$u(x,t) = X(x)T(t), \tag{5.12}$$

then from the heat equation, we obtain

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.13}$$

where λ is a constant. The boundary conditions in (5.11) become, accordingly

$$X'(0) = X'(1) = 0. \tag{5.14}$$

Integrating the X equation in (5.13) gives rise to again three cases depending on the sign of λ but as seen earlier, only the case where $\lambda = -k^2$ for some constant k is relevant. Thus, we have

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.15}$$

Imposing the boundary conditions (5.14) shows that

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0, \tag{5.16}$$

which leads to

$$c_1 = 0, \quad \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots, n\pi, \tag{5.17}$$

where n is an integer. From (5.13), we also deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x,$$

where we have set $c_1 c_3 = a_n$. Using the initial condition gives

$$\begin{aligned} u(x,0) = x - x^2 &= \sum_{n=0}^{\infty} a_n \cos n\pi x \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x. \end{aligned}$$

From the chapter 4, we recognize this as a Fourier cosine series and that the coefficients a_n are chosen such that

$$\begin{aligned} a_0 &= 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{3}, \\ a_n &= 2\int_0^1 (x - x^2) \cos n\pi x\, dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2}\cos n\pi x - \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right)\sin n\pi x\right]\Bigg|_0^1 \\ &= -\frac{2}{n^2\pi^2}(1 + (-1)^n). \end{aligned}$$

Thus, the solution of the PDE as

$$u(x,t) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} - 1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x. \tag{5.18}$$

Figure 2 shows the solution at times $t = 0, 0.25$ and 0.5. It is interesting to note that even though that same initial condition are used for each of the two problems, fixing the boundaries and insulated them gives rise two totally different behaviors after $t \geq 0$.

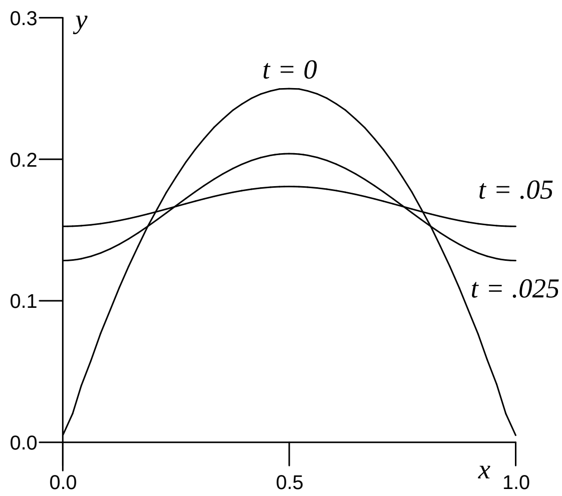


Figure 2. The solution of the heat equation with no flux boundary conditions.

Example 2
Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{5.19}$$

subject to

$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{cases} \quad u(0,t) = u_x(2,t) = 0. \tag{5.20}$$

In this problem, we have a mixture of both fixed and no flux boundary conditions. Again, assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.21}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.22}$$

where λ is a constant. The boundary conditions in (5.20) becomes, accordingly

$$X(0) = X'(1) = 0. \tag{5.23}$$

Integrating the X equation in (5.22) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.24}$$

Imposing the boundary conditions (5.23) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 k \cos 2k - c_2 k \sin 2k = 0,$$

which leads to

$$c_2 = 0, \quad \cos 2k = 0 \ \Rightarrow \ k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots, \frac{(2n-1)\pi}{4},$$

for integer n. From (5.22), we then deduce that

$$T(t) = c_3 e^{-\frac{(2n-1)^2}{16}\pi^2 t},$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{(2n-1)}{4}\pi x,$$

where we have set $c_1 c_3 = b_n$. Using the initial condition give

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)}{4}\pi x.$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned}
b_n &= 2\int_0^1 x \sin \frac{(2n-1)}{4}\pi x \, dx + \int_1^2 (2-x) \sin \frac{(2n-1)}{4}\pi x \, dx \\
&= \left[\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x - \frac{8x}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x\right]\Bigg|_0^1 \\
&+ \left[-\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x + \frac{8(x-2)}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x\right]\Bigg|_1^2 \\
&= \frac{32}{(2n-1)^2\pi^2}\left(\sin \frac{(2n-1)\pi}{4} + \cos n\pi\right).
\end{aligned}$$

Hence, the solution of the PDE is

$$u(x,t) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{(2n-1)\pi}{4} + \cos n\pi\right)}{(2n-1)^2} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{2n-1}{4}\pi x. \tag{5.25}$$

Figure 3 and 4 shows the solution both in short time $t = 0, 0.1, 0.2$ and long time $t = 10, 20, 30$.

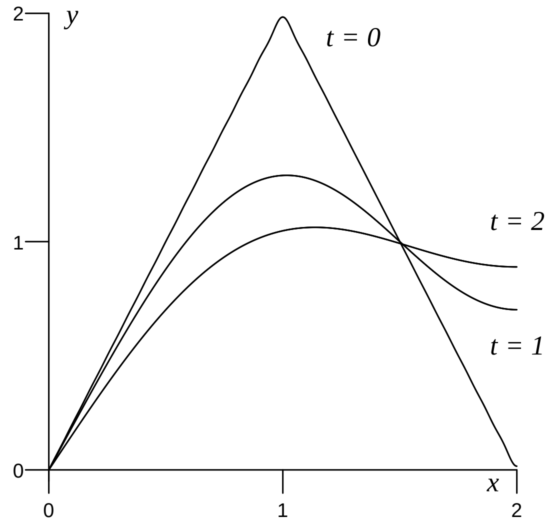


Figure 3. Short time behavior of the solution (5.25).

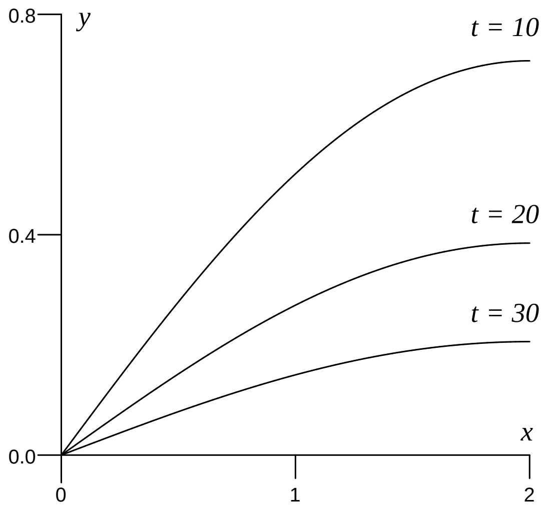


Figure 4. Long time behavior of the solution (5.25).

Example 3
Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{5.26}$$

subject to

$$u(x,0) = 2x - x^2 \quad u(0,t) = 0, \;\; u_x(2,t) = -u(2,t). \tag{5.27}$$

In this problem, we have a fixed left endpoint and a radiating right end point. Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.28}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.29}$$

where λ is a constant. The boundary conditions in (5.27) becomes, accordingly

$$X(0) = 0, \quad X'(2) = -X(2). \tag{5.30}$$

Integrating the X equation in (5.29) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.31}$$

Imposing the first boundary condition of (5.30) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \ \Rightarrow \ c_2 = 0$$

and the second boundary condition of (5.30) gives

$$\tan 2k = -k. \tag{5.32}$$

It is very important that we recognize that the solutions of (5.32) are not equally spaced as seen in earlier problems. In fact, there are an infinite number of solutions of this equation. Figure 5 shows graphically the curves $y = -k$ and $y = \tan 2k$. The first three intersection points are the first three solutions of (5.32).

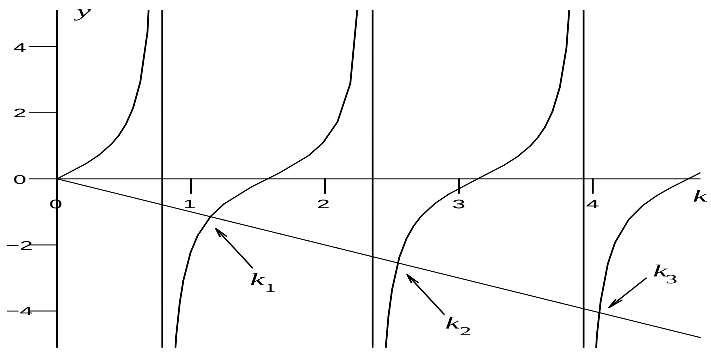


Figure 5. The graph of $y = \tan 2k$ and $y = -k$.

Thus, it is necessary that we solve for k numerically. The first 10 solutions are given in table 1.

n	k_n	n	k_n
1	1.144465	6	8.696622
2	2.54349	7	10.258761
3	4.048082	8	11.823162
4	5.586353	9	13.389044
5	7.138177	10	14.955947

Table 1. The first ten solution of $\tan 2k = -k$.

