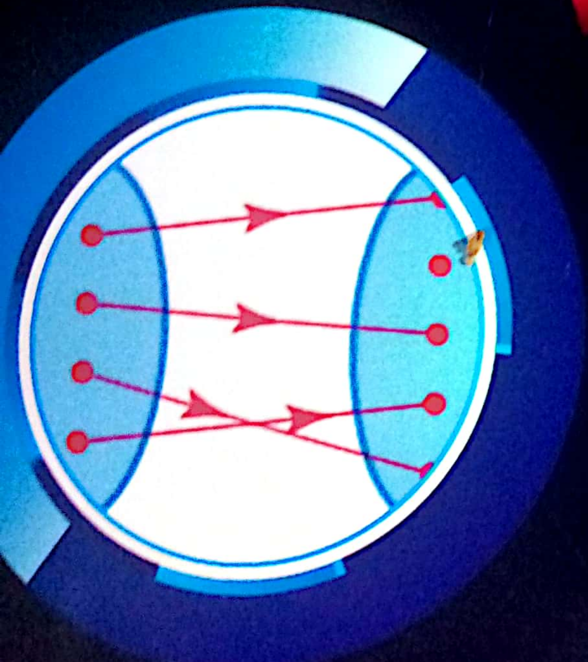
CONCISE GUIDE TO INTRODUCTORY UNIVERSITY MATHEMATICS

STUDENTS' WORKBOOK



DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
FEDERAL UNIVERSITY OYE-EKITI

Contents

1 CHAPTER ONE
- 1.1 FUNCTION OF A REAL VARIABLE
- 1.2 Introduction
- 1.3 Mapping
  - 1.3.1 One-One Mapping
  - 1.3.2 Onto-Mapping
  - 1.3.3 Constant Mapping
  - 1.3.4 Identity Mapping
- 1.4 Commonly used General Functions
- 1.5 Some Special Functions
- 1.6 Graphs

2 *CHAPTER TWO*
- 2.1 LIMITS AND CONTINUITY
- 2.2 Introduction
- 2.3 Definition
- 2.4 Theorems on Limits
- 2.5 Continuity
- 2.6 Comparing Limits and Continuity
- 2.7 Special Limits

3 *CHAPTER THREE*
- 3.1 THE DERIVATIVE AS A LIMIT OF CHANGE
- 3.2 Increments
- 3.3 Average Rate of Change
- 3.4 The Derivative
- 3.5 Maximum and Minimum
- 3.6 Theorems and Definitions
- 3.7 Curve Sketching

4 *CHAPTER FOUR*
- 4.1 **BASIC TECHNIQUES OF DIFFERENTIATION**
- 4.2 Differentiation
- 4.3 Product Rule:
- 4.4 Quotient Rule
- 4.5 Logarithmic Differentiation
- 4.6 Differentiation of a function of a function.
- 4.7 The second Derivative of a Function
- 4.8 Differentiation of implicit function

5 *CHAPTER FIVE*
- 5.1 **INTEGRATION AND IT'S APPLICATIONS**
- 5.2 Introduction
- 5.3 Standard integral forms
- 5.4 Function of a linear function of x.
- 5.5 Integrals of the form $\int \frac{f'(x)}{f(x)}dx$ and $\int f(x)f'(x)dx$
- 5.6 Integration by substitution
- 5.7 Integration by Trigonometric Substitution

5.8 Integration of Trigonometric Functions 78
5.9 Products of Sine and Cosine 80
5.10 Integration by part 81
5.11 Integration by partial fraction 83
5.12 Application of integration 86
5.13 Trapezoidal Rule: 91
5.14 Simpson's Rule 93

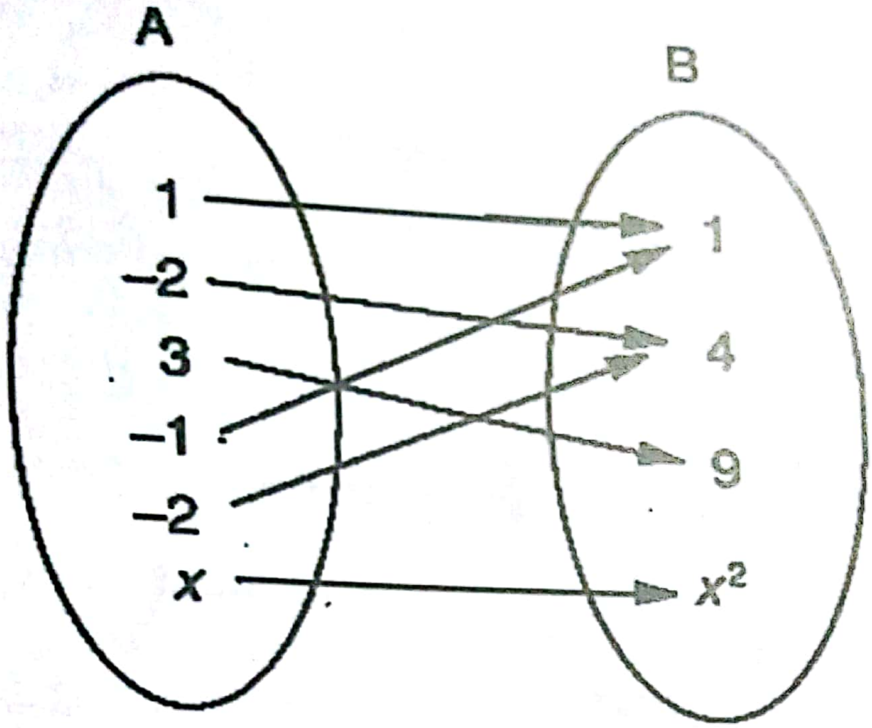


Figure 1: Mapping

under, we consider properties of mapping.

1.3 Mapping

As discussed earlier, mapping is any rule which associates two sets of items. A relation is a mapping if and only if every element in the domain has an image in the co-domain and also, each element in the domain must not have more than one image. Consider the relation between two sets A and B in Figure 1 where x in set A determines x^2 in set B.

Set A is called the domain while set B is the co-domain. The set of all the elements in the co-domain that are images of objects in the domain is referred to as Range.

1.3.1 One-One Mapping

A mapping is one-one if different elements in the domain have different elements in the co-domain. Figures 2and 3 illustrate one-one mapping.

1 CHAPTER ONE

1.1 FUNCTION OF A REAL VARIABLE

1.2 Introduction

The concept of function is one of the most important ideas in the study of Mathematics. Firm understanding of a basic elementary functions and their properties helps in our day-to-day activities. One of the most important aspects of any science is the establishment of correspondences among various types of phenomena. Once a correspondence is known, predictions can be made. We have already had experiences with correspondences in daily living. For example:

to each member of staff in FUOYE there corresponds a monthly income.
to each item in a supermarket there corresponds a price.
to each student in Federal University Oye-Ekiti there corresponds a grade point average.
to each day of the week there corresponds a maximum temperature.
for the manufacture of exercise books there corresponds a cost.
to each square there corresponds an area.

An educationist would like to know the correspondence between class attendance and poor performance in a course (PHY 101). The question then is, what is a function?

A function f is a relation between two sets A and B commonly referred to as the domain or the source set and the codomain or target set respectively. f assigns x, a member of the domain to a unique member of the codomain. The subset of the codomain that takes part in the function is called the image set or simply the image of the function. This rule of correspondence takes the form $y = f(x)$. Thus, we say y is a function of x and $f(x)$ is read as "f of x". As a newly born baby absolutely depends on the mother for development, so also y depends on x for its value(s). The variable x is called the independent variable and y is called the dependent variable. The set of values of the independent variable x is called the domain of the function while the set of values of the dependent variable is called the range of the function. Many real life situations

can be modelled by functions. For example, the perimeter of a squared office in FUOYE is $P = 4L$. In this case, P, the dependent variable is a function of L, the independent variable.

The two words function and mapping are used interchangeably in text. Here

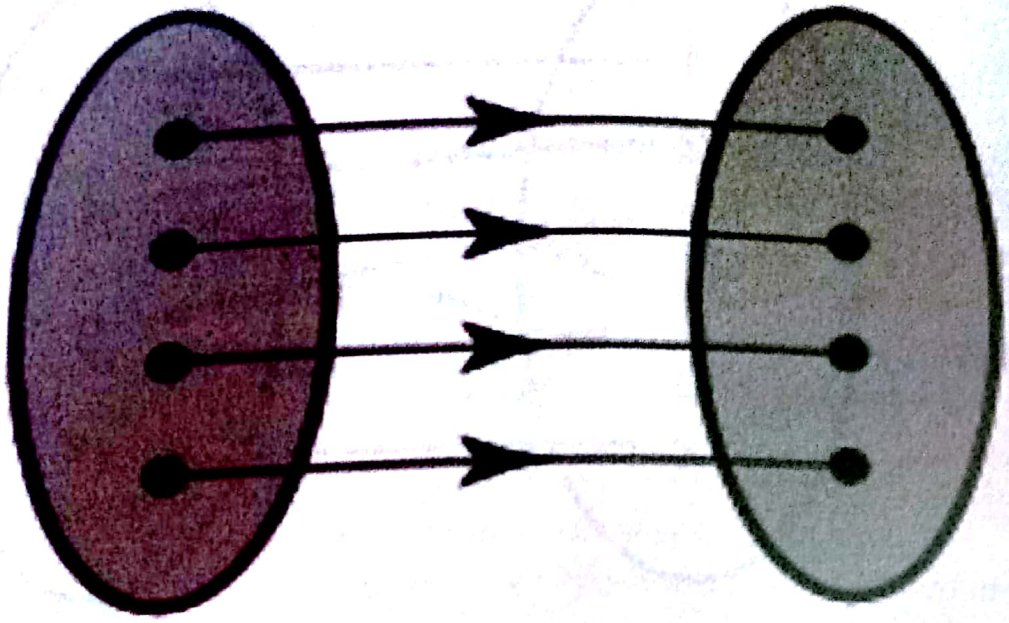


Figure 2: One-One Mapping

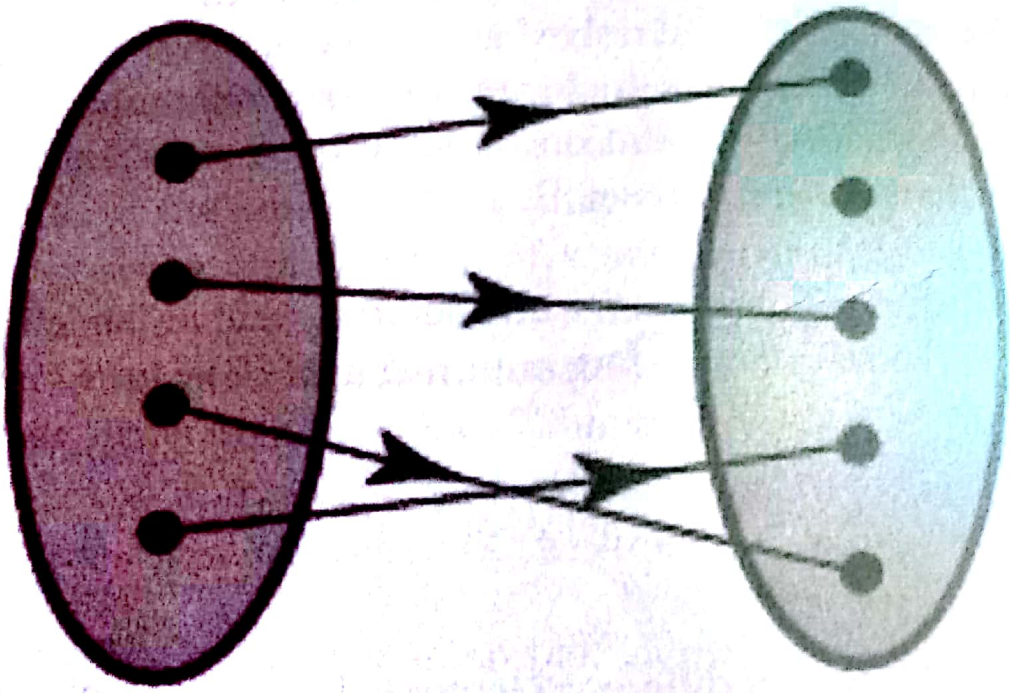


Figure 3: One-One Mapping

The domain of f is $\Re$, the codomain of f is $\Re$ while the range is the set $\{2x : x \in \Re\}$. This function is single valued function as one value of x in the domain generate one value of $f(x)$ in the codomain.
2.A function $f : \Re \to \Re$ defined as $f(x) = \sqrt{x}$ for every $x \in \Re$.
The domain of f is $\Re$, the codomain of f is $\Re$ while the range is the set $\{\pm\sqrt{x} : x \in \Re\}$. This function is not a single valued function as one value of x in the domain generate two values of $f(x)$ in the codomain.
3. Consider Sets A and B given as follow:
$A = \{a, b, c, d\}$ and $B = \{r, s, t, u\}$ Define relations $f, g, h : A \to B$ as
$f(a) = s, f(b) = u, f(c) = t$
$g(a) = s, g(b) = u, g(c) = u, g(d) = t$
$h(a) = s, h(b) = u, h(b) = t, h(c) = u, h(d) = t$
Is f a function, give reason.
Is g a function, give reason.
Is h a function, give reason.
Answer
f is not a function since $f(d)$ is not defined. For a relation to be a function, every element of the source set must be assigned an image in the target set.
g is a function since two or more elements from the codomain can have the same element as their image.
h is not a function since an element from a source set cannot have two different elements as its image in the target set.

1.4 Commonly used General Functions

The following are some of the general functions we come across frequently. ***Polynomial function, logarithmic function, rational function,*** *absolute* ***value function, trigonometric function, exponential function,*** *con-* ***stant function*** etc. (a) ***Polynomial function:*** Any function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad (1)$$

where n = 0, 1, 2, . . . and $a_j's$ are real numbers. If n = 0, then $f(x) = a_0$ is a ***constant function*** and if n = 1, then $f(x) = a_0 + a_1 x$ is a *linear function*.
The following are examples of polynomial functions.
(i) f(x) = 5. . . . . polynomial of order zero (constant function)
(ii) f(x) = x+3 . . . . polynomial of order one (linear function)
(iii) f(x) = $x^2 + 4x - 7$ . . . . polynomial of order two (quadratic function)

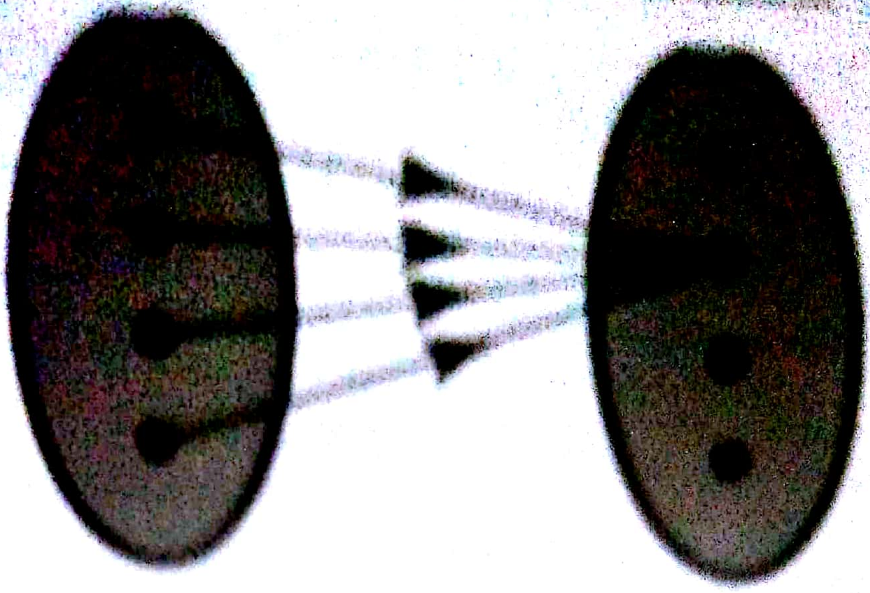


Figure 4: Constant Mapping

1.3.2 Onto Mapping

A mapping is said to be onto if every element in the co-domain is an image of at least one element in the domain. Every onto-mapping has its co-domain equals the range.
In section 1.3.1, Figure 2 illustrates onto-mapping while Figure 3 is not.

1.3.3 Constant Mapping

A mapping where every element in the domain is mapped to a single element in the co-domain is referred to as Constant Mapping. This is illustrated in Figure 4 where every element in the domain is mapped to same element (image) in the co-domain

1.3.4 Identity Mapping

This is a unique mapping where a set, say A is mapped to itself. Consider a set A defines as A=(a, b, d, e). If f maps A to A such that f(a)=a, f(b)=b, f(d)=d and f(e)=e, then the mapping is an identity mapping.

A function can be futher classified into either ***single-valued*** or ***multi-valued function***. If to each value of y there corresponds only one value of x then, the function is ***single-valued***. Otherwise, the function is called ***multi-valued function***.
Here are some examples;
1. A function $f : \Re \rightarrow \Re$ defined as $f(x) = 2x$ for every $x \in \Re$.

are the inverse functions of the trigonometric functions. They are the inverse of tangent, sine, cosine, secant, cosecant and cotangent functions. The following are examples of inverse trigonometric functions:
(i) $y = sin^{-1}x$ (ii) $y = cos^{-1}x$ (iii) $y = tan^{-1}x$ (iv) $y = cosec^{-1}x$ (v) $y = sec^{-1}x$
(vi) $y = cot^{-1}x$ (v) ***Hyperbolic functions***: Hyperbolic functions have similar names to the trigonometric functions, but are defined in terms of exponential functions. Examples are:

(i) sinh x = $\frac{e^x - e^{-x}}{2}$ (ii) cosh x = $\frac{e^x + e^{-x}}{2}$ (iii) tanh x = $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ (iv) sech x = $\frac{2}{e^x + e^{-x}}$
(v) cosech x = $\frac{2}{e^x - e^{-x}}$ (vi) coth x = $\frac{e^x + e^{-x}}{e^x - e^{-x}}$. (vi) ***Inverse hyperbolic functions***: these are the inverse functions of the hyperbolic functions. For example, $\sinh^{-1} x$, $\cosh^{-1} x$, $tanh^{-1}x$, $cosech^{-1}x$, $sech^{-1}x$, $coth^{-1}x$ etc.

1.5 Some Special Functions

(1) **Constant Function** Let $f : A \to \Re$ be a function. If there is a real number $c \in \Re$ such that $f(x) = c$ for every $x \in A$, then f is called a constant function.

Example: Let $A = \{1, 2, 4, 6\}$. Define $f : A \to \Re$ by $f(x) = -2$ for every $x \in A$. f is called a constant function.

Note that:

(i) Every function whose domain is a singleton is a constant function.

(ii) The sum, product and quotient of two constant functions is a constant function.

(iii) The range of a constant function is a singleton.

(2) **Identity Function**

The function from $\Re \to \Re$ whose value at any $x \in \Re$ is x is called an identity function. It is usually denoted by Id. That is

$Id(x) = x$ fr all $x \in \Re$.

Example: Let $A = \{1, 2, 4, 6\}$. Define $f : A \to \Re$ by $f(1) = 1, f(2) = 2, f(4) = 4, f(6) = 6$, then f is an identity function.

(3) **Signum Function**

A function from $\Re \to \Re$ whose value at any $x \in \Re$ is $1, 0, -1$ according to whether x is positive, zero or negative respectively is called a signum function.

(iv) f(x) $= 3x^3 - 7x^2 + x - 1$ .... polynomial of order three (cubic function) etc.

Polynomial function (1) is called a polynomial of degree n if $a_n \neq 0$. (b) ***Algebraic functions***: These are functions of the form $y = f(x)$ satisfying the equation

$$P_0(x)y^n + P_1(x)y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0 \qquad (2)$$

where $P_0(x), \dots, P_n(x)$ are polynomials in x. If the function can be expressed as the quotient of two polynomials, it is called a ***rational algebraic function***; otherwise, it is an ***irrational algebraic function***. A ***rational function*** is any function of the type $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomial functions and g is not a zero polynomial. The following are examples of rational functions.

(i) h(x)$= \frac{2x-1}{x+2}$, $x \neq -2$ (ii) g(x)$= \frac{x^2}{5}$ (iii) f(x)$= \frac{1}{x}$, $x \neq 0$ (iv) f(x) $=$ x etc.

(c) ***Transcendental functions***: are functions which are not algebraic i.e., they do not satisfy equations of the form of Equation (2). These functions can be further classified into ***exponential functions, logarithmic functions, trigonometric functions, Inverse trigonometric functions, Hyperbolic functions, Inverse hyperbolic functions***. (i) ***Logarithmic Functions***: Let $y = f(x)$ then $f(x) = log_a x$ if and only if $x = a^y$ where $x > 0, a > 0$ and $a \neq 1$. The function $f(x) = log_a x$ is a logarithmic function and the number a is fixed and called the base of the logarithm.

