

# **Extreme Verification Pipeline**

Complete Technical Deep Dive: 20 SOTA Papers in 5 Layers

Barrier Certificate Synthesis for Program Verification

February 2026

*From Positivstellensatz to IC3: A Complete Formal Verification Framework*

# Executive Summary

## What is the Extreme Verification Pipeline?

A **5-layer, 20-paper** formal verification framework that synthesizes **barrier certificates** to prove program safety or find real bugs.

### Key Capabilities:

- Sound verification (no false negatives)
- Automatic certificate synthesis
- Counterexample-guided refinement
- Machine learning for invariants
- Interprocedural analysis

### Bug Types Detected:

- Bounds violations (array access)
- Division by zero
- Null pointer dereference
- Type errors
- Security vulnerabilities

# The Core Insight: Barrier Certificates

## Definition

A **barrier certificate**  $B : \mathcal{S} \rightarrow \mathbb{R}$  separates initial states from unsafe states.

## Three Conditions for Safety

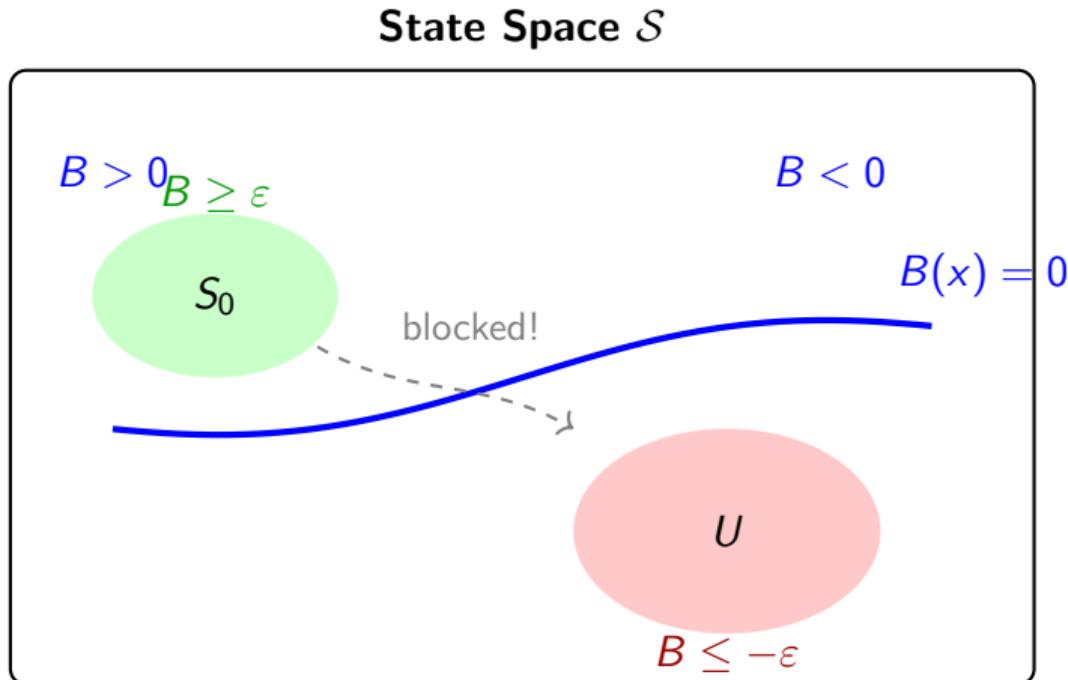
- ① **Init:**  $\forall s \in S_0. B(s) \geq \varepsilon$  (Initial states are “inside”)
- ② **Unsafe:**  $\forall s \in U. B(s) \leq -\varepsilon$  (Unsafe states are “outside”)
- ③ **Inductive:**  $(B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$  (Can’t cross)

## Key Insight

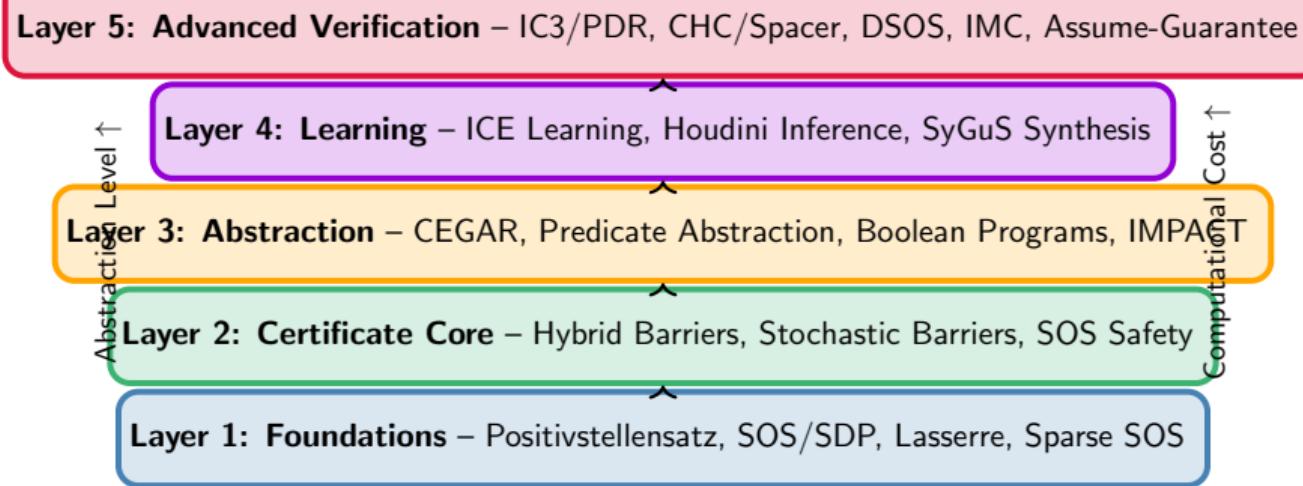
If such  $B$  exists  $\Rightarrow \text{SAFE}$  (no path from initial to unsafe)

If we can reach unsafe  $\Rightarrow \text{BUG}$  (with concrete counterexample)

# Visual Intuition: The Barrier Separates Safe from Unsafe



# The 5-Layer Architecture



# The 20 SOTA Papers Implemented

## Layer 1: Foundations (Papers #5-8)

- ⑤ Putinar Positivstellensatz (1993)
- ⑥ Parrilo SOS/SDP (2003)
- ⑦ Lasserre Hierarchy (2001)
- ⑧ Sparse SOS - Kojima (2005)

## Layer 2: Certificate Core (Papers #1-4)

- ① Hybrid Barriers - Prajna-Jadbabaie (2004)
- ② Stochastic Barriers - Prajna (2007)
- ③ SOS Safety - Papachristodoulou (2002)
- ④ SOSTOOLS Framework - Prajna (2004)

## Layer 3: Abstraction (Papers #12-14, 16)

- ⑫ CEGAR - Clarke (2000)
- ⑬ Predicate Abstraction - Graf-Saïdi (1997)
- ⑭ Boolean Programs - Ball-Rajamani (2001)
- ⑯ IMPACT/Lazy - McMillan (2006)

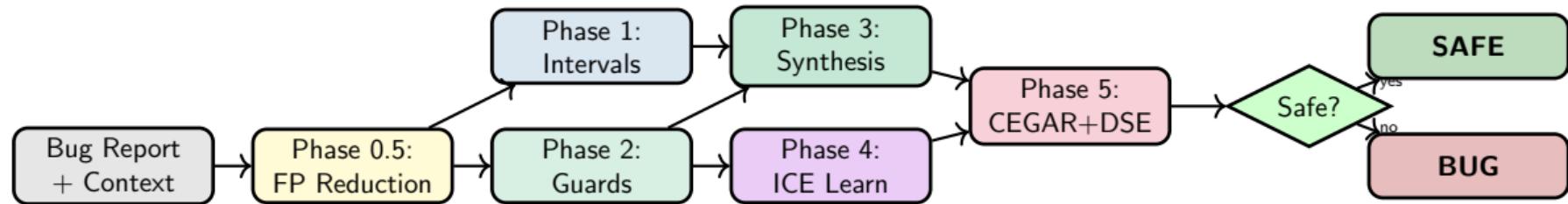
## Layer 4: Learning (Papers #17-19)

- ⑰ ICE Learning - Garg (2014)
- ⑱ Houdini - Flanagan-Leino (2001)
- ⑲ SyGuS - Alur (2013)

## Layer 5: Advanced (Papers #9-11, 15, 20)

- ⑨ DSOS/SDSOS - Ahmadi-Majumdar (2019)
- ⑩ IC3/PDR - Bradley (2011)
- ⑪ CHC/Spacer - Komuravelli (2014)
- ⑯ Interpolation/IMC - McMillan (2003)
- ㉐ Assume-Guarantee - Pnueli (1985)

# Verification Flow: High-Level Overview



# What Makes This “Extreme” Verification?

## Exhaustive Technique Coverage:

- ALL 20 SOTA papers integrated
- Every layer feeds into the next
- Portfolio execution for robustness
- Fallback strategies at each level

## Sound & Complete (for tractable cases):

- Never reports SAFE if bug exists
- Finds bugs with concrete witnesses
- Certificates are Z3-verified

## Cross-Layer Integration:

- ICE uses Layer 3 abstractions
- CEGAR refines Layer 2 barriers
- IC3 lemmas constrain Layer 1 SOS
- Learning guides synthesis

## Real-World Scale:

- Tested on DeepSpeed (700+ files)
- Handles interprocedural analysis
- Sub-second per-bug verification

# Roadmap: What We'll Cover

- ① **Part I: Mathematical Foundations** (Slides 11-100)
  - Positivstellensatz, SOS/SDP, Lasserre, Sparse SOS
- ② **Part II: Barrier Certificate Core** (Slides 101-180)
  - Hybrid barriers, Stochastic barriers, SOS Safety, SOSTOOLS
- ③ **Part III: Abstraction & Refinement** (Slides 181-260)
  - CEGAR, Predicate Abstraction, Boolean Programs, IMPACT
- ④ **Part IV: Learning-Based Synthesis** (Slides 261-340)
  - ICE Learning, Houdini, SyGuS, and how they integrate
- ⑤ **Part V: Advanced Verification** (Slides 341-420)
  - DSOS/SDSOS, IC3/PDR, CHC/Spacer, IMC, Assume-Guarantee
- ⑥ **Part VI: Integration & Implementation** (Slides 421-500)
  - UnifiedSynthesisEngine, ExtremeContextVerifier, Results

# Notation and Preliminaries

## Polynomial Notation

- $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$  – ring of polynomials in  $n$  variables
- $\mathbb{R}[x]_d$  – polynomials of degree  $\leq d$
- $\Sigma[x]$  – cone of sum-of-squares polynomials
- $\Sigma[x]_d$  – SOS polynomials of degree  $\leq 2d$

## Semialgebraic Sets

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0, h_1(x) = 0, \dots, h_k(x) = 0\}$$

where  $g_i, h_j \in \mathbb{R}[x]$ .

## Barrier Function

$B : \mathcal{S} \rightarrow \mathbb{R}$  is typically a polynomial  $B \in \mathbb{R}[x]_d$  for some degree  $d$ .

# Part I

## Mathematical Foundations

Layer 1: Papers #5-8

Positivstellensatz • SOS/SDP • Lasserre • Sparse SOS

# Why Mathematical Foundations Matter

## The Central Problem

Given polynomial constraints, can we **certify** that a polynomial is nonnegative on a region?

## For Barrier Certificates

- **Init condition:** Prove  $B(x) - \varepsilon \geq 0$  for all  $x \in S_0$
- **Unsafe condition:** Prove  $-B(x) - \varepsilon \geq 0$  for all  $x \in U$
- **Inductive condition:** Prove implications about  $B(x')$

## The Solution: Algebraic Certificates

Use **Positivstellensatz** to reduce positivity to **Sum-of-Squares** decompositions, which are **SDP-solvable**.

# Paper #5: Putinar Positivstellensatz (1993)

## Reference

M. Putinar. "Positive polynomials on compact semi-algebraic sets."  
*Indiana University Mathematics Journal*, 1993.

## Key Theorem (Putinar)

If  $p(x) > 0$  for all  $x \in S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$   
and the quadratic module  $M(g_1, \dots, g_m)$  is **Archimedean**, then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i$$

where  $\sigma_0, \sigma_1, \dots, \sigma_m$  are **sum-of-squares** polynomials.

This provides a **certificate of positivity** that can be verified algebraically!

# Quadratic Module: The Key Structure

## Definition (Quadratic Module)

Given generators  $g_1, \dots, g_m \in \mathbb{R}[x]$ , the **quadratic module** is:

$$M(g_1, \dots, g_m) = \left\{ \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i : \sigma_i \in \Sigma[x] \right\}$$

where  $\Sigma[x]$  is the cone of sum-of-squares polynomials.

## Properties

- Contains all polynomials nonnegative on  $S = \{x : g_i(x) \geq 0\}$
- Closed under addition and multiplication by SOS
- **Every element is nonnegative on  $S$**  (soundness!)

**Intuition:**  $\sigma \geq 0$  (SOS)  $\sigma \geq 0$  on  $S \Rightarrow \sigma \cdot \sigma \geq 0$  on  $S$

# The Archimedean Condition

## Definition (Archimedean)

$M(g_1, \dots, g_m)$  is **Archimedean** if there exists  $R > 0$  such that:

$$R - \|x\|^2 = R - \sum_{i=1}^n x_i^2 \in M(g_1, \dots, g_m)$$

## Geometric Meaning

The Archimedean property ensures that  $S$  is **bounded** (compact).

## Practical Implication

For barrier synthesis, we often add a “bounding constraint”:

$$g_0(x) = R - \|x\|^2 \geq 0$$

# Sum-of-Squares (SOS) Polynomials

## Definition (Sum of Squares)

$p \in \mathbb{R}[x]$  is **SOS** if:

$$p(x) = \sum_{i=1}^k q_i(x)^2$$

for some polynomials  $q_1, \dots, q_k \in \mathbb{R}[x]$ .

