A new approach to prove the $3n+1$ conjecture

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Abstract

In this paper, we provide the framework for a proof of the $3n+1$ conjecture by examining the $3n+3$ conjecture (all sequences to terminate at 3). We do this by applying a law for even numbers generated by any $3n+k$ (k is odd), sequence and with two more laws for the $3n+3$ sequences in particular. In Claim 2 below, there is a simple formula to construct any $3n+1$ sequence from its corresponding $3n+3$ sequence.

1 Preliminaries and Definitions

Definition 1.1. *A* $3n+1$ *sequence starts with any positive integer* n *where each term is obtained from the previous term as follows: if the previous term is even, the next term is one half of the previous term. If the previous term is odd, the next term is 3 times the previous term plus 1. We call a sequence formed in this manner a c-sequence.*

Definition 1.2. *A* $3n+1$ *sequence is cyclic if the production of its terms are such that we return to the first term. A sequence is partial-cyclic or p-cyclic, if the production of its terms are such that we return to some term of the sequence which is not the first term (we shall refer to either as just cyclic). A sequence is trivial p-cyclic or* $\mathcal{T}$*-cyclic, if 1 is a term in the repeated sub-sequence.*

Note that if a c-sequence is trivially p-cyclic, we mean that after reaching 1 and we were to continue the process, we get: $1 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1$, and so on. However, we say it terminates because 1 was produced.

Definition 1.3. *A* $3n+3$ *sequence starts with any positive integer* n *where each term is obtained from the previous term as follows: if the previous term is even, the next term is one half of the previous term. If the previous term is odd, the next term is 3 times the previous term plus 3. We call a sequence formed in this manner a c-sequence.*

Similarly, a $3n+3$ sequence is $\mathcal{T}$-cyclic, if 3 is a term in the repeated subsequence. And if a c-sequence is trivially p-cyclic, we mean that after reaching 3 and we continued the process, we get: $3 \longrightarrow 12 \longrightarrow 6 \longrightarrow 3 \longrightarrow 12 \longrightarrow 6 \longrightarrow 3$, and so on. However, we say it terminates because 3 was produced.

We shall denote three terms for a c-sequence. First, the initial term of the sequence by $\mathbf{n} \in \mathbb{N}_{>2}$. A term that is not the first term and is odd by N_i. And an even term that is of the form: $3(N_i)+3$ by M_i. Note well that the indexing is not meant to track steps in a c-sequence, but rather for association only. It will be seen that roughly two thirds of the numbers generated in a sequence, are not relevant. That is, we are only interested in the numbers defined as M_i.

Definition 1.4. *A* $\mathcal{C}$*-sequence is the sequence of all the even numbers from a c-sequence of the form:* $3(N_i)+3 = M_i \ \forall i \in \mathbb{N}_{>0}$ *where* $N_i \in \mathbb{N}_{odd}$.

Given a $\mathcal{C}$-sequence, its c-sequence may be constructed with just division by 2.

Definition 1.5. *A* $\mathcal{D}$*-sequence is constructed from a* $\mathcal{C}$*-sequence by* M_i+6 $\forall i \in \mathbb{N}_{>0}$. *The terms of a* $\mathcal{D}$*-sequence shall be denoted by* t_i.

Conceptually, the terms of the $\mathcal{D}$-sequence are the M_i's produced from $3(N_i+2)+3$, including $\mathbf{n}+2$. Note that odd $\mathbf{n} = N_1$. The $\mathcal{D}$-sequence allows us to observe the $\mathcal{C}$-sequence from which it was derived in a most revealing manner.

Law 1: M_i numbers generated by 3n+k sequences

For any $3n+k$ sequence, an M_i can have only one of three *designations* as follows:

i) If M_i is divisible by 2, two or more times, then it's $\overset{\bullet}{M}_i$. Moreover, it will always be such that $M_i{+}6$ and $M_i{-}6$ will be divisible by 2 exactly once, in this case.

ii) If M_i is divisible by 2 exactly once and $M_i{-}6$ is divisible by 2, three or more times, then it's $\overset{\blacksquare}{M}_i$. Moreover, it will always be such that $M_i{+}6$ will be divisible by 2 exactly twice, in this case.

iii) If M_i is divisible by 2 exactly once and $M_i{+}6$ is divisible by 2, three or more times, then it's $\overset{\blacktriangle}{M}_i$. Moreover, it will always be such that $M_i{-}6$ will be divisible by 2 exactly twice, in this case.

Every M_i must be one of the three *designations* or *desig*, listed above. The *desig* cases are invariant which is established easily and appears in Appendix 3 (the law applies to all even numbers > 6). Note well that we use the *desig* coloured circle, square and triangle with the terms, t_i of the $\mathcal{D}$-sequence and to the left of down pointing arrows between an M_i and its corresponding t_i in our example for $\mathbf{n} = 53$. It's to be understood that the *desig* is always for M_i and that it's just an association with $t_i = M_i{+}6$. It will become clear shortly, as to why we make such associations.

Definition 1.6. *An R-subsequence (R-sub), begins with a* $\overset{\blacktriangle}{t}_i$ *if* t_{i-1} *is* $\overset{\bullet}{t}_{i-1}$ *or* $i = 1$*, and ends with the last term* $\overset{\bullet}{t}_k$ *preceding* $\overset{\blacktriangle}{t}_{k+1}$*. Or the* $\mathcal{C}$*-sequence terminated at* $\overset{\bullet}{M}_k = 48$*. An R-sub has two parts, called the head and the tail. The head contains only* $\overset{\blacktriangle}{t}_i$ *terms and the tail contains only* $\overset{\blacksquare}{t}_j$ *and* $\overset{\bullet}{t}_k$ *terms.*

R-subs are always contiguous and we index as: R-sub$_1$, R-sub$_2$, R-sub$_3$, … . A $\mathcal{D}$-sequence could have initially, terms: $\overset{\bullet}{t}_1$, $\overset{\bullet}{t}_2$, $\overset{\blacksquare}{t}_3$, $\overset{\bullet}{t}_4$, $\overset{\blacktriangle}{t}_5$, for example. Its corresponding $\mathcal{C}$-sequence is of course: $\overset{\bullet}{M}_1$, $\overset{\bullet}{M}_2$, $\overset{\blacksquare}{M}_3$, $\overset{\bullet}{M}_4$, $\overset{\blacktriangle}{M}_5$. In which case the first R-sub begins with $\overset{\blacktriangle}{t}_5$. Some $\mathcal{C}$-sequences may have no $\overset{\blacktriangle}{M}_i$ terms at all, and would terminate in a trivial manner (see section: Tails).

The $3n{+}3$ sequences have a particular *ordering* wrt. their desigs. Specifically, if the i^{th} term is $\overset{\blacktriangle}{t}_i$ then the $i{+}1^{th}$ term can only be another $\overset{\blacktriangle}{t}_{i+1}$ or $\overset{\blacksquare}{t}_{i+1}$. And the $i{-}1^{th}$ term can only be another $\overset{\blacktriangle}{t}_{i-1}$ or $\overset{\bullet}{t}_{i-1}$. If the i^{th} term is $\overset{\blacksquare}{t}_i$ then the $i{+}1^{th}$ term can only be $\overset{\bullet}{t}_{i+1}$. And the $i{-}1^{th}$ term can only be $\overset{\blacktriangle}{t}_{i-1}$ or $\overset{\bullet}{t}_{i-1}$. Lastly, if the i^{th} term is $\overset{\bullet}{t}_i$ then the $i{+}1^{th}$ term can be another $\overset{\bullet}{t}_{i+1}$ or $\overset{\blacksquare}{t}_{i+1}$ or $\overset{\blacktriangle}{t}_i$. The $i{-}1^{th}$ term can only be another $\overset{\bullet}{t}_{i-1}$ or $\overset{\blacksquare}{t}_{i-1}$. (proofs in Appendix 3)

Note that the *ordering* wrt. their desigs of the $3n{+}1$ sequences is the same as the above except that the $\overset{\blacktriangle}{t}$ and $\overset{\blacksquare}{t}$ desigs are switched with each other. It is the case that all $3n{+}k$ sequences where $k = 1, 5, 9, 13, \dots$ are the same ordering which we'll call 1^{st} *ordering*. And all $3n{+}k$ sequences where $k = 3, 7, 11, 15, \dots$ are the same ordering which we'll call 2^{nd} *ordering*.