Therefore, we have

$$X(x) = c_1 \sin k_n x. \tag{5.33}$$

Further, integrating (5.29) for T gives

$$T(t) = c_3 e^{-k_n^2 t} \tag{5.34}$$

and together with X, we have the solution to the PDE as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} \sin k_n x, \tag{5.35}$$

Imposing the boundary conditions (5.27) gives

$$u(0,t) = 2x - x^2 = \sum_{n=1}^{\infty} c_n \sin k_n x, \tag{5.36}$$

It is important to know that the c_n's are **not** given by the formula

$$c_n = \frac{2}{2} \int_0^2 (2x - x^2) \sin k_n x, dx$$

as usual because the k_n's are not equally spaced. So it is necessary to examine (5.36) on its own. Multiplying by $\sin k_m x$ and integrating over [0,2] gives

$$\int_0^2 (2x - x^2) \sin k_m x \, dx = \sum_{n=1}^{\infty} c_n \int_0^2 \sin k_m x \, \sin k_n x \, dx,$$

For $n \neq m$, we have

$$\begin{aligned} \int_0^2 \sin k_m x \, \sin k_n x \, dx &= \frac{k_m \sin 2k_n \cos 2k_m - k_n \sin 2k_m \cos 2k_n}{k_n^2 - k_m^2}, \\ &= \frac{k_m k_n \cos 2k_m \cos 2k_n}{k_n^2 - k_m^2} \left(\frac{\sin 2k_n}{k_n \cos 2k_n} - \frac{\sin 2k_m}{k_m \cos 2k_m} \right), \end{aligned} \tag{5.37}$$

and imposing (5.32) for each of k_m and k_n shows (5.37) to be identically zero. Therefore, we obtain the following when $n = m$

$$\int_0^2 (2x - x^2) \sin k_n x \, dx = c_n \int_0^2 \sin^2 k_n x \, dx,$$

or

$$c_n = \frac{\int_0^2 (2x - x^2) \sin k_n x \, dx}{\int_0^2 \sin^2 k_n x \, dx,} \tag{5.38}$$

Table 2 gives the first ten c_n's that correspond to each k_n.

n	c_n	n	c_n
1	0.873214	6	0.028777
2	0.341898	7	-.016803
3	-.078839	8	0.015310
4	0.071427	9	-.010202
5	-.032299	10	0.009458

Table 2. The coefficients c_n from (5.78).

Having obtain k_n and c_n, the solution to the problem is found in (5.35). Figure 6 show plots at time $t = 0, 1,$ and 2 when 20 terms are used.

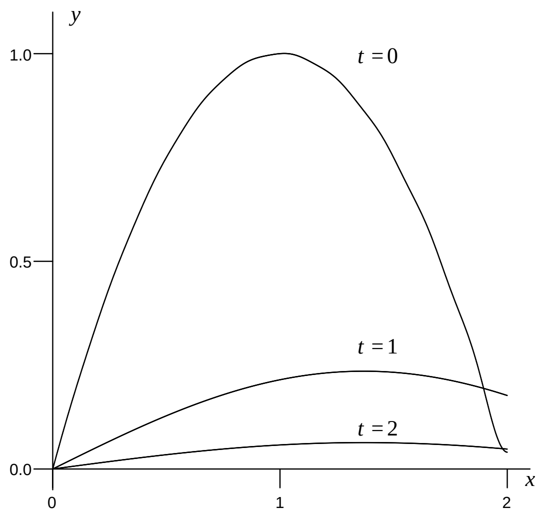


Figure 6. The solution (5.35).

5.1.1 Nonhomogeneous boundary conditions

In the preceding examples, the boundary conditions where either fixed to zero, insulted or radiating. Often, we encounter boundary conditions which are nonstandard or nonhomogeneous. For example, the boundary may be fixed to particular constant or the flux is maintained at a constant value. The following examples illustrate.

Example 4
Solve

$$u_t = u_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{5.39}$$

subject to

$$u(x, 0) = 4x - x^2 \quad u(0, t) = 0, \;\; u(3, t) = 3. \tag{5.40}$$

If we seek separable solutions $u(x,t) = X(x)T(t)$, then from (5.40) we have

$$X(0)T(t) = 0, \quad X(3)T(t) = 3, \tag{5.41}$$

and we have a problem! The second boundary condition doesn't separate. To overcome this we try and transform this problem to one that we know how to solve. As the right boundary condition is not zero, we try a transformation to fix the boundary to zero. If we try $u = v + 3$, this cures the problem, however, in the process it transforms the left boundary condition to -3. The next simplest transformation is to introduce $u = v + ax + b$ and ask, "Can we choose the constants a and b as to fix both boundary conditions to zero. Upon substitution of both boundary conditions (5.40), we obtain

$$0 = v(0,t) + a \times 0 + b, \;\; 3 = v(3,t) + 3a + b. \tag{5.42}$$

Now we require that $v(0,t) = 0$ and $v(3,t) = 0$ which implies that we must choose $a = 1$ and $b = 0$. Therefore, we have

$$u = v + x. \tag{5.43}$$

We notice that under the transformation (5.43), the original equation doesn't change form, *i.e.*

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx},$$

however, the initial condition does change. At $t = 0$, then

$$\begin{aligned} u(x,0) &= v(x,0) + x \\ \Rightarrow \quad 4x - x^2 &= v(x,0) + x \\ \Rightarrow \quad v(x,0) &= 3x - x^2. \end{aligned} \tag{5.44}$$

Thus, we have the new problem to solve

$$v_t = v_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{5.45}$$

subject to

$$v(x,0) = 3x - x^2 \quad v(0,t) = 0, \;\; v(3,t) = 3. \tag{5.46}$$

As usual, we seek separable solutions $v(x,t) = X(x)T(t)$ which lead to the systems $X'' = -k^2 X$ and $T' = -k^2 T$ with the boundary conditions $X(0) = 0$ and $X(3) = 0$. Solving for X gives

$$X(x) = c_1 \sin kx + c_2 \cos kx, \tag{5.47}$$

and imposing both boundary conditions gives

$$X(x) = c_1 \sin \frac{n\pi}{3} x, \tag{5.48}$$

and

$$T(t) = c_3 e^{-\frac{n^2\pi^2}{9} t},$$

where n is an integer. Therefore, we have the solution of (5.45) as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9} t} \sin \frac{n\pi}{3} x.$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 (3x - x^2) \sin \frac{n\pi}{3} x \, dx \\ &= \left[-\frac{6(2x-3)}{n^2\pi^2} \sin \frac{n\pi}{3} x + \frac{3(n^2\pi^2 x^2 - 3n^2\pi^2 x - 18)}{n^3\pi^3} \cos \frac{n\pi}{3} x \right] \Bigg|_0^3 \\ &= \frac{32(1-(-1)^n)}{n^3\pi^3}. \end{aligned}$$

This gives

$$v(x,t) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9} t} \sin \frac{n\pi}{3} x.$$

and since $u = v + x$, we obtain the solution for u as

$$u(x,t) = x + \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9} t} \sin \frac{n\pi}{3} x. \tag{5.49}$$

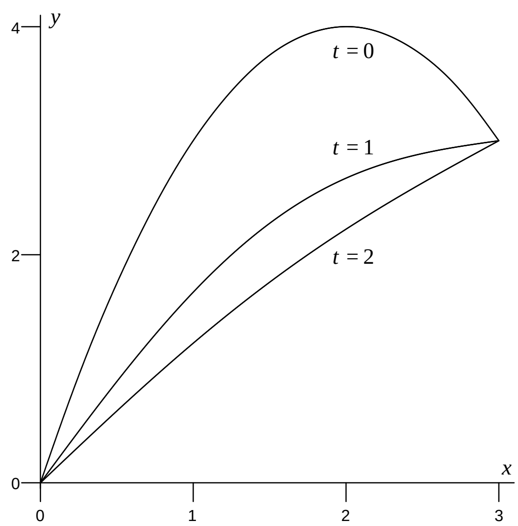


Figure 7. The solution (5.49) at time $t = 0, 1$ and 2.

Example 5
Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.50}$$

subject to

$$u(x,0) = 0 \quad u_x(0,t) = -1, \;\; u_x(1,t) = 0. \tag{5.51}$$

Unfortunately, the trick $u = v + ax + b$ won't work since $u_x = v_x + a$ and choosing a to fix the right boundary to zero only makes the left boundary nonzero. To overcome this we might try $u = v + ax^2 + bx$ but the original equation changes

$$u_t = u_{xx} \;\; \Rightarrow \;\; v_t = v_{xx} + 2a \tag{5.52}$$

As a second attempt, we try

$$u = v + a(x^2 + 2t) + bx \tag{5.53}$$

so now

$$u_t = u_{xx} \;\; \Rightarrow \;\; v_t = v_{xx}. \tag{5.54}$$

Since $u_x = v_x + 2ax + b$, then choosing $a = 1/2$ and $b = -1$ gives the the new boundary conditions as $v_x(0,t) = 0$ and $v_x(1,t) = 0$ and the transformation becomes

$$u = v + \frac{1}{2}(x^2 + 2t) - x, \tag{5.55}$$

Finally, we consider the initial condition. From (5.55), we have

$$v(x,0) = u(x,0) - \frac{1}{2}x^2 + x = x - \frac{x^2}{2}$$

and our problem is transformed to the new problem

$$v_t = v_{xx}, \quad 0 < x < 1, \;\; t > 0, \tag{5.56}$$

subject to

$$v(x,0) = x - \frac{1}{2}x^2, \;\; v_x(0,t) = v_x(1,t) = 0 \tag{5.57}$$

A separation of variables $v = XT$ leads to $X'' = -k^2 X$ and $T' = -k^2 T$ from which we obtain

$$X = c_1 \sin kx + c_2 \cos kx, \quad X' = c_1 k \cos kx - c_2 k \sin kx, \tag{5.58}$$

and imposing the boundary conditions (5.57) gives

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0,$$

from which we obtain

$$c_1 = 0, \quad k = n\pi, \tag{5.59}$$

where n is an integer. This then leads to

$$X(x) = c_2 \cos n\pi x \tag{5.60}$$

and further

$$T(t) = c_3 e^{-n^2\pi^2 t}. \tag{5.61}$$

Finally we arrive at

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x.$$

noting that we have chosen $a_n = c_1 c_3$. Upon substitution of $t = 0$ and using the initial condition (5.57), we have

$$x - \frac{1}{2}x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

a Fourier cosine series. The coefficients are obtained by

$$\begin{aligned}
a_0 &= \frac{2}{1}\int_0^1 \left(x - \frac{1}{2}x^2\right), dx = x^2 - \frac{1}{3}x^3\bigg|_0^1 = \frac{2}{3}, \\
a_n &= \frac{2}{1}\int_0^1 (x - \frac{1}{2}x^2) \cos n\pi x \, dx \\
&= \left[\frac{-2(x-1)}{n^2\pi^2}\cos n\pi x - \left(\frac{(2x - x^2)}{n\pi} + \frac{2}{n^3\pi^3}\right)\sin n\pi x\right]\bigg|_0^1 \\
&= -\frac{2}{n^2\pi^2}.
\end{aligned}$$