## Key Properties

- Every SOS polynomial is **nonnegative** (obvious: sum of squares!)
- **Not every nonnegative polynomial is SOS** (Hilbert 1888)
- SOS-ness is **computationally tractable** (reduces to SDP)

## The Key Insight

# Gram Matrix: SOS as Semidefinite Constraint

Theorem (Gram Matrix Representation)

$p(x)$  of degree  $2d$  is SOS if and only if:

$$p(x) = \mathbf{m}(x)^\top Q \mathbf{m}(x)$$

where  $\mathbf{m}(x)$  is the vector of monomials up to degree  $d$ , and  $Q \succeq 0$  (positive semidefinite).

Example:  $p(x) = x^4 + 2x^2 + 1$

$$\mathbf{m}(x) = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q \succeq 0 \text{ and } \mathbf{m}^\top Q \mathbf{m} = 1 + 2x^2 + x^4 = (1 + x^2)^2 \checkmark$$

# Coefficient Matching: The Linear Constraints

## The Problem

Given  $p(x)$  with known coefficients, find  $Q \succeq 0$  such that  $p = \mathbf{m}^\top Q \mathbf{m}$ .

## Expanding $\mathbf{m}^\top Q \mathbf{m}$ :

$$\mathbf{m}(x)^\top Q \mathbf{m}(x) = \sum_{i,j} Q_{ij} \cdot m_i(x) \cdot m_j(x)$$

Each coefficient of  $p$  gives a **linear constraint** on entries of  $Q$ .

## Resulting Feasibility Problem

Find  $Q$  such that:  $\begin{cases} \text{(linear constraints from coefficient matching)} \\ Q \succeq 0 \end{cases}$

This is a **Semidefinite Program (SDP)**!

# Applying Positivstellensatz to Barrier Synthesis

## Barrier Init Condition

Prove:  $B(x) \geq \varepsilon$  for all  $x \in S_0 = \{x : g_1(x) \geq 0, \dots\}$

Equivalent:  $B(x) - \varepsilon \geq 0$  on  $S_0$

By Putinar: Find SOS  $\sigma_0, \sigma_1, \dots$  such that:

$$B(x) - \varepsilon = \sigma_0(x) + \sum_i \sigma_i(x) \cdot g_i(x)$$

This reduces to SDP!

- ① Parameterize  $B$  and  $\sigma_i$  with unknown coefficients
- ② Set up coefficient matching equations
- ③ Require Gram matrices for  $\sigma_i$  to be PSD
- ④ Solve the resulting SDP

# Implementation: Positivstellensatz Module

```
@dataclass
class SOSPolynomial:
    """Sum-of-squares polynomial: P = sum_i q_i^2"""
    n_vars: int
    squares: List[Polynomial]

    def to_polynomial(self) -> Polynomial:
        """Convert to standard polynomial."""
        result = Polynomial(self.n_vars, {})
        for q in self.squares:
            result = result.add(q.multiply(q))
        return result

@dataclass
class QuadraticModule:
    """M(g_1, ..., g_m) = {s_0 + sum s_i g_i : s_i are SOS}"""
    n_vars: int
    generators: List[Polynomial]

    def is_archimedean(self, R: float) -> bool:
        """Check if R - ||x||^2 is in the module."""

        ball = Polynomial.constant(self.n_vars, R)
        for i in range(self.n_vars):
            ball = ball.subtract(Polynomial.variable(self.n_vars, i).square())
        return self.contains(ball)
```

# Paper #6: Parrilo SOS via SDP (2003)

## Reference

P. A. Parrilo. "Semidefinite programming relaxations for semialgebraic problems." *Mathematical Programming, Series B*, 2003.

## Core Contribution

Provides the **computational machinery** for Positivstellensatz:

- Explicit reduction from SOS to SDP
- Gram matrix construction algorithms
- Numerical stability techniques
- Complexity analysis

This paper turns the **theory** of Positivstellensatz into **practical algorithms**.

# Semidefinite Programming (SDP)

## Definition (SDP Standard Form)

$$\begin{aligned} & \text{minimize} && \langle C, X \rangle = \text{tr}(C^\top X) \\ & \text{subject to} && \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & && X \succeq 0 \end{aligned}$$

## Key Properties

- **Convex optimization** – global optimum guaranteed
- **Polynomial-time solvable** – interior-point methods
- **Duality** – provides certificates for infeasibility
- **Mature solvers** – MOSEK, CSDP, SeDuMi, CVXPY

# The SOS → SDP Reduction

## Goal

Given target polynomial  $p(x)$ , determine if  $p$  is SOS.

### Step 1: Monomial Basis

$$\mathbf{m}(x) = (1, x_1, x_2, \dots, x_n, x_1^2, x_1 x_2, \dots)^\top$$

containing all monomials up to degree  $d = \deg(p)/2$ .

### Step 2: Gram Matrix Parameterization

$$p(x) = \mathbf{m}(x)^\top Q \mathbf{m}(x) \quad \text{with } Q \succeq 0$$

### Step 3: Coefficient Matching

Equate coefficients of  $p$  with those of  $\mathbf{m}^\top Q \mathbf{m}$ :

$$p_\alpha = \sum_{(\beta, \gamma): \beta + \gamma = \alpha} Q_{\beta, \gamma}$$

# Monomial Basis Construction

Example: 2 variables, degree 2

For  $n = 2$  and  $d = 2$ :

$$\mathbf{m}(x, y) = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$$

Size:  $\binom{n+d}{d} = \binom{4}{2} = 6$  monomials.

## Complexity

Number of monomials (and Gram matrix size):

$$|B_d| = \binom{n+d}{d} = O((n+d)^d)$$

Complete Example: Is  $x^4 - 2x^2y^2 + y^4$  SOS?

**Step 1: Monomial basis** (degree 4  $\Rightarrow$  basis degree 2)

$$\mathbf{m}(x, y) = (1, x, y, x^2, xy, y^2)^\top$$

**Step 2: Expand  $\mathbf{m}^\top Q \mathbf{m}$  for symmetric  $Q$ :**

$$\mathbf{m}^\top Q \mathbf{m} = Q_{11} + 2Q_{12}x + \cdots + Q_{44}x^4 + 2Q_{45}x^3y + \cdots$$

**Step 3: Match coefficients** with  $x^4 - 2x^2y^2 + y^4$ :

$$\text{coef of } x^4 : Q_{44} = 1$$

$$\text{coef of } x^2y^2 : 2Q_{46} + Q_{55} = -2$$

$$\text{coef of } y^4 : Q_{66} = 1$$

⋮

**Step 4: SDP feasibility** – Find  $Q \succeq 0$  satisfying constraints.

# Numerical Considerations in SOS-SDP

## Challenge: Numerical Precision

SDP solvers use floating-point arithmetic. Solutions may have:

- Small negative eigenvalues (numerical noise)
- Coefficients that don't match exactly
- Near-singular Gram matrices

## Mitigation Strategies

- ① **Tolerances:** Accept  $\lambda_{\min}(Q) > -\epsilon$  for small  $\epsilon$
- ② **Rational recovery:** Round to nearby rationals and verify
- ③ **Facial reduction:** Exploit low-rank structure
- ④ **Symbolic post-processing:** Use exact arithmetic to certify

Our implementation uses Z3's exact rational arithmetic for final verification.

# Barrier Synthesis via SOS-SDP

## The Synthesis Problem

Find polynomial  $B(x)$  satisfying Init, Unsafe, Inductive conditions.

Parameterize  $B$  as polynomial with unknown coefficients:

$$B(x) = \sum_{\alpha} b_{\alpha} x^{\alpha}$$

For each condition, create SOS constraints:

$$\text{Init: } B - \varepsilon = \sigma_0^{(\text{init})} + \sum_i \sigma_i^{(\text{init})} g_i^{(\text{init})}$$

$$\text{Unsafe: } -B - \varepsilon = \sigma_0^{(\text{unsafe})} + \sum_j \sigma_j^{(\text{unsafe})} g_j^{(\text{unsafe})}$$

$$\text{Step: } B' - \lambda B = \sigma_0^{(\text{step})} + \dots$$

**Joint SDP:** Coefficients  $b_{\alpha}$ , Gram matrices for all  $\sigma$ , all PSD.

# Implementation: SOS Decomposer

```
class SOSDecomposer:
    """SOS decomposition via SDP (Paper \#6)."""

    def __init__(self, n_vars: int, max_degree: int):
        self.n_vars = n_vars
        self.max_degree = max_degree
        self.basis = MonomialBasis(n_vars, max_degree // 2)

    def is_sos(self, p: Polynomial) -> Optional[SOSDecomposition]:
        """Check if p is SOS, return decomposition if so."""

        gram_size = len(self.basis.monomials)
        Q = self._create_gram_matrix(gram_size)

        constraints = self._build_coefficient_constraints(p, Q)

        constraints.append(Q >> 0)

        result = self._solve_sdp(constraints)

        if result.status == OPTIMAL:
            return self._extract_decomposition(result.Q_value)
        return None
```

# Complexity of SOS-SDP

## SDP Size for SOS Check

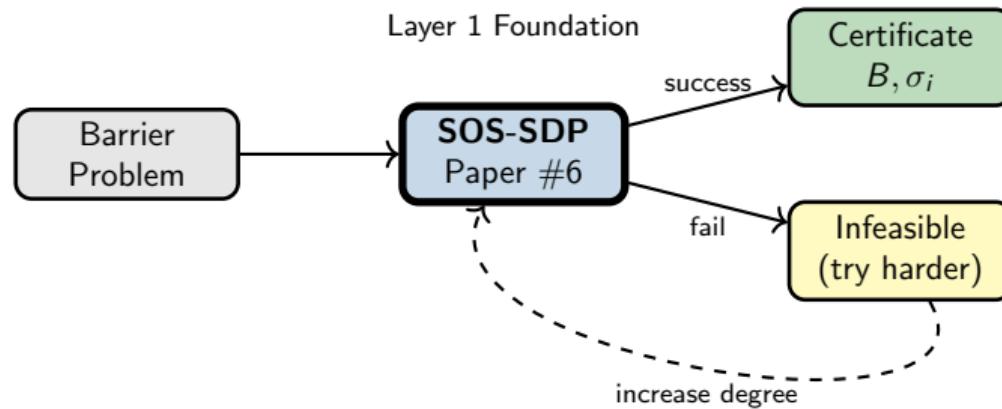
For polynomial  $p$  in  $n$  variables of degree  $2d$ :

- Gram matrix size:  $N = \binom{n+d}{d}$
- SDP has  $O(N^2)$  variables
- $O(N)$  linear constraints (coefficient matching)
- Interior-point method:  $O(N^{4.5})$  per iteration

## Practical Scaling

$n$ (vars)	$d$ (degree)	Gram size	Typical time
2	2	6	<1ms
2	4	15	~10ms
5	4	126	~1s
10	4	1001	~1min

# SOS-SDP in the Verification Pipeline



## Role in Pipeline

- **First attempt** for polynomial barrier synthesis
- Provides **certificates** that can be verified by Z3
- Failure triggers **degree escalation** (Lasserre hierarchy)

# Paper #7: Lasserre Hierarchy (2001)

## Reference

J. B. Lasserre. "Global optimization with polynomials and the problem of moments." *SIAM Journal on Optimization*, 2001.

## Core Contribution

A **hierarchy of SDP relaxations** that:

- Provides increasingly tight bounds
- **Converges** to the global optimum
- Gives **completeness** for polynomial positivity

## Key Insight

If  $p \geq 0$  on  $S$  but not SOS, increase degree  $\Rightarrow$  eventually find SOS certificate.

# Moment-SOS Duality

## Primal (Moment Problem):

Find a probability measure  $\mu$  on  $S$  such that:

$$\min_{\mu} \int_S p(x) d\mu(x)$$

subject to  $\text{supp}(\mu) \subseteq S$ .

Moments:  $y_\alpha = \int x^\alpha d\mu$

## Dual (SOS Problem):

Find maximum  $\gamma$  such that:

$$p(x) - \gamma \in M_d(g_1, \dots, g_m)$$

where  $M_d$  is the degree- $d$  quadratic module.

Provides lower bound on  $p$  over  $S$ .

## Duality

Strong duality holds under mild conditions.

Primal optimal = Dual optimal as  $d \rightarrow \infty$ .

# The Lasserre Hierarchy: Levels

## Definition (Degree- $d$ Relaxation)

The level- $d$  Lasserre relaxation for  $\min_{x \in S} p(x)$ :

$$\gamma_d^* = \max \left\{ \gamma : p - \gamma = \sigma_0 + \sum_i \sigma_i g_i, \deg(\sigma_i g_i) \leq 2d \right\}$$

## Hierarchy Properties

- $\gamma_1^* \leq \gamma_2^* \leq \gamma_3^* \leq \dots \leq p^* = \min_{x \in S} p(x)$
- Each level is an SDP of increasing size
- **Convergence:**  $\gamma_d^* \rightarrow p^*$  as  $d \rightarrow \infty$
- **Finite convergence:** For generic problems, exact at some finite  $d$

# Convergence of the Lasserre Hierarchy

## Theorem (Lasserre 2001)

If  $S$  is compact and non-empty, and  $p(x) > 0$  for all  $x \in S$ , then there exists  $d_0$  such that for all  $d \geq d_0$ :

$$p \in M_d(g_1, \dots, g_m)$$

i.e., the SOS representation exists at some finite level.

## Implication for Barrier Synthesis

If a polynomial barrier **exists**, the Lasserre hierarchy will **find it** at some level.

**Strategy:** Start at  $d = 2$ , increment until success or resource limit.

# Practical Degree Bounds

## When Does Hierarchy Converge?

**Empirical observation:** Most practical problems converge at low degree.

## Barrier Synthesis Experience

Problem Type	Typical $d$	Notes
Linear systems	1-2	Often exact at lowest level
Quadratic dynamics	2-4	Usually tractable
Polynomial (degree 3-4)	4-6	May need sparse techniques
High-degree / many vars	6+	Consider DSOS/SDSOS relaxations

Our implementation tries  $d \in \{2, 4, 6\}$  with timeouts.

