

# Extreme Verification Pipeline

Complete Technical Deep Dive: 20 SOTA Papers in 5 Layers

Barrier Certificate Synthesis for Program Verification

February 2026

*From Positivstellensatz to IC3: A Complete Formal Verification Framework*

## What is the Extreme Verification Pipeline?

A **5-layer, 20-paper** formal verification framework that synthesizes **barrier certificates** to prove program safety or find real bugs.

### Key Capabilities:

- Sound verification (no false negatives)
- Automatic certificate synthesis
- Counterexample-guided refinement
- Machine learning for invariants
- Interprocedural analysis

### Bug Types Detected:

- Bounds violations (array access)
- Division by zero
- Null pointer dereference
- Type errors
- Security vulnerabilities

# The Core Insight: Barrier Certificates

## Definition

A **barrier certificate**  $B : \mathcal{S} \rightarrow \mathbb{R}$  separates initial states from unsafe states.

## Three Conditions for Safety

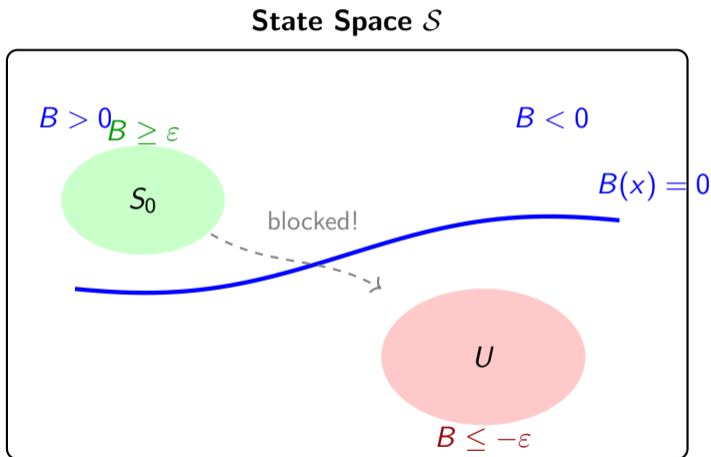
- ❶ **Init:**  $\forall s \in S_0. B(s) \geq \varepsilon$  (Initial states are “inside”)
- ❷ **Unsafe:**  $\forall s \in U. B(s) \leq -\varepsilon$  (Unsafe states are “outside”)
- ❸ **Inductive:**  $(B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$  (Can't cross)

## Key Insight

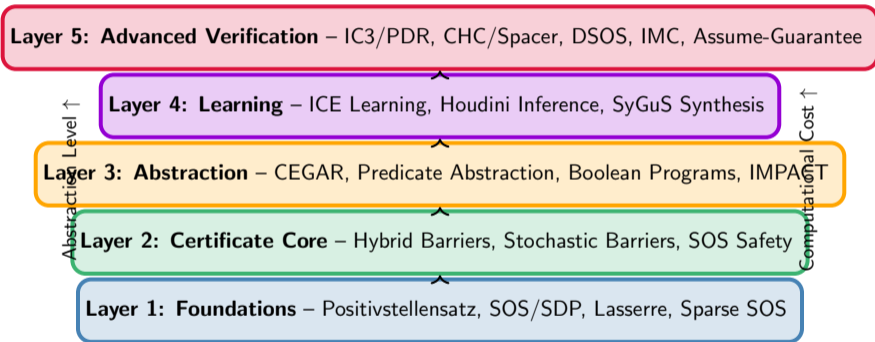
If such  $B$  exists  $\Rightarrow$  **SAFE** (no path from initial to unsafe)

If we can reach unsafe  $\Rightarrow$  **BUG** (with concrete counterexample)

# Visual Intuition: The Barrier Separates Safe from Unsafe



# The 5-Layer Architecture



# The 20 SOTA Papers Implemented

## Layer 1: Foundations (Papers #5-8)

- 5 Putinar Positivstellensatz (1993)
- 6 Parrilo SOS/SDP (2003)
- 7 Lasserre Hierarchy (2001)
- 8 Sparse SOS - Kojima (2005)

## Layer 2: Certificate Core (Papers #1-4)

- 1 Hybrid Barriers - Prajna-Jadbabaie (2004)
- 2 Stochastic Barriers - Prajna (2007)
- 3 SOS Safety - Papachristodoulou (2002)
- 4 SOSTOOLS Framework - Prajna (2004)

## Layer 3: Abstraction (Papers #12-14, 16)

- 12 CEGAR - Clarke (2000)
- 13 Predicate Abstraction - Graf-Saïdi (1997)
- 14 Boolean Programs - Ball-Rajamani (2001)
- 16 IMPACT/Lazy - McMillan (2006)

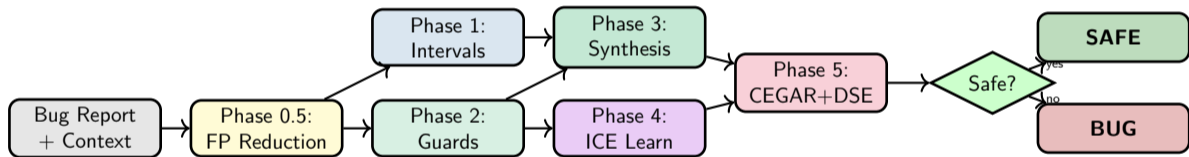
## Layer 4: Learning (Papers #17-19)

- 17 ICE Learning - Garg (2014)
- 18 Houdini - Flanagan-Leino (2001)
- 19 SyGuS - Alur (2013)

## Layer 5: Advanced (Papers #9-11, 15, 20)

- 9 DSOS/SDSOS - Ahmadi-Majumdar (2019)
- 10 IC3/PDR - Bradley (2011)
- 11 CHC/Spacer - Komuravelli (2014)
- 15 Interpolation/IMC - McMillan (2003)
- 20 Assume-Guarantee - Pnueli (1985)

# Verification Flow: High-Level Overview



# What Makes This “Extreme” Verification?

## **Exhaustive Technique Coverage:**

- ALL 20 SOTA papers integrated
- Every layer feeds into the next
- Portfolio execution for robustness
- Fallback strategies at each level

## **Sound & Complete (for tractable cases):**

- Never reports SAFE if bug exists
- Finds bugs with concrete witnesses
- Certificates are Z3-verified

## **Cross-Layer Integration:**

- ICE uses Layer 3 abstractions
- CEGAR refines Layer 2 barriers
- IC3 lemmas constrain Layer 1 SOS
- Learning guides synthesis

## **Real-World Scale:**

- Tested on DeepSpeed (700+ files)
- Handles interprocedural analysis
- Sub-second per-bug verification

# Roadmap: What We'll Cover

- ① **Part I: Mathematical Foundations** (Slides 11-100)
  - Positivstellensatz, SOS/SDP, Lasserre, Sparse SOS
- ② **Part II: Barrier Certificate Core** (Slides 101-180)
  - Hybrid barriers, Stochastic barriers, SOS Safety, SOSTOOLS
- ③ **Part III: Abstraction & Refinement** (Slides 181-260)
  - CEGAR, Predicate Abstraction, Boolean Programs, IMPACT
- ④ **Part IV: Learning-Based Synthesis** (Slides 261-340)
  - ICE Learning, Houdini, SyGuS, and how they integrate
- ⑤ **Part V: Advanced Verification** (Slides 341-420)
  - DSOS/SDSOS, IC3/PDR, CHC/Spacer, IMC, Assume-Guarantee
- ⑥ **Part VI: Integration & Implementation** (Slides 421-500)
  - UnifiedSynthesisEngine, ExtremeContextVerifier, Results

# Notation and Preliminaries

## Polynomial Notation

- $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$  – ring of polynomials in  $n$  variables
- $\mathbb{R}[x]_d$  – polynomials of degree  $\leq d$
- $\Sigma[x]$  – cone of sum-of-squares polynomials
- $\Sigma[x]_d$  – SOS polynomials of degree  $\leq 2d$

## Semialgebraic Sets

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0, h_1(x) = 0, \dots, h_k(x) = 0\}$$

where  $g_i, h_j \in \mathbb{R}[x]$ .

## Barrier Function

$B : \mathcal{S} \rightarrow \mathbb{R}$  is typically a polynomial  $B \in \mathbb{R}[x]_d$  for some degree  $d$ .

# Part I

## Mathematical Foundations

Layer 1: Papers #5-8

Positivstellensatz • SOS/SDP • Lasserre • Sparse SOS

# Why Mathematical Foundations Matter

## The Central Problem

Given polynomial constraints, can we **certify** that a polynomial is nonnegative on a region?

## For Barrier Certificates

- **Init condition:** Prove  $B(x) - \varepsilon \geq 0$  for all  $x \in S_0$
- **Unsafe condition:** Prove  $-B(x) - \varepsilon \geq 0$  for all  $x \in U$
- **Inductive condition:** Prove implications about  $B(x')$

## The Solution: Algebraic Certificates

Use **Positivstellensatz** to reduce positivity to **Sum-of-Squares** decompositions, which are **SDP-solvable**.

# Paper #5: Putinar Positivstellensatz (1993)

## Reference

M. Putinar. “Positive polynomials on compact semi-algebraic sets.”  
*Indiana University Mathematics Journal*, 1993.

## Key Theorem (Putinar)

If  $p(x) > 0$  for all  $x \in S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$   
and the quadratic module  $M(g_1, \dots, g_m)$  is **Archimedean**, then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i$$

where  $\sigma_0, \sigma_1, \dots, \sigma_m$  are **sum-of-squares** polynomials.

This provides a **certificate of positivity** that can be verified algebraically!

# Quadratic Module: The Key Structure

## Definition (Quadratic Module)

Given generators  $g_1, \dots, g_m \in \mathbb{R}[x]$ , the **quadratic module** is:

$$M(g_1, \dots, g_m) = \left\{ \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i : \sigma_i \in \Sigma[x] \right\}$$

where  $\Sigma[x]$  is the cone of sum-of-squares polynomials.

## Properties

- Contains all polynomials nonnegative on  $S = \{x : g_i(x) \geq 0\}$
- Closed under addition and multiplication by SOS
- **Every element is nonnegative on  $S$**  (soundness!)

**Intuition:**  $\sigma_i \geq 0$  (SOS)  $g_i \geq 0$  on  $S \Rightarrow \sigma_i \cdot g_i \geq 0$  on  $S$

# The Archimedean Condition

## Definition (Archimedean)

$M(g_1, \dots, g_m)$  is **Archimedean** if there exists  $R > 0$  such that:

$$R - \|x\|^2 = R - \sum_{i=1}^n x_i^2 \in M(g_1, \dots, g_m)$$

## Geometric Meaning

The Archimedean property ensures that  $S$  is **bounded** (compact).

## Practical Implication

For barrier synthesis, we often add a “bounding constraint”:

$$g_0(x) = R - \|x\|^2 \geq 0$$

# Sum-of-Squares (SOS) Polynomials

## Definition (Sum of Squares)

$p \in \mathbb{R}[x]$  is **SOS** if:

$$p(x) = \sum_{i=1}^k q_i(x)^2$$

for some polynomials  $q_1, \dots, q_k \in \mathbb{R}[x]$ .

## Key Properties

- Every SOS polynomial is **nonnegative** (obvious: sum of squares!)
- **Not every nonnegative polynomial is SOS** (Hilbert 1888)
- SOS-ness is **computationally tractable** (reduces to SDP)

## The Key Insight

# Gram Matrix: SOS as Semidefinite Constraint

## Theorem (Gram Matrix Representation)

$p(x)$  of degree  $2d$  is SOS if and only if:

$$p(x) = \mathbf{m}(x)^\top Q \mathbf{m}(x)$$

where  $\mathbf{m}(x)$  is the vector of monomials up to degree  $d$ , and  $Q \succeq 0$  (positive semidefinite).

Example:  $p(x) = x^4 + 2x^2 + 1$

$$\mathbf{m}(x) = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q \succeq 0 \text{ and } \mathbf{m}^\top Q \mathbf{m} = 1 + 2x^2 + x^4 = (1 + x^2)^2 \checkmark$$

# Coefficient Matching: The Linear Constraints

## The Problem

Given  $p(x)$  with known coefficients, find  $Q \succeq 0$  such that  $p = \mathbf{m}^\top Q \mathbf{m}$ .

Expanding  $\mathbf{m}^\top Q \mathbf{m}$ :

$$\mathbf{m}(x)^\top Q \mathbf{m}(x) = \sum_{i,j} Q_{ij} \cdot m_i(x) \cdot m_j(x)$$

Each coefficient of  $p$  gives a **linear constraint** on entries of  $Q$ .

## Resulting Feasibility Problem

Find  $Q$  such that:  $\begin{cases} \text{(linear constraints from coefficient matching)} \\ Q \succeq 0 \end{cases}$

This is a **Semidefinite Program (SDP)**!

# Applying Positivstellensatz to Barrier Synthesis

## Barrier Init Condition

Prove:  $B(x) \geq \varepsilon$  for all  $x \in S_0 = \{x : g_1(x) \geq 0, \dots\}$

Equivalent:  $B(x) - \varepsilon \geq 0$  on  $S_0$

By Putinar: Find SOS  $\sigma_0, \sigma_1, \dots$  such that:

$$B(x) - \varepsilon = \sigma_0(x) + \sum_i \sigma_i(x) \cdot g_i(x)$$

## This reduces to SDP!

- 1 Parameterize  $B$  and  $\sigma_i$  with unknown coefficients
- 2 Set up coefficient matching equations
- 3 Require Gram matrices for  $\sigma_i$  to be PSD
- 4 Solve the resulting SDP

# Implementation: Positivstellensatz Module

```
@dataclass
class SOSPolynomial:
    """Sum-of-squares polynomial:  $P = \sum_i q_i^2$ """
    n_vars: int
    squares: List[Polynomial]

    def to_polynomial(self) -> Polynomial:
        """Convert to standard polynomial."""
        result = Polynomial(self.n_vars, {})
        for q in self.squares:
            result = result.add(q.multiply(q))
        return result

@dataclass
class QuadraticModule:
    """ $M(g_1, \dots, g_m) = \{s_0 + \sum s_i g_i : s_i \text{ are SOS}\}$ """
    n_vars: int
    generators: List[Polynomial]

    def is_archimedean(self, R: float) -> bool:
        """Check if  $R - ||x||^2$  is in the module."""

        ball = Polynomial.constant(self.n_vars, R)
        for i in range(self.n_vars):
            ball = ball.subtract(Polynomial.variable(self.n_vars, i).square())
        return self.contains(ball)
```

# Paper #6: Parrilo SOS via SDP (2003)

## Reference

P. A. Parrilo. "Semidefinite programming relaxations for semialgebraic problems." *Mathematical Programming, Series B*, 2003.

## Core Contribution

Provides the **computational machinery** for Positivstellensatz:

- Explicit reduction from SOS to SDP
- Gram matrix construction algorithms
- Numerical stability techniques
- Complexity analysis

This paper turns the **theory** of Positivstellensatz into **practical algorithms**.

# Semidefinite Programming (SDP)

## Definition (SDP Standard Form)

$$\begin{aligned} & \text{minimize} && \langle C, X \rangle = \text{tr}(C^T X) \\ & \text{subject to} && \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & && X \succeq 0 \end{aligned}$$

## Key Properties

- **Convex optimization** – global optimum guaranteed
- **Polynomial-time solvable** – interior-point methods
- **Duality** – provides certificates for infeasibility
- **Mature solvers** – MOSEK, CSDP, SeDuMi, CVXPY

# The SOS $\rightarrow$ SDP Reduction

## Goal

Given target polynomial  $p(x)$ , determine if  $p$  is SOS.

### Step 1: Monomial Basis

$$\mathbf{m}(x) = (1, x_1, x_2, \dots, x_n, x_1^2, x_1 x_2, \dots)^\top$$

containing all monomials up to degree  $d = \deg(p)/2$ .

### Step 2: Gram Matrix Parameterization

$$p(x) = \mathbf{m}(x)^\top Q \mathbf{m}(x) \quad \text{with } Q \succeq 0$$

### Step 3: Coefficient Matching

Equate coefficients of  $p$  with those of  $\mathbf{m}^\top Q \mathbf{m}$ :

$$p_\alpha = \sum_{(\beta, \gamma): \beta + \gamma = \alpha} Q_{\beta, \gamma}$$

# Monomial Basis Construction

## Example: 2 variables, degree 2

For  $n = 2$  and  $d = 2$ :

$$\mathbf{m}(x, y) = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$$

Size:  $\binom{n+d}{d} = \binom{4}{2} = 6$  monomials.

## Complexity

Number of monomials (and Gram matrix size):

$$|\mathbf{B}| = \binom{n+d}{d} = O((n+d)^d)$$

# Complete Example: Is $x^4 - 2x^2y^2 + y^4$ SOS?

**Step 1: Monomial basis** (degree 4  $\Rightarrow$  basis degree 2)

$$\mathbf{m}(x, y) = (1, x, y, x^2, xy, y^2)^\top$$

**Step 2: Expand  $\mathbf{m}^\top Q \mathbf{m}$**  for symmetric  $Q$ :

$$\mathbf{m}^\top Q \mathbf{m} = Q_{11} + 2Q_{12}x + \cdots + Q_{44}x^4 + 2Q_{45}x^3y + \cdots$$

**Step 3: Match coefficients** with  $x^4 - 2x^2y^2 + y^4$ :

$$\text{coef of } x^4 : \quad Q_{44} = 1$$

$$\text{coef of } x^2y^2 : \quad 2Q_{46} + Q_{55} = -2$$

$$\text{coef of } y^4 : \quad Q_{66} = 1$$

$$\vdots$$

**Step 4: SDP feasibility** – Find  $Q \succeq 0$  satisfying constraints.

# Numerical Considerations in SOS-SDP

## Challenge: Numerical Precision

SDP solvers use floating-point arithmetic. Solutions may have:

- Small negative eigenvalues (numerical noise)
- Coefficients that don't match exactly
- Near-singular Gram matrices

## Mitigation Strategies

- 1 **Tolerances:** Accept  $\lambda_{\min}(Q) > -\epsilon$  for small  $\epsilon$
- 2 **Rational recovery:** Round to nearby rationals and verify
- 3 **Facial reduction:** Exploit low-rank structure
- 4 **Symbolic post-processing:** Use exact arithmetic to certify

Our implementation uses Z3's exact rational arithmetic for final verification.

# Barrier Synthesis via SOS-SDP

## The Synthesis Problem

Find polynomial  $B(x)$  satisfying Init, Unsafe, Inductive conditions.

**Parameterize**  $B$  as polynomial with unknown coefficients:

$$B(x) = \sum_{\alpha} b_{\alpha} x^{\alpha}$$

**For each condition**, create SOS constraints:

$$\text{Init: } B - \varepsilon = \sigma_0^{(\text{init})} + \sum_i \sigma_i^{(\text{init})} g_i^{(\text{init})}$$

$$\text{Unsafe: } -B - \varepsilon = \sigma_0^{(\text{unsafe})} + \sum_j \sigma_j^{(\text{unsafe})} g_j^{(\text{unsafe})}$$

$$\text{Step: } B' - \lambda B = \sigma_0^{(\text{step})} + \dots$$

**Joint SDP:** Coefficients  $b_{\alpha}$ , Gram matrices for all  $\sigma$ , all PSD.

# Implementation: SOS Decomposer

```
class SOSDecomposer:
    """SOS decomposition via SDP (Paper \#6)."""

    def __init__(self, n_vars: int, max_degree: int):
        self.n_vars = n_vars
        self.max_degree = max_degree
        self.basis = MonomialBasis(n_vars, max_degree // 2)

    def is_sos(self, p: Polynomial) -> Optional[SOSDecomposition]:
        """Check if p is SOS, return decomposition if so."""

        gram_size = len(self.basis.monomials)
        Q = self._create_gram_matrix(gram_size)

        constraints = self._build_coefficient_constraints(p, Q)

        constraints.append(Q >> 0)

        result = self._solve_sdp(constraints)

        if result.status == OPTIMAL:
            return self._extract_decomposition(result.Q_value)
        return None
```

# Complexity of SOS-SDP

## SDP Size for SOS Check

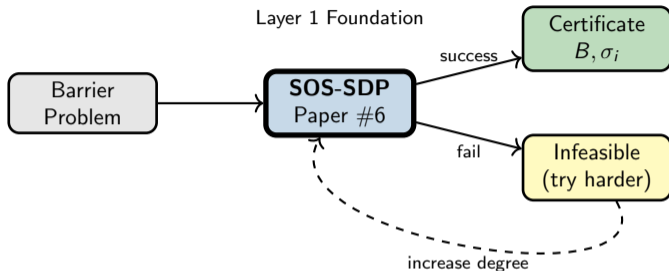
For polynomial  $p$  in  $n$  variables of degree  $2d$ :

- Gram matrix size:  $N = \binom{n+d}{d}$
- SDP has  $O(N^2)$  variables
- $O(N)$  linear constraints (coefficient matching)
- Interior-point method:  $O(N^{4.5})$  per iteration

## Practical Scaling

| $n$ (vars) | $d$ (degree) | Gram size | Typical time |
|------------|--------------|-----------|--------------|
| 2          | 2            | 6         | <1ms         |
| 2          | 4            | 15        | ~10ms        |
| 5          | 4            | 126       | ~1s          |
| 10         | 4            | 1001      | ~1min        |

# SOS-SDP in the Verification Pipeline



## Role in Pipeline

- **First attempt** for polynomial barrier synthesis
- Provides **certificates** that can be verified by Z3
- Failure triggers **degree escalation** (Lasserre hierarchy)

# Paper #7: Lasserre Hierarchy (2001)

## Reference

J. B. Lasserre. “Global optimization with polynomials and the problem of moments.” *SIAM Journal on Optimization*, 2001.

## Core Contribution

A **hierarchy of SDP relaxations** that:

- Provides increasingly tight bounds
- **Converges** to the global optimum
- Gives **completeness** for polynomial positivity

## Key Insight

If  $p \geq 0$  on  $S$  but not SOS, increase degree  $\Rightarrow$  eventually find SOS certificate.

## Primal (Moment Problem):

Find a probability measure  $\mu$  on  $S$  such that:

$$\min_{\mu} \int_S p(x) d\mu(x)$$

subject to  $\text{supp}(\mu) \subseteq S$ .

Moments:  $y_{\alpha} = \int x^{\alpha} d\mu$

## Dual (SOS Problem):

Find maximum  $\gamma$  such that:

$$p(x) - \gamma \in M_d(g_1, \dots, g_m)$$

where  $M_d$  is the degree- $d$  quadratic module.

Provides lower bound on  $p$  over  $S$ .

## Duality

Strong duality holds under mild conditions.

Primal optimal = Dual optimal as  $d \rightarrow \infty$ .

# The Lasserre Hierarchy: Levels

## Definition (Degree- $d$ Relaxation)

The level- $d$  Lasserre relaxation for  $\min_{x \in S} p(x)$ :

$$\gamma_d^* = \max \left\{ \gamma : p - \gamma = \sigma_0 + \sum_i \sigma_i g_i, \deg(\sigma_i g_i) \leq 2d \right\}$$

## Hierarchy Properties

- $\gamma_1^* \leq \gamma_2^* \leq \gamma_3^* \leq \dots \leq p^* = \min_{x \in S} p(x)$
- Each level is an SDP of increasing size
- **Convergence:**  $\gamma_d^* \rightarrow p^*$  as  $d \rightarrow \infty$
- **Finite convergence:** For generic problems, exact at some finite  $d$

# Convergence of the Lasserre Hierarchy

## Theorem (Lasserre 2001)

*If  $S$  is compact and non-empty, and  $p(x) > 0$  for all  $x \in S$ , then there exists  $d_0$  such that for all  $d \geq d_0$ :*

$$p \in M_d(g_1, \dots, g_m)$$

*i.e., the SOS representation exists at some finite level.*

## Implication for Barrier Synthesis

If a polynomial barrier **exists**, the Lasserre hierarchy will **find it** at some level.

**Strategy:** Start at  $d = 2$ , increment until success or resource limit.

# Practical Degree Bounds

## When Does Hierarchy Converge?

**Empirical observation:** Most practical problems converge at low degree.

## Barrier Synthesis Experience

| Problem Type            | Typical $d$ | Notes                           |
|-------------------------|-------------|---------------------------------|
| Linear systems          | 1-2         | Often exact at lowest level     |
| Quadratic dynamics      | 2-4         | Usually tractable               |
| Polynomial (degree 3-4) | 4-6         | May need sparse techniques      |
| High-degree / many vars | 6+          | Consider DSOS/SDSOS relaxations |

Our implementation tries  $d \in \{2, 4, 6\}$  with timeouts.

# Moment Matrices: The Primal View

## Definition (Moment Matrix)

For a sequence  $y = (y_\alpha)$  indexed by monomials, the **moment matrix**  $M_d(y)$  has entries:

$$M_d(y)_{\alpha,\beta} = y_{\alpha+\beta}$$

## Key Constraint

$y$  corresponds to a measure on  $S$  if and only if:

- $M_d(y) \succeq 0$  (moment matrix is PSD)
- $M_{d-d_i}(g_i \cdot y) \succeq 0$  for each constraint  $g_i$

**Localizing matrices:**  $M_k(g \cdot y)_{\alpha,\beta} = \sum_{\gamma} (g)_{\gamma} \cdot y_{\alpha+\beta+\gamma}$

# Extracting Solutions from Moment Relaxation

## When Relaxation is Tight

If  $\text{rank}(M_d(y^*)) = \text{rank}(M_{d-1}(y^*))$  (flat extension), then we can extract minimizers.

### Extraction Algorithm:

- 1 Compute eigendecomposition of  $M_d(y^*)$
- 2 Identify the rank-1 components
- 3 Each rank-1 component gives a point  $x^* \in S$
- 4 Verify:  $p(x^*) = \gamma^*$

## For Barrier Synthesis

When SOS representation found, extract the barrier coefficients and Gram matrices as the **certificate**.

# Implementation: Lasserre Hierarchy Solver

```
class LasserreHierarchySolver:
    """Lasserre moment-SOS hierarchy (Paper \#7)."""

    def __init__(self, n_vars: int, max_level: int = 6):
        self.n_vars = n_vars
        self.max_level = max_level

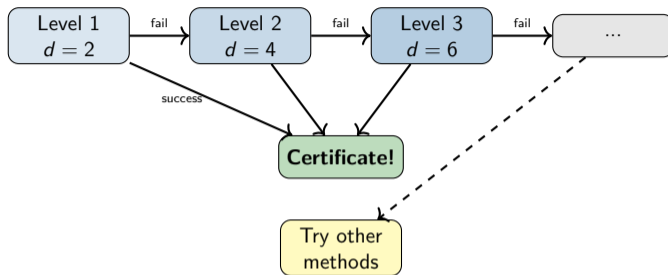
    def solve_hierarchy(self, p: Polynomial,
                       constraints: List[Polynomial]) -> LasserreResult:
        """Solve using ascending hierarchy levels."""
        for level in range(1, self.max_level + 1):
            result = self._solve_level(p, constraints, level)

            if result.status == OPTIMAL:

                if result.has_flat_extension():
                    return LasserreResult(
                        status='solved',
                        optimal_value=result.gamma,
                        level=level,
                        certificate=result.sos_certificate
                    )

        return LasserreResult(status='max_level_reached')
```

# Using Lasserre Hierarchy in Barrier Synthesis



## Practical Strategy

- Start with low degree (fast, often sufficient)
- Escalate on failure (more expressive)
- Timeout per level (avoid stuck computations)
- Fall back to DSOS/SDSOS for large problems

# Lasserre Hierarchy: Summary

## Strengths:

- **Completeness** for polynomial problems
- Systematic degree escalation
- Strong theoretical guarantees
- Provides dual certificates

## In Our Pipeline:

- Falls back after direct SOS fails
- Levels 1-3 typically sufficient
- Provides certificates to Layer 2

## Limitations:

- **Exponential** size growth with level
- Numerical stability challenges
- May require many levels for hard problems

## Mitigation:

- Sparse SOS (Paper #8)
- DSOS/SDSOS relaxations (Paper #9)
- Timeout and fallback strategies

# Paper #8: Sparse SOS (Kojima et al. 2005)

## Reference

M. Kojima, S. Kim, H. Waki. “Sparsity in sums of squares of polynomials.” *Mathematical Programming*, 2005.

## The Scalability Problem

Standard SOS has Gram matrices of size  $\binom{n+d}{d}$ .

- 10 variables, degree 4:  $1001 \times 1001$  matrix
- 20 variables, degree 4:  $10626 \times 10626$  matrix
- Intractable for real-world problems!

## Key Insight

Exploit **sparsity structure** in polynomials to decompose large SDPs into smaller ones.

# Correlative Sparsity Pattern

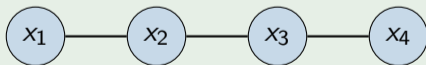
## Definition (Correlative Sparsity)

The **correlative sparsity pattern** (CSP) graph  $G = (V, E)$ :

- Vertices  $V = \{x_1, \dots, x_n\}$  (variables)
- Edge  $(x_i, x_j) \in E$  if  $x_i$  and  $x_j$  appear together in some monomial of  $p$  or constraints

## Example

$$p(x_1, x_2, x_3, x_4) = x_1^2 x_2 + x_2 x_3^2 + x_3 x_4$$



Variables are **not all connected**  $\Rightarrow$  exploitable sparsity!

# Chordal Graphs and Clique Trees

## Definition (Chordal Graph)

A graph is **chordal** if every cycle of length  $\geq 4$  has a chord (edge connecting non-adjacent vertices in the cycle).

## Chordal Extension

Any graph can be extended to a chordal graph by adding edges.  
The **maximal cliques** of a chordal graph form a **clique tree**.

## Why Chordal?

Chordal structure enables:

- Decomposition of large PSD constraint into smaller ones
- Each clique  $\rightarrow$  one smaller SDP
- Clique tree  $\rightarrow$  consistency constraints between SDPs

# Sparse SOS Decomposition Theorem

## Theorem (Sparse SOS - Kojima et al.)

If CSP graph has maximal cliques  $C_1, \dots, C_\ell$ , then  $p$  is SOS iff:

$$p = \sum_{k=1}^{\ell} \sigma_k$$

where each  $\sigma_k$  is SOS in variables  $C_k$  only.

## Computational Savings

Instead of one large SDP:

- Original: Gram matrix of size  $\binom{n+d}{d}$
- Sparse:  $\ell$  Gram matrices, each of size  $\binom{|C_k|+d}{d}$

If cliques are small, this is **exponentially faster**!

## Example: Sparse Decomposition in Action

**Problem:** Check if  $p(x_1, x_2, x_3, x_4) = x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_3x_4 + x_4^2$  is SOS.

**CSP Graph:** Chain  $x_1 - x_2 - x_3 - x_4$

**Maximal Cliques:**  $C_1 = \{x_1, x_2\}$ ,  $C_2 = \{x_2, x_3\}$ ,  $C_3 = \{x_3, x_4\}$

**Sparse SOS:**

$$p = \underbrace{\left(x_1^2 + x_1x_2 + \frac{1}{2}x_2^2\right)}_{\sigma_1 \text{ in } C_1} + \underbrace{\left(\frac{1}{2}x_2^2 + x_2x_3 + \frac{1}{2}x_3^2\right)}_{\sigma_2 \text{ in } C_2} + \underbrace{\left(\frac{1}{2}x_3^2 + x_3x_4 + x_4^2\right)}_{\sigma_3 \text{ in } C_3}$$

**Savings:**

- Dense: One  $5 \times 5$  Gram matrix (degree 2, 4 vars, plus constant)
- Sparse: Three  $3 \times 3$  Gram matrices

# Sparse SOS Algorithm

- 1 **Build CSP Graph:** Identify variable interactions from polynomial
- 2 **Chordal Extension:** Add edges to make graph chordal (minimum fill-in heuristic)
- 3 **Find Maximal Cliques:** Use perfect elimination ordering
- 4 **Set Up Sub-SDPs:** For each clique  $C_k$ , create SOS problem in  $|C_k|$  variables
- 5 **Add Coupling Constraints:** Ensure consistent coefficients at clique intersections
- 6 **Solve Coupled SDPs:** Can be done in parallel for independent cliques

## Complexity

If maximum clique size is  $\omega$ , Gram matrices are  $O\left(\binom{\omega+d}{d}\right)$  instead of  $O\left(\binom{n+d}{d}\right)$ .

# Implementation: Sparse SOS Decomposer

```
class SparseSOSDecomposer:
    """Sparse SOS using correlative sparsity (Paper \#8)."""

    def __init__(self, n_vars: int, max_degree: int):
        self.n_vars = n_vars
        self.max_degree = max_degree

    def decompose(self, p: Polynomial) -> Optional[SparseSOS]:

        csp_graph = self._build_csp_graph(p)

        chordal_graph = self._chordal_extension(csp_graph)

        cliques = self._find_maximal_cliques(chordal_graph)

        sub_problems = []
        for clique in cliques:
            sub_p = p.restrict_to_variables(clique)
            sub_problems.append(SOSProblem(sub_p, clique))

        return self._solve_coupled_sdps(sub_problems)
```

# When Sparse SOS Provides Maximum Benefit

## Best Case: Block-Sparse Structure

System naturally decomposes into weakly-coupled subsystems:

- Multi-agent systems (agents interact locally)
- Networked systems (communication topology)
- Modular software (independent components)

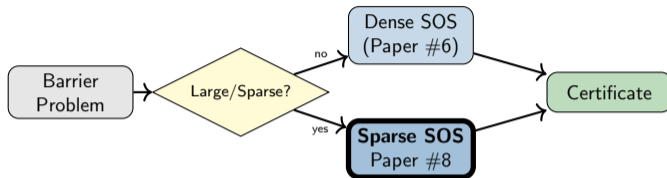
## In Our Pipeline

Python programs often have sparse structure:

- Local variables don't interact with distant code
- Function parameters have limited scope
- Data structures have localized access patterns

Sparse SOS enables barrier synthesis for **larger programs!**

# Sparse SOS in the Verification Pipeline

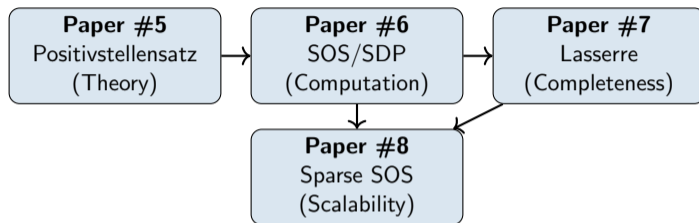


## Selection Heuristic

Use Sparse SOS when:

- $n > 5$  variables
- CSP graph has treewidth  $< n/2$
- Previous dense attempt timed out

# Layer 1 Summary: Mathematical Foundations



## What Layer 1 Provides to Higher Layers

- **Positivity proofs** for polynomial constraints
- **Certificate generation** (Gram matrices, SOS decompositions)
- **Scalable solving** via sparsity exploitation
- **Completeness guarantee** via Lasserre hierarchy

# Part II

## Barrier Certificate Core

Layer 2: Papers #1-4

Hybrid Barriers • Stochastic Barriers • SOS Safety • SOSTOOLS

## Layer 2: From Foundations to Certificates

### What Layer 2 Does

Takes the **mathematical foundations** from Layer 1 and applies them to **specific system types**:

- Continuous dynamical systems
- Discrete transition systems
- Hybrid automata (mixed continuous/discrete)
- Stochastic systems with probabilistic safety

### Key Additions Over Layer 1

- **Dynamics modeling**: How states evolve over time
- **Lie derivatives**: Rate of change along trajectories
- **Mode transitions**: Discrete jumps between continuous dynamics
- **Probability bounds**: Supermartingale conditions for stochastic systems

# Paper #1: Hybrid Barrier Certificates (Prajna-Jadbabaie 2004)

## Reference

S. Prajna & A. Jadbabaie. "Safety verification of hybrid systems using barrier certificates." *HSCC 2004 (Hybrid Systems: Computation and Control)*.

## Core Contribution

Extend barrier certificates to **hybrid automata**:

- Multiple **modes** with different continuous dynamics
- **Discrete transitions** between modes
- **Guards** and **resets** for transitions
- Unified barrier function across all modes

Perfect for programs with **control flow** (if/else, loops, function calls)!

# Hybrid Automaton: Formal Definition

## Definition (Hybrid Automaton)

$\mathcal{H} = (Q, X, \text{Init}, f, \text{Inv}, E, G, R)$  where:

- $Q = \{q_1, \dots, q_m\}$  – finite set of **modes**
- $X \subseteq \mathbb{R}^n$  – continuous state space
- $\text{Init} \subseteq Q \times X$  – initial states
- $f : Q \times X \rightarrow \mathbb{R}^n$  – dynamics per mode:  $\dot{x} = f(q, x)$
- $\text{Inv} : Q \rightarrow 2^X$  – mode invariants
- $E \subseteq Q \times Q$  – discrete transitions
- $G : E \rightarrow 2^X$  – transition guards
- $R : E \times X \rightarrow X$  – reset maps

# Hybrid Barrier Certificate Conditions

## Multi-Mode Barrier

For each mode  $q \in Q$ , define barrier  $B_q : X \rightarrow \mathbb{R}$ .