Thus, we obtain the solution for v as

$$v(x,t) = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x.$$

and this, together with the transformation (5.55) gives

$$u(x,t) = \frac{1}{2}(x^2 + 2t) - x + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x. \tag{5.62}$$

Figure 8 shows plots at time $t = 0.01, 0.5, 1.0$ and 1.5. It is interesting to note that at the left boundary $u_x = -1$ and since the flux $\phi = -ku_x$ implies that $\phi = k > 0$ which gives that heat is being added at the left boundary. Hence the profile increase at the left while insulated at the right boundary (no flux).

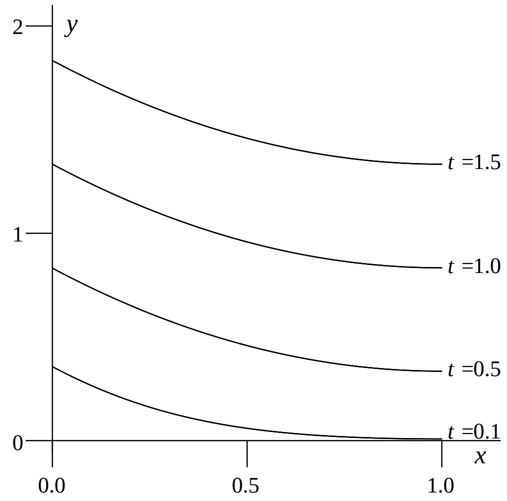


Figure 8. The solution (5.62) at time $t = 0, 1$ and 2.

A natural question is, can we transform

$$u_t = u_{xx}, \;\; 0 < x < L, \;\; t > 0, \tag{5.63}$$

$$u(x,0) = f(x), \;\; u(0,t) = p(t), \;\; u(L,t) = q(t) \tag{5.64}$$

to a problem with homogeneous boundary conditions. The answer is yes. We seek a transformation of the form

$$u = v + A(t)x + B(t) \tag{5.65}$$

as to transform the nonhomogeneous boundary conditions to homogeneous ones. On substitution of the u and v boundary conditions, we obtain

$$p(t) = 0 + A(t)0 + B(t), \;\; q(t) = 0 + A(t)L + B(t) \tag{5.66}$$

and solving for $A(t)$ and $B(t)$ gives

$$A(t) = \frac{q(t) - p(t)}{L}, \;\; B(t) = p(t), \tag{5.67}$$

which results in the transformation

$$\begin{aligned} u &= v + \frac{q(t) - p(t)}{L} x + p(t) \\ &= v + q(t)\frac{x}{L} + p(t)\frac{(L-x)}{L}. \end{aligned} \tag{5.68}$$

However in doing so, we change not only the original equation but also the initial condition. They becomes, respectively

$$\begin{aligned} v_t &= v_{xx} - q'(t)\frac{x}{L} + p'(t)\frac{(L-x)}{L}, \\ v(x,0) &= f(x) - q(0)\frac{x}{L} - p(0)\frac{(L-x)}{L}. \end{aligned}$$

The new initial condition doesn't pose a problem but how do we solve the heat equation with a term added to the equation. This is the topic of the next section.

5.1.2 Nonhomogeneous equations

We now focus our attention to solving the heat equation with a source term

$$u_t = u_{xx} + Q(x), \quad 0 < x < L, \quad t > 0, \tag{5.69}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x).$$

To investigate this problem, we will considered a particular example where $L = 2$, $f(x) = 2x - x^2$ and $Q(x) = 1 - |x - 1|$. If we were to consider this problem without a source term and use the separation of variables technique, we would have obtained the solution

$$u(x,t) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}e^{-\frac{n^2\pi^2}{4}t}\sin\frac{n\pi x}{2}. \tag{5.70}$$

Suppose that we looked for solutions in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{2}. \tag{5.71}$$

noting that both boundary conditions are satisfied. Substituting into the heat equation, and isolating coefficients of $\sin\frac{n\pi x}{2}$, would lead to

$$T_n'(t) = -\frac{n^2\pi^2}{4}T_n(t).$$

Solving this would give

$$T_n(t) = \frac{16(1-(-1)^n)}{n^3\pi^3}e^{-\frac{n^2\pi^2}{4}t} \tag{5.72}$$

and (5.72), together with (5.71) we would have obtained (5.70). With this idea, we try and solve the heat equation with a source term by looking for solutions of the form (5.71). However, in order that this technique works, it is also necessary to expand the source term also in terms of a Fourier sine series, *i.e.*

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{2}. \tag{5.73}$$

For

$$Q(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2-x & \text{if } 1 < x < 2, \end{cases} \tag{5.74}$$

where

$$\begin{aligned} q_n &= \int_0^2 Q(x)\sin\frac{n\pi x}{2}dx. \\ &= \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}dx, \\ &= \left[\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} - \frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]\Bigg|_0^1 \\ &+ \left[-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} + \frac{2x-4}{n\pi}\cos\frac{n\pi x}{2}\right]\Bigg|_1^2, \\ &= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned} \tag{5.75}$$

Substituting both (5.70) and (5.73) into (5.69) gives

$$\sum_{n=1}^{\infty} T_n'(t)\sin\frac{n\pi x}{2} = \sum_{n=1}^{\infty} -\left(\frac{n\pi x}{2}\right)^2 T_n(t)\sin\frac{n\pi x}{2} + \sum_{n=1}^{\infty} q_n\sin\frac{n\pi x}{2},$$

and re-grouping and isolating the coefficients of $\sin\frac{n\pi x}{2}$ gives

$$T_n'(t) + \frac{n^2\pi^2}{4}T_n(t) = q_n, \tag{5.76}$$

a linear ODE in $T_n(t)$! On solving (5.76) we obtain

$$T_n(t) = \frac{4}{n^2\pi^2}q_n + b_n e^{-(\frac{n\pi}{2})^2 t},$$

giving the final solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + b_n e^{-(\frac{n\pi}{2})^2 t} \right) \sin \frac{n\pi x}{2}. \tag{5.77}$$

Imposing the initial condition gives (5.70)

$$2x - x^2 = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + b_n \right) \sin \frac{n\pi x}{2}.$$

If we set

$$c_n = \frac{4}{n^2\pi^2} q_n + b_n,$$

then we have

$$2x - x^2 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{2}.$$

a regular Fourier sine series. Therefore

$$\begin{aligned} c_n &= \int_0^2 \left(2x - x^2\right) \sin \frac{n\pi x}{2} dx. \\ &= \frac{16}{n^3\pi^3} \left(1 - \cos \frac{n\pi}{2}\right), \end{aligned} \tag{5.78}$$

which in turn, gives

$$b_n = c_n - \frac{4}{n^2\pi^2} q_n,$$

and finally, the solution as

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} q_n + \left(c_n - \frac{4}{n^2\pi^2} q_n \right) e^{-(\frac{n\pi}{2})^2 t} \right) \sin \frac{n\pi x}{2},$$

where q_n and c_n are given in (5.75) and (5.78), giving

$$u(x,t) = \sum_{n=1}^{\infty} \left[\begin{array}{c} \frac{32}{n^4\pi^4} \left(1 - e^{-\frac{n^2\pi^2}{4} t}\right) \sin \frac{n\pi}{2} \\ + \frac{16}{n^2\pi^3} \left(1 - \cos \frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2}{4} t} \end{array} \right] \sin \frac{n\pi x}{2}. \tag{5.79}$$

Typical plots are given in figure 9 at times $t = 0, 1, 2$ and 3.

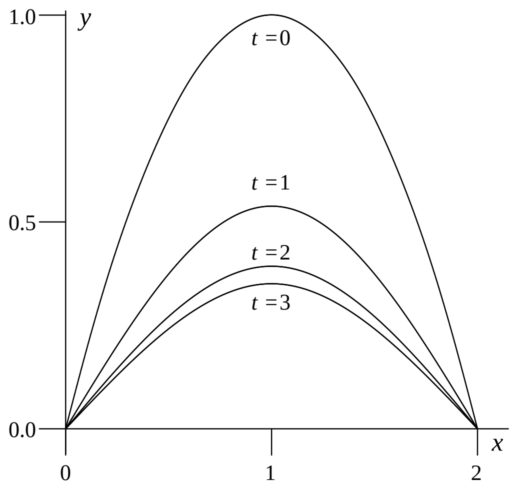


Figure 9. The solution (5.79) at time $t = 0, 1, 2$ and 3.

