Introduction to Bosonic String Theory

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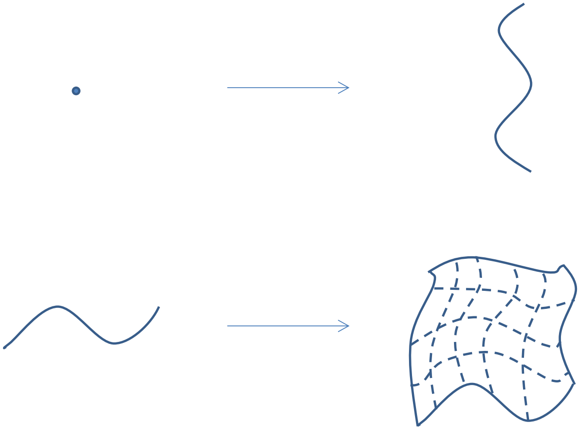
Requirements for a Physical Theory

1. Lorentz Invariance
2. Quantisation
3. Gravity

Worldlines and Worldsheets



String Theory as a Theory of Everything

1. Lorentz Invariance - Holds by construction
2. Quantisation - Holds by construction
3. Gravity - Appears as emergent phenomenon

GGRT: The Seminal Paper

Peter Goddard, Jeffrey Goldstone, Claudio Rebbi, Charles Thorn, *Quantum Dynamics of the Massless Relativistic String*, 1972, Nuclear Physics B56 109-135

Equations of Motion

- Classical, non-relativistic action
  - fails to satisfy Lorentz invariance
- Classical, relativistic action
  - leads to the equations of motion

$$
\begin{aligned}
&\frac{\partial}{\partial\tau}\left(\frac{T_0}{c}\left(\frac{(\dot{X}.X')X'_\mu-(X')^2\dot{X}_\mu}{\sqrt{(\dot{X}.X')^2-(\dot{X})^2(X')^2}}\right)\right)\\
&+\frac{\partial}{\partial\sigma}\left(\frac{T_0}{c}\left(\frac{(\dot{X}.X')\dot{X}_\mu-(X')^2X'_\mu}{\sqrt{(\dot{X}.X')^2-(\dot{X})^2(X')^2}}\right)\right) \quad = \quad 0
\end{aligned}
$$

The Light-Cone Gauge

- A preferred choice of direction
- $+$ Direction
  - $(1, 1, 0 \ldots 0)$
- - Direction
  - $(1, -1, 0 \ldots 0)$
- Can rewrite x^{μ} by $(x^{+}, x^{-}, x^{2}, x^{3} \ldots x^{d-1})$

Classical Equations of Motion

$$
\begin{aligned}
X^{+}(\tau,\sigma) &= 2\alpha' p^{+}\tau \\
X^{-}(\tau,\sigma) &= x_0^{-} + \sqrt{2\alpha'}\alpha_0^{-}\tau + i\sqrt{2\alpha'}\sum_{n=0}^{\infty}\frac{1}{n}\alpha_n^{-}e^{-in\tau}\cos n\sigma \\
X^{i}(\tau,\sigma) &= x_0^{i} + \sqrt{2\alpha'}\alpha_0^{i}\tau + i\sqrt{2\alpha'}\sum_{n=0}^{\infty}\frac{1}{n}\alpha_n^{i}e^{-in\tau}\cos n\sigma \\
\text{where } \alpha_n^{-} &= \frac{1}{2p^{+}\sqrt{2\alpha'}}\sum_{a\in\mathbb{Z}}\sum_{i=2}^{d-1}\alpha_{n-a}^{i}\alpha_a^{i}.
\end{aligned}
$$

Quantum Equations of Motion

$$
\begin{aligned}
\hat{X}^+(\tau,\sigma) &= 2\alpha' \hat{p}^+ \tau \\
\hat{X}^-(\tau,\sigma) &= \hat{x}_0^- + \sqrt{2\alpha'}\hat{\alpha}_0^- \tau + i\sqrt{2\alpha'} \sum_{n=0}^{\infty} \frac{1}{n} \hat{\alpha}_n^- e^{-in\tau} \cos n\sigma \\
\hat{X}^i(\tau,\sigma) &= \hat{x}_0^i + \sqrt{2\alpha'}\hat{\alpha}_0^i \tau + i\sqrt{2\alpha'} \sum_{n=0}^{\infty} \frac{1}{n} \hat{\alpha}_n^i e^{-in\tau} \cos n\sigma \\
\text{where } \hat{\alpha}_n^- &= \frac{1}{2\hat{p}^+\sqrt{2\alpha'}} \sum_{a\in\mathbb{Z}} \sum_{i=2}^{d-1} \hat{\alpha}_{n-a}^i \hat{\alpha}_a^i .
\end{aligned}
$$

Possible Existence of Hidden Dimensions

- Bosonic string theory predicts 26 spacetime dimensions
- More modern string theories predict 10 or 11 spacetime dimensions
- Trapped subspace
- Compact dimensions

The Square Well Problem

- One dimensional space
- Infinite potential outside $(0, a)$
- Zero potential inside $(0, a)$
- Schrödinger equation
  - $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$
- Solutions
  - $\psi^p(x) = b_p \sin\frac{p\pi x}{a}$
- Energies
  - $E_p = \frac{\hbar^2 p^2 \pi^2}{2ma^2}$

Small compact dimensions

- Two-dimensional spacetime
    - One observed, normal dimension, x
    - One compact hidden dimension, y, of length $2\pi r$
- Zero potential for $x \in (0, a)$
- Infinite potential for $x \notin (0, a)$

Small compact dimensions

- Schrödinger equation
  - $-\frac{\hbar^2}{2m}(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}) = E\psi$
- Assume $\psi(x, y) = \phi_1(x)\phi_2(y)$
- By separation of variables
  - $-\frac{\hbar^2}{2m}\frac{d^2\phi_1}{dx^2} = E_1\phi_1(x)$
  - $-\frac{\hbar^2}{2m}\frac{d^2\phi_2}{dy^2} = E_2\phi_2(y)$
- Solutions
  - $\phi_1^p(x) = b_p \sin \frac{p\pi x}{a}$
  - $\phi_2^q(y) = c_q \sin \frac{qy}{r} + d_q \cos \frac{qy}{r}$

Energies

- $q = 0$
  - $E_{p0} = \frac{\hbar^2 p^2 \pi^2}{2ma^2}$
- q $= 1$
  - $E_{p1} = \frac{\hbar^2}{2m}(\frac{p^2\pi^2}{a^2} + \frac{1}{r^2})$
- Scales based off scale of a and r

References

- Peter Goddard, Jeffrey Goldstone, Claudio Rebbi, Charles Thorn, *Quantum Dynamics of the Massless Relativistic String*, 1972, Nuclear Physics B56 109-135
- Barton Zwiebach, *A First Course in String Theory*, 2009, Cambridge University Press