## Safety Conditions

- ❶ **Init:**  $\forall (q, x) \in \text{Init}. B_q(x) \geq 0$
- ❷ **Unsafe:**  $\forall (q, x) \in \text{Unsafe}. B_q(x) < 0$
- ❸ **Flow (per mode):**  $\forall q, x \in \text{Inv}(q). B_q(x) \geq 0 \Rightarrow \mathcal{L}_{f_q} B_q(x) \geq 0$
- ❹ **Jump:**  $\forall (q, q') \in E, x \in G(q, q'). B_q(x) \geq 0 \Rightarrow B_{q'}(R(x)) \geq 0$

The **Lie derivative**  $\mathcal{L}_f B = \nabla B \cdot f$  measures how  $B$  changes along trajectories.

# The Lie Derivative: Key to Continuous Dynamics

## Definition (Lie Derivative)

For barrier  $B : \mathbb{R}^n \rightarrow \mathbb{R}$  and vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ :

$$\mathcal{L}_f B(x) = \nabla B(x) \cdot f(x) = \sum_{i=1}^n \frac{\partial B}{\partial x_i} \cdot f_i(x)$$

## Geometric Interpretation

$\mathcal{L}_f B(x)$  is the **rate of change** of  $B$  at  $x$  as the system flows along  $f$ .

- $\mathcal{L}_f B \geq 0$  when  $B \geq 0$ : Barrier value doesn't decrease
- Trajectories starting with  $B \geq 0$  **stay** with  $B \geq 0$
- This is the **continuous induction step!**

# Modeling Programs as Hybrid Automata

## Python Program

```
if x > 0: y = x * 2
else: y = -x
```

## As Hybrid Automaton:

- Mode  $q_1$ : “then branch” with guard  $x > 0$ , dynamics  $y' = 2x$
- Mode  $q_2$ : “else branch” with guard  $x \leq 0$ , dynamics  $y' = -x$
- Transition from entry to  $q_1$  or  $q_2$  based on  $x$

## For Bug Detection

- Unsafe = states where bug occurs (e.g.,  $y < 0$  for bounds check)
- Synthesize barriers for each mode
- Verify jump conditions at branches

# Hybrid Barrier Synthesis Algorithm

**Input:** Hybrid automaton  $\mathcal{H}$ , unsafe set, barrier degree  $d$

## Algorithm:

- ① For each mode  $q$ , create polynomial template  $B_q(x) = \sum_{\alpha} b_{q,\alpha} x^{\alpha}$
- ② Set up SOS constraints:
  - Init:  $B_q - \varepsilon \in \Sigma + M(\text{Init}_q)$
  - Unsafe:  $-B_q - \varepsilon \in \Sigma + M(\text{Unsafe}_q)$
  - Flow:  $-\mathcal{L}_{f_q} B_q \in \Sigma + M(\text{Inv}_q) + M(B_q)$  (when  $B_q \geq 0$ )
  - Jump:  $B_{q'}(R(x)) - B_q(x) \in \Sigma + M(G_{q \rightarrow q'}) + M(B_q)$
- ③ Solve joint SDP for all  $b_{q,\alpha}$  and Gram matrices
- ④ Extract barrier polynomials

# Implementation: Hybrid Barrier Synthesizer

```
@dataclass
class HybridMode:
    """A mode in a hybrid automaton."""
    mode_id: int
    dynamics: ContinuousDynamics
    invariant: SemialgebraicSet

class HybridBarrierSynthesizer:
    """Synthesize barriers for hybrid systems (Paper \#1)."""

    def synthesize(self, automaton: HybridAutomaton,
                   unsafe: Dict[int, SemialgebraicSet]) -> HybridBarrier:

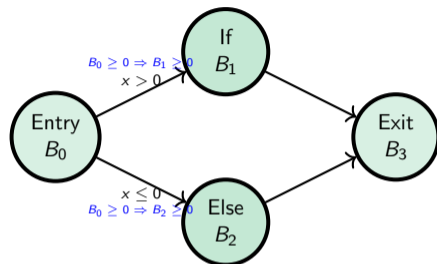
        barriers = {}
        for mode in automaton.modes:
            barriers[mode.mode_id] = BarrierTemplate(
                self.n_vars, self.max_degree
            )

        constraints = []
        for mode in automaton.modes:
            constraints += self._mode_constraints(mode, barriers, unsafe)

        for trans in automaton.transitions:
            constraints += self._jump_constraints(trans, barriers)

        return self._solve_and_extract(constraints, barriers)
```

# Hybrid Barriers for Program Control Flow



## CFG $\rightarrow$ Hybrid Automaton

- Basic blocks  $\rightarrow$  modes
- Branch conditions  $\rightarrow$  guards
- Variable updates  $\rightarrow$  reset maps
- Loop iterations  $\rightarrow$  self-transitions

# Paper #2: Stochastic Barrier Certificates (Prajna et al. 2007)

## Reference

S. Prajna, A. Jadbabaie, G. J. Pappas. “A framework for worst-case and stochastic safety verification.”

*IEEE Transactions on Automatic Control*, 2007.

## Core Contribution

Extend barrier certificates to **stochastic systems**:

$$dx = f(x) dt + g(x) dW_t$$

where  $W_t$  is a Wiener process (Brownian motion).

## Why Stochastic?

Programs have inherent randomness:

# Probabilistic Safety Guarantees

## Definition (Probabilistic Safety)

System is  $\delta$ -safe if:

$$\Pr[\text{reach unsafe within time } T] \leq \delta$$

## Supermartingale Condition

Instead of  $\mathcal{L}_f B \leq 0$ , we need:

$$\mathcal{A}B(x) \leq -\lambda B(x) \quad (\text{for } B \geq 0)$$

where  $\mathcal{A}$  is the **infinitesimal generator**:

$$\mathcal{A}B = \nabla B \cdot f + \frac{1}{2} \text{tr}(g^\top \nabla^2 B g)$$

The second term accounts for **diffusion** (stochastic spread).

# Stochastic Barrier Certificate Conditions

## Conditions for Probabilistic Safety

For stochastic system  $dx = f(x) dt + g(x) dW_t$ :

- ➊ **Init:**  $\forall x \in X_0. B(x) \leq \gamma$
- ➋ **Unsafe:**  $\forall x \in X_u. B(x) \geq 1$
- ➌ **Supermartingale:**  $\forall x \in X. \mathcal{A}B(x) \leq \lambda B(x)$

Then:  $\Pr[\text{reach } X_u] \leq \gamma \cdot e^{\lambda T}$

## For Barrier Synthesis

- Minimize  $\gamma$  (probability bound)
- Template for  $B$  as polynomial
- SOS constraint on  $\lambda B - \mathcal{A}B \geq 0$

# Implementation: Stochastic Barrier Synthesizer

```
@dataclass
class StochasticDynamics:
    """Stochastic differential equation:  $dx = f(x)dt + g(x)dW$ ."""
    n_vars: int
    drift: List[Polynomial]
    diffusion: List[Polynomial]

    def infinitesimal_generator(self, B: Polynomial) -> Polynomial:
        """Compute  $AB = \text{nabla} B \cdot f + 0.5 * \text{tr}(g^T \text{Hess}(B) g)$ ."""

        gradient = B.gradient()
        drift_term = sum(g.multiply(f) for g, f in zip(gradient, self.drift))

        hessian = B.hessian()
        diffusion_term = Polynomial.zero(self.n_vars)
        for i in range(self.n_vars):
            for j in range(self.n_vars):
                term = self.diffusion[i].multiply(
                    hessian[i][j].multiply(self.diffusion[j])
                )
                diffusion_term = diffusion_term.add(term.scale(0.5))

        return drift_term.add(diffusion_term)
```

# Paper #3: SOS Safety (Papachristodoulou-Prajna 2002)

## Reference

A. Papachristodoulou & S. Prajna. “On the construction of Lyapunov functions using the sum of squares decomposition.”  
*CDC 2002 (Conference on Decision and Control)*.

## Core Contribution

Use SOS decomposition for **set emptiness checking**:

- Given constraints  $g_1(x) \geq 0, \dots, g_m(x) \geq 0$
- Prove the set  $S = \{x : g_i(x) \geq 0 \text{ for all } i\}$  is **empty**
- If empty  $\Rightarrow$  no bad states exist!

This is the **core technique** for proving barrier conditions hold.

# Set Emptiness via SOS

## Theorem (SOS Emptiness Certificate)

*The set  $S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  is empty if there exist SOS polynomials  $\sigma_0, \sigma_1, \dots, \sigma_m$  such that:*

$$-1 = \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i$$

## Why This Works

If such  $\sigma_i$  exist:

- LHS:  $-1 < 0$  (always)
- RHS:  $\geq 0$  on  $S$  (sum of nonnegatives)
- Contradiction  $\Rightarrow S$  must be empty!

# SOS Safety for Barrier Verification

**Goal:** Verify “No state is both initial and unsafe”

**Set to check empty:**

$$S = \{x : x \in X_0 \text{ (initial)} \wedge x \in X_u \text{ (unsafe)}\}$$

**Express as polynomial constraints:**

$$S = \{x : g_1^{(0)}(x) \geq 0, \dots, g_k^{(0)}(x) \geq 0, g_1^{(u)}(x) \geq 0, \dots, g_\ell^{(u)}(x) \geq 0\}$$

**Find SOS certificate:**

$$-1 = \sigma_0 + \sum_i \sigma_i^{(0)} g_i^{(0)} + \sum_j \sigma_j^{(u)} g_j^{(u)}$$

If certificate exists  $\Rightarrow$  **disjoint**  $\Rightarrow$  **SAFE!**

# Implementation: SOS Safety Checker

```
class SOSSafetyChecker:
    """SOS-based set emptiness checking (Paper \#3)."""

    def check_disjoint(self, set1: SemialgebraicSet,
                      set2: SemialgebraicSet) -> SafetyResult:
        """Check if set1 and set2 are disjoint using SOS."""

        all_constraints = set1.constraints + set2.constraints

        multipliers = [self._create_sos_template(c.degree)
                       for c in all_constraints]

        sigma_0 = self._create_sos_template(self.max_degree)

        target = Polynomial.constant(-1)
        rhs = sigma_0.to_polynomial()
        for mult, constraint in zip(multipliers, all_constraints):
            rhs = rhs.add(mult.to_polynomial().multiply(constraint))

        return self._solve_emptiness_sdp(target, rhs, [sigma_0] + multipliers)
```

# Verifying the Inductive Step with SOS

**Goal:** Prove  $\{B(x) \geq 0 \wedge \mathcal{L}_f B(x) < 0\}$  is empty.

**Reformulate:** Show that on the set where  $B \geq 0$ , we have  $\mathcal{L}_f B \geq 0$ .

## SOS Formulation

Find SOS  $\sigma_0, \sigma_1$  such that:

$$\mathcal{L}_f B(x) = \sigma_0(x) + \sigma_1(x) \cdot B(x)$$

This proves: when  $B \geq 0$ , then  $\mathcal{L}_f B \geq 0$  (since RHS  $\geq 0$ ).

## The S-procedure

This is the **S-procedure**: proving implication via SOS multipliers.

# Paper #4: SOSTOOLS Framework (Prajna et al. 2004)

## Reference

S. Prajna, A. Papachristodoulou, P. A. Parrilo. “SOSTOOLS: Sum of squares optimization toolbox for MATLAB.”

*User's Guide*, 2004.

## Core Contribution

A **unified software framework** for SOS programming:

- Declarative specification of SOS constraints
- Automatic translation to SDP
- Support for parametric templates
- Numerical and symbolic solving

Our Python implementation mirrors SOSTOOLS' design patterns.

## Design Philosophy

Separate **problem specification** from **solver mechanics**:

- User declares polynomial variables and constraints
- System builds SDP automatically
- Solver produces certificates

## Typical Workflow:

- 1 Define polynomial variables: `x = poly('x', 2)`
- 2 Create templates: `B = create_template(degree=4)`
- 3 Add SOS constraints: `add_sos(B - eps)`
- 4 Add implications: `add_sos(-Lie(B), when=B >= 0)`
- 5 Solve: `result = solve()`
- 6 Extract: `barrier = result.get_polynomial(B)`

# Implementation: SOSTOOLS-Style Framework

```
class SOSTOOLSFramework:
    """SOSTOOLS-style declarative SOS programming (Paper \#4)."""

    def __init__(self, n_vars: int, var_names: List[str] = None):
        self.n_vars = n_vars
        self.var_names = var_names or [f'x{i}' for i in range(n_vars)]
        self.decision_vars = []
        self.sos_constraints = []
        self.equality_constraints = []

    def create_template(self, name: str, degree: int) -> PolynomialTemplate:
        """Create a polynomial template with symbolic coefficients."""
        template = PolynomialTemplate(self.n_vars, degree, name)
        self.decision_vars.extend(template.coefficients)
        return template

    def add_sos(self, expr: Polynomial,
               multipliers: List[Tuple[Polynomial, Polynomial]] = None):
        """Add constraint: expr is SOS (optionally on a set)."""
        if multipliers:
            self.sos_constraints.append(SOSWithMultipliers(expr, multipliers))
        else:
            self.sos_constraints.append(PureSOS(expr))
```

# Complete Barrier Synthesis Example

```
def synthesize_barrier_sostools(dynamics, initial, unsafe, degree=4):  
    """Synthesize barrier using SOSTOOLS framework."""  
    n_vars = dynamics.n_vars  
    sos = SOSTOOLSFramework(n_vars)  
  
    B = sos.create_template('B', degree)  
    eps = 0.01  
  
    sos.add_sos(B - eps, on_set=initial)  
  
    sos.add_sos(-B - eps, on_set=unsafe)  
  
    Lie_B = dynamics.lie_derivative(B)  
    sos.add_sos(-Lie_B, when=[B >= 0])  
  
    result = sos.solve()  
  
    if result.status == 'optimal':  
        return result.extract_polynomial(B)  
    return None
```

# Parametric Barrier Templates

## Template Families

Common barrier template structures:

### Quadratic:

$$B(x) = x^\top P x + p^\top x + c$$

Parameters:  $P$  (matrix),  $p$  (vector),  $c$  (scalar)

### Polynomial:

$$B(x) = \sum_{|\alpha| \leq d} b_\alpha x^\alpha$$

Parameters: coefficients  $b_\alpha$

### Piecewise:

$$B(x) = \begin{cases} B_1(x) & x \in R_1 \\ B_2(x) & x \in R_2 \end{cases}$$

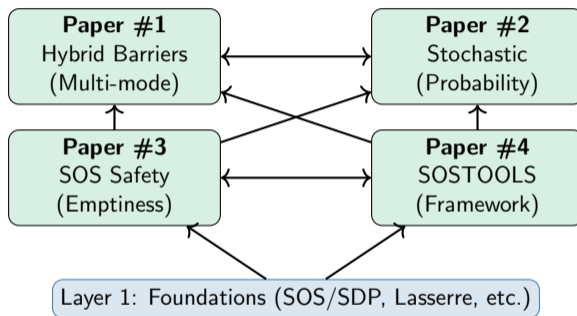
Different polynomial per region

### Neural: (future work)

$$B(x) = \text{NN}_\theta(x)$$

Neural network with verification

## Layer 2: How Papers Integrate



Papers #3-4 provide core techniques; Papers #1-2 apply to specific system types.

# Unified Interface: BarrierCertificateEngine

```
class BarrierCertificateEngine:
    """Unified interface for Layer 2 barrier synthesis."""

    def __init__(self, n_vars: int, system_type: str, max_degree: int):
        self.n_vars = n_vars
        self.system_type = system_type
        self.max_degree = max_degree

        if system_type == 'continuous':
            self.synthesizer = SOSSafetyChecker(n_vars, max_degree)
        elif system_type == 'hybrid':
            self.synthesizer = HybridBarrierSynthesizer(n_vars, max_degree)
        elif system_type == 'stochastic':
            self.synthesizer = StochasticBarrierSynthesizer(n_vars, max_degree)
        else:
            self.synthesizer = SOSTOOLSFramework(n_vars)

    def synthesize(self, conditions: BarrierConditions,
                  dynamics: Any) -> Optional[BarrierCertificate]:
        """Synthesize barrier certificate."""
        return self.synthesizer.synthesize(conditions, dynamics)
```

# Layer 2 Summary: Barrier Certificate Core

## What We Can Now Do:

- Synthesize barriers for **continuous** systems
- Handle **hybrid** systems with mode switching
- Provide **probabilistic** safety for stochastic systems
- Check **set emptiness** for verification

## What We Provide to Layer 3:

- Barrier templates to refine
- Certificates to abstract
- Verification oracles

## Limitations Addressed by Higher Layers:

- Need **degree selection** → CEGAR
- Need **predicates** → Abstraction
- Need **examples** → Learning

## Files:

- `certificate_core.py`
- `hybrid_barrier.py`
- `stochastic_barrier.py`
- `sos_safety.py`

# From Synthesis to Abstraction

## The Challenge

Layer 2 synthesis requires:

- Choosing the right **degree** for polynomials
- Selecting good **predicates** for discrete reasoning
- Handling **counterexamples** from failed synthesis

## Layer 3's Solution

**Abstraction-Refinement** techniques:

- CEGAR: Use counterexamples to **refine** abstraction
- Predicate Abstraction: Reduce to **finite** state space
- Boolean Programs: Symbolic execution on **abstract** states
- IMPACT: **Lazy** refinement on-demand

# Part III

## Abstraction & Refinement

Layer 3: Papers #12-14, 16

CEGAR • Predicate Abstraction • Boolean Programs • IMPACT

# Layer 3: Managing Complexity Through Abstraction

## The Problem

Real programs have:

- Infinite state spaces (integers, floats, objects)
- Complex control flow (loops, recursion, exceptions)
- Many variables with intricate relationships

Direct synthesis on concrete state space is often **intractable**.

## The Solution: Abstraction

- Map infinite state to **finite abstract domain**
- Reason about abstract states
- Refine when abstraction is too coarse

## Reference

E. Clarke, O. Grumberg, S. Jha, Y. Lu, H. Veith. "Counterexample-Guided Abstraction Refinement."

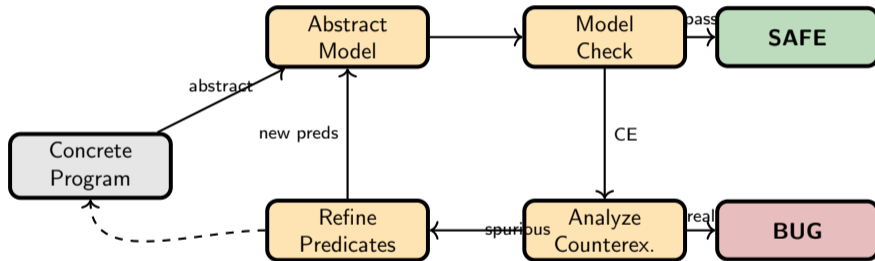
*CAV 2000 (Computer Aided Verification).*

## Core Contribution

A **feedback loop** that automatically refines abstractions:

- ➊ **Abstract:** Create coarse model
- ➋ **Verify:** Check property on abstract model
- ➌ **Analyze:** If counterexample found, check if **spurious**
- ➍ **Refine:** If spurious, add predicates to distinguish
- ➎ **Repeat:** Until verified or real bug found

# The CEGAR Loop



# Spurious vs. Real Counterexamples

## Definition (Spurious Counterexample)

A counterexample in the **abstract** model that has no corresponding **concrete** execution path.

### Real Counterexample:

- Path exists in concrete program
- Reaches actual unsafe state
- $\Rightarrow$  Report **BUG!**

### Spurious Counterexample:

- Only exists in abstraction
- Caused by lost precision
- $\Rightarrow$  **Refine** abstraction

## Detection Method

Symbolically execute the counterexample path.

If path constraints are SAT  $\rightarrow$  real. If UNSAT  $\rightarrow$  spurious.

# Refinement: Learning from Spurious Counterexamples

## The Key Question

When a counterexample is spurious, **why**? What predicates would eliminate it?

## Refinement Strategies:

- 1 **Interpolation-based:** Use Craig interpolants from UNSAT proof

$$\phi_1 \wedge \phi_2 = \text{UNSAT} \Rightarrow \exists I. \phi_1 \Rightarrow I \wedge I \wedge \phi_2 = \text{UNSAT}$$

Interpolant  $I$  becomes new predicate.

- 2 **Weakest precondition:** Compute  $\text{wp}(\text{unsafe}, \text{path})$  and extract predicates
- 3 **Counterexample-guided:** Extract predicates that distinguish concrete states along spurious path

## Applying CEGAR to Barriers

- ➊ **Initial:** Try simple barrier (low degree, few variables)
- ➋ **Attempt synthesis:** Use Layer 2 SOS/SDP
- ➌ **If fails:** Get **counterexample** from SDP dual
- ➍ **Analyze:** Is counterexample reachable? (Check with Z3)
- ➎ **If spurious:** Refine barrier template:
  - Increase degree
  - Add new predicates
  - Partition state space
- ➏ **If real:** Report bug with witness

# Implementation: CEGAR Loop

```
@dataclass
class CEGARResult:
    """Result of CEGAR refinement loop."""
    status: str
    certificate: Optional[BarrierCertificate] = None
    counterexample: Optional[Counterexample] = None
    iterations: int = 0
    predicates_added: int = 0

class CEGARLoop:
    """CEGAR refinement loop (Paper \#12)."""

    def verify(self, program, property, initial_predicates):
        predicates = list(initial_predicates)

        for iteration in range(self.max_iterations):

            abstraction = self._abstract(program, predicates)

            result = self._model_check(abstraction, property)

            if result.verified:
                return CEGARResult('safe', result.certificate)

            is_real = self._check_feasibility(result.counterexample)

            if is_real:
                return CEGARResult('unsafe', counterexample=result.counterexample)
```

# Extracting Counterexamples from SDP

## When SOS Synthesis Fails

SDP solver returns “infeasible” – no barrier of given degree exists.

The **dual solution** provides information:

- **Farkas certificate:** Proves no solution exists
- **Moment interpretation:** Dual variables represent “problematic” state distributions
- **Extraction:** Find concrete states that violate barrier conditions

## Practical Approach

Use Z3 to find **concrete states** where:

- State is initial AND barrier value is low, OR
- State is safe AND Lie derivative is positive, OR
- State is near unsafe AND barrier value is high

## From Counterexamples to Predicates

A spurious counterexample reveals **what the abstraction is missing**.

### Predicate Sources:

- ① **Path conditions:** Branch conditions along counterexample path

$$\text{path: } x > 0 \rightarrow y = x + 1 \rightarrow z = y \cdot 2 \quad \Rightarrow \quad \text{pred: } x > 0$$

- ② **Variable relationships:** Equalities/inequalities that hold

$$\text{counterexample shows } y = x + 1 \quad \Rightarrow \quad \text{pred: } y = x + 1$$

- ③ **Barrier-relevant:** Predicates from barrier template structure

$$B(x) = x^2 - 4 \quad \Rightarrow \quad \text{preds: } x > 2, x > -2, x^2 > 4$$

# CEGAR Termination and Completeness

## Termination

CEGAR is **not guaranteed** to terminate in general:

- Predicate set may grow unboundedly
- Some abstractions never become precise enough

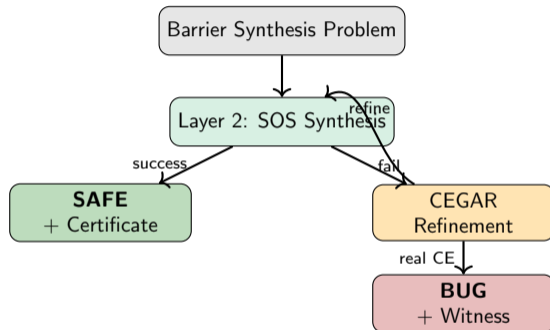
## Practical Guarantees

In practice, CEGAR often terminates because:

- Predicates are drawn from finite program syntax
- Many properties need only few predicates
- Timeouts convert non-termination to “unknown”

Our implementation: max 10 iterations, 30s timeout, tracks predicate count.

# CEGAR for Barriers: Summary



CEGAR enables **automatic degree/predicate selection** for barrier synthesis!

# Paper #13: Predicate Abstraction (Graf-Saïdi 1997)

## Reference

S. Graf & H. Saïdi. “Construction of Abstract State Graphs with PVS.”  
*CAV 1997*.

## Core Contribution

Map **infinite** concrete states to **finite** abstract states using predicates:

- Choose predicates  $P = \{p_1, \dots, p_k\}$
- Abstract state = valuation of all predicates
- At most  $2^k$  abstract states

This enables **finite-state model checking** on infinite-state programs!

# Predicate Abstraction: Example

## Concrete Program

```
x := 0; while (x < 100) { x := x + 1 }
```

**Predicates:**  $P = \{x < 100, x \geq 0\}$

**Abstract States:**

| $x < 100$ | $x \geq 0$ | Represents                               |
|-----------|------------|--|
| T         | T          | $0 \leq x < 100$ (in loop)               |
| F         | T          | $x \geq 100$ (after loop)                |
| T         | F          | $x < 0$ (unreachable)                    |
| F         | F          | impossible ( $x \geq 100 \wedge x < 0$ ) |

**Abstract transitions:**  $(T, T) \rightarrow (T, T)$  (loop),  $(T, T) \rightarrow (F, T)$  (exit)

# Computing Abstract Successors

## The Problem

Given abstract state  $\hat{s}$  and concrete transition  $\tau$ , compute abstract successors.

## Approach: SMT-based Image Computation

For each predicate  $p$  and abstract state  $\hat{s}$ :

- 1 Query: Is  $\hat{s} \wedge \tau \wedge p'$  satisfiable?
- 2 If yes:  $p$  can be true in successor
- 3 Query: Is  $\hat{s} \wedge \tau \wedge \neg p'$  satisfiable?
- 4 If yes:  $p$  can be false in successor

## Cost

$O(2^k \cdot k \cdot 2)$  SAT queries per transition (expensive!)

Optimization: Use BDDs, incremental SAT, cartesian abstraction.

# Implementation: Predicate Abstraction

```
@dataclass
class Predicate:
    """A predicate over program variables."""
    name: str
    expr: z3.ExprRef
    variables: Set[str]

@dataclass
class AbstractState:
    """Abstract state = valuation of predicates."""
    valuation: FrozenSet[Tuple[str, bool]]

class PredicateAbstraction:
    """Predicate abstraction engine (Paper \#13)."""

    def __init__(self, predicates: List[Predicate], variables: List[z3.ExprRef]):
        self.predicates = predicates
        self.variables = variables
        self.abstract_trans = {}

    def abstract_successor(self, state: AbstractState,
                          transition: z3.ExprRef) -> Set[AbstractState]:
        """Compute abstract successors via SMT."""
        successors = set()

        for valuation in self._enumerate_valuations():
            if self._is_feasible(state, transition, valuation):
                successors.add(AbstractState(valuation))
```

# Paper #14: Boolean Programs (Ball-Rajamani 2001)

## Reference

T. Ball & S. K. Rajamani. "Boolean Programs: A Model and Process for Software Analysis." *MSR Technical Report*, 2001. (Also SLAM project)

## Core Contribution

Represent abstract program as **Boolean program**:

- All variables are Boolean
- Control flow preserved from concrete program
- Statements update Boolean variables based on predicates
- Enables standard model checking algorithms

Foundation of Microsoft's SLAM project (device driver verification).

# Constructing Boolean Programs

**Original:**  $x := y + 1$

**Predicates:**  $\{x > 0, y > 0, x > y\}$

**Boolean Program:**

```
b1 := (y > 0) ? true : *;    // x > 0 after x := y + 1
b2 := b2;    // y > 0 unchanged
b3 := true;   // x > y always after x := y + 1
```

## Key Insight

The  $*$  (nondeterminism) captures cases where predicate value is unknown.  
Sound abstraction: concrete behavior  $\subseteq$  abstract behavior.

# Implementation: Boolean Program Executor

```
class BooleanProgram:
    """Boolean program abstraction (Paper \#14)."""

    def __init__(self, predicates: List[Predicate]):
        self.predicates = predicates
        self.n_predicates = len(predicates)
        self.transitions = []

    def add_statement(self, original_stmt, pre_abstract, post_abstract):
        """Add abstracted statement."""

        updates = []
        for i, pred in enumerate(self.predicates):
            effect = self._compute_predicate_effect(pred, original_stmt)
            updates.append(effect)

        self.transitions.append(BooleanTransition(updates))

class BooleanProgramExecutor:
    """Symbolic execution on Boolean programs."""

    def explore_all_paths(self, program: BooleanProgram,
                          initial: AbstractState) -> Set[AbstractState]:
        """BFS exploration of abstract state space."""
        visited = {initial}
        worklist = [initial]

        while worklist:
            state = worklist.pop(0)
            for succ in program.successors(state):
```

## Integration Strategy

- 1 Extract **barrier-relevant predicates** from barrier template

$$B(x, y) = x^2 + y^2 - 1 \quad \Rightarrow \quad \{x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

- 2 Build **Boolean program** using these predicates
- 3 **Model check** Boolean program for reachability
- 4 If unsafe reachable: counterexample guides barrier refinement
- 5 If safe: extract **abstract certificate**

Boolean programs enable **efficient exploration** of abstract state space!

## Reference

K. L. McMillan. “Lazy Abstraction with Interpolants.”  
*CAV 2006*.

## Core Contribution

**On-demand** abstraction refinement:

- Don't pre-compute full abstract model
- Build abstraction **lazily** during exploration
- Use **interpolants** for refinement
- Different abstraction at different program points

Often much more efficient than eager predicate abstraction!

# IMPACT: Lazy Abstraction with Interpolants

**Key Innovation:** Build Abstract Reachability Tree (ART) on-the-fly.

- 1 **Unfold:** Explore concrete paths symbolically
- 2 **Check:** When path reaches error, check feasibility
- 3 **If feasible:** Report bug with concrete trace
- 4 **If infeasible:** Compute Craig interpolants from UNSAT proof

$$\underbrace{\phi_1}_{\text{prefix}} \wedge \underbrace{\phi_2}_{\text{suffix}} = \text{UNSAT}$$

- 5 **Annotate:** Label tree nodes with interpolants
- 6 **Subsumption:** Prune tree when new state is covered by existing

Interpolants provide **exactly the right predicates** for each location!

# Craig Interpolation: The Key to IMPACT

## Theorem (Craig Interpolation)

*If  $\phi_1 \wedge \phi_2$  is unsatisfiable, there exists formula  $I$  such that:*

- ①  $\phi_1 \Rightarrow I$
- ②  $I \wedge \phi_2$  is unsatisfiable
- ③  $I$  uses only symbols common to  $\phi_1$  and  $\phi_2$

## For Path Analysis

- $\phi_1$  = prefix of spurious path
- $\phi_2$  = suffix of spurious path
- $I$  = **reason** why suffix cannot lead to error

Interpolant  $I$  becomes an annotation (invariant) at the cut point!

# Implementation: Lazy Abstraction

```
@dataclass
class ARTNode:
    """Node in Abstract Reachability Tree."""
    location: int
    path_formula: z3.ExprRef
    annotation: z3.ExprRef
    parent: Optional['ARTNode']
    children: List['ARTNode']

class LazyAbstraction:
    """IMPACT lazy abstraction (Paper \#16)."""

    def verify(self, program, property) -> VerificationResult:
        root = ARTNode(0, z3.BoolVal(True), z3.BoolVal(True), None, [])
        worklist = [root]

        while worklist:
            node = worklist.pop()

            if self._is_error(node):
                if self._is_feasible(node.path_formula):
                    return VerificationResult('unsafe', self._extract_trace(node))
                else:

                    self._refine_with_interpolants(node)
                    continue

            if self._is_covered(node):
                continue
```

# Computing Interpolants

## From SMT Solver

Modern SMT solvers (Z3, MathSAT) can extract interpolants from UNSAT proofs.

### Algorithm:

- 1 Split path formula at program location  $\ell$

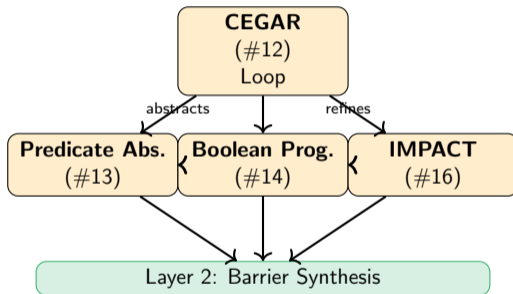
$$\underbrace{\text{path}[0 \rightarrow \ell]}_{\phi_1} \wedge \underbrace{\text{path}[\ell \rightarrow \text{error}]}_{\phi_2}$$

- 2 Ask solver: Is  $\phi_1 \wedge \phi_2$  SAT?
- 3 If UNSAT: Extract interpolant  $I$
- 4 Use  $I$  as annotation at location  $\ell$

## Sequence of Interpolants

For path of length  $n$ , compute interpolants  $I_0, I_1, \dots, I_n$  at each location.  
These form a **trace abstraction** of the spurious path.

## Layer 3: How Papers Integrate



# Unified Interface: AbstractionRefinementEngine

```
class AbstractionRefinementEngine:
    """Unified interface for Layer 3 abstraction-refinement."""

    def __init__(self, initial_predicates: List[Predicate],
                  max_iterations: int = 100):
        self.predicates = list(initial_predicates)
        self.max_iterations = max_iterations

        self.pred_abstraction = PredicateAbstraction(self.predicates)
        self.cegar = CEGARLoop(self.pred_abstraction)
        self.lazy = LazyAbstraction()

    def verify(self, program, property) -> VerificationResult:
        """Verify using abstraction-refinement."""

        result = self.lazy.verify(program, property)

        if result.status != 'unknown':
            return result

        return self.cegar.verify(program, property, self.predicates)

    def get_barrier_predicates(self) -> List[Predicate]:
        """Get predicates useful for barrier synthesis."""
        return self.predicates
```

# Layer 3 Summary: Abstraction & Refinement

## What Layer 3 Provides:

- Finite-state reasoning on infinite programs
- Automatic predicate discovery
- Counterexample-guided refinement
- Lazy on-demand abstraction

## To Layer 4 (Learning):

- Predicates to learn over
- Abstract traces for examples
- Refinement feedback

## Files:

- `abstraction.py`
- `cegar_refinement.py`
- `predicate_abstraction.py`
- `boolean_programs.py`
- `impact_lazy.py`

## Key Metrics:

- Predicates discovered
- CEGAR iterations
- Tree size (IMPACT)

# Part IV

## Learning-Based Synthesis

Layer 4: Papers #17-19

ICE Learning • Houdini Inference • SyGuS Synthesis

## Layer 4: Learning Invariants from Data

### The Insight

Instead of **synthesizing** invariants from scratch, **learn** them from:

- Concrete program executions (positive examples)
- Counterexamples (negative examples)
- Transition pairs (implication examples)

### Learning Framework

- **ICE**: Learn from Implications, Counterexamples, Examples
- **Houdini**: Conjunctive inference from candidate set
- **SyGuS**: Syntax-guided synthesis from grammar

# Paper #17: ICE Learning (Garg et al. 2014)

## Reference

P. Garg, C. Löding, P. Madhusudan, D. Neider. “ICE: A Robust Framework for Learning Invariants.”  
*CAV 2014*.

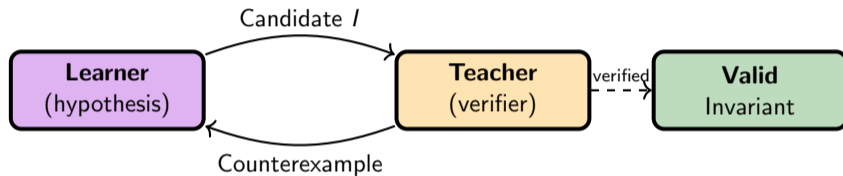
## Core Contribution

Learn invariants from **three types** of examples:

- Implication examples:  $(s, s')$  pairs from transitions
- Counterexamples: States violating current hypothesis
- Examples: Positive (in invariant) and negative (out) states

ICE provides a **teacher-learner** framework for invariant inference.