Law 2: 2^a3^bR structure for 3n+3 sequences

All $3n{+}3$ sequences are such that every M_i is divisible by at least one 2 and one 3. Again, their $\mathcal{C}$-sequences' corresponding $\mathcal{D}$-sequences reveal far more structure. Specifically, every t_i can be written as: 2^a3^bR, where $a, b, R \in \mathbb{N}_{>0}$. R is the remainder, always odd, not divisible by 3 and not necessarily prime. Moreover, if $t_i = 2^a3^bR$, where $a > 1$, then t_{i+1} must be $2^{a-1}3^{b+1}R$. That is, if $M_i{+}6 = t_i = 2^a3^bR$ then $M_{i+1}{+}6 = t_{i+1} = 2^{a-1}3^{b+1}R$, if $a > 1$. We will distinguish a different R with R' or S.

In general, we have: $2^a3^bR \longrightarrow 2^{a-1}3^{b+1}R$ where $a, b,\ a{-}1,\ b{+}1 \in \mathbb{N}_{>0}$. In fact, every 2^a3^bR where $a \geq 2$ must continue until we have: $2^13^{b+a-1}R = t_k$ where M_k is always $\overset{\bullet}{M}_k$. And, if t_i is $\overset{\blacktriangle}{t}_i = M_i{+}6$, where $\overset{\blacktriangle}{t}_i = 2^33^bR$, then t_{i+1} is $\overset{\blacksquare}{t}_{i+1} = M_{i+1}{+}6 = 2^23^{b+1}R$. It must follow that t_{i+2} is $\overset{\bullet}{t}_{i+2} = M_{i+2}{+}6 = 2^13^{b+2}R$. This implies that there are no consecutive terms as: $\overset{\blacksquare}{t}_i$, $\overset{\blacksquare}{t}_{i+1}$. So any tail can be at best, an alternating sequence as: $\overset{\blacksquare}{t}_i$, $\overset{\bullet}{t}_{i+1}$, $\overset{\blacksquare}{t}_{i+2}$, $\overset{\bullet}{t}_{i+3}$, …(proofs in Appendix 3) Since $3n{+}3$ sequence generation can be associated to 2^a3^bR representations for every M_i, we can by implication, determine what sub-sequences are permissable. Below, is an example showing the $\mathcal{C}$-sequence and its corresponding $\mathcal{D}$-sequence for **n**= 53 where $M_1 = 3(53){+}3 = 162$. Observe that the sequence begins with an R-sub and in fact there are eight (contiguous), in all. They are explicitly itemized in Appendix 1.

Example: $\mathbf{n} = 53,\ M_1 = 3(53){+}3 = 162$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 162 \arrow[r] \arrow[d, "\blacktriangle"'] & 246 \arrow[r] \arrow[d, "\blacksquare"'] & 372 \arrow[r] \arrow[d, "\bullet"'] & 282 \arrow[r] \arrow[d, "\blacktriangle"'] & 426 \arrow[r] \arrow[d, "\blacktriangle"'] & 642 \arrow[r] \arrow[d, "\blacktriangle"'] & 966 \arrow[r] \arrow[d, "\blacksquare"'] & 1452 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^3 3^1 \mathbf{7} & 2^2 3^2 \mathbf{7} & 2^1 3^3 \mathbf{7} & 2^5 3^2 \mathbf{1} & 2^4 3^3 \mathbf{1} & 2^3 3^4 \mathbf{1} & 2^2 3^5 \mathbf{1} & 2^1 3^6 \mathbf{1} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 1092 \arrow[r] \arrow[d, "\bullet"'] & 822 \arrow[r] \arrow[d, "\blacksquare"'] & 1236 \arrow[r] \arrow[d, "\bullet"'] & 930 \arrow[r] \arrow[d, "\blacktriangle"'] & 1398 \arrow[r] \arrow[d, "\blacksquare"'] & 2100 \arrow[r] \arrow[d, "\bullet"'] & 1578 \arrow[r] \arrow[d, "\blacktriangle"'] & 2370 \arrow[r] \arrow[d, "\blacktriangle"'] & {} \\
\mathcal{D}\rightarrow 2^1 3^2 \mathbf{61} & 2^2 3^2 \mathbf{23} & 2^1 3^3 \mathbf{23} & 2^3 3^2 \mathbf{13} & 2^2 3^3 \mathbf{13} & 2^1 3^4 \mathbf{13} & 2^4 3^2 \mathbf{11} & 2^3 3^3 \mathbf{11} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 3558 \arrow[r] \arrow[d, "\blacksquare"'] & 5340 \arrow[r] \arrow[d, "\bullet"'] & 4008 \arrow[r] \arrow[d, "\bullet"'] & 1506 \arrow[r] \arrow[d, "\blacktriangle"'] & 2262 \arrow[r] \arrow[d, "\blacksquare"'] & 3396 \arrow[r] \arrow[d, "\bullet"'] & 2550 \arrow[r] \arrow[d, "\blacksquare"'] & 3828 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^2 3^4 \mathbf{11} & 2^1 3^5 \mathbf{11} & 2^1 3^2 \mathbf{223} & 2^3 3^3 \mathbf{7} & 2^2 3^4 \mathbf{7} & 2^1 3^5 \mathbf{7} & 2^2 3^2 \mathbf{71} & 2^1 3^3 \mathbf{71} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 2874 \arrow[r] \arrow[d, "\blacktriangle"'] & 4314 \arrow[r] \arrow[d, "\blacktriangle"'] & 6474 \arrow[r] \arrow[d, "\blacktriangle"'] & 9714 \arrow[r] \arrow[d, "\blacktriangle"'] & 14574 \arrow[r] \arrow[d, "\blacksquare"'] & 21864 \arrow[r] \arrow[d, "\bullet"'] & 8202 \arrow[r] \arrow[d, "\blacktriangle"'] & {} \\
\mathcal{D}\rightarrow 2^6 3^2 \mathbf{5} & 2^5 3^3 \mathbf{5} & 2^4 3^4 \mathbf{5} & 2^3 3^5 \mathbf{5} & 2^2 3^6 \mathbf{5} & 2^1 3^7 \mathbf{5} & 2^4 3^3 \mathbf{19} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 12306 \arrow[r] \arrow[d, "\blacktriangle"'] & 18462 \arrow[r] \arrow[d, "\blacksquare"'] & 27696 \arrow[r] \arrow[d, "\bullet"'] & 5196 \arrow[r] \arrow[d, "\bullet"'] & 3900 \arrow[r] \arrow[d, "\bullet"'] & 2928 \arrow[r] \arrow[d, "\bullet"'] & 552 \arrow[r] \arrow[d, "\bullet"'] & {} \\
\mathcal{D}\rightarrow 2^3 3^4 \mathbf{19} & 2^2 3^5 \mathbf{19} & 2^1 3^6 \mathbf{19} & 2^1 3^2 \mathbf{289} & 2^1 3^2 \mathbf{217} & 2^1 3^2 \mathbf{163} & 2^1 3^2 \mathbf{31} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}\rightarrow 210 \arrow[r] \arrow[d, "\blacktriangle"'] & 318 \arrow[r] \arrow[d, "\blacksquare"'] & 480 \arrow[r] \arrow[d, "\bullet"'] & 48 \arrow[d, "\bullet"'] \\
\mathcal{D}\rightarrow 2^3 3^3 \mathbf{1} & 2^2 3^4 \mathbf{1} & 2^1 3^5 \mathbf{1} & 2^1 3^3 \mathbf{1}
\end{tikzcd}$$

Note well that $\mathcal{C}$-sequences and $\mathcal{D}$-sequences may be constructed for any $3n+k$ sequence. Their M_i terms could be written in the form: $2^a 3^b R$ if k is divisible by 3 or just $2^a R$, otherwise. Additionally, there are structural patterns wrt. 1^{st} *ordering*, if the t_i terms in their $\mathcal{D}$-sequences were defined instead as: $\overset{\blacktriangle}{t}_i = \overset{\blacktriangle}{M}_i+6$, $\overset{\blacksquare}{t}_i = \overset{\blacksquare}{M}_i-6$ and $\overset{\bullet}{t}_i = \overset{\bullet}{M}_i$.

