CAPE Laboratory

Spring Semester 2025 - 2026

Assignment – 4

Objective: Numerical solution of a Partial Differential Equation.

Problem

Consider the following unsteady-state heat conduction problem in a one-dimensional slab of 1 m thickness.

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \qquad T(x, t = 0) = 350\ K,\ \ T(x = 0, t) = 300\ K,\ \ T(x = 1\ \text{m}, t) = 400\ K$$

Determine the unsteady temperature distribution $T(x,t)$ in the slab at different times (t = 1, 5, 10, 50, 100 s) for three different values of thermal diffusivity (α = 1, 10, 100 m^2/s).

Use: Explicit discretization, Implicit discretization, and Crank-Nicholson discretization.

Also solve the above problem using MATLAB function `pdepe` also and compare your results with the output of `pdepe` and analytical solution.

In the following description of discretization, n stands for time and i stands for space.

Explicit discretization:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n + T_{i-1}^n - 2T_i^n}{(\Delta x)^2}$$

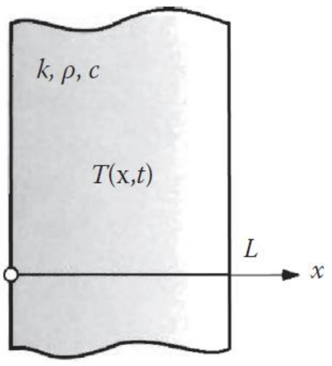
Implicit discretization:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^{n+1} + T_{i-1}^{n+1} - 2T_i^{n+1}}{(\Delta x)^2}$$

Crank-Nicholson discretization:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{T_{i+1}^{n+1} + T_{i-1}^{n+1} - 2T_i^{n+1}}{(\Delta x)^2} + \frac{T_{i+1}^n + T_{i-1}^n - 2T_i^n}{(\Delta x)^2}\right]$$

Analytical Solution:



$$T(x,0) = f(x)$$

$$T(0,t) = T_1, \qquad T(L,t) = T_2$$

Solution:

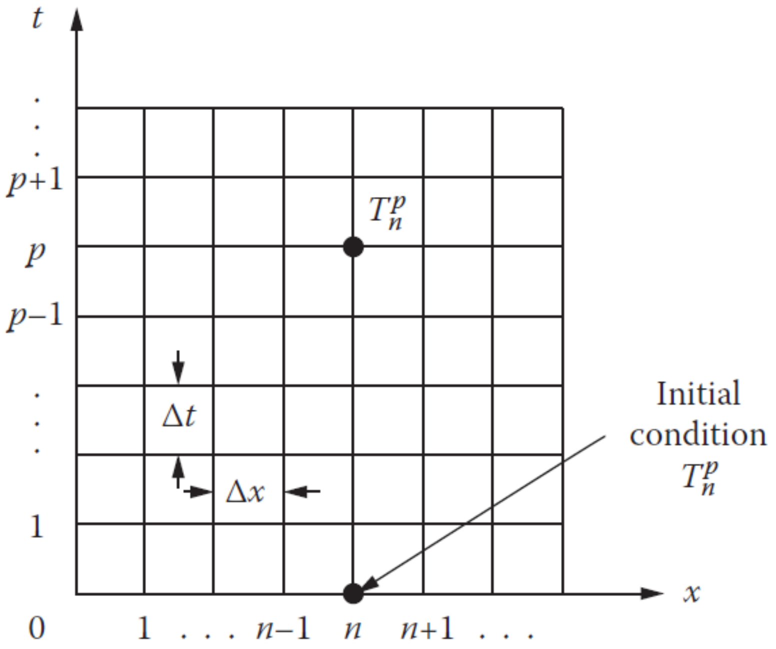
$$T(x,t) = T_1 - (T_1 - T_2)\frac{x}{L} - \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t} \sin \lambda_n x$$
$$\times \left\{ \frac{T_1 - (-1)^n T_2}{\lambda_n} - \int_0^L f(x') \sin \lambda_n x' \, dx' \right\}$$

Here,

$$\lambda_n = n\pi/L.$$

Explicit Finite Difference Method

Let us form a network of grid points by dividing x and t domains into small intervals of Δx and Δt



In this description of discretization, p stands for time and n stands for space. Thus, T_n^p represents the temperature at location $x = n\,\Delta x$ at time $t = p\,\Delta t$.

Using Explicit Finite Difference method, the Heat Equation can be approximated at time t (= $p\,\Delta t$) as

$$\frac{T_{n+1}^{p} + T_{n-1}^{p} - 2T_{n}^{p}}{(\Delta x)^2} = \frac{1}{\alpha}\,\frac{T_n^{p+1} - T_n^{p}}{\Delta t}$$

$$T_n^{p+1} = \left[1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right] T_n^p + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{n+1}^p + T_{n-1}^p\right)$$

For stable solution, we must have

$$\frac{\alpha\Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

This formulation is called *explicit* because it is possible to write the temperature T_n^{p+1} (at location n, at time $t + \Delta t$) explicitly in terms of the temperatures at time t and at locations n – l, n, and n + l.

Thus, if the temperatures of the grid locations are known at any particular time t, the temperatures after a time increment Δt may be calculated by writing an equation like above for each grid location, and obtaining the values of T_n^{p+1}.

The calculation proceeds directly from one time increment to the next until the temperature distribution is obtained at the desired time.