# Moment Matrices: The Primal View

## Definition (Moment Matrix)

For a sequence  $y = (y_\alpha)$  indexed by monomials, the **moment matrix**  $M_d(y)$  has entries:

$$M_d(y)_{\alpha,\beta} = y_{\alpha+\beta}$$

## Key Constraint

$y$  corresponds to a measure on  $S$  if and only if:

- $M_d(y) \succeq 0$  (moment matrix is PSD)
- $M_{d-d_i}(g_i \cdot y) \succeq 0$  for each constraint  $g_i$

**Localizing matrices:**  $M_k(g \cdot y)_{\alpha,\beta} = \sum_\gamma (g)_\gamma \cdot y_{\alpha+\beta+\gamma}$

# Extracting Solutions from Moment Relaxation

## When Relaxation is Tight

If  $\text{rank}(M_d(y^*)) = \text{rank}(M_{d-1}(y^*))$  (flat extension), then we can extract minimizers.

### Extraction Algorithm:

- ① Compute eigendecomposition of  $M_d(y^*)$
- ② Identify the rank-1 components
- ③ Each rank-1 component gives a point  $x^* \in S$
- ④ Verify:  $p(x^*) = \gamma^*$

## For Barrier Synthesis

When SOS representation found, extract the barrier coefficients and Gram matrices as the **certificate**.

# Implementation: Lasserre Hierarchy Solver

```
class LasserreHierarchySolver:
    """Lasserre moment-SOS hierarchy (Paper \#7)."""

    def __init__(self, n_vars: int, max_level: int = 6):
        self.n_vars = n_vars
        self.max_level = max_level

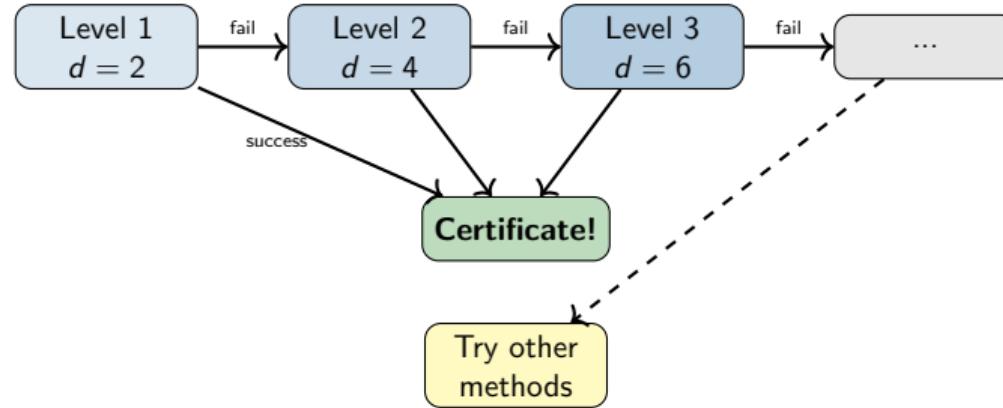
    def solve_hierarchy(self, p: Polynomial,
                        constraints: List[Polynomial]) -> LasserreResult:
        """Solve using ascending hierarchy levels."""
        for level in range(1, self.max_level + 1):
            result = self._solve_level(p, constraints, level)

            if result.status == OPTIMAL:

                if result.has_flat_extension():
                    return LasserreResult(
                        status='solved',
                        optimal_value=result.gamma,
                        level=level,
                        certificate=result.sos_certificate
                    )

        return LasserreResult(status='max_level_reached')
```

# Using Lasserre Hierarchy in Barrier Synthesis



## Practical Strategy

- Start with low degree (fast, often sufficient)
- Escalate on failure (more expressive)
- Timeout per level (avoid stuck computations)
- Fall back to DSOS/SDSOS for large problems

# Lasserre Hierarchy: Summary

## Strengths:

- Completeness for polynomial problems
- Systematic degree escalation
- Strong theoretical guarantees
- Provides dual certificates

## In Our Pipeline:

- Falls back after direct SOS fails
- Levels 1-3 typically sufficient
- Provides certificates to Layer 2

## Limitations:

- Exponential size growth with level
- Numerical stability challenges
- May require many levels for hard problems

## Mitigation:

- Sparse SOS (Paper #8)
- DSOS/SDSOS relaxations (Paper #9)
- Timeout and fallback strategies

# Paper #8: Sparse SOS (Kojima et al. 2005)

## Reference

M. Kojima, S. Kim, H. Waki. "Sparsity in sums of squares of polynomials." *Mathematical Programming*, 2005.

## The Scalability Problem

Standard SOS has Gram matrices of size  $\binom{n+d}{d}$ .

- 10 variables, degree 4:  $1001 \times 1001$  matrix
- 20 variables, degree 4:  $10626 \times 10626$  matrix
- Intractable for real-world problems!

## Key Insight

Exploit **sparsity structure** in polynomials to decompose large SDPs into smaller ones.

# Correlative Sparsity Pattern

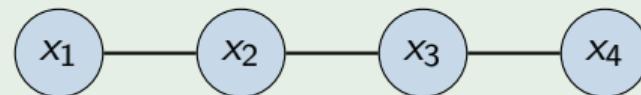
## Definition (Correlative Sparsity)

The **correlative sparsity pattern** (CSP) graph  $G = (V, E)$ :

- Vertices  $V = \{x_1, \dots, x_n\}$  (variables)
- Edge  $(x_i, x_j) \in E$  if  $x_i$  and  $x_j$  appear together in some monomial of  $p$  or constraints

## Example

$$p(x_1, x_2, x_3, x_4) = x_1^2 x_2 + x_2 x_3^2 + x_3 x_4$$



Variables are **not all connected**  $\Rightarrow$  exploitable sparsity!

# Chordal Graphs and Clique Trees

## Definition (Chordal Graph)

A graph is **chordal** if every cycle of length  $\geq 4$  has a chord (edge connecting non-adjacent vertices in the cycle).

## Chordal Extension

Any graph can be extended to a chordal graph by adding edges.

The **maximal cliques** of a chordal graph form a **clique tree**.

## Why Chordal?

Chordal structure enables:

- Decomposition of large PSD constraint into smaller ones
- Each clique  $\rightarrow$  one smaller SDP
- Clique tree  $\rightarrow$  consistency constraints between SDPs

# Sparse SOS Decomposition Theorem

Theorem (Sparse SOS - Kojima et al.)

If CSP graph has maximal cliques  $C_1, \dots, C_\ell$ , then  $p$  is SOS iff:

$$p = \sum_{k=1}^{\ell} \sigma_k$$

where each  $\sigma_k$  is SOS in variables  $C_k$  only.

## Computational Savings

Instead of one large SDP:

- Original: Gram matrix of size  $\binom{n+d}{d}$
- Sparse:  $\ell$  Gram matrices, each of size  $\binom{|C_k|+d}{d}$

If cliques are small, this is **exponentially faster!**

## Example: Sparse Decomposition in Action

**Problem:** Check if  $p(x_1, x_2, x_3, x_4) = x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_3x_4 + x_4^2$  is SOS.

**CSP Graph:** Chain  $x_1 - x_2 - x_3 - x_4$

**Maximal Cliques:**  $C_1 = \{x_1, x_2\}$ ,  $C_2 = \{x_2, x_3\}$ ,  $C_3 = \{x_3, x_4\}$

**Sparse SOS:**

$$p = \underbrace{(x_1^2 + x_1x_2 + \frac{1}{2}x_2^2)}_{\sigma_1 \text{ in } C_1} + \underbrace{(\frac{1}{2}x_2^2 + x_2x_3 + \frac{1}{2}x_3^2)}_{\sigma_2 \text{ in } C_2} + \underbrace{(\frac{1}{2}x_3^2 + x_3x_4 + x_4^2)}_{\sigma_3 \text{ in } C_3}$$

**Savings:**

- Dense: One  $5 \times 5$  Gram matrix (degree 2, 4 vars, plus constant)
- Sparse: Three  $3 \times 3$  Gram matrices

# Sparse SOS Algorithm

- ① **Build CSP Graph:** Identify variable interactions from polynomial
- ② **Chordal Extension:** Add edges to make graph chordal (minimum fill-in heuristic)
- ③ **Find Maximal Cliques:** Use perfect elimination ordering
- ④ **Set Up Sub-SDPs:** For each clique  $C_k$ , create SOS problem in  $|C_k|$  variables
- ⑤ **Add Coupling Constraints:** Ensure consistent coefficients at clique intersections
- ⑥ **Solve Coupled SDPs:** Can be done in parallel for independent cliques

## Complexity

If maximum clique size is  $\omega$ , Gram matrices are  $O\left(\binom{\omega+d}{d}\right)$  instead of  $O\left(\binom{n+d}{d}\right)$ .

# Implementation: Sparse SOS Decomposer

```
class SparseSOSDecomposer:
    """Sparse SOS using correlative sparsity (Paper \#8)."""

    def __init__(self, n_vars: int, max_degree: int):
        self.n_vars = n_vars
        self.max_degree = max_degree

    def decompose(self, p: Polynomial) -> Optional[SparseSOS]:
        csp_graph = self._build_csp_graph(p)

        chordal_graph = self._chordal_extension(csp_graph)

        cliques = self._find_maximal_cliques(chordal_graph)

        sub_problems = []
        for clique in cliques:
            sub_p = p.restrict_to_variables(clique)
            sub_problems.append(SOSProblem(sub_p, clique))

        return self._solve_coupled_sdps(sub_problems)
```

# When Sparse SOS Provides Maximum Benefit

## Best Case: Block-Sparse Structure

System naturally decomposes into weakly-coupled subsystems:

- Multi-agent systems (agents interact locally)
- Networked systems (communication topology)
- Modular software (independent components)

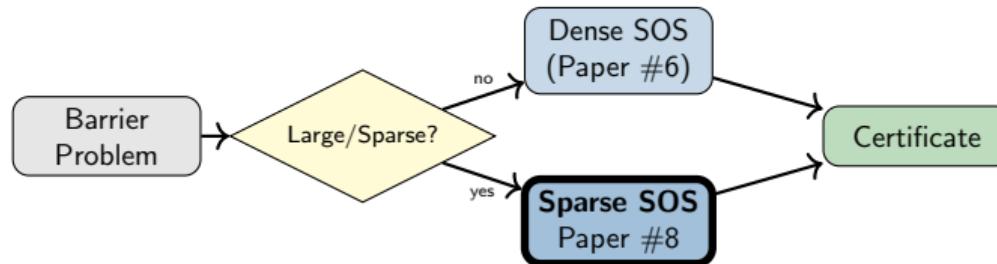
## In Our Pipeline

Python programs often have sparse structure:

- Local variables don't interact with distant code
- Function parameters have limited scope
- Data structures have localized access patterns

Sparse SOS enables barrier synthesis for **larger programs!**

# Sparse SOS in the Verification Pipeline

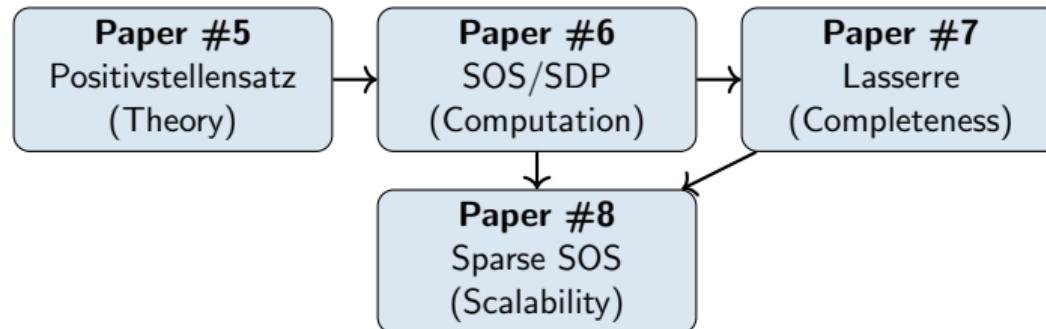


## Selection Heuristic

Use Sparse SOS when:

- $n > 5$  variables
- CSP graph has treewidth  $< n/2$
- Previous dense attempt timed out

# Layer 1 Summary: Mathematical Foundations



## What Layer 1 Provides to Higher Layers

- **Positivity proofs** for polynomial constraints
- **Certificate generation** (Gram matrices, SOS decompositions)
- **Scalable solving** via sparsity exploitation
- **Completeness guarantee** via Lasserre hierarchy

# **Part II**

## Barrier Certificate Core

Layer 2: Papers #1-4

Hybrid Barriers • Stochastic Barriers • SOS Safety • SOSTOOLS

## Layer 2: From Foundations to Certificates

### What Layer 2 Does

Takes the **mathematical foundations** from Layer 1 and applies them to **specific system types**:

- Continuous dynamical systems
- Discrete transition systems
- Hybrid automata (mixed continuous/discrete)
- Stochastic systems with probabilistic safety

### Key Additions Over Layer 1

- **Dynamics modeling:** How states evolve over time
- **Lie derivatives:** Rate of change along trajectories
- **Mode transitions:** Discrete jumps between continuous dynamics
- **Probability bounds:** Supermartingale conditions for stochastic systems

## Reference

S. Prajna & A. Jadbabaie. "Safety verification of hybrid systems using barrier certificates." *HSCC 2004 (Hybrid Systems: Computation and Control)*.

## Core Contribution

Extend barrier certificates to **hybrid automata**:

- Multiple **modes** with different continuous dynamics
- **Discrete transitions** between modes
- **Guards** and **resets** for transitions
- Unified barrier function across all modes

Perfect for programs with **control flow** (if/else, loops, function calls)!