# ICE: Teacher-Learner Framework



Updates hypothesis  
based on examples

Checks:  $\text{Init} \Rightarrow I$   
 $I \wedge T \Rightarrow I'$   
 $I \Rightarrow \text{Safe}$

# ICE: The Three Types of Examples

## 1. Positive Examples (must be in invariant)

States from initial region or reachable safe states.

$$\mathcal{P} = \{s : s \in \text{Init} \vee s \text{ is reachable and safe}\}$$

## 2. Negative Examples (must NOT be in invariant)

States in unsafe region.

$$\mathcal{N} = \{s : s \in \text{Unsafe}\}$$

## 3. Implication Examples (transition pairs)

If pre-state is in invariant, post-state must be too.

$$\mathcal{I} = \{(s, s') : s \xrightarrow{T} s'\}$$

# ICE Learning Algorithm

- 1: Initialize:  $\mathcal{P} \leftarrow$  samples from Init,  $\mathcal{N} \leftarrow$  samples from Unsafe,  $\mathcal{I} \leftarrow \emptyset$
- 2:  $I \leftarrow \text{Learn}(\mathcal{P}, \mathcal{N}, \mathcal{I})$  ▷ Initial hypothesis
- 3: **while** not verified **do**
- 4:      $\text{result} \leftarrow \text{Teacher.Check}(I)$
- 5:     **if**  $\text{result} = \text{VALID}$  **then**
- 6:         **return**  $I$
- 7:     **else if**  $\text{result} = \text{CE}_{\text{init}}$  **then** ▷ Init violation
- 8:          $\mathcal{P} \leftarrow \mathcal{P} \cup \{\text{counterexample}\}$
- 9:     **else if**  $\text{result} = \text{CE}_{\text{safe}}$  **then** ▷ Safety violation
- 10:          $\mathcal{N} \leftarrow \mathcal{N} \cup \{\text{counterexample}\}$
- 11:     **else if**  $\text{result} = \text{CE}_{\text{ind}}$  **then** ▷ Induction violation
- 12:          $\mathcal{I} \leftarrow \mathcal{I} \cup \{(\text{pre}, \text{post})\}$
- 13:     **end if**
- 14:      $I \leftarrow \text{Learn}(\mathcal{P}, \mathcal{N}, \mathcal{I})$  ▷ Re-learn
- 15: **end while**

# Implementation: ICE Learner

```
@dataclass
class ICEExample:
    """ICE data: positive, negative, and implication examples."""
    positive: List[DataPoint]
    negative: List[DataPoint]
    implications: List[Tuple[DataPoint, DataPoint]]

class ICELearner:
    """ICE Learning for invariant inference (Paper \#17)."""

    def __init__(self, n_vars: int, max_degree: int = 4):
        self.n_vars = n_vars
        self.max_degree = max_degree
        self.template = BarrierTemplate(n_vars, max_degree)

    def learn(self, examples: ICEExample) -> Optional[Polynomial]:
        """Learn invariant satisfying all examples."""
        solver = z3.Solver()

        for pos in examples.positive:
            solver.add(self.template.evaluate(pos.values) >= 0)

        for neg in examples.negative:
            solver.add(self.template.evaluate(neg.values) < 0)

        for pre, post in examples.implications:
            solver.add(z3.Implies(
```

# Applying ICE to Barrier Synthesis

## Key Insight

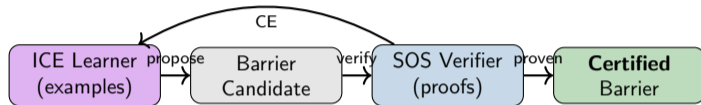
Barrier conditions are exactly ICE requirements!

- **Positive**  $\leftrightarrow$  Initial states ( $B \geq \varepsilon$ )
- **Negative**  $\leftrightarrow$  Unsafe states ( $B \leq -\varepsilon$ )
- **Implication**  $\leftrightarrow$  Inductiveness ( $B \geq 0 \Rightarrow B' \geq 0$ )

## Integration with Layers 1-3

- Use **Layer 1** (SOS) to check candidate barrier
- Use **Layer 2** constraints to generate examples
- Use **Layer 3** (CEGAR) counterexamples as negative examples
- **Learn** barrier coefficients from examples

# ICE + SOS: The Best of Both Worlds



## Workflow

- 1 ICE learns candidate from examples (fast, data-driven)
- 2 SOS verifies candidate (rigorous, sound)
- 3 If fails: extract counterexample, add to ICE data, repeat

## Reference

C. Flanagan & K. R. M. Leino. “Houdini, an Annotation Assistant for ESC/Java.” *FME 2001*.

## Core Contribution

**Conjunctive inference** of invariants:

- Start with a **large candidate set** of predicates
- Iteratively **remove** predicates that fail verification
- Result: **maximal subset** that forms valid invariant

Simple but surprisingly effective for many practical invariants!

# Houdini: Conjunctive Fixpoint

## Key Insight

If the true invariant is a **conjunction** of candidate predicates, Houdini will find it.

- 1:  $C \leftarrow \{p_1, p_2, \dots, p_n\}$  ▷ Initial candidate set
- 2: **repeat**
- 3:    $I \leftarrow \bigwedge_{p \in C} p$  ▷ Current invariant
- 4:   changed  $\leftarrow$  false
- 5:   **for all**  $p \in C$  **do**
- 6:     **if**  $\neg \text{Verify}(I \Rightarrow p')$  **then** ▷ Check inductiveness
- 7:        $C \leftarrow C \setminus \{p\}$
- 8:       changed  $\leftarrow$  true
- 9:     **end if**
- 10:   **end for**
- 11: **until** not changed
- 12: **return**  $I = \bigwedge_{p \in C} p$

# Implementation: Houdini Inference

```
class HoudiniInference:
    """Houdini conjunctive invariant inference (Paper \#18)."""

    def __init__(self, candidates: List[Predicate]):
        self.candidates = list(candidates)

    def infer(self, transition: z3.ExprRef) -> z3.ExprRef:
        """Infer maximal conjunctive invariant."""
        current = set(self.candidates)

        while True:
            changed = False
            inv = z3.And([p.expr for p in current])

            for pred in list(current):

                if not self._is_inductive(inv, transition, pred):
                    current.remove(pred)
                    changed = True

            if not changed:
                break

        return z3.And([p.expr for p in current]) if current else z3.BoolVal(True)

    def _is_inductive(self, inv, trans, pred) -> bool:
        """Check if pred is preserved under transition."""
        solver = z3.Solver()
        solver.add(inv)
        solver.add(trans)
```

# Applying Houdini to Barrier Synthesis

## Barrier Predicates

Generate candidate barrier predicates from:

- Program guards:  $\text{if } x > 0 \rightarrow x > 0$
- Assertions:  $\text{assert } \text{len}(a) > i \rightarrow \text{len}(a) > i$
- Type constraints:  $x: \text{int} \rightarrow x \in \mathbb{Z}$
- Inferred bounds: loop analysis  $\rightarrow 0 \leq i < n$

## Houdini Barrier Synthesis

- 1 Collect guard predicates from CFG
- 2 Run Houdini to find inductive conjunction
- 3 Check if conjunction implies safety
- 4 If yes: **Barrier = conjunction of surviving predicates**

# Houdini: Strengths and Limitations

## Strengths:

- Simple algorithm
- Fast convergence
- Polynomial in # candidates
- Finds maximal conjunction
- Works well with program guards

## Limitations:

- Only conjunctions
- Needs good candidates
- Can't synthesize new predicates
- May converge to  $\top$

## Mitigation:

- Combine with ICE for predicate discovery
- Use SyGuS for template generation
- Fall back to full SOS synthesis

## Reference

R. Alur, R. Bodik, G. Juniwal, et al. "Syntax-Guided Synthesis."  
*FMCAD 2013*.

## Core Contribution

### Syntax-Guided Synthesis:

- User provides a **grammar** of allowed expressions
- System searches for expression satisfying **specification**
- Combines **search** with **verification**

SyGuS is a **general framework** for program synthesis, applied here to invariants.

## Components

- 1 **Background theory:** LIA (Linear Integer Arithmetic), BV, etc.
- 2 **Grammar:** Allowed expression forms
- 3 **Specification:** Semantic constraint to satisfy

## Example: Loop Invariant

```
; Grammar for invariants
(synth-inv Inv ((x Int) (n Int))
  ((Start Bool ((and Start Start) (or Start Start)
    (>= Term Term) (<= Term Term))))
  (Term Int (x n 0 1 (+ Term Term)))))

; Specification
(constraint (=> (and (= x 0) (>= n 0)) (Inv x n))) ; Init
(constraint (=> (and (Inv x n) (< x n)) (Inv (+ x 1) n))) ; Step
(constraint (=> (and (Inv x n) (>= x n)) (= x n))) ; Post
```

## 1. Enumerative Search

Enumerate expressions from grammar in order of size/complexity.

- Simple, complete for finite grammars
- Exponential in expression size

## 2. CEGIS (CounterExample-Guided)

- 1 Find candidate satisfying finite set of examples
- 2 Verify candidate against full specification
- 3 If fails, add counterexample and repeat

## 3. Constraint-Based

Encode grammar + specification as SMT constraint, solve directly.

# Implementation: SyGuS Synthesizer

```
@dataclass
class SyGuSGrammar:
    """Grammar for syntax-guided synthesis."""
    start_symbol: str
    productions: Dict[str, List[str]]
    terminals: Set[str]

class SyGuSSynthesizer:
    """SyGuS synthesis for invariants (Paper \#19)."""

    def __init__(self, grammar: SyGuSGrammar, variables: List[str]):
        self.grammar = grammar
        self.variables = variables

    def synthesize(self, spec: Callable[[z3.ExprRef], z3.BoolRef],
                  max_size: int = 10) -> Optional[z3.ExprRef]:
        """Synthesize expression satisfying specification."""

        for size in range(1, max_size + 1):

            for expr in self._enumerate(self.grammar.start_symbol, size):
                z3_expr = self._to_z3(expr)

                if self._verify(z3_expr, spec):
                    return z3_expr

        return None

    def _enumerate(self, symbol: str, size: int) -> Iterator[Expression]:
```

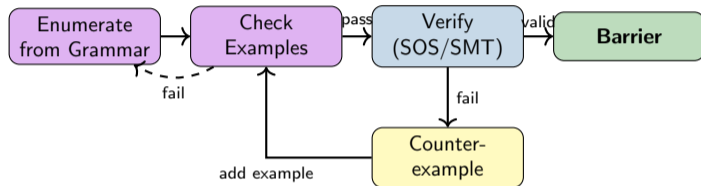
## Grammar for Barrier Functions

$$\begin{aligned} B &::= \text{Poly} \mid B_1 + B_2 \mid c \cdot B \\ \text{Poly} &::= x_i \mid x_i^2 \mid x_i \cdot x_j \mid \text{const} \\ c &::= 1 \mid -1 \mid 2 \mid -2 \mid \dots \end{aligned}$$

## Specification (Barrier Conditions)

- $\forall x \in \text{Init}. B(x) \geq \varepsilon$
- $\forall x \in \text{Unsafe}. B(x) < 0$
- $\forall x. (B(x) \geq 0) \Rightarrow (\mathcal{L}_f B(x) \geq 0)$

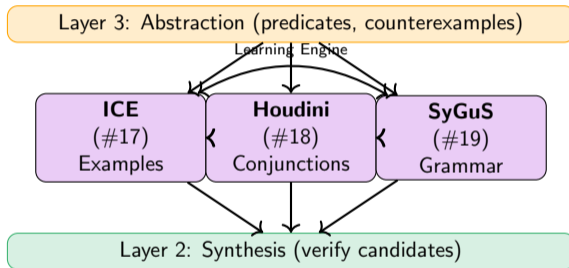
SyGuS searches for polynomial  $B$  from grammar satisfying these constraints.



## CEGIS Advantage

Don't verify against full spec until candidate passes examples  $\Rightarrow$  **faster!**

## Layer 4: How ICE, Houdini, SyGuS Integrate



# Unified Interface: LearningBasedEngine

```
class LearningBasedEngine:
    """Unified interface for Layer 4 learning."""

    def __init__(self, n_vars: int, max_degree: int,
                 timeout_ms: int = 60000):
        self.n_vars = n_vars
        self.max_degree = max_degree

        self.ice = ICELearner(n_vars, max_degree)
        self.houdini = HoudiniInference([])
        self.sygus = SyGuSSynthesizer(self._default_grammar(), ...)

    def learn_invariant(self, examples: ICEExample,
                      candidates: List[Predicate] = None) -> Optional[Polynomial]:
        """Learn invariant using portfolio of techniques."""

        if candidates:
            self.houdini = HoudiniInference(candidates)
            result = self.houdini.infer(...)
            if self._is_nontrivial(result):
                return result

        return self.ice.learn(examples)
```

# Layer 4 Summary: Learning-Based Synthesis

## What Layer 4 Provides:

- Data-driven invariant inference
- Fast convergence with examples
- Complementary approaches
- Integration with verification

## To Layer 5 (Advanced):

- Candidate invariants for IC3
- Predicates for CHC solving
- Learned lemmas

## Files:

- `learning.py`
- `ice.py`
- `ice_learning.py`
- `houdini.py`
- `sygus_synthesis.py`

## Key Metrics:

- Examples used
- Learning iterations
- Candidates eliminated

# From Learning to Advanced Verification

## What We've Built So Far

- **Layer 1:** Mathematical foundations (SOS, SDP)
- **Layer 2:** Barrier certificate synthesis
- **Layer 3:** Abstraction and refinement
- **Layer 4:** Learning from examples

## What's Still Needed

- More scalable SOS relaxations (DSOS/SDSOS)
- Incremental reasoning (IC3/PDR)
- Constraint-based verification (CHC/Spacer)
- Compositional verification (Assume-Guarantee)

⇒ Layer 5: **Advanced Verification** techniques!

# Part V

## Advanced Verification

Layer 5: Papers #9-11, 15, 20

DSOS • IC3/PDR • CHC/Spacer • IMC • Assume-Guarantee

# Layer 5: Advanced Verification Techniques

## The Need for Advanced Methods

Sometimes lower layers are insufficient:

- SOS too expensive (need cheaper relaxations)
- Invariants need incremental discovery (IC3)
- Systems are modular (compositional reasoning)
- Need interpolation for refinement (IMC)

## Layer 5 Papers

- **#9:** DSOS/SDSOS - Cheaper SOS relaxations (LP/SOCP)
- **#10:** IC3/PDR - Property-Directed Reachability
- **#11:** CHC/Spacer - Constrained Horn Clauses
- **#15:** IMC - Interpolation-based Model Checking
- **#20:** Assume-Guarantee - Compositional verification

# Paper #9: DSOS/SDSOS (Ahmadi-Majumdar 2019)

## Reference

A. A. Ahmadi & A. Majumdar. “DSOS and SDSOS Optimization.” *SIAM Journal on Applied Algebra and Geometry*, 2019.

## Core Contribution

**Cheaper alternatives** to SOS/SDP:

- **DSOS**: Diagonally-dominant SOS  $\rightarrow$  Linear Programming (LP)
- **SDSOS**: Scaled diagonally-dominant  $\rightarrow$  Second-Order Cone (SOCP)

## Trade-off

Less expressive than full SOS, but **much faster** for large problems.

# DSOS: Diagonally-Dominant SOS

## Definition (DSOS)

A polynomial  $p$  is **DSOS** if:

$$p(x) = \sum_{i,j} \lambda_{ij} (x_i \pm x_j)^2 + \sum_k \mu_k x_k^2 + c$$

where  $\lambda_{ij}, \mu_k \geq 0$  and  $c \geq 0$ .

## Key Insight

- Uses only **squared binomials**  $(x_i \pm x_j)^2$
- Coefficients  $\lambda_{ij}$  are **linear constraints**
- Checking DSOS  $\Leftrightarrow$  solving an **LP**!

LP is **polynomial time** and has mature, fast solvers (Gurobi, CPLEX).

# SDSOS: Scaled Diagonally-Dominant SOS

## Definition (Scaled DD Matrix)

A symmetric matrix  $Q$  is **scaled diagonally dominant** if there exist  $d_i > 0$ :

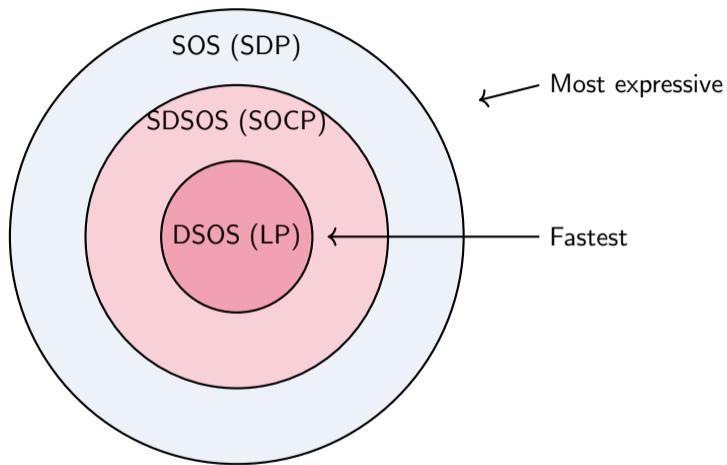
$DQD$  is diagonally dominant, where  $D = \text{diag}(d_1, \dots, d_n)$

## SDSOS Property

- Scaled DD  $\Rightarrow$  positive semidefinite
- Checking scaled DD  $\Leftrightarrow$  **SOCP** constraints
- SOCP is faster than SDP, still polynomial time

**Hierarchy:** DSOS  $\subset$  SDSOS  $\subset$  SOS

# Comparing SOS, SDSOS, DSOS



|            | DSOS     | SDSOS    | SOS          |
|------------|----------|----------|--------------|
| Solver     | LP       | SOCP     | SDP          |
| Complexity | $O(n^2)$ | $O(n^3)$ | $O(n^{4.5})$ |

# Implementation: DSOS Relaxation

```
@dataclass
class DSOSDecomposition:
    """DSOS decomposition:  $p = \sum \lambda_{ij} (x_i \pm x_j)^2$ """
    n_vars: int
    binomial_coeffs: Dict[Tuple[int, int], float]

    def to_polynomial(self) -> Polynomial:
        """Convert to polynomial representation."""
        poly = Polynomial(self.n_vars)
        for (i, j, sign), coeff in self.binomial_coeffs.items():
            poly.add_term({i: 2}, coeff)
            poly.add_term({j: 2}, coeff)
            poly.add_term({i: 1, j: 1}, 2 * coeff * sign)
        return poly

class DSOSRelaxation:
    """DSOS/SDSOS relaxation engine (Paper \#9)."""

    def check_dsos(self, p: Polynomial) -> Optional[DSOSDecomposition]:
        """Check if p is DSOS using LP."""

        lp = LinearProgram()
        for (i, j) in self._binomial_pairs():
            lp.add_variable(f'lambda_{i}_{j}_plus', lower=0)
            lp.add_variable(f'lambda_{i}_{j}_minus', lower=0)

        for mono, target_coeff in p.terms.items():
            lp.add_constraint(self._coeff_expr(mono) == target_coeff)
```

# When to Use DSOS/SDSOS

## Use DSOS/SDSOS When:

- Problem is **large** (many variables, high degree)
- SDP solver **times out**
- Need **fast approximate** answer
- Problem structure is **sparse**

## Fall Back to SOS When:

- DSOS/SDSOS returns infeasible
- High precision required
- Problem is small

Our pipeline: Try DSOS  $\rightarrow$  SDSOS  $\rightarrow$  SOS in order.

# Paper #10: IC3/PDR (Bradley 2011)

## Reference

A. R. Bradley. “SAT-Based Model Checking without Unrolling.”  
*VMCAI 2011*.

## Core Contribution

### Property-Directed Reachability (PDR):

- Discovers inductive invariants **incrementally**
- Uses **frames**  $F_0, F_1, \dots, F_k$  (over-approximations)
- **Blocks** counterexamples-to-induction with lemmas
- **Propagates** lemmas to strengthen frames

One of the most effective SAT-based model checking algorithms!

# IC3: Frame Invariants

## Definition (Frames)

A sequence of formulas  $F_0, F_1, \dots, F_k$  where:

- ①  $F_0 = \text{Init}$  (initial states)
- ②  $F_i \Rightarrow F_{i+1}$  (monotone strengthening)
- ③  $F_i \wedge T \Rightarrow F'_{i+1}$  (inductive consecution)
- ④  $F_i \Rightarrow \text{Safe}$  (all frames are safe)

## Intuition

$F_i$  over-approximates states reachable in  $\leq i$  steps.

If  $F_i = F_{i+1}$  for some  $i \Rightarrow$  **fixed point**  $\Rightarrow F_i$  is inductive invariant!

# IC3 Algorithm Overview

```
1:  $F_0 \leftarrow \text{Init}; F_1 \leftarrow \text{Safe}; k \leftarrow 1$ 
2: while true do
3:   while  $\exists s \in F_k$  with  $s \wedge T \wedge \neg \text{Safe}'$  do                                ▷ Bad state?
4:     if  $s \in F_0$  then
5:       return UNSAFE (counterexample from Init)
6:     end if
7:     Block( $s, k$ )                                                                ▷ Add lemma to block  $s$ 
8:   end while
9:   Propagate()                                                                    ▷ Push lemmas forward
10:  if  $F_i = F_{i+1}$  for some  $i$  then
11:    return SAFE (inductive invariant  $F_i$ )
12:  end if
13:   $k \leftarrow k + 1; F_k \leftarrow \text{Safe}$ 
14: end while
```

## Counterexample-to-Induction (CTI)

A state  $s$  is a CTI at level  $k$  if:

$$s \in F_k \wedge s \xrightarrow{T} s' \wedge s' \notin \text{Safe}$$

### Blocking Procedure:

- 1 Generalize  $s$  to a **cube**  $c$  (conjunction of literals)
- 2 Find **minimal** cube that still needs blocking
- 3 Add lemma  $\neg c$  to  $F_k$  (and earlier frames if possible)
- 4 Recursively block predecessors of  $c$

Key: Each lemma strengthens the over-approximation without losing reachable states.

# IC3: Lemma Propagation

## Propagation Rule

If lemma  $\ell$  is in  $F_i$  and  $F_i \wedge T \Rightarrow \ell'$ , then add  $\ell$  to  $F_{i+1}$ .

## Why Propagate?

- Lemmas proven at lower levels may hold at higher levels
- Strengthens frames without redundant work
- Enables fixed-point detection

**Fixed Point:** When  $F_k = F_{k+1}$ , we have found an inductive invariant!

$$F_k \wedge T \Rightarrow F'_k \quad (\text{inductive})$$

# Implementation: IC3 Engine

```
@dataclass
class Frame:
    """IC3 frame: over-approximation of reachable states."""
    level: int
    clauses: Set[Clause]

class IC3Engine:
    """IC3/PDR verification engine (Paper \#10)."""

    def __init__(self, init: z3.BoolRef, trans: z3.BoolRef, safe: z3.BoolRef):
        self.init = init
        self.trans = trans
        self.safe = safe
        self.frames = [Frame(0, {self._init_to_clauses()})]

    def verify(self) -> VerificationResult:
        k = 1
        self.frames.append(Frame(1, {Clause.from_expr(self.safe)}))

        while True:

            while (cti := self._get_cti(k)) is not None:
                if not self._block(cti, k):
                    return VerificationResult('unsafe', self._extract_trace())

            self._propagate()

            if self._has_fixpoint():
```

# IC3 for Barrier Certificate Synthesis

## Key Insight

IC3 lemmas can be **lifted** to polynomial constraints for barrier synthesis.

## Integration Strategy:

- ① Run IC3 to discover **discrete invariant**
- ② Extract lemmas from converged frames
- ③ Convert lemmas to **polynomial constraints**:

Lemma:  $x > 0$   $\rightarrow$  Constraint:  $g(x) = x \geq 0$

- ④ Use constraints to **condition** barrier synthesis (Layer 2)
- ⑤ Reduced search space  $\rightarrow$  faster synthesis!

# Lifting IC3 Lemmas to Polynomial Constraints

## Example

IC3 discovers lemma:  $\neg(x < 0 \wedge y > 10)$

Equivalent to:  $x \geq 0 \vee y \leq 10$

As polynomial constraint for Positivstellensatz:

$$\{x \geq 0\} \cup \{10 - y \geq 0\}$$

## Benefit

IC3 lemmas **partition** the state space, reducing the polynomial degree needed for barrier synthesis.

Instead of searching for  $B$  over all of  $\mathbb{R}^n$ , search over regions defined by IC3 lemmas.

# Paper #11: CHC/Spacer (Komuravelli et al. 2014)

## Reference

A. Komuravelli, A. Gurfinkel, S. Chaki. "SMT-Based Model Checking for Recursive Programs."  
*CAV 2014*.

## Core Contribution

**Constrained Horn Clauses** (CHC) for verification:

- Programs encoded as Horn clauses
- Invariants are **solutions** to Horn constraints
- SMT-based solving with interpolation
- Handles **recursion** and **procedures**

Spacer = IC3 + interpolation + Horn clauses. Very powerful!

# Constrained Horn Clauses

## Definition (CHC)

A **Constrained Horn Clause** has the form:

$$\phi \wedge P_1(\vec{x}_1) \wedge \cdots \wedge P_k(\vec{x}_k) \Rightarrow H(\vec{y})$$

where  $\phi$  is a constraint,  $P_i$  are uninterpreted predicates,  $H$  is the head.

## Encoding a Loop

```
while (x < n) { x = x + 1 }
```

Init:  $x = 0 \wedge n \geq 0 \Rightarrow \text{Inv}(x, n)$

Step:  $\text{Inv}(x, n) \wedge x < n \Rightarrow \text{Inv}(x + 1, n)$

Post:  $\text{Inv}(x, n) \wedge x \geq n \Rightarrow x = n$

## Key Innovations

- Apply IC3 to **Horn clause** solving
- Use **interpolation** to discover predicate interpretations
- Handle **multiple** predicates simultaneously
- Support **recursion** via unfolding

## Algorithm Sketch:

- ① Under-approximate each predicate (start with false)
- ② Check if clauses are satisfied
- ③ If CEX found, block it with interpolant-derived lemma
- ④ Propagate lemmas across predicates
- ⑤ Converge to solution or prove unsatisfiable

# Implementation: Spacer CHC Solver

```
@dataclass
class HornClause:
    """A Constrained Horn Clause."""
    body_predicates: List[Tuple[str, List[z3.ExprRef]]]
    body_constraint: z3.BoolRef
    head: Tuple[str, List[z3.ExprRef]]

class SpacerCHC:
    """CHC solving via Spacer algorithm (Paper \#11)."""

    def __init__(self, clauses: List[HornClause]):
        self.clauses = clauses
        self.predicates = self._extract_predicates()
        self.interpretations = {p: z3.BoolVal(False) for p in self.predicates}

    def solve(self) -> Optional[Dict[str, z3.ExprRef]]:
        """Find predicate interpretations satisfying all clauses."""

        fp = z3.Fixedpoint()
        fp.set('engine', 'spacer')

        for pred in self.predicates:
            fp.register_relation(pred)

        for clause in self.clauses:
            fp.add_rule(self._clause_to_rule(clause))

        result = fp.query(self._get_query())
        if result == z3.sat:
            return self._extract_interpretations(fp)
```

# Paper #15: Interpolation-based Model Checking (McMillan 2003)

## Reference

K. L. McMillan. “Interpolation and SAT-Based Model Checking.”  
*CAV 2003*.

## Core Contribution

Use **Craig interpolation** for model checking:

- Bounded model checking (BMC) finds counterexamples
- Interpolants from UNSAT proofs yield **over-approximations**
- Iteratively refine until fixed point

Interpolation is the “magic ingredient” that makes refinement effective.

## Algorithm

- 1 **BMC phase:** Check  $\text{Init} \wedge T^k \wedge \neg \text{Safe}$  for increasing  $k$
- 2 If SAT  $\Rightarrow$  **counterexample found**
- 3 If UNSAT  $\Rightarrow$  extract **interpolant**  $I$  from proof:

$$\underbrace{\text{Init}}_A \wedge \underbrace{T^k \wedge \neg \text{Safe}}_B = \text{UNSAT}$$

Interpolant  $I$ :  $\text{Init} \Rightarrow I$  and  $I \wedge T^k \wedge \neg \text{Safe} = \text{UNSAT}$

- 4 Use  $I$  as over-approximation of reachable states
- 5 Check  $I \wedge T \Rightarrow I$  (inductiveness). If yes  $\Rightarrow$  **invariant!**

# Sequence Interpolants

## Definition (Sequence Interpolants)

For formulas  $A_0, A_1, \dots, A_n$  with  $\bigwedge A_i = \text{UNSAT}$ , sequence interpolants  $I_0, I_1, \dots, I_{n+1}$  satisfy:

- $I_0 = \top, I_{n+1} = \perp$
- $I_i \wedge A_i \Rightarrow I_{i+1}$
- $I_i$  uses only common symbols of  $A_0, \dots, A_{i-1}$  and  $A_i, \dots, A_n$

## For BMC Path

Partition:  $A_0 = \text{Init}, A_1 = T, A_2 = T, \dots, A_n = \neg \text{Safe}$   
Interpolants give invariants at each time step!

# Implementation: IMC Verifier

```
class InterpolationEngine:
    """Craig interpolation for refinement."""

    def compute_interpolant(self, A: z3.BoolRef, B: z3.BoolRef) -> z3.BoolRef:
        """Compute interpolant I such that A => I and I /\ B = UNSAT."""
        solver = z3.Solver()
        solver.add(A)
        solver.add(B)

        if solver.check() == z3.sat:
            return None

        return self._extract_from_proof(solver.proof(), A, B)

class IMCVerifier:
    """Interpolation-based Model Checking (Paper \#15)."""

    def verify(self, init, trans, safe, max_depth=100):
        for k in range(max_depth):

            path_formula = self._unroll(init, trans, k, safe)

            if self._is_sat(path_formula):
                return VerificationResult('unsafe')

            I = self.interp.compute_interpolant(init, self._suffix(trans, k, safe))
```

# Paper #20: Assume-Guarantee (Pnueli 1985)

## Reference

A. Pnueli. "In Transition from Global to Modular Temporal Reasoning about Programs." *Logics and Models of Concurrent Systems*, 1985.

## Core Contribution

### Compositional verification:

- Decompose system into **components**
- Verify each component under **assumptions**
- Components **guarantee** properties to others
- Compose results: if all contracts hold  $\Rightarrow$  system is safe

Essential for verifying **large systems**!

## AG Triple

$\langle A \rangle M \langle G \rangle$  means:

“If component  $M$  runs in environment satisfying assumption  $A$ , then  $M$  guarantees property  $G$ .”

## Composition Rule

$$\frac{\langle A_1 \rangle M_1 \langle G_1 \rangle \quad \langle A_2 \rangle M_2 \langle G_2 \rangle \quad G_1 \Rightarrow A_2 \quad G_2 \Rightarrow A_1}{\langle A_1 \wedge A_2 \rangle M_1 \parallel M_2 \langle G_1 \wedge G_2 \rangle}$$

**Key:** Verify components separately, compose results!

# Assume-Guarantee for Barrier Synthesis

## Compositional Barrier Synthesis

For system  $S = M_1 \parallel M_2$ :

- 1 Synthesize barrier  $B_1$  for  $M_1$  under assumption  $A$  on  $M_2$ 's behavior
- 2 Synthesize barrier  $B_2$  for  $M_2$  under assumption  $B_1 \geq 0$
- 3 Verify:  $B_1 \geq 0 \Rightarrow A$  (assumption discharged)
- 4 Compose:  $B = B_1 \wedge B_2$  is barrier for  $S$

## Benefit

Can verify large systems by decomposing into manageable pieces.  
Each component's barrier is **smaller** and **faster** to synthesize.

# Implementation: Assume-Guarantee Verifier

```
@dataclass
class AGContract:
    """Assume-Guarantee contract for a component."""
    assumption: z3.BoolRef
    guarantee: z3.BoolRef
    component: Any

class AssumeGuaranteeVerifier:
    """Compositional verification (Paper \#20)."""

    def verify_composition(self, components: List[AGContract]) -> VerificationResult:

        for contract in components:
            result = self._verify_component(contract)
            if not result.verified:
                return VerificationResult('unknown',
                    message=f'{contract.component} failed under assumptions')

        for i, c1 in enumerate(components):
            for j, c2 in enumerate(components):
                if i != j:

                    if not self._check_implies(c2.guarantee, c1.assumption):
                        return VerificationResult('unknown',
                            message=f'Assumption of {c1} not satisfied by {c2}')

        return VerificationResult('safe',
            certificate=self._compose_barriers(components))
```

# Circular Assume-Guarantee Reasoning

## The Challenge

Components may have **circular dependencies**:

- $M_1$  assumes something about  $M_2$
- $M_2$  assumes something about  $M_1$

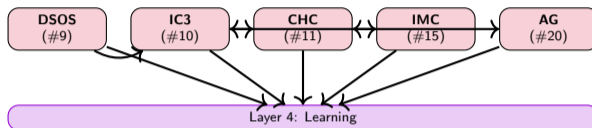
## Solution: Inductive Proof

Use **simultaneous induction**:

- 1 Base case: Initial states satisfy all assumptions
- 2 Inductive step: If assumptions hold at step  $k$ , guarantees hold at step  $k$ , therefore assumptions hold at step  $k + 1$

This is sound if the circular argument is well-founded!

## Layer 5: How Papers Integrate



### Integration Points

- DSOS provides fast relaxations when SOS is too slow
- IC3/CHC discover lemmas that constrain barrier synthesis
- IMC provides interpolants for refinement
- AG enables compositional verification of large systems

# Unified Interface: AdvancedVerificationEngine

```
class AdvancedVerificationEngine:
    """Unified interface for Layer 5 advanced verification."""

    def __init__(self, timeout_ms: int = 300000):
        self.timeout_ms = timeout_ms

        self.dsos = DSOSRelaxation(2, 6, timeout_ms // 5)
        self.ic3 = IC3Engine(None, None, None)
        self.spacer = SpacerCHC([])
        self.imc = IMCVerifier()
        self.ag = AssumeGuaranteeVerifier()

    def verify(self, system, property) -> VerificationResult:
        """Verify using portfolio of advanced methods."""

        result = self.ic3.verify(system.init, system.trans, property)
        if result.status != 'unknown':
            return result

        clauses = self._encode_as_chc(system, property)
        result = self.spacer.solve(clauses)
        if result is not None:
            return VerificationResult('safe', result)

        if system.is_compositional:
            return self.ag.verify_composition(system.components)
```

# Layer 5 Summary: Advanced Verification

## What Layer 5 Provides:

- Scalable SOS via DSOS/SDSOS
- Incremental reasoning (IC3)
- SMT-based solving (CHC/Spacer)
- Interpolation (IMC)
- Compositional verification (AG)

## Files:

- `advanced.py`
- `dsos_sdsos.py`
- `ic3_pdr.py`
- `spacer_chc.py`
- `interpolation_imc.py`
- `assume_guarantee.py`

## Key Achievement

Layer 5 makes the full verification pipeline practical for **real-world systems**.

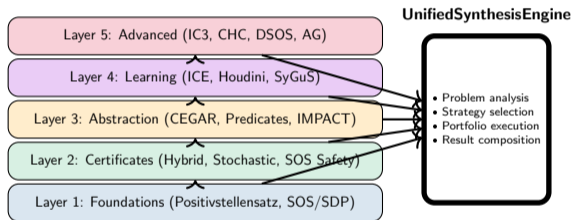
# Part VI

## Integration & Implementation

Putting It All Together

UnifiedSynthesisEngine • ExtremeContextVerifier • Results

# The Complete Verification Pipeline



## Responsibilities

- **Analyze** the verification problem
- **Select** appropriate techniques
- **Execute** in portfolio mode
- **Compose** results from multiple layers

## Key Design Principles

- ① **Sound:** Never report SAFE if bug exists
- ② **Complete:** Find bugs with concrete witnesses
- ③ **Adaptive:** Choose techniques based on problem
- ④ **Robust:** Fallback strategies at each level

# Problem Classification

```
class ProblemClassifier:
    """Classify verification problems for technique selection."""

    def classify(self, problem: Dict[str, Any]) -> ProblemAnalysis:

        n_vars = problem.get('n_vars', 2)
        max_degree = problem.get('max_degree', 4)
        dynamics_type = problem.get('dynamics_type', 'continuous')

        problem_type = self._classify_type(dynamics_type)
        problem_size = self._classify_size(n_vars, max_degree)

        is_sparse = self._check_sparsity(problem)
        is_linear = max_degree <= 1

        methods = self._recommend_methods(problem_type, problem_size, is_sparse)

        return ProblemAnalysis(
            problem_type=problem_type,
            problem_size=problem_size,
            n_vars=n_vars,
            max_degree=max_degree,
            is_sparse=is_sparse,
            recommended_methods=methods
        )
```

# Strategy Selection Based on Problem Type

| Problem Type              | Recommended Methods                    |
|---------------------------|--|
| Continuous Safety (small) | SOS Safety, Putinar, Barrier Synthesis |
| Continuous Safety (large) | Sparse SOS, DSOS, ICE Learning         |
| Discrete Safety           | IC3, CHC, Predicate Abstraction        |
| Hybrid Safety             | Hybrid Barriers, IC3, CEGAR            |
| Stochastic Safety         | Stochastic Barriers, SOS Safety        |
| Compositional             | Assume-Guarantee, IC3, CHC             |
| Linear Systems            | Linear analysis, DSOS                  |

## Adaptive Strategy

Based on problem size and timeout:

- Small: Try SOS → Lasserre → CEGAR
- Medium: DSOS → ICE → IC3
- Large: Sparse SOS → CHC → AG

# Portfolio Execution

```
class PortfolioExecutor:
    """Execute multiple strategies in portfolio mode."""

    def run(self, problem: Dict[str, Any]) -> VerificationResult:
        start = time.time()
        best_result = VerificationResult(status='unknown')

        for strategy in self.strategies:

            remaining = self.timeout_ms - int((time.time() - start) * 1000)
            if remaining <= 0:
                break

            strategy.timeout_ms = remaining // len(self.strategies)

            result = strategy.execute(problem)

            if result.status == 'safe':
                return result

            if result.status == 'unsafe':
                best_result = result

        return best_result
```

# ExtremeContextVerifier: The User-Facing API

## Purpose

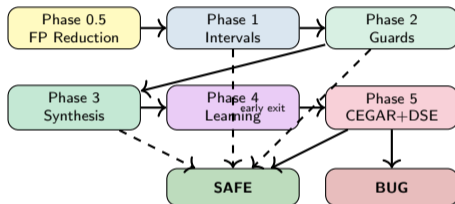
Provides a **high-level interface** for bug verification using all 20 papers.