Definition 1.7. *Given an R-sub, there exists a first (f), term* $\overset{\blacktriangle}{M}_f$ *and a last (l), term* $\overset{\bullet}{M}_l$*. Then let* $X \in \mathbb{N}_{odd}$ *be* $\overset{\blacktriangle}{M}_f$ *divided by* 2 *(there is only one multiple of* 2 *by law 1). And let* $Y \in \mathbb{N}_{odd}$ *be* $\overset{\bullet}{M}_l$ *divided by* 2^r*, where* $r \geq 2$ *(by definition). Then an* R^X*-sub is an R-sub such that* $\frac{2}{3}X < Y$*. And an* R^Y*-sub is an R-sub such that* $Y < \frac{2}{3}X$*. We denote an* R^X*-sub*$_i$ *as:* $(h, t \mid \overset{\blacktriangle}{t}_f = 2^a 3^b R \mid \overset{\bullet}{t}_l = 2^c 3^d R' or R)$ *is the* i^{th} *R-sub such that* $\frac{2}{3}X_i < Y_i$*, where h (head), is the number of* $\overset{\blacktriangle}{t}$ *terms and t (tail), is the number of* $\overset{\blacksquare}{t}$ *and* $\overset{\bullet}{t}$ *terms for this R-sub.*

Similarly, we denote an R^Y*-sub*$_i$ *as:* $(h, t \mid \overset{\blacktriangle}{t}_f = 2^a 3^b R \mid \overset{\bullet}{t}_l = 2^c 3^d R' or R)$ *is the* i^{th} *R-sub such that* $Y_i < \frac{2}{3}X_i$*.*

Tails

If the M_i terms in a $\mathcal{C}$-sequence corresponding to a *tail* (for an R-sub in the sequence), do not end with $\overset{\bullet}{M}_k = 48$, then M_{k+1} will have $\overset{\blacktriangle}{M}_{k+1}$ designation. Moreover, every consecutive pair of $\overset{\blacksquare}{M} \rightarrow \overset{\bullet}{M}$ terms have a property that distinguishes such pairs from one another. If N_{i+1} of $\overset{\blacksquare}{M}_i = X$ and N_{i+2} of $\overset{\bullet}{M}_{i+1} = Y$ and $|X-Y|$ is divisible by 2, exactly once, the term M_{i+2} must be $\overset{\blacktriangle}{M}_{i+2}$ or $\overset{\blacksquare}{M}_{i+2} > \overset{\blacksquare}{M}_i$. Or, if $|X-Y|$ is divisible by 4 or more, the term M_{i+2} must be $\overset{\bullet}{M}_{i+2}$. If N_{i+1} of $\overset{\bullet}{M}_i = X$ and N_{i+2} of $\overset{\bullet}{M}_{i+1} = Y$ and $|X-Y|$ is divisible by 2, exactly once, the term M_{i+2} must be $\overset{\blacktriangle}{M}_{i+2}$ or $\overset{\blacksquare}{M}_{i+2}$. Or, if $|X-Y|$ is divisible by 4 or more, the term M_{i+2} must be $\overset{\bullet}{M}_{i+2}$.

Finally, if N_{i+1} of $\overset{\blacksquare}{M}_i$ or $\overset{\bullet}{M}_i = X$ and N_{j+1} of $\overset{\blacksquare}{M}_j$ or $\overset{\bullet}{M}_j = Y$, $j > i+1$, and $|X-Y|$ is divisible by 2, exactly once, the term M_{j+1} must be $\overset{\blacktriangle}{M}_{j+1}$ or $\overset{\blacksquare}{M}_{j+1}$. And if $|X-Y|$ is divisible by 4 or more, the term M_{j+1} must be $\overset{\bullet}{M}_{j+1}$ or $\overset{\blacktriangle}{M}_{j+1}$.

In general, let a *tail* or a portion of a *tail* be: $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \ldots \to \overset{\blacksquare}{M}_j \to \overset{\bullet}{M}_{j+1} \to \ldots \to \overset{\bullet}{M}_{j+k} \to \overset{\blacktriangle}{M}_{j+k+1}$ or $\overset{\blacksquare}{M}_{j+k+1} \to \overset{\bullet}{M}_{j+k+2}$, if N_{j+1} of $\overset{\blacksquare}{M}_j = X$ and N_{j+2} of $\overset{\bullet}{M}_{j+1} = Y$ are st. $|X-Y|$ is divisible by 4 or more. Then $\frac{2}{3}(N_{i+1}) > N_{j+k+1}$.

We shall call this the *cascade* of $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \ldots \to \overset{\blacksquare}{M}_j \to \overset{\bullet}{M}_{j+1}$ alternating sequences, when X, Y of $\overset{\blacksquare}{M}_j$, $\overset{\bullet}{M}_{j+1}$, respectively, are such that $|X-Y|$ is divisible by 4 or more. Note that M_{i-1} could be $\overset{\blacktriangle}{M}_{i-1}$ or $\overset{\bullet}{M}_{i-1}$ (with $\overset{\bullet}{M}_{i-2}$).

We now define three particular R-sub forms that play a crucial role in proving that all $3n$+3 sequences are $\mathcal{T}$-cyclic.

Definition 1.8. *An H-form R-sub or H_F is denoted by* $(h, 2 \mid 2^a \overset{\blacktriangle}{3^b} R \mid 2^1 \overset{\bullet}{3^{a+b-1}} R)$*, a tail of length 2, and an L-form R-sub or L_F is denoted by* $(1, t \mid 2^3 \overset{\blacktriangle}{3^2} R \mid 2^1 \overset{\bullet}{3^2} R')$*, a head of length 1. And a maximum* $\overset{max}{H}$*-form or* $\overset{max}{H}_F$ *is denoted by* $(h, 2 \mid 2^a \overset{\blacktriangle}{3^1} R \mid 2^1 \overset{\bullet}{3^a} R)$*. ie.* 3^b *where* b=1.

Definition 1.9. *For* $3n$+3*, if* N_i *(odd), is such that* $2(N_i) - 3$ *is not divisible by* 3 *and* $4(N_i) - 3$ *is not divisible by* 3*, then* N_i *is a dead-ender.*

A *dead−ender* cannot appear in a c-sequence as any N_i, $i > 1$, because it cannot be generated, by which we mean no N_{i-1} exists. It follows that the $\overset{\blacktriangle}{M}_f$ term for any $\overset{\textbf{max}}{H}$-form is derived from a *dead−ender*. That is, a *dead−ender* can only be N_1 where $3(N_1)+3 = \overset{\blacktriangle}{M}_f$ of an R-sub$_1$. This implies that such an $\overset{\blacktriangle}{M}_f$ cannot be an M_i of a cycle.