# Hybrid Automaton: Formal Definition

## Definition (Hybrid Automaton)

$\mathcal{H} = (Q, X, \text{Init}, f, \text{Inv}, E, G, R)$  where:

- $Q = \{q_1, \dots, q_m\}$  – finite set of **modes**
- $X \subseteq \mathbb{R}^n$  – continuous state space
- $\text{Init} \subseteq Q \times X$  – initial states
- $f : Q \times X \rightarrow \mathbb{R}^n$  – dynamics per mode:  $\dot{x} = f(q, x)$
- $\text{Inv} : Q \rightarrow 2^X$  – mode invariants
- $E \subseteq Q \times Q$  – discrete transitions
- $G : E \rightarrow 2^X$  – transition guards
- $R : E \times X \rightarrow X$  – reset maps

# Hybrid Barrier Certificate Conditions

## Multi-Mode Barrier

For each mode  $q \in Q$ , define barrier  $B_q : X \rightarrow \mathbb{R}$ .

## Safety Conditions

- ① **Init:**  $\forall (q, x) \in \text{Init}. B_q(x) \geq 0$
- ② **Unsafe:**  $\forall (q, x) \in \text{Unsafe}. B_q(x) < 0$
- ③ **Flow (per mode):**  $\forall q, x \in \text{Inv}(q). B_q(x) \geq 0 \Rightarrow \mathcal{L}_{f_q} B_q(x) \geq 0$
- ④ **Jump:**  $\forall (q, q') \in E, x \in G(q, q'). B_q(x) \geq 0 \Rightarrow B_{q'}(R(x)) \geq 0$

The **Lie derivative**  $\mathcal{L}_f B = \nabla B \cdot f$  measures how  $B$  changes along trajectories.

# The Lie Derivative: Key to Continuous Dynamics

## Definition (Lie Derivative)

For barrier  $B : \mathbb{R}^n \rightarrow \mathbb{R}$  and vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ :

$$\mathcal{L}_f B(x) = \nabla B(x) \cdot f(x) = \sum_{i=1}^n \frac{\partial B}{\partial x_i} \cdot f_i(x)$$

## Geometric Interpretation

$\mathcal{L}_f B(x)$  is the **rate of change** of  $B$  at  $x$  as the system flows along  $f$ .

- $\mathcal{L}_f B \geq 0$  when  $B \geq 0$ : Barrier value doesn't decrease
- Trajectories starting with  $B \geq 0$  **stay** with  $B \geq 0$
- This is the **continuous induction step!**

# Modeling Programs as Hybrid Automata

## Python Program

```
if x > 0:  y = x * 2  
else:    y = -x
```

## As Hybrid Automaton:

- Mode  $q_1$ : “then branch” with guard  $x > 0$ , dynamics  $y' = 2x$
- Mode  $q_2$ : “else branch” with guard  $x \leq 0$ , dynamics  $y' = -x$
- Transition from entry to  $q_1$  or  $q_2$  based on  $x$

## For Bug Detection

- Unsafe = states where bug occurs (e.g.,  $y < 0$  for bounds check)
- Synthesize barriers for each mode
- Verify jump conditions at branches

# Hybrid Barrier Synthesis Algorithm

**Input:** Hybrid automaton  $\mathcal{H}$ , unsafe set, barrier degree  $d$

**Algorithm:**

- ① For each mode  $q$ , create polynomial template  $B_q(x) = \sum_{\alpha} b_{q,\alpha}x^{\alpha}$
- ② Set up SOS constraints:
  - Init:  $B_q - \varepsilon \in \Sigma + M(\text{Init}_q)$
  - Unsafe:  $-B_q - \varepsilon \in \Sigma + M(\text{Unsafe}_q)$
  - Flow:  $-\mathcal{L}_{f_q} B_q \in \Sigma + M(\text{Inv}_q) + M(B_q)$  (when  $B_q \geq 0$ )
  - Jump:  $B_{q'}(R(x)) - B_q(x) \in \Sigma + M(G_{q \rightarrow q'}) + M(B_q)$
- ③ Solve joint SDP for all  $b_{q,\alpha}$  and Gram matrices
- ④ Extract barrier polynomials

# Implementation: Hybrid Barrier Synthesizer

```
@dataclass
class HybridMode:
    """A mode in a hybrid automaton."""
    mode_id: int
    dynamics: ContinuousDynamics
    invariant: SemialgebraicSet

class HybridBarrierSynthesizer:
    """Synthesize barriers for hybrid systems (Paper \#1)."""

    def synthesize(self, automaton: HybridAutomaton,
                  unsafe: Dict[int, SemialgebraicSet]) -> HybridBarrier:

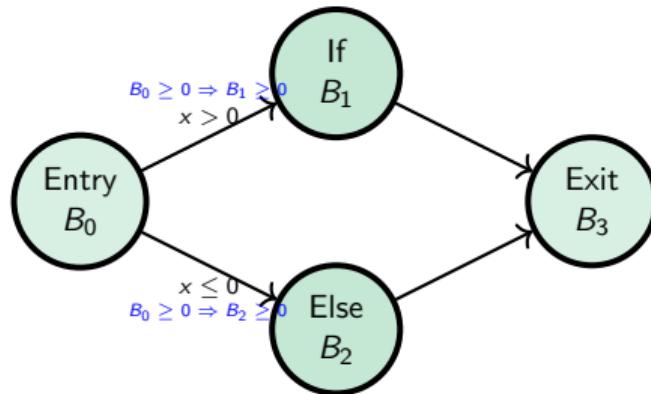
        barriers = {}
        for mode in automaton.modes:
            barriers[mode.mode_id] = BarrierTemplate(
                self.n_vars, self.max_degree
            )

        constraints = []
        for mode in automaton.modes:
            constraints += self._mode_constraints(mode, barriers, unsafe)

        for trans in automaton.transitions:
            constraints += self._jump_constraints(trans, barriers)

        return self._solve_and_extract(constraints, barriers)
```

# Hybrid Barriers for Program Control Flow



## CFG → Hybrid Automaton

- Basic blocks → modes
- Branch conditions → guards
- Variable updates → reset maps
- Loop iterations → self-transitions

## Paper #2: Stochastic Barrier Certificates (Prajna et al. 2007)

### Reference

S. Prajna, A. Jadbabaie, G. J. Pappas. "A framework for worst-case and stochastic safety verification." *IEEE Transactions on Automatic Control*, 2007.

### Core Contribution

Extend barrier certificates to **stochastic systems**:

$$dx = f(x) dt + g(x) dW_t$$

where  $W_t$  is a Wiener process (Brownian motion).

### Why Stochastic?

Programs have inherent randomness:

# Probabilistic Safety Guarantees

## Definition (Probabilistic Safety)

System is  $\delta$ -safe if:

$$\Pr[\text{reach unsafe within time } T] \leq \delta$$

## Supermartingale Condition

Instead of  $\mathcal{L}_f B \leq 0$ , we need:

$$\mathcal{A}B(x) \leq -\lambda B(x) \quad (\text{for } B \geq 0)$$

where  $\mathcal{A}$  is the **infinitesimal generator**:

$$\mathcal{A}B = \nabla B \cdot f + \frac{1}{2}\text{tr}(g^\top \nabla^2 B g)$$

The second term accounts for **diffusion** (stochastic spread).

# Stochastic Barrier Certificate Conditions

## Conditions for Probabilistic Safety

For stochastic system  $dx = f(x) dt + g(x) dW_t$ :

- ① **Init:**  $\forall x \in X_0. B(x) \leq \gamma$
- ② **Unsafe:**  $\forall x \in X_u. B(x) \geq 1$
- ③ **Supermartingale:**  $\forall x \in X. \mathcal{A}B(x) \leq \lambda B(x)$

Then:  $\Pr[\text{reach } X_u] \leq \gamma \cdot e^{\lambda T}$

## For Barrier Synthesis

- Minimize  $\gamma$  (probability bound)
- Template for  $B$  as polynomial
- SOS constraint on  $\lambda B - \mathcal{A}B \geq 0$

# Implementation: Stochastic Barrier Synthesizer

```
@dataclass
class StochasticDynamics:
    """Stochastic differential equation: dx = f(x)dt + g(x)dW."""
    n_vars: int
    drift: List[Polynomial]
    diffusion: List[Polynomial]

    def infinitesimal_generator(self, B: Polynomial) -> Polynomial:
        """Compute AB = nabla B . f + 0.5 * tr(g^T Hess(B) g)."""

        gradient = B.gradient()
        drift_term = sum(g.multiply(f) for g, f in zip(gradient, self.drift))

        hessian = B.hessian()
        diffusion_term = Polynomial.zero(self.n_vars)
        for i in range(self.n_vars):
            for j in range(self.n_vars):
                term = self.diffusion[i].multiply(
                    hessian[i][j].multiply(self.diffusion[j]))
            diffusion_term = diffusion_term.add(term.scale(0.5))

        return drift_term.add(diffusion_term)
```

# Paper #3: SOS Safety (Papachristodoulou-Prajna 2002)

## Reference

A. Papachristodoulou & S. Prajna. "On the construction of Lyapunov functions using the sum of squares decomposition."  
*CDC 2002 (Conference on Decision and Control)*.

## Core Contribution

Use SOS decomposition for **set emptiness checking**:

- Given constraints  $g_1(x) \geq 0, \dots, g_m(x) \geq 0$
- Prove the set  $S = \{x : g_i(x) \geq 0 \text{ for all } i\}$  is **empty**
- If empty  $\Rightarrow$  no bad states exist!

This is the **core technique** for proving barrier conditions hold.

# Set Emptiness via SOS

## Theorem (SOS Emptiness Certificate)

The set  $S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  is empty if there exist SOS polynomials  $\sigma_0, \sigma_1, \dots, \sigma_m$  such that:

$$-1 = \sigma_0 + \sum_{i=1}^m \sigma_i \cdot g_i$$

## Why This Works

If such  $\sigma_i$  exist:

- LHS:  $-1 < 0$  (always)
- RHS:  $\geq 0$  on  $S$  (sum of nonnegatives)
- Contradiction  $\Rightarrow S$  must be empty!

# SOS Safety for Barrier Verification

**Goal:** Verify “No state is both initial and unsafe”

**Set to check empty:**

$$S = \{x : x \in X_0 \text{ (initial)} \wedge x \in X_u \text{ (unsafe)}\}$$

**Express as polynomial constraints:**

$$S = \{x : g_1^{(0)}(x) \geq 0, \dots, g_k^{(0)}(x) \geq 0, g_1^{(u)}(x) \geq 0, \dots, g_\ell^{(u)}(x) \geq 0\}$$

**Find SOS certificate:**

$$-1 = \sigma_0 + \sum_i \sigma_i^{(0)} g_i^{(0)} + \sum_j \sigma_j^{(u)} g_j^{(u)}$$

If certificate exists  $\Rightarrow$  **disjoint**  $\Rightarrow$  **SAFE!**

# Implementation: SOS Safety Checker

```
class SOSSafetyChecker:
    """SOS-based set emptiness checking (Paper \#3)."""

    def check_disjoint(self, set1: SemialgebraicSet,
                       set2: SemialgebraicSet) -> SafetyResult:
        """Check if set1 and set2 are disjoint using SOS."""

        all_constraints = set1.constraints + set2.constraints

        multipliers = [self._create_sos_template(c.degree)
                      for c in all_constraints]

        sigma_0 = self._create_sos_template(self.max_degree)

        target = Polynomial.constant(-1)
        rhs = sigma_0.to_polynomial()
        for mult, constraint in zip(multipliers, all_constraints):
            rhs = rhs.add(mult.to_polynomial().multiply(constraint))

        return self._solve_emptiness_sdp(target, rhs, [sigma_0] + multipliers)
```

# Verifying the Inductive Step with SOS

**Goal:** Prove  $\{B(x) \geq 0 \wedge \mathcal{L}_f B(x) < 0\}$  is empty.

**Reformulate:** Show that on the set where  $B \geq 0$ , we have  $\mathcal{L}_f B \geq 0$ .

## SOS Formulation

Find SOS  $\sigma_0, \sigma_1$  such that:

$$\mathcal{L}_f B(x) = \sigma_0(x) + \sigma_1(x) \cdot B(x)$$

This proves: when  $B \geq 0$ , then  $\mathcal{L}_f B \geq 0$  (since RHS  $\geq 0$ ).

## The S-procedure

This is the **S-procedure**: proving implication via SOS multipliers.

# Paper #4: SOSTOOLS Framework (Prajna et al. 2004)

## Reference

S. Prajna, A. Papachristodoulou, P. A. Parrilo. "SOSTOOLS: Sum of squares optimization toolbox for MATLAB." *User's Guide*, 2004.

## Core Contribution

A **unified software framework** for SOS programming:

- Declarative specification of SOS constraints
- Automatic translation to SDP
- Support for parametric templates
- Numerical and symbolic solving

Our Python implementation mirrors SOSTOOLS' design patterns.