It is interesting to note that if we let $t \to \infty$ the solution approaches the same curve at $t = 3$. This is what is called steady state (no changes in time). It is natural to ask "Can we find this steady state solution.?" The answer is yes. For the steady state, $u_t \to 0$ as $t \to \infty$ and the original PDE becomes

$$u_{xx} + Q(x) = 0. \tag{5.80}$$

Integrating twice with $Q(x)$ given in (5.74) gives

$$u = \begin{cases} -\frac{x^3}{6} + c_1 x + c_2 & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + k_1 x + k_2 & \text{if } 1 < x < 2, \end{cases} \tag{5.81}$$

where c_1, c_2, k_1 and k_2 are constants of integration. Imposing that the solution and its first derivative are continuous at $x = 1$ and that the solution is zero at the endpoints gives

$$c_1 - k_1 + 1 = 0, \;\; c_1 + c_2 - k_1 - k_2 + \frac{2}{3} = 0,$$

$$c_2 = 0, \quad 2k_1 + k_2 - \frac{8}{3} = 0,$$

which gives, upon solving

$$c_1 = \frac{1}{2}, \;\; c_2 = 0, \;\; k_1 = \frac{3}{2}, \;\; k_2 = -\frac{1}{3} \tag{5.82}$$

This, in turn gives the steady state solution as

$$u = \begin{cases} -\frac{x^3}{6} + \frac{x}{2} & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + \frac{3x}{2} - \frac{1}{3} & \text{if } 1 < x < 2. \end{cases} \tag{5.83}$$

5.1.3 Equations with a solution dependent source term

We now consider the heat equation with a solution dependent source term. For simplicity we will consider a source term that is linear. Take, for example

$$\begin{aligned} u_t = u_{xx} + \alpha u, \quad 0 < x < 1, \;\; t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = x - x^2, \end{aligned} \tag{5.84}$$

where α is some constant. We could try a separation of variables to obtain solutions for this problem, but for more complicated source terms like $Q(x,t)u$, a separation of variables is unsuccessful. Therefore, we try a different technique. Here we will try and transform the PDE to one that has no source term. In attempting to do so, we seek a transformation of the form

$$u(x,t) = A(x,t)v \tag{5.85}$$

and ask "Is is possible to find A such that the source term in (5.84) can be removed?" Substituting of (5.85) in (5.84) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx}v + \alpha A v.$$

Dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + 2\frac{A_x}{A}v_x + \left(\frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha\right)v.$$

In order to target the standard heat equation, we choose

$$A_x = 0, \quad \frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha = 0. \tag{5.86}$$

From the first equation in (5.86), we obtain that $A = A(t)$ and from the second we obtain $A' = \alpha A$ which has the solution $A(t) = A_0 e^{\alpha t}$ for some constant A_0 leading to $u = A_0 e^{\alpha t} v$. The boundary conditions become

$$\begin{aligned} u(0,t) = 0 \;\; \Rightarrow \;\; A_0 e^{\alpha t} v(0,t) = 0 \;\; \Rightarrow \;\; v(0,t) = 0, \\ v(1,t) = 0 \;\; \Rightarrow \;\; A_0 e^{\alpha t} v(1,t) = 0 \;\; \Rightarrow \;\; v(1,t) = 0, \end{aligned}$$

so the boundary conditions are unchanged. Next, we consider the initial condition, so

$$u(x,0) = x - x^2 \;\; \Rightarrow \;\; A_0 e^{\alpha 0} v(x,0) = x - x^2 \;\; \Rightarrow \;\; A_0 v(x,0) = x - x^2.$$

To leave the initial condition unchanged, choose $A_0 = 1$. Thus, under the transformation

$$u = e^{\alpha t} v \tag{5.87}$$

problem (5.84) becomes

$$\begin{gathered} v_t = v_{xx}, \quad 0 < x < 1, \;\; t > 0, \\ v(x,0) = 0, \quad v(1,t) = 0, \quad v(x,0) = x - x^2, \end{gathered} \tag{5.88}$$

This particular problem was considered at the beginning of this chapter, (5.1) where the solution was given in (5.25) by

$$v(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x,$$

and so, from (5.87) we obtain the solution of (5.84) as

$$u(x,t) = \frac{4}{\pi^3} e^{\alpha t} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \tag{5.89}$$

Figures 10 and 11 shows plots at times $t = 0, 0.1$ and 0.2 when $\alpha = 5$ and 12. It is interesting to note that in the case where $\alpha = 5$, the diffusion is slower in comparison with no source term (*i.e* $\alpha = 0$ see Figure 1) and there is no diffusion at all when $\alpha = 12$.

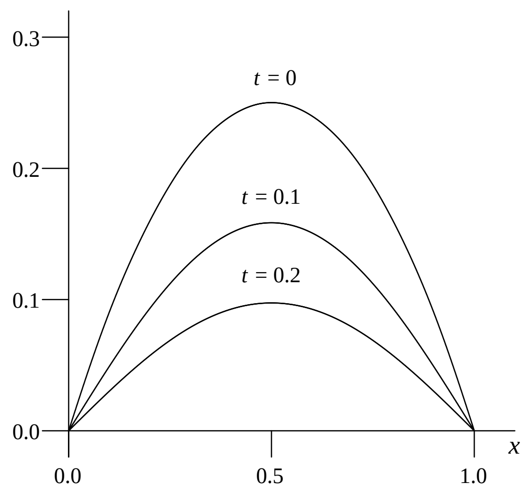


Figure 10. The solution of the heat equation with a source (5.84) with $\alpha = 5$.

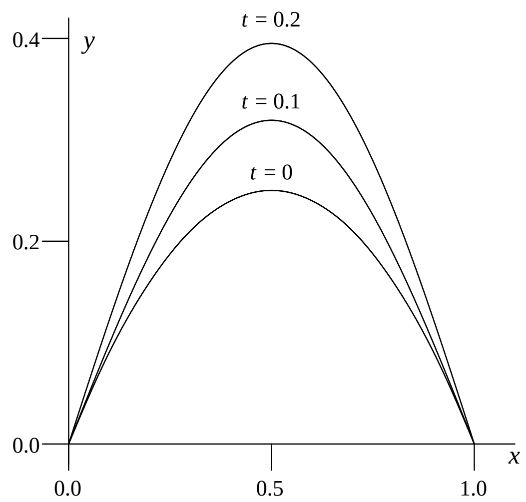


Figure 11. The solution of the heat equation with a source (5.84) with $\alpha = 12$.

It is natural to ask, for what value of α do we achieve a steady state solution. To answer this consider the first few terms of the solution (5.89)

$$u = \frac{8}{\pi^3} e^{\alpha t} \left(e^{-\pi^2 t} \sin \pi x + \frac{1}{27} e^{-9\pi^2 t} \sin 3\pi x + \dots \right). \tag{5.90}$$

Now clearly the exponential terms in (5.90) will decay to zero with the first term decaying the slowest. Therefore, it is the balance between $e^{\alpha t}$ and $e^{-\pi^2 t}$ which determine whether the solution will decay to zero or not. It is equality $\alpha = \pi^2$ that gives the steady state solution.

Example 6
Solve

$$\begin{aligned} u_t &= u_{xx} + \alpha u, \quad 0 < x < 2, \quad t > 0 \\ u(x,0) &= 4x - x^3 \quad u_x(0,t) = 0, \;\; u_x(2,t) = 0. \end{aligned} \tag{5.91}$$

It was already established that the transformation, (5.87), will transform the heat equation with a solution dependent source term to the heat equation and will leave the initial condition and fixed boundary conditions unchanged. It is now necessary to determine what happens to no flux boundary conditions. Using (5.87) we have

$$u_x(0,t) = 0 \;\; \Rightarrow \;\; A_0 e^{\alpha t} v_x(0,t) = 0 \;\; \Rightarrow \;\; v_x(0,t) = 0, \tag{5.92}$$

$$u_x(2,t) = 0 \;\; \Rightarrow \;\; A_0 e^{\alpha t} v_x(2,t) = 0 \;\; \Rightarrow \;\; v_x(2,t) = 0, \tag{5.93}$$

and so the insulted boundary conditions also remain insulated! Thus, problem (5.91) reduces to

$$
\begin{gathered}
v_t = v_{xx}, \quad 0 < x < 2, \;\; t > 0 \\
v(x,0) = 4x - x^2, \quad v_x(0,t) = 0, \quad v_x(2,t) = 0.
\end{gathered} \tag{5.94}
$$

Using a separation of variables and imposing the boundary conditions gives (see example 1)

$$
v = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}t} \cos\frac{n\pi}{2}x, \tag{5.95}
$$

where

$$
\begin{aligned}
a_0 &= \frac{2}{2}\int_0^2 (4x - x^3)\,dx = \left[2x^2 - \frac{x^4}{4}\right]\Bigg|_0^2 = 4, \\
a_n &= \frac{2}{2}\int_0^2 (4x - x^3)\cos\frac{n\pi}{2}x\,dx \\
&= \left[\left(\frac{2(4x - x^3)}{n\pi} + \frac{48}{n^3\pi^3}\right)\sin\frac{n\pi}{2}x + \left(\frac{4(4 - 3x^2)}{n^2\pi^2} + \frac{96}{n^4\pi^4}\right)\cos\frac{n\pi}{2}x\right]\Bigg|_0^2 \\
&= -\frac{16}{n^2\pi^2} - \frac{96}{n^4\pi^4} + \left(\frac{6}{n\pi} + \frac{48}{n^3\pi^3}\right)\sin\frac{n\pi}{2} + \left(\frac{4}{n^2\pi^2} + \frac{96}{n^4\pi^4}\right)\cos\frac{n\pi}{2}.
\end{aligned} \tag{5.96}
$$

Together with the transformation (5.87), the solution of (5.91) is

$$
u = 2e^{\alpha t} + e^{\alpha t}\sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}t} \cos\frac{n\pi}{2}x, \tag{5.97}
$$

where a_n is given in (5.96). Figures 11 and 12 show plots at $t = 0, 0.2, 0.4$ and 0.6 when $\alpha = -2$ and 2. It is interesting to note that the sign of α will determine whether the solution will grow or decay exponentially.