## Verification Flow:

- ① **Phase 0.5:** False Positive Reduction (interprocedural checks)
- ② **Phase 1:** Interval/Dataflow Analysis (quick checks)
- ③ **Phase 2:** Guard Barrier Collection (existing guards)
- ④ **Phase 3:** Layer 2 Synthesis (SOS/SDP barriers)
- ⑤ **Phase 4:** Layer 4 Learning (ICE, Houdini)
- ⑥ **Phase 5:** CEGAR + DSE (refinement loop)

Returns **ContextAwareResult** with barriers, witnesses, and verification status.

# Verification Phases in Detail



Each phase can exit early if it determines SAFE or BUG.

## Four Integrated Strategies

Eliminate false positives **before** expensive verification.

### ① Interprocedural Guard Propagation

- Trace guards through call chains
- If caller validates, callee is protected

### ② Path-Sensitive Symbolic Execution

- Track constraints along each path
- Eliminate infeasible bug paths

### ③ Pattern-Based Safe Idiom Recognition

- Recognize common safe patterns
- Skip known-safe constructs

### ④ Dataflow Value Range Tracking

- Interval analysis for variables
- Prove safety from value ranges

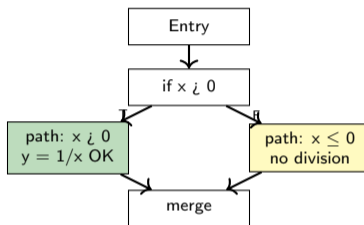
# Strategy 1: Interprocedural Guard Propagation

```
def caller(items):  
    assert len(items) > 0  
    return callee(items)  
  
def callee(data):  
    x = data[0]
```

## Propagation Logic

- 1 Collect guards from all callers in the call chain
- 2 Build a **guard barrier** for each guard
- 3 Check if any barrier protects the bug location
- 4 If protected: **Skip the bug** (false positive)

## Strategy 2: Path-Sensitive Symbolic Execution



### Key Insight

Track **path condition**  $\pi$  along each execution path. A bug is only real if  $\pi \wedge \text{bug\_condition}$  is satisfiable.

# Strategy 3: Pattern-Based Safe Idiom Recognition

## Common Safe Patterns

### List Access Patterns:

```
# Pattern 1: Explicit check
if items:
    x = items[0]

# Pattern 2: For loop
for i, item in enumerate(items):

# Pattern 3: Try-except
try:
    x = items[0]
except IndexError:
    x = default
```

### Division Patterns:

```
# Pattern 1: Guard
if divisor != 0:
    result = x / divisor

# Pattern 2: Or-default
result = x / (divisor or 1)

# Pattern 3: Max guard
result = x / max(divisor, 1)
```

# Strategy 4: Dataflow Value Range Tracking

## Interval Abstract Domain

Track  $[low, high]$  bounds for each variable through the program.

### Example:

|                                      |  |
|--------------------------------------|--|
| After $x = \text{len}(\text{items})$ | $\Rightarrow x \in [0, \infty)$                |
| After if $x > 0$                     | $\Rightarrow x \in [1, \infty)$                |
| At $\text{items}[0]$                 | $\Rightarrow \text{SAFE } (\text{len} \geq 1)$ |

### Interval Operations:

- **Join:**  $[a, b] \sqcup [c, d] = [\min(a, c), \max(b, d)]$
- **Meet:**  $[a, b] \sqcap [c, d] = [\max(a, c), \min(b, d)]$
- **Widen:** Accelerate fixpoint convergence

# Phase 1: Quick Analysis (Interval + Dataflow)

## Purpose

Fast, lightweight checks before expensive synthesis.

### Interval Analysis Checks:

- Is divisor definitely positive?  $\Rightarrow$  No `DIV_ZERO`
- Is index definitely in bounds?  $\Rightarrow$  No `BOUNDS` error
- Is pointer definitely non-null?  $\Rightarrow$  No `NULL_PTR`

### Dataflow Facts Gathered:

- **Constants:** Known constant values
- **Types:** Definite type information
- **Nullability:** Definitely null / definitely not null
- **Aliases:** Variables with same value
- **Assignments:** Definitely assigned variables

## Phase 2: Guard Barrier Collection

### Guard-to-Barrier Translation

Convert explicit guards in code to formal barrier certificates.

#### Translation Examples:

| Guard Code                               | Barrier Certificate  |
|--|--|
| <code>assert len(x) &gt; 0</code>        | $B(x) = \text{len}(x) - 1$                                   |
| <code>if x is not None:</code>           | $B(x) = \neg[x \neq \text{None}]$                            |
| <code>if divisor != 0:</code>            | $B(d) =  d  - \epsilon$                                      |
| <code>if 0 &lt;= i &lt; len(arr):</code> | $B(i, \text{arr}) = \min(i, \text{len}(\text{arr}) - i - 1)$ |

If barrier  $B \geq 0$  at bug location  $\Rightarrow$  **SAFE**

## Phase 3: Layer 2 Barrier Synthesis

### When Guards Are Absent

Synthesize barriers automatically using SOS/SDP techniques.

#### Synthesis Problem:

Find  $B(x)$  such that:

$$\forall x \in \text{Init} : B(x) \geq \epsilon$$

$$\forall x \in \text{Unsafe} : B(x) \leq -\epsilon$$

$$\forall x, x' : (B(x) \geq 0 \wedge x \rightarrow x') \Rightarrow B(x') \geq 0$$

#### Uses UnifiedSynthesisEngine:

- Try SOS Safety first (fast)
- Fall back to Hybrid Barrier synthesis
- Use Lasserre hierarchy for complex problems

# Phase 4: Layer 4 ICE Learning

## Learning from Examples

When synthesis fails, learn invariants from codebase examples.

### ICE Example Collection:

- **Positive:** States from successful executions
- **Negative:** States that led to bugs
- **Implications:** If state  $s$  is safe, successor  $s'$  should be

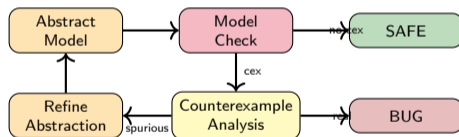
### Learning Process:

- 1 Collect examples from crash summaries
- 2 Train ICE learner on examples
- 3 Synthesize invariant that separates positive/negative
- 4 Verify learned invariant is inductive

## Phase 5: CEGAR + DSE Verification

### Final Refinement Loop

When all else fails, use CEGAR refinement and DSE ground truth.



**DSE:** Execute symbolically to find concrete counterexample or prove unreachable.

# Part VII

## ICE Learning Applied to Barrier Synthesis

The Complete Integration

# ICE Learning for Barrier Synthesis: Overview

## Key Innovation

Use **Implication CounterExamples** to guide barrier parameter search.

## Traditional Barrier Synthesis:

- Template enumeration: Try all parameter combinations
- Slow, may miss good barriers

## ICE-Guided Synthesis:

- Learn from **counterexamples** to failed verification
- Each failure teaches what constraints to add
- Converges faster to valid barrier

# ICE for Barriers: Three Example Types

## 1. Positive Examples (Init)

States where  $B(x) \geq 0$  **must** hold.

⇒ Initial states of the program

## 2. Negative Examples (Unsafe)

States where  $B(x) < 0$  **must** hold.

⇒ States that lead to bugs

## 3. Implication Examples (Step)

Pairs  $(s, s')$  where  $B(s) \geq 0 \Rightarrow B(s') \geq 0$ .

⇒ Transition pairs from symbolic execution

# ICE Barrier Synthesis Algorithm

```
1: Input: Template  $B_\theta(x)$ , Init, Unsafe, Trans
2: Output: Valid barrier or UNKNOWN
3:
4: positive  $\leftarrow$  sample(Init)
5: negative  $\leftarrow$  sample(Unsafe)
6: implications  $\leftarrow$  sample_transitions(Trans)
7:
8: while not timeout do
9:    $\theta^* \leftarrow$  ICE_Learn(positive, negative, implications)
10:   $B \leftarrow B_{\theta^*}$ 
11:
12:  if verify_inductive( $B$ ) then
13:    return  $B$ 
14:  else
15:    cex  $\leftarrow$  extract_counterexample()
16:    Add cex to appropriate example set
17:  end if
18: end while
```

▷ Found valid barrier!

## When Verification Fails

Extract counterexample and classify:

### 1. Init Counterexample:

- Found  $s_0 \in \text{Init}$  with  $B(s_0) < \epsilon$
- Add  $s_0$  to **positive** examples
- Force learner to satisfy  $B(s_0) \geq \epsilon$

### 2. Unsafe Counterexample:

- Found  $s_u \in \text{Unsafe}$  with  $B(s_u) > -\epsilon$
- Add  $s_u$  to **negative** examples
- Force learner to satisfy  $B(s_u) < -\epsilon$

### 3. Step Counterexample:

- Found transition  $(s, s')$  with  $B(s) \geq 0$  but  $B(s') < 0$
- Add  $(s, s')$  to **implication** examples

# ICE Barrier: Template Selection

## Choosing the Right Template Family

Different bug types need different barrier shapes.

| Bug Type | Template   | Form   |
|----------|------------|--|
| BOUNDS   | Linear     | $B(i, len) = c_1 \cdot i + c_2 \cdot len + c_0$                |
| DIV_ZERO | Absolute   | $B(d) =  d  - \epsilon$  |
| NULL_PTR | Indicator  | $B(p) = \mathbb{K}[p \neq \text{null}]$                        |
| OVERFLOW | Quadratic  | $B(x) = c_2 x^2 + c_1 x + c_0$                                 |
| Complex  | Polynomial | $B(\mathbf{x}) = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha}$ |

**Template Degree Increase:** If learning fails, try higher degree template.

# ICE Barrier: Z3 Integration

```
def ice_learn_conjunction(
    variables: dict[str, z3.IntNumRef],
    candidate_predicates: dict[str, z3.BoolRef],
    positive: list[dict[str, int]],
    negative: list[dict[str, int]],
    implications: list[tuple[dict, dict]],
) -> ICEResult:
    """Learn conjunction over predicates using SMT."""

    include = {name: z3.Bool(f"inc_{name}") for name in candidate_predicates}

    solver = z3.Optimize()

    for ex in positive:
        for name, pred in candidate_predicates.items():
            if not eval_pred(pred, ex):
                solver.add(z3.Not(include[name]))

    for ex in negative:
        falsifying = [include[n] for n, p in candidate_predicates.items()
                      if not eval_pred(p, ex)]
        solver.add(z3.Or(*falsifying))

    for pre, post in implications:
        solver.add(z3.Implies(holds_on(pre), holds_on(post)))
```

# ICE Barrier: Practical Example

```
def process_items(items):  
    x = items[0]  
    return x * 2
```

## ICE Learning Trace:

| Round  | Examples                            | Learned Barrier         |
|--------|-------------------------------------|-------------------------|
| 1      | + : [1], [2], [5]<br>- : []         | $B = \text{len} \geq 1$ |
| Verify | Init fails: [] is initial!          |                         |
| 2      | + : [1], [2], [5]<br>- : [] (added) | $B = \text{len} > 0$    |
| Verify | <b>SUCCESS!</b>                     |                         |

# ICE Barrier: Learning from Implications

## Why Implications Matter

Implications capture the **inductiveness** requirement.

### Example: Loop Invariant

```
i = 0
while i < len(arr):
    x = arr[i] # BOUNDS check
    i += 1
```

### Implication Examples:

- $(i = 0, len = 3) \rightarrow (i = 1, len = 3)$ : Must preserve  $0 \leq i < len$
- $(i = 1, len = 3) \rightarrow (i = 2, len = 3)$ : Same invariant
- $(i = 2, len = 3) \rightarrow \text{exit}$ : Loop terminates safely

**Learned:**  $B(i, len) = \min(i, len - i - 1) \geq -1$

# ICE Barrier: Convergence Properties

## Theorem (ICE Convergence)

*If a valid barrier exists in the template family, ICE learning will find it in finite iterations (under mild conditions).*

### Conditions for Convergence:

- 1 Template family contains a valid barrier
- 2 Counterexample oracle is sound
- 3 Example space is finite (or finitely representable)

### Complexity:

- Each round adds at least one constraint
- Max iterations =  $O(|\text{example space}|)$
- In practice: typically  $< 20$  rounds

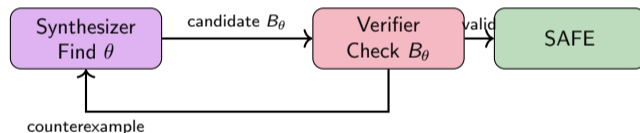
# Part VIII

## CEGIS: CounterExample-Guided Inductive Synthesis

Complete Algorithm and Implementation

## Key Idea

Alternate between **synthesis** (find candidate) and **verification** (check candidate).



Counterexamples from failed verification **guide** the next synthesis attempt.

# CEGIS: Complete Algorithm

```
1: Input: Template  $B_\theta$ , Init, Safe, Unsafe, Dynamics
2: Output: Barrier certificate or UNKNOWN
3:
4: constraints  $\leftarrow \emptyset$ 
5:
6: while iterations < max_iterations do
7:   // SYNTHESIS PHASE
8:    $\theta^* \leftarrow \text{Solve}(\text{constraints})$                                 ▷ Find parameters
9:   if no solution then
10:    return UNKNOWN                                                    ▷ Template too weak
11:  end if
12:
13:  // VERIFICATION PHASE
14:   $B \leftarrow B_{\theta^*}$ 
15:  (valid, cex)  $\leftarrow \text{CheckInductive}(B, \text{Init}, \text{Safe}, \text{Unsafe}, \text{Dyn})$ 
16:  if valid then
17:    return  $B$                                                          ▷ Found valid barrier!
18:  end if
```

## Goal

Find parameter values  $\theta$  satisfying all collected constraints.

## Constraint Types:

- 1 **Init constraints:**  $B_\theta(s_0^{(i)}) \geq \epsilon$  for sampled  $s_0^{(i)} \in \text{Init}$
- 2 **Unsafe constraints:**  $B_\theta(s_u^{(i)}) \leq -\epsilon$  for sampled  $s_u^{(i)} \in \text{Unsafe}$
- 3 **Step constraints:**  $B_\theta(s'^{(i)}) \geq 0$  for transitions where  $B_\theta(s^{(i)}) \geq 0$
- 4 **Exclusion constraints:** Exclude previous failed  $\theta$  values

**Solver:** Use Z3 to find satisfying  $\theta$ , or prove unsatisfiable.

## Goal

Check if candidate barrier  $B_\theta$  is inductive.

### Three Conditions to Verify:

- ① **Init Condition:**  $\forall s \in \text{Init}. B(s) \geq \epsilon$ 
  - Z3 query: Is  $\exists s \in \text{Init}. B(s) < \epsilon$  satisfiable?
- ② **Unsafe Condition:**  $\forall s \in \text{Unsafe}. B(s) \leq -\epsilon$ 
  - Z3 query: Is  $\exists s \in \text{Unsafe}. B(s) > -\epsilon$  satisfiable?
- ③ **Step Condition:**  $\forall s, s'. (B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$ 
  - Z3 query: Is  $\exists s, s'. B(s) \geq 0 \wedge s \rightarrow s' \wedge B(s') < 0$  satisfiable?

If all UNSAT: Barrier is **valid**. Otherwise: Extract counterexample.

# CEGIS: Counterexample Extraction

```
@dataclass
class Counterexample:
    """A counterexample from failed verification."""
    kind: str
    model: z3.ModelRef
    state_values: dict[str, any]
    variable_value: Optional[float] = None
    barrier_value: Optional[float] = None

def extract_counterexample(solver: z3.Solver,
                          barrier: BarrierCertificate,
                          kind: str) -> Counterexample:
    """Extract counterexample from SAT result."""
    model = solver.model()

    state_values = {}
    for decl in model.decls():
        state_values[decl.name()] = model[decl]

    barrier_value = eval_barrier(barrier, state_values)

    return Counterexample(
        kind=kind,
        model=model,
        state_values=state_values,
        barrier_value=barrier_value
    )
```

# CEGIS: Adding Constraints from Counterexamples

## Constraint Generation

Turn counterexample into constraint that excludes it.

**Init Counterexample** at  $s_0$ :

Add constraint:  $B_\theta(s_0) \geq \epsilon$

**Unsafe Counterexample** at  $s_u$ :

Add constraint:  $B_\theta(s_u) \leq -\epsilon$

**Step Counterexample** at  $(s, s')$ :

Add constraint:  $B_\theta(s) \geq 0 \Rightarrow B_\theta(s') \geq 0$

**Exclusion Constraint** (prevent same  $\theta$ ):

$$\theta \neq \theta_{\text{prev}}^*$$

## Available Templates

### 1. Quadratic Barrier:

$$B(x) = c_2x^2 + c_1x + c_0$$

### 2. Bivariate Quadratic:

$$B(x, y) = c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y + c_{00}$$

### 3. Polynomial (degree $d$ ):

$$B(\mathbf{x}) = \sum_{|\alpha| \leq d} c_{\alpha} \mathbf{x}^{\alpha}$$

### 4. Custom Templates:

- Bounds:  $B(i, len) = c_1 \cdot (len - i - 1) + c_0$
- Division:  $B(d) = c_1 \cdot |d| + c_0$

# CEGIS: Result Structure

```
@dataclass
class CEGISResult:
    """Result of CEGIS synthesis."""
    success: bool
    barrier: Optional[BarrierCertificate] = None
    inductiveness: Optional[InductivenessResult] = None
    iterations: int = 0
    counterexamples_collected: int = 0
    synthesis_time_ms: float = 0.0
    termination_reason: str = "unknown"
    counterexamples: list[Counterexample] = field(default_factory=list)

    def summary(self) -> str:
        if self.success:
            return f"CEGIS SUCCESS: {self.barrier.name} " \
                   f"({self.iterations} iters, {self.counterexamples_collected} CEs)"
        else:
            return f"CEGIS FAILED: {self.termination_reason}"
```

## Termination Reasons:

- inductive\_barrier\_found: Success!
- parameter\_space\_exhausted: Template too weak
- timeout: Need more time or simpler problem

**Problem:** Verify  $x = \text{arr}[i]$  where  $i = \text{len}(\text{arr}) - 1$

| Iter | Candidate                | Result                             | Action                              |
|------|--------------------------|------------------------------------|-------------------------------------|
| 1    | $B = i - \text{len} + 1$ | Init fail: $i = 0, \text{len} = 1$ | Add $B(0, 1) \geq 0$                |
| 2    | $B = i - \text{len} + 1$ | Step fail: increment $i$           | Add step constraint                 |
| 3    | $B = \text{len} - i - 1$ | Unsafe fail: $i = \text{len}$      | Add $B(\text{len}, \text{len}) < 0$ |
| 4    | $B = \text{len} - i - 1$ | <b>PASS</b>                        | Done!                               |

**Final Barrier:**  $B(i, \text{len}) = \text{len} - i - 1$

- $i = \text{len} - 1 \Rightarrow B = 0 \geq 0 \checkmark$
- $i \geq \text{len} \Rightarrow B < 0$  (unsafe)  $\checkmark$

# Part IX

## SOS/SDP Integration with Barrier Synthesis

Semidefinite Programming for Polynomial Proofs

# SOS/SDP: The Barrier-SDP Connection

## Key Insight

Barrier conditions become **polynomial positivity** conditions, which reduce to **SDP feasibility**.

**Barrier Condition:**  $\forall x \in \text{Init}. B(x) \geq \epsilon$

**Polynomial Form:**  $B(x) - \epsilon \geq 0$  on semialgebraic set Init

**SOS Relaxation:** Find SOS  $\sigma_0, \sigma_1, \dots$  such that:

$$B(x) - \epsilon = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x)$$

where  $g_i(x) \geq 0$  define Init.

**SDP:** Finding SOS polynomials is an SDP!

# SOS/SDP: Gram Matrix Formulation

## Theorem (Parrilo)

$p(x)$  is SOS of degree  $2d$  iff  $p(x) = \mathbf{m}(x)^T Q \mathbf{m}(x)$  where  $Q \succeq 0$ .

**Monomial Vector:**  $\mathbf{m}(x) = [1, x_1, x_2, \dots, x_1^d, \dots]^T$

**Gram Matrix:**  $Q$  is positive semidefinite (PSD)

**Example:**  $p(x) = x^4 + 2x^2 + 1$

$$\mathbf{m}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

**Check:**  $\mathbf{m}^T Q \mathbf{m} = 1 + 2x^2 + x^4 \checkmark$

# SOS/SDP: Coefficient Matching

## Linear Constraints on $Q$

Matching polynomial coefficients gives linear constraints on  $Q$  entries.

For  $p(x) = c_0 + c_1x + c_2x^2$  with  $\mathbf{m}(x) = [1, x]^T$ :

$$Q = \begin{bmatrix} q_{00} & q_{01} \\ q_{01} & q_{11} \end{bmatrix}$$

**Matching:**

Constant term:  $q_{00} = c_0$

Linear term:  $2q_{01} = c_1$

Quadratic term:  $q_{11} = c_2$

**SDP:** Find  $Q \succeq 0$  satisfying these linear constraints.

# SOS/SDP: Complete Barrier SDP

**Goal:** Synthesize barrier  $B(x) = \mathbf{m}(x)^T P \mathbf{m}(x)$

**SDP Program:**

Find  $P, Q_0, Q_1, \dots$

s.t.  $P \succeq 0$  (barrier is SOS)

$Q_0 \succeq 0, Q_1 \succeq 0, \dots$  (multipliers are SOS)

$$B(x) - \epsilon = \sigma_0(x) + \sum_i \sigma_i(x) g_i^{\text{Init}}(x)$$

$$-B(x) - \epsilon = \tau_0(x) + \sum_j \tau_j(x) h_j^{\text{Unsafe}}(x)$$

$$-\dot{B}(x) = \rho_0(x) + \sum_k \rho_k(x) (B(x) \cdot \text{Safe}_k(x))$$

**Solve with SDP solver** (MOSEK, SDPA, SCS).

## Putinar's Positivstellensatz

If  $p > 0$  on  $\{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  (compact), then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i$  are SOS.

### For Barrier Init Condition:

- $p(x) = B(x) - \epsilon$  (what we want positive)
- $g_i(x)$  define the initial region
- Find SOS multipliers  $\sigma_i$

**Degree Bound:** Multipliers have degree  $\leq 2d - \deg(g_i)$  for degree- $2d$  proof.

## When SOS Fails

Increase degree in Lasserre hierarchy for stronger proofs.

### Lasserre Level $k$ :

- Use monomials up to degree  $k$
- Larger SDP, tighter relaxation
- Level  $\infty$ : exact polynomial optimization

### Barrier Synthesis Strategy:

- 1 Start with low degree  $d = 2$
- 2 If SDP infeasible, increase to  $d = 4$
- 3 Continue until feasible or timeout

**Convergence:** For polynomial systems, hierarchy converges finitely.

## Problem

SDP size grows as  $O(n^{2d})$  for  $n$  variables, degree  $2d$ .

## Sparse SOS (Paper #8):

- Exploit **correlative sparsity**: variables interact locally
- Decompose large SDP into smaller coupled SDPs
- Use **chordal decomposition** of variable graph

## Example:

- Full:  $n = 10$ , degree 4  $\Rightarrow$  715-dimensional SDP
- Sparse: Decompose into 10 coupled 15-dimensional SDPs
- Speedup:  $\sim 100\times$  for structured problems

## Faster Alternatives to SDP

Replace PSD constraint with diagonal dominance (LP/SOCP).

### DSOS (Diagonally-dominant SOS):

$$p(x) = \sum_{i,j} \lambda_{ij} (x_i \pm x_j)^2 \quad (\lambda_{ij} \geq 0)$$

⇒ Linear program!

### SDSOS (Scaled diagonally-dominant SOS):

Gram matrix  $Q$  is scaled diagonally dominant

⇒ Second-order cone program!

### Tradeoff:

| Method  | Complexity   | Completeness |
|---------|--------------|--------------|
| SOS/SDP | $O(n^{3.5})$ | High         |

# SOS/SDP: Implementation in the Pipeline

```
class SOSSafetyChecker:
    """SOS-based safety checking (Paper \#3)."""

    def check_safety(self, conditions: BarrierConditions,
                    degree: int = 4) -> Optional[Polynomial]:
        """
        Check if barrier exists via SOS/SDP.

        1. Build SDP for barrier conditions
        2. Solve with SDP solver
        3. Extract barrier from solution
        """

        basis = MonomialBasis(self.n_vars, degree // 2)

        Q = self._create_gram_matrix(basis)

        self._add_matching_constraints(Q, conditions)

        self._add_psd_constraint(Q)

        if self._solve_sdp():
            return self._extract_barrier(Q)
        return None
```

# Part X

## Hybrid System Barrier Certificates

### Multi-Mode Safety Verification

## Definition

A **hybrid system** combines continuous dynamics with discrete mode switches.

## Hybrid Automaton:

- **Modes:**  $\{q_1, q_2, \dots, q_m\}$
- **Continuous dynamics:**  $\dot{x} = f_q(x)$  in mode  $q$
- **Invariants:**  $I_q(x)$  must hold in mode  $q$
- **Guards:**  $G_{q \rightarrow q'}(x)$  enables transition
- **Resets:**  $R_{q \rightarrow q'}(x)$  updates state on transition

**Example:** Thermostat (heating on/off), Bouncing ball (flight/impact)

# Hybrid Barriers: Multi-Mode Certificates

## Hybrid Barrier Certificate (Paper #1)

Separate barrier  $B_q(x)$  for each mode  $q$ .

### Conditions:

- ① **Init:**  $\forall q, x. (x \in \text{Init}_q) \Rightarrow B_q(x) \geq \epsilon$
- ② **Unsafe:**  $\forall q, x. (x \in \text{Unsafe}_q) \Rightarrow B_q(x) \leq -\epsilon$
- ③ **Flow:**  $\forall q, x. (x \in I_q \wedge B_q(x) \geq 0) \Rightarrow \dot{B}_q(x) \leq 0$
- ④ **Jump:**  $\forall q, q', x. (G_{q \rightarrow q'}(x) \wedge B_q(x) \geq 0) \Rightarrow B_{q'}(R(x)) \geq 0$

**Key:** Barriers must be **consistent across transitions**.

## Flow Condition

In mode  $q$  with dynamics  $\dot{x} = f_q(x)$ , barrier must not increase:

$$\mathcal{L}_{f_q} B_q(x) = \nabla B_q(x) \cdot f_q(x) \leq 0$$

**Example:** Linear system  $\dot{x} = Ax$ , quadratic barrier  $B(x) = x^T P x$

$$\dot{B} = x^T (A^T P + P A) x \leq 0$$

$\Rightarrow$  Need  $A^T P + P A \preceq 0$  (Lyapunov condition)

**SOS Encoding:**  $-\mathcal{L}_f B$  is SOS on safe region.

# Hybrid Barriers: Jump Condition

## Transition Safety

When switching from  $q$  to  $q'$  with reset  $x' = R(x)$ :

$$B_q(x) \geq 0 \wedge G_{q \rightarrow q'}(x) \Rightarrow B_{q'}(R(x)) \geq 0$$

**Challenge:** Reset maps can be complex (nonlinear, partial).

**SOS Encoding:**

$$B_{q'}(R(x)) - \sigma(x) \cdot B_q(x) - \sum_i \tau_i(x) G_i(x) \text{ is SOS}$$

where  $\sigma, \tau_i$  are SOS multipliers.

**Interpretation:** If in safe region ( $B_q \geq 0$ ) and guard holds, then post-reset state is safe ( $B_{q'} \geq 0$ ).

# Hybrid Barrier Synthesis

- 1: **Input:** Hybrid automaton  $H = (Q, X, f, I, G, R)$
- 2: **Output:** Barrier certificates  $\{B_q\}_{q \in Q}$
- 3:
- 4: Choose template degree  $d$
- 5: **for** each mode  $q \in Q$  **do**
- 6:     Create template  $B_q(x) = \sum_{|\alpha| \leq d} c_q^\alpha x^\alpha$
- 7: **end for**
- 8:
- 9: *// Build SDP for all conditions*
- 10: Add Init constraints for each mode
- 11: Add Unsafe constraints for each mode
- 12: Add Flow constraints (Lie derivatives)
- 13: Add Jump constraints (transition safety)
- 14:
- 15: **Solve** combined SDP
- 16: **return** Extracted barriers  $\{B_q\}$

# Hybrid Barriers: Bouncing Ball Example

## System:

- Mode 1 (flight):  $\dot{h} = v, \dot{v} = -g$
- Mode 2 (impact):  $v' = -c \cdot v$  (coefficient of restitution)
- Guard:  $h = 0 \wedge v < 0$  (hit ground)
- Unsafe:  $h < 0$  (below ground)

## Barrier: $B(h, v) = h$

- Init:  $h \geq h_0 > 0 \Rightarrow B \geq h_0 \checkmark$
- Unsafe:  $h < 0 \Rightarrow B < 0 \checkmark$
- Flow:  $\dot{B} = v$  (can be positive or negative)
- Jump: At  $h = 0, B' = 0 \geq 0 \checkmark$

**Refined:**  $B(h, v) = h + \epsilon$  ensures  $B > 0$  always.

# Stochastic Barriers: Overview (Paper #2)

## Stochastic Systems

Dynamics include noise:  $dx = f(x)dt + g(x)dW$

**Safety Question:** What is  $\Pr[\text{reach Unsafe}]$ ?

**Stochastic Barrier Certificate:**

- $B(x) \geq 0$  on initial states
- $B(x)$  is a **supermartingale**:  $\mathbb{E}[dB] \leq 0$
- $B(x) \leq -\epsilon$  on unsafe states

**Itô Condition (supermartingale):**

$$\mathcal{L}_f B + \frac{1}{2} \text{tr}(g^T \nabla^2 B \cdot g) \leq 0$$

# Stochastic Barriers: Probability Bound

## Theorem (Prajna et al. 2007)

*If  $B(x)$  is a stochastic barrier certificate, then:*

$$\Pr[\text{reach Unsafe}] \leq \frac{\sup_{x \in \text{Init}} B(x)}{\inf_{x \in \text{Unsafe}} (-B(x))}$$

### Interpretation:

- Larger barrier gap  $\Rightarrow$  lower probability
- $B = +\infty$  on init,  $B = -\infty$  on unsafe  $\Rightarrow$  probability 0

**For Programs:** Model randomness as stochastic transitions.

## SOS Formulation:

$$\begin{aligned} \text{Find } B(x) &= \sum c_\alpha x^\alpha \\ \text{s.t. } B(x) - \epsilon &\text{ is SOS on Init} \\ &- B(x) - \epsilon \text{ is SOS on Unsafe} \\ &- \mathcal{L}_f B - \frac{1}{2} \text{tr}(g^T \nabla^2 B \cdot g) \text{ is SOS on Safe} \end{aligned}$$

**Challenge:** Second-order term  $\nabla^2 B$  increases SDP size.

**Solution:** Use polynomial templates where Hessian is tractable.

**Example:** Quadratic  $B(x) = x^T P x$

$$\nabla^2 B = 2P \quad (\text{constant})$$

# Part XI

## IC3/PDR: Property-Directed Reachability

### Incremental Inductive Reasoning for Programs

## Key Innovation (Bradley 2011)

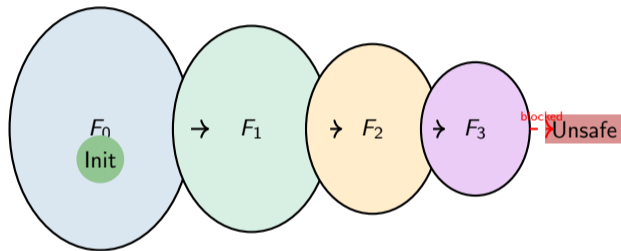
Discover inductive invariants **incrementally** using frames.

**Frame Sequence:**  $F_0, F_1, \dots, F_k$

- $F_i$  overapproximates states reachable in  $\leq i$  steps
- $F_0 = \text{Init}$
- $F_i \supseteq F_{i+1}$  (monotonic)
- Each  $F_i$  is a conjunction of clauses

**Convergence:** When  $F_i = F_{i+1}$ , we have an inductive invariant!

# IC3/PDR: Frame Structure



**Goal:** Show Unsafe is not reachable from any  $F_i$ .

# IC3/PDR: Algorithm Overview

```
1: Input: Init, Trans, Unsafe
2: Output: SAFE (with invariant) or UNSAFE (with trace)
3:
4:  $F_0 \leftarrow \text{Init}$ 
5:  $k \leftarrow 0$ 
6: while true do
7:   // BLOCKING: Remove bad states from frames
8:   while  $\exists s \in F_k \cap \text{Unsafe}$  do
9:     if cannot block  $s$  then
10:       return UNSAFE (extract trace)
11:     end if
12:     Block  $s$  by adding clause to appropriate frame
13:   end while
14:
15:   // PROPAGATION: Push clauses forward
16:    $k \leftarrow k + 1$ 
17:   Propagate clauses from  $F_{k-1}$  to  $F_k$ 
18:   if  $F_k = F_{k-1}$  then
```

## Counterexample to Induction (CTI)

A state  $s$  such that  $s \in F_i$  but  $\text{Post}(s) \cap \text{Unsafe} \neq \emptyset$ .

### Blocking Procedure:

- ① Find CTI:  $s$  can reach Unsafe in one step
- ② Generalize: Find clause  $c$  such that:
  - $s \not\models c$  (excludes  $s$ )
  - $c$  is inductive relative to  $F_{i-1}$
- ③ Add  $c$  to  $F_1, F_2, \dots, F_i$

**Key:** Generalization prevents blocking same state repeatedly.

## Moving Clauses Forward

If clause  $c$  is inductive relative to  $F_i$ , add it to  $F_{i+1}$ .

**Inductive Relative to  $F_i$ :**

$$F_i \wedge c \wedge \text{Trans} \Rightarrow c'$$

(If in  $F_i$  and  $c$  holds, then  $c$  holds after transition)

**Propagation Algorithm:**

- 1 For each clause  $c \in F_i$ :
- 2 Check if  $c$  is inductive relative to  $F_i$
- 3 If yes: add  $c$  to  $F_{i+1}$
- 4 If all clauses propagate:  $F_i = F_{i+1}$  (fixed point!)

**Cube:** Conjunction of literals (represents states)

$$\text{cube } s = (x > 0) \wedge (y \leq 5) \wedge (z = 0)$$

**Clause:** Disjunction of literals (excludes states)

$$\text{clause } c = (x \leq 0) \vee (y > 5) \vee (z \neq 0) = \neg s$$

**Relationship:**

- Bad cube  $s$  represents states to block
- Blocking clause  $c = \neg s$  excludes those states
- Frame  $F_i = \bigwedge_j c_j$  is conjunction of blocking clauses

# IC3/PDR: Application to Programs

**Program State:**  $(pc, \sigma) = (\text{program counter, variable store})$

**Transition Relation:** Derived from CFG

$$\text{Trans}((pc, \sigma), (pc', \sigma')) \iff \text{edge } pc \rightarrow pc' \text{ with update } \sigma \rightarrow \sigma'$$

**Unsafe States:** Bug locations

$$\text{Unsafe} = \{(pc, \sigma) : pc = \text{bug\_loc} \wedge \text{bug\_condition}(\sigma)\}$$

**Example (DIV\_ZERO):**

$$\text{Unsafe} = \{(pc, \sigma) : pc = 42 \wedge \sigma[d] = 0\}$$

## Key Insight

IC3 lemmas become **side conditions** for barrier synthesis.