The following are examples of logarithmic functions: (i) $Log_2 x$ (ii) log_5^x (iii) $log_{\frac{1}{3}}^x$. function.

Properties of Logarithmic Functions

Let $g(x) = a^x$ and $f(z) = log_a^z$. If x and z are positive real numbers and p is a real number, then for any base a, a ¿ 0 and $a \neq 1$, we have (a) $g(f(z)) = a^{log_a^z} = z$ (b) $f(g(x)) = log_a^{a^x} = x$ (ii) ***Exponential Functions***: The function $f(x) = a^x$ is a general representation of exponential function, where a is any fixed positive number and x is any real number. For example, 7^x is an exponential function.

(iii) ***Trigonometric functions (Circular functions)***: These are functions of an angle. They relate the angles of a triangle to the length of its sides. Examples are *tangent, sine, cosine, secant, cosecant* and *cotangent* function.

(iv) ***Inverse trigonometric functions (Cyclometric functions)*** these

: The Floor of 5 is 5 and the Ceiling of 5 is 5.

What is the floor and ceiling of 7.92? Ans.: The floor and ceiling of 7.92 ' and 8 respectively.

Ceiling Function: For every real number x the value of $\lceil x \rceil$ is the smallest ger that is greater than or equal to x.

n the table below, the floor and ceiling of different values are obtained.

x	Floor	Ceiling
-9	-9	-9
-9.5	-10	-9
9.5	9	10
9.99	9	10
0.4	0	1
8	8	8

Even Function: A function f is said to be an even function if f(-x) = f(x) all x in the domain of f. It is assumed that for every x in the domain of f,) is also in the domain.

amples

f(x) = x^2

) f(x) = e^{-x^2}

) f(x) = $|x|$

6 Graphs

ie most common method for visualizing a function is its graph. If f is a nction with domain A, then its graph is the set of ordered pairs $\{(x, f(x))$ $\in A$. In other words, the graph of f consists of all points (x,y) in the ordinate plane such that $y = f(x)$ and x is in the domain of f. The graph a function gives us a useful picture of the behavior or *"life history"* of a nction. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, e can read the value of $f(x)$ from the graph as being the height of the graph

$$sgn(x) = \begin{cases} 1 & if x > 0, \\ 0 & if x = 0, \\ -1 & if x < 0 \end{cases}$$

The domain of sgn is $\Re$ while the range is $\{-1, 0, 1\}$. (4) **Heaviside function**: The function from $\Re \to \Re$ whose value at any positive real number is 1 and whose value at all other real number is 0 is called Heaviside function or Heaviside step function denoted by H.

$$H(x) = \begin{cases} 1 & if x > 0, \\ 0 & if x \leq 0 \end{cases}$$

The domain of H is $\Re$ while the range is $\{0, 1\}$.

(5)**Modulus or Absolute Value Function**:

The modulus or absolute value function on $\Re$, denoted by mod or $||$ is defined as

$$| x |= \begin{cases} x & if x > 0, \\ 0 & if x = 0, \\ -x & if x < 0 \end{cases}$$

Note that for all $x \in \Re$,

(i) $| x | \geq 0$

(ii) $| x |= 0$ if and only if $x = 0$

(iii) $| -x |=| x |$.

Other functions are:

(1) **Floor Function**: For every real number x, the value of $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x.

Examples

(i) What is the floor and ceiling of 5?

2 CHAPTER TWO

2.1 LIMITS AND CONTINUITY

2.2 Introduction

Limit is an important aspect in calculus. Without it, calculus would not exist. Let L be some real numbers, c be a number and f be a function defined at all numbers x near c but not necessarily at c itself. We say that the limit of f(x) as x tends to c is L and write

$$\lim_{x \to c} f(x) = L$$

Take for example $\lim_{x\to 1} 5x + 9 = 14$. As x approaches 1, $5x$ approaches 5 and $5x + 9$ approaches 14.

Limits are further categorized into two – ***left*** and ***right limits***. Those that lie to the left of c and those that lie to the right of c.

We write

$$\lim_{x\to c} f(x) = L$$

to indicate that as$x$ approaches c from the left, $f(x)$ approaches L.

We write

$$\lim_{x\to c^+} f(x) = L$$

to indicate that as x approaches c from the right, $f(x)$ approaches L.

2.3 Definition

To aid the better understanding of the concept of limit, the following definitions are considered.

1. The function f is said to be continuous at c from the right if $f(c)$ is defined, and

$$\lim_{x\to c^+} f(x) = L$$

2. The function f is said to be continuous at c from the left if f(c) is defined, and

$$\lim_{x\to c^-} f(x) = L$$

above the point x. The graph of f also allows us to picture the domain of f on the $x-axis$ and its range on the $y-axis$.
Consider an exponential function $f(x) = a^x$, where **a** is any fixed positive number and x is any real number. The graphs of $f(x) = (\frac{1}{2})^x$, $f(x) = (\frac{1}{3})^x$, $f(x) = 3^x$, $f(x) = 2^x$ and $f(x) = 1^x$ are as shown in Figure 5. Figures 5 to 13 are graphs of eight other basic functions.

Exercise

(1) If f(x) $= x^2$ and g(x)$= x^2 + 1$, find

(i) g(g(0))

(ii) f(g(2)).

(2) If f(x)$= 2^x$, find and simplify

(i) $f(x^2)f(1)$

(ii) $\frac{f(x+3)}{f(-x)}$

(iii) $f(\sqrt{x})$

(iv) $f(x^2 + 1)$

(3) Given that $f(x) = 10^x$. Find and simplify

(i) f(x) + f(2+x)

(ii) f(x)f(2+x)

(iii) $\frac{f(x)}{f(2+x)}$

(iv) f(f(2+x))

(v) $f(sin^2x)f(cos^2x)$

(vi) $\frac{f(cos^2x)}{f(sin^y)}$

(4) Solve for a if (i) $log_a^{216} = 3$ (ii) $log_a^{625} = 4$

(iii) $log_a^{\frac{1}{4a}} = -2$ (iv) $log_{14}^{x} = \frac{1}{4}$

(5) Solve for x if (a) $4^x = 7$ (b) $3^x = 6^{x+3}$

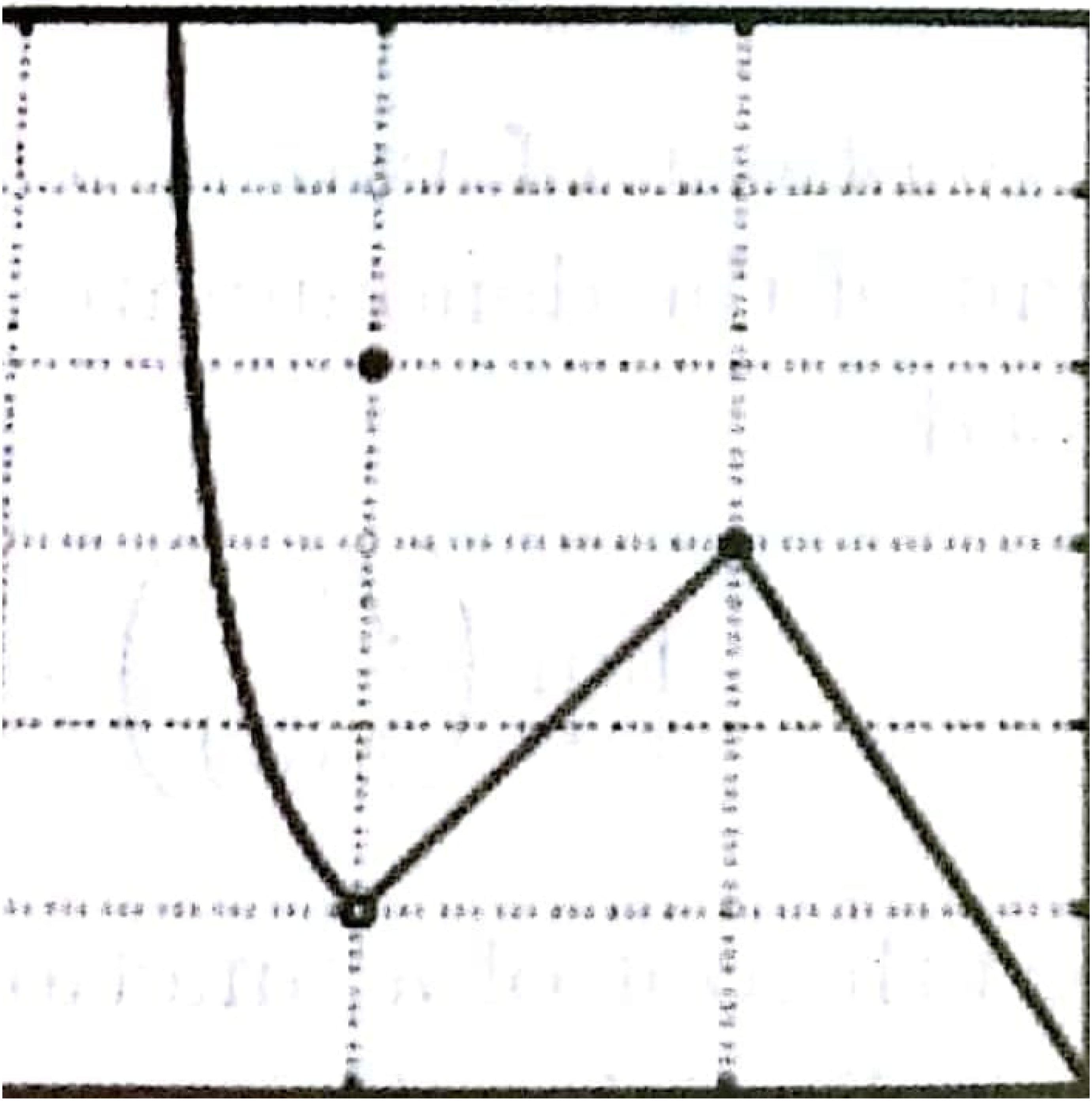
(c) $5^{x+1} = 9$ (d) $2^{x-1} = 5^{2x+1}$

(e) $8^{x+2} = 3^{2x-1}$ (d) $7^x = 4^{2x-1}$

(6) Find the values of the floor and ceiling of the functions represented by the values in the table below.

) , that is defined at x

its and Continuity



3. The function f is said to be (two-sided) continuous at c if f(c) is defined, and

$$\lim_{x \to c} f(x) = L$$

2.4 Theorems on Limits

The following theorems hold on limit with assumption that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$

1. The limit of a constant function is a constant. If f(x) = c, where c is a constant, then

$$\lim_{x \to a} f(x) = c$$

2. The limit of the product of a constant and a function is the product of the constant and the limit of the function. That is

$$\lim_{x \to a} c.f(x) = c \lim_{x \to a} f(x)$$

3. The limit of a sum (difference) of two functions is the sum (difference) of the respective limits.

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

4. The limit of the product of two functions is the product of their respective limits.

$$\lim_{x \to a} [f(x)\ g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

5. The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is nonzero, else the quotient of their limits is undefined.

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \lim_{x \to a} g(x) \neq 0$$

6. The limit of the nth root of a function is the nth root of its limit provided the nth root of its limit is a finite real number.

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{provided } \sqrt[n]{\lim_{x \to a} f(x)} \text{ is}$$

defined.

(a) Infinite: if n is any positive integer,then

$$\lim_{x \to a} \frac{1}{X^n} = 0$$

Solution

$$(a).\ \lim_{x\to 0}\frac{x^2+7x}{x}=\lim_{x\to 0}\frac{x(x+7)}{x}$$
$$=\lim_{x\to 0}(x+7)$$
$$=7$$

$$(b).\ \lim_{x\to -3}\frac{x^3+27}{x+3}=\lim_{x\to -3}\frac{(x+3)(x^2-3x+9)}{(x+3)}$$
$$=\lim_{x\to -3}(x^2-3x+9)$$
$$=27$$

$$(c).\ \lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)=\frac{(x-2)(x-1)}{(x-3)(x-1)}$$
$$=\lim_{x\to 1}\frac{(x-2)}{(x-3)}$$
$$=\frac{1}{2}$$

$$\lim_{x \to 1} f(x) = 1$$

At $x = 2$, the function is continuous with $f(2) = 3$, which also means that the limit exists. At all non-integer values of x the function is continuous (hence its limit exists). We will see that the derivative only exists at these non-integer values of x.

2.7 Special Limits

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

3. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

4. $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

5. $\lim\limits_{x \to 0} \left(\dfrac{e^x - 1}{x}\right) = 1$

6. $\lim\limits_{x \to 0} \left(\dfrac{x - 1}{lnx}\right) = 1$

Example 1

1. Evaluate the following limits

(a) $\lim\limits_{x \to 0} \dfrac{x^2 + 7x}{x}$

(b) $\lim\limits_{x \to -3} \dfrac{x^3 + 27}{x + 3}$

(c) $\lim\limits_{x \to 1} \left(\dfrac{x^2 - 3x + 2}{x^2 - 4x + 3}\right)$

(d) $\lim\limits_{x \to 1} \left(\dfrac{x - 1}{x^2 + 2x - 3}\right)$

(e) $\lim\limits_{x \to 2} \left\{\sqrt[3]{2x^3 - 3x - 18}\right\}$

(f) $\lim\limits_{x \to 0} ((x^2 - 2)(x^2 + 1))$

Example 2:

1. if $f(x) = \dfrac{3x + 1}{x - 2}$, $x \neq 2$, find:

(a) $\dfrac{5f(-1) - 2f(0) + 3f(5)}{6}$

(b) $\left\{f\left(\dfrac{-1}{2}\right)\right\}^2$

Solution

(a) Given that $f(x) = \dfrac{3x + 1}{x - 2}$, $x \neq 2$

$$f(-1) = \frac{3(-1) + 1}{-1 - 2} = \frac{2}{3}$$

$$f(0) = \frac{3(0) + 1}{0 - 2} = -\frac{1}{2}$$

$$f(5) = \frac{3(5) + 1}{5 - 2} = -\frac{16}{3}$$

therefore,

$$\frac{5f(-1) - 2f(0) + 3f(5)}{6} = \frac{5\left(\frac{2}{3}\right) - 2\left(\frac{-1}{2}\right) + 3\left(\frac{16}{3}\right)}{6}$$

$$= \frac{61}{48}$$

$$(b)\ f\left(-\frac{1}{2}\right) = \frac{3\left(-\frac{1}{2}\right) + 1}{-\frac{1}{2} - 1}$$

$$= \frac{1}{3}$$

therefore,

$$(d).\ \lim_{x\to 1}\left(\frac{x-1}{x^2+2x-3}\right)=\frac{x-1}{(x+3)(x-1)}$$
$$=\frac{1}{x+3}$$
$$=\frac{1}{4}$$

$$(e).\ \lim_{x\to 2}\left\{\sqrt[3]{2x^3-3x-18}\right\}=\left\{\sqrt[3]{\lim_{x\to 2}2x^3-3x-18}\right\}$$
$$=\sqrt[3]{-8}$$
$$=-2$$

$$(f).\ \lim_{x\to 0}\left((x^2-2)(x^2+1)\right)=-2$$

a. $\lim_{x\to\frac{1}{2}}\left\{\frac{2x^2-1}{(3x+2)(5x-3)}-\frac{2-3x}{x^2-5x+3}\right\}$

b. $\lim_{x\to\infty}\frac{(3x-1)(2x+3)}{(5x-3)(4x+5)}$

Solution

a.

$$\lim_{x\to\frac{1}{2}}\left\{\frac{2x^2-1}{(3x+2)(5x-3)}-\frac{2-3x}{x^2-5x+3}\right\}=\lim_{x\to\frac{1}{2}}\left\{\frac{2x^2-1}{(3x+2)(5x-3)}\right\}-\lim_{x\to\frac{1}{2}}\left\{\frac{2-3x}{x^2-5x+3}\right\}$$

$$\lim_{x\to\frac{1}{2}}\left\{\frac{2x^2-1}{(3x+2)(5x-3)}\right\}=\frac{2(\frac{1}{2})^2-1}{\{3(\frac{1}{2})+2\}\{5(\frac{1}{2})-3\}}=\frac{2}{7}$$

$$\lim_{x\to\frac{1}{2}}\left\{\frac{2-3x}{x^2-5x+3}\right\}=\frac{2-3(\frac{1}{2})}{\{(\frac{1}{2})^2-5(\frac{1}{2})+3\}}=\frac{2}{3}$$

therefore, $\lim_{x\to\frac{1}{2}}\left\{\frac{2x^2-1}{(3x+2)(5x-3)}\right\}-\lim_{x\to\frac{1}{2}}\left\{\frac{2-3x}{x^2-5x+3}\right\}=\frac{2}{7}-\frac{2}{3}=-\frac{8}{21}$

b.

$$\lim_{x\to\infty}\frac{(3x-1)(2x+3)}{(5x-3)(4x+5)}=\frac{\lim_{x\to\infty}(6x^2+7x-3)}{\lim_{x\to\infty}(20x^2+13x+5)}$$

$$=\frac{\lim_{x\to\infty}\left(6+\frac{7}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(20+\frac{13}{x}+\frac{5}{x^2}\right)}$$

$$=\frac{\lim_{x\to\infty}6+\lim_{x\to\infty}\frac{7}{x}-\lim_{x\to\infty}\frac{3}{x^2}}{\lim_{x\to\infty}20+\lim_{x\to\infty}\frac{13}{x}+\lim_{x\to\infty}+\lim_{x\to\infty}\frac{5}{x^2}}$$

$$\left\{f\left(-\frac{1}{2}\right)\right\}^2 = \left(\frac{1}{3}\right)^2$$
$$= \frac{1}{9}$$

Example 3

Evaluate the following limits.

I. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

II. $\lim_{h \to 0} \frac{(2 + h)^4 - 16}{h}$

Solution

I. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$

$\lim_{x \to 2}(x + 2) = 4$

II.

$$\begin{aligned}
\lim_{h \to 0} \frac{(2 + h)^4 - 16}{h} &= \lim_{h \to 0} \frac{[(2 + h)^2]^2 - 4^2}{h} \\
&= \lim_{h \to 0} \frac{[(2 + h)^2 - 4][(2 + h)^2 + 4]}{h} \\
&= \lim_{h \to 0} \frac{[h^2 + 4h + 4 - 4][h^2 + 4h + 4 + 4]}{h} \\
&= \lim_{h \to 0} \frac{[h^2 + 4h][h^2 + 4h + 8]}{h} \\
&= \lim_{h \to 0} \frac{h[h + 4][h^2 + 4h + 8]}{h} \\
&= \lim_{h \to 0}[h + 4][h^2 + 4h + 8] \\
&= 32
\end{aligned}$$

Example 4

Evaluate each of the following using the theorems on limits.

The derivative of a function f is the limit of (3) as $\Delta x \to 0$ if the limit exists, that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} \qquad (4)$$

$$= \frac{f(x_0 + \Delta x) - f(x_0)}{x_1 - x_0}.$$

Observe that $\Delta x \to 0$ implies $x_1 \to x_0$, so that one can define the derivative of f as

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

Likewise, we can still have the derivative of f defined as $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$, that is $\Delta x = h$.

Definition: Let $f : I \to R$. The first derivative of f at $x_0 \in I$ denoted by $f'(x_0)$ is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x)}{\Delta x},$$

if the limit exists.

The derivative can be expressed in different forms such as

$$f'(x_0), Df(x_0), \frac{df}{dx}(x_0), y'(x_0), Dy(x_0), \frac{dy}{dx}(x_0) \text{ or } D_x f(x_0)$$

If there is no ambiguity in the argument, we can simply write, for the derivative of f as $f', Df, \frac{df}{dx}, y', or D_x f$. The process of finding the derivatives is called differentiation.

Examples. Find the derivatives of the following functions.

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = sin x$
4. $f(x) = cot(x)$

Solution

1. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

2. $f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right) = \lim_{\Delta x \to 0} \left(\frac{(x + \Delta x)^2 - x^2)}{\Delta x} \right)$

Scanned by CamScanner

3 CHAPTER THREE

3.1 THE DERIVATIVE AS A LIMIT OF CHANGE

The derivative is a concept that serves as the fundamental element of the differential calculus. Consider a function defined in an open interval I , that is, $f : I \to R$. The value of f at any point $x \in I$ is denoted by $y = f(x)$. For any two points $x_0, x_1 \in I$, et $y_0 = f(x_0)$ and $y_1 = f(x_1)$. The following definitions are therefore considered.

3.2 Increments

The increment Δx of a variable x is the change in x as x increases or decreases from one value $x = x_0$ to another value $x = x_1$ in its domain.
Also let change in y be denoted by Δy. Then $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$

Now, $\Delta x = x_1 - x_0 \Rightarrow x_1 = x_0 + \Delta x$ (1)
$\Delta y = y_1 - y_0 = f(x_1) - f(x_0)$

3.3 Average Rate of Change

If the variable x is given as an increment Δx from $x = x_0$ to $x = x_1$(that is, if x changes from $x = x_0$ to $x = x_0 + \Delta x$) and a function y = f(x) is thereby given an increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ from $y = f(x_0)$, the quotient

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$$

is called the average rate of change of the function on the interval between $x = x_0$ and $x = x_0 + \Delta x$

3.4 The Derivative

The derivative of a function y=f(x) with respect to x at the point $x = x_0$ is defined as

$$\Delta y = y_1 - y_0 = f(x_1) - f(x_0), (since\ y_0 = f(x_0)\ and\ y_1 = f(x_1)) \quad (2)$$

$$= f(x_0 + \Delta x) - f(x_0)$$

by using (1) and (2).