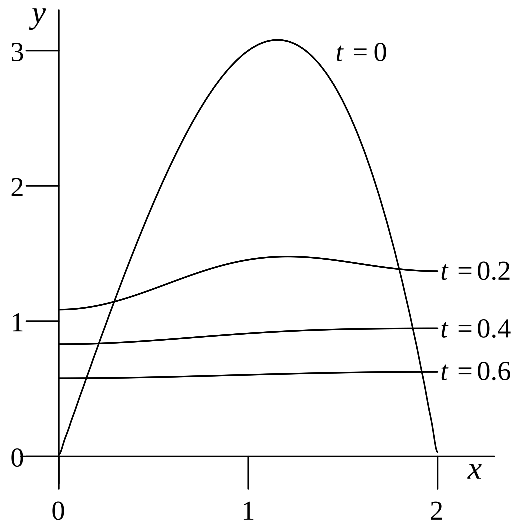


Figure 11. The solution of the heat equation with a source (5.91) with no flux boundary condition with $\alpha = -2$.

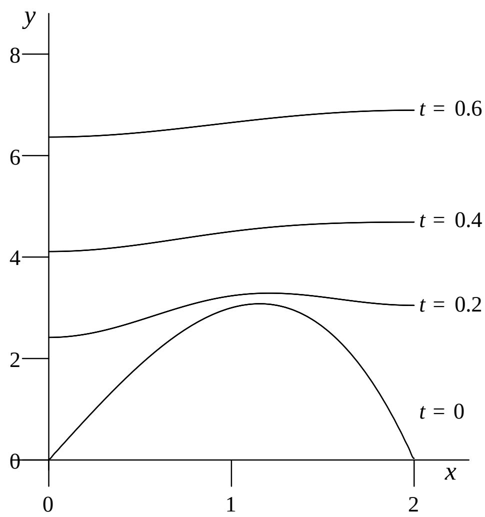


Figure 12. The solution of the heat equation with a source with no flux boundary condition with $\alpha = 2$.

5.1.4 Equations with a solution dependent convective term

We now consider the heat equation with a solution dependent linear convection term, *i.e.*

$$
\begin{gathered}
u_t = u_{xx} + \beta u_x, \quad 0 < x < 1, \;\; t > 0 \\
u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = x - x^2,
\end{gathered}
\tag{5.98}
$$

where β is some constant. We consider the same initial and boundary conditions as in the previous section as it provides a means of comparing the two respective problems. Again, we could try a separation of variables to obtain solutions for this problem, but for more complicated convection terms like $P(x,t)u_x$, a separation of variable is unsuccessful. Therefore, we again try and transform the PDE to one that has no convection term using a transformation of the form

$$u(x,t) = A(x,t)v. \tag{5.99}$$

"Is is possible to find A such that the convection term can be removed?" Substituting of (5.99) in (5.98) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx}v + \beta\left(Av_x + A_x v\right).$$

Dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + \frac{2A_x + \beta A}{A}\, v_x + \frac{A_{xx} - A_t + \beta A_x}{A}\, v. \tag{5.100}$$

In order to obtain the standard heat equation, choose

$$2A_x + \beta A = 0, \quad A_{xx} - A_t + \beta A_x = 0. \tag{5.101}$$

From the first of (5.101) we obtain that $A(x,t) = C(t)e^{-\frac{1}{2}\beta x}$ and from the second of (5.101), we obtain $C' + \frac{\beta^2}{4}C = 0$ which has the solution $C(t) = C_0 e^{-\frac{1}{4}\beta^2 t}$ for some constant C_0. This then gives

$$A(x,t) = C_0 e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} \tag{5.102}$$

The boundary conditions becomes

$$u(0,t) = 0 \quad \Rightarrow \quad C_0 e^{-\frac{1}{4}\beta^2 t} v(0,t) = 0 \quad \Rightarrow \quad v(0,t) = 0, \tag{5.103a}$$

$$u(1,t) = 0 \quad \Rightarrow \quad C_0 e^{-\frac{1}{2}-\frac{1}{4}\beta^2 t} v(1,t) = 0 \quad \Rightarrow \quad v(1,t) = 0, \tag{5.103b}$$

so the boundary conditions become unchanged. Next, we consider the initial condition, so

$$u(x,0) = x - x^2 \;\; \Rightarrow \;\; u(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}, \tag{5.104}$$

where we have chosen $C_0 = 1$. So here, the initial condition actually changes. Thus, under the transformation

$$u = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v, \tag{5.105}$$

the problem (5.98) becomes

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0,$$
$$v(x,0) = 0, \quad v(1,t) = 0, \quad v(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}. \tag{5.106}$$

As in the previous section, the solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.107}$$

where b_n is now given by

$$b_n = \frac{2}{1}\int_0^1 (x - x^2)e^{\frac{1}{2}\beta x} \sin n\pi x \, dx, \tag{5.108}$$

and from (5.105)

$$u(x,t) = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.109}$$

At this point we consider two particular examples: $\beta = 6$ and $\beta = -12$.
For $\beta = 6$

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)e^{3x} \sin n\pi x \, dx \\ &= -4n\pi \frac{27 + n^2\pi^2 + 2n^2\pi^2 e^3 \cos n\pi}{(9 + n^2\pi^2)^3}. \end{aligned} \tag{5.110}$$

For $\beta = -12$

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)e^{-6x} \sin n\pi x \, dx \\ &= 2n\pi \frac{7n^2\pi^2 + 108 + (5n^2\pi^2 + 324)e^{-6} \cos n\pi}{(36 + n^2\pi^2)^3}. \end{aligned} \tag{5.111}$$

The respective solutions for each are

$$\begin{aligned} u(x,t) &= 4\pi\, e^{-3x-9t} \\ &\times \sum_{n=1}^{\infty} n \frac{27 + n^2\pi^2 + 2n^2\pi^2 e^3 \cos n\pi}{(9 + n^2\pi^2)^3} e^{-n^2\pi^2 t} \sin n\pi x, \end{aligned} \tag{5.112}$$

$$\begin{aligned} u(x,t) &= 2\pi\, e^{6x+36t} \\ &\times \sum_{n=1}^{\infty} n \frac{7n^2\pi^2 + 108 + (5n^2\pi^2 + 324)e^{-6} \cos n\pi}{(36 + n^2\pi^2)^3} e^{-n^2\pi^2 t} \sin n\pi x. \end{aligned} \tag{5.113}$$

Figures 13 and 14 shows graphs at a variety of times for $\beta = 6$ and $\beta = -12$.

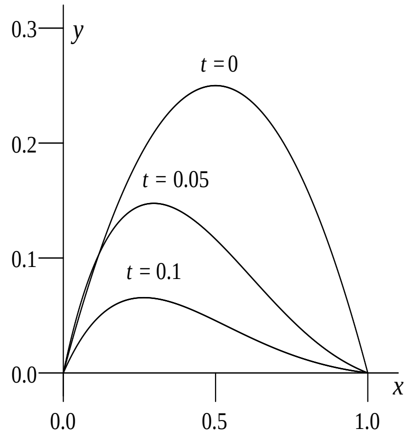


Figure 13. The solution of the heat equation with convection with fixed boundary conditions with $\beta = 6$.

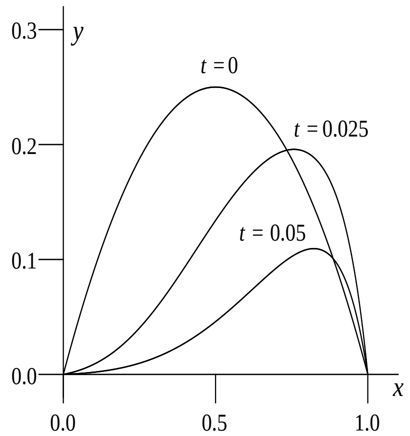


Figure 14. The solution of the heat equation with convection with fixed boundary conditions with $\beta = -12$.

Example 7
As a final example, we consider

$$u_t = u_{xx} + \beta u_x, \quad 0 < x < 2, \quad t > 0 \tag{5.114}$$

subject to

$$u(x,0) = 4x - x^3 \quad u_x(0,t) = 0, \;\; u_x(2,t) = 0. \tag{5.115}$$

This problem has the same initial condition as example 6 but with insulated boundary conditions. Here, we will simply transform the problem to one that is in standard form to contrast the differences between the two problems. The transformation (5.105) transforms (5.114) to the heat equation so we will primarily focus on the boundary and initial conditions. For the boundary conditions, upon differentiating (5.105) with respect to x gives

$$u_x = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v_x - \frac{\beta}{2} e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v, \tag{5.116}$$

and so

$$u_x(0,t) = 0 \;\; \Rightarrow \;\; v_x(0,t) - \frac{\beta}{2} v(0,t) = 0, \tag{5.117}$$

$$u_x(2,t) = 0 \;\; \Rightarrow \;\; v_x(2,t) - \frac{\beta}{2} v(2,t) = 0. \tag{5.118}$$

Thus, the insulated boundary condition become radiating boundary conditions. As for the initial condition

$$u(x,0) = 4x - x^3 \;\; \Rightarrow \;\; v(x,0) = (4x - x^3) e^{\frac{1}{2}\beta x}, \tag{5.119}$$

so again, the initial condition changes.