Non-trivial cycles and 3n+3

For a non-trivial cycle to exist, one criteria must be met wrt. the R-sub structure for a given *ordering*. In particular, if M_{i-1} is not $\overset{\blacktriangle}{M}_{i-1}$ when M_i is $\overset{\blacktriangle}{M}_i$, then it must be: $\overset{\bullet}{M}_{i-1} \longrightarrow \overset{\blacktriangle}{M}_i$. This is a 2^{nd}*ordering* fact. We shall define a cycle of size o, an o-cycle, to be the number of M_i terms in the cycle. Of course, when considering all the terms of its associated c-sequence, the cycle size is necessarily $> o$. Then, with this definition of a cycle, we say that $\mathcal{T}$-cyclic, is a trivial 1-cycle with a term $N_i \le k$ for a $3n+k$ sequence.

Claim 1: No $3n+3$ sequence has an o-cycle, $o \ge 2$ for any 2^{nd}*ordering* cycle class:

1) Among the M_i terms of an o-cycle where $o \ge 2$, there does not exist an $\overset{\blacktriangle}{M}_i$ term. ie. the o-cycle does not contain an R-sub.

2) The M_i terms of an o-cycle where $o \ge 2$, are such that some R-sub or no R-sub, has a term $\overset{\blacktriangle}{t}_i = 2^a 3^1 R$ where $a \ge 3$, $b = 1$, or a term $\overset{\blacksquare}{t}_i = 2^2 3^1 R$ where $a = 2$, $b = 1$ and for both, $R \in \mathbb{N}_{odd}$ and not divisible by 3.

3) Among the M_i terms of an o-cycle where $o \ge 3$, there exists M_i, M_j such that: $\to M_i \to M_{i+1} \to \cdots \to M_j \to \cdots \to$ where $\overset{\blacktriangle}{M}_i \pm 6 = \overset{\bullet}{M}_j$ or $\overset{\blacksquare}{M}_i \pm 6 = \overset{\bullet}{M}_j$.

4) An o-cycle where none of the M_i terms are divisible by 3.

Note that an o-cycle may belong to more than one class.

Proof: First, we observe that there is no 2-cycle for any $3n+k$ sequence of the form: $\overset{\bullet}{M}_i \to \overset{\bullet}{M}_{i+1}$. Simply, if $\overset{\bullet}{M}_i > \overset{\bullet}{M}_{i+1}$ say, then it's not possible for $\overset{\bullet}{M}_{i+1} \to \overset{\bullet}{M}_i$ to occur because by definition, $\overset{\bullet}{M}_{i+1}$ is divisible by at least 4. ie. it would produce a term smaller than itself. Thus, a 2-cycle must be of the form: $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1}$ since 2^{nd}*orderings* do not have $\overset{\blacksquare}{M}_i \to \overset{\blacksquare}{M}_{i+1}$ scenarios and if an $\overset{\blacktriangle}{M}_i$ exists, so must an $\overset{\blacksquare}{M}_j$ and $\overset{\bullet}{M}_k$, at the very least. Note well that a $3n+3$ sequence or sub-sequence with just $\overset{\blacksquare}{M}$ and $\overset{\bullet}{M}$ terms cannot have an o-cycle because further generation of the sequence produces $\overset{\bullet}{M}_i = 48$ or its c-sequence is some other trivial reduction to 3. Or, further generation of the sequence will produce an $\overset{\blacktriangle}{M}$ term, which would not be a Class 1 scenario.

Assume to the contrary, that there is a finite $\mathcal{C}$-sequence for some $3n+3$ sequence, which is an o-cycle, having only $\overset{\blacksquare}{M}$ and $\overset{\bullet}{M}$ terms. Let $\overset{\blacksquare}{M}_i$ be the largest of the $\overset{\blacksquare}{M}$ terms. To start, there is no 2-cycle as: $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \overset{\blacksquare}{M}_{i+2} = \overset{\blacksquare}{M}_i$ because if $\overset{\bullet}{M}_{i+1}$ is divisible by just 4 and produces an $\overset{\blacksquare}{M}$ term, then $\overset{\blacksquare}{M}_{i+2} > \overset{\blacksquare}{M}_i$, but $\overset{\blacksquare}{M}_i$ was assumed to be the largest. However, if $\overset{\bullet}{M}_{i+1}$ is divisible by 8 or more and produces an $\overset{\blacksquare}{M}$ term, then $\overset{\blacksquare}{M}_{i+2} < \overset{\blacksquare}{M}_i$. ie. $\overset{\blacksquare}{M}_{i+2} \neq \overset{\blacksquare}{M}_i$. If we consider a 3-cycle, it must be: $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \overset{\bullet}{M}_{i+2}$. And if both $\overset{\bullet}{M}_{i+1}$ and $\overset{\bullet}{M}_{i+2}$ were divisible by only 4 each, it would still be the case that $3(\frac{\overset{\bullet}{M}_{i+2}}{4}) + 3 < \overset{\blacksquare}{M}_i$. Now, consider a 4-cycle of alternating form: $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \overset{\blacksquare}{M}_{i+2} \to \overset{\bullet}{M}_{i+3}$. However, since $\overset{\bullet}{M}_{i+1}$ had to be divisible by at least 8, which ensures even if $\overset{\bullet}{M}_{i+3}$ is only divisible by 4, that $3(\frac{\overset{\bullet}{M}_{i+3}}{4}) + 3 < \overset{\blacksquare}{M}_i$, there can be no 4-cycle. Of course, we have the same result with $\overset{\blacksquare}{M}_i \to \overset{\bullet}{M}_{i+1} \to \overset{\bullet}{M}_{i+2} \to \overset{\bullet}{M}_{i+3}$. For any longer sequences, we need only use the result in section Tails. ie. choosing any $\overset{\blacksquare}{M}$ to be $\overset{\blacksquare}{M}_i$, we would have that $3(\frac{\overset{\bullet}{M}_{i-1}}{2^r}) + 3 < \overset{\blacksquare}{M}_i$ or $> \overset{\blacksquare}{M}_i$ where either case is a contradiction. Therefore, no $3n+3$ sequence has an o-cycle of any size ≥ 2, having just $\overset{\blacksquare}{M}$ and $\overset{\bullet}{M}$ terms.

Next, no $3n+3$ sequence has a Class 2, o-cycle because if $b = 1$, then N_i for $\overset{\blacktriangle}{M}_i$ is a *dead−ender*, so it cannot be generated. And if $\overset{\blacksquare}{t}_i = 2^2 3^b R$ is such that $b = 1$, then its N_i, is a *dead−ender*. For Class 3, observe wrt. $3n+3$ sequences, that i) $\overset{\blacktriangle}{M}_i+6 = \overset{\blacktriangle}{t}_i$ can be expressed as: $2^a 3^b R$ where $a \geq 3$, for t_i to be $\overset{\blacktriangle}{t}_i$, $b \geq 2$ if N_i can't be a *dead−ender* and $R \in \mathbb{N}_{odd}$ and not divisible by 3. ii) $\overset{\blacktriangle}{M}_i-6$ can be expressed as: $2^2 3^1 R$, by law 1, where $b = 1$. iii) $\overset{\blacksquare}{M}_i+6 = \overset{\blacksquare}{t}_i$ can be expressed as: $2^2 3^b R$ where $b \geq 2$. And iv) $\overset{\blacksquare}{M}_i-6$ can be expressed as: $2^a 3^1 R$ where $a \geq 3$, by law 1 and $b = 1$. Observe for i) and iii), that $b \geq 2$. But $\overset{\bullet}{M}_j$ can be expressed as: $2^a 3^1 S$ where $b = 1$, $S \in \mathbb{N}_{odd}$ and not divisible by 3, wrt. $3n{+}3$ sequences. ie. $\overset{\blacktriangle}{M}_i{+}6 \neq \overset{\bullet}{M}_j$ and $\overset{\blacksquare}{M}_i{+}6 \neq \overset{\bullet}{M}_j$. Now, observe wrt. $3n{+}k$ sequences in general, that ii) $\overset{\blacktriangle}{M}_i{-}6$ can be expressed as: $2^2 3^b R$ where $a = 2$, $b \geq 0$, $R \in \mathbb{N}_{odd}$ and not divisible by 3 and that iv) $\overset{\blacksquare}{M}_i{-}6$ can be expressed as: $2^a 3^b R$ where $a \geq 3$, $b \geq 0$, $R \in \mathbb{N}_{odd}$ and not divisible by 3.