# SOSTOOLS: Declarative SOS Programming

## Design Philosophy

Separate **problem specification** from **solver mechanics**:

- User declares polynomial variables and constraints
- System builds SDP automatically
- Solver produces certificates

## Typical Workflow:

- ① Define polynomial variables: `x = poly('x', 2)`
- ② Create templates: `B = create_template(degree=4)`
- ③ Add SOS constraints: `add_sos(B - eps)`
- ④ Add implications: `add_sos(-Lie(B), when=B >= 0)`
- ⑤ Solve: `result = solve()`
- ⑥ Extract: `barrier = result.get_polynomial(B)`

# Implementation: SOSTOOLS-Style Framework

```
class SOSTOOLSFramework:
    """SOSTOOLS-style declarative SOS programming (Paper \#4)."""

    def __init__(self, n_vars: int, var_names: List[str] = None):
        self.n_vars = n_vars
        self.var_names = var_names or [f'x{i}' for i in range(n_vars)]
        self.decision_vars = []
        self.sos_constraints = []
        self.equality_constraints = []

    def create_template(self, name: str, degree: int) -> PolynomialTemplate:
        """Create a polynomial template with symbolic coefficients."""
        template = PolynomialTemplate(self.n_vars, degree, name)
        self.decision_vars.extend(template.coefficients)
        return template

    def add_sos(self, expr: Polynomial,
               multipliers: List[Tuple[Polynomial, Polynomial]] = None):
        """Add constraint: expr is SOS (optionally on a set)."""
        if multipliers:
            self.sos_constraints.append(SOSWithMultipliers(expr, multipliers))
        else:
            self.sos_constraints.append(PureSOS(expr))
```

# Complete Barrier Synthesis Example

```
def synthesize_barrier_sostools(dynamics, initial, unsafe, degree=4):
    """Synthesize barrier using SOSTOOLS framework."""
    n_vars = dynamics.n_vars
    sos = SOSITOOLSFramework(n_vars)

    B = sos.create_template('B', degree)
    eps = 0.01

    sos.add_sos(B - eps, on_set=initial)

    sos.add_sos(-B - eps, on_set=unsafe)

    Lie_B = dynamics.lie_derivative(B)
    sos.add_sos(-Lie_B, when=[B >= 0])

    result = sos.solve()

    if result.status == 'optimal':
        return result.extract_polynomial(B)
    return None
```

# Parametric Barrier Templates

## Template Families

Common barrier template structures:

### Quadratic:

$$B(x) = x^\top Px + p^\top x + c$$

Parameters:  $P$  (matrix),  $p$  (vector),  $c$  (scalar)

### Polynomial:

$$B(x) = \sum_{|\alpha| \leq d} b_\alpha x^\alpha$$

Parameters: coefficients  $b_\alpha$

### Piecewise:

$$B(x) = \begin{cases} B_1(x) & x \in R_1 \\ B_2(x) & x \in R_2 \end{cases}$$

Different polynomial per region

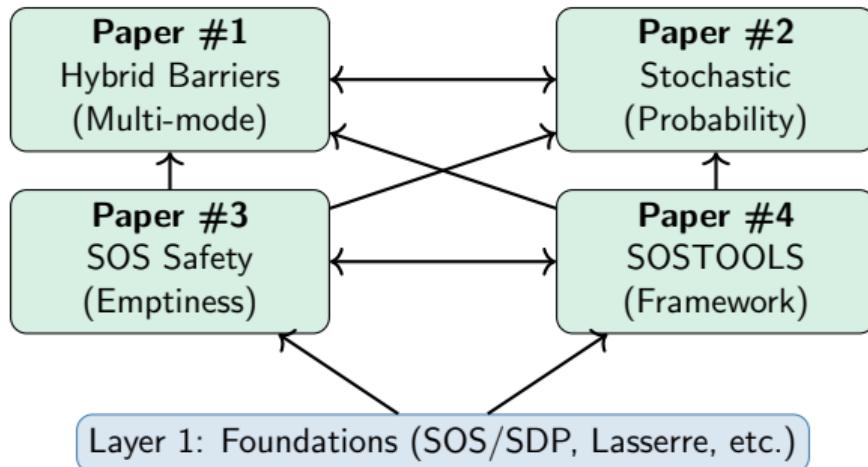
### Neural:

 (future work)

$$B(x) = \text{NN}_\theta(x)$$

Neural network with verification

## Layer 2: How Papers Integrate



Papers #3-4 provide core techniques; Papers #1-2 apply to specific system types.

# Unified Interface: BarrierCertificateEngine

```
class BarrierCertificateEngine:
    """Unified interface for Layer 2 barrier synthesis."""

    def __init__(self, n_vars: int, system_type: str, max_degree: int):
        self.n_vars = n_vars
        self.system_type = system_type
        self.max_degree = max_degree

        if system_type == 'continuous':
            self.synthesizer = SOSSafetyChecker(n_vars, max_degree)
        elif system_type == 'hybrid':
            self.synthesizer = HybridBarrierSynthesizer(n_vars, max_degree)
        elif system_type == 'stochastic':
            self.synthesizer = StochasticBarrierSynthesizer(n_vars, max_degree)
        else:
            self.synthesizer = SOSTOOLSFramework(n_vars)

    def synthesize(self, conditions: BarrierConditions,
                  dynamics: Any) -> Optional[BarrierCertificate]:
        """Synthesize barrier certificate."""
        return self.synthesizer.synthesize(conditions, dynamics)
```

# Layer 2 Summary: Barrier Certificate Core

## What We Can Now Do:

- Synthesize barriers for **continuous** systems
- Handle **hybrid** systems with mode switching
- Provide **probabilistic** safety for stochastic systems
- Check **set emptiness** for verification

## What We Provide to Layer 3:

- Barrier templates to refine
- Certificates to abstract
- Verification oracles

## Limitations Addressed by Higher Layers:

- Need **degree selection** → CEGAR
- Need **predicates** → Abstraction
- Need **examples** → Learning

## Files:

- certificate\_core.py
- hybrid\_barrier.py
- stochastic\_barrier.py
- sos\_safety.py

# From Synthesis to Abstraction

## The Challenge

Layer 2 synthesis requires:

- Choosing the right **degree** for polynomials
- Selecting good **predicates** for discrete reasoning
- Handling **counterexamples** from failed synthesis

## Layer 3's Solution

**Abstraction-Refinement** techniques:

- CEGAR: Use counterexamples to **refine** abstraction
- Predicate Abstraction: Reduce to **finite** state space
- Boolean Programs: Symbolic execution on **abstract** states
- IMPACT: **Lazy** refinement on-demand

# Part III

## Abstraction & Refinement

Layer 3: Papers #12-14, 16

CEGAR • Predicate Abstraction • Boolean Programs • IMPACT

# Layer 3: Managing Complexity Through Abstraction

## The Problem

Real programs have:

- Infinite state spaces (integers, floats, objects)
- Complex control flow (loops, recursion, exceptions)
- Many variables with intricate relationships

Direct synthesis on concrete state space is often **intractable**.

## The Solution: Abstraction

- Map infinite state to **finite abstract domain**
- Reason about abstract states
- Refine when abstraction is too coarse

# Paper #12: CEGAR (Clarke et al. 2000)

## Reference

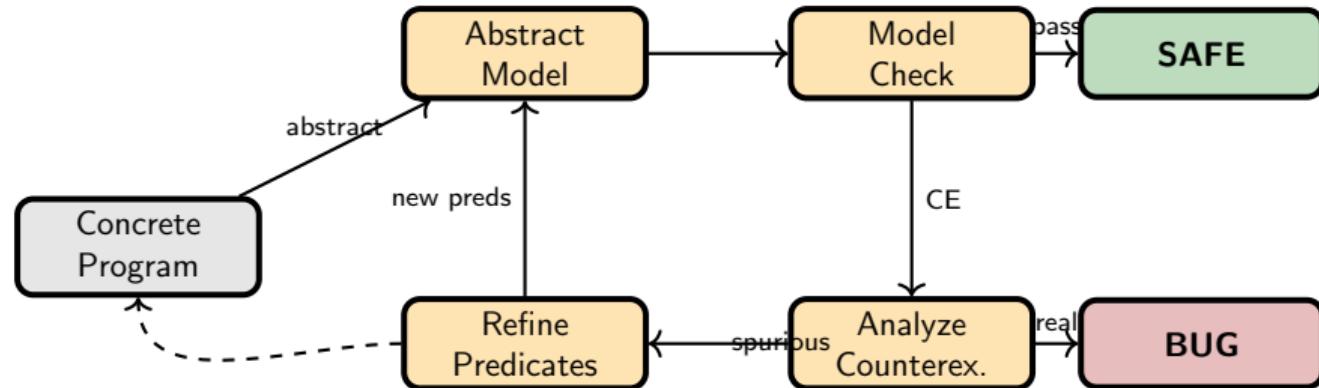
E. Clarke, O. Grumberg, S. Jha, Y. Lu, H. Veith. "Counterexample-Guided Abstraction Refinement." *CAV 2000 (Computer Aided Verification)*.

## Core Contribution

A **feedback loop** that automatically refines abstractions:

- ① **Abstract:** Create coarse model
- ② **Verify:** Check property on abstract model
- ③ **Analyze:** If counterexample found, check if **spurious**
- ④ **Refine:** If spurious, add predicates to distinguish
- ⑤ **Repeat:** Until verified or real bug found

# The CEGAR Loop



# Spurious vs. Real Counterexamples

## Definition (Spurious Counterexample)

A counterexample in the **abstract** model that has no corresponding **concrete** execution path.

### Real Counterexample:

- Path exists in concrete program
- Reaches actual unsafe state
- ⇒ Report **BUG!**

### Spurious Counterexample:

- Only exists in abstraction
- Caused by lost precision
- ⇒ **Refine** abstraction

## Detection Method

Symbolically execute the counterexample path.

If path constraints are SAT → real. If UNSAT → spurious.

# Refinement: Learning from Spurious Counterexamples

## The Key Question

When a counterexample is spurious, **why?** What predicates would eliminate it?

## Refinement Strategies:

- ① **Interpolation-based:** Use Craig interpolants from UNSAT proof

$$\phi_1 \wedge \phi_2 = \text{UNSAT} \Rightarrow \exists I. \phi_1 \Rightarrow I \wedge I \wedge \phi_2 = \text{UNSAT}$$

Interpolant  $I$  becomes new predicate.

- ② **Weakest precondition:** Compute  $\text{wp}(\text{unsafe}, \text{path})$  and extract predicates
- ③ **Counterexample-guided:** Extract predicates that distinguish concrete states along spurious path

## Applying CEGAR to Barriers

- ① **Initial:** Try simple barrier (low degree, few variables)
- ② **Attempt synthesis:** Use Layer 2 SOS/SDP
- ③ **If fails:** Get **counterexample** from SDP dual
- ④ **Analyze:** Is counterexample reachable? (Check with Z3)
- ⑤ **If spurious:** Refine barrier template:
  - Increase degree
  - Add new predicates
  - Partition state space
- ⑥ **If real:** Report bug with witness

# Implementation: CEGAR Loop

```
@dataclass
class CEGARResult:
    """Result of CEGAR refinement loop."""
    status: str
    certificate: Optional[BarrierCertificate] = None
    counterexample: Optional[Counterexample] = None
    iterations: int = 0
    predicates_added: int = 0

class CEGARLoop:
    """CEGAR refinement loop (Paper \#12)."""

    def verify(self, program, property, initial_predicates):
        predicates = list(initial_predicates)

        for iteration in range(self.max_iterations):

            abstraction = self._abstract(program, predicates)

            result = self._model_check(abstraction, property)

            if result.verified:
                return CEGARResult('safe', result.certificate)

            is_real = self._check_feasibility(result.counterexample)

            if is_real:
                return CEGARResult('unsafe', counterexample=result.counterexample)
```

# Extracting Counterexamples from SDP

## When SOS Synthesis Fails

SDP solver returns “infeasible” – no barrier of given degree exists.

The **dual solution** provides information:

- **Farkas certificate:** Proves no solution exists
- **Moment interpretation:** Dual variables represent “problematic” state distributions
- **Extraction:** Find concrete states that violate barrier conditions

## Practical Approach

Use Z3 to find **concrete states** where:

- State is initial AND barrier value is low, OR
- State is safe AND Lie derivative is positive, OR
- State is near unsafe AND barrier value is high

# Discovering New Predicates

## From Counterexamples to Predicates

A spurious counterexample reveals **what the abstraction is missing**.

### Predicate Sources:

- ① **Path conditions:** Branch conditions along counterexample path

$$\text{path: } x > 0 \rightarrow y = x + 1 \rightarrow z = y \cdot 2 \Rightarrow \text{pred: } x > 0$$

- ② **Variable relationships:** Equalities/inequalities that hold

$$\text{counterexample shows } y = x + 1 \Rightarrow \text{pred: } y = x + 1$$

- ③ **Barrier-relevant:** Predicates from barrier template structure

$$B(x) = x^2 - 4 \Rightarrow \text{preds: } x > 2, x > -2, x^2 > 4$$

## Termination

CEGAR is **not guaranteed** to terminate in general:

- Predicate set may grow unboundedly
- Some abstractions never become precise enough

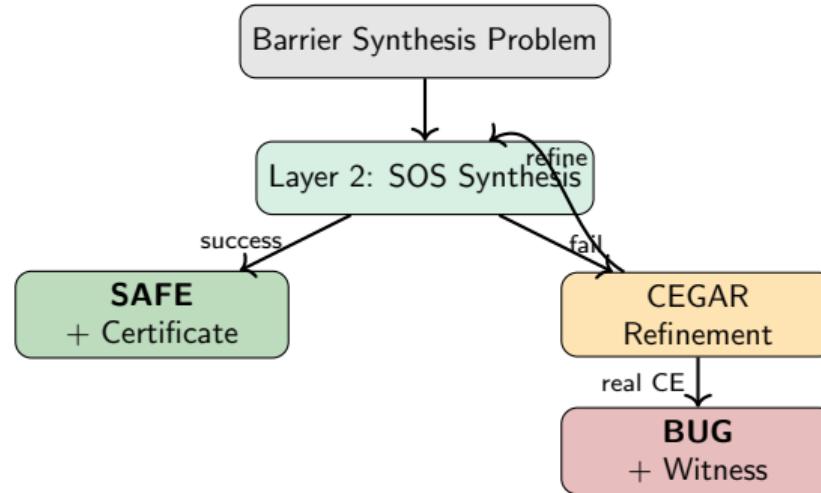
## Practical Guarantees

In practice, CEGAR often terminates because:

- Predicates are drawn from finite program syntax
- Many properties need only few predicates
- Timeouts convert non-termination to “unknown”

Our implementation: max 10 iterations, 30s timeout, tracks predicate count.

# CEGAR for Barriers: Summary



CEGAR enables **automatic degree/predicate selection** for barrier synthesis!