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} \quad (3)$$

$$=\frac{6+0+0}{20+0+0}=\frac{3}{10}.$$

Exercise

1. State the point of discontinuity of the following functions.
(i) f(x)= $\frac{(x^2+x-6)}{(x+7)}$ (ii) g(x)=$\frac{(x^2-4)}{(x-1)}$ (iii) h(x)=$\frac{(2+x)(x+1)}{(9+x)}$
2. Evaluate each of the following.
(a) $\lim_{x\to 2}(x^3+2x-6)$ (b) $\lim_{x\to\infty}\frac{4x^3-x^2+x-2}{x^3+3x^2-3x+1}$
3. Is the function h(x)=$\frac{x^3+24}{x-2}$ continuous at x=1?
4. For what values of x in the domain of definition is each of the following functions continuous?
(i) f(x)=$\frac{5}{x+3}$ (ii) f(x)=$\frac{3+cosx}{7}$
5. If f(x)=$\frac{x-1}{x^2+x-2}$, find the following limits if they exist
(a) $\lim\limits_{x\to 0} f(x)$ (b) $\lim\limits_{x\to\infty} f(x)$
6. Find $\lim\limits_{x\to 0}\dfrac{f(x)-f(a)}{x-a}$ and $\lim\limits_{t\to 0}\dfrac{f(a+t)-f(a)}{t}$ for:
(i) $f(x)=x^2$, a=3
(ii) $f(x)=x^2+1$, a=2
(iii) $f(x)=3x^2-x$, a=0
7. Find the following if they exist.
(i) $\lim_{x\to a}\frac{x^4-a^4}{x-a}$ (ii) $\lim_{h\to 0}\frac{4(x+h)^3-4x^3}{h}$ (iii) $\lim_{x\to 0}\frac{1-2^{2x}}{1+2^x}$ (iv) $\lim_{x\to 0}\frac{x^2-x}{x}$ (v) $\lim_{x\to 10}(1-$

Exercise. Find the derivatives of the following functions

1. $f(x) = \frac{1}{x^2}$
2. $f(x) = x^2 + 1$
3. $f(x) = sin2x$
4. $f(x) = cos2x$
5. $f(x) = x^2 + 3x + 1$
6. $f(x) = \frac{x}{x+1}$
7. $f(x) = tan2x$
8. $f(x) = x(x^2 + 2x)$

3.5 Maximum and Minimum

A graph of a function is the coordinate of the domain and co-domain, that is $(x_0, f(x_0))$. For instance, the function $y = 2x + 1$ has its graph as points of $(x, f(x))$. If $x = 1$, then $y(1) = 2(1) + 1 = 3$; so that $(1, 3)$ is an element of the graph of $y = 2x + 1$. The collection of all such points is called the graph of the function $y = 2x + 1$. Alternatively, a graph of a function f consists of two finite non empty sets and such that $G = \{(x_j, y_0) : x_0 \in X \; and \; y_0 \in Y\}$.

A function f is said to have a relative maximum at a point $x = c$ if the value of the function at $x = c$, called a local (or relative) maximum value, is greater than its value at all points close to c. If this happens then the point $(c, f(c))$ is called a local (or relative) maximum point. If f is such that its value at $x = c$ is greater than all other values of f at all other points in the domain of f, then f is said to have a global (or absolute) maximum value at $x = c$; and if this be the case then $(c, f(c))$ is called the global (or absolute) maximum point.

In the same vein, a function is said to have a local (or relative) minimum value at a point $x = c$ if the value of the function at $x = c$, called a local

$$= \lim_{\Delta x \to 0} \left(\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \right) = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

3. $f'(x) = \lim_{\Delta x \to 0} \left[\frac{sin(x + \Delta x) - sinx}{\Delta x} \right] = \lim_{\Delta x \to 0} \left[\frac{sinxcos\Delta x + sin\Delta xcosx - sinx}{\Delta x} \right]$

$\lim_{\Delta x \to 0} \left[\frac{sinx(cos\Delta x - 1)}{\Delta x} + \frac{sin\Delta xcosx}{\Delta x} \right]$

$$= \lim_{\Delta x \to 0} \frac{sinx(cos\Delta x - 1)}{\Delta x} + \lim_{\Delta x \to 0} cosx \frac{sin\Delta x}{\Delta x}$$

$$= sinx \lim_{\Delta x \to 0} \frac{(cos\Delta x) - 1}{\Delta x} + cosx \lim_{\Delta x \to 0} \frac{sin\Delta x}{\Delta x}$$

$$= sinx \text{ x } 0 + cosx \text{ x } 1 = cosx$$

4. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{cot(x + \Delta x) - cotx}{\Delta x}$

$$= \lim_{\Delta x \to x} \left[\frac{cosxcos\Delta x - sinxsin\Delta x}{sinxcos\Delta x + sin\Delta xcosx} - cotx \right] \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{cotx - tan\Delta x}{1 + tan\Delta xcotx} - cotx \right] \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{cotx - tan\Delta x - cotx - tan\Delta xcot^2x}{1 + tan\Delta xcotx} \right] \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{-tan\Delta x(1 + cot^2x)}{1 + tan\Delta xcotx} \right] \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{-tan\Delta xcosec^2x}{1 + tan\Delta xcotx} \right] \frac{1}{\Delta x}$$

$$= -cosec^2x \lim_{\Delta x \to 0} \frac{sin\Delta x}{\Delta x} \lim_{\Delta x \to 0} \frac{1}{cos\Delta x} \lim_{\Delta x \to 0} \left(\frac{1}{1 + tan\Delta xcotx} \right)$$

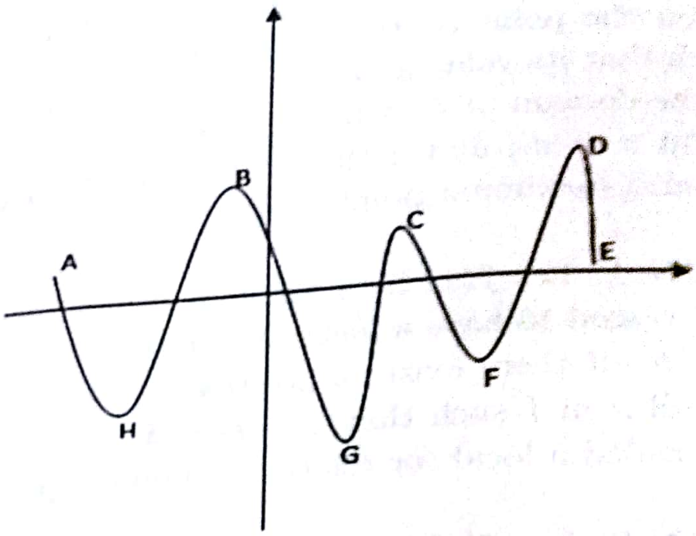
$$= -cosec^2x$$

5. Show that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sum_{r=0}^{n} \binom{n}{r} x^{n-r} (\Delta x)^r - x^n}{\Delta x} \text{ (Using Binomial expansion)}$$



In fig 1 above, points A, B, C, D are local maximum points but D is the global maximum whereas points E, F, G, H are local minimum points but G is the global minimum.

Examples on local extremum.

1. Find all extremum points of

   (a) $y = x^2 - x$

   (b) $y = cos(2x) - 1$

   (c) $y = 2 + 3x - x^3$

Solution

1. (a) $y' = 2x - 1. y' = 0 \Rightarrow x = \frac{1}{2}$

   An interval containing $x = \frac{1}{2}$ is $(0, 1)$, so we can consider the interval [0,1].

   $y(\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}; y(0) = 0$ *and* $y(1) = 0$.It is obvious from these that there must be a local minimum at $x = \frac{1}{2}$ because $y\left(\frac{1}{2}\right) < y(x_0)$ *for all* $x_0 \in [0, 1]$.

   (b) $= cos(2x) - x \Rightarrow y' = -2sin(2x) - 1.$
   $y' = 0 \Rightarrow 2x = -\frac{\pi}{6} \pm 2k\pi, k = 0, 1, 2, \cdots \Rightarrow x = -\frac{\pi}{12} \pm k\pi.$
   Choose the interval $(-\pi, \pi)$ and evaluate appropriately.
   $y(-\pi) = 1 + \pi, y\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{12}$ *and* $y(\pi) = 1 - \pi$. from these we see that $y(-\pi) > y\left(\frac{\pi}{12}\right) > y(\pi)$.

(or relative) minimum value, is less than its value at all points close to c . If this happens then the point $(c, f(c))$ is called a local (or relative) minimum point. If f is such that its value at $x = c$ is less than all other values of f at all other points in the domain of f , then f is said to have a global (or absolute) minimun value at $x = c$; and if this be the case then $(c, f(c))$ is called the global (or absolute) maximum point. Formally,

DEFINITION 1: Let $f(x)$ be a function defined in an interval $I = (a, b)$. The function f is said to have a local (or relative) minimum at a point $x = c$, where $a < c < b$, if there exists a positive number δ $(i.e\ \delta > 0)$ such that $f(c) \leq f(x)$ for all x in I such that $|x - c| < \delta$ $(i.e. c - \delta < x < c + \delta)$. The point $(c, f(c))$ is called a local (or relative) minimum point.

DEFINITION 2: Let $f(x)$ be a function defined in an interval $I = (a, b)$. The function f is said to have a local (or relative) maximum at a point $x = c$, where $a < c < b$, if there exists a positive number δ $(i.e.\ \delta > 0)$ such that $f(c) \geq f(x)$ for all x in I such that $|x - c| < \delta$ $(i.e.\ c - \delta < x < c + \delta)$. The point $(c, f(c))$ is called a local (or relative) maximum point.

DEFINITION 3: Let $f(x)$ be a function defined in an interval $I = (a, b)$. The function f is said to have a global (or absolute) maximum at a point $x = c$, where $a < c < b$, if $f(c) \geq f(x)$ for all x in I . The point $(c, f(c))$ is called a global (or absolute) maximum point.

DEFINITION 4: Let $f(x)$ be a function defined in an interval $I = (a, b)$. The function f is said to have a global (or absolute) minimum at a point $x = c$

3. $f(x) = x^2 + 5x + 6, \quad -3 \leq x \leq -2$

4. $f(x) = \dfrac{x-1}{x} \quad -1 \leq x \leq 1$

Solution

1). f is continuous on $[1, 2]$, differentiable on $(1, 2)$ and $f(1) = 1^2 - 3(1) + 2 = 0$, $f(2) = 2^2 - 3(2) + 2 = 0$ implies that $f(1) = f(2)$; thus f satisfies the condition of Rolle's theorem.

2). f is continuous on $[0, 2]$, differentiable on $(0, 2)$ but $f(0) = 3 \neq f(2) = 7$; hence f fails to satisfy the conditions of Rolle's theorem.

3). f is continuous on $[-3, -2]$, differentiable on $(-3, -2)$ and $f(-3) = (-3)^2 + 5(-3) + 6 = 15 - 15 = 0$, $\quad f(-2) = (-2)^2 + 5(-2) + 6 = 10 - 10 = 0$, implies that $f(-3) = f(-2)$; thus f satisfies the condition of Rolle's theorem.

4). f is not continuous in $[-1, 1]$, not differentiable in $(-1, 1)$ and $f(-1) = 2 \neq f(1) = 0$; hence f fails to satisfy the conditions of Rolle's theorem.

Example 2.

For the functions above that satisfy Rolle's Theorem in example 1, find all possible points at which the extremum values occur:

a). (2) satisfies the conditions of Rolle's Theorem. $f'(c) = 0 \Rightarrow 2c - 3 = 0 \Rightarrow c = \frac{3}{2}$. Thus the required point is $c = \frac{3}{2}$.

b). (3) also satisfies the conditions of Rolle's Theorem. $f'(c) = 0 \Rightarrow 2c + 5 = 0 \Rightarrow c = -\frac{5}{2}$, thus the required point is $c = -\frac{5}{2}$.

(2) **The Extreme Value Theorem (EVT)**: If a function is continuous on a closed and bounded interval $[a, b]$, then the function attains a maximum and a minimum value on $[a, b]$.

Fermat's Theorem:

Let f be a continuous function defined in $[a, b]$. If f has an extreme value at a point c strictly between a and b , and if f is differentiable at $x = c$, then

Based on our method and knowledge so far we cannot make a decision about an extremum value at $x = -\frac{5}{12}$.

(c) $y = 2 + 3x - x^3 \Rightarrow y' = 3 - 3x^2$ and $y' = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$. For the point $x = 1$, consider any interval around $x = 1$ such that neither of the end points of the interval is farther away than the nearest stationary points. So we consider the interval $[0, 1]$. Now, $f(1) = 4$, $f(0) = 2 < 4$ and $f(2) = 0 < 4$. This shows that there must be a local extremum at $x = 1$. In a like manner, a local maximum must occur at $x = -1$ because $f(0) = 2 > 0 = f(-1)$ and $f(-2) = 4 > 0 = f(-1)$, considering the interval $[-2, 0]$

Exercise.

Analyze the functions (a)$y = x^3 - 9x^2 + 24x$ (b)$y = x^4 - 2x^2 + 3$, for extremum.

DEFINITION 5:

A stationary point of a function f is a point $x = x_0$ at which $f'(x_0) = 0$.

Determining the extremum of functions is not always an easy task. One sure way of determining these extrema of functions is the use of derivatives. To do this effectively, we shall assemble necessary and useful tools to enable us in determining these extrema. These tools are theorems that tell us how to go about these tasks.

3.6 Theorems and Definitions

(1) **Rolle's Theorem:** Let f be a continuous function defined on the closed interval $a \leq x \leq b$ (that is [a,b]), differentiable in the open interval $I = (a, b)$ and $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$. This theorem says that if the conditions on f is satisfied, then there is a stationary point of f which lies between a and b.

Example 1.

Which of the following functions satisfy the conditions of Rolle's theorem?

1. $f(x) = x^2 - 3x + 2, \quad 1 \leq x \leq 2$
2. $f(x) = 5x - 3, \quad 0 \leq x \leq 2$

ii) a singular point, and by singular point is meant a point where a function is not defined.

iii) a stationary point

If $x = c$ is a critical point of f , then $f(c)$ is a critical value of f .

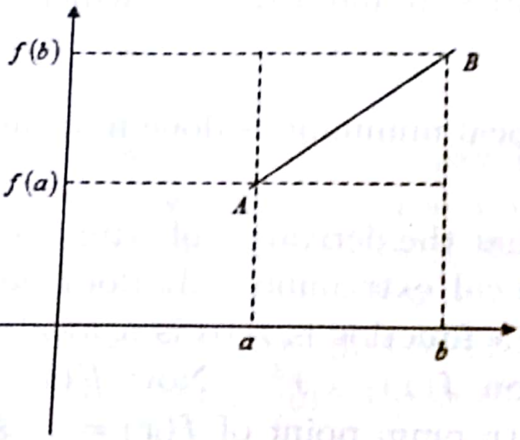
Theorem:

Let f be continuous on a closed interval $[a, b]$, then there exists numbers m and M such that $m \leq x \leq M$ for each x *in* $[a, b]$ and there are points α and β in $[a, b]$ where $f(\alpha) = M$ and $f(\beta) = m$

(3) **The Mean Value Theorem (MVT) Let f be continuous on a closed** interval $[a, b]$ and differentiable in (a, b) . **Then there exists at least a point** c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Before the proof of this theorem, let us explain what the theorem says. Consider the graph shown below



If $y = f(x)$ then $y_1 = f(a)$ at $x = a$ and $y_2 = f(b)$ at x=b . The slope of line AB given as m is

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

. This shows that $\frac{f(b) - f(a)}{b - a}$ is the slope of the line AB . Also observe that line AB is the line connecting the points $(a, f(a))$ and $(b, f(b))$.

The geometric interpretation of the MVT is that for some point, which

$f'(c) = 0$.

Proof: We shall proof the case $x = c$ is a local maximum.

Let $\delta > 0$ be such that $f(x) \le f(c)$ for $|x - c| < \delta$. Consider the quotient $\frac{f(x) - f(c)}{x - c}$.

Since f is differentiab e at $x = c$, then $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists . But the limit of a function exists at a point $x = c$ if the left and right limits are equal. Thus,

$\lim\limits_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c)$. Since $f(x) \le f(c)$ for $x < c$ in the δ-neighbourhood of c , $\frac{f(x) - f(c)}{x - c} \le 0$ because $f(x) - f(c) < 0$ and therefore,

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'(c) \le 0 \qquad \text{(i)}$$

Also for $x > c$ in the δ-neighbourhood of c , $\frac{f(x) - f(c)}{x - c} \ge 0$ and therefore,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \ge 0 \qquad \text{(ii)}$$

From (i) and (ii) we obtain that $f'(c) \le 0$ and $f'(c) \ge 0$ which is possible only if $f'(c) = 0$.

The proof of Fermat's Theorem for local minimum is done in a similar manner.

What this theorem actually says is that the derivative of a function vanishes at a point where the function has a local extremum . It does not say that any point where the first derivative of a functios is zero is a local extremum. Consider for an illustration the function $f(x) = x^3$. Now $f'(x) = 3x^2$ and $f'(c) = 0 \Rightarrow 3c^2 = 0$. $c = 0$ is not an extremum point of $f(x) = x^3$ since f(x) is not differentiable in any interval containing 0

Definition :

A point $x = c$ in an interval I is called a critical point of a function f if it is any of the following:

i) an end point of the interval I

Solution

By MVT we have that 'average rate of change of $f(x)$ in $[a,b]$ = instantaneous change of $f(x)$ at $x = c$.

This implies that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ (1)

Now, $f(x) = x^2 \Rightarrow f(b) = b^2$, $f(1) = 1^2 = 1$,
$f'(x) = 2x$ *and* $f'(2) = 2(2) = 4$ (2)

Using (2) in (1) yields

$\dfrac{b^2 - 1}{b - 1} = 4$

$\Rightarrow b^2 - 1 = 4b - 4$

$\Rightarrow b^2 - 4b + 3 = 0$

$\Rightarrow (b-3)(b-1) = 0 \Rightarrow b = 3$ or b=1. Since $b \neq 1$, *then* $b = 3$.

Example 2:

Let $f(t) = t - t^3$. Does this function satisfy the hypothesis of the MVT in $[-2, 0]$. If it does, find all numbers c that satisfy the conclusion of the MVT.

Solution

$f(t) = t - t^3$ is a polynomial of degree three (3) and such is continuous everywhere including $[-2, 0]$.

$f'(t) = 1 - 3t^2$ is a polynomial of degree two (2) and such is continuous everywhere including in $[-2, 0]$. So the function satisfies the hypothesis of MVT.

To find all numbers c that satisfy the MVT, we compute $f(-2)$,$f(0)$ and $f'(c)$. $f(-2) = (-2) - (-2)^3 = 6$; $f(0) = (0) - (0)^3 = 0$ and $f'(c) = 1 - 3c^2$.
With the values obtained substituted for in the MVT one obtains $1 - 3c^2 = \dfrac{0-6}{2} - 3$

$\Rightarrow 3c^2 = 4 \Rightarrow = \pm\dfrac{2\sqrt{3}}{3}$. The point $\dfrac{2\sqrt{3}}{3}$ is not a point in [-2,0]; thus $c = -\dfrac{2\sqrt{3}}{3}$.

not be unique, between a and b the tangent (slope) line at a point $p(c, f(c))$ to a graph of a function differentiable at $x = c$ is the same as or equal to the slope of the tangent line connecting the points AB on the graph.

Now the proof:

The equation of a line through a point $p(x_1, y_1)$ with slope of the line m is given by $y - y_1 = m(x - x_1)$.

But $y_1 = f(a)$, $m = \dfrac{f(b) - f(a)}{b - a}$ and $y_2 = f(b)$ from the graph shown above, so the required equation becomes

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \text{ implying that } y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

.

Now let $g(x) = f(x) - \left(f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)\right)$. Observe that g as defined is continuous in $[a, b]$ and differentiable in (a, b) because $f(x)$ is. In addition,

$$g(a) = f(a) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(a - a)\right) = 0 \text{ and },$$

$$g(b) = f(b) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right) = 0 \text{ showing that } g(a) = g(b)$$

. Therefore g satisfies the conditions of Rolle's Theorem and by the same theorem there exists a point c in (a, b) such that $g'(c) = 0$. But $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$ so that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \text{ . This implies that } f'(c) = \frac{f(b) - f(a)}{b - a} .$$

In essence what the MVT implies is that if f is continuous in $[a, b]$ and differentiable at every point strictly in (a, b) (that is between a and b), then there exists a point $x = c$ (and may be more than one of such point) strictly between a *and* b such that the average rate of change of f in $[a, b]$ equals the instantaneous rate of change of f at $x = c$.