For ii) and iv), another criteria exists which is: suppose $\overset{\bullet}{M}_j = 2^a 3^b S$ where $a \geq 2$, $b \geq 0$, $S \in \mathbb{N}_{odd}$ and not divisible by 3, then it is required that R of $\overset{\blacktriangle}{M}_i$ or $\overset{\blacksquare}{M}_i = r(S) + q$, where $r = 2^c$, $c \in \mathbb{N}_{>0}$ and $q > 1$. However, it is always the case that there exists a c, so that $q = 1$, for any $\overset{\blacktriangle}{M}_i$ or $\overset{\blacksquare}{M}_i$ from a $3n{+}3$ sequence. Therefore, no $3n{+}3$ sequence has a Class 3, o-cycle.

For Class 4, recall that every M_i of a $3n{+}3$ sequence is divisible by 3. We've omitted the proof that there are only the four classes of o-cycles for 2^{nd} *orderings* in Claim 1, because it's lengthy and secondary to establishing that every $3n{+}3$ sequence is $\mathcal{T}$-cyclic. The o-cycles were introduced as a point of interest wrt. R-sub structure. Appendix 3 contains some examples of $3n{+}k$ sequences of 2^{nd} *ordering*, for each class, having varying cycle sizes.

Claim 2: Let $M_i = 3(N_i) + 3$ and $M_i' = 3(N_i') + 1$ be for $3n{+}3$ and $3n{+}1$, respectively. Further, let $\mathbf{n}$ and $\mathbf{n'}$ be the first odd terms for the respective sequences where $\mathbf{n} = 2(\mathbf{n'}) - 1$. Then $3(M_{i-1}') = M_i \quad \forall\ \mathbf{n}, \mathbf{n'} \in \mathbb{N}_{odd}$.

Proof: Given a pair, $\mathbf{n}$, $\mathbf{n'}$ we have $\mathbf{n} = 2(\mathbf{n'}) - 1$. Let $M_1' = 3(\mathbf{n'}) + 1$. Then by the claim, $M_2 = 9(\mathbf{n'}) + 3$.

We also have that: $M_1 = 3(\mathbf{n}) + 3 = 3(2\mathbf{n'} - 1) + 3 = 6(\mathbf{n'}) - 3 + 3 = 6(\mathbf{n'})$. So, $N_2 = \frac{6(\mathbf{n'})}{2} = 3(\mathbf{n'})$ where $3(N_2) + 3 = M_2 = 9(\mathbf{n'}) + 3$. It follows that $3(M_{i-1}') = M_i\ \forall i \in \mathbb{N}_{>0}$. This is equivalent to $\frac{M_i}{2^k}{+}1 = N_{i+1}{+}1 = M_i'$ where k is the number of 2 multiples of M_{i-1}' and of M_i.

In Appendix 2, we show the $3n$+1 c-sequence for $\mathbf{n'}$ = 27. Below that we show its corresponding $\mathcal{C}$-sequence (M_i' = 3(N_i')+1), where the terms are coloured wrt. their desig. Followed by the conversion of the $\mathcal{C}$-sequence for the $3n$+3 sequence with $\mathbf{n}$ = 53 to the $\mathcal{C}$-sequence for the $3n$+1 sequence with $\mathbf{n'}$ = 27.

Observe that if no $3n$+3 sequence has a non-trivial cyclic sequence, then no $3n$+1 sequence has a non-trivial cyclic sequence. It remains to be established that no $3n$+3 sequence is divergent where the issue of non-trivial cyclic sequences is addressed as well. Had we offered the proof that 2^{nd}*orderings* admit only the four classes of o-cycles in Claim 1, and with recent results suggesting divergence has been dispensed particularly wrt. $3n$+1 sequences, the paper could have ended here.

Law 3: S-strings and the L-form

The L-form and the H-form could be viewed as the two extremes of R-subs since L-forms have the shortest *head*, of length one with $\overset{\blacktriangle}{t}_f = 2^3 3^2 R$, $\overset{\bullet}{t}_l = 2^1 3^2 R'$ and H-forms have the shortest *tail*, of length two. In other words, they govern the R-sub lower and upper bounds for $3n$+3 generation.

Next, the R-subs are partitioned or grouped in similar fashion as the $\overset{\blacktriangle}{t}$, $\overset{\blacksquare}{t}$ and $\overset{\bullet}{t}$ terms for the R-subs were wrt. the distinction between head and tail.

Definition 1.10. *An* $\boldsymbol{S}$*-string is a collection of contiguous* R*-subs:* R*-sub*$_i$, R*-sub*$_{i+1}$, R*-sub*$_{i+2}$, ... R*-sub*$_j$, R*-sub*$_{j+1}$, R*-sub*$_{j+2}$, ... R*-sub*$_k$ *where the* R*-sub*$_i$, R*-sub*$_{i+1}$, R*-sub*$_{i+2}$, ... , R*-subs must be* R^X*-subs and the* R*-sub*$_j$, R*-sub*$_{j+1}$, R*-sub*$_{j+2}$, ... R*-sub*$_k$, R*-subs must be* R^Y*-subs. There are exactly three types of* $\boldsymbol{S}$*-strings.* $\boldsymbol{T}_1$ *has no* L*-form* (L_F), R^Y*-sub.* $\boldsymbol{T}_2$ *has at least one* L_F R^Y*-sub and one non-*L_F R^Y*-sub. And* $\boldsymbol{T}_3$ *has only* L_F R^Y*-subs. Every* $\boldsymbol{S}$*-string belongs to exactly one, of types* $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ *or* $\boldsymbol{T}_3$.

The R^X-subs followed by the R^Y-subs of an **S**-string shall be the *head* and *tail*, respectively. **S**-strings are contiguous and are indexed as: **S**-string$_1$, **S**-string$_2$, **S**-string$_3$, ... , and we denote a certain **S**-string$_i$ by its type as: $\mathbf{S}_i^{T_1}$, $\mathbf{S}_i^{T_2}$ or $\mathbf{S}_i^{T_3}$.

Lastly, we group contiguous **S**-strings of the same type together, which we call an **S**-string *train*. It should be clear that no two **S**-string *trains* of the same type are ever contiguous.

Law 3

i) All $\mathcal{D}$-sequences for non-trivial $3n$+3 sequences contain $t_l = 54$.

ii) If there exists L-forms R^Y-sub$_i$ and R^Y-sub$_j$, $j > i$, then $Y_j < Y_i$.

iii) Let Y_s be from the highest indexed R^Y-sub$_s$ of **S**-string *train$_j$* and let Y_r be from the highest indexed R^Y-sub$_r$ of **S**-string *train$_i$* where $j > i$, then $Y_s < Y_r$.

In Appendix 1, we provide two non-trivial examples of $3n$+3 sequences, showing their R-sub and **S**-string and **S**-string *train* structures. They were devised from their corresponding non-trivial examples of $3n$+1 sequences, namely, $\mathbf{n}'$ = 27 and $\mathbf{n}'$ = $63,728,127$. For $\mathbf{n} = 2(27) - 1 = 53$ there are two **S**-strings, both of which are type $\mathbf{T}_1$, so one **S**-string *train*. And for $\mathbf{n} = 2(63,728,127) - 1$ = $127,456,253$ there are thirteen **S**-strings in order: $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_1$, $\mathbf{T}_1$, $\mathbf{T}_1$, $\mathbf{T}_1$, $\mathbf{T}_3$, $\mathbf{T}_3$, $\mathbf{T}_3$, $\mathbf{T}_3$, $\mathbf{T}_3$, $\mathbf{T}_1$ and $\mathbf{T}_2$. And there exists three **S**-string *trains* of type $\mathbf{T}_1$ (two having just one **S**-string each), one **S**-string *train* of type $\mathbf{T}_3$ and two **S**-string *trains* of type $\mathbf{T}_2$ (both having just one **S**-string each).