# Paper #13: Predicate Abstraction (Graf-Saïdi 1997)

## Reference

S. Graf & H. Saïdi. "Construction of Abstract State Graphs with PVS."  
CAV 1997.

## Core Contribution

Map **infinite** concrete states to **finite** abstract states using predicates:

- Choose predicates  $P = \{p_1, \dots, p_k\}$
- Abstract state = valuation of all predicates
- At most  $2^k$  abstract states

This enables **finite-state model checking** on infinite-state programs!

# Predicate Abstraction: Example

## Concrete Program

```
x := 0; while (x < 100) { x := x + 1 }
```

**Predicates:**  $P = \{x < 100, x \geq 0\}$

**Abstract States:**

$x < 100$	$x \geq 0$	Represents
T	T	$0 \leq x < 100$ (in loop)
F	T	$x \geq 100$ (after loop)
T	F	$x < 0$ (unreachable)
F	F	impossible ( $x \geq 100 \wedge x < 0$ )

**Abstract transitions:**  $(T, T) \rightarrow (T, T)$  (loop),  $(T, T) \rightarrow (F, T)$  (exit)

# Computing Abstract Successors

## The Problem

Given abstract state  $\hat{s}$  and concrete transition  $\tau$ , compute abstract successors.

## Approach: SMT-based Image Computation

For each predicate  $p$  and abstract state  $\hat{s}$ :

- ① Query: Is  $\hat{s} \wedge \tau \wedge p'$  satisfiable?
- ② If yes:  $p$  can be true in successor
- ③ Query: Is  $\hat{s} \wedge \tau \wedge \neg p'$  satisfiable?
- ④ If yes:  $p$  can be false in successor

## Cost

$O(2^k \cdot k \cdot 2)$  SAT queries per transition (expensive!)

Optimization: Use BDDs, incremental SAT, cartesian abstraction.

# Implementation: Predicate Abstraction

```
@dataclass
class Predicate:
    """A predicate over program variables."""
    name: str
    expr: z3.ExprRef
    variables: Set[str]

@dataclass
class AbstractState:
    """Abstract state = valuation of predicates."""
    valuation: FrozenSet[Tuple[str, bool]]

class PredicateAbstraction:
    """Predicate abstraction engine (Paper \#13)."""

    def __init__(self, predicates: List[Predicate], variables: List[z3.ExprRef]):
        self.predicates = predicates
        self.variables = variables
        self.abstract_trans = {}

    def abstract_successor(self, state: AbstractState,
                          transition: z3.ExprRef) -> Set[AbstractState]:
        """Compute abstract successors via SMT."""
        successors = set()

        for valuation in self._enumerate_valuations():
            if self._is_feasible(state, transition, valuation):
                successors.add(AbstractState(valuation))

    def _is_feasible(self, state: AbstractState, transition: z3.ExprRef,
                    valuation: Tuple[str, bool]) -> bool:
        ...
```

# Paper #14: Boolean Programs (Ball-Rajamani 2001)

## Reference

T. Ball & S. K. Rajamani. "Boolean Programs: A Model and Process for Software Analysis." *MSR Technical Report*, 2001. (Also SLAM project)

## Core Contribution

Represent abstract program as **Boolean program**:

- All variables are Boolean
- Control flow preserved from concrete program
- Statements update Boolean variables based on predicates
- Enables standard model checking algorithms

Foundation of Microsoft's SLAM project (device driver verification).

# Constructing Boolean Programs

**Original:**  $x := y + 1$

**Predicates:**  $\{x > 0, y > 0, x > y\}$

**Boolean Program:**

```
b1 := (y > 0) ? true : *;    // x > 0 after x := y + 1
b2 := b2;      // y > 0 unchanged
b3 := true;    // x > y always after x := y + 1
```

## Key Insight

The \* (nondeterminism) captures cases where predicate value is unknown.

Sound abstraction: concrete behavior  $\subseteq$  abstract behavior.

# Implementation: Boolean Program Executor

```
class BooleanProgram:
    """Boolean program abstraction (Paper \#14)."""

    def __init__(self, predicates: List[Predicate]):
        self.predicates = predicates
        self.n_predicates = len(predicates)
        self.transitions = []

    def add_statement(self, original_stmt, pre_abstract, post_abstract):
        """Add abstracted statement."""

        updates = []
        for i, pred in enumerate(self.predicates):
            effect = self._compute_predicate_effect(pred, original_stmt)
            updates.append(effect)

        self.transitions.append(BooleanTransition(updates))

class BooleanProgramExecutor:
    """Symbolic execution on Boolean programs."""

    def explore_all_paths(self, program: BooleanProgram,
                          initial: AbstractState) -> Set[AbstractState]:
        """BFS exploration of abstract state space."""
        visited = {initial}
        worklist = [initial]

        while worklist:
            state = worklist.pop(0)
            for succ in program.successors(state):
```

## Integration Strategy

- ① Extract **barrier-relevant predicates** from barrier template

$$B(x, y) = x^2 + y^2 - 1 \quad \Rightarrow \quad \{x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

- ② Build **Boolean program** using these predicates
- ③ **Model check** Boolean program for reachability
- ④ If unsafe reachable: counterexample guides barrier refinement
- ⑤ If safe: extract **abstract certificate**

Boolean programs enable **efficient exploration** of abstract state space!

## Reference

K. L. McMillan. "Lazy Abstraction with Interpolants."  
*CAV 2006.*

## Core Contribution

**On-demand** abstraction refinement:

- Don't pre-compute full abstract model
- Build abstraction **lazily** during exploration
- Use **interpolants** for refinement
- Different abstraction at different program points

Often much more efficient than eager predicate abstraction!

# IMPACT: Lazy Abstraction with Interpolants

**Key Innovation:** Build Abstract Reachability Tree (ART) on-the-fly.

- ① **Unfold:** Explore concrete paths symbolically
- ② **Check:** When path reaches error, check feasibility
- ③ **If feasible:** Report bug with concrete trace
- ④ **If infeasible:** Compute Craig interpolants from UNSAT proof

$$\underbrace{\phi_1}_{\text{prefix}} \wedge \underbrace{\phi_2}_{\text{suffix}} = \text{UNSAT}$$

- ⑤ **Annotate:** Label tree nodes with interpolants
- ⑥ **Subsumption:** Prune tree when new state is covered by existing

Interpolants provide **exactly the right predicates** for each location!

# Craig Interpolation: The Key to IMPACT

## Theorem (Craig Interpolation)

If  $\phi_1 \wedge \phi_2$  is unsatisfiable, there exists formula  $I$  such that:

- ①  $\phi_1 \Rightarrow I$
- ②  $I \wedge \phi_2$  is unsatisfiable
- ③  $I$  uses only symbols common to  $\phi_1$  and  $\phi_2$

## For Path Analysis

- $\phi_1$  = prefix of spurious path
- $\phi_2$  = suffix of spurious path
- $I$  = **reason** why suffix cannot lead to error

Interpolant  $I$  becomes an annotation (invariant) at the cut point!

# Implementation: Lazy Abstraction

```
@dataclass
class ARTNode:
    """Node in Abstract Reachability Tree."""
    location: int
    path_formula: z3.ExprRef
    annotation: z3.ExprRef
    parent: Optional['ARTNode']
    children: List['ARTNode']

class LazyAbstraction:
    """IMPACT lazy abstraction (Paper \#16)."""

    def verify(self, program, property) -> VerificationResult:
        root = ARTNode(0, z3.BoolVal(True), z3.BoolVal(True), None, [])
        worklist = [root]

        while worklist:
            node = worklist.pop()

            if self._is_error(node):
                if self._is_feasible(node.path_formula):
                    return VerificationResult('unsafe', self._extract_trace(node))
                else:

                    self._refine_with_interpolants(node)
                    continue

            if self._is_covered(node):
                continue

            # Add children to worklist
```

# Computing Interpolants

## From SMT Solver

Modern SMT solvers (Z3, MathSAT) can extract interpolants from UNSAT proofs.

### Algorithm:

- ① Split path formula at program location  $\ell$

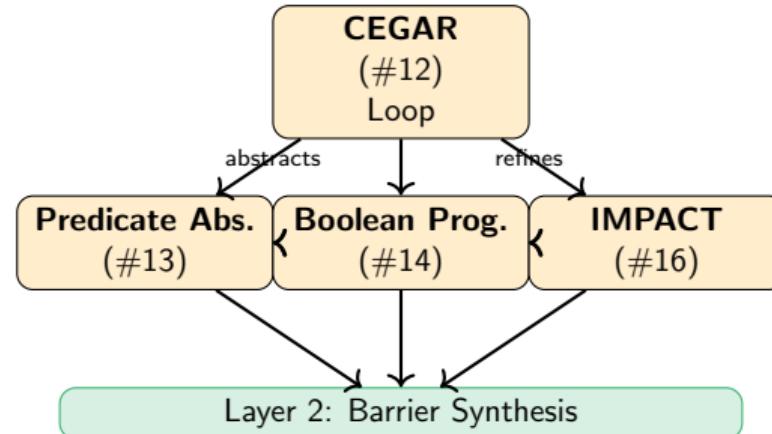
$$\underbrace{\text{path}[0 \rightarrow \ell]}_{\phi_1} \wedge \underbrace{\text{path}[\ell \rightarrow \text{error}]}_{\phi_2}$$

- ② Ask solver: Is  $\phi_1 \wedge \phi_2$  SAT?
- ③ If UNSAT: Extract interpolant  $I$
- ④ Use  $I$  as annotation at location  $\ell$

## Sequence of Interpolants

For path of length  $n$ , compute interpolants  $I_0, I_1, \dots, I_n$  at each location.  
These form a **trace abstraction** of the spurious path.

# Layer 3: How Papers Integrate



# Unified Interface: AbstractionRefinementEngine

```
class AbstractionRefinementEngine:  
    """Unified interface for Layer 3 abstraction-refinement."""  
  
    def __init__(self, initial_predicates: List[Predicate],  
                 max_iterations: int = 100):  
        self.predicates = list(initial_predicates)  
        self.max_iterations = max_iterations  
  
        self.pred_abstraction = PredicateAbstraction(self.predicates)  
        self.cegar = CEGARLoop(self.pred_abstraction)  
        self.lazy = LazyAbstraction()  
  
    def verify(self, program, property) -> VerificationResult:  
        """Verify using abstraction-refinement."""  
  
        result = self.lazy.verify(program, property)  
  
        if result.status != 'unknown':  
            return result  
  
        return self.cegar.verify(program, property, self.predicates)  
  
    def get_barrier_predicates(self) -> List[Predicate]:  
        """Get predicates useful for barrier synthesis."""  
        return self.predicates
```

# Layer 3 Summary: Abstraction & Refinement

## What Layer 3 Provides:

- Finite-state reasoning on infinite programs
- Automatic predicate discovery
- Counterexample-guided refinement
- Lazy on-demand abstraction

## To Layer 4 (Learning):

- Predicates to learn over
- Abstract traces for examples
- Refinement feedback

## Files:

- `abstraction.py`
- `cegar_refinement.py`
- `predicate_abstraction.py`
- `boolean_programs.py`
- `impact_lazy.py`

## Key Metrics:

- Predicates discovered
- CEGAR iterations
- Tree size (IMPACT)

# Part IV

## Learning-Based Synthesis

Layer 4: Papers #17-19

ICE Learning • Houdini Inference • SyGuS Synthesis

# Layer 4: Learning Invariants from Data

## The Insight

Instead of **synthesizing** invariants from scratch, **learn** them from:

- Concrete program executions (positive examples)
- Counterexamples (negative examples)
- Transition pairs (implication examples)

## Learning Framework

- **ICE**: Learn from Implications, Counterexamples, Examples
- **Houdini**: Conjunctive inference from candidate set
- **SyGuS**: Syntax-guided synthesis from grammar

# Paper #17: ICE Learning (Garg et al. 2014)

## Reference

P. Garg, C. Löding, P. Madhusudan, D. Neider. "ICE: A Robust Framework for Learning Invariants." *CAV 2014*.

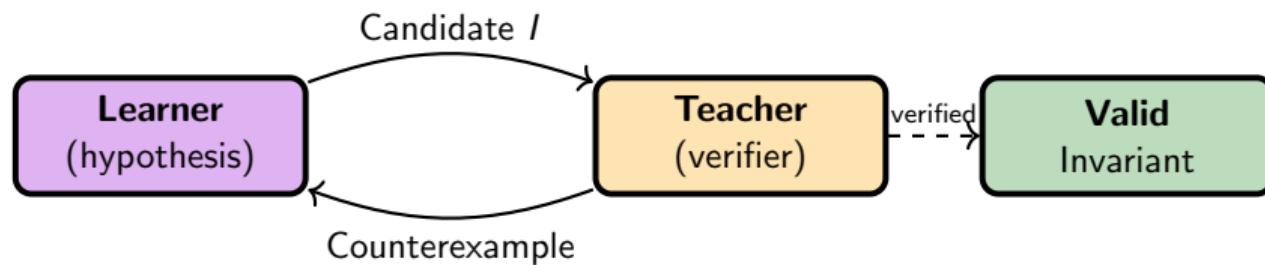
## Core Contribution

Learn invariants from **three types** of examples:

- Implication examples:  $(s, s')$  pairs from transitions
- Counterexamples: States violating current hypothesis
- Examples: Positive (in invariant) and negative (out) states

ICE provides a **teacher-learner** framework for invariant inference.