5.2 Laplace's equation

The two dimensional Laplace's equation is

$$u_{xx} + u_{yy} = 0. \tag{5.120}$$

To this we attach the boundary conditions

$$u(x,0) = 0, \;\; u(x,1) = x - x^2 \tag{5.121a}$$

$$u(0,y) = 0, \;\; u(1,0) = 0. \tag{5.121b}$$

We will show that separation of variables also works for this equation. If we assume solutions of the form

$$u(x,y) = X(x)Y(y), \tag{5.122}$$

then substituting this into (5.120) gives

$$X''Y + XY'' = 0. \tag{5.123}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{5.124}$$

and since each term is only a function of x or y, then each must be constant giving

$$\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda. \tag{5.125}$$

From the first of (5.121a) and both of (5.121b) we deduce the boundary conditions

$$Y(0) = 0, \quad X(0) = 0, \quad X(1) = 0. \tag{5.126}$$

The remaining boundary condition in (5.121a) will be used later. As we saw in previous section, in order to solve the X equation in (5.125) subject to the boundary conditions (5.126), it is necessary to set $\lambda = -k^2$. The X equation (5.125) as the general solution

$$X = c_1 \sin kx + c_2 \cos kx \tag{5.127}$$

To satisfy the boundary conditions in (5.126) it is necessary to have $c_2 = 0$ and $k = n\pi, \ k \in \mathbb{Z}^+$ so

$$X(x) = c_1 \sin n\pi x. \tag{5.128}$$

From (5.125), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y \tag{5.129}$$

Since $Y(0) = 0$ this implies $c_4 = 0$ so

$$X(x)Y(y) = a_n \sin n\pi x \sinh n\pi y \tag{5.130}$$

where we have chosen $a_n = c_1 c_3$. Therefore, we obtain the solution to (5.120) subject to three of the four boundary conditions in (5.121)

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y. \tag{5.131}$$

The remaining boundary condition is (5.121a) now needs to be satisfied, thus

$$u(x,1) = x - x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi. \tag{5.132}$$

This looks like a Fourier sine series and if we let $A_n = a_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} A_n \sin n\pi x = x - x^2. \tag{5.133}$$

which is precisely a Fourier sine series. The coefficients A_n are given by

$$A_n = \frac{2}{1}\int_0^1 \left(x - x^2\right) \sin n\pi x \, dx$$
$$= \frac{4}{n^3\pi^3}(1 - \cos n\pi), \tag{5.134}$$

and since $A_n = a_n \sinh n\pi$, this gives

$$a_n = \frac{4(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}. \tag{5.135}$$

Thus, the solution to Laplace's equation with the boundary conditions given in (5.121) is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}. \tag{5.136}$$

Figures 15 and 16 show both a top view and a $3-D$ view of the solution.

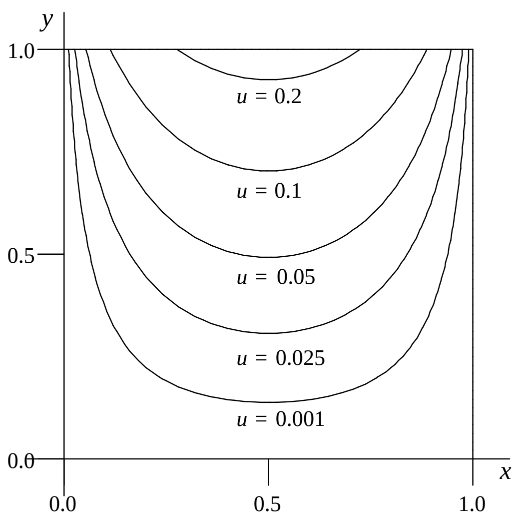


Figure 15. The solution (5.120) with the boundary conditions (5.121)

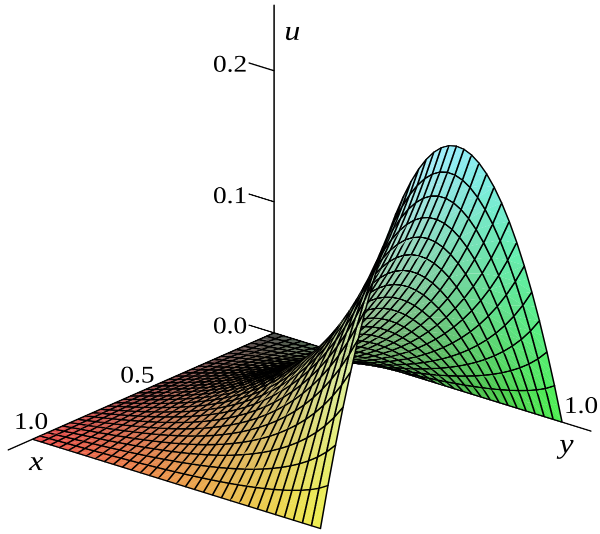


Figure 16. The solution (5.120) with the boundary conditions (5.121)

Example 8
Solve

$$u_{xx} + u_{yy} = 0, \tag{5.137}$$

subject to

$$u(x,0) = 0, \quad u(x,1) = 0, \tag{5.138a}$$
$$u(0,y) = 0, \quad u(1,y) = y - y^2. \tag{5.138b}$$

Assume separable solutions of the form

$$u(x,y) = X(x)Y(y) \tag{5.139}$$

Then substituting this into (5.137) gives

$$X''Y + XY'' = 0. \tag{5.140}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{5.141}$$

from which we obtain

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda \tag{5.142}$$

From (5.138) we deduce the boundary conditions

$$X(0) = 0, \quad Y(0) = 0, \quad Y(1) = 0. \tag{5.143}$$

The remaining boundary condition in (5.138) will be used later. As seen the in previous problem, in order to solve the Y equation in (5.142) subject to the boundary conditions (5.143), it is necessary to set $\lambda = k^2$. The Y equation (5.142) as the general solution

$$Y = c_1 \sin ky + c_2 \cos ky \tag{5.144}$$

To satisfy the boundary conditions in (5.143) it is necessary to have $c_2 = 0$ and $k = n\pi$ so

$$Y(y) = c_1 \sin n\pi y. \tag{5.145}$$

From (5.142), we obtain the solution to the X equation

$$X(x) = c_3 \sinh n\pi x + c_4 \cosh n\pi x \tag{5.146}$$

Since $X(0) = 0$ this implies $c_4 = 0$. This gives

$$X(x)Y(y) = a_n \sinh n\pi x \sin n\pi y \tag{5.147}$$

where we have chosen $a_n = c_1 c_3$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} a_n \sinh n\pi x \sin n\pi y. \tag{5.148}$$

The remaining boundary condition is (5.138) now needs to be satisfied, thus

$$u(1, y) = y - y^2 = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin n\pi y. \tag{5.149}$$

If we let $A_n = a_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} A_n \sin n\pi y = y - y^2. \tag{5.150}$$

Comparing with the previous problem, we find that interchanging x and y interchanges the two problems and so we conclude that

$$A_n = \frac{16}{n^3 \pi^3}(1 - \cos n\pi). \tag{5.151}$$

Therefore the solution to Laplace's equation subject to (5.138) is

$$u = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \frac{\sinh n\pi x}{\sinh n\pi} \sin n\pi y. \tag{5.152}$$

Figures 17 and 18 show both a top view and a $3 - D$ view of the solution.

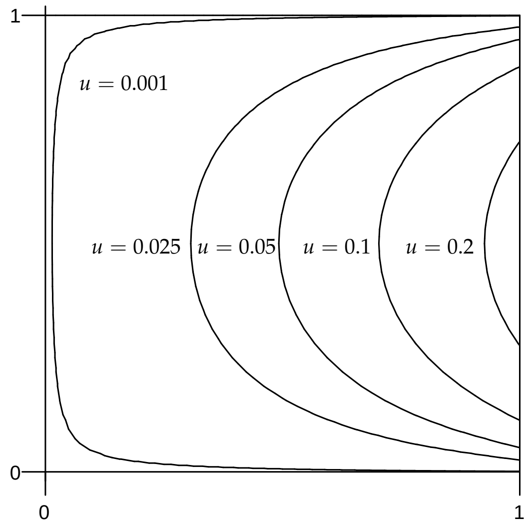


Figure 17. Top view of the solution of Laplaces equation with the boundary conditions (5.138).

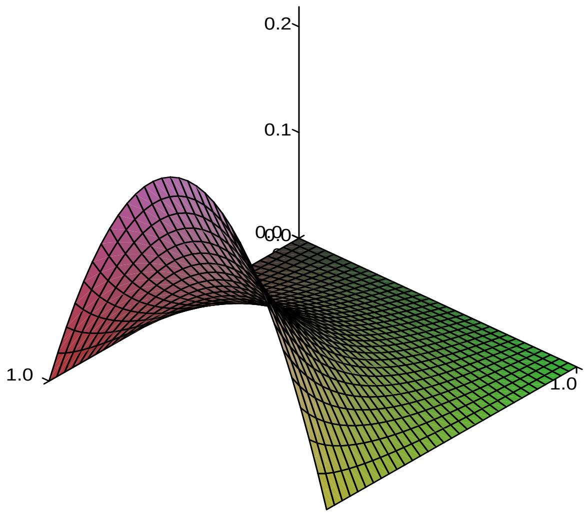


Figure 18. The solution of Laplaces equation with the boundary conditions (5.138)

In comparing the solutions (5.136) and (5.152) shows that if we interchange x and y they are the same. This should not be surprising because if we consider Laplace equations with the boundary conditions given in (5.121) and (5.138), that if we interchange x and y, the problems are transformed to each other.