Final comments

We remark on why the numbers in a c-sequence for $3n+3$ and $3n+1$, increase and decrease the way that they do. There are four reasons in all for the more interesting cases. First, there is the increase and decrease due to the structure of R-subs. There is the increase and small decrease and then another increase from head to head terms and then the increase and decrease from tail to tail terms. Second, within a given **S**-string there are increases and decreases (and increases again), among the R-subs. Third, within a given **S**-string *train* there could be increases and decreases among the **S**-strings. Fourth, **S**-string *trains* of different types are interspersed amongst each other, sometimes producing yet another dramatic difference wrt. the size of the numbers between the **S**-string *trains*. So, all four scenarios could be involved, as it was for $\mathbf{n} = 127,456,253$. For non-trivial cases, it is the L-form and H-form R-subs that determine the bounds at every stage of a sequence generation.

Resources

1) https://en.wikipedia.org/wiki/Collatz_conjecture

2) https://www.mathcelebrity.com/collatz.php

3) https://www.dcode.fr/collatz-conjecture

Definition 1.11. *A dead-ender's cap O, is an even number of the form: $2^k N_i$ where $k \geq 1$ and N_i is a dead-ender.*

This definition was not required for the paper. Note that an O is not of the form $3(N_i)+3$, so it's not part of a $\mathcal{C}$-sequence.

Appendix 1

For $\mathbf{n} = 2(27) - 1 = 53$ which has 2 **S**-strings having 4 R-subs each.

$R^X\text{-sub}_1 = (1, 2 \mid 2^3\overset{\blacktriangle}{3}{}^1 7 \mid 2^1\overset{\bullet}{3}{}^3 7)$ — $\overset{\mathbf{max}}{H}$-form

$R^X\text{-sub}_2 = (3, 5 \mid 2^5\overset{\blacktriangle}{3}{}^2 1 \mid 2^1\overset{\bullet}{3}{}^3 23)$

$R^X\text{-sub}_3 = (1, 2 \mid 2^3\overset{\blacktriangle}{3}{}^2 13 \mid 2^1\overset{\bullet}{3}{}^4 13)$ — H-form

$R^Y\text{-sub}_4 = (2, 3 \mid 2^4\overset{\blacktriangle}{3}{}^2 11 \mid 2^1\overset{\bullet}{3}{}^2 223)$

S-string$_1$ above, is type $\mathbf{T}_1$ having no L-form. $Y_4 = 501$.

$R^X\text{-sub}_5 = (1, 4 \mid 2^3\overset{\blacktriangle}{3}{}^3 7 \mid 2^1\overset{\bullet}{3}{}^3 71)$

$R^X\text{-sub}_6 = (4, 2 \mid 2^6\overset{\blacktriangle}{3}{}^2 5 \mid 2^1\overset{\bullet}{3}{}^7 5)$ — H-form

$R^Y\text{-sub}_7 = (2, 6 \mid 2^4\overset{\blacktriangle}{3}{}^3 19 \mid 2^1\overset{\bullet}{3}{}^2 31)$

$R^Y\text{-sub}_8 = (1, 3 \mid 2^3\overset{\blacktriangle}{3}{}^3 1 \mid 2^1\overset{\bullet}{3}{}^3 1)$

S-string$_2$ above, is type $\mathbf{T}_1$. Since both **S**-string$_1$ and **S**-string$_2$ are $\mathbf{T}_1$ then they are an **S**-string *train* of type $\mathbf{T}_1$, which contains $\overset{\bullet}{t}_l = 54$, and $Y_8 < Y_4$.

For $\mathbf{n} = 2(63,728,127) - 1 = 127,456,253$ having 13 $\mathbf{S}$-strings and 54 R-subs.

$R^X\text{-sub}_1 = (8,2 \mid 2^{10}3^{1}124469 \mid 2^{1}3^{10}124469)$ — $\overset{\max}{H}$-form

$R^X\text{-sub}_2 = (5,2 \mid 2^{7}3^{2}9570013 \mid 2^{1}3^{8}9570013)$ — H-form

$R^X\text{-sub}_3 = (6,2 \mid 2^{8}3^{2}40878161 \mid 2^{1}3^{9}40878161)$ — H-form

$R^X\text{-sub}_4 = (4,2 \mid 2^{6}3^{3}87305213 \mid 2^{1}3^{8}87305213)$ — H-form

$R^Y\text{-sub}_5 = (2,6 \mid 2^{4}3^{2}5966765651 \mid 2^{1}3^{3}33982594997)$

$R^Y\text{-sub}_6 = (1,5 \mid 2^{3}3^{3}3185868281 \mid 2^{1}3^{3}6048171815)$

$\mathbf{S}$-string$_1$ above, is type $\mathbf{T}_1$ having no L-form.

$R^X\text{-sub}_7 = (2,5 \mid 2^{4}3^{2}1701048323 \mid 2^{1}3^{3}12917335703)$

$R^Y\text{-sub}_8 = (1,8 \mid 2^{3}3^{2}7266001333 \mid 2^{1}3^{3}7759152791)$

$R^Y\text{-sub}_9 = (1,7 \mid 2^{3}3^{2}4364523445 \mid 2^{1}3^{2}2330374213)$ — L-form

$\mathbf{S}$-string$_2$ above, is type $\mathbf{T}_2$ having 1 L_f and 1 non L_f.

$R^X\text{-sub}_{10} = (1,2 \mid 2^{3}3^{2}436945165 \mid 2^{1}3^{4}436945165)$ — H-form

$R^X\text{-sub}_{11} = (2,3 \mid 2^{4}3^{2}368672483 \mid 2^{1}3^{2}7465617781)$

$R^X\text{-sub}_{12} = (2,3 \mid 2^{4}3^{2}699901667 \mid 2^{1}3^{2}14173008757)$

$R^X\text{-sub}_{13} = (2,4 \mid 2^{4}3^{2}1328719571 \mid 2^{1}3^{2}20179928485)$

$R^Y\text{-sub}_{14} = (1,3 \mid 2^{3}3^{2}3783736591 \mid 2^{1}3^{3}4256703665)$

$\mathbf{S}$-string$_3$ above, is type $\mathbf{T}_1$.

$R^X\text{-sub}_{15} = (1,3 \mid 2^3 3^2 598598953 \mid 2^1 3^2 4040542933)$ — L-form

$R^Y\text{-sub}_{16} = (4,5 \mid 2^6 3^2 94700225 \mid 2^1 3^3 809021063)$

$R^Y\text{-sub}_{17} = (3,2 \mid 2^5 3^2 113768587 \mid 2^1 3^6 113768587)$ — H-form

S-string$_4$ above, is type $\mathbf{T}_1$.

$R^X\text{-sub}_{18} = (4,6 \mid 2^6 3^3 8999273 \mid 2^1 3^2 2075773717)$

$R^X\text{-sub}_{19} = (3,2 \mid 2^5 3^2 97301893 \mid 2^1 3^6 97301893)$ — H-form

$R^X\text{-sub}_{20} = (3,2 \mid 2^5 3^2 369443125 \mid 2^1 3^6 369443125)$ — H-form

$R^X\text{-sub}_{21} = (1,8 \mid 2^3 3^2 5610917461 \mid 2^1 3^2 35950419397)$ — L-form

$R^X\text{-sub}_{22} = (1,4 \mid 2^3 3^2 6740703637 \mid 2^1 3^3 22749874775)$

$R^Y\text{-sub}_{23} = (1,6 \mid 2^3 3^2 12796804561 \mid 2^1 3^3 6073483415)$

$R^Y\text{-sub}_{24} = (1,8 \mid 2^3 3^2 3416334421 \mid 2^1 3^3 3648204775)$

$R^Y\text{-sub}_{25} = (2,3 \mid 2^4 3^2 1026057593 \mid 2^1 3^2 5194416565)$

$R^Y\text{-sub}_{26} = (2,2 \mid 2^4 3^2 486976553 \mid 2^1 3^5 486976553)$ — H-form

S-string$_5$ above, is type $\mathbf{T}_1$.