Example 1: Let $f(x) = x^2$. Find the value of b such that the average rate of change of $f(x)$ from 1 to b equals the instantaneous rate of change of $f(x)$ at $t = 2$.

$\frac{f(b)-f(a)}{b-a} = f'(c)$. This implies $f(b) - f(a) = f'(c)(b-a)$. . Also $b > a \Rightarrow b - a > 0$. Now $f'(c)(b-a) < 0$ because $f' < 0$ and $b - a > 0$. This implies $f(b) - f(a) < 0 \equiv f(b) < f(a)$.

Examples

Find the intervals where the following functions are increasing or decreasing.

(a) $f(x) = x^2 - 8x + 2$ (b) $f(x) = 3x^4 - 4x^3 + 8$.

Solution.

(a) $f'(x) = 2x - 8$. $f'(x) > 0 \Rightarrow x > 4$, so that $I = (4, \infty)$. Also $f'(x) < 0 \Rightarrow x < 4$, so that I=$(-\infty, 4)$

(b) $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$. $f'(x) > 0 \Rightarrow x > 0$ and $x > 1$ or $x < 0$ and $x < 1$. This shows that f is increasing in $(-\infty, 0)$ and $(1, \infty)$. Also $f'(x) < 0 \Rightarrow x > 0$ and $x < 1$ or $x < 0$ and $x > 1$. Thus f is decreasing in $(0, 1)$.

Exercise.

Find the intervals where the following functions are monotone.

(1) $f(x) = x^3 - 6x^2 + 12x - 8$ (2) $f(x) = 3x^3 - 6x^2$ (3) $f(x) = x^2 - 6x$
(4) $f(x) = x^4 - 8x^3 + 10x^2 + 40$.

Definition

Let f be a function defined in an interval I . f(c) is the global (absolute) maximum value of f if $f(c) \geq f(x)$ for all x in I . If $f(c) \leq f(x)$ for all x in I , then $f(c)$ is said to be the global (absolute) minimum.

Definition

A point where a function is not defined is called a singular points of the function.

Definition

A critical point $x = c$ of a function f is a point in an interval of definition of f such that c is one of the following:

Example 3:

Consider the function $y = \dfrac{x^2 - 1}{x}$. Find the intervals and values of c for which this function satisfies the hypothesis of MVT.

Solution

The function $y = \dfrac{x^2 - 1}{x}$ is not defined at $x = 0$ and as such is not continuous at that point i.e $x = 0$.

Similarly, $y' = \dfrac{x^2 + 1}{x^2}$ is also not define at $x = 0$ and as a result y is not differentiable at that point. From these we see that y will satisfy the hypothesis of the MVT in any interval not containing $x = 0$, i.e $(-\infty, 0) \cup (0, \infty)$.

Exercises:

1). Find all c such that $y = \dfrac{x^2}{x + 1}$ satisfy the MVT in [1,2].

2). Determine which of the following satisfies the conditions of the MVT.
(a) $x^2 - x^3$, $-2 \leq x \leq 1$ (b) $g(x) = 3x^3 - 9x^2 - 3x$, $-2 \leq x \leq 3$

Definitions

1. **Monotone Functions:** A function $f(x)$ is said to be monotone in an interval I if $f(x)$ is either increasing or decreasing in I.

2. A function $f(x)$ is said to be non-decreasing in an interval I if for every pair of point x_1, x_2 in I such that $x_1 < x_2$, $f(x_2) \geq f(x_1)$. If $f(x_2) > f(x_1)$, then we say that $f(x)$ is strictly increasing in I.

3. A function $f(x)$ is said to be non-increasing in an interval I if for every pair of points x_1, x_2 in I such that $x_1 < x_2$, $f(x_2) \leq f(x_1)$. If $f(x_2) < f(x_1)$, then we say that $f(x)$ is strictly decreasing in I.

Theorem: Let $f(x)$ be differentiable in an open interval I. Then if
(i) $f' > 0$ in I, f is increasing in I
(ii) $f' < 0$ in I, f is decreasing in I

Proof: Let's proof the second condition (ii) and leave (i) as an exercise.
Let f be differentiable in I with $f' < 0$ in I. Let a, b be two points in I such that $a < b$. By the MVT there exists a point $c \in [a, b]$ such that

Concave Function: Let f be differentiable in an open interval I. If,

(i) f' is increasing in I then the function f is said to be concave up in I;

(ii) f' is decreasing in I then the function f is said to be concave down in I

A point where the concavity of a function changes is called an inflection point.

Note that at a point of inflection the derivative of a function changes sign, either from positive to negative or vice versa. This in other words means that if at $x = x_0$, $f'(x_0) > 0$ changes to $f'(x_0) < 0$ then $(x_0, f(x_0))$ is a point of inflection.

On the alternative we can define the concavity of f with respect to the second derivative of f as follows. Let $f' = g$ be defined in an open interval I. Let g be differentiable in an open interval I. Then by monotone theorem of functions, (i) if $g' > 0$ in I then g is increasing; and (ii) if $g' < 0$ in I then g is decreasing in I. But we had earlier set $g = f'$ so that $g' = (f')' = f''$. Observe that g is increasing in I implies f' is increasing in I and g is decreasing in I implies f is decreasing in I. In summary we have demonstrated the following.

Definition.

Suppose f is twice differentiable in an open interval I. Then,

(i) if $f'' > 0$ in I then f' is increasing in I and f is concave up in I

(ii) if $f'' < 0$ in I then f' is decreasing in I and f is concave down in I.

If f'' exist in I then $f'' = 0$ at a point of inflection. This means that the second derivative of f at an inflection point must vanish, note that the vanishing of the second derivative at a point makes the point an inflection point.

Example

Consider the functions given below.

(1) $f(x) = x^3 - 6x^2 + 12x - 8$ (2) $f(x) = \dfrac{(x+2)(x-1)}{x}$

Find the intervals where each is monotone increasing or decreasing, concave up or concave down and point(s) of inflection. Solution.

(1) $f(x) = x^3 - 6x^2 + 12x - 8$, $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2 > 0$ for all x f is, therefore, nondecreasing. $f''(x) = 6x - 12 = 6(x-2)$ $f''(x) > 0 \Rightarrow x > 2$ and $f''(x) < 0 \Rightarrow x < 2$, thus f is concave up in $(2, \infty)$ and concave down in

Example

Find the critical points of the following functions and test them for local extrema.

(a)$g(x) = x^3 - x^2, -2 \leq x \leq 1$ (b)$f(x) = 2 + x - x^3, -2 \leq x \leq 3$.

Solution.

(a) $g'(x) = 3x^2 - 2x$. $g'(x) > 0 \Rightarrow 3x - 2 > 0$ and $x > 0$ or $3x - 2 < 0$ and $x < 0$, that is $x > \frac{2}{3}$ and $x > 0$ or $x < \frac{2}{3}$ and $x < 0$. So $g'(x) > 0$ in the interval (-2,0) and $(\frac{2}{3}, 1)$. Similarly, $g'(x) < 0$ in $(0, \frac{2}{3})$. Since $g'(x) > 0$ in $(-2, 0)$ and $g'(x) < 0$ in $(0, \frac{2}{3})$, $x = 0$ is local maximum. Also $g'(x) < 0$ in $(0, \frac{2}{3})$ and $g'(x) > 0$ in $(\frac{2}{3}, 1)$, therefore $x = \frac{2}{3}$ is a local minimum.

(b) $f'(x) = 1 - 3x^2 = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{3}$. Observe that the point $\frac{\sqrt{3}}{3}$ does not lie in the interval $-2 \leq x \leq 3$. So we are to consider only $x = -\frac{\sqrt{3}}{3}$. Following the way of previous analyses, we find out that $f'(x) > 0$ in $\left(-2, -\frac{\sqrt{3}}{3}\right)$ and $f'(x) < 0$ in $\left(-\frac{\sqrt{3}}{3}, 0\right)$, showing that there is a local maximum at $x = \frac{-\sqrt{3}}{3}$. Likewise $f'(x) < 0$ in $\left(0, -\frac{\sqrt{3}}{3}\right)$ and $f'(x) > 0$ in $\left(-\frac{\sqrt{3}}{3}, 0\right)$ implies there is a local minimum at $x = 0$.

Example

For the points where extremum occur in the above problems, find the local extremum values and points.

Solution.

(a) For $g(x) = x^3 - x^2, -2 \leq x \leq 1$, a local maximum occurs at $x = 0$ and a local minimum occurs at $x = \frac{2}{3}$. Therefore, local maximum value is $g(0) = 0^3 - 0^2 = 0$ and local minimum value is $g\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2 = -\frac{4}{27}$. Thus, the local minimum point is $\left(\frac{2}{3}, -\frac{4}{27}\right)$ and the local maximum point is $(0, 0)$.

(b) For $f(x) = 2 + x - x^3, -2 \leq x \leq 3$, a local minimum occur at $x = 0$ whereas a local maximum occur at $x = -\frac{\sqrt{3}}{3}$. The local minimum value for $x = 0$ is $f(0) = 2 + 0 - 0^3 = 2$ and local maximum value is $f\left(-\frac{\sqrt{3}}{3}\right) = 2 + \left(-\frac{\sqrt{3}}{3}\right) - \left(-\frac{\sqrt{3}}{3}\right)^3 = 2 - \frac{2\sqrt{3}}{9}$. Thus, the local minimum point is $(0, 0)$ and the local maximum point is $\left(-\frac{\sqrt{3}}{3}, \frac{18-2\sqrt{3}}{9}\right)$.

(i) an end point

(ii) a singular point

(iii) a stationary point.

This tells us that local or global extremum occur at any of these points. In other words, we investigate these points for extrema of functions.

Example

Find the critical points of the functions

(a) $f(x) = x^2 - 6x + 5, 0 \leq x \leq 3$ (b) $f(x) = \dfrac{x-1}{x^2-2x}, -1 \leq x \leq 5$.

Solution.

(a) end points of f are 0 and 3 , no singular point (why?) and stationary point is 3 , so the critical points are $-1, 0, 2$ *and* 5.

b) End points are -1 and 5 , no stationary, singular points are $x^2 - 2x = 0 \Rightarrow x = 0, 2$. Thus the critical points are $-1, 0, 2$ *and* 5 .

Theorem (FIRST DERIVATIVE TEST).

Let f be continuous in the interval $[c-\delta, c+\delta]$ containing the point c .

(i) If $f'(x) > 0$ in $[c-\delta, c]$ and $f'(x) < 0$ in $[\delta, c+\delta]$, then f has a local maximum at c .

(ii) If $f'(x) < 0$ in $[c-\delta, c]$ and $f'(x) > 0$ in $[\delta, c+\delta]$, then f has a local minimum at c .

Proof (Part ii)

$f'(x) < 0$ in $[c-\delta, c]$ implies f is decreasing in $[c-\delta, c]$. By definition of decreasing function, then, $f(x) \geq f(c)$ for all $x \in [c-\delta, c]$. Again $f(x) > f(c)$ in $[\delta, c+\delta]$ implies f is increasing in $[\delta, c+\delta]$. Then by definition of increasing functions, $f(x) \geq f(c)$ for all $x \in [\delta, c+\delta]$. Thus $f(x) \geq f(c)$ for all x in $[c-\delta] \cup [c, c+\delta] = [c-\delta, c+\delta]$.

Proof of part (i) is left as an exercise for the reader.

$(-\infty, 2)$. Also there is an inflecton point at $x = 2$, i.e $(2, 0)$ is an inflection point.

(2) $f'(x) = \dfrac{-(x^2+2)}{x^2} < 0$ for all x , thus f is decreasing for all x. $f''(x) = \dfrac{4(x^2+1)}{x^3} < 0$ if $x < 0$ and $f''(x) = \dfrac{4(x^2+1)}{x^3} > 0$ if $x > 0$. Therefore, f is concave up in $(0, \infty)$ and concave down in $(-\infty, 0)$. f has no point of inflection because there is no x in R for which $x^2 + 1 = 0$.

Theorem. (Second Derivative Test).

Suppose f' and f'' exist in an open interval I containing the number c and $f'(c) = 0$. Then

(i)$f'(c) > 0$ implies $(c, f(c))$ is a relative minimum point of f

(ii)$f'(c) < 0$ implies $(c, f(c))$ is a relative maximum point of f.

Proof (part (i)).

Let f' and f'' exist in an open interval I containing the number c , $f'(c) = 0$ and $f'(c) > 0$. We then prove that $(c, f'(c))$ is a relative minimum point of f . Now by MVT

$f''(c) = \lim\limits_{k\to 0} \dfrac{f'(c+h) - f'(c)}{h}$. Since $f'(c) = 0$ and $f''(c) > 0$ (by the hypothesis), then $\lim\limits_{k\to 0} \dfrac{f'(c+h) - f'(c)}{h} = \lim\limits_{k\to 0} \dfrac{f'(c+h)}{h} > 0$. This implies that there exists an interval $0 < |h| < \delta$ within which $\dfrac{f'(c+h)}{h} > 0$. If $h > 0$, then $f'(c+h) > 0$ while if $h < 0$, then$f'(c+h) < 0$. By first derivative test f has a local minimum at $(c, f(c))$.

Example.

Find the critical points of the following functions and test them for local extrema.

1). $f(x) = 2x^3 - 3x^2 + 1$ 2). $f(x) = x^3 - 6x^2 + 12x - 8$ Solution.

1). $f'(x) = 6x^2 - 6x = 0 \Rightarrow x = 0, 1$. Critical points are 0 and 1

$f''(x) = 12x - 6 = 6(2x - 1)$. $f''(0) = -6 < 0$. implies there is a local maximum at 0 , thus $(0, 1)$ is a local maximum point. Also $f''(1) = 6 > 0$

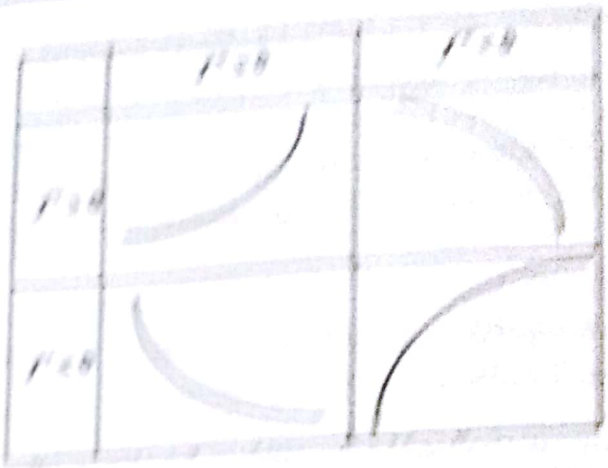


Figure showing the monotonicity and concavity of functions.

Examples

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

(1) $y = x^5 - 5x^4 + 5x^3$ (2) $y = x^3 - 3x^2 - 9x + 5$

Solution

$y' = 5x^4 - 20x^3 + 15x^2 = 5x^2(x-3)(x-1)$. Critical points are 0, 1 and 3
$y' = 5x^2(x-3)(x-1)$. $y' > 0 \Rightarrow 5x > 0, (x-3) > 0$ and $(x-1) > 0$ or $x > 0, (x-3) < 0$ and $(x-1) < 0$ or $x < 0, (x-3) < 0$ and $(x-1) > 0$ or

$y' > 0 \Rightarrow x > 0, x < 1$ *and* $x < 3$, *or* $x < 0, x > 1$ and $x < 3$, or $x < 0, x < 1$ *and* $x > 3$, *or* $x > 0, x > 1$ *and* $x > 3$. Therefore y is increasing in $(0, 1)$ and $(3, \infty)$.

Similarly,

$y' < 0 \Rightarrow x < 0, x > 1$ *and* $x > 3$, *or* $x > 0, x < 1$ *and* $x > 3$, *or* $x > 0, x > 1$ *and* $x < 3$, *or* $x < 0, x < 1$ *and* $x < 3$, Therefore y is decreasing in $(-\infty, 0)$ and $(1, 3)$.

$y'' = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$ Following a similar analysis as above one finds out that $y'' > 0$ in $\left(0, \frac{3-\sqrt{3}}{2}\right)$ and $\left(\frac{3+\sqrt{3}}{2}, \infty\right)$ and $y'' < 0$ in the interval $(-\infty, 0)$ and $\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$.

Intercepts are $(0, 0)$, $\left(\frac{3-\sqrt{3}}{2}, 0\right)$ and $\left(\frac{3+\sqrt{3}}{2}, 0\right)$.

implies a local minimum at 1 , thus (1, 0) is a local minimum point.

2). $f(x) = x^3 - 6x^2 + 12x - 8$, $f'(x) = 3(x-2)^2 = 0 \Rightarrow x = 2$ thus Critical points are 2 . $f''(x) = 6x - 12$ implies $f''(2) = 0$. No conclusion can be reached about the local extrema at 2 . This shows that if $f''(c) = 0$, then no conclusion can be drawn about the existence of local extrema at $x = c$.

Definition

1. **Vertical Asymptotes**: The line $x = a$ is a vertical asymptote if at least one of the following holds: $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

2. **Horizontal Asymptotes**: The line $y = L$ is a horizontal asymptote if either (or both) of the following holds: $\lim_{x \to +\infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = L$.

3. **Symmetry**: If $f(x) = f(-x)$ then f is called an even function. The graph of such functions is symmetric with respect to the y - axis. Some examples of even functions are: x^{2n} , $\cos x, \sin^2 x$. If $f(-x) = -f(x)$ then f is called an odd function. Their graph is symmetry with respect to the origin. Examples of odd functions include: x^{2n+1}, $\sin x$, e.t.c. On the other hand, some functions are neither even nor odd. Some of these include the exponential function. These functions do not have any particular symmetry properties.

3.7 Curve Sketching

To sketch the curve of a graph, we are going to look out for the following properties of functions studied so far:

First derivative of f , second derivative of f , local extremum points, inflection points or none of f and the intercepts of f . Why do we need these? We need the first derivative to tell us where the graph of a function goes up or down. The graph of a function goes up in the interval where a function is increasing and the graph goes down in the interval where the function is decreasing. The second derivative of f tells us the interval where a function turns up or down. If the second derivative is greater than zero at a point the function bends up; but if it is negative, then the function bends down. The inflection point is where the sign of a function changes from positive to negative and vice versa.

4 CHAPTER FOUR

4.1 BASIC TECHNIQUES OF DIFFERENTIATION

The basic techniques of differentiation are as follows,

1. Product Rule
2. Quotient Rule
3. Logarithmic differentiation
4. Differentiation a function of a function
5. The second derivation of a function
6. Differentiation of implicit function

We shall thus need to make some basic definitions that will stand as aids to understanding the topic.

Constant: A constant, say C is a quantity the value of which does not change, i.e. it carries no independent variable. Example are 1,2,3, ...

Variable : A variable is a quantity which may take different values in mathematical operation.

An Independent Variable: say x , is a variable which may assume any value assigned to it. It is sometimes called the predictor variable.

A dependent Variable: say y is one in which change in value occur because of change in the value of another variable. For example, in $y = x^2$, x is an independent variable which may take negative , zero and positive values. The dependent variable y , on the contrary will result in non-negative values for all values of x .

Function: A function of $\{x, z, or\ even\ t\}$is a quantity the value of which depends on the x or any of the variable so defined. For example, $y = f(x) = x^2$ is a function of x.

The following symbols are used to denote a function: $f(x)$, $F(x)$,$\phi(x)$ all of which are read as $(f\ of\ x)$ or $(\phi\ of\ x)$

4.2 Differentiation

The process of finding a general expression for the gradient of a curve at any point is known as differentiation. This gradient function is derived from

Inflection points is $(0, 0)$

Local maximum point is $(1, 3)$ while local minimum point is $(3, -27)$. The function is not symmetric.

A summary of the above is tabulated below.

	$(-\infty, 0)$	0	$(0,1)$	1	$(1,3)$	3	$(3, \infty)$
y'	$-$	0	$+$	0	$-$	0	$+$
y''	$-$	0	$+$	$+$	$-$	$-$	$+$

$u = e^{2x}, \ v = ln5x$

Let $\frac{du}{dx} = 2e^{2x}, \ \frac{dv}{dx} = \frac{1}{5x}.5$

$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= ln5x.2e^{2x} + e^{2x}\frac{1}{5x}.5$

$= e^{2x}(2ln5x + \frac{1}{x})$

3)Differentiate $f(x) = (x^3 - 2)(4x + 1)$

Let $u = (x^3 - 2), \quad v = (4x + 1), \ du = 3x^2, \ dv = 4$

$$\begin{aligned}\frac{df(x)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= (4x + 1)(3x^3) + (x^3 - 2) + (4)\\ &= 16x^3 + 3x^2 - 8\end{aligned}$$

4)Differentiate $(5x^2)(\frac{6}{x})$

let $v = 5x^2, \ u = \frac{6}{x}, \ du = -6x^{-2}, \ dv = 10x$

$$\begin{aligned}\frac{df(x)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= (5x^2).\left(-\frac{6}{x}\right) + \left(\frac{6}{x}\right).(10x)\\ &= -30 + 60\\ &= 30\end{aligned}$$

the given function; hence it is also called the derived function or simply the derivative.