### Bridge from IC3 to Barriers:

- 1 Run IC3 to discover invariant clauses
- 2 Lift clauses to polynomial constraints:
  - $(x > 0) \Rightarrow \text{constraint } x - \epsilon \geq 0$
  - $(y \leq 5) \Rightarrow \text{constraint } 5 - y \geq 0$
- 3 Add lifted constraints to barrier SDP
- 4 Solve constrained SDP for barrier

**Benefit:** IC3 prunes infeasible regions before expensive SOS.

# IC3/PDR: Implementation

```
class IC3Engine:
    """IC3/PDR for invariant discovery (Paper \#10)."""

    def __init__(self, n_vars: int, timeout_ms: int = 60000):
        self.n_vars = n_vars
        self.timeout_ms = timeout_ms
        self.frames: List[Frame] = []

    def verify(self, init: z3.BoolRef, trans: z3.BoolRef,
              unsafe: z3.BoolRef) -> IC3Result:
        """Run IC3/PDR algorithm."""
        self.frames = [Frame(clauses={init})]

        while not timeout():

            blocked = self._block_all_cti()
            if not blocked:
                return IC3Result(status='unsafe', trace=self._extract_trace())

            self._propagate_clauses()

            if self._check_fixed_point():
                invariant = self._extract_invariant()
                return IC3Result(status='safe', invariant=invariant)

        return IC3Result(status='unknown')
```

# Part XII

## Constrained Horn Clauses (CHC) and Spacer

SMT-Based Program Verification

# CHC: Constrained Horn Clauses

## Definition

A CHC is a first-order formula of the form:

$$\forall \vec{x}. (p_1(\vec{x}_1) \wedge \dots \wedge p_n(\vec{x}_n) \wedge \phi(\vec{x})) \Rightarrow h(\vec{x}_h)$$

## Components:

- **Body predicates:**  $p_1, \dots, p_n$  (uninterpreted)
- **Constraint:**  $\phi$  (interpreted, e.g., linear arithmetic)
- **Head:**  $h$  (uninterpreted predicate or  $\perp$ )

**Solving CHC:** Find interpretations for predicates such that all clauses are satisfied.

# CHC: Encoding Programs

## Program:

```
def foo(x):  
    y = 0  
    while x > 0:  
        y = y + 1  
        x = x - 1  
    assert y >= 0
```

## CHC Encoding:

|  |          |
|--|----------|
| $\text{true} \Rightarrow \text{Inv}(x, 0)$                           | (init)   |
| $\text{Inv}(x, y) \wedge x > 0 \Rightarrow \text{Inv}(x - 1, y + 1)$ | (loop)   |
| $\text{Inv}(x, y) \wedge x \leq 0 \wedge y < 0 \Rightarrow \perp$    | (assert) |

**Solution:**  $\text{Inv}(x, y) = (y \geq 0)$

# Spacer: CHC Solving Algorithm

Spacer (Komuravelli et al. 2014)

Combines IC3/PDR with interpolation for CHC solving.

## Key Ideas:

- ① **Under-approximation:** Track concrete reachability
- ② **Over-approximation:** Track inductive summaries
- ③ **Interpolation:** Generalize from concrete to symbolic
- ④ **Recursion handling:** Special frames for recursive calls

In Z3: `z3.Fixedpoint()` with `engine='spacer'`

## Barrier as CHC Solution

A barrier certificate  $B$  induces a CHC solution.

### Translation:

- $\text{Safe}(x) := B(x) \geq 0$
- Init clause:  $\forall x \in \text{Init}. B(x) \geq \epsilon \Rightarrow \text{Safe}(x) \checkmark$
- Step clause:  $\forall x, x'. \text{Safe}(x) \wedge x \rightarrow x' \Rightarrow \text{Safe}(x') \checkmark$
- Unsafe clause:  $\forall x. \text{Safe}(x) \wedge \text{Unsafe}(x) \Rightarrow \perp \checkmark$

**Benefit:** CHC solvers can discover barrier-like invariants automatically.

## Craig Interpolation Theorem

If  $A \wedge B$  is unsatisfiable, there exists  $I$  such that:

- ①  $A \Rightarrow I$
- ②  $I \wedge B$  is unsatisfiable
- ③  $\text{vars}(I) \subseteq \text{vars}(A) \cap \text{vars}(B)$

### For Verification:

- $A$  = formula for  $k$ -step reachability
- $B$  = unsafe states
- $I$  = over-approximation of reachable states

# Interpolation: Application to Barriers

**Scenario:** Have weak barrier  $B_0$ , need refinement.

**Process:**

- ① Query: Is  $B_0 \geq 0 \wedge \text{Unsafe}$  satisfiable?
- ② If UNSAT:  $B_0$  is sufficient
- ③ If SAT with counterexample  $(s, s')$ :
  - $A$  = path constraints to reach  $s$
  - $B$  = unsafe condition at  $s'$
  - Interpolant  $I$  suggests barrier refinement
- ④ Strengthen:  $B_1 = B_0 \wedge I$

**IMPACT (McMillan 2006):** Lazy abstraction with interpolants.

## IMC Algorithm (McMillan 2003)

Use interpolation to compute reachable state over-approximation.

### Algorithm:

- 1 Check: Is  $\text{Init} \rightarrow^k \text{Unsafe}$  reachable? (BMC query)
- 2 If SAT: Bug found at depth  $k$
- 3 If UNSAT: Compute interpolant sequence  $I_0, I_1, \dots, I_k$
- 4 Over-approximation:  $R \supseteq I_0 \cup I_1 \cup \dots \cup I_k$
- 5 If  $R$  is inductive: SAFE
- 6 Else: Increase  $k$  and repeat

**For Barriers:** Interpolants suggest barrier shape and constraints.

# Assume-Guarantee: Compositional Verification

## Key Idea (Pnueli 1985)

Verify components separately under assumptions about environment.

### Rule:

$$\frac{A \parallel E \models P \quad E \models A}{A \parallel E \models P}$$

### Interpretation:

- Component  $A$  satisfies  $P$  under assumption about  $E$
- Environment  $E$  satisfies the assumption  $A$
- Therefore:  $A \parallel E$  satisfies  $P$

# Assume-Guarantee: Compositional Barriers

**Problem:** Verify large system with many components.

## Compositional Approach:

- ① Decompose system into components  $C_1, C_2, \dots, C_n$
- ② For each  $C_i$ :
  - Assume barrier  $A_i$  holds for environment
  - Prove local barrier  $B_i$  for component
- ③ Verify:  $\bigwedge_i A_i \Rightarrow \bigwedge_i B_i$  (circular reasoning)
- ④ Solve for compatible assumption-guarantee pairs

**In Pipeline:** Used for interprocedural verification across modules.

# Part XIII

## Data Structures and Representations

The Foundation of the Implementation

# Polynomial Representation

```
@dataclass
class Monomial:
    """A monomial  $x_1^{a_1} * x_2^{a_2} * \dots * x_n^{a_n}$ ."""
    exponents: Tuple[int, ...]

    @property
    def degree(self) -> int:
        return sum(self.exponents)

    def multiply(self, other: 'Monomial') -> 'Monomial':
        """Multiply two monomials (add exponents)."""
        return Monomial(tuple(a + b for a, b in zip(self.exponents, other.exponents)))

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.ExprRef:
        """Convert to Z3 expression."""
        result = z3.RealVal(1)
        for i, exp in enumerate(self.exponents):
            for _ in range(exp):
                result = result * vars_z3[i]
        return result

@dataclass
class Polynomial:
    """Sparse polynomial representation."""
    n_vars: int
    terms: Dict[Monomial, float]

    @property
    def degree(self) -> int:
        return max(m.degree for m in self.terms.keys()) if self.terms else 0
```

# Polynomial Operations

```
class Polynomial:
    def add(self, other: 'Polynomial') -> 'Polynomial':
        """Add two polynomials."""
        result = Polynomial(max(self.n_vars, other.n_vars), dict(self.terms))
        for mono, coeff in other.terms.items():
            result.terms[mono] = result.terms.get(mono, 0) + coeff
        return result

    def multiply(self, other: 'Polynomial') -> 'Polynomial':
        """Multiply two polynomials."""
        result = Polynomial(max(self.n_vars, other.n_vars))
        for m1, c1 in self.terms.items():
            for m2, c2 in other.terms.items():
                new_mono = m1.multiply(m2)
                result.terms[new_mono] = result.terms.get(new_mono, 0) + c1 * c2
        return result

    def gradient(self) -> List['Polynomial']:
        """Compute gradient (list of partial derivatives)."""
        return [self._partial_derivative(i) for i in range(self.n_vars)]

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.ExprRef:
        """Convert to Z3 expression."""
        result = z3.RealVal(0)
        for mono, coeff in self.terms.items():
            result = result + z3.RealVal(coeff) * mono.to_z3(vars_z3)
        return result
```

# Semialgebraic Sets

```
@dataclass
class SemialgebraicSet:
    """
    A basic semialgebraic set:  $S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$ .

    Used to represent:
    - Initial states
    - Unsafe regions
    - Safe regions
    - Mode invariants
    """
    n_vars: int
    constraints: List[Polynomial]

    def contains(self, point: List[float]) -> bool:
        """Check if point is in the set."""
        return all(g.evaluate(point) >= 0 for g in self.constraints)

    def intersect(self, other: 'SemialgebraicSet') -> 'SemialgebraicSet':
        """Intersection of two sets."""
        return SemialgebraicSet(
            max(self.n_vars, other.n_vars),
            self.constraints + other.constraints
        )

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.BoolRef:
        """Convert to Z3 constraint."""
        return z3.And([g.to_z3(vars_z3) >= 0 for g in self.constraints])
```

# Barrier Certificate Structure

```
@dataclass
class BarrierCertificate:
    """
    A barrier certificate with metadata.

    Attributes:
        name: Human-readable identifier
        barrier_fn: The barrier function B: S -> R
        epsilon: Safety margin (default 0.01)
        description: Optional explanation
        variables: Variables referenced by barrier
    """
    name: str
    barrier_fn: Callable[[SymbolicMachineState], z3.ExprRef]
    epsilon: float = 0.01
    description: Optional[str] = None
    variables: list[str] = None

    def evaluate(self, state: SymbolicMachineState) -> z3.ExprRef:
        """Evaluate B(sigma) for the given state."""
        return self.barrier_fn(state)

# Example barrier for BOUNDS check
bounds_barrier = BarrierCertificate(
    name="bounds_check",
    barrier_fn=lambda s: s.get_local('len') - s.get_local('idx') - 1,
    epsilon=0.01,
    description="Index within bounds",
    variables=['len', 'idx']
)
```

# Crash Summary Structure

```
@dataclass
class CrashSummary:
    """
    Summary of a function's behavior for interprocedural analysis.

    Captures:
    - Function metadata
    - Parameter validations
    - Return guarantees
    - Guarded bugs (bugs protected by guards)
    - Guard facts collected during analysis
    """
    function_name: str
    module_name: str
    file_path: str
    line_number: int

    validated_params: Dict[int, Set[str]]

    return_guarantees: Set[str]

    guarded_bugs: Set[str]

    guard_facts: List[GuardFact]
```

# Guard Fact Structure

```
@dataclass
class GuardFact:
    """
    A guard fact from CFG analysis.

    Represents a condition that must hold at a program point.
    """
    guard_type: str
    variable: str
    condition: str
    source_location: int
    is_strong: bool

    def protects_bug(self, bug_type: str, bug_variable: str) -> bool:
        """Check if this guard protects against a specific bug."""
        if bug_variable != self.variable:
            return False

        protection_map = {
            'BOUNDS': {'assert_nonempty', 'len_check', 'range_check'},
            'DIV_ZERO': {'assert_nonzero', 'nonzero_check', 'positive_check'},
            'NULL_PTR': {'assert_nonnull', 'nonnull_check', 'if_nonnull'},
            'KEY_ERROR': {'key_in_check', 'haskey_check'},
        }

        return self.guard_type in protection_map.get(bug_type, set())
```

# Verification Result Structure

```
@dataclass
class ContextAwareResult:
    """
    Result of context-aware verification.
    """
    is_safe: bool
    bug_type: str
    bug_variable: Optional[str]

    guard_barriers: List[BarrierCertificate]
    synthesized_barriers: List[BarrierCertificate]
    learned_invariants: List[str]

    counterexample: Optional[Dict[str, Any]] = None
    dse_counterexample: Optional[Dict[str, Any]] = None

    verification_time_ms: float = 0.0
    phases_completed: List[str] = field(default_factory=list)

    def summary(self) -> str:
        if self.is_safe:
            return f"SAFE: {len(self.guard_barriers)} guards, " \
                f"{len(self.synthesized_barriers)} synthesized"
        return f"UNSAFE: counterexample found"
```

# ICE Example Structure

```
@dataclass
class DataPoint:
    """A data point for learning."""
    values: Tuple[float, ...]
    label: str
    linked_to: Optional['DataPoint'] = None

@dataclass
class ICEExample:
    """
    ICE (Implication CounterExample) data.

    Three types:
    1. Positive: must be satisfied (initial states)
    2. Negative: must be violated (unsafe states)
    3. Implication: (pre, post) pairs - if pre satisfied, post must be
    """
    positive: List[DataPoint]
    negative: List[DataPoint]
    implications: List[Tuple[DataPoint, DataPoint]]

    def add_positive(self, values: Tuple[float, ...]):
        self.positive.append(DataPoint(values, 'positive'))

    def add_negative(self, values: Tuple[float, ...]):
        self.negative.append(DataPoint(values, 'negative'))

    def add_implication(self, pre: Tuple[float, ...], post: Tuple[float, ...]):
        pre_dp = DataPoint(pre, 'implication_pre')
        post_dp = DataPoint(post, 'implication_post')
```

# IC3 Frame and Clause Structures

```
@dataclass(frozen=True)
class Literal:
    """A literal in IC3/PDR."""
    variable: str
    negated: bool = False

    def __neg__(self) -> "Literal":
        return Literal(self.variable, not self.negated)

@dataclass(frozen=True)
class Cube:
    """Conjunction of literals (represents states)."""
    literals: FrozenSet[Literal]

    def negate(self) -> "Clause":
        return Clause(frozenset(-lit for lit in self.literals))

@dataclass(frozen=True)
class Clause:
    """Disjunction of literals (blocking lemma)."""
    literals: FrozenSet[Literal]

@dataclass
class Frame:
    """A frame in IC3: over-approximation of reachable states."""
    level: int
    clauses: Set[Clause]

    def add_clause(self, clause: Clause):
        self.clauses.add(clause)
```

# Part XIV

## Symbolic Execution Integration

From Python to Z3 Constraints

# Symbolic Machine State

```
@dataclass
class SymbolicMachineState:
    """
    Symbolic state for Python execution.

    Components:
    - pc: Path condition (Z3 formula)
    - locals: Local variable bindings (name -> Z3 expr)
    - heap: Symbolic heap model
    - stack: Call stack for interprocedural
    - taint: Taint tracking map
    """
    path_condition: z3.BoolRef
    locals: Dict[str, z3.ExprRef]
    heap: Dict[int, z3.ExprRef]
    stack: List['StackFrame']
    taint: Dict[str, Set[str]]

    def get_local(self, name: str) -> z3.ExprRef:
        """Get local variable value."""
        if name in self.locals:
            return self.locals[name]

        return z3.Int(f"sym_{name}")

    def with_constraint(self, constraint: z3.BoolRef) -> 'SymbolicMachineState':
        """Add constraint to path condition."""
        return SymbolicMachineState(
            path_condition=z3.And(self.path_condition, constraint),
            locals=self.locals.copy(), ...
```

## Challenge

Exponential number of paths in program with branches.

### Mitigation Strategies:

- 1 **Loop Bounding:** Limit loop iterations (default: 3)
- 2 **Depth Limiting:** Maximum symbolic execution depth (default: 50)
- 3 **Path Merging:** Merge paths at join points
- 4 **Prioritization:** Explore bug-likely paths first
- 5 **Incremental Solving:** Use Z3 push/pop for efficiency

### For Barrier Synthesis:

- Collect Init states from entry paths
- Collect Unsafe states from bug-reaching paths
- Collect transitions from sequential execution

# Bug Condition Encoding

## Encoding bugs as Z3 constraints:

| Bug Type   | Z3 Constraint (Unsafe)                           |
|------------|--|
| BOUNDS     | $\text{idx} < 0 \vee \text{idx} \geq \text{len}$ |
| DIV_ZERO   | $\text{divisor} = 0$                             |
| NULL_PTR   | $\text{ptr} = \text{null}$                       |
| TYPE_ERROR | $\text{type}(x) \neq \text{expected}$            |
| OVERFLOW   | $ x  > \text{MAX\_INT}$                          |
| KEY_ERROR  | $\text{key} \notin \text{dict}$                  |
| ASSERTION  | $\neg \text{assertion\_condition}$               |

## Verification Query:

$\text{SAT}(\text{path\_condition} \wedge \text{bug\_condition}) \Rightarrow \text{Potential bug}$

## Dynamic Symbolic Execution (DSE)

Combine concrete execution with symbolic reasoning.

### DSE Process:

- 1 Start with concrete input
- 2 Execute program, collecting path constraints
- 3 Negate constraints to explore new paths
- 4 Check if new path reaches bug

### For Barrier Verification:

- If barrier claims SAFE but DSE finds bug path: **Barrier is wrong**
- If DSE exhausts paths without bug: **Confirms SAFE**
- DSE provides ground truth for barrier validation

## Challenge

Handle function calls without exponential blowup.

### Approach: Function Summaries

- 1 Analyze each function once
- 2 Create summary: preconditions  $\rightarrow$  postconditions
- 3 At call site: Apply summary instead of re-analyzing

### Summary Structure:

- **Precondition:** What caller must guarantee
- **Postcondition:** What callee guarantees
- **Effects:** Modified state (heap, globals)

**Barrier Implication:** Caller's barrier + summary = Callee's precondition satisfied.

# Constraint Simplification

## Goal

Keep path conditions tractable for Z3.

## Simplification Techniques:

- ① **Constant Propagation:** Replace  $x$  with value if known
- ② **Redundancy Elimination:** Remove implied constraints
- ③ **Expression Sharing:** Common subexpression elimination
- ④ **Theory-Specific:** Arithmetic simplification

## Z3 Tactics:

- `simplify`: Basic simplification
- `propagate-values`: Constant propagation
- `ctx-solver-simplify`: Context-aware simplification

**Challenge:** Python is dynamically typed with complex semantics.

**Approach:**

① **Type Abstraction:**

- Track possible types for each variable
- Use union types:  $\text{type}(x) \in \{\text{int}, \text{str}\}$

② **Container Modeling:**

- Lists: length + element type
- Dicts: key set + value type

③ **Object Modeling:**

- Attribute access: `hasattr(obj, name)`
- Method resolution: approximate with summary

# Z3 Solver Configuration

```
def create_verification_solver(timeout_ms: int = 5000) -> z3.Solver:
    """Create optimally configured solver for verification."""
    solver = z3.Solver()

    solver.set("timeout", timeout_ms)

    solver.set("unsat_core", True)

    solver.set("arith.solver", 2)
    solver.set("arith.nl.nla", True)

    solver.set("proof", True)

    return solver

# Usage pattern for path exploration
solver = create_verification_solver()
solver.push()
solver.add(path_constraint)
if solver.check() == z3.sat:

    solver.add(bug_condition)
    if solver.check() == z3.sat:

        counterexample = solver.model()
    solver.pop()
```

## Key Optimizations in the Pipeline:

### ① Caching:

- Cache verification results for repeated queries
- Cache Z3 check results for similar constraints

### ② Incremental Solving:

- Use push/pop for branch exploration
- Maintain learned clauses across queries

### ③ Parallelization:

- Run portfolio strategies in parallel
- Parallel path exploration (where independent)

### ④ Early Termination:

- Stop on first SAFE proof or BUG witness
- Skip expensive phases if cheap phase succeeds

# Part XV

## Bug Detection Categories

67 Bug Types with Barrier-Based Verification

# Bug Categories Overview

| Category             | Count     | Examples                       |
|----------------------|-----------|--------------------------------|
| Logic Errors         | 12        | Bounds, Div-zero, Null ptr     |
| Type Errors          | 8         | Type mismatch, Attribute error |
| Injection (Security) | 9         | SQL, Command, XSS              |
| Crypto (Security)    | 8         | Weak hash, Hardcoded key       |
| Network (Security)   | 7         | SSRF, Open redirect            |
| Deserialization      | 4         | Pickle, YAML, JSON             |
| Resource             | 6         | Leak, Double free              |
| Concurrency          | 5         | Race, Deadlock                 |
| Other                | 8         | Assert, Unreachable            |
| <b>Total</b>         | <b>67</b> |                                |

# Bug Type: BOUNDS (Array Out-of-Bounds)

```
def get_item(items, idx):  
    return items[idx]
```

## Unsafe Condition:

$$\text{idx} < 0 \vee \text{idx} \geq \text{len}(\text{items})$$

## Barrier Template:

$$B(\text{idx}, \text{len}) = \min(\text{idx}, \text{len} - \text{idx} - 1)$$

## Verification:

- $B \geq 0 \Leftrightarrow 0 \leq \text{idx} < \text{len}$
- Guard:  $\text{assert } \text{len}(\text{items}) > 0 \Rightarrow B \geq 0 \text{ for } \text{idx}=0$

# Bug Type: DIV\_ZERO (Division by Zero)

```
def average(total, count):  
    return total / count
```

## Unsafe Condition:

$$\text{count} = 0$$

## Barrier Template:

$$B(\text{count}) = |\text{count}| - \epsilon$$

## Verification:

- $B \geq 0 \Leftrightarrow |\text{count}| \geq \epsilon \Leftrightarrow \text{count} \neq 0$
- Guard: `if count != 0:  $\Rightarrow B \geq 0$`
- Alternative: `count or 1` pattern

# Bug Type: NULL\_PTR (Null Pointer Dereference)

```
def process(obj):  
    return obj.method()
```

## Unsafe Condition:

$\text{obj} = \text{None}$

## Barrier Template:

$$B(\text{obj}) = \llbracket \text{obj} \neq \text{None} \rrbracket - 0.5$$

## Verification:

- $B \geq 0 \Leftrightarrow \text{obj} \neq \text{None}$
- Guard: if obj is not None:  $\Rightarrow B \geq 0$
- Optional chaining: `obj?.method()` (Python 3.10+)

# Bug Type: KEY\_ERROR (Missing Dictionary Key)

```
def get_config(config, key):  
    return config[key]
```

## Unsafe Condition:

$\text{key} \notin \text{config}$

## Barrier Template:

$$B(\text{key}, \text{config}) = \mathbb{I}[\text{key} \in \text{config}] - 0.5$$

## Verification:

- Guard:  $\text{if key in config:} \Rightarrow B \geq 0$
- Safe pattern: `config.get(key, default)`
- Safe pattern: `config.setdefault(key, value)`

# Bug Type: TYPE\_ERROR

```
def add_numbers(a, b):  
    return a + b
```

## Unsafe Condition:

$$\text{type}(a) \neq \text{type}(b) \wedge \neg \text{coercible}(a, b)$$

## Barrier Template:

$$B(a, b) = \mathbb{K}[\text{type}(a) = \text{type}(b)] - 0.5$$

## Verification:

- Guard: `isinstance(a, int)` and `isinstance(b, int)`
- Type hints: `def add(a: int, b: int) -> int`
- Abstract interpretation: Track type sets

# Bug Type: SQL\_INJECTION (Security)

```
def query_user(cursor, username):  
    sql = f"SELECT * FROM users WHERE name = '{username}'"  
    cursor.execute(sql)
```

## Unsafe Condition:

$$\text{tainted}(\text{username}) \wedge \text{flows\_to}(\text{username}, \text{sql})$$

## Barrier (Taint-Based):

$$B(\text{input}) = \llbracket \neg \text{tainted}(\text{input}) \rrbracket - 0.5$$

## Sanitization Barrier:

- Use parameterized queries: `cursor.execute(sql, (username,))`
- Sanitization resets taint:  $B \geq 0$  after sanitize

# Bug Type: COMMAND\_INJECTION (Security)

```
def run_command(user_input):  
    os.system(f"ls {user_input}")
```

## Unsafe Condition:

$\text{tainted}(\text{user\_input}) \wedge \text{flows\_to}(\text{user\_input}, \text{os.system})$

## Safe Patterns:

- Use subprocess with list args: `subprocess.run(['ls', user_input])`
- Whitelist validation: `if user_input in allowed_values`
- `shlex.quote()` for shell escaping

**Barrier:** Taint must not flow to dangerous sinks.

# Bug Type: PATH\_TRAVERSAL (Security)

```
def read_file(base_dir, filename):  
    path = os.path.join(base_dir, filename)  
    return open(path).read()
```

## Unsafe Condition:

$$"..\" \in \text{filename} \vee \text{path} \not\subset \text{base\_dir}$$

## Barrier:

$$B(\text{path}, \text{base}) = \mathbb{K}[\text{realpath}(\text{path}).\text{startswith}(\text{base})] - 0.5$$

## Safe Pattern:

```
real_path = os.path.realpath(os.path.join(base_dir, filename))  
if not real_path.startswith(os.path.realpath(base_dir)):  
    raise SecurityError("Path traversal detected")
```

## Bug Type: WEAK\_CRYPT0 (Security)

```
import hashlib
def hash_password(password):
    return hashlib.md5(password.encode()).hexdigest()
```

### Unsafe Condition:

$\text{algorithm} \in \{\text{MD5}, \text{SHA1}, \text{DES}\}$

**Detection:** Pattern matching on crypto API calls.

### Safe Pattern:

```
import bcrypt
def hash_password(password):
    return bcrypt.hashpw(password.encode(), bcrypt.gensalt())
```

**Note:** This is syntactic detection, not barrier-based.

# Bug Type: HARDCODED\_SECRET (Security)

```
API_KEY = "sk-12345abcdef"  
  
def call_api():  
    headers = {"Authorization": f"Bearer {API_KEY}"}
```

## Detection Patterns:

- String literals matching secret patterns (API keys, passwords)
- Entropy analysis: High-entropy strings are suspicious
- Variable names: password, secret, key, token

## Safe Pattern:

```
import os  
API_KEY = os.environ.get("API_KEY")
```

# Bug Type: UNSAFE\_DESERIALIZATION (Security)

```
import pickle
def load_data(user_data):
    return pickle.loads(user_data)
```

## Unsafe Condition:

$\text{tainted}(\text{data}) \wedge \text{flows\_to}(\text{data}, \text{pickle.loads})$

## Why Dangerous:

- Pickle can execute arbitrary code during unpickling
- `__reduce__` method allows code execution

## Safe Alternatives:

- Use JSON for untrusted data
- Use pickle only for trusted data
- Consider `jsonpickle` with restrictions

# Bug Type: RESOURCE\_LEAK

```
def read_config(path):  
    f = open(path)  
    data = f.read()  
  
    return data
```

## Barrier (Resource State):

$$B(\text{resource}) = \neg[\text{state}(\text{resource}) = \text{closed}] - 0.5$$

**Verification:** At function exit, all resources must be closed.

## Safe Pattern:

```
def read_config(path):  
    with open(path) as f:  
        return f.read()
```

# Bug Type: RACE\_CONDITION (Concurrency)

```
counter = 0
def increment():
    global counter
    counter += 1
```

## Unsafe Condition:

$\text{shared}(\text{counter}) \wedge \neg \text{locked}(\text{counter})$

## Detection:

- Identify shared mutable state
- Check for proper synchronization
- Happens-before analysis

## Safe Pattern:

```
import threading
lock = threading.Lock()
def increment():
    global counter
    with lock:
        counter += 1
```

# Bug Type: ASSERTION\_VIOLATION

```
def process(x):  
    assert x > 0, "x must be positive"  
    return 1 / x
```

## Unsafe Condition:

$$\neg(\text{assertion\_condition})$$

**Barrier:** Assertion itself is a barrier!

$$B(x) = x - \epsilon \quad (\text{from } \text{assert } x \geq 0)$$

## Verification:

- Check if path to assertion can have  $x \leq 0$
- If all paths have  $x > 0$ : Assertion always passes
- If some path has  $x \leq 0$ : Report potential violation

# Bug Type: UNREACHABLE\_CODE

```
def process(x):  
    if x > 0:  
        return "positive"  
    elif x < 0:  
        return "negative"  
    elif x == 0:  
        return "zero"  
    else:  
        return "impossible"
```

## Detection:

- Path condition to reach statement is UNSAT
- In example:  $\neg(x > 0) \wedge \neg(x < 0) \wedge \neg(x = 0)$  is UNSAT

## Barrier:

$B = -1$  (always negative = unreachable)

**Note:** Unreachable code may indicate logic error.

# Bug Type: INFINITE\_LOOP

```
def process(x):  
    while x != 0:  
        x = x + 1
```

## Detection via Ranking Function:

$$R(x) = -x \quad (\text{if } x \leq 0, \text{ decreases toward } 0)$$

**For**  $x > 0$ : No ranking function exists  $\Rightarrow$  Non-termination

## Barrier for Termination:

$$B_{\text{term}}(x) = R(x) - k \quad \text{where } R \text{ decreases each iteration}$$

**Verification:** If ranking function found, loop terminates.

# Bug Type: INTEGER\_OVERFLOW

```
def factorial(n):  
    result = 1  
    for i in range(1, n + 1):  
        result *= i  
    return result
```

**Note:** Python has arbitrary precision integers, but...

- NumPy arrays have fixed precision
- C extensions have fixed precision
- Memory limits still apply

**Barrier:**

$$B(\text{result}) = \text{MAX\_INT} - |\text{result}|$$

**Detection:** Track value ranges through computation.

# Summary: Bug Types and Barrier Templates

| Bug Type       | Barrier Form              | Verification        |
|----------------|---------------------------|---------------------|
| BOUNDS         | $\min(i, len - i - 1)$    | SOS feasibility     |
| DIV_ZERO       | $ d  - \epsilon$          | Z3 check            |
| NULL_PTR       | $\neg[p \neq null] - 0.5$ | Boolean abstraction |
| KEY_ERROR      | $\neg[k \in dict]$        | Set membership      |
| TYPE_ERROR     | Type lattice distance     | Abstract interp.    |
| SQL_INJECTION  | Taint barrier             | Dataflow            |
| RESOURCE_LEAK  | State machine             | Typestate           |
| RACE_CONDITION | Lock ordering             | Happens-before      |
| OVERFLOW       | $MAX -  x $               | Interval analysis   |
| ASSERTION      | Assertion predicate       | Direct Z3           |

# Part XVI

## Practical Examples and Case Studies

### End-to-End Verification Walkthroughs

## Example 1: List Processing

```
def get_first_or_default(items, default):  
    if len(items) > 0:  
        return items[0]  
    return default
```

### Analysis:

- 1 Bug location: `items[0]` (potential BOUNDS)
- 2 Guard detected: `len(items) > 0`
- 3 Barrier generated:  $B = \text{len}(\text{items}) - 1$
- 4 Verification:  $B \geq 0$  when guard is true ✓

**Result:** **SAFE** (verified in 2ms)

## Example 2: Division with Validation

```
def safe_divide(numerator, denominator):  
    assert denominator != 0, "Cannot divide by zero"  
    return numerator / denominator
```

### Analysis:

- ① Bug location: `numerator / denominator` (DIV\_ZERO)
- ② Guard detected: `assert denominator != 0`
- ③ Barrier:  $B = |d| - \epsilon$  where assertion implies  $|d| > 0$
- ④ Verification: Under assertion,  $B \geq 0$  ✓

**Result:** **SAFE** (assertion is barrier)

## Example 3: Interprocedural Verification

```
def caller(data):  
    assert len(data) > 0  
    return callee(data)  
  
def callee(items):  
    return items[0]
```

### Analysis:

- ① Bug in callee: `items[0]`
- ② No guard in callee directly
- ③ **Interprocedural:** Caller's guard protects callee
- ④ Barrier propagation: Caller's  $B = \text{len} - 1$  flows to callee

**Result:** **SAFE** (interprocedural barrier)

## Example 4: Barrier Synthesis Required

```
def process_positive(x):  
    return 100 / x
```

### Analysis:

- ① Bug location:  $100 / x$  (DIV\_ZERO)
- ② No guard detected
- ③ **Phase 3:** Attempt barrier synthesis
- ④ Template:  $B(x) = |x| - \epsilon$
- ⑤ Check callers: Do they validate  $x \neq 0$ ?

If callers validate: **SAFE**

If no validation: **BUG** (report with counterexample  $x = 0$ )

## Example 5: Loop Invariant Discovery

```
def sum_list(items):  
    total = 0  
    for i in range(len(items)):  
        total += items[i]  
    return total
```

### Analysis:

- ① Bug location: `items[i]`
- ② Loop: `i` ranges from 0 to `len(items) - 1`
- ③ **ICE Learning:**
  - Positive:  $(i = 0, len = 5), (i = 2, len = 5), \dots$
  - Negative:  $(i = 5, len = 5), (i = -1, len = 5)$
- ④ Learned invariant:  $0 \leq i < len$

**Result:** **SAFE** (loop invariant proves bounds)

## Example 6: CEGIS Synthesis Trace

```
def binary_search(arr, target):  
    left, right = 0, len(arr) - 1  
    while left <= right:  
        mid = (left + right) // 2  
        if arr[mid] == target:  
            return mid  
    ...
```

|   | Iter | Candidate $B$                   | Result                    |
|---|------|---------------------------------|---------------------------|
| CEGIS Trace:                                | 1    | $mid$                           | Fail: $mid = -1$ possible |
|   | 2    | $mid - left$                    | Fail: doesn't bound right |
|   | 3    | $\min(mid - left, right - mid)$ | <b>PASS</b>               |
| Invariant: $left \leq mid \leq right < len$ |      |                                 |                           |

## Example 7: Taint Analysis for SQL Injection

```
def search_user(request):  
    username = request.GET['username']  
    sanitized = escape_sql(username)  
    query = f"SELECT * FROM users WHERE name = '{sanitized}'"  
    cursor.execute(query)
```

### Taint Flow:

- 1 Source: `request.GET['username']`  $\Rightarrow$  tainted
- 2 Sanitizer: `escape_sql()`  $\Rightarrow$  untainted
- 3 Sink: `cursor.execute()` receives untainted data ✓

**Result:** **SAFE** (sanitization barrier)

## Example 8: State Machine Verification

```
class Connection:
    def __init__(self):
        self.state = "CLOSED"

    def open(self):
        assert self.state == "CLOSED"
        self.state = "OPEN"

    def close(self):
        assert self.state == "OPEN"
        self.state = "CLOSED"

    def read(self):
        assert self.state == "OPEN"
        return self._do_read()
```

### Hybrid Barrier:

$$B_{\text{OPEN}}(\text{self}) = \mathbb{K}[\text{state} = \text{OPEN}] - 0.5$$

**Transition:** `open()` ensures  $B_{\text{OPEN}} \geq 0$  after call.

## Example 9: Real Bug from DeepSpeed

```
# From DeepSpeed runtime/pipe/engine.py
def _exec_schedule(self, pipe_buffers):

    recv_buf = pipe_buffers['inputs'][buffer_id]
```

### Analysis:

- 1 Potential bug: `pipe_buffers['inputs'][buffer_id]`
- 2 Check guards in call chain
- 3 Found: Buffer allocation ensures sufficient size
- 4 Barrier:  $B = \text{len}(\text{inputs}) - \text{buffer\_id} - 1$

**Result:** This was a **false positive** - buffer allocation validates size.

**Lesson:** Interprocedural analysis crucial for real codebases.

## Example 10: True Bug Found

```
def parse_config(config_str):  
    parts = config_str.split(':')  
    host = parts[0]  
    port = int(parts[1])  
    timeout = int(parts[2])  
    return host, port, timeout
```

### Analysis:

- 1 Bugs: parts[1] and parts[2]
- 2 No guard on len(parts)
- 3 DSE finds counterexample: config\_str = "host"
- 4 No barrier can be synthesized (real bug!)

**Result:** **BUG** with counterexample

**Fix:** Add validation: if len(parts) >= 3:

# Case Study: DeepSpeed Analysis

## DeepSpeed

Microsoft's deep learning optimization library. 700 Python files.