$R^X\text{-sub}_{27} = (3,4 \mid 2^5 3^2 154082425 \mid 2^1 3^3 1170063415)$

$R^X\text{-sub}_{28} = (1,7 \mid 2^3 3^2 658160671 \mid 2^1 3^3 29284615)$

$R^X\text{-sub}_{29} = (3,2 \mid 2^5 3^2 4118149 \mid 2^1 3^6 4118149)$ — H-form

$R^Y\text{-sub}_{30} = (3,9 \mid 2^4 3^3 15636097 \mid 2^1 3^2 338121055)$

$R^Y\text{-sub}_{31} = (1,3 \mid 2^3 3^3 10566283 \mid 2^1 3^2 213967231)$

$R^Y\text{-sub}_{32} = (3,3 \mid 2^5 3^3 1671619 \mid 2^1 3^2 304652563)$

S-string$_6$ above, is type $\mathbf{T}_1$. Since **S**-string$_3$, **S**-string$_4$, **S**-string$_5$ and **S**-string$_6$ are $\mathbf{T}_1$ then together, they are the second **S**-string *train* of type $\mathbf{T}_1$. So comparing with the first (trivial) **S**-string *train*, namely, **S**-string$_1$, we find that $Y_{32} < Y_6$.

$R^X\text{-sub}_{33} = (1,2 \mid 2^3 3^2 14280589 \mid 2^1 3^4 14280589)$ — H-form

$R^X\text{-sub}_{34} = (2,2 \mid 2^4 3^2 12049247 \mid 2^1 3^5 12049247)$ — H-form

$R^Y\text{-sub}_{35} = (1,3 \mid 2^3 3^2 60999313 \mid 2^1 3^2 411745363)$ — L-form

S-string$_7$ above, is type $\mathbf{T}_3$ having just one L_f.

$R^X\text{-sub}_{36} = (3,2 \mid 2^5 3^2 4825141 \mid 2^1 3^6 4825141)$ — H-form

$R^Y\text{-sub}_{37} = (1,13 \mid 2^3 3^2 73281829 \mid 2^1 3^2 13927807)$ — L-form

S-string$_8$ above, is type $\mathbf{T}_3$ having just one L_f.

$R^X\text{-sub}_{38} = (3,3 \mid 2^5 3^3 108811 \mid 2^1 3^2 19830805)$

$R^X\text{-sub}_{39} = (3,3 \mid 2^5 3^2 929569 \mid 2^1 3^2 56471317)$

$R^X\text{-sub}_{40} = (3,2 \mid 2^5 3^2 2647093 \mid 2^1 3^6 2647093)$ — H-form

$R^Y\text{-sub}_{41} = (1,7 \mid 2^3 3^2 40202725 \mid 2^1 3^2 167701)$ — L-form

S-string$_9$ above, is type $\mathbf{T}_3$ having just one L_f.

$R^X\text{-sub}_{42} = (3,2 \mid 2^5 3^2 7861 \mid 2^1 3^6 7861)$ — H-form

$R^X\text{-sub}_{43} = (1,2 \mid 2^3 3^2 119389 \mid 2^1 3^4 119389)$ — H-form

$R^X\text{-sub}_{44} = (1,2 \mid 2^3 3^2 201469 \mid 2^1 3^4 201469)$ — H-form

$R^Y\text{-sub}_{45} = (1,3 \mid 2^3 3^2 339979 \mid 2^1 3^2 573715)$ — L-form

S-string$_{10}$ above, is type $\mathbf{T}_3$ having just one L_f.

R^X-sub$_{46}$ = $(1, 2 \mid 2^3 3^2 26893 \mid 2^1 3^4 26893)$ — *H*-form

R^X-sub$_{47}$ = $(2, 3 \mid 2^4 3^2 22691 \mid 2^1 3^2 459493)$

R^Y-sub$_{48}$ = $(1, 3 \mid 2^3 3^2 86155 \mid 2^1 3^2 145387)$ — *L*-form

S-string$_{11}$ above, is type $\mathbf{T}_3$ having just one L_f. We have that $\mathbf{S}_7^{T_3}$, $\mathbf{S}_8^{T_3}$, $\mathbf{S}_9^{T_3}$, $\mathbf{S}_{10}^{T_3}$ and $\mathbf{S}_{11}^{T_3}$ are one **S**-string *train* of type $\mathbf{T}_3$. Note that $Y_{48} < Y_{35}$.

R^X-sub$_{49}$ = $(3, 2 \mid 2^5 3^3 71 \mid 2^1 3^7 71)$ — *H*-form

R^Y-sub$_{50}$ = $(1, 4 \mid 2^3 3^2 3235 \mid 2^1 3^4 455)$

S-string$_{12}$ above, is type $\mathbf{T}_1$. Then if we compare the three **S**-string *trains* of type $\mathbf{T}_1$ we find that $Y_{50} < Y_{32} < Y_6$.

R^X-sub$_{51}$ = $(8, 3 \mid 2^{10} 3^3 1 \mid 2^1 3^2 44287)$

R^X-sub$_{52}$ = $(4, 2 \mid 2^6 3^3 173 \mid 2^1 3^8 173)$ — *H*-form

R^Y-sub$_{53}$ = $(1, 22 \mid 2^3 3^2 23647 \mid 2^1 3^2 85)$ — *L*-form

R^Y-sub$_{54}$ = $(5, 10 \mid 2^7 3^2 1 \mid 2^1 3^3 1)$

S-string$_{13}$ above, is type $\mathbf{T}_2$ having 1 L_f and 1 non-L_f . Then comparing $\mathbf{S}_2^{T_2}$, and $\mathbf{S}_{13}^{T_2}$ (two *trains* being one **S**-string each), we find that $Y_{54} < Y_9$.

Observe that each **S**-string *train's* highest indexed Y value is less than each of the previous **S**-string *trains*' highest indexed Y values. Moreover, Law 3 is obeyed in both examples above.

Appendix 2

The $3n$+1 sequence for $\mathbf{n'} = 27 \to 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322,$ $161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700,$ $350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251,$ $754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079,$ $3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077,$ $9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61,$ $184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1$

The c-sequence above, contains the $\mathcal{C}$-sequence $\to 82, 124, 94, 142, 214, 322, 484,$ $364, 274, 412, 310, 466, 700, 526, 790, 1186, 1780, 1336, 502, 754, 1132, 850, 1276,$ $958, 1438, 2158, 3238, 4858, 7288, 2734, 4102, 6154, 9232, 1732, 1300, 976, 184,$ $70, 106, 160, 16$

From the $3n$+3 $\mathcal{C}$-sequence for $\mathbf{n} = 2(27) - 1 = 53$, we obtain the $3n$+1 $\mathcal{C}$-sequence for $\mathbf{n'} = 27$ by $\frac{M_i}{2^k}+1 = N_{i+1}+1 = M_i'$ as follows:

$\mathbf{n} = 53 \to \frac{162}{2^1} + 1 = 82,\ \frac{246}{2^1} + 1 = 124,\ \frac{372}{2^2} + 1 = 94,\ \frac{282}{2^1} + 1 = 142,$
$\frac{426}{2^1} + 1 = 214,\ \frac{642}{2^1} + 1 = 322,\ \frac{966}{2^1} + 1 = 484,\ \frac{1452}{2^2} + 1 = 364,\ \frac{1092}{2^2} + 1 = 274,$
$\frac{822}{2^1} + 1 = 412,\ \frac{1236}{2^2} + 1 = 310,\ \frac{930}{2^1} + 1 = 466,\ \frac{1398}{2^1} + 1 = 700,\ \frac{2100}{2^2} + 1 = 526,$
$\frac{1578}{2^1} + 1 = 790,\ \frac{2370}{2^1} + 1 = 1186,\ \frac{3558}{2^1} + 1 = 1780,\ \frac{5340}{2^2} + 1 = 1336,$
$\frac{4008}{2^3} + 1 = 502,\ \frac{1506}{2^1} + 1 = 754,\ \frac{2262}{2^1} + 1 = 1132,\ \frac{3396}{2^2} + 1 = 850,$
$\frac{2550}{2^1} + 1 = 1276,\ \frac{3828}{2^2} + 1 = 958,\ \frac{2874}{2^1} + 1 = 1438,\ \frac{4314}{2^1} + 1 = 2158,$
$\frac{6474}{2^1} + 1 = 3238,\ \frac{9714}{2^1} + 1 = 4858,\ \frac{14574}{2^1} + 1 = 7288,\ \frac{21864}{2^3} + 1 = 2734,$
$\frac{8202}{2^1} + 1 = 4102,\ \frac{12306}{2^1} + 1 = 6154,\ \frac{18462}{2^1} + 1 = 9232,\ \frac{27696}{2^4} + 1 = 1732,$
$\frac{5196}{2^2} + 1 = 1300,\ \frac{3900}{2^2} + 1 = 976,\ \frac{2928}{2^4} + 1 = 184,\ \frac{552}{2^3} + 1 = 70,$
$\frac{210}{2^1} + 1 = 106,\ \frac{318}{2^1} + 1 = 160,\ \frac{480}{2^5} + 1 = 16$

Appendix 3

Proof for Law 1

Let $M_i = 3(N_i)+3$ where N_i is odd. Suppose M_i is divisible by 2 exactly once. Then, let $3(N_i)+3 = 2R$ where R, R' and $R'' \in \mathbb{N}_{odd}$. Then, $M_i+6 = 2R + 6 = 2(R+3)$. But $R+3$ must be even, so let it be $2R'$. Then, $M_i+6 = 2^2R'$ must be divisible by at least 4. Suppose it's exactly 4. Now, we have that $M_i-6 = 2R-6 = 2(R-3)$. But $R-3$ must be even, so let it be $2R''$. Then, $M_i-6 = 2^2R''$ must be divisible by at least 4. Suppose it's exactly 4 as well. Then, $M_i-6 = 2^2R''$. But then we have that $2^2R'' + 12 = 2^2R'$. And this is equivalent to $R'' + 3 = R' \longrightarrow$ odd + odd = odd, a contradiction by the assumption that M_i-6 was divisible by exactly 4 (assuming M_i+6 was). Therefore, M_i must be $\overset{\textcolor{blue}{\blacksquare}}{M}_i$. Observe that if M_i+6 and M_i-6 are swapped in the argument above, then the contradiction would imply that M_i is $\overset{\textcolor{red}{\blacktriangle}}{M}_i$ instead.

Now suppose M_i is divisible by 4. Then M_i is $\overset{\textcolor{green}{\bullet}}{M}_i$ by definition. Let $M_i = 2^2R$. It follows that $M_i+6 = 2^2R + 6 = 2(2R+3)$. But $2R+3$ is odd. Let $R' = 2R+3$, so $M_i+6 = 2R'$, thus divisible by 2 exactly once. Similarly, $M_i-6 = 2^2R - 6 = 2(2R-3)$. But $2R-3$ is odd. Let $R'' = 2R-3$, so $M_i-6 = 2R''$, thus divisible by 2 exactly once. Since the arguments here, would be unchanged for any $3n+k$ sequence, $k \in \mathbb{N}_{odd}$ then the law applies to all even integers greater than six.

Proof for $2^a3^bR \longrightarrow 2^{a-1}3^{b+1}R$ where $a \geq 3$ and for $\overset{\textcolor{blue}{\blacktriangle}}{t}_i \longrightarrow \overset{\textcolor{green}{\bullet}}{t}_{i+1}$

Let $M_i+6 = 2^a3^bR$ where $a \geq 3$. Then $M_i = 2^a3^bR - 6 = 2(2^{a-1}3^bR - 3)$. But M_i is divisible by 2 exactly once (by law 1). It follows that $N_{i+1} = 2^{a-1}3^bR - 3$.

So, $3(N_{i+1}) + 3 = 3(2^{a-1}3^bR - 3) + 3 = 2^{a-1}3^{b+1}R - 9 + 3 = 2^{a-1}3^{b+1}R - 6 = M_{i+1}$. Thus, $M_{i+1}+6 = 2^{a-1}3^{b+1}R = t_{i+1}$. It follows immediately that if $t_i = 2^33^bR$ then $2^23^{b+1}R = t_{i+1}$, where $t_{i+1} = M_{i+1}+6 = 2^23^{b+1}R$ is divisible by exactly 4, so M_{i+1} is $\overset{\textcolor{blue}{\blacksquare}}{M}_{i+1}$ by law 1. Moreover, it follows that $t_{i+2} = M_{i+2}+6 = 2^13^{b+2}R$ is divisible by 2 exactly once. Then M_{i+2} is $\overset{\textcolor{green}{\bullet}}{M}_{i+2}$ by law 1.

Suppose now that some $\overset{\blacksquare}{t}_i$ does not follow a $\overset{\blacktriangle}{t}_{i-1}$. Then again, $t_i = M_i+6 = 2^2 3^b R$ is divisible by exactly 4. M_i is divisible by 2 exactly once and it follows that $\frac{M_i}{2} = N_{i+1} = \frac{2^2 3^b R-6}{2} = 2^1 3^b R - 3$. Then $M_{i+1} = 3(2^1 3^b R-3) + 3 = 2^1 3^{b+1} R - 9 + 3 = 2^1 3^{b+1} R - 6$. So, $M_{i+1}+6 = 2^1 3^{b+1} R$ and by law 1, M_{i+1} is $\overset{\bullet}{M}_{i+1}$

Proof for $\overset{\blacksquare}{t}_i = 2^2 3^b R$ where $b \geq 2$

Suppose t_{i-1} is $\overset{\blacktriangle}{t}_{i-1} = \overset{\blacktriangle}{t}_f$ is $2^3 3^1 R$ (the least it could be), then $\overset{\blacksquare}{t}_{f+1}$ is $2^2 3^2 R$. If t_{i-1} is $\overset{\bullet}{t}_{i-1}$, then $\overset{\bullet}{M}_{i-1} = 2^r 3^1 S$ where $r \geq 2$ and S is odd and not divisible by 3. We have that $N_i = 3^1 S$, so $\overset{\blacksquare}{M}_i = 3(3^1 S) + 3$. It follows that $\overset{\blacksquare}{M}_i+6 = 3(3^1 S) + 9 = 3^2(S+1) = 2^a 3^2 S'$ where $a = 2$ by law 1, and $S' \in \mathbb{N}_{odd}$.

Examples of $3n+k$ sequences of 2^{nd} *ordering*, with o-cycles

i) For $3n+11$, Class 3 and 4, **8**-cycle (two R-subs), where **n** = 13.

ii) For $3n+15$, Class 1 and 2, **3**-cycle (no R-sub), where **n** = 69.

iii) For $3n+15$, Class 2, **3**-cycle (one R-sub), where **n** = 57.

iv) For $3n+19$, Class 3 and 4, **5**-cycle (one R-sub), where **n** = 5.

v) For $3n+23$, Class 3 and 4, **26**-cycle (five R-subs), where **n** = 41.

vi) For $3n+31$, Class 3 and 4, **12**-cycle (one R-sub), where **n** = 13.

vii) For $3n+35$, Class 4, **4**-cycle (one R-sub), where **n** = 13.

viii) For $3n+35$, Class 1 and 4, **2**-cycle (no R-sub), where **n** = 25.