4.3 Product Rule:

Let y=uv, where u and v are functions of x. Then $\frac{dy}{dx} = \frac{vdu}{dx}$.

Proof

Let x increase by a small amount, say dx. Let du,dv denote the corresponding increase in u, v. Let the resulting increase in y be dy.

Then; $y + \partial y = (u + \partial u)(v + \partial v)$

Subtract y from both sides of the equation

$$(y + \partial y) - y = \partial y$$

$$dy = v\partial u + u\partial v + \partial u\partial v$$

$$\frac{\partial y}{\partial x} = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + \frac{\partial u\partial v}{\partial x}$$

Let $dx \to 0$ then

$$\frac{\partial y}{\partial x} \to \frac{dy}{dx}, \frac{\partial u}{\partial x} \to \frac{du}{dx}, \frac{\partial v}{\partial x} \to \frac{dv}{dx}, \frac{\partial u\partial v}{\partial x} \to 0$$

Since $\partial v \to 0$ which is a constant term.

$$\therefore \frac{dy}{dx} = \frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Examples: 1. Find the derivative of $y = x^3 \sin 3x$. Solution $u = x^3$, $\frac{du}{dx} = 3x^2$

$$v = \sin 3x, \frac{dv}{dx} = 3\cos 3x$$

$$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$=\sin 3x(3x^2) + x^3(3\cos 3x)$$

$$=3x^2 \sin 3x + 3x^3(\cos 3x)$$

$$=3x^2(x\cos 3x + \sin 3x)$$

2) Differentiate $y = e^{2x} ln5x$.

Example 1:

Differentiate $y = \dfrac{\sin 3x}{x+1}$

Let $u = \sin 3x$

$v = x + 1$

$du = 3\cos 3x$

$dv = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(x+1)(3\cos 3x) - (\sin 3x)1}{(x+1)^2}$$

Example 2:

Differentiate with respect to x

$y = \dfrac{ln\ x}{e^{2x}}$

Solution

Let $u = ln\ x,\ du = \dfrac{1}{x},\ \ v = e^{2x},\ du = 2e^{2x}$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{e^{2x}.\frac{1}{x} - ln\ x.2e^{2x}}{(e^{2x})^2}$$

$$= \frac{e^{2x}\left(\frac{1}{x} - 2ln\ x\right)}{e^{4x}}$$

$$= \frac{\left(\frac{1}{x} - 2ln\ x\right)}{e^{2x}}$$

Example 3

if $y = \dfrac{cos2x}{x^2}$

Solution

$u = cos2x,\quad du = -2\sin 2x,\quad v = x^2,\quad dv = 2x$

Exercises 1

1)$y = e^{5x}(3x + 1)$

2)$y = x\cos 2x$

3)$y = x^3 \sin 5x^2$

4)$y = x^2 \cos^2 x$

4.4 Quotient Rule

Let $y = \frac{u}{v}$, where u *and* v are functions of x and $v \neq 0$

Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Proof: let x increase by a small amount∂x , let the corresponding increase in u and v be ∂u and∂v and let the resulting increase in y be ∂y .

Then

$y + \partial y = \frac{u + \partial u}{v + \partial u}$

$$\begin{aligned}\partial y &= (y + \partial y) - y \\ &= \frac{u + \partial u}{v + \partial v} - \frac{u}{v} \\ &= \frac{v(u + \partial u) - u(v + \partial v)}{v(v + \partial v)} \\ &= \frac{uv + v\partial u - uv + u\partial v}{v^2 + v\partial v}\end{aligned}$$

$\therefore \frac{\partial y}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2 - v\partial v}$

Let $\partial x \to 0$

Then $\frac{\partial y}{\partial x} \to \frac{dy}{dx}, \frac{\partial u}{\partial x} \to \frac{du}{dx}, \frac{\partial v}{\partial x} \to \frac{dv}{dx}, \partial v \to 0$

since $\partial y \to 0$

hence $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ for $v \neq 0$

The rules for differentiation of a product or a quotient that we have revised are used when there are just two factor functions. i.e uv or $\frac{u}{v}$. Where there are more than two functions in any arrangement top or bottom, the derivative is best found by the Logarithmic Differentiation. It all depends on the basic fact that $\frac{d}{dx}\{ln\ x\} = \frac{1}{x}$ and if x is replaced by function F, then $\frac{d}{dx}\{ln\ F\} = \frac{1}{F}.\frac{dF}{dx}$

With this fact, let us consider the case where $y = \frac{uv}{w}$ where u, v and w and also y are functions of x

First Take log to the base e

$ln\ y = ln\ u + ln\ v - ln\ w$

Now differentiate each side with respect to x, remembering that u, v, w and y are all function of x

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} - \frac{1}{w}.\frac{dw}{dx}$$

Hence

$$\frac{dy}{dx} = \frac{uv}{w}\left\{\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} - \frac{1}{w}.\frac{dw}{dx}\right\}$$

Example 1:

if $y = \frac{x^2 \sin x}{\cos 2x}$.Find $\frac{dy}{dx}$

Solution:

$$y = \frac{x^2 \sin x}{\cos 2x}$$

$\therefore ln\ y = ln(x^2) + ln(\sin x) - ln(\cos 2x)$

Now differentiate both sides with respect to x remembering that

$$\frac{d}{dx}\{ln\ F\} = \frac{1}{F}.\frac{dF}{dx}$$

$$\therefore \frac{1}{y}.\frac{dy}{dx} = \frac{1}{x^2}.2x + \frac{1}{\sin x}.\cos x - \frac{1}{cos 2x}(-2\sin 2x)$$

$$= \frac{2}{x} - \cot x + 2\tan 2x \quad \frac{dy}{dx} = y\left\{\frac{2}{x} + \cot x + 2\tan 2x\right\}$$

$$\frac{dy}{dx} = \frac{x^2 \sin x}{\cos 2x}\left\{\frac{2}{x} + \cot x + 2\tan 2x\right\}$$

Exercises 2

Differentiate the following with respect to x

1) $\dfrac{\sin x}{\cos 9x}$

2) $\dfrac{\sin 2x}{2x+5}$

3) $\dfrac{(3x+1)\cos 2x}{e^{2x}}$

4) $\dfrac{x\sin x}{1+\cos x}$

5) $\dfrac{e^{4x}\sin x}{x\cos 2x}$

6) $\dfrac{x^4}{(x+1)^2}$

4.6 Differentiation of a function of a function.

If the function $y = f(x)$ has the derivative $f'(x)$ and the function $u = g(x)$ the derivative $g'(f(x))$, then the function $y = g(f(x))$ has at x the derivative $y'(x) = g'(f(x))\, f'(x)$.

Generally If $y = f(u)$ and $u = F = (x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ i.e if $y = ln\ F$.

Where F is a function of x,

then $\frac{dy}{dx} = \frac{dy}{dF}.\frac{dF}{dx} = \frac{1}{F}.\frac{dF}{dx}$.

Example 1

$y = ln \ \sin x$

Solution

Basic standard form is $y = lnx$, $\frac{dy}{dx} = \frac{1}{x}$.

Therefore, $\frac{dy}{dx} = \frac{1}{sinx}.cosx$

$= (\frac{1}{sinx})(cosx)$

=cotx

$\frac{1}{F} = \frac{1}{\sin x}$ $\quad \frac{dF}{dx} = \frac{d(\sin x)}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{1}{\sin x}.\cos x = \cot x$$

Example 2

$y = \tan(5x - 4)$

Solution

Note: the basic standard form is

$y = \tan x$ $\quad \frac{dy}{dx} = \sec^2 x$ in this case $(5x - 4)$ replaces the single x.

$\therefore \frac{dy}{dx} = \sec^2(5x - 4)$ x the derivative of the function $(5x - 4)$.

$= \sec^2(5x - 4) \times 5$

$= 5\sec^2(5x - 4)$

Example 3

$y = (4x - 3)^5$

Solution

Basic standard form is $y = x^5$, $\frac{dy}{dx} = 5x^4$ here $(4x - 3)$ replaces single x.

$\therefore \frac{dy}{dx} = 5(4x - 3)^4 \times$ the derivative of $(4x - 3)$

Example 2:

If $y = x^4 e^{3x} \tan x$ Solution $y = x^4 e^{3x} \tan x \therefore \ln y = \ln(x^4) + \ln(e^{3x}) + \ln(\tan x)$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x^4} . 4x^3 + \frac{1}{e^{3x}} . 3e^{3x} + \frac{1}{\tan x} . \sec^2 x$$

$$= \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}$$

$$\therefore \frac{dy}{dx} = x^4 e^{3x} \tan x \left\{ \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x} \right\}$$

Exercises 3

Find $\frac{dy}{dx}$ given that

1) $y = \dfrac{e^{4x}}{x^3 \cosh 3x}$

2) $y = \dfrac{(3x+1)\cos 2x}{e^{2x}}$

3) $y = x^5 \sin 2x \cos 4x$

4) $y = \dfrac{(x^3 - 1)\sin 5x}{x^6}$

5) $y = \dfrac{\sin 2x \cos 3x}{\cos 4x}$

$= 5(4x - 3)^4 \times 4$

$= 20(4x - 3)^4$

4.7 The second Derivative of a Function

Let $y = f(x)$ be a differentiable function. The derivative $\frac{dy}{dx} = f'(x)$ is also a function of x.

Note: $f'(x)$ may or may not be differentiable. In case $f'(x)$ is differentiable, we call the derivative of $f'(x)$ the second derivative of f , that is , the derivative of the derivative of f ; denoted by $f''(x)$ or $\frac{d}{dx}\left\{\frac{dy}{dx}\right\}$ or $\frac{d^2y}{dx^2}$

Example 1

Find the second derivative of the function $f(x) = x^3$

Solution:

$$f'(x) = \frac{df}{dx} = 3x^2$$

$$\therefore (f')' = f'(x) = \frac{d}{dx}\left\{\frac{dy}{dx}\right\} = 6x$$

Exercises 2

Find $\frac{d^2y}{dx^2}$ when

1. $y = (1 + 2x^5)$

2. $y = \left(3x^2 - 4\right)^4$

3. $y = x^4 - 8x^2 + 1$

4. $y = x^3 + 7$

5. $y = 2x^3 + 3x^2 - 12x + 20$

Exercises 4

Differentiate

1. $y = \cos(7x + 2)$

2. $y = (4x - 5)^6$

3. $y = e^{3-x}$

4. $y = \sin 2x$

5. $y = \cos(x^2)$

6. $y = ln(3 - 4\cos x)$

7. $y = e^{\sin 2x}$

8. $y = \sin 2x$

9. $y = \cos^3(3x)$

10. $y = ln \cos 3x$

5 CHAPTER FIVE

5.1 INTEGRATION AND IT'S APPLICATIONS

5.2 Introduction

Integration is the reverse of differentiation: In integration the function from which the differential coefficient is gotten is derived.

In general

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

The constant C, in the general solution is called an arbitrary constant of integration, it can only be determined if additional information is given.

5.3 Standard integral forms

$$if\ \frac{d}{dx}[G(x) + c] = g(x)$$

Then $\int g(x)dx = G(x) + c$

a. $\frac{d}{dx}(x^n + 1) = (n+1)x^n \quad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c; (n \neq 1)$

b. $\frac{d}{dx}[\cos x] = -\sin x \quad \therefore \int \sin x dx = -\cos x + c$

c. $\frac{d}{dx}[\sin x] = \cos x \quad \therefore \int \cos x dx = \sin x + c$

d. $\frac{d}{dx}[\tan x] = \sec^2 x \quad \therefore \int \sec^2 x dx = \tan x + c$

e. $\frac{d}{dx}[\cosh x] = \sinh x \quad \therefore \int \sinh x dx = \cosh x + c$

f. $\frac{d}{dx}[\sinh x] = \cosh x \quad \therefore \int \cosh x dx = \sinh x + c$

g. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{(1-x^2)}} \quad \therefore \int \frac{1}{\sqrt{(1-x^2)}} dx = \frac{d}{dx}[\sin^{-1} x] + c$

Exercises 6

1. find $\frac{dy}{dx}$ when $x^3 + y^3 - 3xy = 8$

2. If $x^2 + y^3 - 4x + 4y = 26$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

3. Find an expression for $\frac{dy}{dx}$ when $x^3 + y^3 + 4xy^2 = 5$

4. Find $\frac{dy}{dy}$ when $x^2 + y^2 - 5xy^3 + 9 = 0$.

4.8 Differentiation of implicit function

The equation of the function $y = f(x)$, is a function in which y is given as an explicit function of x. Sometimes, however, equations governing practical problems turn out in this form $f(x, y) = c$ Where $f(x, y)$ is a function of x and y which may involve x and y in any manner whatsoever, y is then said to be an implicit function of x.

The technique for finding $\frac{dy}{dx}$ is to differentiate both sides of the equation with respect to x. the equation thus obtained may then be solved for $\frac{dy}{dx}$.

Example 1

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 2cxy$ where c is a constant.

Solution

Differentiate both sides, term by term, with respect to x ; then

$$2x + 2y\frac{dy}{dx} = 2c\frac{d}{dx}(xy)$$

Using the method of differentiating a function of a function for y^2 and using differentiation technique for a product on (xy). We have

$$2x + 2y\frac{dy}{dx} = 2c\left(y + x\frac{dy}{dx}\right)$$

Collecting like terms. $\frac{2ydy}{dx} - \frac{2cxdy}{dx} = 2cy - 2x$

$$\frac{2dy}{dx}(y - cx) = 2(cy - x)$$

$$\therefore \frac{dy}{dx} = \frac{cy - x}{y - cx}$$

Example 2

If $x^2 + y^2 - 2x - 6y + 5 = 0$ find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} \Rightarrow 2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$$

$$= 2x - 2 + (2y - 6)\frac{dy}{dx} = 0$$

$$= (2y - 6)\frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{2y - 6}$$

Recall $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{3}{x^4} dx = \frac{3x^{-4+1}}{-4+1} + c = -x^{-3} + c$$

c. $\displaystyle\int (x^3 + 2)\, dx$

Recall $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

$$\int x^3\, dx + \int 2\, dx = \frac{x^{3+1}}{3+1} + 2x + c = \frac{1}{4}x^4 + 2x + c$$

d. $\displaystyle\int (6x^3 + 8x^2 - x + 9)\, dx$

Recall $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int (6x^3 + 8x^2 - x + 9)\, dx = \int 6x^3\, dx + \int 8x^2\, dx - \int x\, dx + \int 9\, dx + c$$

$$= \frac{6x^{3+1}}{3+1} + \frac{8x^{2+1}}{2+1} - \frac{x^{1+1}}{1+1} + 9x + c$$

$$= \frac{6x^4}{4} + \frac{8x^3}{3} - \frac{x^2}{2} + 9x + c$$

$$= \frac{3x^4}{2} + \frac{8x^3}{3} - \frac{x^2}{2} + 9x + c$$

Example 2:

Evaluate

a. $\displaystyle\int e^{4x} dx$

b. $\displaystyle\int \sqrt[4]{x} dx$

c. $\displaystyle\int \frac{2}{x} dx$

d. $\displaystyle\int \frac{1}{1+x^2} dx$

h. $\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{(1-x^2)}}$ $\therefore \int \frac{-1}{\sqrt{(1-x^2)}}dx = \frac{d}{dx}[\cos^{-1}x] + c$

i. $\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1-x^2}$ $\therefore \int \frac{1}{1-x^2}dx = \frac{d}{dx}[\tan^{-1}x] + c$

j. $\frac{d}{dx}[lnx] = \frac{1}{x}dx$ $\therefore \int \frac{1}{x} = lnx + c$

k. $\frac{d}{dx}[e^x] = e^x$ $\therefore \int e^x\,dx = e^x + c$

l. $\frac{d}{dx}[e^{kx}] = ke^x$ $\therefore \int e^{kx}\,dx = \frac{e^{kx}}{k} + c$

m. $\frac{d}{dx}[a^x] = a^x lna$ $\therefore \int a^x\,dx = \frac{a^x}{lna} + c$

Examples

Evaluate

a. $\int x^{\frac{3}{2}}dx$

b. $\int \frac{3}{x^4}dx$

c. $\int (x^3+2)\,dx$

d. $\int (6x^3+8x^2-x+9)\,dx$

Solution

a. $\int x^{\frac{3}{2}}dx$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int x^{3/2}dx = \frac{x^{3/2+1}}{\frac{3}{2}+1} - \frac{x^{5/2}}{\frac{5}{2}} = \frac{2x^{5/2}}{5} + c$$

b. $\int \frac{3}{x^4}dx$

e. $\int 6^x dx$

Solution

a. $\int e^{4x} dx = \frac{e^{4x}}{4} + c$

b. $\int \sqrt[4]{x} dx = \frac{x^{1/4+1}}{1/4+1} + c = \frac{x^{5/4}}{5/4} + c = \frac{4}{5} x^{5/4} + c$

c. $\int \frac{2}{x} dx = 2ln\ x + c$

d. $\int 6^x\ dx = \frac{6^x}{ln6} + c$

5.4 Function of a linear function of x.

Integrate $\int f(ax+b)\,dx$ at $k = ax + b$

Then

$$\int f(ax+b)\,dx = \int f(k)\,dx = \int f(k)\,\frac{dk}{dk}.dx$$

$$= \int f(k)\,\frac{dx}{dk}.dx$$

Then, $\int f(ax+b)\,dx = \int f(k)\,\frac{dx}{dk}.dk$

$$= \frac{dk}{dx} = a$$

Taking the inverse of both sides

$$\frac{dx}{dk} = \frac{1}{a}$$

$$\int f(k)\,\frac{1}{a}dk = \frac{1}{a}\int f(k)\,dk$$

Example

Evaluate

(a) $\int (3x+5)^3\, dx$

(b) $\int (\cos 9x\ dx)$

(c) $\int \left(e^{-x^2}+2\right) dx$

Solution

(a) $\int (3x+5)^3\, dx$

Let $k = 3x+5$

Then, $\int (3x+5)^3\, dx = \int k^3 dx = \int k^3 \frac{dk}{dk} dx = \int k^3 \frac{dx}{dk} dk$

$$\Rightarrow \int (3x+5)^3\, dx = \int k^3 \frac{dx}{dk} dk \qquad (1)$$

Then, $\frac{dk}{dx} = 3 \qquad (2)$

Taking the inverse of both sides

$$\frac{dk}{dx} = \frac{1}{3}$$

sub (2) into (1)

$$\Rightarrow \int (3x+5)^3\, dx = \int k^3 \frac{1}{3} dk$$

$$= \frac{1}{3} \int k^3 dk = \frac{1}{3} \cdot \frac{k^{3+1}}{3+1} + c$$

$$= \frac{k^4}{12} + c$$

Recall $k = 3x+5$

Then

$$\int (3x+5)^3\, dx = \frac{k^4}{12} + c = \frac{(3x+5)^4}{12} + c$$

b) $\int (\cos 9x\ dx)$

Let $k = 9x$

$$\Rightarrow \int (\cos 9x\ dx) = \int (\cos k)\, dx = \int (\cos k)\, \frac{dx}{dk} dk$$

$$\frac{dk}{dx} = 9 \Rightarrow \frac{dx}{dk} = \frac{1}{9}$$

Then (1) becomes

$$\int (\cos 9x\ dx) = \cos k \frac{dx}{dk} dk = \frac{1}{9} \int \cos k\ dk = \frac{1}{9} \sin k + c = 1$$

Recall $k = 9x$

Then $\int (\cos 9x\ dx) = \frac{1}{9} \sin 9x + c$

(c) $\int \left(e^{-x^2} + 2\right) dx$

$$\int \left(e^{-x^2} + 2\right) dx = \int e^{-x^2} dx + \int 2dx$$

$$\int \left(e^{-x^2} + 2\right) dx = \int e^{k} \frac{dx}{dk} dk + \int 2dx$$

Where $k = -x^2$

$$\frac{dk}{dx} = -2x$$

$$\frac{dx}{dk} = \frac{-1}{2x}$$

$$\int \left(e^{-x^2} + 2\right) dx = \int e^{k} dx + \int 2dx$$

$$= -e^{-x^2}/2x + 2x + c$$

5.5 Integrals of the form $\int \frac{f'(x)}{f(x)} dx$ and $\int f(x) f'(x) dx$

Let $k = f(x)$

Then $\frac{dk}{dx} = f'(x)$

$$\Rightarrow dk = f'(x) dx$$

$$\frac{dk}{k} = \frac{f'(x)}{f(x)} dx$$

c. $\int \frac{2\sec^2 2x}{\tan 2x}$

Let $k = \tan 2x$ $\frac{dk}{dx} = 2\sec^2 2x$

Then

$$\int \frac{2\sec^2 2x}{\tan 2x} = \log_e(\tan 2x) + c$$

ii. $\int f(x) f'(x) dx$

Let $k = f(x)$

$\frac{dk}{dx} = f'(x)$

$\int k \frac{dk}{dx} dx$

$= \int -1$

$\int k dx = \frac{k^2}{2} + c = \frac{(f(x))^2}{2} + c$

Example

Evaluate

a. $\int ln \ \cos 3x \tan 3x dx$

b. $\int \sin 2x \cos 2x \ dx$

c. $\int (2x^2 + 8x + 5)(4x + 8)\, dx$

d. $\int \cos(x)^2 x \sin x^2 dx$

Solution

$$\int \frac{1}{k}dk = \int \frac{f'(x)}{f(x)}dx$$

$$\int \frac{1}{k}dk = ln\ k + c$$

$$ln\ k = \int \frac{f'(x)}{f(x)}dx$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \log_e f(x) + c$$