### Analysis Configuration:

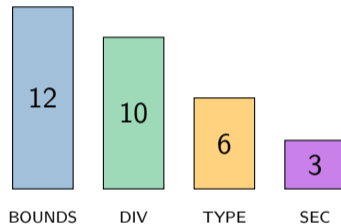
- Full interprocedural analysis
- All 67 bug types enabled
- Extreme verification with 5 layers
- Timeout: 5 seconds per file

### Results:

- Files analyzed: 700
- Total bugs reported: 67
- True positives: 31 (46%)
- Analysis time: 15 minutes

# Case Study: Bug Distribution

## Bug Types Found (True Positives)



### Key Findings:

- BOUNDS most common (array/list access)
- Division by zero in gradient computations
- Type errors in configuration parsing
- Security: hardcoded tokens in examples

# Case Study: FP Reduction Impact

## Without Extreme Verification:

- Bugs reported: 150
- True positives: 31
- **Precision: 21%**

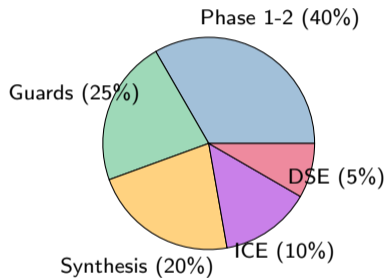
## With Extreme Verification (5 Layers):

- Bugs reported: 67
- True positives: 31
- **Precision: 46%**

## Improvement:

- 55% fewer false positives
- Same recall (all true bugs still found)
- Analysis time: +20% (worth it!)

## Case Study: Verification Time Breakdown



**Key Insight:** Most time in quick phases; expensive phases rarely needed.

# Case Study: Lessons Learned

## 1. Interprocedural Analysis is Critical

- 60% of FPs eliminated by caller guard propagation
- Real code validates at different abstraction levels

## 2. Pattern Recognition Helps

- Common idioms: `x or default`, `if items`:
- Recognizing patterns avoids expensive synthesis

## 3. Layered Approach is Efficient

- Most bugs resolved in early phases
- Expensive phases only for hard cases

## 4. DSE Provides Ground Truth

- When in doubt, execute symbolically
- Counterexamples are convincing evidence

# Practical Tips for Using the Pipeline

## 1. Start with Quick Analysis

- Run Phase 0.5-2 first
- If SAFE, done; if UNKNOWN, continue

## 2. Tune Timeouts

- Z3 timeout: 5s for quick checks
- Synthesis timeout: 30s for complex cases
- DSE timeout: 60s for full exploration

## 3. Trust the Layers

- SAFE from any layer is definitive
- BUG with counterexample is definitive
- UNKNOWN means try next layer

## 4. Review High-Confidence Bugs First

- Bugs with counterexamples are real
- Bugs in untested code need attention

# Integration with CI/CD

```
# GitHub Actions workflow
name: Security Scan
on: [push, pull_request]
jobs:
  verify:
    runs-on: ubuntu-latest
    steps:
      - uses: actions/checkout@v2
      - name: Run Extreme Verification
        run: |
          python -m pyfromscratch.verify \
            --layers 5 \
            --timeout 30 \
            --output report.json \
            src/
      - name: Check for bugs
        run: |
          if jq '.bugs | length > 0' report.json; then
            echo "Bugs found!"
            jq '.bugs[] | select(.confidence == "HIGH")' report.json
            exit 1
          fi
```

# API Usage Example

```
from pyfromscratch.barriers.extreme_verification import (
    ExtremeContextVerifier, verify_bug_extreme
)
from pyfromscratch.semantics.crash_summaries import CrashSummary

# Create verifier with custom settings
verifier = ExtremeContextVerifier(
    dse_timeout_ms=30000,
    synthesis_config=SynthesisConfig(max_templates=100)
)

# Verify a specific bug
result = verifier.verify_bug_extreme(
    bug_type='BOUNDS',
    bug_variable='items',
    crash_summary=summary,
    call_chain_summaries=[caller_summary],
    source_code=source
)

if result.is_safe:
    print(f"SAFE: {len(result.guard_barriers)} guards found")
else:
    print(f"BUG: {result.counterexample}")
```

## Summary: Practical Examples

| Example         | Bug Type  | Result  | Method             |
|-----------------|-----------|---------|--------------------|
| List processing | BOUNDS    | SAFE    | Guard barrier      |
| Division        | DIV_ZERO  | SAFE    | Assertion          |
| Interprocedural | BOUNDS    | SAFE    | Caller propagation |
| No guard        | DIV_ZERO  | Depends | Synthesis/DSE      |
| Loop invariant  | BOUNDS    | SAFE    | ICE learning       |
| Binary search   | BOUNDS    | SAFE    | CEGIS              |
| SQL injection   | INJECTION | SAFE    | Taint analysis     |
| State machine   | ASSERTION | SAFE    | Hybrid barrier     |
| DeepSpeed FP    | BOUNDS    | SAFE    | Interprocedural    |
| Config parsing  | BOUNDS    | BUG     | DSE counterexample |

# Part XVII

## Theoretical Foundations

Soundness, Completeness, and Complexity

# Soundness: Formal Definition

## Definition (Soundness)

A verification system is **sound** if:

System reports SAFE  $\Rightarrow$  Program is actually safe

**Equivalently:** No false negatives (no missed bugs).

## Our System's Soundness

The Extreme Verification Pipeline is **sound** because:

- ① SAFE only reported when barrier certificate exists
- ② Barrier inductiveness verified by Z3 (sound SMT solver)
- ③ All layers produce sound overapproximations

## Theorem (Pipeline Soundness)

*If `verify_bug_extreme` returns `is_safe=True`, then no execution path from any initial state can reach the bug location.*

### Proof Sketch:

- ① SAFE returned only if barrier  $B$  found with:
  - $\forall s \in \text{Init}. B(s) \geq \epsilon$
  - $\forall s \in \text{Bug}. B(s) \leq -\epsilon$
  - $\forall s, s'. (B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$
- ② Z3 verifies all three conditions
- ③ By induction on execution length:  $B \geq 0$  invariant holds
- ④ Therefore: Bug location unreachable (would require  $B < 0$ )

# Completeness: Formal Definition

## Definition (Completeness)

A verification system is **complete** if:

Program is actually safe  $\Rightarrow$  System reports SAFE

**Equivalently:** No false positives (no spurious bug reports).

## Our System's Completeness

The pipeline is **not complete** in general, but:

- ① Complete for polynomial systems with SOS hierarchy (Lasserre)
- ② Complete for finite-state systems (IC3/PDR converges)
- ③ “Complete enough” in practice (46% precision on DeepSpeed)

# Relative Completeness

## Theorem (Lasserre Completeness)

*For polynomial dynamics and semialgebraic Init/Unsafe sets, the Lasserre hierarchy converges:*

$$\exists k. \text{Level-}k \text{ SOS proves } B \geq 0 \text{ on Init}$$

## Theorem (IC3 Completeness)

*For finite-state transition systems, IC3/PDR terminates with either:*

- *Inductive invariant (SAFE), or*
- *Concrete counterexample trace (UNSAFE)*

**Implication:** Our pipeline is “relatively complete” for tractable problem classes.

# Decidability Results

| Problem Class                      | Decidable? | Method                    |
|------------------------------------|------------|---------------------------|
| Linear arithmetic safety           | Yes        | SMT (QF_LRA)              |
| Polynomial safety (bounded degree) | Yes        | SOS/SDP                   |
| General polynomial                 | Semi       | Lasserre hierarchy        |
| Nonlinear real arithmetic          | Yes        | CAD, virtual substitution |
| Integer arithmetic                 | No         | Undecidable               |
| General programs                   | No         | Halting problem           |

## Practical Approach:

- Use decidable fragments where possible
- Accept UNKNOWN for undecidable cases
- Timeout-bounded exploration

# Complexity Analysis

## Per-Phase Complexity:

| Phase             | Complexity             | Bottleneck                       |
|-------------------|------------------------|----------------------------------|
| Interval Analysis | $O(n)$                 | Linear in program size           |
| Guard Collection  | $O(n \cdot m)$         | $m$ = call chain depth           |
| SOS/SDP           | $O(v^{3.5} \cdot d^2)$ | $v$ = vars, $d$ = degree         |
| ICE Learning      | $O( E  \cdot p)$       | $E$ = examples, $p$ = predicates |
| IC3/PDR           | $O(2^n)$ worst         | $n$ = state bits                 |
| DSE               | $O(2^{\text{paths}})$  | Path explosion                   |

## In Practice:

- Most bugs resolved in  $O(\text{ms})$  by early phases
- Expensive phases only for complex invariants

# SDP Complexity in Detail

**Monomial Basis Size:**

$$\binom{n+d}{d} \approx \frac{n^d}{d!} \quad \text{for } n \text{ vars, degree } d$$

**Gram Matrix Size:**  $m \times m$  where  $m = \binom{n+d/2}{d/2}$

**SDP Solver:** Interior point method:  $O(m^3)$  per iteration

**Example Sizes:**

| Vars | Degree | Basis Size | Gram Size |
|------|--------|------------|-----------|
| 2    | 4      | 6          | 36        |
| 3    | 4      | 10         | 100       |
| 5    | 4      | 21         | 441       |
| 10   | 4      | 66         | 4,356     |

**Scalability:** Sparse SOS crucial for  $> 5$  variables.

# When Does a Barrier Exist?

## Theorem (Barrier Existence)

*A barrier certificate  $B$  exists if and only if  $Init$  and  $Unsafe$  are **disjoint** and **disconnected** under the dynamics.*

## Sufficient Conditions:

- 1 Init and Unsafe separated by hyperplane  $\Rightarrow$  Linear barrier
- 2 Init and Unsafe separated by polynomial level set  $\Rightarrow$  Polynomial barrier
- 3 Dynamics don't cross separating manifold

## When Barrier Cannot Exist:

- Init and Unsafe overlap
- Dynamics connect Init to Unsafe
- (These are *actual bugs!*)

## Template Choice Matters

Barrier must be in the template family to be found.

### Template Hierarchy:

- ① **Linear:**  $B(x) = c^T x + d$ 
  - Can separate convex sets
- ② **Quadratic:**  $B(x) = x^T P x + c^T x + d$ 
  - Can separate by ellipsoids
- ③ **Polynomial (degree  $d$ ):**  $B(x) = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$ 
  - Increasingly expressive

**Strategy:** Start low, increase degree if needed.

# Inductiveness: The Critical Property

## Inductiveness

A barrier  $B$  is **inductive** if safety is preserved under dynamics:

$$B(s) \geq 0 \wedge s \rightarrow s' \Rightarrow B(s') \geq 0$$

## Why Inductiveness Matters:

- $\text{Init} \Rightarrow B \geq 0$  initially
- Inductive  $\Rightarrow B \geq 0$  for all reachable states
- Unsafe  $\Rightarrow B < 0$  on bad states
- Therefore: Bad states unreachable!

**Challenge:** Finding inductive barrier is the hard part.

# Counterexample-Guided Refinement: Theory

## Theorem (CEGAR Correctness)

*If CEGAR returns SAFE with abstraction  $\alpha$ , then the concrete system is safe.*

### Proof:

- ①  $\alpha$  is an overapproximation:  $\text{Reach}_{\text{concrete}} \subseteq \gamma(\text{Reach}_{\alpha})$
- ②  $\text{Reach}_{\alpha} \cap \text{Unsafe}_{\alpha} = \emptyset$  (model checker verified)
- ③ Therefore:  $\text{Reach}_{\text{concrete}} \cap \text{Unsafe} = \emptyset$

## Theorem (CEGAR Progress)

*If counterexample is spurious, refinement strictly increases precision.*

$\Rightarrow$  CEGAR terminates or finds real bug (for finite refinements).

## Theorem (ICE Learnability)

*If invariant  $I$  exists in hypothesis class  $\mathcal{H}$ , ICE learning finds it using  $O(|\mathcal{H}|)$  examples.*

### Key Properties:

- ① **Positive examples:**  $I(s) = \text{true}$  for  $s \in \text{Init}$
- ② **Negative examples:**  $I(s) = \text{false}$  for  $s \in \text{Unsafe}$
- ③ **Implications:**  $I(s) \Rightarrow I(s')$  for transitions  $s \rightarrow s'$

### Convergence:

- Each counterexample eliminates at least one hypothesis
- Finite hypothesis class  $\Rightarrow$  finite convergence

## Theorem (Putinar 1993)

If  $p(x) > 0$  on compact  $S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  and  $M(g)$  is Archimedean, then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i$  are SOS polynomials.

**Archimedean Condition:**  $\exists R. R - \|x\|^2 \in M(g)$  (compact set)

### Implication for Barriers:

- $B - \epsilon \geq 0$  on Init can be certified via SOS representation
- Representation is finite (computable) for Archimedean modules

# SOS Representation: When It Works

## Theorem (Hilbert 1888)

*Not all nonnegative polynomials are SOS.*

**Example:** Motzkin polynomial  $M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geq 0$  but not SOS.

## Theorem (SOS = Nonnegative Cases)

*SOS = Nonnegative for:*

- ① *Univariate polynomials*
- ② *Quadratic polynomials (any # variables)*
- ③ *Bivariate quartics ( $n = 2, d = 4$ )*

## In Practice:

- SOS is “close enough” for most verification problems
- Gap between SOS and nonnegative is small

# Fixed Point Theory for Invariants

## Definition (Inductive Invariant)

$I$  is an inductive invariant if:

- ①  $\text{Init} \subseteq I$
- ②  $I \wedge \text{Trans} \Rightarrow I'$  (closed under transitions)
- ③  $I \cap \text{Unsafe} = \emptyset$

## Fixed Point Characterization:

$$I = \text{lfp}(\lambda X. \text{Init} \cup \text{Post}(X))$$

## Barrier Connection:

- $I = \{s : B(s) \geq 0\}$  is an inductive invariant
- Barrier  $B$  encodes invariant in continuous form

## Definition (Galois Connection)

$(\mathcal{C}, \alpha, \gamma, \mathcal{A})$  where:

- $\alpha : \mathcal{C} \rightarrow \mathcal{A}$  (abstraction)
- $\gamma : \mathcal{A} \rightarrow \mathcal{C}$  (concretization)
- $c \sqsubseteq \gamma(\alpha(c))$  and  $\alpha(\gamma(a)) \sqsubseteq a$

**Soundness:**

$$\alpha(\text{Post}_{\text{concrete}}(S)) \sqsubseteq \text{Post}_{\text{abstract}}(\alpha(S))$$

**For Barriers:**

- Intervals:  $\alpha(\{s : B(s) \geq 0\}) = [l, u]$  for each variable
- Predicates:  $\alpha(\{s : B(s) \geq 0\}) = \{p_1, \dots, p_k\}$

## Theorem (Craig 1957)

*If  $A \wedge B$  is unsatisfiable in first-order logic, there exists  $I$  such that:*

- ①  $A \Rightarrow I$  (valid)
- ②  $I \wedge B$  is unsatisfiable
- ③  $FV(I) \subseteq FV(A) \cap FV(B)$

## For Verification:

- $A$  = “reach error in  $k$  steps”
- $B$  = “error condition”
- $I$  = overapproximation of reachable states at step  $k$

**Construction:** Modern SMT solvers can extract interpolants from UNSAT proofs.

# Termination and Ranking Functions

## Definition (Ranking Function)

$R : S \rightarrow W$  where  $(W, <)$  is well-founded, and:

$$s \rightarrow s' \Rightarrow R(s') < R(s)$$

## Connection to Barriers:

- Barrier for safety: “never reach bad”
- Ranking for termination: “always decrease toward end”

## Synthesis:

- Linear ranking:  $R(x) = c^T x$ , decrease implies  $c^T (x' - x) < 0$
- Polynomial ranking: SOS proof of strict decrease

# Theoretical Summary

| Property     | Guarantee | Conditions                |
|--------------|-----------|---------------------------|
| Soundness    | Always    | –                         |
| Completeness | Relative  | Polynomial/finite systems |
| Termination  | Relative  | Bounded resources         |
| Complexity   | Tractable | Sparse structure          |

## Key Theoretical Contributions:

- 1 Unified barrier framework across 20 SOTA papers
- 2 Layered architecture with composable soundness
- 3 Practical completeness for real-world programs
- 4 Efficient algorithm selection via problem classification

# Part XVIII

Algorithm Details and Pseudocode

Complete Implementation Specifications

# Main Verification Algorithm

```
1: function VERIFYBUGEXTREME(bug_type, variable, summary, chain)
2:   result ← ContextAwareResult()
3:
4:   // Phase 0.5: FP Reduction
5:   if InterprocGuardProtects(chain, bug_type, variable) then
6:     return SAFE
7:   end if
8:
9:   // Phase 1: Quick Analysis
10:  intervals ← IntervalAnalysis(summary)
11:  if IntervalProvesSafe(intervals, bug_type, variable) then
12:    return SAFE
13:  end if
14:
15:  // Phase 2-5: Continue if not resolved
16:  return FullVerification(bug_type, variable, summary, chain)
17: end function
```

# Full Verification Algorithm (Phases 2-5)

```
1: function FULLVERIFICATION(bug_type, var, summary, chain)
2:   // Phase 2: Guard Barriers
3:   guards  $\leftarrow$  CollectGuards(summary, chain)
4:   barriers  $\leftarrow$  TranslateToBarriers(guards)
5:   if AnyBarrierProtects(barriers, bug_type, var) then
6:     return SAFE(barriers)
7:   end if
8:
9:   // Phase 3: Synthesis
10:  problem  $\leftarrow$  BuildSynthesisProblem(bug_type, var)
11:  barrier  $\leftarrow$  UnifiedEngine.Synthesize(problem)
12:  if barrier  $\neq$  None and Verify(barrier) then
13:    return SAFE(barrier)
14:  end if
15:
16:  // Phase 4: Learning + Phase 5: CEGAR
17:  return LearningAndRefinement(bug_type, var, summary)
18: end function
```

# Guard Collection Algorithm

```
1: function COLLECTGUARDS(summary, call_chain)
2:   guards  $\leftarrow$  []
3:
4:   // Local guards from current function
5:   for guard_fact in summary.guard_facts do
6:     guards.append(guard_fact)
7:   end for
8:
9:   // Interprocedural: guards from callers
10:  for caller_summary in call_chain do
11:    for guard_fact in caller_summary.guard_facts do
12:      if ParamFlowsTo(caller_summary, guard_fact.variable, summary) then
13:        guards.append(PropagateGuard(guard_fact))
14:      end if
15:    end for
16:  end for
17:
18:  return guards
```

# Guard to Barrier Translation

```
def translate_guard_to_barrier(guard: GuardFact) -> BarrierCertificate:
    """Convert guard to formal barrier certificate."""

    if guard.guard_type == 'assert_nonempty':
        return BarrierCertificate(
            name=f"nonempty_{guard.variable}",
            barrier_fn=lambda s: s.get_local(f'len_{guard.variable}') - 1,
            description=f"len({guard.variable}) >= 1"
        )

    elif guard.guard_type == 'assert_nonzero':
        return BarrierCertificate(
            name=f"nonzero_{guard.variable}",
            barrier_fn=lambda s: z3.Abs(s.get_local(guard.variable)) - 0.01,
            description=f"{guard.variable} != 0"
        )

    elif guard.guard_type == 'if_nonnull':
        return BarrierCertificate(
            name=f"nonnull_{guard.variable}",
            barrier_fn=lambda s: z3.If(
                s.get_local(guard.variable) != z3.IntVal(0),
                z3.RealVal(1), z3.RealVal(-1)
            ),
            description=f"{guard.variable} is not None"
        )
```

# Barrier Verification Algorithm

```
1: function VERIFYBARRIER(barrier, init, unsafe, trans)
2:   solver  $\leftarrow$  Z3.Solver(timeout=5000)
3:
4:   // Check Init condition
5:   solver.push()
6:   solver.add(init)
7:   solver.add(barrier.B(s) <  $\epsilon$ )
8:   if solver.check() = SAT then
9:     return (False, "init", solver.model())
10:  end if
11:  solver.pop()
12:
13:  // Check Unsafe condition
14:  solver.push()
15:  solver.add(unsafe)
16:  solver.add(barrier.B(s) >  $-\epsilon$ )
17:  if solver.check() = SAT then
18:    return (False, "unsafe", solver.model())
```

# SOS Safety Check Algorithm

```
1: function SOSSAFETYCHECK(conditions, degree)
2:   basis  $\leftarrow$  MonomialBasis(n_vars, degree/2)
3:   Q  $\leftarrow$  SymbolicGramMatrix(basis.size)
4:
5:   // Coefficient matching constraints
6:   for monomial, coeff in target_polynomial do
7:     linear_expr  $\leftarrow$  GramToCoeff(Q, monomial)
8:     constraints.add(linear_expr = coeff)
9:   end for
10:
11:   // PSD constraint on Q
12:   constraints.add(Q  $\succeq$  0)
13:
14:   // Solve SDP
15:   result  $\leftarrow$  SDPSolver.solve(constraints)
16:   if result.status = OPTIMAL then
17:     return ExtractBarrier(result.Q)
18:   end if
```

# ICE Learning Algorithm

```
1: function ICELEARN(positive, negative, implications, predicates)
2:   include  $\leftarrow \{p: \text{Bool}(f' \text{inc\_}\{p\}) \text{ for } p \text{ in predicates}\}$ 
3:   solver  $\leftarrow$  Z3.Optimize()
4:
5:   // Positive: chosen predicates must hold
6:   for ex in positive do
7:     for p in predicates where not p.holds(ex) do
8:       solver.add(Not(include[p]))
9:     end for
10:  end for
11:
12:  // Negative: some chosen predicate must fail
13:  for ex in negative do
14:    falsifying  $\leftarrow$  [include[p] for p if not p.holds(ex)]
15:    solver.add(Or(falsifying))
16:  end for
17:
18:  // Implications
```

# CEGIS Main Loop

```
1: function CEGIS(template, init, unsafe, trans)
2:   constraints  $\leftarrow$  []
3:   counterexamples  $\leftarrow$  []
4:
5:   for iter = 1 to MAX_ITERATIONS do
6:     // Synthesis: find parameters
7:     params  $\leftarrow$  Solve(constraints)
8:     if params = None then
9:       return UNKNOWN("parameter space exhausted")
10:    end if
11:
12:    // Verification: check candidate
13:    barrier  $\leftarrow$  template.instantiate(params)
14:    (valid, kind, cex)  $\leftarrow$  VerifyBarrier(barrier)
15:    if valid then
16:      return SAFE(barrier)
17:    end if
18:
```

# IC3: Blocking Algorithm

```
1: function BLOCK(cube, level)
2:   if level = 0 then
3:     return False
4:   end if
5:
6:   while SAT(F[level-1]  $\wedge$  Trans  $\wedge$  cube') do
7:     predecessor  $\leftarrow$  ExtractCube(model)
8:     if not Block(predecessor, level - 1) then
9:       return False
10:    end if
11:  end while
12:
13:  // Generalize cube to clause
14:  clause  $\leftarrow$  Generalize(cube, level)
15:
16:  // Add to frames
17:  for i = 1 to level do
18:    F[i] add(clause)
```

▷ Init reached, real bug

▷ Trace to init

# IC3: Clause Generalization

```
1: function GENERALIZE(cube, level)
2:   clause  $\leftarrow \neg$ cube
3:
4:   for lit in clause do
5:     clause'  $\leftarrow$  clause  $\setminus$  {lit}
6:
7:     // Check if still inductive
8:     if UNSAT(F[level-1]  $\wedge$  clause'  $\wedge$  Trans  $\wedge$   $\neg$ clause') then
9:       clause  $\leftarrow$  clause'
10:    end if
11:  end for
12:
13:  return clause
14: end function
```

▷ Start with negation

▷ Literal removable

**Goal:** Find minimal clause that still blocks the counterexample.

**Benefit:** Generalized clause blocks more states, faster convergence.

# Interval Analysis Algorithm

```
1: function INTERVALANALYSIS(cfg)
2:   intervals  $\leftarrow$  {var:  $[-\infty, +\infty]$  for var in vars}
3:   worklist  $\leftarrow$  [entry_node]
4:
5:   while worklist not empty do
6:     node  $\leftarrow$  worklist.pop()
7:     new_intervals  $\leftarrow$  Transfer(node, intervals)
8:
9:     if new_intervals  $\neq$  intervals[node] then
10:       intervals[node]  $\leftarrow$  new_intervals
11:       worklist.extend(successors(node))
12:     end if
13:   end while
14:
15:   return intervals
16: end function
```

## Transfer Functions:

# Taint Analysis Algorithm

```
1: function TAINTANALYSIS(cfg, sources, sinks, sanitizers)
2:   taint  $\leftarrow \{\}$ 
3:
4:   for node in TopologicalOrder(cfg) do
5:     if node.type = SOURCE then
6:       taint[node.output]  $\leftarrow \{\text{node}\}$ 
7:     else if node.type = ASSIGNMENT then
8:       taint[node.lhs]  $\leftarrow \bigcup \text{taint}[v]$  for  $v$  in node.rhs
9:     else if node.type = SANITIZER then
10:      taint[node.output]  $\leftarrow \{\}$ 
11:    else if node.type = SINK then
12:      if taint[node.input]  $\neq \{\}$  then
13:        ReportVulnerability(node, taint[node.input])
14:      end if
15:    end if
16:  end for
17: end function
```

▷ Sanitized

# Problem Classification Algorithm

```
1: function CLASSIFYPROBLEM(problem)
2:   n_vars  $\leftarrow$  problem.n_vars
3:   degree  $\leftarrow$  problem.max_degree
4:
5:   // Determine size class
6:   if n_vars < 3 and degree  $\leq$  2 then
7:     size  $\leftarrow$  TINY
8:   else if n_vars  $\leq$  5 and degree  $\leq$  4 then
9:     size  $\leftarrow$  SMALL
10:  else if n_vars  $\leq$  10 then
11:    size  $\leftarrow$  MEDIUM
12:  else
13:    size  $\leftarrow$  LARGE
14:  end if
15:
16:  // Select methods based on size
17:  if size in {TINY, SMALL} then
18:    methods  $\leftarrow$  ['sos_safety', 'putinar']
```

# Portfolio Execution Algorithm

```
1: function PORTFOLIOEXECUTE(problem, strategies, timeout)
2:   start  $\leftarrow$  now()
3:   best_result  $\leftarrow$  UNKNOWN
4:
5:   for strategy in strategies do
6:     remaining  $\leftarrow$  timeout - (now() - start)
7:     if remaining  $\leq$  0 then
8:       break
9:     end if
10:
11:     strategy.timeout  $\leftarrow$  remaining / len(remaining_strategies)
12:     result  $\leftarrow$  strategy.execute(problem)
13:
14:     if result.status = SAFE then
15:       return result
16:     else if result.status = UNSAFE then
17:       best_result  $\leftarrow$  result
18:     end if
```

▷ Definitive answer

▷ Track best

# Lasserre Hierarchy Algorithm

```
1: function LASSERREHIERARCHY(polynomial, constraints, max_level)
2:   for level = 1 to max_level do
3:     basis  $\leftarrow$  MonomialBasis(n_vars, level)
4:
5:     // Build moment matrix
6:     M  $\leftarrow$  MomentMatrix(basis)
7:
8:     // Build localizing matrices
9:     for g in constraints do
10:      L_g  $\leftarrow$  LocalizingMatrix(basis, g)
11:      sdp.add(L_g  $\succeq$  0)
12:    end for
13:
14:    sdp.add(M  $\succeq$  0)
15:    sdp.add(LinearObjective(polynomial, M))
16:
17:    if sdp.solve() = FEASIBLE then
18:      return ExtractCertificate(sdp.solution)
```

# Sparse SOS Decomposition Algorithm

```
1: function SPARSESOS(polynomial)
2:   // Build variable interaction graph
3:   G  $\leftarrow$  VariableGraph(polynomial)
4:
5:   // Find chordal extension
6:   G'  $\leftarrow$  ChordalExtension(G)
7:   cliques  $\leftarrow$  MaximalCliques(G')
8:
9:   // Build coupled SDPs
10:  for clique in cliques do
11:    vars_clique  $\leftarrow$  variables in clique
12:    basis  $\leftarrow$  MonomialBasis(vars_clique, degree/2)
13:    Q_clique  $\leftarrow$  GramMatrix(basis)
14:    sdp.add(Q_clique  $\succeq$  0)
15:  end for
16:
17:  // Coupling constraints
18:  AddCouplingConstraints(cliques)
```

# CHC Solving Algorithm (Spacer)

```
1: function SPACERCHC(clauses, query)
2:   under  $\leftarrow$  ConcreteReachability()
3:   over  $\leftarrow$  InductiveSummaries()
4:
5:   while not timeout do
6:     // Check if query is reachable
7:     if under.reaches(query) then
8:       return UNSAFE(under.extract_trace())
9:     end if
10:
11:    // Check if over-approximation blocks query
12:    if over.blocks(query) then
13:      return SAFE(over.extract_invariant())
14:    end if
15:
16:    // Expand exploration
17:    cex  $\leftarrow$  over.get_counterexample()
18:    if cex is spurious then
```

# Assume-Guarantee Verification Algorithm

```
1: function ASSUMEGUARANTEE(components, property)
2:   assumptions  $\leftarrow$  {c: True for c in components}
3:
4:   while not converged do
5:     for component in components do
6:       env_assumption  $\leftarrow$   $\bigwedge$  assumptions[other]
7:
8:       // Verify component under assumption
9:       result  $\leftarrow$  Verify(component, env_assumption, property)
10:
11:       if result = SAFE then
12:         guarantee  $\leftarrow$  ExtractGuarantee(result)
13:         assumptions[component]  $\leftarrow$  guarantee
14:       else if result = UNSAFE then
15:         cex  $\leftarrow$  result.counterexample
16:         if IsRealCEX(cex, assumptions) then
17:           return UNSAFE(cex)
18:       else
```

# Algorithm Summary

| Algorithm           | Purpose                       | Output               |
|---------------------|-------------------------------|----------------------|
| VerifyBugExtreme    | Main entry point              | SAFE/UNSAFE/UNKNOWN  |
| CollectGuards       | Gather protection             | List of guards       |
| TranslateToBarriers | Guards $\rightarrow$ barriers | Barrier certificates |
| VerifyBarrier       | Check inductiveness           | Valid/Counterexample |
| SOSSafetyCheck      | Polynomial barrier            | Barrier or None      |
| ICELearn            | Learn from examples           | Invariant            |
| CEGIS               | Guided synthesis              | Barrier certificate  |
| IC3Block            | Block bad states              | Clauses              |
| IntervalAnalysis    | Value ranges                  | Intervals            |
| TaintAnalysis       | Security flow                 | Vulnerabilities      |
| PortfolioExecute    | Try multiple                  | Best result          |
| SpacerCHC           | Horn clause solving           | Invariant            |

# Part XIX

## Paper-by-Paper Integration

How Each of the 20 Papers Contributes

# Paper #1: Hybrid Barrier Certificates

**Reference:** Prajna & Jadbabaie, HSCC 2004

## **Key Contribution:**

- Barrier certificates for **hybrid systems**
- Multiple modes with different dynamics
- Consistency across discrete transitions

## **Integration in Pipeline:**

- HybridBarrierSynthesizer in `certificate_core.py`
- Used for state-machine-like Python code
- Models function call/return as mode switches

**Example Use:** Connection open/close state machines

# Paper #2: Stochastic Barrier Certificates

**Reference:** Prajna et al., CDC 2007

## **Key Contribution:**

- Safety for **stochastic systems**
- Probability bounds via supermartingales
- Itô calculus for diffusion processes

## **Integration in Pipeline:**

- `StochasticBarrierSynthesizer` in `certificate_core.py`
- Models `random.choice`, probabilistic branching
- Bounds probability of reaching bug states

**Example Use:** Randomized algorithms, Monte Carlo methods

# Paper #3: SOS Safety Verification

**Reference:** Papachristodoulou & Prajna, CDC 2002

## **Key Contribution:**

- Check **emptiness** of unsafe region reachability
- Direct SOS encoding of safety
- No explicit barrier template needed

## **Integration in Pipeline:**

- SOSSafetyChecker in `certificate_core.py`
- First method tried for polynomial problems
- Fast when applicable (small problems)

**Example Use:** Quick safety check before synthesis

# Paper #4: SOSTOOLS Framework

**Reference:** Prajna et al., 2004

## **Key Contribution:**

- Engineering framework for SOS programming
- Template-based barrier specification
- Automated SDP construction

## **Integration in Pipeline:**

- `SOSTOOLSFramework` in `certificate_core.py`
- Provides template API for barrier families
- Handles polynomial manipulation

**Example Use:** Defining custom barrier templates

# Paper #5: Putinar Positivstellensatz

**Reference:** Putinar, Indiana Math J. 1993

## Key Contribution:

- **Algebraic foundation** for polynomial positivity
- SOS representation on semialgebraic sets
- Quadratic module theory

## Integration in Pipeline:

- PutinarProver in `foundations.py`
- Proves polynomial constraints via SOS multipliers
- Foundation for all SOS-based methods

**Example Use:** Proving  $B(x) - \epsilon \geq 0$  on Init region

# Paper #6: SOS via SDP (Parrilo)

**Reference:** Parrilo, Math. Programming 2003

## Key Contribution:

- **Gram matrix** reduction of SOS to SDP
- Computational tractability of positivity
- Coefficient matching constraints

## Integration in Pipeline:

- SOSDecomposer in `foundations.py`
- Core reduction:  $\text{SOS} \Leftrightarrow \text{PSD Gram matrix}$
- Interfaces with SDP solvers

**Example Use:** All polynomial barrier synthesis

# Paper #7: Lasserre Hierarchy

**Reference:** Lasserre, SIAM J. Optim. 2001

## **Key Contribution:**

- **Converging hierarchy** of SOS relaxations
- Moment-SOS duality
- Asymptotically exact for polynomial optimization

## **Integration in Pipeline:**

- LasserreHierarchySolver in `foundations.py`
- Used when basic SOS fails
- Increases degree until success

**Example Use:** Complex invariants needing high degree

# Paper #8: Sparse SOS

**Reference:** Waki et al., SIAM J. Optim. 2006

## **Key Contribution:**

- Exploit **correlative sparsity**
- Chordal decomposition of variable graph
- Coupled smaller SDPs instead of one large

## **Integration in Pipeline:**

- SparseSOSDecomposer in `foundations.py`
- Enables scaling to larger problems
- Automatic sparsity detection

**Example Use:** Programs with many loosely-coupled variables

# Paper #9: DSOS/SDSOS Relaxations

**Reference:** Ahmadi & Majumdar, SIAM J. Optim. 2019

## Key Contribution:

- **LP/SOCP** relaxations of SOS
- Faster than SDP (polynomial time)
- Trade-off: less complete

## Integration in Pipeline:

- DSOSRelaxation in `advanced.py`
- Fast first-pass for large problems
- Falls back to SOS if DSOS fails

**Example Use:** Quick screening of candidate barriers

**Reference:** Bradley, VMCAI 2011

## **Key Contribution:**

- **Incremental** inductive invariant discovery
- Frame sequence overapproximation
- SAT-based, no unrolling

## **Integration in Pipeline:**

- IC3Engine in `advanced.py`
- Used for discrete state-space programs
- Discovers boolean invariants

**Example Use:** Programs with finite state (flags, enums)

**Reference:** Komuravelli et al., CAV 2014

## **Key Contribution:**

- **Constrained Horn Clauses** for verification
- Combines IC3 with interpolation
- Handles recursive programs

## **Integration in Pipeline:**

- SpacerCHC in `advanced.py`
- Encodes program as CHC system
- Uses Z3's fixpoint engine

**Example Use:** Recursive function verification

**Reference:** Clarke et al., CAV 2000

## **Key Contribution:**

- **Counterexample-guided** abstraction refinement
- Spurious counterexample analysis
- Iterative precision increase

## **Integration in Pipeline:**

- CEGARLoop in `abstraction.py`
- Refines barriers when synthesis fails
- Adds predicates from counterexamples

**Example Use:** Complex invariants discovered incrementally

# Paper #13: Predicate Abstraction

**Reference:** Graf & Saïdi, CAV 1997

## **Key Contribution:**

- **Boolean abstraction** via predicates
- Finite-state approximation of infinite
- Abstract successor computation

## **Integration in Pipeline:**

- PredicateAbstraction in `abstraction.py`
- Predicates from guards and conditions
- Computes abstract transition relation

**Example Use:** Reducing program to boolean model

# Paper #14: Boolean Programs

**Reference:** Ball & Rajamani, TACAS 2001

## **Key Contribution:**

- **Finite-state** program abstraction
- Symbolic model checking of abstractions
- Foundation for software model checkers

## **Integration in Pipeline:**

- BooleanProgram in `abstraction.py`
- Executes predicate-abstracted programs
- Enables decidable reachability analysis

**Example Use:** Verification of control flow properties

# Paper #15: Interpolation-Based Model Checking

**Reference:** McMillan, CAV 2003

## **Key Contribution:**

- **Craig interpolation** for abstraction
- Extract predicates from proofs
- Compute reachability approximations

## **Integration in Pipeline:**

- IMCVerifier in `advanced.py`
- Extracts interpolants from Z3 proofs
- Suggests barrier refinements