# ICE: Teacher-Learner Framework



Updates hypothesis  
based on examples

Checks:  
 $\text{Init} \Rightarrow I$   
 $I \wedge T \Rightarrow I'$   
 $I \Rightarrow \text{Safe}$

# ICE: The Three Types of Examples

## 1. Positive Examples (must be in invariant)

States from initial region or reachable safe states.

$$\mathcal{P} = \{s : s \in \text{Init} \vee s \text{ is reachable and safe}\}$$

## 2. Negative Examples (must NOT be in invariant)

States in unsafe region.

$$\mathcal{N} = \{s : s \in \text{Unsafe}\}$$

## 3. Implication Examples (transition pairs)

If pre-state is in invariant, post-state must be too.

$$\mathcal{I} = \{(s, s') : s \xrightarrow{T} s'\}$$

# ICE Learning Algorithm

```
1: Initialize:  $\mathcal{P} \leftarrow$  samples from Init,  $\mathcal{N} \leftarrow$  samples from Unsafe,  $\mathcal{I} \leftarrow \emptyset$ 
2:  $I \leftarrow \text{Learn}(\mathcal{P}, \mathcal{N}, \mathcal{I})$  ▷ Initial hypothesis
3: while not verified do
4:   result  $\leftarrow \text{Teacher.Check}(I)$ 
5:   if result = VALID then
6:     return  $I$ 
7:   else if result = CEinit then ▷ Init violation
8:      $\mathcal{P} \leftarrow \mathcal{P} \cup \{\text{counterexample}\}$ 
9:   else if result = CEsafe then ▷ Safety violation
10:     $\mathcal{N} \leftarrow \mathcal{N} \cup \{\text{counterexample}\}$ 
11:   else if result = CEind then ▷ Induction violation
12:      $\mathcal{I} \leftarrow \mathcal{I} \cup \{(pre, post)\}$ 
13:   end if
14:    $I \leftarrow \text{Learn}(\mathcal{P}, \mathcal{N}, \mathcal{I})$  ▷ Re-learn
15: end while
```

# Implementation: ICE Learner

```
@dataclass
class ICEExample:
    """ICE data: positive, negative, and implication examples."""
    positive: List[DataPoint]
    negative: List[DataPoint]
    implications: List[Tuple[DataPoint, DataPoint]]


class ICELearner:
    """ICE Learning for invariant inference (Paper \#17)."""

    def __init__(self, n_vars: int, max_degree: int = 4):
        self.n_vars = n_vars
        self.max_degree = max_degree
        self.template = BarrierTemplate(n_vars, max_degree)

    def learn(self, examples: ICEExample) -> Optional[Polynomial]:
        """Learn invariant satisfying all examples."""
        solver = z3.Solver()

        for pos in examples.positive:
            solver.add(self.template.evaluate(pos.values) >= 0)

        for neg in examples.negative:
            solver.add(self.template.evaluate(neg.values) < 0)

        for pre, post in examples.implications:
            solver.add(z3.Implies(
```

# Applying ICE to Barrier Synthesis

## Key Insight

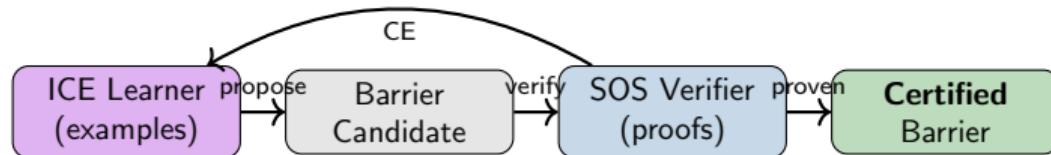
Barrier conditions are exactly ICE requirements!

- **Positive**  $\leftrightarrow$  Initial states ( $B \geq \varepsilon$ )
- **Negative**  $\leftrightarrow$  Unsafe states ( $B \leq -\varepsilon$ )
- **Implication**  $\leftrightarrow$  Inductiveness ( $B \geq 0 \Rightarrow B' \geq 0$ )

## Integration with Layers 1-3

- Use **Layer 1** (SOS) to check candidate barrier
- Use **Layer 2** constraints to generate examples
- Use **Layer 3** (CEGAR) counterexamples as negative examples
- **Learn** barrier coefficients from examples

# ICE + SOS: The Best of Both Worlds



## Workflow

- ① ICE learns candidate from examples (fast, data-driven)
- ② SOS verifies candidate (rigorous, sound)
- ③ If fails: extract counterexample, add to ICE data, repeat

# Paper #18: Houdini (Flanagan-Leino 2001)

## Reference

C. Flanagan & K. R. M. Leino. "Houdini, an Annotation Assistant for ESC/Java." *FME 2001*.

## Core Contribution

### Conjunctive inference of invariants:

- Start with a **large candidate set** of predicates
- Iteratively **remove** predicates that fail verification
- Result: **maximal subset** that forms valid invariant

Simple but surprisingly effective for many practical invariants!

# Houdini: Conjunctive Fixpoint

## Key Insight

If the true invariant is a **conjunction** of candidate predicates, Houdini will find it.

```
1:  $C \leftarrow \{p_1, p_2, \dots, p_n\}$                                 ▷ Initial candidate set
2: repeat
3:    $I \leftarrow \bigwedge_{p \in C} p$                                 ▷ Current invariant
4:   changed  $\leftarrow$  false
5:   for all  $p \in C$  do
6:     if  $\neg \text{Verify}(I \Rightarrow p')$  then                                ▷ Check inductiveness
7:        $C \leftarrow C \setminus \{p\}$ 
8:       changed  $\leftarrow$  true
9:     end if
10:   end for
11: until not changed
12: return  $I = \bigwedge_{p \in C} p$ 
```

# Implementation: Houdini Inference

```
class HoudiniInference:
    """Houdini conjunctive invariant inference (Paper \#18)."""

    def __init__(self, candidates: List[Predicate]):
        self.candidates = list(candidates)

    def infer(self, transition: z3.ExprRef) -> z3.ExprRef:
        """Infer maximal conjunctive invariant."""
        current = set(self.candidates)

        while True:
            changed = False
            inv = z3.And([p.expr for p in current])

            for pred in list(current):
                if not self._is_inductive(inv, transition, pred):
                    current.remove(pred)
                    changed = True

            if not changed:
                break

        return z3.And([p.expr for p in current]) if current else z3.BoolVal(True)

    def _is_inductive(self, inv, trans, pred) -> bool:
        """Check if pred is preserved under transition."""
        solver = z3.Solver()
        solver.add(inv)
        solver.add(trans)
```

# Applying Houdini to Barrier Synthesis

## Barrier Predicates

Generate candidate barrier predicates from:

- Program guards:  $\text{if } x > 0 \rightarrow x > 0$
- Assertions:  $\text{assert } \text{len}(a) > i \rightarrow \text{len}(a) > i$
- Type constraints:  $x: \text{int} \rightarrow x \in \mathbb{Z}$
- Inferred bounds: loop analysis  $\rightarrow 0 \leq i < n$

## Houdini Barrier Synthesis

- ① Collect guard predicates from CFG
- ② Run Houdini to find inductive conjunction
- ③ Check if conjunction implies safety
- ④ If yes: **Barrier = conjunction of surviving predicates**

# Houdini: Strengths and Limitations

## Strengths:

- Simple algorithm
- Fast convergence
- Polynomial in # candidates
- Finds maximal conjunction
- Works well with program guards

## Limitations:

- Only conjunctions
- Needs good candidates
- Can't synthesize new predicates
- May converge to  $\top$

## Mitigation:

- Combine with ICE for predicate discovery
- Use SyGuS for template generation
- Fall back to full SOS synthesis

# Paper #19: SyGuS (Alur et al. 2013)

## Reference

R. Alur, R. Bodik, G. Juniwal, et al. "Syntax-Guided Synthesis."  
*FMCAD 2013.*

## Core Contribution

### Syntax-Guided Synthesis:

- User provides a **grammar** of allowed expressions
- System searches for expression satisfying **specification**
- Combines **search** with **verification**

SyGuS is a **general framework** for program synthesis, applied here to invariants.

# SyGuS: Problem Specification

## Components

- ① **Background theory:** LIA (Linear Integer Arithmetic), BV, etc.
- ② **Grammar:** Allowed expression forms
- ③ **Specification:** Semantic constraint to satisfy

## Example: Loop Invariant

```
; Grammar for invariants
(synth-inv Inv ((x Int) (n Int))
  ((Start Bool ((and Start Start) (or Start Start)
                (>= Term Term) (<= Term Term)))
   (Term Int (x n 0 1 (+ Term Term)))))

; Specification
(constraint (=> (and (= x 0) (>= n 0)) (Inv x n))) ; Init
(constraint (=> (and (Inv x n) (< x n)) (Inv (+ x 1) n))) ; Step
(constraint (=> (and (Inv x n) (>= x n)) (= x n))) ; Post
```

# SyGuS Solving Strategies

## 1. Enumerative Search

Enumerate expressions from grammar in order of size/complexity.

- Simple, complete for finite grammars
- Exponential in expression size

## 2. CEGIS (CounterExample-Guided)

- ① Find candidate satisfying finite set of examples
- ② Verify candidate against full specification
- ③ If fails, add counterexample and repeat

## 3. Constraint-Based

Encode grammar + specification as SMT constraint, solve directly.

# Implementation: SyGuS Synthesizer

```
@dataclass
class SyGuSGrammar:
    """Grammar for syntax-guided synthesis."""
    start_symbol: str
    productions: Dict[str, List[str]]
    terminals: Set[str]

class SyGuSSynthesizer:
    """SyGuS synthesis for invariants (Paper \#19)."""

    def __init__(self, grammar: SyGuSGrammar, variables: List[str]):
        self.grammar = grammar
        self.variables = variables

    def synthesize(self, spec: Callable[[z3.ExprRef], z3.BoolRef],
                  max_size: int = 10) -> Optional[z3.ExprRef]:
        """Synthesize expression satisfying specification."""

        for size in range(1, max_size + 1):

            for expr in self._enumerate(self.grammar.start_symbol, size):
                z3_expr = self._to_z3(expr)

                if self._verify(z3_expr, spec):
                    return z3_expr

        return None

    def _enumerate(self, symbol: str, size: int) -> Iterator[Expression]:
```

## Grammar for Barrier Functions

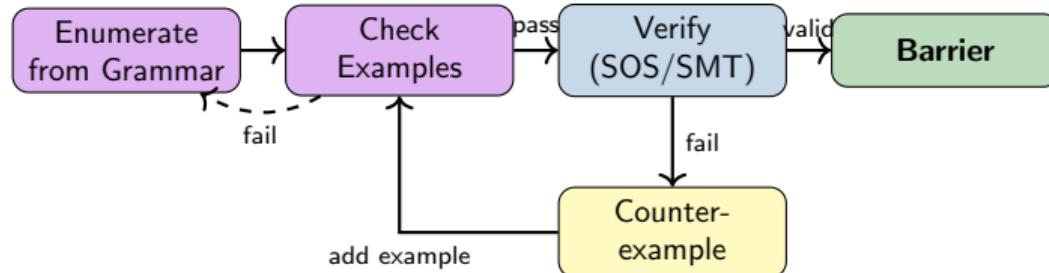
$$\begin{aligned} B &::= \text{Poly} \mid B_1 + B_2 \mid c \cdot B \\ \text{Poly} &::= x_i \mid x_i^2 \mid x_i \cdot x_j \mid \text{const} \\ c &::= 1 \mid -1 \mid 2 \mid -2 \mid \dots \end{aligned}$$

## Specification (Barrier Conditions)

- $\forall x \in \text{Init}. B(x) \geq \varepsilon$
- $\forall x \in \text{Unsafe}. B(x) < 0$
- $\forall x. (B(x) \geq 0) \Rightarrow (\mathcal{L}_f B(x) \geq 0)$

SyGuS searches for polynomial  $B$  from grammar satisfying these constraints.

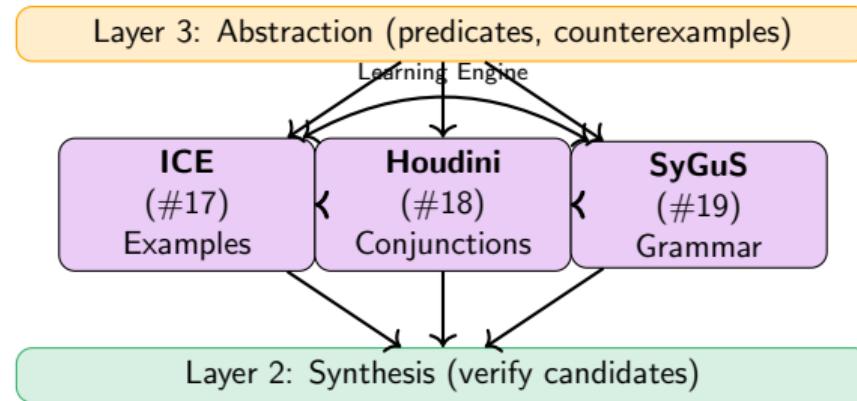
# SyGuS with CEGIS Loop



## CEGIS Advantage

Don't verify against full spec until candidate passes examples  $\Rightarrow$  faster!

# Layer 4: How ICE, Houdini, SyGuS Integrate



# Unified Interface: LearningBasedEngine

```
class LearningBasedEngine:
    """Unified interface for Layer 4 learning."""

    def __init__(self, n_vars: int, max_degree: int,
                 timeout_ms: int = 60000):
        self.n_vars = n_vars
        self.max_degree = max_degree

        self.ice = ICELearner(n_vars, max_degree)
        self.houdini = HoudiniInference([])
        self.sygus = SyGuSSynthesizer(self._default_grammar(), ...)

    def learn_invariant(self, examples: ICEExample,
                        candidates: List[Predicate] = None) -> Optional[Polynomial]:
        """Learn invariant using portfolio of techniques."""

        if candidates:
            self.houdini = HoudiniInference(candidates)
            result = self.houdini.infer(...)
            if self._is_nontrivial(result):
                return result

        return self.ice.learn(examples)
```

# Layer 4 Summary: Learning-Based Synthesis

## What Layer 4 Provides:

- Data-driven invariant inference
- Fast convergence with examples
- Complementary approaches
- Integration with verification

## To Layer 5 (Advanced):

- Candidate invariants for IC3
- Predicates for CHC solving
- Learned lemmas

## Files:

- learning.py
- ice.py
- ice\_learning.py
- houdini.py
- sygus\_synthesis.py

## Key Metrics:

- Examples used
- Learning iterations
- Candidates eliminated

# From Learning to Advanced Verification

## What We've Built So Far

- **Layer 1:** Mathematical foundations (SOS, SDP)
- **Layer 2:** Barrier certificate synthesis
- **Layer 3:** Abstraction and refinement
- **Layer 4:** Learning from examples

## What's Still Needed

- More scalable SOS relaxations (DSOS/SDSOS)
- Incremental reasoning (IC3/PDR)
- Constraint-based verification (CHC/Spacer)
- Compositional verification (Assume-Guarantee)

⇒ Layer 5: **Advanced Verification** techniques!