Example

Evaluate

a. $\displaystyle\int \frac{12x^2 + 10}{4x^3 + 10x + 5}dx$

b. $\displaystyle\int \frac{\cos x e^{\sin x}}{e^{\sin x}}dx$

c. $\displaystyle\int \frac{2\sec^2 2x}{\tan 2x}$

Solution

a. $\displaystyle\int \frac{12x^2 + 10}{4x^3 + 10x + 5}dx$

Let $k = 4x^3 + 10x + 5$

$$\frac{dk}{dx} = 12x^2 + 10$$

$$\Rightarrow \int \frac{12x^2 + 10}{4x^3 + 10x + 5}dx = \log_e \left(4x^3 + 10x + 5\right) + c$$

b. $\displaystyle\int \frac{\cos x e^{\sin x}}{e^{\sin x}}dx$

Let $k = e^{\sin x}$

$$\frac{dk}{dx} = \cos x e^{\sin x}$$

$$\int \frac{\cos x e^{\sin x}}{e^{\sin x}}dx = \log_e \left(e^{\sin x}\right) + c$$

a. $\int ln \ \cos 3x \tan 3x dx$

Let $k = ln \cos 3x$

$\frac{dk}{dx} = \frac{-3\sin 3x}{\cos 3x} \quad -3\tan 3x dx$

This implies that

$\int ln \ \cos 3x \tan 3x dx = \frac{1}{3}\int k dx = \frac{-1}{3}.\frac{k^2}{2} + c$

$= \frac{-k^2}{6} + c = -\frac{(ln\cos 3x)^2}{6} + c$

b. $\int \sin 2x \cos 2x \ dx$

Let $k = \sin 2x$

$\frac{dk}{dx} = 2\cos 2x$

$$\int \sin 2x \cos 2x \ dx = \frac{1}{2}\int \sin 2x\,(2\cos 2x)\,dx = \frac{1}{2}\left[\frac{(\sin 2x)^2}{2}\right] + c$$

$= \frac{\sin^2 2x}{4} + c$

d. $\int \cos(x)^2 x \sin x^2 dx$

Let $k = \cos\left(x^2\right)$

$\frac{dk}{dx} = -2x\sin x^2$

Then

$$\int \cos(x)^2 x \sin x^2 dx = -\frac{1}{2}\int \cos(x)^2 x\left(-2x\sin x^2\right)dx = -\frac{1}{2}\left[\frac{\left(\cos^2 x^2\right)}{2}\right] +$$

$c = -\frac{\cos^2 x^2}{4} + c$

5.6 Integration by substitution

Integrals which are not is standard form can be converted to standard form by making appropriate algebraic substitution.

$$\frac{dk}{dx} = 12x^3 \Rightarrow dx = \frac{dk}{12x^3}$$

$$\int x^3 \left(8 + 3x^4\right)^{1/2} dx$$

$$= \int x^3 k^{1/2} \frac{dk}{12x^3} = \frac{1}{12} \int k^{1/2} dk$$

$$= \left[\frac{1}{12} \times \frac{k^{1/2+1}}{1/2+1}\right] + c = \left[\frac{1}{12} k^{3/2} \times \frac{2}{3}\right] + c = \frac{k^3/2}{18} + c = \frac{\left(8 + 3x^4\right)^{3/2}}{18} + c$$

5.7 Integration by Trigonometric Substitution

Here,we consider integral of the form

a. $\sqrt{a^2 - x^2}$

b. $\frac{1}{\sqrt{a^2 - x^2}}$

b. $\frac{1}{a^2 + x^2}$

Example

Evaluate

a. $\int \sqrt{a^2 - x^2} dx$

b. $\int \frac{1}{\sqrt{a^2 + x^2}} dx$

c. $\int \frac{1}{a^2 + x^2} dx$

Solution

Example 1

Evaluate a. $\int x^2 \tan 2x^3 dx$

$$\text{Let } k = 2x^3$$
$$\frac{dk}{dx} = 6x^2$$
$$\frac{dx}{dk} = \frac{1}{6x^2}$$
$$dx = \frac{dk}{6x^2}$$

Then

$$\int x^2 \tan 2x^3 dx = \int x^2 \tan k \frac{dk}{6x^2}$$
$$= \frac{1}{6}\int \tan k dk = \frac{1}{6}\sec^2 k + c$$
$$= \frac{\sec^2 2x^3}{6} + c$$

Example 2

Evaluate

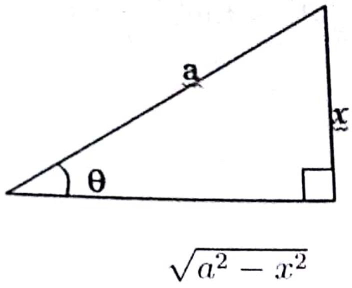
a. $\int x^2 e^{-x^3} dx$

$$\text{Let } k = -x^3$$
$$\frac{dk}{dx} = -3x^2$$
$$dx = -\frac{dk}{3x^2}$$
$$\int x^2 e^{x^3} dx = -\int x^2 e^k \frac{dk}{3x^2}$$
$$= \frac{1}{3}e^k + c = -\frac{1}{3}e^{-x^3} + c$$

b. $\int x^3 \left(8 + 3x^4\right)^{1/2} dx$

Let $k = 8 + 3x^4$

b. $\int \frac{1}{\sqrt{a^2 - x^2}} dx$



$\sin\theta = \frac{x}{a}, \qquad \theta = \sin^{-1}\frac{x}{a}, \qquad x = a\sin\theta$

$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a} = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$\int a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\left(1 - \sin^2\theta\right)$

$a^2 - x^2 = a^2\cos^2\theta$

$\sqrt{a^2 - x^2} = a\cos\theta$

$\Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{a\cos\theta} a\cos\theta d\theta$

$= \int d\theta = \theta + c$

$\Rightarrow \int \frac{1}{\sqrt{a^2 + x^2}} dx = \theta + c = \sin^{-1}\frac{x}{a} + c$

c. $\int \frac{1}{a^2 + x^2} dx$

Let $x = a\tan\theta \qquad \theta = \tan^{-1} x/a$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta d\theta$

$a^2 + x^2 = a^2 + a^2\tan^2\theta$

$a^2 + x^2 = a^2\left(1 + \tan^2\theta\right)$

$a^2 + x^2 = \sec^2\theta$

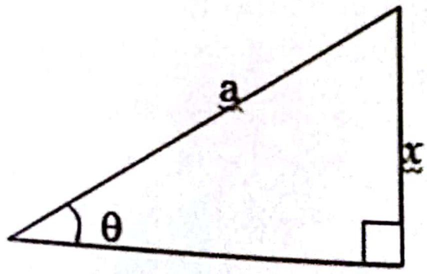
$\int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a\sec^2\theta} a\sec^2\theta d\theta$

.a $\int \sqrt{a^2 - x^2}dx$

Put $x = a\sin\theta$, $\theta = \sin^{-1}\frac{x}{a}$

$$\frac{dx}{d\theta} = a\cos\theta$$
$$dx = a\cos\theta d\theta$$
$$a^2 - x^2 = a^2 - a^2\sin^2\theta$$
$$a^2 - x^2 = a^2\left(1 - \sin^2\theta\right)$$
$$\sqrt{a^2 - x^2} = \sqrt{a^2\cos^2\theta}$$
$$\sqrt{a^2 - x^2} = a\cos\theta$$
$$\Rightarrow \int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2\cos^2\theta}.a\cos\theta d\theta$$
$$= \int a^2\cos^2\theta d\theta$$
$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$= \int a^2\left[\frac{\cos 2\theta}{2} + \frac{1}{2}\right]d\theta$$
$$= \int \left[a^2\frac{\cos 2\theta}{2} + \frac{a^2}{2}\right]d\theta$$
$$= \frac{a^2\sin 2\theta}{4} + \frac{a^2\theta}{2} + c$$
$$= \frac{a^2}{2}\left[\frac{2\sin\theta\cos\theta}{2} + \theta\right] + c$$
$$= \frac{a^2}{2}\left[\sin\theta\cos\theta + \theta\right] + c$$



$\sin\theta = x/a$ $\qquad \cos\theta = \frac{\sqrt{a^2 - x^2}}{a} = \frac{a^2}{2}\frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a} + \sin^{-1}\frac{x}{a} + c$

$$\Rightarrow \sqrt{a^2 - x^2} = \frac{x\sqrt{a^2 - x^2}}{2} + \sin^{-1}\frac{x}{a} + c$$

$$\int \sin^2 x dx = \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + c = \frac{1}{2}\left[x - \sin\ 2x\right] + c$$

(ii) $\int \cos^2 x dx$

Recall

$$\cos^2 x = 2\cos 2x - 1$$

$$2\cos 2x = \cos^2 x + 1$$

$$\cos^2 x = \frac{\cos 2x}{2} + \frac{1}{2}$$

$$\int \cos^2 x dx = \int \frac{\cos 2x}{2} dx + \int \frac{1}{2} dx$$

$$= \frac{\sin 2x}{4} + \frac{x}{2} + c$$

Example

Evaluate

a. $\int \sin^3 x dx$

b. $\int \cos^3 x dx$

Solution

a. $\int \sin^3 x dx = \int \sin^2 x \sin dx$

$$= \int \left(1 - \cos^2 x\right) \sin x dx = \int \sin x dx - \int \cos^2 x \sin x dx = -\cos x - \frac{\cos^3 x}{3} + c$$

b. $\int \cos^3 x dx = \int \cos^2 x \cos x dx$

$$\int \left(1 - \sin^2 x\right) \cos x dx$$

$$= \int \left(\cos x - \sin^2 x \cos x\right) dx$$

$$= \int \cos x dx - \int \sin^2 x \cos x dx = sinx - \frac{sin^3 x}{3} + c \quad \sin x - \frac{\sin^3 x}{3} + c$$

$\Rightarrow \int d\theta = \theta + c$

$$\int \frac{1}{a^2 + x^2} dx = \tan^{-1} \frac{x}{a} + c$$

Example

Evaluate

$$\int \sqrt{a - x^2}$$

Solution

$$\int \sqrt{a - x^2} =$$

$$\frac{x\sqrt{3 - x^2}}{2} + \sin^{-1} \frac{x}{3} + c$$

5.8 Integration of Trigonometric Functions

a. Powers of $\sin x$ *and* $\cos x$

$\int \sin x dx = -\cos x + c$

$= \sin x + c$

Evaluate

$\int \sin^2 x dx$

$\int \cos^2 x dx$

Solution

Recall $\cos 2x = 1 - 2\sin^2 x$ $\quad \sin^2 x = \frac{(1 - \cos 2x)}{2}$

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

$$uv = \int udv + \int vdu$$

$$\therefore \int udv = uv - \int vdu$$

Example

a. $\int x^2 \sin 2x dx$

b. $\int \frac{lnx}{\sqrt{x}}dx$

Solution

a. $\int x^2 \sin 2x dx$

$udv = uv - \int vdu$

Let $u = x^2$, $du = 2xdx$

$dv = \sin 2x \quad v = -\frac{\cos 2x}{2}$

$$\int x^2 \sin 2x dx = -x^2\frac{\cos 2x}{2} + \int \frac{\cos 2x}{2}2xdx$$

$$= -x^2\frac{\cos 2x}{2} + x\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}dx$$

$$= \frac{x\sin 2x}{2} - x^2\frac{\cos 2x}{2} + \frac{\cos 2x}{4} + c$$

b. $\int \frac{lnx}{\sqrt{x}}dx$

$\int udv = uv - \int vdu$

Let $u = lnx$, $\quad du = \frac{1}{x}dx$

$dv = x^{-\frac{1}{2}} \quad v = \frac{x^{-1/2+1}}{-1/2+1} = 2x^{1/2}$

$$= \frac{1}{2}\int \cos 4x dx - \frac{1}{2}\int \cos 12x dx$$

$$= \frac{\sin 4x}{8} - \frac{\sin 12x}{24} + c$$

c. $\int \sin 3x \cos x dx$

$$\int \sin 3x \cos x dx = \int \frac{1}{2}\left[2\sin 3x \cos x\right] dx$$

$$= \frac{1}{2}\left[\int (\sin(3x+x))\, dx + \int \sin(3x-x)\, dx\right]$$

$$= \frac{1}{2}\int \sin 4x dx + \frac{1}{2}\int \sin 2x dx$$

$$= -\frac{\cos 4x}{8} - \frac{\cos 2x}{4} + c$$

d. $\int \cos 7x \sin 4x dx$

$$\int \cos 7x \sin 4x dx = \frac{1}{2}(2\cos 7x \sin 4x)$$

$$= \frac{1}{2}\left[\int \sin(7x+4x)\, dx = \frac{1}{2}\int (2\sin 7x \sin 4x)\right]$$

$$= \frac{1}{2}\int \sin 11x dx - \frac{1}{2}\int \sin 3x dx$$

$$= -\frac{\cos 11x}{22} + \frac{\cos 3x}{6} + c$$

$$= \frac{\cos 3x}{6} - \frac{\cos 11x}{22} + c$$

5.10 Integration by part

If u *and* v are functions of x then using product rule

$$\frac{d}{du}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x.

5.9 Products of Sine and Cosine

Evaluate

a. $\int \cos 6x \cos 2x dx$

b. $\int \sin 8x \sin 4x dx$

c. $\int \sin 3x \cos x dx$

d. $\int \cos 7x \sin 4x dx$

Solution

Recall the identity

$$\sin(A+B) + \sin(A-B) = 2\sin A \ \cos B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \ \sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A \ \cos B$$

$$\cos(A-B) - \cos(A+B) = 2\sin A \ \sin B$$

a. $\int \cos 6x \cos 2x dx$

$$\int \cos 6x \cos 2x dx = \int \frac{1}{2}(2\cos 6x \cos 2x)$$

$$= \frac{1}{2}\int (\cos(6x+2x) + \cos(6x-2x))\, dx$$

$$= \frac{1}{2}\left[\int \cos 8x dx + \int \cos 4x dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + c$$

$$= \frac{\sin 8x}{16} + \frac{\sin 4x}{8} + c$$

b. $\int \sin 8x \sin 4x dx$

$$\int \sin 8x \sin 4x dx = \int \frac{1}{2}(2\sin 8x \sin 4x)\, dx$$

$$= \frac{1}{2}\left[\int [\cos 8x - 4x]\, dx - \frac{1}{2}\int [\cos 8x + 4x]\, dx\right]$$

Put $x = -4$ in (1)

$-4 - 2 = A(-4 + 6)$

$-6 = 2A \quad \Rightarrow A = -3$

$\frac{x - 2}{x^2 \quad 10x + 24}dx = \int \frac{4}{x + 6}dx - \frac{-3}{x + 4}dx$

$= 4ln(x + 6) - 4ln(x + 4) + c$

2. $\int \frac{6x^2 + x + 12}{x(x^2 + 4)}dx$

Express in partial fraction

2. $\int \frac{6x^2 + x + 12}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$

$\Rightarrow 6x^2 + x + 12 = A(x^2 + 4) + (Bx + C)x$

$6x^2 + x + 12 = Ax^2 + 4A + Bx^2 + Cx$

$6x^2 + x + 12 = (A + B)x^2 + Cx + 4A$

Equating coefficient

$6 = A + B; \qquad C = 1; \qquad 4A = 12 \Rightarrow A = 3$

$\Rightarrow B = 3$ and $C = 1$

$\Rightarrow \frac{6x^2 + x + 12}{x(x^2 + 4)} = \frac{3}{x} + \frac{3x + 1}{x^2 + 4}$

$\Rightarrow \int \frac{6x^2 + x + 12}{x(x^2 + 4)} = dx = \int \frac{3}{x}dx + \int \frac{3x + 1}{x^2 + 4}dx$

$\Rightarrow = 3lnx + \frac{3x}{x^2 + 4}dx + \int \frac{1}{x^2 + 4}dx$

$= 3lnx + \frac{3}{2}ln(x^2 + 4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

3. $\frac{5x^2 + 22x + 19}{(x + 3)^3} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3}$

$5x^2 + 22x + 19 = A(x + 3)^2 + B(x + 3) + C$

$5x^2 + 22x + 19 = Ax^2 + 6xA + 9A + Bx + 3B + C$

$\int lnx x^{-1/2}dx = 2lnx x^{1/2} - 2\int(1/x)(x^{1/2})dx$

$= 2x^{1/2}lnx - 2\int x^{-1/2}dx$

$= 2x^{1/2}lnx - 4x^{1/2} + c$

$= 2\sqrt{x}lnx - 4\sqrt{x} + c$

5.11 Integration by partial fraction

If an algebraic fraction is not in standard integral form, it is converted to standard form by expressing in partial fraction.

Example

1. $\int \frac{x-2}{x^2+10x+24}dx$

2. $\int \frac{6x^2+x+12}{x(x^2+4)}dx$

3. $\frac{5x^2+22x+19}{(x+3)^3}$

Solution

1. $\int \frac{x-2}{x^2+10x+24}dx$

Express in partial fraction

$$\frac{x-2}{x^2+10x+24} = \frac{A}{(x+4)} + \frac{B}{(x+6)}$$

$$x - 2 = A(x+6) + B(x+4) \qquad (3)$$

Put $x = -6$ in (1)

$-6 - 2 = -2B \quad \Rightarrow B = 4$

$$= 5ln(x+3) + \frac{8}{(x+3)} + \frac{1}{(x+3)^2} + c$$

5.12 Application of integration

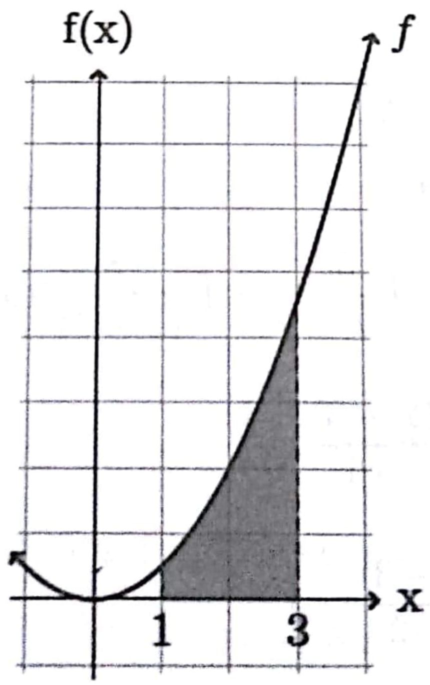
Application to *areas, volumes (including approximate integration)*, and *trapezium* and *Simpson's rules.*

1. *Areas*

Integral calculus has interesting applications; one of these very important applications is that: it can be readily employed to obtain/calculate exactly the areas enclosed by curves. Consider a curve described by a function $y = f(x)$. Integration helps to obtain exactly the area under the curve $y = f(x)$ say from, to $x = a$ to $x = b$. This simple description takes us to what is known as the ***Definite integral***. Definite in the sense that the area covered by the curve i.e. $x = a$ to $x = b$, is clearly defined.

Example 1: Find the area bounded by $f(x) = \frac{x^2}{4}$ and the x-axis along the Interval $[1, 3]$.

Solution: consider the figure below



Collecting like terms

$5x^2 + 22x + 19 = Ax^2 + (6A + B)x + 9A + 3B + C$

Equating coefficients

$$5 = A \tag{4}$$

$$6A + B = 22 \tag{5}$$

$$9A + 3B + C = 19 \tag{6}$$

If $A = B$ from (4) then from (5)

$B = 22 - 6(5)$

$B = 22 - 30 = B = -8$

From (6) C becomes

$9(5) + 3(-8) + C = 19$

$C = 19 - 45 + 24$

$C = -2$

$A = 5; \quad B = -8, C = -2$

Substituting this result

$5x^2 + 22x + 19$. $5dx$ 8 . 2

Area under the curve(A)

$$A = \int_1^3 \frac{x^2}{4}dx = \frac{3^3}{12} - \frac{1^3}{12} = \frac{13}{6}$$

Example 2: Find the area of the region bounded by the positive y-axis, $y = 4$ and $f(x) = x^2$ In this example, how do we know what our limits of integration should be? All we need to do is to find the points of intersection which begin and end the region. We are given that the region is bounded by the positive y-axis, hence, we start from $x = 0$ We can then find the second endpoint by finding the point of intersection between $f(x) = x^2$ and the vertical line $y = 4$.