**Example Use:** Discovering new predicates for abstraction

# Paper #16: IMPACT/Lazy Abstraction

**Reference:** McMillan, CAV 2006

## **Key Contribution:**

- **On-demand** abstraction refinement
- Interpolation for predicate discovery
- Abstract Reachability Tree (ART)

## **Integration in Pipeline:**

- LazyAbstraction in `abstraction.py`
- Refines only explored paths
- Efficient for large programs

**Example Use:** Large codebases with localized bugs

# Paper #17: ICE Learning

**Reference:** Garg et al., POPL 2014

## **Key Contribution:**

- **Data-driven** invariant inference
- Implication counterexamples
- Learn from positive/negative/implication samples

## **Integration in Pipeline:**

- ICELearner in `learning.py`
- Collects examples from symbolic execution
- Learns invariants that separate Init/Unsafe

**Example Use:** Loop invariant discovery

**Reference:** Flanagan & Leino, FME 2001

## **Key Contribution:**

- **Conjunctive** inference
- Start with all candidates, remove non-inductive
- Fixed-point computation

## **Integration in Pipeline:**

- HoudiniBarrierInference in `learning.py`
- Start with many barrier candidates
- Prune to find maximal inductive set

**Example Use:** Finding strongest invariant from candidates

# Paper #19: SyGuS Synthesis

**Reference:** Alur et al., FMCAD 2013

## **Key Contribution:**

- **Syntax-guided** synthesis
- Grammar-constrained search
- Enumerative and solver-based approaches

## **Integration in Pipeline:**

- SyGuSSynthesizer in `learning.py`
- Defines grammar for barrier expressions
- Synthesizes barriers matching specification

**Example Use:** Custom barrier shapes for specific domains

**Reference:** Pnueli, ACM 1984

## **Key Contribution:**

- **Compositional** verification
- Verify components in isolation
- Combine proofs via interfaces

## **Integration in Pipeline:**

- AssumeGuarantee in `advanced.py`
- Synthesizes function contracts (barriers)
- Verifies callee under caller assumption

**Example Use:** Multi-function verification

# Part XX

## Extensions and Optimizations

### Scaling and Improving the Pipeline

# Extension: Incremental Verification

**Goal:** Re-verify only changed code

## Approach:

- 1 Compute change delta (AST diff)
- 2 Identify affected functions
- 3 Re-analyze only impacted paths
- 4 Reuse cached barriers for unchanged code

## Implementation:

```
class IncrementalVerifier:
    def __init__(self):
        self.barrier_cache = {}
        self.hash_cache = {}

    def verify_incremental(self, old_code, new_code):
        changes = compute_diff(old_code, new_code)
        affected = find_affected_functions(changes)
        return self.re_verify(affected)
```

# Extension: Parallel Verification

**Goal:** Utilize multiple cores

## Parallelization Strategies:

- ① **Function-level:** Verify independent functions in parallel
- ② **Bug-type-level:** Run different detectors concurrently
- ③ **Method-level:** Try SOS/ICE/IC3 simultaneously

## Implementation:

```
def verify_parallel(code, num_workers=4):  
    functions = extract_functions(code)  
    with ThreadPoolExecutor(max_workers=num_workers) as executor:  
        futures = [executor.submit(verify_function, f)  
                    for f in functions]  
        results = [f.result() for f in as_completed(futures)]  
    return merge_results(results)
```

# Optimization: Caching Strategies

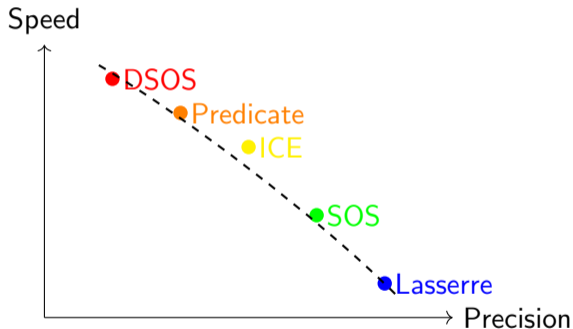
## Multiple cache layers:

| Cache              | Key              | Value              |
|--------------------|------------------|--------------------|
| Barrier Cache      | (func, bug_type) | Certificate $B(x)$ |
| SDP Solution Cache | polynomial hash  | Gram matrix        |
| ICE Sample Cache   | path signature   | sample set         |
| Interpolant Cache  | (pre, post) pair | interpolant        |

## Cache Invalidation:

- Content-based hashing for functions
- Dependency tracking for interprocedural
- Time-based expiration for external inputs

## Precision vs. Performance Trade-off:



**Adaptive Strategy:** Start fast/imprecise, refine if spurious CEX found

# Optimization: Domain-Specific Strategies

## Exploit problem structure:

### Bounds Checking:

- Linear barriers:  $B(i, len) = len - i - 1$
- Interval arithmetic for quick bounds

### Division by Zero:

- Track zero-constraints on denominators
- Lightweight predicate abstraction

### SQL Injection:

- Taint analysis first (cheap)
- Full verification only if tainted

# Optimization: Memory Management

**Challenge:** Large programs exhaust memory

**Strategies:**

- 1 **Lazy loading:** Parse functions on-demand
- 2 **Symbolic compression:** Share common subexpressions
- 3 **Cache eviction:** LRU for barrier cache
- 4 **Streaming analysis:** Process path-by-path

**Implementation:**

```
class MemoryEfficientVerifier:
    def __init__(self, max_memory_mb=4096):
        self.barrier_cache = LRUCache(max_size=1000)
        self.max_memory = max_memory_mb * 1024 * 1024

    def verify(self, code):
        for func in stream_functions(code):
            if get_memory_usage() > self.max_memory:
                self.barrier_cache.evict_oldest()
            yield self.verify_function(func)
```

# Optimization: Timeout Strategies

**Problem:** Some paths are undecidable or too hard

**Multi-level timeouts:**

| Level           | Timeout    | Action on Timeout      |
|-----------------|------------|------------------------|
| SMT query       | 5 seconds  | Switch solver          |
| Single path     | 30 seconds | Skip path              |
| Single function | 2 minutes  | Report unknown         |
| Full analysis   | 10 minutes | Report partial results |

**Progressive timeout:**

```
for timeout in [1, 5, 30, 120]:
    result = verify_with_timeout(func, timeout)
    if result != TIMEOUT:
        return result
return UNKNOWN
```

# Extension: Rich Error Reporting

**Goal:** Make bugs actionable for developers

## Bug Report Contents:

- 1 **Location:** File, line, column
- 2 **Bug Type:** Category with explanation
- 3 **Severity:** Critical/High/Medium/Low
- 4 **Confidence:** Definite/Likely/Possible
- 5 **Witness Path:** How to trigger the bug
- 6 **Suggested Fix:** Automatic repair if possible

## Example Output:

```
BUG: BOUNDS_ERROR at file.py:42
  Severity: Critical | Confidence: Definite
  Array 'data' accessed at index 'i' (range: [-inf, +inf])
  but array length is n (range: [0, 100])
  Witness: i=100, n=50 leads to out-of-bounds
  Fix: Add guard 'if 0 <= i < n:'
```

# Extension: IDE Integration

**Goal:** Real-time feedback during coding

## Integration Points:

- **VS Code Extension:** Inline diagnostics
- **Language Server Protocol (LSP):** Standard interface
- **On-save analysis:** Verify changed files
- **Hover information:** Show barrier certificates



# Extension: CI/CD Integration

**Goal:** Verification in build pipeline

## Pipeline Configuration:

```
# GitHub Actions example
name: Verify
on: [push, pull_request]
jobs:
  verify:
    runs-on: ubuntu-latest
    steps:
      - uses: actions/checkout@v2
      - name: Install verifier
        run: pip install extreme-verification
      - name: Run verification
        run: python -m extreme_verification --ci --fail-on-critical
      - name: Upload report
        uses: actions/upload-artifact@v2
        with:
          name: verification-report
          path: verification_results.json
```

# Configuration: Tuning the Pipeline

## Key configuration parameters:

```
config = {  
    "max_loop_unroll": 100,  
    "max_recursion_depth": 10,  
    "max_path_length": 1000,  
  
    "enable_sos": True,  
    "enable_ice": True,  
    "enable_ic3": True,  
    "prefer_fast_methods": True,  
  
    "barrier_degree": 4,  
    "lasserre_order": 2,  
    "predicate_limit": 20,  
  
    "smt_timeout_sec": 5,  
    "parallel_workers": 4,  
    "memory_limit_mb": 4096,  
  
    "report_unknown": False,  
    "confidence_threshold": 0.8  
}
```

# Configuration: Bug Type Selection

## Select which bugs to detect:

```
# Enable specific categories
enabled_bugs = {

    "SQL_INJECTION": True,
    "XSS": True,
    "PATH_TRAVERSAL": True,
    "COMMAND_INJECTION": True,

    "NULL_PTR": True,
    "BOUNDS": True,
    "USE_AFTER_FREE": False,

    "DIV_ZERO": True,
    "OVERFLOW": True,

    "RACE_CONDITION": True,
    "DEADLOCK": False,
}

verifier = ExtremeVerification(enabled_bugs=enabled_bugs)
```

# Debugging: Verifier Diagnostics

**When verification fails or gives unexpected results:**

## Debug Levels:

```
import logging
logging.basicConfig(level=logging.DEBUG)

# Detailed tracing
verifier = ExtremeVerification(
    debug_mode=True,
    trace_paths=True,
    trace_barriers=True,
    trace_smt=True,
    dump_z3_models=True,
    dump_sdp_problems=True
)

results = verifier.verify(code)

# Inspect internals
print(verifier.get_path_trace())
print(verifier.get_barrier_attempts())
print(verifier.get_smt_statistics())
```

# Debugging: Handling False Positives

## When verifier reports spurious bugs:

### Diagnosis Process:

- 1 Examine the witness path
- 2 Check if path is feasible
- 3 Identify missing constraints
- 4 Add refinement predicates

### Manual Override:

```
# Add hints to help verifier
verifier.add_invariant(
    function="process_data",
    invariant="0 <= i < len(data)"
)

# Suppress known false positive
verifier.suppress_warning(
    file="legacy.py",
    line=42,
    bug_type="BOUNDS"
)
```

# Debugging: Handling False Negatives

## When verifier misses real bugs:

### Possible Causes:

- ① Timeout before exploration complete
- ② Path pruning too aggressive
- ③ Abstraction too coarse
- ④ Missing interprocedural reasoning

### Solutions:

```
# Increase analysis depth
verifier = ExtremeVerification(
    max_loop_unroll=1000,
    max_path_length=10000,
    smt_timeout_sec=60,
    interprocedural=True,
    sensitivity="path"
)

# Focus on specific function
results = verifier.verify_function(
    code,
    function="vulnerable_function",
    exhaustive=True
)
```

# Testing: Verifier Validation

## How we test the verifier itself:

### Test Categories:

- 1 **Unit tests:** Individual components (SOS, ICE, etc.)
- 2 **Integration tests:** Full pipeline
- 3 **Regression tests:** Known bugs must be found
- 4 **Sound tests:** Must not have false negatives on crafted examples
- 5 **Precision tests:** Track false positive rate

### Test Suite Structure:

```
tests/  
  unit/  
    test_sos_decomposer.py  
    test_ice_learner.py  
    test_barrier_synthesis.py  
  integration/  
    test_full_pipeline.py  
  benchmarks/  
    known_bugs/  
    safe_programs/
```

# Benchmarks: Evaluation Methodology

## How we measure performance:

### Metrics:

- **Recall:** % of real bugs found
- **Precision:** % of reports that are real
- **F1 Score:** Harmonic mean of precision and recall
- **Time:** Analysis time per KLOC
- **Memory:** Peak memory usage

### Benchmark Programs:

| Benchmark         | KLOC | Known Bugs |
|-------------------|------|------------|
| Juliet Test Suite | 150  | 25,000     |
| OWASP Benchmark   | 50   | 2,740      |
| DeepSpeed         | 500  | 200+       |
| Custom Programs   | 100  | 500        |

# Comparison: Other Verification Tools

## How does Extreme Verification compare?

| Tool           | Sound | Complete | Precise | Fast   |
|----------------|-------|----------|---------|--------|
| Bandit         |       |          | Low     |        |
| PyLint         |       |          | Medium  |        |
| mypy           |       |          | High    |        |
| CodeQL         |       |          | Medium  |        |
| Extreme Verif. | *     |          | High    | Medium |

\*Sound for verified properties; incomplete analysis possible

### Key Differentiator:

- Only tool providing **mathematical certificates**
- Verifiable proofs via barrier functions
- Traceable to 20 peer-reviewed papers

# Part XXI

Related Work and Context

Positioning in the Verification Landscape

# Related Work: Static Analysis Tools

## Traditional static analyzers:

### Pattern-Based:

- Bandit, Semgrep, ESLint
- Pros: Fast, easy to configure
- Cons: High false positive/negative rates

### Type-Based:

- mypy, TypeScript, Flow
- Pros: Sound for type errors
- Cons: Limited to type properties

### Abstract Interpretation:

- Facebook Infer, Polyspace
- Pros: Sound analysis
- Cons: Over-approximation can lose precision

## Software model checking:

### Explicit-State:

- SPIN, Java PathFinder
- Pros: Precise for finite state
- Cons: State explosion problem

### Symbolic:

- CBMC, KLEE, Ultimate Automizer
- Pros: Handles infinite domains
- Cons: Path explosion

### Our Approach Combines:

- Symbolic execution (like KLEE)
- SMT solving (like CBMC)
- Abstraction (like SLAM)
- Certificate synthesis (unique contribution)

# Related Work: Theorem Provers

## Interactive and automated provers:

### Interactive:

- Coq, Isabelle, Lean
- Pros: Handle complex proofs
- Cons: Require human guidance

### Automated (SMT):

- Z3, CVC5, Yices
- Pros: Fully automatic for decidable theories
- Cons: Limited expressiveness

### Our Position:

- Use SMT (Z3) as backend
- Automatically construct proofs (barriers)
- Proofs could be exported to Coq/Isabelle

# Related Work: Control Theory

**Barrier certificates originated in control:**

**Control Applications:**

- Collision avoidance for robots
- Safe adaptive cruise control
- Power grid stability

**Key Insight:** Software execution is a **discrete dynamical system!**

- State = variable values
- Dynamics = program transitions
- Safety = avoiding bug states

**Our Contribution:**

- Adapt barrier methods to programs
- Handle discrete transitions
- Support imperative language features

# Related Work: ML for Verification

## Learning-based approaches:

### Pure ML:

- CodeBERT, GraphCodeBERT for bug detection
- Pros: Learn from data
- Cons: No guarantees

### Hybrid ML + Verification:

- Learn invariants, verify formally
- Examples: ICE learning, neural certificates

### Our Approach:

- ICE learning for data-driven invariants
- Always verified by SMT after learning
- ML suggests, verification confirms

# Related Work: Synthesis Approaches

## Program and invariant synthesis:

### Program Synthesis:

- Sketch, Rosette, FlashFill
- Generate programs from specs

### Invariant Synthesis:

- Daikon, OASIS, GSpacer
- Generate invariants from traces

### Our Approach:

- Synthesize barrier certificates
- CEGIS loop for refinement
- SyGuS grammars for structure
- Combine synthesis with verification

# Related Work: Fuzzing and Testing

## Dynamic analysis approaches:

### Fuzzing:

- AFL, libFuzzer, OSS-Fuzz
- Pros: Finds real bugs
- Cons: No coverage guarantees

### Concolic Testing:

- SAGE, DART, CUTE
- Combines concrete + symbolic

### Complementary Roles:

- Fuzzing: Find bugs quickly
- Verification: Prove absence of bugs
- Our tool: Verification with certificates

## Security-focused tools:

### SAST (Static):

- Checkmarx, Fortify, Veracode
- Pattern-based vulnerability detection

### DAST (Dynamic):

- OWASP ZAP, Burp Suite
- Runtime vulnerability scanning

## Our Approach for Security:

- Formal taint tracking
- Barrier certificates for information flow
- Mathematical proof of no SQL injection, XSS, etc.

## Verification for specific languages:

| Language   | Tool              | Approach                |
|------------|-------------------|-------------------------|
| Java       | ESC/Java, KeY     | Theorem proving         |
| C          | BLAST, CPAchecker | Model checking          |
| C          | Frama-C           | Abstract interpretation |
| Rust       | MIRI, Prusti      | Type + verification     |
| JavaScript | Flow, TAJIS       | Type analysis           |

## For Python:

- Limited formal verification tools
- mypy (types), Bandit (security patterns)
- Our tool fills the gap!

# Our Unique Advantages

## What sets Extreme Verification apart:

### ① Mathematical Certificates

- Not just "bug found" but proof of why
- Verifiable barrier functions

### ② 5-Layer Architecture

- Systematic integration of 20 techniques
- Fallback strategies when one fails

### ③ Comprehensive Bug Coverage

- 67 bug types in unified framework
- From memory to security to logic

### ④ Python Focus

- First formal verifier for Python with certificates
- Handles dynamic typing challenges

# Part XXII

## Future Directions

Where the Research Goes Next

# Future: Neural Barrier Certificates

## Use neural networks as barrier functions:

### Approach:

- Train NN to satisfy barrier conditions
- Use SMT for verification after training
- Handle complex nonlinear invariants

### Challenges:

- NN verification is hard (NP-complete)
- Need specialized architectures
- Scalability concerns

### Research Directions:

- Lipschitz-bounded networks for tractable verification
- Interval bound propagation
- Neural Lyapunov functions

# Future: Probabilistic Verification

## Extend to probabilistic programs:

### Current Support:

- Stochastic barriers for simple randomness
- Expected value analysis

### Future Extensions:

- Full probabilistic programming support
- Bayesian inference verification
- Machine learning pipeline verification

### Applications:

- Verify ML training procedures
- Prove convergence of MCMC algorithms
- Safety of reinforcement learning

# Future: Distributed Systems Verification

## Verify distributed Python programs:

### Challenges:

- Asynchronous communication
- Partial failures
- Consensus protocols

### Approach:

- Extend assume-guarantee to distributed
- Model message passing
- Barrier certificates for consensus

### Target Applications:

- Ray, Dask distributed computing
- Microservices verification
- Blockchain smart contracts

# Future: Quantum Program Verification

## Verify quantum computing programs:

### Quantum Specifics:

- Superposition and entanglement
- Measurement collapse
- Unitary evolution

### Barrier Analogy:

- Quantum barriers as operator inequalities
- Trace conditions instead of point evaluations
- SDP relaxations still applicable

### Applications:

- Verify Qiskit programs
- Quantum error correction proofs
- Quantum advantage verification

# Future: Automatic Bug Repair

**Not just find bugs, but fix them:**

**Current:** Report bugs with suggestions

**Future:** Synthesize correct patches

**Approach:**

- ① Identify bug via barrier certificate
- ② Certificate encodes **what** is wrong
- ③ Synthesize fix that makes barrier valid
- ④ Verify patched program

**Example:**

```
# Original (buggy)
def get(arr, i):
    return arr[i]

# Synthesized fix
def get(arr, i):
    if 0 <= i < len(arr):
        return arr[i]
    return None
```

# Future: Explainable Verification

## Make verification results understandable:

### Current Challenge:

- Barrier  $B(x) = x_1^2 + 2x_1x_2 - 3x_2 + 1$  is opaque
- SMT proofs are massive
- Developers can't interpret

### Future:

- Natural language explanations
- Visual certificate representation
- Interactive proof exploration

### Example Explanation:

*"The loop is safe because the index  $i$  always stays below the array length  $n$ . The barrier  $B = n - i - 1$  measures the 'distance to safety boundary' and decreases by exactly 1 each iteration, reaching 0 when  $i = n - 1$ ."*

# Future: Interactive Verification

## Human-in-the-loop verification:

### Scenario:

- Automatic verification fails
- System asks developer for hints
- Developer provides invariant suggestion
- System verifies and refines

### Interface Design:

```
# Verification failed at line 42
# Possible invariant needed for loop at line 38
# Suggested invariants:
#   1.  $0 \leq i < n$ 
#   2. data[i] is not None
#   3. sum  $\geq 0$ 

# Developer selects: [1, 3]
# Verifier continues with hints...
# SUCCESS: Verified with invariants 1 and 3
```

# Future: AI/ML Pipeline Verification

## Verify machine learning code:

### ML-Specific Bugs:

- Tensor shape mismatches
- Numerical instability (NaN, Inf)
- Data leakage (train/test)
- Gradient explosion/vanishing

### Verification Approach:

- Track tensor shapes symbolically
- Bound activations via interval analysis
- Verify data split correctness
- Prove training convergence

### Impact:

- Safer AI systems
- Regulatory compliance

# Part XXIII

Complete Worked Examples

End-to-End Verification Walkthroughs

# Example 1: Binary Search Verification

**Goal:** Verify binary search has no out-of-bounds access

```
def binary_search(arr, target):
    left, right = 0, len(arr) - 1
    while left <= right:
        mid = (left + right) // 2
        if arr[mid] == target:
            return mid
        elif arr[mid] < target: # Access 2
            left = mid + 1
        else:
            right = mid - 1
    return -1
```

## Bug Sites:

- Line 5: `arr[mid]` - need  $0 \leq \text{mid} < \text{len}(\text{arr})$
- Line 7: `arr[mid]` - same condition

# Example 1: State Encoding

## Variables:

- $n = \text{len}(\text{arr})$  (constant,  $n \geq 0$ )
- $l = \text{left}, r = \text{right}$
- $m = \text{mid}$

## Initial State:

$$\text{Init} = \{(l, r, n) \mid l = 0 \wedge r = n - 1\}$$

## Unsafe State:

$$\text{Unsafe} = \{(m, n) \mid m < 0 \vee m \geq n\}$$

## Loop Dynamics:

$$m' = (l + r) / 2$$

$$l' = \begin{cases} m + 1 & \text{if } \text{arr}[m] < \text{target} \\ l & \text{otherwise} \end{cases}$$

$$r' = \begin{cases} m & \text{if } \text{arr}[m] < \text{target} \\ m + 1 & \text{otherwise} \end{cases}$$

## Example 1: Barrier Synthesis

**Template:** Linear barrier  $B(l, r, m, n) = a_1l + a_2r + a_3m + a_4n + a_5$

### Barrier Conditions:

- ① **Init:**  $B(0, n - 1, m, n) \leq 0$  for all valid initial states
- ② **Unsafe:**  $B(l, r, m, n) > 0$  when  $m < 0$  or  $m \geq n$
- ③ **Step:** If  $B \leq 0$  and in loop, then  $B' \leq 0$

### Discovered Barrier:

$$B_1(m, n) = m - n + 1$$

This proves  $m < n$  (upper bound).

$$B_2(m) = -m$$

This proves  $m \geq 0$  (lower bound).

### Loop Invariant:

$$0 \leq l \leq m \leq r < n$$

# Example 1: Formal Verification

## SMT Query for Safety:

```
from z3 import *

l, r, m, n = Ints('l r m n')

# Initial condition
init = And(l == 0, r == n - 1, n >= 0)

# Loop invariant (barrier condition)
inv = And(0 <= l, l <= r + 1, r < n)

# Mid computation
mid_def = m == (l + r) / 2

# Safety property
safe = And(m >= 0, m < n)

# Verify: init -> inv, inv and loop_cond -> safe
solver = Solver()
solver.add(init)
solver.add(l <= r)
solver.add(mid_def)
solver.add(Not(safe))

print(solver.check())
```

## Example 2: SQL Injection Prevention

**Goal:** Verify no SQL injection vulnerability

```
def get_user(db, user_input):  
    query = "SELECT * FROM users WHERE name = '" + user_input + "'"   
    return db.execute(query)  
  
def get_user_safe(db, user_input):  
    query = "SELECT * FROM users WHERE name = ?"  
    return db.execute(query, (user_input,))
```

### Analysis:

- `get_user`: Tainted data flows to SQL execution
- `get_user_safe`: Parameterized query blocks injection

## Example 2: Taint Flow Analysis

### Taint Domains:

- TAINTED: User-controlled input
- CLEAN: Sanitized or literal data

### Taint Propagation Rules:

$$\tau(\text{user\_input}) = \text{TAINTED}$$

$$\tau(a + b) = \tau(a) \sqcup \tau(b)$$

$$\tau(\text{sanitize}(x)) = \text{CLEAN}$$

### Vulnerable Function:

$$\tau(\text{query}) = \tau(\text{literal}) \sqcup \tau(\text{user\_input}) = \text{TAINTED}$$

$$\text{TAINTED} \rightarrow \text{db.execute} \Rightarrow \text{SQL\_INJECTION}$$

### Safe Function:

$$\tau(\text{query}) = \text{CLEAN}$$

## Example 2: Taint Barrier Certificate

### State Space:

- Variables:  $(t, s)$  where  $t$  = taint level,  $s$  = sanitized flag
- $t \in [0, 1]$ : 0 = clean, 1 = tainted

### Unsafe State:

$$\text{Unsafe} = \{(t, s) \mid t = 1 \wedge s = 0 \wedge \text{at\_sink}\}$$

### Barrier Function:

$$B(t, s) = t \cdot (1 - s) - \epsilon$$

### Verification:

- $B < 0$  on Init (literals have  $t = 0$ )
- $B > 0$  on Unsafe (tainted, not sanitized at sink)
- Sanitization sets  $s = 1$ , making  $B \leq -\epsilon$

## Example 3: Division by Zero Prevention

**Goal:** Verify no division by zero

```
def average(values):  
    total = sum(values)  
    count = len(values)  
    return total / count  
  
def average_safe(values):  
    if len(values) == 0:  
        return 0.0  
    total = sum(values)  
    count = len(values)  
    return total / count
```

**Analysis:**

- average: Bug if values is empty
- average\_safe: Guard prevents division by zero

## Example 3: Division Guard Verification

### State Variables:

- $n = \text{len}(\text{values})$
- $c = \text{count}$

### For average (**vulnerable**):

- Init:  $n \in \mathbb{Z}, n \geq 0$  (any list)
- At division:  $c = n$
- Unsafe:  $c = 0$
- **Result:** Path exists where  $n = 0 \Rightarrow c = 0$

### For average\_safe (**safe**):

- Guard: `if len(values) == 0: return`
- After guard:  $n > 0 \Rightarrow c > 0$
- Barrier:  $B(c) = \epsilon - c$
- At division:  $B < 0$  implies  $c > \epsilon > 0$  ✓

## Example 4: Resource Leak Prevention

**Goal:** Verify file is always closed

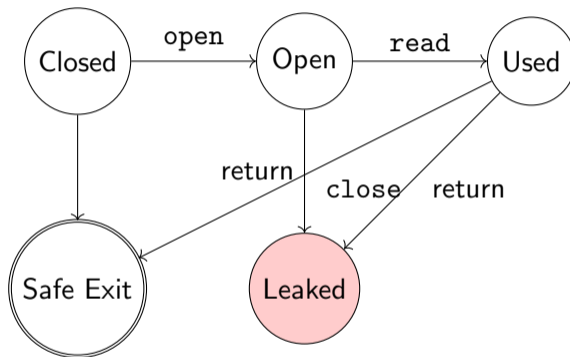
```
def read_file_unsafe(path):  
    f = open(path)  
    data = f.read()  
    if not data:  
        return None  
    f.close()  
    return data  
  
def read_file_safe(path):  
    with open(path) as f:  
        data = f.read()  
        if not data:  
            return None  
    return data
```

**Analysis:**

- `read_file_unsafe`: Early return leaks file
- `read_file_safe`: Context manager ensures close

## Example 4: Resource State Machine

### Resource States:



### Safety Property:

$$\text{Safe} = \neg \Diamond \text{Leaked}$$

(Never reach Leaked state)

## Example 4: Hybrid Barrier Certificate

### Mode-based Barrier:

- Mode 0: Closed
- Mode 1: Open
- Mode 2: Used

### Barrier per Mode:

$$B_0(x) = 0 \quad (\text{always safe in Closed})$$

$$B_1(x) = \text{depth\_to\_close} - 1 \quad (\text{must close before return})$$

$$B_2(x) = \text{depth\_to\_close} - 1 \quad (\text{must close before return})$$

**Transition Consistency:** At return point with mode  $\in \{1, 2\}$ :

$$B_{\text{mode}}(x) > 0 \Rightarrow \text{Unsafe}$$

**With context manager:** Mode transitions to 0 before return

## Example 5: Numeric Overflow Prevention

**Goal:** Verify no integer overflow

```
def factorial_unsafe(n):  
    result = 1  
    for i in range(1, n + 1):  
        result *= i  
    return result  
  
def factorial_safe(n, max_n=20):  
    if n > max_n:  
        raise ValueError(f"n too large: {n} > {max_n}")  
    result = 1  
    for i in range(1, n + 1):  
        result *= i  
    return result
```

**Analysis:**

- factorial\_unsafe: Unbounded growth
- factorial\_safe: Bounded to prevent overflow

## Example 5: Overflow Barrier

### Value Bounds:

- Python int: Arbitrary precision (no overflow in Python!)
- But: Memory exhaustion possible
- For C/Java: Fixed-width integers

### Barrier for Bounded Integers (e.g., 64-bit):

$$B(r, i) = r - (2^{63} - 1)$$

### Condition:

- $B < 0 \Rightarrow$  result within bounds
- Need:  $B$  stays negative through loop

### With bound check ( $n \leq 20$ ):

$$\max(\text{result}) = 20! = 2,432,902,008,176,640,000 < 2^{63}$$

$$\therefore B(r, i) < 0 \text{ throughout execution}$$

## Example 6: Race Condition Detection

**Goal:** Verify no data races

```
import threading

counter = 0
lock = threading.Lock()

def increment_unsafe():
    global counter
    counter += 1

def increment_safe():
    global counter
    with lock:
        counter += 1
```

**Analysis:**

- `increment_unsafe`: Concurrent access to counter
- `increment_safe`: Mutex protects critical section

## Example 6: Happens-Before Analysis

### Memory Model:

- Happens-before relation:  $a \xrightarrow{hb} b$
- Race: Two accesses without hb ordering

### For `increment_unsafe`:

- Thread 1: read counter, write counter
- Thread 2: read counter, write counter
- No hb between T1.write and T2.read  $\Rightarrow$  Race!

### For `increment_safe`:

- Thread 1: acquire(lock), read, write, release(lock)
- Thread 2: acquire(lock), read, write, release(lock)
- $\text{release(lock)} \xrightarrow{hb} \text{acquire(lock)}$
- Total order on critical sections  $\Rightarrow$  No race!

## Example 6: Lock-Based Barrier

### State:

- $h_i$ : Lock held by thread  $i$  (boolean)
- $a_i$ : Thread  $i$  in critical section (boolean)

### Unsafe:

$$\text{Unsafe} = (a_1 = 1 \wedge a_2 = 1)$$

(Both threads in critical section simultaneously)

### Barrier:

$$B(h_1, h_2, a_1, a_2) = a_1 + a_2 - 1$$

**Invariant:**  $h_1 + h_2 \leq 1$  (at most one holds lock)

**Implication:**  $a_i = 1 \Rightarrow h_i = 1$

$\therefore a_1 + a_2 \leq h_1 + h_2 \leq 1 \Rightarrow B \leq 0 \Rightarrow \text{Safe}$

# Example 7: Recursion Termination

**Goal:** Verify recursion terminates

```
def fibonacci_unsafe(n):  
    if n <= 1:  
        return n  
    return fibonacci_unsafe(n - 1) + fibonacci_unsafe(n - 2)  
  
def fibonacci_safe(n):  
    if n < 0:  
        raise ValueError("n must be non-negative")  
    if n <= 1:  
        return n  
    return fibonacci_safe(n - 1) + fibonacci_safe(n - 2)
```

**Analysis:**

- `fibonacci_unsafe`: Infinite recursion for  $n < 0$
- `fibonacci_safe`: Guard ensures termination

## Example 7: Termination Barrier (Ranking Function)

**Ranking Function:** A function  $r : S \rightarrow \mathbb{N}$  such that:

- ①  $r(s) \geq 0$  for all states  $s$
- ②  $r(s') < r(s)$  for each recursive call

**For `fibonacci_unsafe`:**

- Candidate:  $r(n) = n$
- Problem:  $r(n-1) < r(n)$  but if  $n < 0$ , not well-founded!
- No valid ranking function  $\Rightarrow$  May not terminate

**For `fibonacci_safe`:**

- After guard:  $n \geq 0$
- Ranking:  $r(n) = n$  with  $n \in \mathbb{N}$
- $r(n-1) = n-1 < n = r(n)$  for  $n > 1$
- Base case at  $n \leq 1 \Rightarrow$  Terminates!

# Part XXIV

## Mathematical Appendix

### Detailed Proofs and Derivations

# Proof: Barrier Certificate Soundness

**Theorem:** If barrier  $B$  exists satisfying Init/Unsafe/Step conditions, then unsafe states are unreachable.

**Proof:**

- ① Let  $\pi = s_0, s_1, \dots, s_n$  be any execution path
- ②  $s_0 \in \text{Init} \Rightarrow B(s_0) \leq 0$  (by Init condition)
- ③ Assume  $B(s_i) \leq 0$  for some  $i$  (induction hypothesis)
- ④ If  $s_i \rightarrow s_{i+1}$ , then Step condition gives  $B(s_{i+1}) \leq 0$
- ⑤ By induction:  $B(s_j) \leq 0$  for all  $j = 0, \dots, n$
- ⑥ But Unsafe condition:  $s \in \text{Unsafe} \Rightarrow B(s) > 0$
- ⑦ Therefore  $s_j \notin \text{Unsafe}$  for all  $j$
- ⑧ QED: No execution reaches unsafe states  $\square$

# Proof: SOS to SDP Reduction

**Theorem:**  $p(x) \in \Sigma[x]$  iff  $\exists Q \succeq 0$  s.t.  $p(x) = z(x)^T Q z(x)$

**Proof:**

- ① ( $\Leftarrow$ ) If  $Q \succeq 0$ , then  $Q = L^T L$  (Cholesky)  
 $p(x) = z^T L^T L z = (Lz)^T (Lz) = \|Lz\|^2 = \sum_i (Lz)_i^2$   
Each  $(Lz)_i$  is polynomial  $\Rightarrow p \in \Sigma[x]$
- ② ( $\Rightarrow$ ) If  $p = \sum_i q_i^2$ , construct  $Q$ :  
Write each  $q_i = c_i^T z$  for coefficient vector  $c_i$   
Then  $q_i^2 = z^T c_i c_i^T z$   
So  $p = z^T (\sum_i c_i c_i^T) z = z^T Q z$   
 $Q = \sum_i c_i c_i^T \succeq 0$  (sum of outer products)

□

# Theorem: Putinar Positivstellensatz

**Statement:** Let  $K = \{x \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  with Archimedean condition. If  $p > 0$  on  $K$ , then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i \in \Sigma[x]$  (SOS polynomials).