Example 9

Solve

$$u_{xx} + u_{yy} = 0 \tag{5.153}$$

subject to

$$u(x,0) = x - x^2, \;\; u(x,1) = 0 \tag{5.154a}$$
$$u(0,y) = 0, \;\; u(1,y) = 0. \tag{5.154b}$$

Again, assuming separable solutions of the form

$$u(x,y) = X(x)Y(y) \tag{5.155}$$

leads to

$$X''Y + XY'' = 0, \tag{5.156}$$

when substituted into (5.153). Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \tag{5.157}$$

Since each term is only a function of x or y then each must be constant giving

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda \tag{5.158}$$

From the first of (5.154a) and both of (5.154b) we deduce the boundary conditions

$$X(0) = 0, \quad X(1) = 0, \quad Y(1) = 0, \tag{5.159}$$

noting that the last boundary condition is different than the boundary condition considered at the beginning of this section (*i.e.* $Y(0) = 0$). The remaining boundary condition in (5.154a) will be used later. In order to solve the X equation in (5.158) subject to the boundary conditions (5.159), it is necessary to set $\lambda = -k^2$. The X equation (5.158) as the general solution

$$Y = c_1 \sin kx + c_2 \cos kx \tag{5.160}$$

To satisfy the boundary conditions in (5.159) it is necessary to have $c_2 = 0$ and $k = n\pi$ so

$$X(x) = c_1 \sin n\pi x. \tag{5.161}$$

From (5.158), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y \tag{5.162}$$

Since $Y(1) = 0$ this implies

$$c_3 \sinh n\pi + c_4 \cosh n\pi = 0 \quad \Rightarrow \quad c_4 = -c_2 \frac{\sinh n\pi}{\cosh n\pi}. \tag{5.163}$$

From (5.164) we have

$$\begin{aligned} Y(y) &= c_3 \sinh n\pi y - c_3 \cosh n\pi y \frac{\sinh n\pi}{\cosh n\pi} \\ &= -c_3 \frac{\sinh n\pi(1-y)}{\sinh n\pi}. \end{aligned} \tag{5.164}$$

so

$$X(x)Y(y) = a_n \sin n\pi x \sinh n\pi(1-y) \tag{5.165}$$

where we have chosen $a_n = -c_1c_3/\sinh n\pi$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y). \tag{5.166}$$

The remaining boundary condition is (5.154a) now needs to be satisfied, thus

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi. \tag{5.167}$$

At this point we recognize that this problem is now identically to the first problem in this section where we obtained

$$a_n = \frac{4(1-(-1)^n)}{n^3\pi^3 \sinh n\pi} \tag{5.168}$$

so the solution to Laplace's equation with the boundary conditions given in (5.154) is

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi(1-y)}{\sinh n\pi}. \tag{5.169}$$

Figure 19 shows the solution.

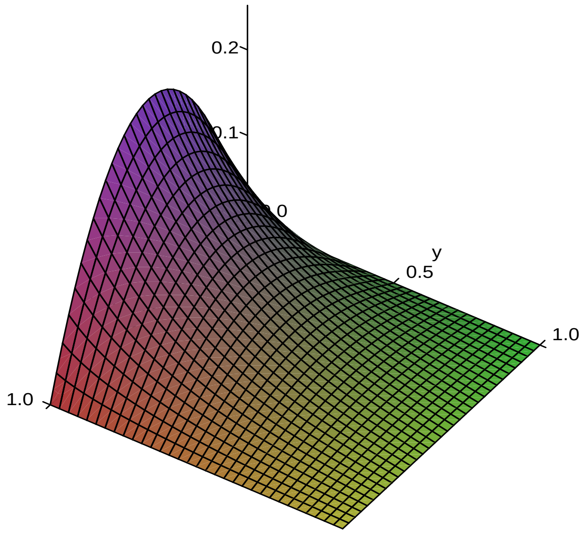


Figure 19. The solution of Laplaces equation with the boundary conditions (5.154)

Example 10
As the final example, we consider

$$u_{xx} + u_{yy} = 0 \tag{5.170}$$

subject to

$$u(x,0) = 0, \quad u(x,1) = 0 \tag{5.171a}$$
$$u(0,y) = y - y^2, \quad u(1,y) = 0. \tag{5.171b}$$

We could go through a separation of variables to obtain the solution but we can avoid many of the steps by considering the previous three problems. One will notice to transform between the first and second problem, the variables x and y only need to be interchanged. This can also be seen in their respective solutions. One will also notice to transform between this problem and problem 9, we only need to transform x and y again. Thus, to obtain the solution for this final problem, we will transform the solution given in (5.169) giving

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \frac{\sinh n\pi(1-x)}{\sinh n\pi} \sin n\pi y. \tag{5.172}$$

Figure 20 shows the solution.

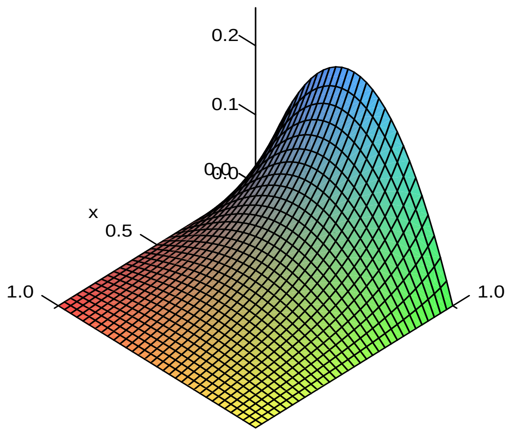


Figure 20. The solution of Laplaces equation with the boundary conditions (5.171)

5.2.1 Laplaces equation on a arbitrary rectangular domain

In this section, we solve Laplaces equation on an arbitrary domain $[0, L_x] \times [0, L_y]$. In analogy to the problems #7 - #10, we will solve this equation when one of the 4 possible boundaries is nonzero and the other boundaries are all zero. Thus, we will solve 4 different problems corresponding to a top, bottom, right and a left boundary. We consider each separately.

Top Boundary

In general, using separation of variables, we wish to find the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.173}$$

subject to

$$\begin{aligned} &u(x,0) = 0, \;\; u(x, L_y) = f(x), \\ &u(0,y) = 0, \;\; u(L_x, y) = 0. \end{aligned}$$

The solution is given by is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \frac{n\pi y}{L_x}}{\sinh \frac{n\pi L_y}{L_x}} \tag{5.175}$$

where

$$b_n = \frac{2}{L_x} \int_0^{L_x} f(x) \sin \frac{n\pi x}{L_x}\, dx \tag{5.176}$$

Bottom Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.177}$$

subject to

$$\begin{aligned} &u(x,0) = f(x), \;\; u(x, L_y) = 0 \\ &u(0,y) = 0, \;\; u(L_x, y) = 0. \end{aligned}$$

The solution is given by

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \frac{n\pi (L_y - y)}{L_x}}{\sinh \frac{n\pi L_y}{L_x}} \tag{5.179}$$

where

$$b_n = \frac{2}{L_x}\int_0^{L_x} f(x)\sin\frac{n\pi x}{L_x}\,dx \tag{5.180}$$

Right Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.181}$$

subject to

$$\begin{aligned} u(x,0) &= 0, \;\; u(x,L_y) = 0 \\ u(0,y) &= 0, \;\; u(L_x,y) = g(y). \end{aligned}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{L_y}\,\frac{\sinh\frac{n\pi x}{L_y}}{\sinh\frac{n\pi L_x}{L_y}} \tag{5.183}$$

where

$$b_n = \frac{2}{L_y}\int_0^{L_y} g(y)\sin\frac{n\pi y}{L_y}\,dy \tag{5.184}$$

Left Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.185}$$

subject to

$$\begin{aligned} u(x,0) &= 0, \;\; u(x,L_y) = 0 \\ u(0,y) &= g(y), \;\; u(L_x,y) = 0. \end{aligned}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{L_y}\,\frac{\sinh\frac{n\pi(L_x-x)}{L_y}}{\sinh\frac{n\pi L_x}{L_y}}. \tag{5.187}$$

where

$$b_n = \frac{2}{L_y}\int_0^{L_y} g(y)\sin\frac{n\pi y}{L_y}\,dy \tag{5.188}$$

Often, we are asked to solve Laplace's equation subject to 4 different nonzero boundary condition. For example

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.189}$$

subject to

$$u(x,0) = 2x, \;\; u(x,2) = 2 - 2x \tag{5.190a}$$

$$u(0,y) = y, \;\; u(1,y) = 2 - y. \tag{5.190b}$$

Here, we use the principle of superposition. We consider 4 subproblems. In each subproblem, $L_x = 1$, $L_y = 2$ and we take zero on three of the four boundaries and then take one of the four boundary conditions in (5.190). Thus, if we consider the solutions of the four subproblems as u_1, u_2, u_3 and u_4 then

$$u = u_1 + u_2 + u_3 + u_4 \tag{5.191}$$

is the solution of the problem in general.