$x^2 = 4$

$x = 2$

We will integrate over the interval $[0, 2]$, this gives

$$Area = \int_0^2 (4 - x^2)dx = \left[4(2) - \frac{(2)^3}{3}\right] - \left[4(0) - \frac{(0)^3}{3}\right] = \frac{16}{3}$$

Note: When finding the area between two curves we subtract the bottom function from the top function within the integral.

Example 3: Find the area bounded by the curves $f(x) = 2 - x^2$ and $g(x) = x$

Solution: It is clear that $f(x)$ is the top function while $g(x)$ is the bottom function. We proceed by finding the limit of integration.

$2 - x^2 = x$

$(x + 2)(x - 1) = 0$

$x = -2$ and $x = 1$

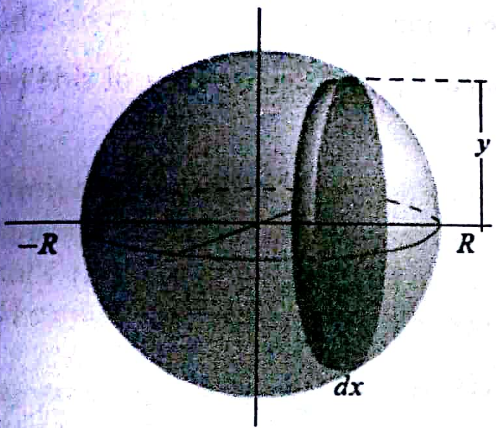
The area A, bounded by the curves is given as

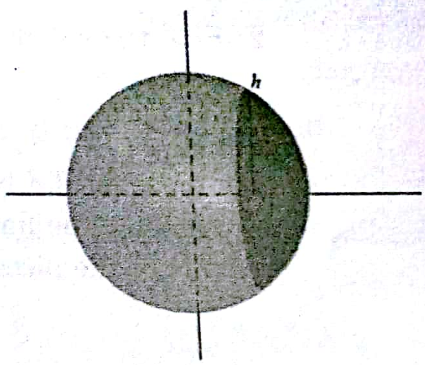
$$A = \int_{-2}^1 [f(x) - g(x)]dx = \int_{-2}^1 [(2 - x^2) - x]dx = \left(\frac{-1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{-9}{2}$$

Scanned by CamScanner

2. Volumes

This section will illustrate how integration can be a very valuable tool in the determination of volumes. As an illustrative example, let us obtain the volume of a sphere of radius R. We have to decide first on a direction of accumulation; and to do that we need a particular coordinate representation of the sphere. Starting from the plane bounded by the x-axis and the curve:$C_1 = x^2 + y^2 = R^2$. If we rotate this region in space about the x-axis, we obtain the sphere of radius (R) (figure 2.1). We now accumulate the volume of the sphere by moving along the axis of rotation, (the x-axis), starting at$x = -R$, and ending at $x = R$. Where $V(x)$ is the accumulated volume up to point x. Then the piece added when we move a small distance dx further along the axis is a cylinder of width dx and of radius y, the length of the line from the x-axis to the curve C_1) (see figure 2.1).





The volume of this piece is thus $dV(x) = \pi y^2 dx$.

Along ,$C_1 = \sqrt{R^2 - x^2}$, from this, we can derive $dV = \pi(R^2 - x^2)dx$, the volume is thus the integral of the differential.

$$V = \int dV = \int_{-R}^{R} \pi(R^2 - x^2)dx = \frac{4\pi R^3}{3}$$

Remark: In general, we can calculate the volume of a solid by integration if we can find a way of sweeping out the solid by a family of surfaces, and we can calculate, or already know the area of those surfaces. Then we calculate the volume by integrating the area along the direction of sweep. In the above example we swept out the sphere by moving along the x-

Remark: The first step is to make a sketch of the curves to determine which one is above the other. When the graph of the curve falls below the horizontal (x-axis), the definite integral will have a negative sign.

Exercises

1. Find the area between the curve $y = x^2 - x$ and the x-axis from $x = 0$ to $x = 1$.

2. Make a sketch of the area common to the parabolas $y = x^2 - 9$ and $y = 4 - 7x^2$. Hence obtain the area enclosed.

3. Find the area of the region between the graphs of $f(x) = 3x^2 - x^2 - 10x$ and $g(x) = -x^3 + 2x$.

4. Obtain the area of the segment cut off from curve: $y = 4 - 3x - x^2$ by a straight line: $2x + y + 2 = 0$

since there are no elementary functions that are anti-derivatives of such integrands. Examples of such functions are $\sqrt{1 - x^3}$ and e^{-x^2}. i.e. $\int_0^1 \sqrt{1 - x^3}$ and $\int_0^1 e^{-x^2}$ can only be evaluated using approximate or numerical methods. There are a number of such numerical/approximate methods, in this text, our discussion will be limited to Trapezoidal and the Simpson's rules.

5.13 Trapezoidal Rule:

The trapezoidal rule is a numerical method that approximates the value of a definite integral. We consider the definite integral

$$\int_a^b f(x)dx$$

We assume that $f(x)$ is continuous on $[a, b]$ and we then divide $[a, b]$ into n sub-intervals of equal length.

Hence, $\Delta x = \dfrac{b - a}{n}$, using $n + 1$ points.

$x_0 = a,\ x_1 = a + \delta x,\ x_2 + 2\delta x, ..., x_n + n\delta x = b$

We can obtain the value of $f(x)$ at these points

$y_0 = f(x_0),\quad y_1 = f(x_2)\quad y_2 = f(x_2), ...,\quad y_n = f(x_n)$

We approximate the integral by using n trapezoids found by using straight line segments between the points (x_{i-1}, y_{i-1}) and x_i, y_i for $1 \leq i \leq n$ as shown in the figure below.

axis, and associating to each point x the area of the disc which is the perpendicular cross-section of the sphere at x.

Generally, we can find the volume of a solid by sketching the region under consideration. Choose a direction in which to accumulate the volume. Then write down the expression for the differential increment in volume,

$$dV(x) = A(x)dx$$

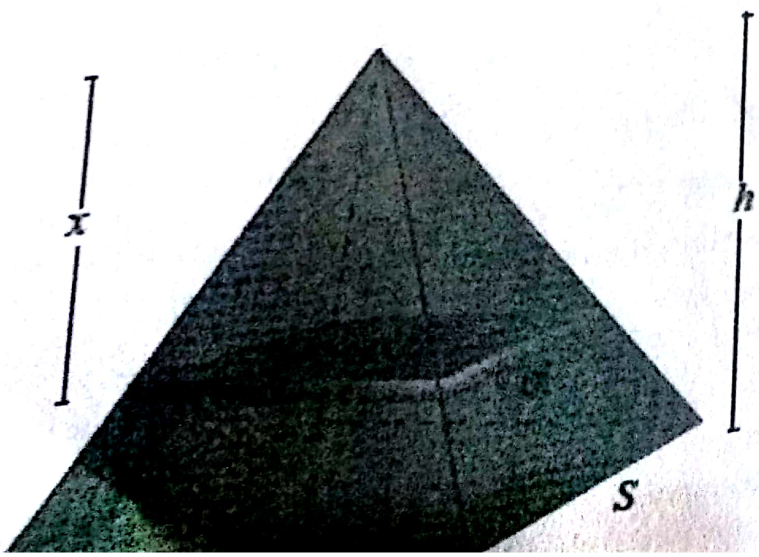
where dx is an infinitesimal increment in the direction of accumulation, and $A(x)$ is the area of the section of the solid at the point x. Of course, if the solid is highly irregular, finding $A(x)$ may still be a problem. Here we intentionally restrict attention to those cases where $A(x)$ is known or is easily found.

Exercises

1. Find the volume of a cone of base radius r and height h.

2. Find the volume of a pyramid of height h and square base of side length s.

3. Let Rbe the region in the plane bounded by the curves $y = x^2$, $y = -x^2$. Let K be a solid lying over this region whose cross-section is a semicircle of diameter the line segment between the curves (see figure 2.3) Find the volume of the piece of K lying between $x = 1$ and $x = 2$.



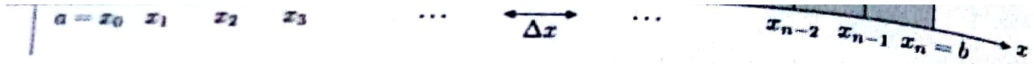


Figure 2.4

By adding the area of the n trapezoids, we obtain the approximation

$$\int_a^b f(x)dx \approx \frac{\delta x}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$

Which is known as the Trapezoidal rule formula.

Example

Use the trapezoidal rule with n=8 to estimate $\int_0^5 \sqrt{(1 - x^2)}dx$

Solution:for $n = 8$, we have $\delta x = \dfrac{5-1}{8} = 0.5$, we compute the values of $y_0, y_1, y_2, \ldots y_8$ as shown in the following Table:

x	1	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$\sqrt{1 - x^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

Table 1

$$\int_0^5 \sqrt{1 - x^2}dx \approx \frac{0.5}{2}\left(2\sqrt{2} + 2\sqrt{3.25} + 2\sqrt{5} + 2\sqrt{7.5} + 2\sqrt{10} + 2\sqrt{13.25} + 2\sqrt{17} + 2\sqrt{21.25} + 2\sqrt{26}\right) \approx$$
12.76

Exercises

1. The following points were found empirically.

x	2.1	2.4	2.5	3.0	3.3	3.6
y	3.2	[illegible]	2.9	3.5	4.1	5.2

Table 2

Use the trapezoidal rule to estimate $\int_{2.1}^{3.6} y dx$.

2. Approximate $\int_{1}^{2}$ using the Trapezoidal rule for $n = 8$

5.14 Simpson's Rule

Another numerical method which can be easily employed in the approximation of integrals is the Simpson's rule, which is denoted as s_n and given by

$$\int_a^b f(x) \approx \frac{\delta x}{2}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 ... + 4y_{n-1} + y_n) \text{ where } \delta x = \frac{b - a}{n}.$$

Note: The proof of Simpson's rule is left out intentionally

Example 5. Use Simpson's rule with $n = 6$ to estimate $\int_1^4 \sqrt{(1 - x^2)}dx$.

Solution: for $n = 6$, we have $\delta x = \frac{4 - 1}{6} = 0.5$, we now compute the values of $y_0, y_1, y_2, ... y_6$.

Consider the table below;

x	1	1.5	2.0	2.5	3.0	3.5	4.0
$y = \sqrt{(1 - x^2)}$	$\sqrt{2}$	$\sqrt{4.375}$	$\sqrt{3}$	$\sqrt{16.625}$	$\sqrt{28}$	$\sqrt{43.875}$	$\sqrt{65}$

Table: 3

therefore

$$\int_0^5 \sqrt{(1 - x^2)}dx \approx \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2(3) + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43.875} + \sqrt{65}\right) \approx$$
12.871

Exercises

1. Use Simpson's rule with 6 sub-intervals to estimate the following integrals

i. $\int_0^4 x^2 dx$

ii. $\int_0^2 e^{x^2} dx$

iii. $\int_0^2 e2x dx$

iv. $\int_0^1 \sqrt{1 - x^4} dx$

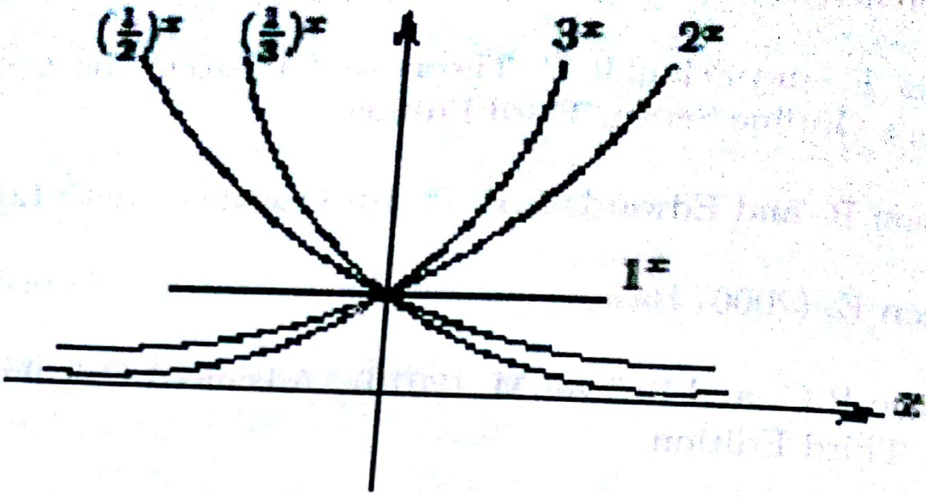


Figure 5: Exponential Function

The Graphs of Special Functions

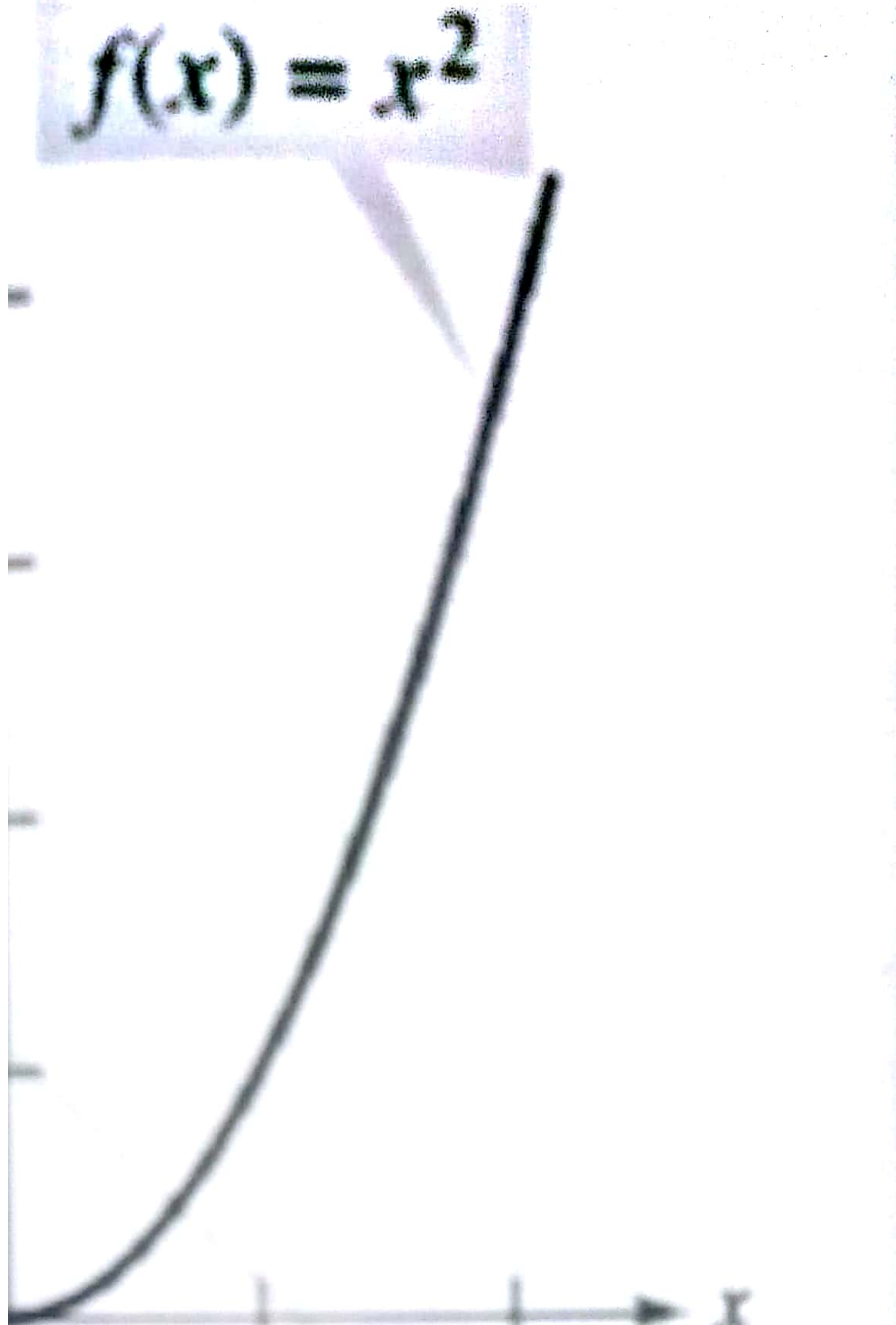
References

Ayres F. and Schmidt P. Theory and Problems on College Mathematics. Schaum's Outline Series, Third Edition.

Larson R. and Edwards B.H. (2010) Calculus. Ninth Edition.

Zakon E. (2006) Basics concept of mathematics. Second Edition

Wrede R.C. and Spiegel M. (2010). Advanced Calculus. Schaum's Outline Series, Third Edition.