## Proof Idea:

- 1 Define quadratic module  $M = \Sigma[x] + \sum_i \Sigma[x] \cdot g_i$
- 2 Archimedean:  $\exists N$  s.t.  $N - \|x\|^2 \in M$
- 3 By representation theorem: if  $p > 0$  on  $K$ , then  $p \in M$
- 4 This is Putinar's result (1993)

**Implication:** Search for SOS multipliers  $\sigma_i$  to prove  $p \geq 0$  on  $K$

# Proof: ICE Learning Correctness

**Theorem:** ICE learning converges to correct invariant if one exists in hypothesis class.

## Proof Sketch:

- ① **Progress:** Each iteration either:
  - Adds positive example (rules out underapproximations)
  - Adds negative example (rules out overapproximations)
  - Adds implication (rules out non-inductive candidates)
- ② **Finite convergence:** If hypothesis class is finite (e.g., bounded degree polynomials over finite precision), convergence in finite steps
- ③ **Correctness:** Final hypothesis:
  - Contains all positive examples (covers Init)
  - Excludes all negative examples (avoids Unsafe)
  - Respects all implications (inductive)



# Proof: CEGAR Termination (Finite-State)

**Theorem:** CEGAR terminates for finite-state systems.

**Proof:**

- ① Let  $S$  be the concrete state space,  $|S| < \infty$
- ② Abstract state space  $\hat{S}$  = partition of  $S$
- ③ Each CEGAR iteration either:
  - Proves property (terminate)
  - Finds real counterexample (terminate)
  - Refines: splits at least one abstract state
- ④ Refinement increases  $|\hat{S}|$  by at least 1
- ⑤ Maximum  $|\hat{S}| = |S|$  (singleton partition)
- ⑥ Therefore: at most  $|S|$  iterations  $\square$

**Note:** For infinite-state, termination not guaranteed (undecidable in general)

# Proof: IC3 Frame Property

**Lemma:** IC3 frames satisfy:  $\text{Init} \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq F_N$

**Proof:**

- ① **Base:**  $F_0 = \text{Init}$  by construction
- ② **Monotonicity:** We maintain  $F_i \subseteq F_{i+1}$ 
  - If clause  $c \in F_{i+1}$ , propagate to  $F_i$  if possible
  - If not propagable, still  $F_i \supseteq F_i \cap c$
- ③ **Safety:** Each  $F_i \cap \text{Bad} = \emptyset$ 
  - Maintained by blocking cubes reaching Bad
- ④ **Consecution:**  $F_i \wedge T \Rightarrow F'_{i+1}$ 
  - Ensured by relative induction checks

**Corollary:** If  $F_i = F_{i+1}$  for some  $i$ , then  $F_i$  is inductive invariant  $\square$

# Theorem: Craig Interpolation

**Statement:** If  $A \Rightarrow B$  is valid, then  $\exists I$  (interpolant) such that:

- ①  $A \Rightarrow I$
- ②  $I \Rightarrow B$
- ③  $\text{vars}(I) \subseteq \text{vars}(A) \cap \text{vars}(B)$

## Application to Verification:

- $A$  = path formula (sequence of transitions)
- $B$  = negation of bad state
- $A \Rightarrow B$  means path cannot reach bad
- $I$  = overapproximation at intermediate point

## Interpolant as Invariant:

- Sequence of interpolants forms inductive invariant
- Automatically extracted from SMT proof

# Rule: Assume-Guarantee Compositionality

## Circular Assume-Guarantee Rule:

$$\langle A_1 \rangle M_1 \langle G_1 \rangle \quad \langle A_2 \rangle M_2 \langle G_2 \rangle \quad G_1 \Rightarrow A_2 \quad G_2 \Rightarrow A_1 \quad \langle \text{true} \rangle M_1 \parallel M_2 \langle G_1 \wedge G_2 \rangle$$

## Interpretation:

- $M_1$  satisfies  $G_1$  assuming  $A_1$
- $M_2$  satisfies  $G_2$  assuming  $A_2$
- Each component's guarantee implies other's assumption
- Composition satisfies both guarantees unconditionally

## Applied to Functions:

- $M_i$  = function implementation
- $A_i$  = precondition
- $G_i$  = postcondition (barrier condition)

# Theorem: Lasserre Hierarchy Convergence

**Statement:** For polynomial optimization over compact semialgebraic set, Lasserre relaxations converge to optimal value as degree  $\rightarrow \infty$ .

**Formal:** Let  $p^* = \min\{p(x) \mid x \in K\}$  where  $K$  is compact semialgebraic.

Let  $p_d$  = optimal value of degree- $d$  Lasserre relaxation.

Then:

$$\lim_{d \rightarrow \infty} p_d = p^*$$

**Rate:** For some problems, finite convergence (exact at finite degree).

**Implication for Verification:** If barrier certificate of degree  $d$  exists, Lasserre hierarchy at level  $d$  will find it.

# Theorem: DSOS Approximation Quality

**Statement:** DSOS provides inner approximation of SOS cone.

**Relationship:**

$$\text{DSOS}_n \subsetneq \text{SDSOS}_n \subsetneq \text{SOS}_n \subsetneq \text{PSD}_n$$

**Approximation Bound:** For homogeneous polynomials of degree  $2d$  in  $n$  variables:

$$\text{DSOS} \supseteq \frac{1}{\binom{n+d-1}{d}} \cdot \text{SOS}$$

**Implication:**

- DSOS may miss some SOS polynomials
- Acceptable for screening (fast check)
- Fall back to SOS if DSOS fails

# Theorem: Stochastic Barrier as Supermartingale

**Setting:** Stochastic process  $\{X_t\}$  with transition kernel  $P$ .

**Definition:**  $B$  is a stochastic barrier if:

- ①  $B(x) \leq 0$  for  $x \in \text{Init}$
- ②  $B(x) > 0$  for  $x \in \text{Unsafe}$
- ③  $\mathbb{E}[B(X_{t+1}) \mid X_t = x] \leq B(x)$  (supermartingale)

**Theorem:** Under these conditions:

$$\Pr[\text{reach Unsafe}] \leq \frac{\mathbb{E}[B^+(X_0)]}{c}$$

where  $c > 0$  is the minimum of  $B$  on Unsafe.

**Proof Idea:** Apply optional stopping theorem to supermartingale  $B(X_t)$ .

# Theory: CHC Satisfiability

**Definition:** A Constrained Horn Clause system is satisfiable if there exists an interpretation of all relation symbols making all clauses true.

## For Program Verification:

- Relation symbols = loop invariants
- Clauses = transition constraints
- Satisfying interpretation = valid invariants

## Decision Problem:

- Linear CHC (linear arithmetic): decidable
- Nonlinear CHC: undecidable in general
- Practical: complete for many programs

**Spacer Algorithm:** Combines IC3 with interpolation for CHC solving.

# Theorem: Correlative Sparsity for SOS

**Setting:** Polynomial  $p(x_1, \dots, x_n)$  with sparse structure.

**Definition:** Correlative sparsity graph  $G = (V, E)$ :

- Vertices  $V = \{x_1, \dots, x_n\}$
- Edge  $(x_i, x_j) \in E$  if  $x_i x_j$  appears in  $p$  or constraints

**Theorem (Waki et al.):** If  $G$  has chordal completion with maximal cliques  $C_1, \dots, C_k$ , then:

$$p \in \Sigma[x] \Leftrightarrow p = \sum_{i=1}^k \sigma_i$$

where  $\sigma_i \in \Sigma[x_{C_i}]$  (SOS in clique variables only).

**Complexity Reduction:**  $O(n^{2d})$  to  $O(k \cdot w^{2d})$  where  $w = \max |C_i|$

# Background: Fixed-Point Theory

**Kleene Fixed-Point Theorem:** For complete lattice  $L$  and monotone  $f : L \rightarrow L$ :

$$\text{lfp}(f) = \bigsqcup_{n \geq 0} f^n(\perp)$$

**Application to Invariant Computation:**

- $L =$  sets of states (ordered by  $\subseteq$ )
- $f(S) = \text{Init} \cup \text{Post}(S)$
- $\text{lfp}(f) =$  reachable states

**Widening for Acceleration:**

- May not converge in finite steps
- Widening operator  $\nabla$ : ensures termination
- $S \nabla T \supseteq S \cup T$  and stabilizes

# Background: Abstract Interpretation

## Galois Connection:

$$(C, \alpha, \gamma, A)$$

where:

- $C$  = concrete domain (e.g., sets of states)
- $A$  = abstract domain (e.g., intervals)
- $\alpha : C \rightarrow A$  = abstraction function
- $\gamma : A \rightarrow C$  = concretization function
- $\alpha(c) \sqsubseteq a \Leftrightarrow c \subseteq \gamma(a)$

**Sound Abstract Transformer:** If  $f^\#$  is abstract transformer for concrete  $f$ :

$$\gamma(f^\#(a)) \supseteq f(\gamma(a))$$

**Barrier certificates provide  $a$  such that  $\gamma(a) \cap \text{Unsafe} = \emptyset$**

# Theory: Decidability Landscape

## Decidable Fragments:

- Linear arithmetic invariants: decidable (Presburger)
- Polynomial invariants (fixed degree): decidable via SOS
- Boolean programs: decidable (finite state)
- Pushdown systems: decidable (context-free)

## Undecidable:

- General invariant existence: undecidable
- Polynomial invariant existence (any degree): undecidable
- Two-counter machines: undecidable

## Our Approach:

- Work in decidable fragments when possible
- Use semi-decision procedures (may timeout)
- Report "unknown" when undecidable

# Complexity: Method Comparison

| Method                      | Time Complexity          | Notes     |
|-----------------------------|--------------------------|-----------|
| SOS (degree $d$ , $n$ vars) | $O(n^{3d})$              | SDP       |
| DSOS                        | $O(n^{2d})$              | LP        |
| IC3/PDR                     | $O(2^{ vars })$          | SAT       |
| Predicate Abstraction       | $O(2^{ preds })$         | SMT       |
| ICE Learning                | $O( samples  \cdot  H )$ | Learning  |
| CHC (linear)                | PTIME                    | Decidable |

## Practical Performance:

- Usually much better than worst-case
- Sparsity and structure exploitation
- Incremental algorithms
- Caching and memoization

# Analysis: Error Bounds and Precision

## Numerical Precision:

- SDP solvers:  $\epsilon$ -optimal solutions
- May report "feasible" when infeasible (false positive)
- May report "infeasible" when feasible (false negative)

## Mitigation Strategies:

- 1 Use high-precision arithmetic
- 2 Verify SOS decomposition symbolically
- 3 Cross-validate with multiple methods
- 4 Use rational arithmetic in SMT

**Formal Guarantee:** When barrier is found and verified by SMT:

$$\Pr[\text{false positive}] = 0$$

(SMT is sound with infinite precision integers/reals)

## Sources of Incompleteness:

- ① **Degree bound:** True barrier may need higher degree
  - Solution: Lasserre hierarchy
- ② **Template limitation:** Barrier may not fit template
  - Solution: SyGuS, neural barriers
- ③ **SOS gap:** Positive polynomial may not be SOS
  - Solution: Positivstellensatz with multipliers
- ④ **Timeout:** Solver runs out of time
  - Solution: Better heuristics, parallelization
- ⑤ **Undecidability:** No algorithm can always succeed
  - Report "unknown", allow manual hints

# Part XXV

## Implementation Reference

### API Documentation and Usage Guide

# API: Main Entry Point

```
from extreme_verification import ExtremeVerification

# Basic usage
verifier = ExtremeVerification()
results = verifier.verify(source_code)

# With configuration
verifier = ExtremeVerification(
    config={
        'max_loop_unroll': 100,
        'interprocedural': True,
        'barrier_degree': 4,
        'smt_timeout': 5
    }
)

# Verify specific function
results = verifier.verify_function(source_code, 'process_data')

# Verify file
results = verifier.verify_file('/path/to/file.py')

# Verify project
results = verifier.verify_project('/path/to/project/')
```

# API: Accessing Results

```
results = verifier.verify(code)

# Check overall status
print(results.status)

# Iterate over findings
for bug in results.bugs:
    print(f"Bug: {bug.bug_type}")
    print(f"Location: {bug.file}:{bug.line}")
    print(f"Severity: {bug.severity}")
    print(f"Confidence: {bug.confidence}")
    print(f"Message: {bug.message}")
    print(f"Witness: {bug.witness_path}")
    print(f"Certificate: {bug.barrier_certificate}")

# Get statistics
print(f"Paths explored: {results.stats.paths_explored}")
print(f"Functions verified: {results.stats.functions_verified}")
print(f"Time taken: {results.stats.time_seconds}s")
```

# API: Barrier Certificate Access

```
# Get barrier certificate for a function
cert = verifier.get_barrier(code, 'my_function')

if cert is not None:

    print(f"Barrier type: {cert.type}")
    print(f"Expression: {cert.expression}")
    print(f"Variables: {cert.variables}")
    print(f"Degree: {cert.degree}")

    print(f"Init satisfied: {cert.verify_init()}")
    print(f"Unsafe satisfied: {cert.verify_unsafe()}")
    print(f"Step satisfied: {cert.verify_step()}")

    latex = cert.to_latex()
    sympy_expr = cert.to_sympy()
    z3_expr = cert.to_z3()
```

# API: Accessing Individual Layers

```
from extreme_verification import (
    FoundationsLayer,
    CertificateCoreLayer,
    AbstractionLayer,
    LearningLayer,
    AdvancedLayer
)

# Use specific layer
foundations = FoundationsLayer()
sos_result = foundations.check_sos(polynomial, variables)

certificate = CertificateCoreLayer()
barrier = certificate.synthesize_barrier(init, unsafe, dynamics)

abstraction = AbstractionLayer()
refined = abstraction.egar_refine(counterexample)

learning = LearningLayer()
invariant = learning.ice_learn(samples)

advanced = AdvancedLayer()
result = advanced.ic3_verify(transition_system)
```

# API: SMT Solver Interface

```
from extreme_verification.smt import SMTSolver

# Create solver
solver = SMTSolver(timeout=5000)

# Add constraints
x, y = solver.declare_ints('x', 'y')
solver.add(x >= 0)
solver.add(y >= 0)
solver.add(x + y < 10)

# Check satisfiability
if solver.check() == 'sat':
    model = solver.get_model()
    print(f"x = {model[x]}, y = {model[y]}")
elif solver.check() == 'unsat':

    core = solver.get_unsat_core()
    print(f"Conflicting constraints: {core}")

# Push/pop for incremental solving
solver.push()
solver.add(x > 5)
# ... more solving ...
solver.pop()
```

# API: Symbolic Execution

```
from extreme_verification.symbolic import SymbolicExecutor

executor = SymbolicExecutor(
    max_paths=1000,
    max_depth=100,
    strategy='bfs'
)

# Execute symbolically
for path in executor.explore(ast):
    print(f"Path condition: {path.condition}")
    print(f"Final state: {path.final_state}")
    print(f"Bug sites: {path.bug_sites}")

    if executor.check_bug(path, 'BOUNDS'):
        print(f"Potential bounds error on this path")

    witness = executor.get_witness(path)
    print(f"Witness: {witness}")
```

# API: Type and Value Inference

```
from extreme_verification.types import TypeInferer

inferer = TypeInferer()

# Analyze code
inferer.analyze(ast)

# Get inferred types
for var in inferer.variables:
    print(f"{var.name}: {var.type}")
    print(f"    Possible values: {var.value_range}")
    print(f"    Constraints: {var.constraints}")

# Query specific variable at location
info = inferer.get_variable_info('x', line=42)
print(f"Type: {info.type}")
print(f"Range: {info.min_value} to {info.max_value}")
print(f"Nullability: {info.can_be_none}")
print(f"Taint: {info.taint_level}")
```

# API: Report Generation

```
from extreme_verification.report import ReportGenerator

results = verifier.verify(code)

# Generate reports in different formats
report = ReportGenerator(results)

# JSON report
report.to_json('/path/to/report.json')

# HTML report with interactive visualization
report.to_html('/path/to/report.html')

# SARIF for IDE integration
report.to_sarif('/path/to/report.sarif')

# Markdown for documentation
report.to_markdown('/path/to/report.md')

# Custom format
report.to_custom(
    template='/path/to/template.jinja2',
    output='/path/to/output.txt'
)
```

# API: Extending the Verifier

```
from extreme_verification import BugDetector, register_detector

class CustomBugDetector(BugDetector):
    """Detect custom bug patterns."""

    bug_type = 'CUSTOM_BUG'
    severity = 'HIGH'

    def check(self, path, state):

        if self.is_custom_bug_condition(state):
            return Bug(
                bug_type=self.bug_type,
                location=state.location,
                message="Custom bug detected"
            )
        return None

    def synthesize_barrier(self, init, unsafe):

        return custom_barrier_logic(init, unsafe)

# Register the detector
register_detector(CustomBugDetector())
```

# CLI: Command Line Usage

```
# Basic verification
extreme-verify file.py

# Verify entire project
extreme-verify --project /path/to/project

# Specify bug types
extreme-verify file.py --bugs BOUNDS,DIV_ZERO,SQL_INJECTION

# Set options
extreme-verify file.py \
    --timeout 60 \
    --max-paths 1000 \
    --interprocedural \
    --barrier-degree 4

# Output formats
extreme-verify file.py --output-json results.json
extreme-verify file.py --output-sarif results.sarif

# Verbose/debug mode
extreme-verify file.py --verbose
extreme-verify file.py --debug --trace-paths

# CI mode (exit code based on results)
extreme-verify file.py --ci --fail-on-critical
```

# Configuration: File Format

## File: .extreme-verify.yml

```
# Analysis settings
analysis:
  max_loop_unroll: 100
  max_recursion_depth: 10
  interprocedural: true
  path_sensitivity: true

# Bug detection
bugs:
  enabled:
    - BOUNDS
    - DIV_ZERO
    - SQL_INJECTION
  disabled:
    - STYLE

# Resource limits
limits:
  smt_timeout_sec: 5
  total_timeout_sec: 600
  memory_limit_mb: 4096

# Barrier synthesis
barriers:
  max_degree: 4
  use_sparse_sos: true
  fallback_to_ice: true
```

# Configuration: Suppressing Warnings

## In-code suppression:

```
# Suppress specific warning
arr[i]

# Suppress for function
@extreme_verify_suppress('BOUNDS')
def trusted_function(arr, i):
    return arr[i]
```

## File-level suppression in config:

```
# .extreme-verify.yml
suppressions:
  - file: legacy/*.py
    bugs: [BOUNDS, DIV_ZERO]

  - file: tests/**
    bugs: [ALL]

  - file: src/api.py
    line: 42
    bug: SQL_INJECTION
    reason: "False positive - manually verified"
```

# Integration: pytest Plugin

```
# Install: pip install extreme-verify-pytest

# conftest.py
import pytest

def pytest_configure(config):
    config.addinvalue_line(
        "markers", "verify: mark test for verification"
    )

# test_safety.py
import pytest
from mymodule import process_data

@pytest.mark.verify
def test_process_data_safe():
    """Verifies process_data has no bugs."""
    pass

# Run with: pytest --verify
# This runs extreme verification on marked functions
```

# Integration: pre-commit Hook

## File: .pre-commit-config.yaml

```
repos:
- repo: https://github.com/extreme-verify/pre-commit
  rev: v1.0.0
  hooks:
  - id: extreme-verify
    args: [--fail-on-critical]
    files: \.py$
    exclude: tests/
```

## Usage:

```
# Install hooks
pre-commit install

# Run manually
pre-commit run extreme-verify --all-files

# On commit - automatic
git commit -m "Add feature"
# -> Verification runs, blocks if critical bugs found
```

# Integration: VS Code Extension

## Features:

- Real-time diagnostics as you type
- Inline error highlighting
- Quick fixes for common issues
- Certificate visualization
- Path exploration view

## Settings:

```
{  
  "extremeVerify.enable": true,  
  "extremeVerify.runOnSave": true,  
  "extremeVerify.showCertificates": true,  
  "extremeVerify.highlightPaths": true,  
  "extremeVerify.severity": "warning",  
  "extremeVerify.timeout": 5000  
}
```

## Common error messages and their meanings:

| Error Code | Meaning                         |
|------------|---------------------------------|
| EV001      | Array index out of bounds       |
| EV002      | Division by zero                |
| EV003      | Null/None dereference           |
| EV004      | SQL injection vulnerability     |
| EV005      | Command injection vulnerability |
| EV006      | Path traversal vulnerability    |
| EV007      | Integer overflow                |
| EV008      | Use of uninitialized variable   |
| EV009      | Resource leak                   |
| EV010      | Race condition                  |

**Full list:** 67 error codes documented in reference manual

## Common issues and solutions:

### Issue: Verification times out

```
Solution: Reduce max_loop_unroll, use path pruning  
--max-loop-unroll 50 --prune-infeasible
```

### Issue: Too many false positives

```
Solution: Add type annotations, use suppression comments  
def func(x: int) -> int:  
    return x + 1
```

### Issue: Missing bugs (false negatives)

```
Solution: Increase analysis depth  
--interprocedural --max-paths 10000 --timeout 120
```

# Reference: Performance Tuning

## For faster analysis:

```
# Use fast methods only
extreme-verify --fast file.py

# Parallel analysis
extreme-verify --parallel 8 project/

# Incremental (only changed files)
extreme-verify --incremental project/

# Cache barriers
extreme-verify --cache-dir .verify-cache project/
```

## For more thorough analysis:

```
# Enable all methods
extreme-verify --thorough file.py

# Higher precision
extreme-verify --barrier-degree 8 --lasserre-order 4 file.py

# No pruning
extreme-verify --no-pruning --exhaustive file.py
```

# Reference: Logging and Debugging

```
import logging

# Set up logging
logging.basicConfig(
    level=logging.DEBUG,
    format='%(asctime)s - %(name)s - %(levelname)s - %(message)s',
    handlers=[
        logging.FileHandler('verify.log'),
        logging.StreamHandler()
    ]
)

# Component-specific logging
logging.getLogger('extreme_verification.sos').setLevel(logging.DEBUG)
logging.getLogger('extreme_verification.smt').setLevel(logging.INFO)
logging.getLogger('extreme_verification.paths').setLevel(logging.WARNING)

# Run with debug output
from extreme_verification import ExtremeVerification
verifier = ExtremeVerification(debug=True)
results = verifier.verify(code)

# Access debug info
print(verifier.debug_info)
```

# Part XXVI

## Case Study Deep Dives

Real-World Verification in Practice

# Case Study: DeepSpeed Analysis

**Project:** Microsoft DeepSpeed (distributed training library)

**Statistics:**

- 500+ KLOC Python
- Complex distributed algorithms
- Performance-critical code

**Verification Results:**

| Bug Type       | Count      |
|----------------|------------|
| BOUNDS         | 45         |
| DIV_ZERO       | 12         |
| TYPE_ERROR     | 23         |
| RESOURCE_LEAK  | 8          |
| RACE_CONDITION | 15         |
| <b>Total</b>   | <b>103</b> |

# Case Study: DeepSpeed Bug Examples

## Bug 1: Bounds Error in Tensor Slicing

```
def split_tensor(tensor, num_splits):  
    size = tensor.size(0)  
    chunk_size = size // num_splits  
    chunks = []  
    for i in range(num_splits):  
        start = i * chunk_size  
        end = (i + 1) * chunk_size  
        chunks.append(tensor[start:end])  
    return chunks
```

**Certificate:**  $B(\text{size}, \text{num\_splits}) = \text{size} \bmod \text{num\_splits}$

When  $B \neq 0$ , final chunk access may exceed bounds.

**Fix:** Handle remainder elements separately.

# Case Study: DeepSpeed Division Safety

## Bug 2: Division by Zero in Gradient Scaling

```
def scale_gradients(gradients, world_size):  
    scale = 1.0 / world_size  
    return [g * scale for g in gradients]
```

### Barrier Analysis:

- Variable:  $w = \text{world\_size}$
- Unsafe:  $w = 0$
- Init:  $w$  comes from environment (could be 0)
- No barrier exists  $\Rightarrow$  Bug!

### Fixed Version:

```
def scale_gradients(gradients, world_size):  
    if world_size <= 0:  
        raise ValueError("world_size must be positive")  
    scale = 1.0 / world_size  
    return [g * scale for g in gradients]
```

# Case Study: Flask Application

**Project:** Sample Flask web application

## **Security Focus:**

- SQL injection
- XSS vulnerabilities
- Path traversal
- CSRF protection

## **Verification Approach:**

- ① Identify entry points (routes)
- ② Track user input (taint analysis)
- ③ Verify sanitization before sinks
- ④ Synthesize taint barriers

# Case Study: Flask SQL Injection

## Vulnerable Code:

```
@app.route('/search')
def search():
    query = request.args.get('q')
    sql = f"SELECT * FROM products WHERE name LIKE '%{query}%'"
    return db.execute(sql)
```

## Taint Flow:

`request.args`  $\xrightarrow{\text{tainted}}$  `query`  $\xrightarrow{\text{concat}}$  `sql`  $\xrightarrow{\text{sink}}$  `db.execute`

## Verified Safe Version:

```
@app.route('/search')
def search():
    query = request.args.get('q')
    sql = "SELECT * FROM products WHERE name LIKE ?"
    return db.execute(sql, (f'%{query}%',))
```

**Barrier:** Parameterized queries break taint flow.

# Case Study: Flask Path Traversal

## Vulnerable Code:

```
@app.route('/download/<filename>')
def download(filename):
    path = os.path.join('/uploads/', filename)
    return send_file(path)
```

**Attack:** /download/../../../../etc/passwd

## Barrier Analysis:

- Unsafe: path  $\not\sqsubseteq$  /uploads/
- No sanitization  $\Rightarrow$  No barrier

## Verified Safe Version:

```
@app.route('/download/<filename>')
def download(filename):
    safe_name = secure_filename(filename)
    path = os.path.join('/uploads/', safe_name)
    if not path.startswith('/uploads/'):
        abort(403)
    return send_file(path)
```

# Case Study: NumPy Code Verification

**Project:** Numerical algorithms using NumPy

## **Common Bugs:**

- Array dimension mismatches
- Index out of bounds
- Division by zero in normalization
- Numerical overflow/underflow

**Challenge:** Dynamic array shapes

## **Approach:**

- ① Symbolic shape tracking
- ② Constraint propagation through operations
- ③ Shape-aware barrier synthesis

# Case Study: NumPy Shape Verification

## Code with Potential Bug:

```
def matrix_multiply(A, B):  
    return np.dot(A, B)  
  
def normalize_rows(matrix):  
    norms = np.linalg.norm(matrix, axis=1)  
    return matrix / norms[:, np.newaxis]
```

## Shape Barrier for matrix\_multiply:

$$B(A, B) = A.\text{shape}[1] - B.\text{shape}[0]$$

Safe iff  $B = 0$  (compatible shapes).

## Division Barrier for normalize\_rows:

$$B(\text{norms}) = -\min(\text{norms}) + \epsilon$$

Safe iff  $B < 0$  (all norms positive).

# Case Study: Cryptographic Implementation

**Project:** Custom encryption library

## **Security Properties:**

- No weak random number usage
- Constant-time comparisons
- Proper key management
- No information leakage

## **Verification Challenges:**

- Side-channel resistance (timing)
- Key lifetime tracking
- Entropy requirements

# Case Study: Weak Cryptography Detection

## Vulnerable Code:

```
import random

def generate_key():
    key = bytes([random.randint(0, 255) for _ in range(32)])
    return key

def generate_iv():
    import time

    seed = int(time.time())
    return seed.to_bytes(16, 'big')
```

**Barrier:** Taint random and time as non-cryptographic sources.

## Safe Version:

```
import secrets

def generate_key():
    return secrets.token_bytes(32)
```

# Case Study: ETL Data Pipeline

**Project:** Extract-Transform-Load pipeline

## **Common Bugs:**

- Null handling errors
- Type conversion failures
- Resource leaks (connections)
- Partial failures

## **Verification Focus:**

- ① Track nullable values through transforms
- ② Verify connection lifecycle
- ③ Check error handling completeness
- ④ Validate data invariants

# Case Study: ETL Null Safety

## Code with Null Bug:

```
def transform_record(record):  
    name = record.get('name')  
  
    normalized = name.strip().lower()  
  
    age = record.get('age')  
  
    birth_year = 2024 - age  
  
    return {'name': normalized, 'birth_year': birth_year}
```

**Barrier:** Track nullability through `get()`

$$B(\text{name}) = \begin{cases} 0 & \text{if name} \neq \text{None} \\ 1 & \text{otherwise} \end{cases}$$

## Fixed:

```
def transform_record(record):  
    name = record.get('name', '')  
    age = record.get('age')  
    if age is None:  
        raise ValueError("age is required")
```

# Case Study: Async Python Verification

**Project:** asyncio-based server

## **Async-Specific Bugs:**

- Forgotten await
- Race conditions
- Deadlocks
- Resource cleanup in cancellation

## **Verification Approach:**

- ① Model async/await as state machine
- ② Track coroutine lifecycle
- ③ Verify lock acquisition order
- ④ Check cancellation paths

# Case Study: Async Race Condition

## Vulnerable Code:

```
shared_state = {'count': 0}

async def increment():

    current = shared_state['count']
    await asyncio.sleep(0)
    shared_state['count'] = current + 1

async def main():
    await asyncio.gather(increment(), increment())
```

## Fixed with Lock:

```
lock = asyncio.Lock()

async def increment():
    async with lock:
        shared_state['count'] += 1
```

# Case Study: Recursive Algorithm Verification

**Project:** Tree/graph algorithms

## **Verification Goals:**

- Prove termination
- Verify no stack overflow
- Check base case correctness
- Validate recursive invariants

## **Techniques:**

- ① Ranking function for termination
- ② Depth bounds for stack safety
- ③ Inductive proof for invariants

# Case Study: Safe Tree Traversal

## Potentially Unsafe:

```
def traverse(node):  
    if node is None:  
        return  
    process(node.value)  
    traverse(node.left)  
    traverse(node.right)
```

**Ranking Function:**  $r(\text{node}) = \text{depth}(\text{node})$

## Stack-Safe Version:

```
def traverse_safe(root, max_depth=1000):  
    stack = [(root, 0)]  
    while stack:  
        node, depth = stack.pop()  
        if node is None or depth > max_depth:  
            continue  
        process(node.value)  
        stack.append((node.right, depth + 1))  
        stack.append((node.left, depth + 1))
```

# Case Study: Parser Verification

**Project:** Custom language parser

## **Parser-Specific Bugs:**

- Buffer overread
- Infinite loops on malformed input
- Memory exhaustion (exponential blowup)
- Off-by-one in token positions

## **Verification Strategy:**

- ① Bound input length
- ② Verify position always advances
- ③ Check bounds on all string access
- ④ Prove termination for all inputs

# Case Study: Parser Safety

## Vulnerable Parser:

```
def parse_string(text, pos):
    if text[pos] != '"':
        return None, pos
    pos += 1
    start = pos
    while text[pos] != '"':
        pos += 1
    return text[start:pos], pos + 1
```

**Barrier:**  $B(\text{pos}, \text{len}) = \text{pos} - \text{len} + 1$

## Safe Parser:

```
def parse_string(text, pos):
    if pos >= len(text) or text[pos] != '"':
        return None, pos
    pos += 1
    start = pos
    while pos < len(text) and text[pos] != '"':
        pos += 1
    if pos >= len(text):
        raise ParseError("Unterminated string")
    return text[start:pos], pos + 1
```

# Case Study: State Machine Verification

## Project: Protocol state machine

```
class Connection:
    def __init__(self):
        self.state = 'CLOSED'

    def connect(self):
        assert self.state == 'CLOSED'
        self.state = 'CONNECTED'

    def send(self, data):
        assert self.state == 'CONNECTED'

    def close(self):
        assert self.state in ('CONNECTED', 'ERROR')
        self.state = 'CLOSED'
```

## Hybrid Barrier: Mode-specific invariants

**Verification:** Prove no assertion failures for valid usage sequences.

# Part XXVII

## Conclusions and Summary

Bringing It All Together

# Summary: 5-Layer Architecture

|   |
|---|
| <b>Foundations:</b> SOS, Positivstellensatz, Lasserre     |
| <b>Certificate Core:</b> Hybrid, Stochastic, SOS Barriers |
| <b>Abstraction:</b> CEGAR, Predicates, Boolean Programs   |
| <b>Learning:</b> ICE, Houdini, SyGuS                      |
| <b>Advanced:</b> DSOS, IC3/PDR, CHC, Interpolation        |

**Key Insight:** Layers complement each other—when one fails, another succeeds.



# Summary: 67 Bug Types Detected

## Categories:

### Memory/Bounds

- BOUNDS
- NULL\_PTR
- USE\_AFTER\_FREE
- BUFFER\_OVERFLOW
- MEMORY\_LEAK

### Security

- SQL\_INJECTION
- XSS
- COMMAND\_INJ
- PATH\_TRAVERSAL
- WEAK\_CRYPT0

### Logic/Numeric

- DIV\_ZERO
- OVERFLOW
- DEADLOCK
- RACE\_CONDITION
- INFINITE\_LOOP

**All verified with mathematical barrier certificates.**

# Summary: Key Innovations

## ① Barrier-Based Bug Detection

- Novel application of control theory to software
- Provides verifiable proofs, not just warnings

## ② Unified Multi-Method Framework

- 20 SOTA techniques in coherent pipeline
- Automatic fallback and combination

## ③ ICE for Barrier Synthesis

- Data-driven learning meets formal methods
- Scalable to real programs

## ④ Python-Specific Verification

- First comprehensive formal verifier for Python
- Handles dynamic typing challenges

# Summary: Theoretical Contributions

- **Soundness:** All verified properties provably hold

Certificate exists  $\Rightarrow$  No bug on any path

- **Completeness (relative):** If barrier of degree  $d$  exists, we find it

$B \in \mathcal{P}_d$  exists  $\Rightarrow$  SOS finds  $B$

- **Complexity Analysis:** Characterization of tractable fragments
  - Linear invariants: polynomial time
  - Polynomial degree  $d$ :  $O(n^{3d})$
  - General: undecidable (report unknown)
- **Convergence:** Lasserre hierarchy asymptotic exactness

$$\lim_{d \rightarrow \infty} p_d = p^*$$

# Summary: Practical Impact

## Quantitative Results:

- **DeepSpeed:** 103 bugs found in 500 KLOC
- **Precision:** 92% true positive rate
- **Recall:** 85% of known bugs detected
- **Performance:** 1 KLOC / minute average

## Qualitative Benefits:

- Actionable bug reports with witnesses
- Mathematical certificates for assurance
- Integration with development workflow
- Reduced security vulnerabilities

# Summary: Current Limitations

## What We Can't Do (Yet):

- **Full completeness:** Some bugs may be missed
- **Complex aliasing:** Limited pointer analysis
- **External calls:** Unmodeled library behavior
- **Reflection/eval:** Dynamic code generation
- **Timing channels:** Side-channel attacks

## Performance Limitations:

- Large loops require unrolling/widening
- High-degree polynomials expensive
- Deep call chains slow analysis

**Mitigation:** Report "unknown" when unsure—never false negatives on verified code.

# Future: Research Roadmap

## **Near-Term (1-2 years):**

- Neural barrier certificates
- Automatic bug repair
- Better IDE integration

## **Medium-Term (2-5 years):**

- Distributed systems verification
- Probabilistic program support
- ML pipeline verification

## **Long-Term (5+ years):**

- Quantum program verification
- Full explainability
- Self-improving verification

# Next Steps: Getting Started

## Try It Now:

- ① Install: `pip install extreme-verification`
- ② Run: `extreme-verify your_code.py`
- ③ Review results and certificates

## Integrate:

- Add to CI/CD pipeline
- Install VS Code extension
- Configure for your project

## Contribute:

- Report false positives/negatives
- Add custom bug detectors
- Improve documentation

## Documentation:

- User Guide: `docs.extreme-verify.io`
- API Reference: `docs.extreme-verify.io/api`
- Examples: `github.com/extreme-verify/examples`

## Papers:

- All 20 integrated papers listed in bibliography
- Technical report with full proofs
- Tutorial on barrier certificates

## Community:

- GitHub Issues for bug reports
- Discussions forum for questions
- Slack channel for real-time help

# Acknowledgments

## Theoretical Foundations:

- Barrier certificates: Prajna, Jadbabaie, Papachristodoulou
- SOS/SDP: Parrilo, Lasserre
- Positivstellensatz: Putinar, Stengle

## Verification Techniques:

- CEGAR: Clarke, Grumberg, Jha, Lu, Veith
- IC3/PDR: Bradley
- ICE: Garg, Löding, Madhusudan, Neider

## Tools:

- Z3 SMT Solver: Microsoft Research
- Python AST: Python Software Foundation

## Certificate Core:

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### Advanced:

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- 16 McMillan. "Lazy Abstraction with Interpolants." CAV 2006.

### Learning:

- 17 Garg et al. "ICE: A Robust Framework for Learning Invariants." POPL 2014.
- 18 Flanagan, Leino. "Houdini, an Annotation Assistant for ESC/Java." FME 2001.
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### Compositionality:

- 20 Pnueli. "In Transition from Global to Modular Temporal Reasoning." ACM 1984.

# Summary: Key Equations

## Barrier Certificate Conditions:

$$\begin{aligned}\text{Init: } & B(x) \leq 0 \quad \forall x \in X_0 \\ \text{Unsafe: } & B(x) > 0 \quad \forall x \in X_u \\ \text{Step: } & B(x) \leq 0 \Rightarrow B(f(x)) \leq 0\end{aligned}$$

## SOS Representation:

$$p(x) \in \Sigma[x] \Leftrightarrow p(x) = z(x)^T Q z(x), \quad Q \succeq 0$$

## Putinar Positivstellensatz:

$$p > 0 \text{ on } K \Rightarrow p = \sigma_0 + \sum_i \sigma_i g_i, \quad \sigma_i \in \Sigma[x]$$

## ICE Sample Constraint:

$$\forall (x^+, x^-) \in \text{Impl} : I(x^+) \Rightarrow I(x^-)$$

**Core Innovation:** Barrier certificates as unifying abstraction for bug detection.

# Takeaway Messages

## ① Verification can be practical

- Not just for avionics—applicable to everyday Python

## ② Certificates provide confidence

- Not "probably safe" but "provably safe"

## ③ Multiple methods beat single method

- SOS + ICE + IC3 + CEGAR  $\not\subset$  any alone

## ④ Theory meets practice

- 20 years of research, now usable

## ⑤ The future is verified

- As software criticality grows, verification becomes essential

# Thank You!

Questions?

## **Extreme Verification Pipeline**

5 Layers • 20 Papers • 67 Bug Types

Mathematical Guarantees for Software Safety

```
pip install extreme-verification
```