Subproblem 1

$$u_{1xx} + u_{1yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.192}$$

subject to

$$\begin{aligned} &u_1(x,0) = 0, \;\; u_1(x,2) = 2 - 2x, \\ &u_1(0,y) = 0, \;\; u_1(1,y) = 0. \end{aligned}$$

From (5.176) we have

$$b_n = \frac{2}{1}\int_0^1 (2-2x)\sin n\pi x \, dx = \frac{4}{n\pi}, \tag{5.194}$$

and from (5.175) we have

$$u_1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x \, \frac{\sinh n\pi y}{\sinh 2n\pi}. \tag{5.195}$$

Subproblem 2

$$u_{2xx} + u_{2yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.196}$$

subject to

$$\begin{aligned} &u_2(x,0) = 2x, \;\; u_2(x,2) = 0, \\ &u_2(0,y) = 0, \;\; u_2(1,y) = 0. \end{aligned}$$

From (5.180) we obtain

$$b_n = 2\int_0^1 2x \sin n\pi x \, dx = \frac{4(-1)^{n+1}}{n\pi} \tag{5.198}$$

and from (5.179) we obtain

$$u_2 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin n\pi x \frac{\sinh n\pi(2-y)}{\sinh 2n\pi}. \tag{5.199}$$

Subproblem 3

$$u_{3xx} + u_{3yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.200}$$

subject to

$$\begin{aligned} u_3(x,0) &= 0, \;\; u_3(x,2) = 0, \\ u_3(0,y) &= 0, \;\; u_3(1,y) = 2-y. \end{aligned}$$

From (5.184) we obtain

$$b_n = \int_0^2 (2-y)\sin\frac{n\pi y}{2}\, dy = \frac{4}{n\pi}, \tag{5.202}$$

and from (5.183) we obtain

$$u_3 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\frac{\sinh n\pi x}{\sinh n\pi} \sin\frac{n\pi y}{2} \tag{5.203}$$

where
Subproblem 4

$$u_{4xx} + u_{4yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.204}$$

subject to

$$\begin{aligned} u_4(x,0) &= 0, \;\; u_4(x,2) = 0, \\ u_4(0,y) &= y, \;\; u_4(1,y) = 0. \end{aligned}$$

From (5.188) we obtain

$$b_n = \int_0^2 y\sin\frac{n\pi y}{2}\, dy = \frac{4(-1)^{n+1}}{n} \tag{5.206}$$

and from (5.187) we obtain

$$u_4 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\frac{\sinh\frac{n\pi(1-x)}{2}}{\sinh\frac{n\pi}{2}} \sin\frac{n\pi y}{2}. \tag{5.207}$$

Adding (5.195), (5.199), (5.203) and (5.207) gives the solution of our problem. Figure 21 shows the solution.

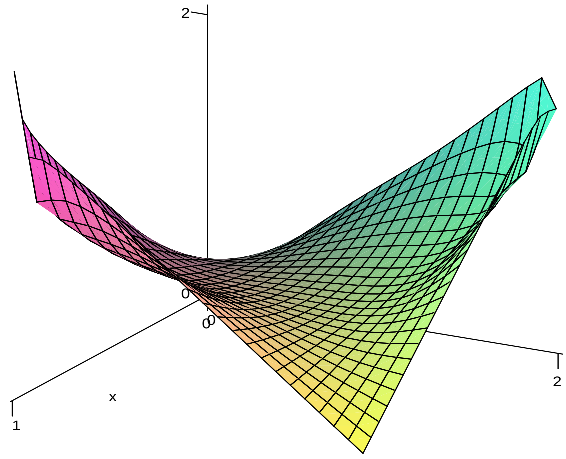


Figure 21. The solution of Laplaces equation with the boundary conditions (5.190)

It is interesting to note the jumps at two of the corners. The reason for this is that for each subproblem there is a jump discontinuity and Gibb's phenomena is experienced.

5.3 The Wave Equation

We end the chapter with the wave equation. We saw in the previous chapter the D'Alembert solution and in this section we obtain separable solutions and show the D'Alembert solution emerging. Here, we will consider a particular example.

$$u_{tt} - u_{xx} = 0, \tag{5.208}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(3,t) = 0, \tag{5.209a}$$

$$u(x,0) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ -(x-1)(x-2) & \text{if } 1 \le x \le 2, \\ 0 & \text{if } 2 < x < 3, \end{cases} \tag{5.209b}$$

$$u_t(x,0) = 0. \tag{5.209c}$$

Assuming solutions of the form

$$u(x,t) = X(x)T(t), \tag{5.210}$$

then substituting this into (5.120) gives

$$T''X - TX'' = 0. \tag{5.211}$$

Dividing by TX and expanding gives

$$\frac{T''}{T} - \frac{X''}{X} = 0, \tag{5.212}$$

and since each term is only a function of x or t, then each must be constant giving

$$\frac{T''}{T} = \lambda, \quad \frac{X''}{X} = \lambda. \tag{5.213}$$

From (5.209a)

$$X(0) = 0, \quad X(3) = 0. \tag{5.214}$$

As we have seen previously, of the three possible cases for λ ($\lambda < 0$, $\lambda = 0$ or $\lambda > 0$) the only case that can satisfy the boundary conditions (5.214) is $\lambda < 0$. Therefore, the X solution of (5.213) subject to (5.214) is

$$X = c_1 \sin \frac{n\pi x}{3} \tag{5.215}$$

noting that $\lambda = -\frac{n^2\pi^2}{9}$. From (5.213) we solve for T giving

$$T = c_2\, sin \frac{n\pi t}{3} + c_3\, cos \frac{n\pi t}{3}. \tag{5.216}$$

Thus, the solution of (5.208) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{n\pi t}{3} + b_n \cos \frac{n\pi t}{3} \right) \sin \frac{n\pi x}{3} \tag{5.217}$$

where we have chosen $a_n = c_1c_2$ and $b_n = c_1c_3$. In order to find a_n and b_n, it is necessary to use the initial conditions (5.209b) and (5.209c). Substituting these into (5.217) gives

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} = \begin{cases} 0 & \text{if } 0 < x < 1, \\ -(x-1)(x-2) & \text{if } 1 \leq x \leq 2, \\ 0 & \text{if } 2 < x < 3, \end{cases} \tag{5.218a}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} a_n \frac{n\pi}{3} \sin \frac{n\pi x}{3} = 0. \tag{5.218b}$$

These we recognize as Fourier Sine series from which we obtain $a_n = 0$ and

$$b_n = \frac{36}{n^3\pi^3}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right) - \frac{6}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right). \tag{5.219}$$

and thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi t}{3}\sin\frac{n\pi x}{3}. \tag{5.220}$$

Different time snap shots are shown in Figure 22.

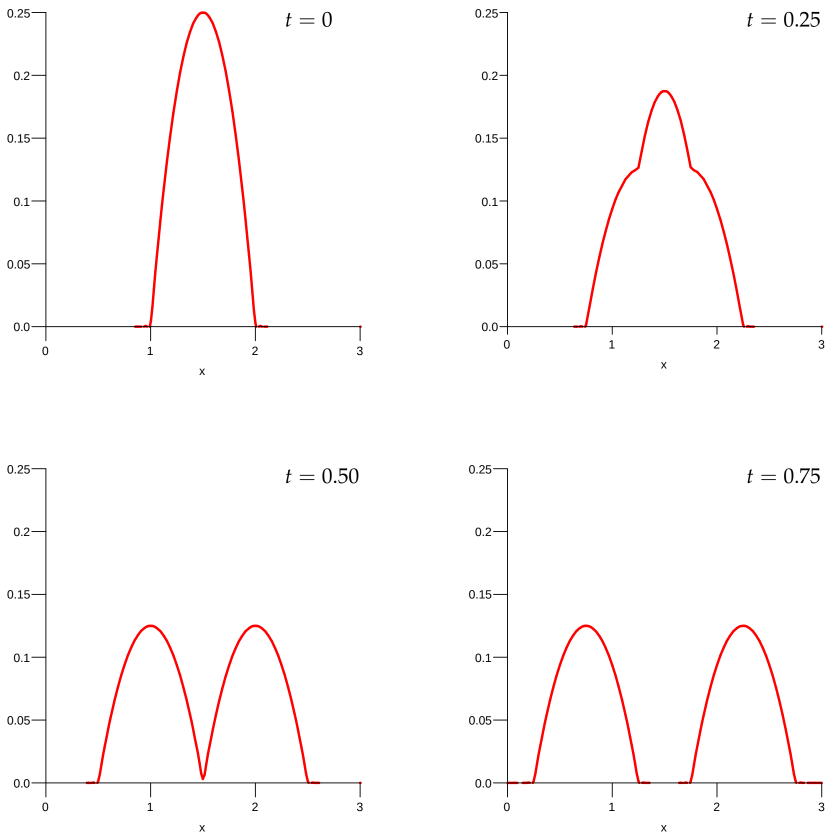


Figure 22. The solution of (5.217) at times $t = 0$, $t = 0.25$, $t = .5$ and $t = .75$.

It is interesting to note that a closer examination of (5.220) shows that if we use the

$$2\sin A\cos B = \sin A + B + \sin A - B,$$

then we can re-write (5.220)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{b_n}{2}\left(\sin\frac{n\pi(x+t)}{3} + \sin\frac{n\pi(x-t)}{3}\right) \tag{5.221}$$

which can be seen as the addition of two wave $-$ one moving left and one moving right. If we recall, the D'Alembert solution for the wave equation with

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \tag{5.222}$$

is

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) \tag{5.223}$$

which we can see emerging from (5.221).

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