$f(x) = x^2$
x

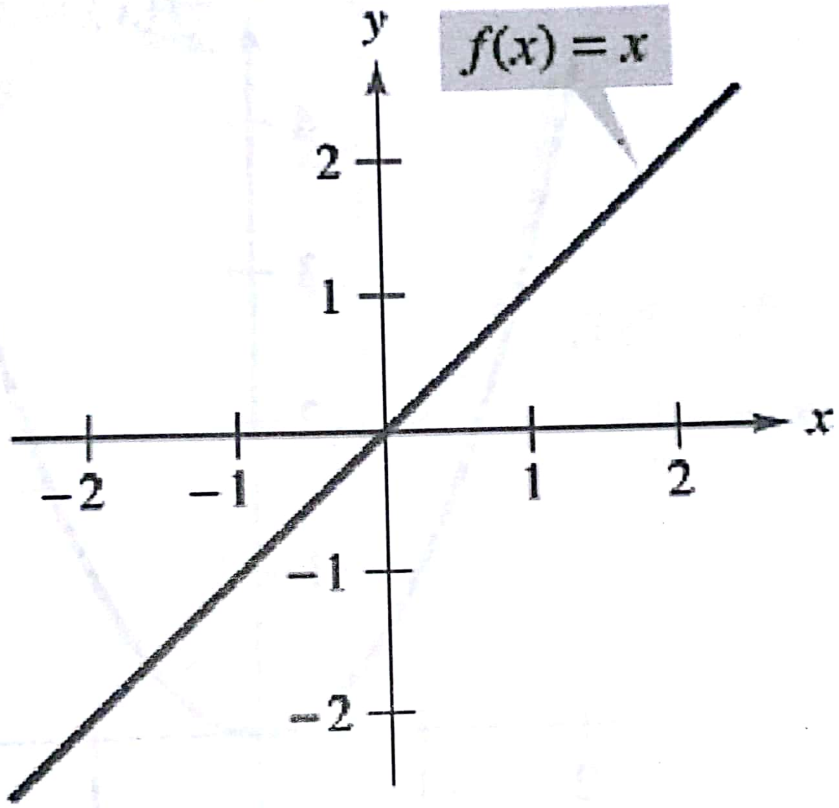


Figure 6: Identity Function

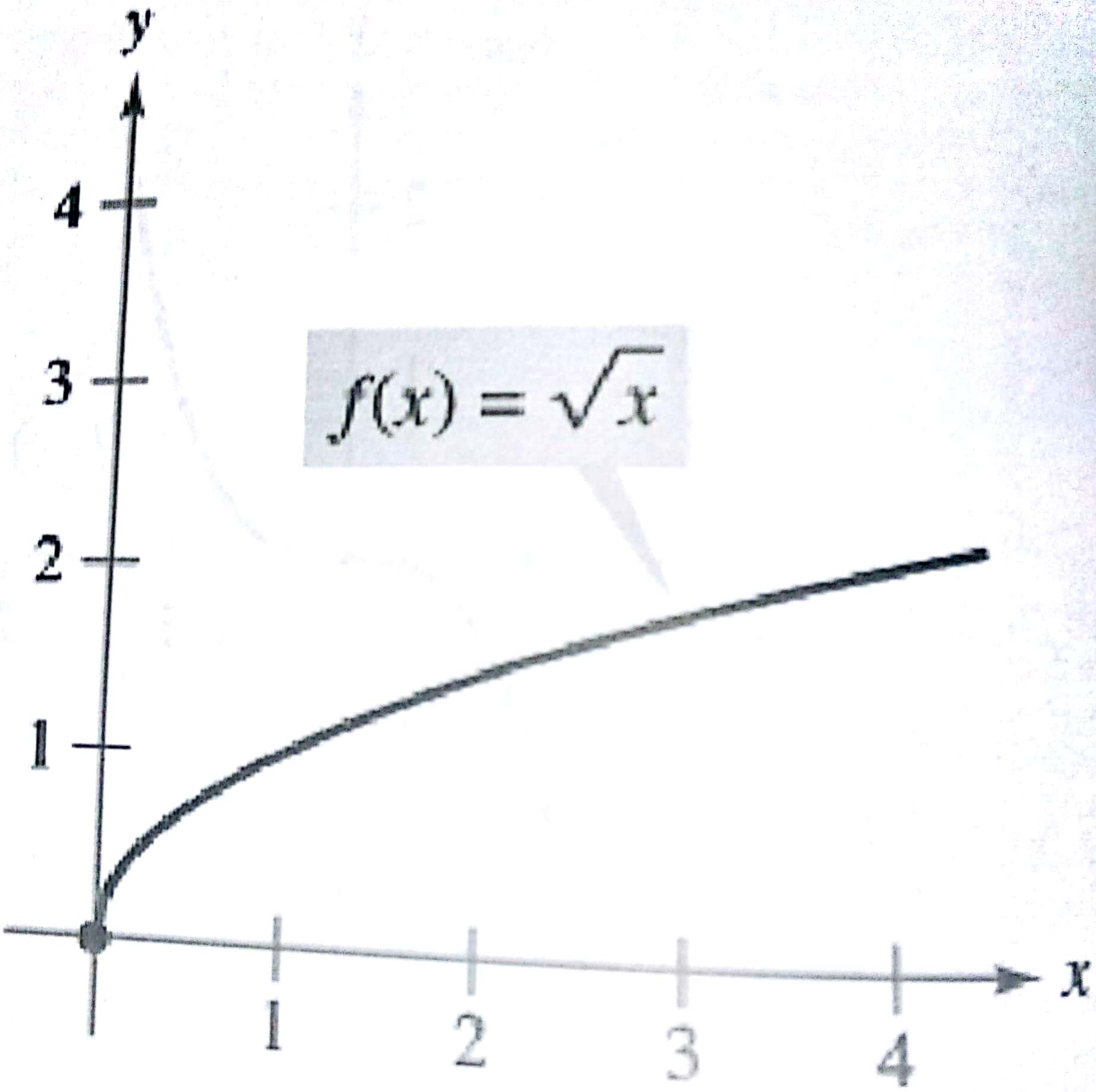


Figure 9: Square Root Function

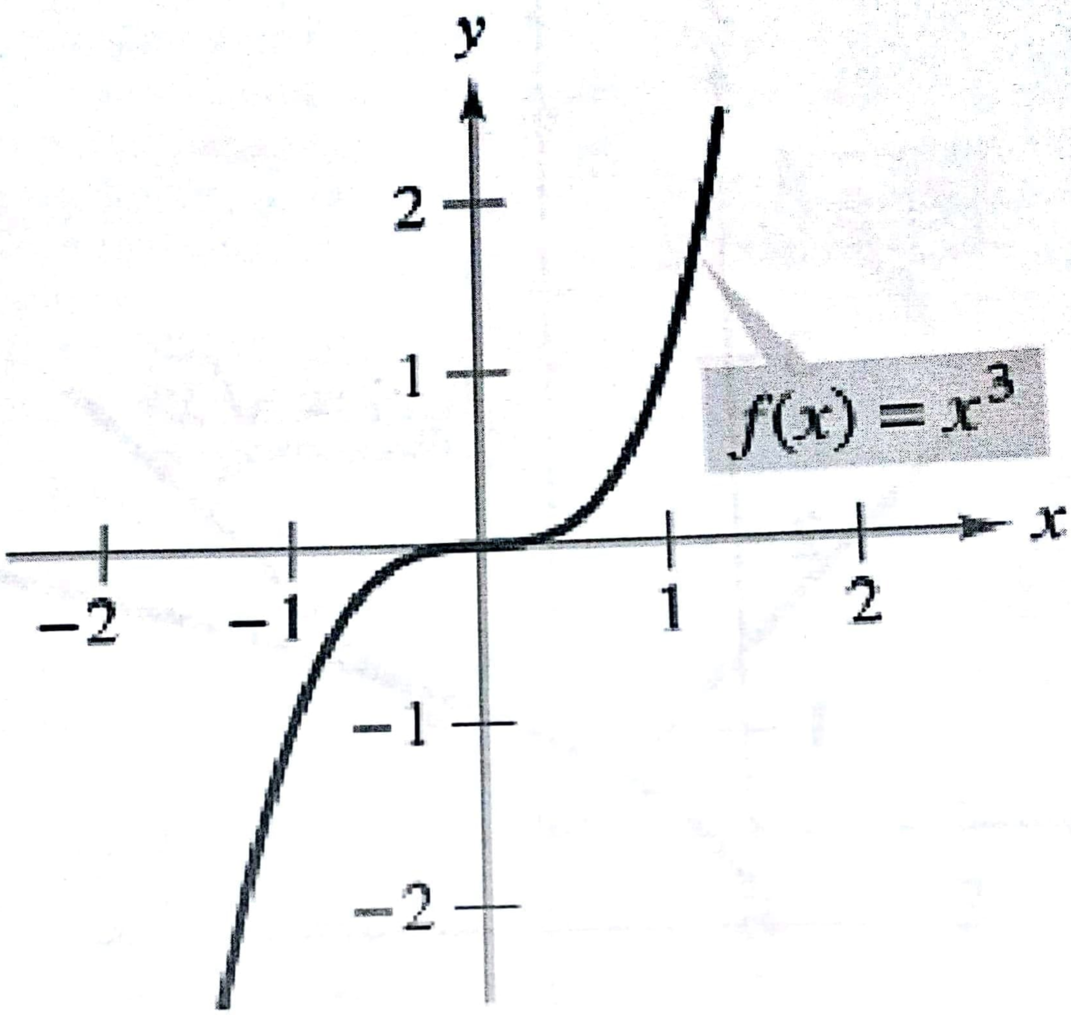
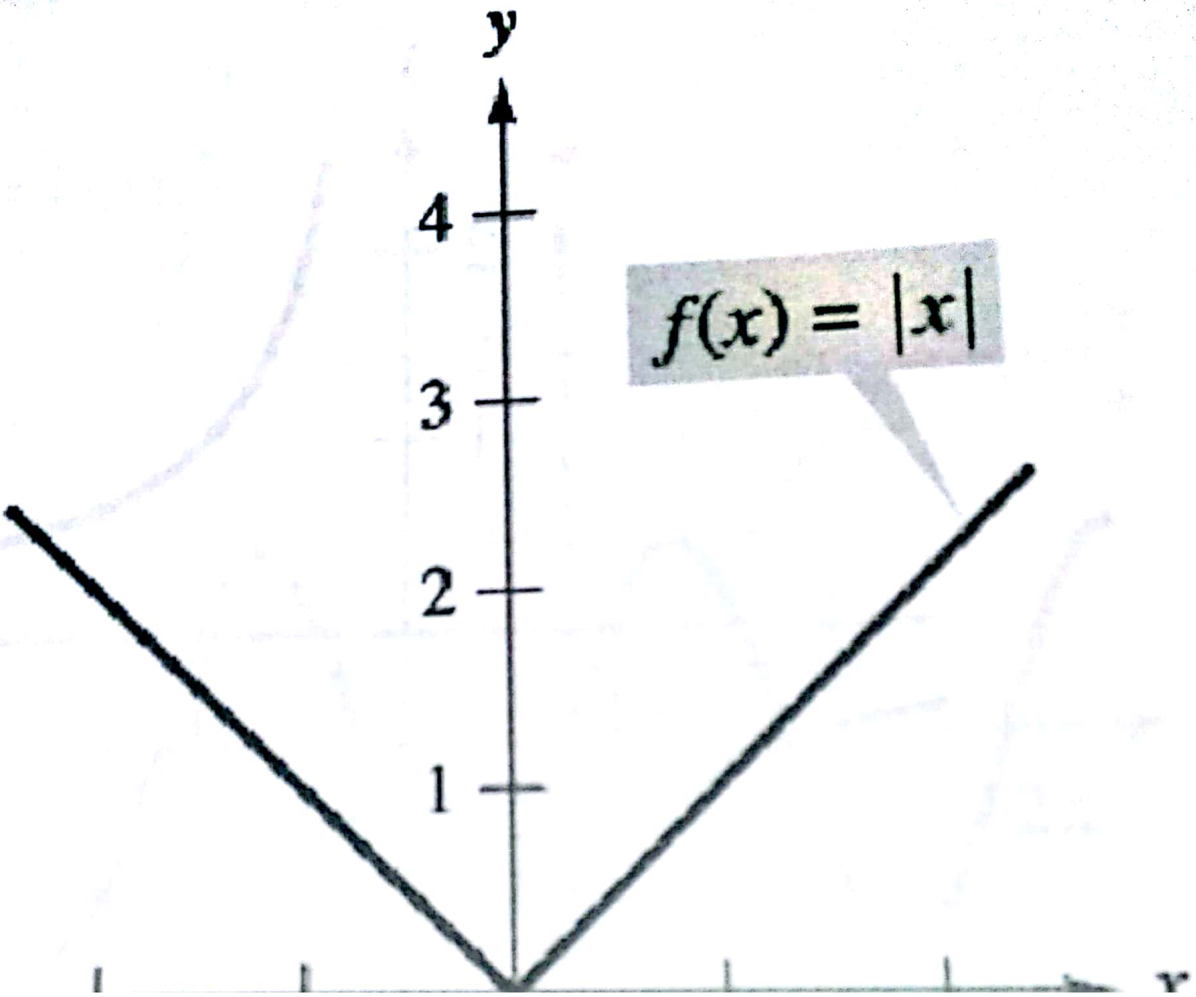
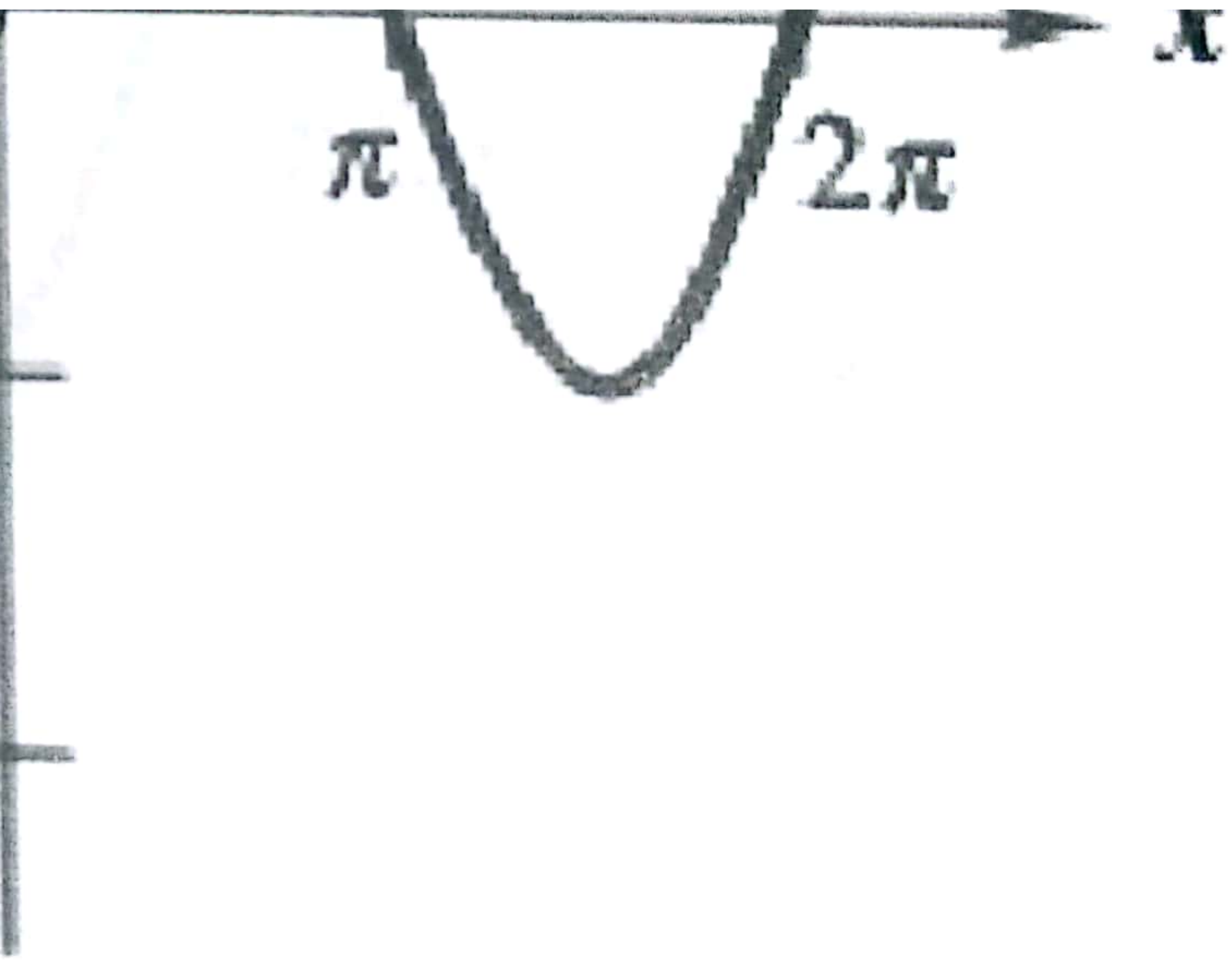


Figure 8: Cubing Function



y
4
3
2
1
$f(x) = |x|$
x



: Sine Function

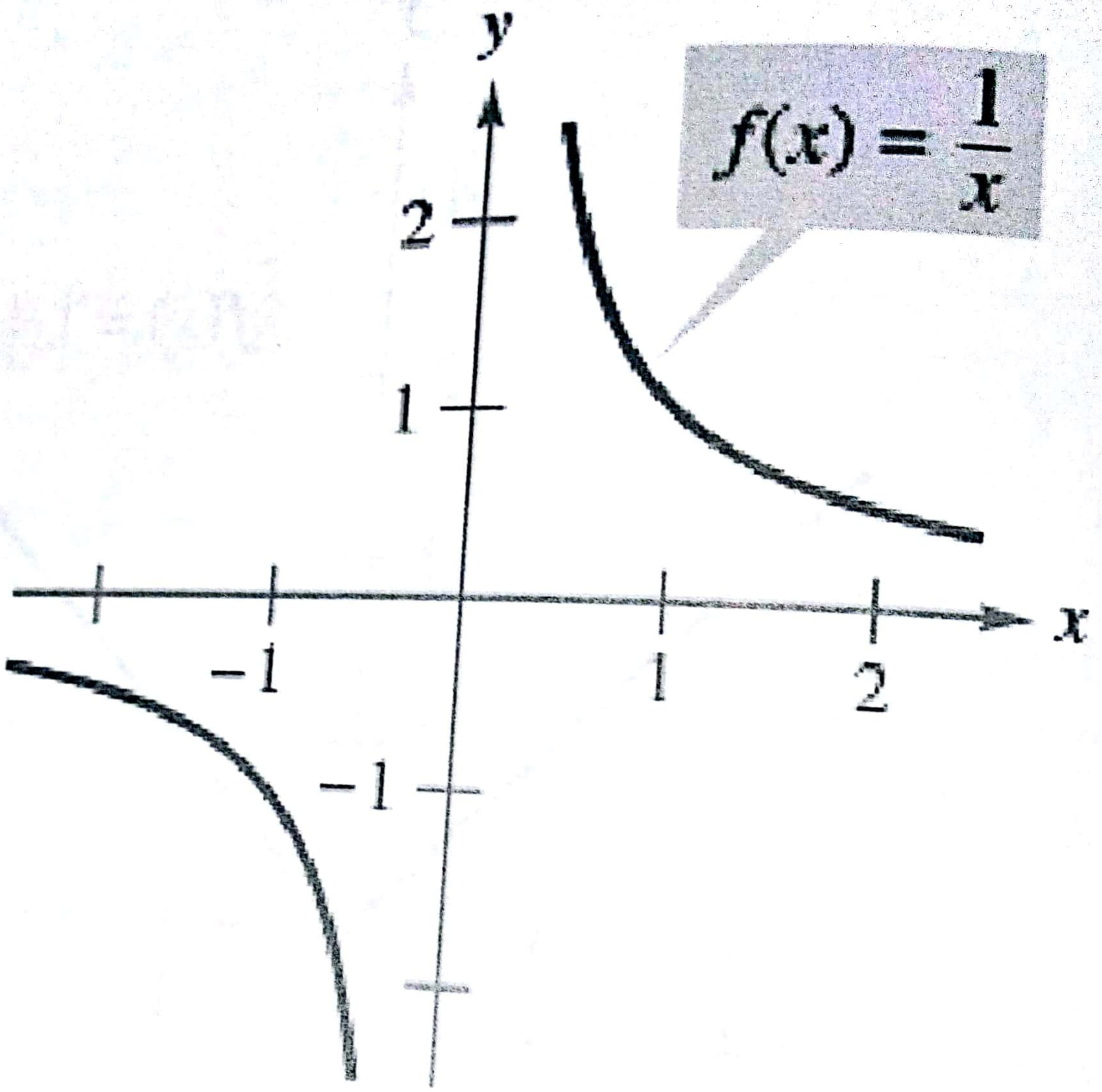


Figure 11: Rational Function

Scanned by CamScanner

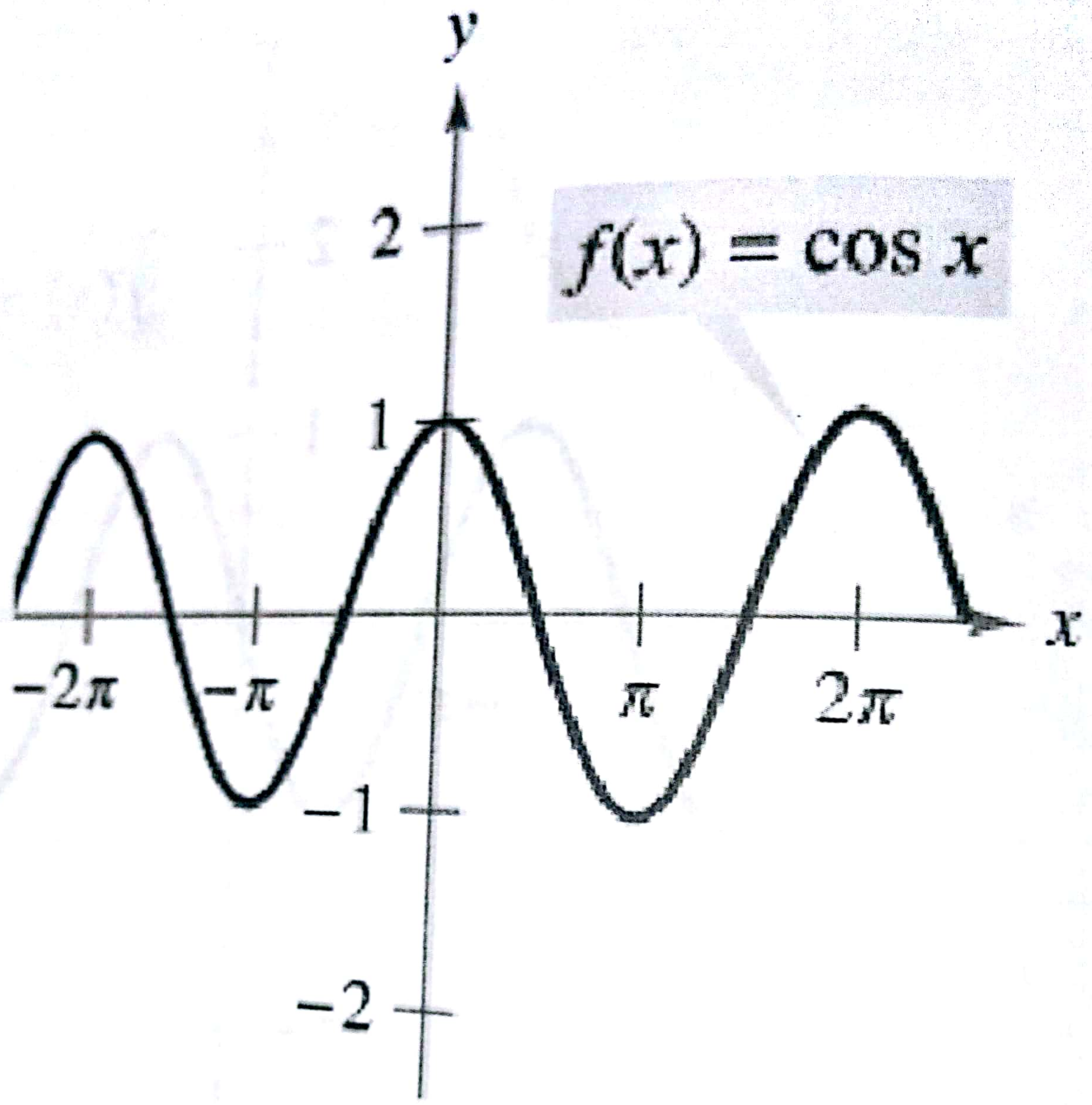


Figure 13: Cosine Function