# Part V

## Advanced Verification

Layer 5: Papers #9-11, 15, 20

DSOS • IC3/PDR • CHC/Spacer • IMC • Assume-Guarantee

# Layer 5: Advanced Verification Techniques

## The Need for Advanced Methods

Sometimes lower layers are insufficient:

- SOS too expensive (need cheaper relaxations)
- Invariants need incremental discovery (IC3)
- Systems are modular (compositional reasoning)
- Need interpolation for refinement (IMC)

## Layer 5 Papers

- #9: DSOS/SDSOS - Cheaper SOS relaxations (LP/SOCP)
- #10: IC3/PDR - Property-Directed Reachability
- #11: CHC/Spacer - Constrained Horn Clauses
- #15: IMC - Interpolation-based Model Checking
- #20: Assume-Guarantee - Compositional verification

# Paper #9: DSOS/SDSOS (Ahmadi-Majumdar 2019)

## Reference

A. A. Ahmadi & A. Majumdar. "DSOS and SDSOS Optimization."  
*SIAM Journal on Applied Algebra and Geometry*, 2019.

## Core Contribution

**Cheaper alternatives** to SOS/SDP:

- **DSOS:** Diagonally-dominant SOS → Linear Programming (LP)
- **SDSOS:** Scaled diagonally-dominant → Second-Order Cone (SOCP)

## Trade-off

Less expressive than full SOS, but **much faster** for large problems.

# DSOS: Diagonally-Dominant SOS

## Definition (DSOS)

A polynomial  $p$  is **DSOS** if:

$$p(x) = \sum_{i,j} \lambda_{ij}(x_i \pm x_j)^2 + \sum_k \mu_k x_k^2 + c$$

where  $\lambda_{ij}, \mu_k \geq 0$  and  $c \geq 0$ .

## Key Insight

- Uses only **squared binomials**  $(x_i \pm x_j)^2$
- Coefficients  $\lambda_{ij}$  are **linear constraints**
- Checking DSOS  $\Leftrightarrow$  solving an **LP**!

LP is **polynomial time** and has mature, fast solvers (Gurobi, CPLEX).

# SDSOS: Scaled Diagonally-Dominant SOS

## Definition (Scaled DD Matrix)

A symmetric matrix  $Q$  is **scaled diagonally dominant** if there exist  $d_i > 0$ :

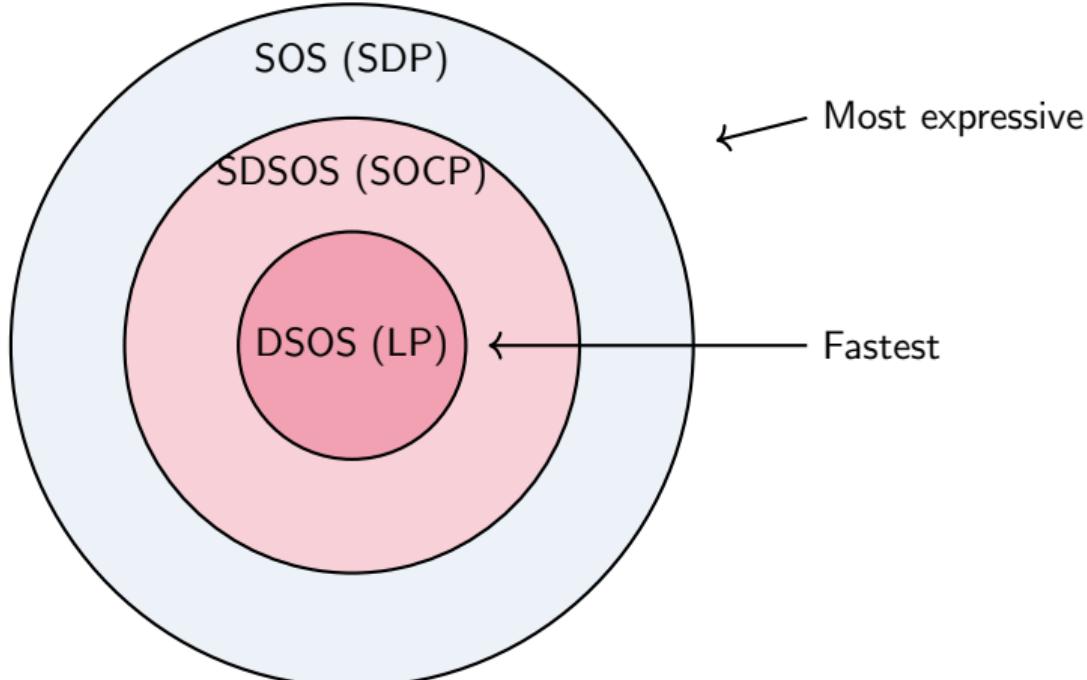
$DQD$  is diagonally dominant, where  $D = \text{diag}(d_1, \dots, d_n)$

## SDSOS Property

- Scaled DD  $\Rightarrow$  positive semidefinite
- Checking scaled DD  $\Leftrightarrow$  **SOCP** constraints
- SOCP is faster than SDP, still polynomial time

**Hierarchy:** DSOS  $\subset$  SDSOS  $\subset$  SOS

# Comparing SOS, SDSOS, DSOS



	DSOS	SDSOS	SOS
Solver Complexity	LP $O(n^2)$	SOCP $O(n^3)$	SDP $O(n^{4.5})$

# Implementation: DSOS Relaxation

```
@dataclass
class DSOSDecomposition:
    """DSOS decomposition: p = sum lambda_ij (xi +/- xj)^2"""
    n_vars: int
    binomial_coeffs: Dict[Tuple[int, int], float]

    def to_polynomial(self) -> Polynomial:
        """Convert to polynomial representation."""
        poly = Polynomial(self.n_vars)
        for (i, j, sign), coeff in self.binomial_coeffs.items():

            poly.add_term({i: 2}, coeff)
            poly.add_term({j: 2}, coeff)
            poly.add_term({i: 1, j: 1}, 2 * coeff * sign)
        return poly

class DSOSRelaxation:
    """DSOS/SDSOS relaxation engine (Paper \#9)."""

    def check_dsos(self, p: Polynomial) -> Optional[DSOSDecomposition]:
        """Check if p is DSOS using LP."""

        lp = LinearProgram()
        for (i, j) in self._binomial_pairs():
            lp.add_variable(f'lambda_{i}_{j}_plus', lower=0)
            lp.add_variable(f'lambda_{i}_{j}_minus', lower=0)

            for mono, target_coeff in p.terms.items():
                lp.add_constraint(self.coef_expr(mono) == target_coeff)
```

# When to Use DSOS/SDSOS

## Use DSOS/SDSOS When:

- Problem is **large** (many variables, high degree)
- SDP solver **times out**
- Need **fast approximate** answer
- Problem structure is **sparse**

## Fall Back to SOS When:

- DSOS/SDSOS returns infeasible
- High precision required
- Problem is small

Our pipeline: Try DSOS → SDSOS → SOS in order.

## Reference

A. R. Bradley. "SAT-Based Model Checking without Unrolling."  
VMCAI 2011.

## Core Contribution

### Property-Directed Reachability (PDR):

- Discovers inductive invariants **incrementally**
- Uses **frames**  $F_0, F_1, \dots, F_k$  (over-approximations)
- **Blocks** counterexamples-to-induction with lemmas
- **Propagates** lemmas to strengthen frames

One of the most effective SAT-based model checking algorithms!

# IC3: Frame Invariants

## Definition (Frames)

A sequence of formulas  $F_0, F_1, \dots, F_k$  where:

- ①  $F_0 = \text{Init}$  (initial states)
- ②  $F_i \Rightarrow F_{i+1}$  (monotone strengthening)
- ③  $F_i \wedge T \Rightarrow F'_{i+1}$  (inductive consecution)
- ④  $F_i \Rightarrow \text{Safe}$  (all frames are safe)

## Intuition

$F_i$  over-approximates states reachable in  $\leq i$  steps.

If  $F_i = F_{i+1}$  for some  $i \Rightarrow \text{fixed point} \Rightarrow F_i$  is inductive invariant!

# IC3 Algorithm Overview

```
1:  $F_0 \leftarrow \text{Init}$ ;  $F_1 \leftarrow \text{Safe}$ ;  $k \leftarrow 1$ 
2: while true do
3:   while  $\exists s \in F_k$  with  $s \wedge T \wedge \neg \text{Safe}'$  do                                ▷ Bad state?
4:     if  $s \in F_0$  then
5:       return UNSAFE (counterexample from  $\text{Init}$ )
6:     end if
7:      $\text{Block}(s, k)$                                          ▷ Add lemma to block  $s$ 
8:   end while
9:    $\text{Propagate}()$                                          ▷ Push lemmas forward
10:  if  $F_i = F_{i+1}$  for some  $i$  then
11:    return SAFE (inductive invariant  $F_i$ )
12:  end if
13:   $k \leftarrow k + 1$ ;  $F_k \leftarrow \text{Safe}$ 
14: end while
```

## Counterexample-to-Induction (CTI)

A state  $s$  is a CTI at level  $k$  if:

$$s \in F_k \wedge s \xrightarrow{T} s' \wedge s' \notin \text{Safe}$$

## Blocking Procedure:

- ① Generalize  $s$  to a **cube**  $c$  (conjunction of literals)
- ② Find **minimal** cube that still needs blocking
- ③ Add lemma  $\neg c$  to  $F_k$  (and earlier frames if possible)
- ④ Recursively block predecessors of  $c$

Key: Each lemma strengthens the over-approximation without losing reachable states.

# IC3: Lemma Propagation

## Propagation Rule

If lemma  $\ell$  is in  $F_i$  and  $F_i \wedge T \Rightarrow \ell'$ , then add  $\ell$  to  $F_{i+1}$ .

## Why Propagate?

- Lemmas proven at lower levels may hold at higher levels
- Strengthens frames without redundant work
- Enables fixed-point detection

**Fixed Point:** When  $F_k = F_{k+1}$ , we have found an inductive invariant!

$$F_k \wedge T \Rightarrow F'_k \quad (\text{inductive})$$

# Implementation: IC3 Engine

```
@dataclass
class Frame:
    """IC3 frame: over-approximation of reachable states."""
    level: int
    clauses: Set[Clause]

class IC3Engine:
    """IC3/PDR verification engine (Paper \#10)."""

    def __init__(self, init: z3.BoolRef, trans: z3.BoolRef, safe: z3.BoolRef):
        self.init = init
        self.trans = trans
        self.safe = safe
        self.frames = [Frame(0, {self._init_to_clauses()})]

    def verify(self) -> VerificationResult:
        k = 1
        self.frames.append(Frame(1, {Clause.from_expr(self.safe)}))

        while True:

            while (cti := self._get_cti(k)) is not None:
                if not self._block(cti, k):
                    return VerificationResult('unsafe', self._extract_trace())

            self._propagate()

            if self._has_fixpoint():

    if self._has_fixpoint():
```

## Key Insight

IC3 lemmas can be **lifted** to polynomial constraints for barrier synthesis.

## Integration Strategy:

- ① Run IC3 to discover **discrete invariant**
- ② Extract lemmas from converged frames
- ③ Convert lemmas to **polynomial constraints**:

$$\text{Lemma: } x > 0 \rightarrow \text{Constraint: } g(x) = x \geq 0$$

- ④ Use constraints to **condition** barrier synthesis (Layer 2)
- ⑤ Reduced search space → faster synthesis!

# Lifting IC3 Lemmas to Polynomial Constraints

## Example

IC3 discovers lemma:  $\neg(x < 0 \wedge y > 10)$

Equivalent to:  $x \geq 0 \vee y \leq 10$

As polynomial constraint for Positivstellensatz:

$$\{x \geq 0\} \cup \{10 - y \geq 0\}$$

## Benefit

IC3 lemmas **partition** the state space, reducing the polynomial degree needed for barrier synthesis.

Instead of searching for  $B$  over all of  $\mathbb{R}^n$ , search over regions defined by IC3 lemmas.

# Paper #11: CHC/Spacer (Komuravelli et al. 2014)

## Reference

A. Komuravelli, A. Gurfinkel, S. Chaki. "SMT-Based Model Checking for Recursive Programs." CAV 2014.

## Core Contribution

**Constrained Horn Clauses (CHC)** for verification:

- Programs encoded as Horn clauses
- Invariants are **solutions** to Horn constraints
- SMT-based solving with interpolation
- Handles **recursion** and **procedures**

Spacer = IC3 + interpolation + Horn clauses. Very powerful!

# Constrained Horn Clauses

## Definition (CHC)

A **Constrained Horn Clause** has the form:

$$\phi \wedge P_1(\vec{x}_1) \wedge \cdots \wedge P_k(\vec{x}_k) \Rightarrow H(\vec{y})$$

where  $\phi$  is a constraint,  $P_i$  are uninterpreted predicates,  $H$  is the head.

## Encoding a Loop

```
while (x < n) { x = x + 1 }
```

Init:  $x = 0 \wedge n \geq 0 \Rightarrow \text{Inv}(x, n)$

Step:  $\text{Inv}(x, n) \wedge x < n \Rightarrow \text{Inv}(x + 1, n)$

Post:  $\text{Inv}(x, n) \wedge x \geq n \Rightarrow x = n$

# Spacer: Solving CHC with IC3

## Key Innovations

- Apply IC3 to **Horn clause** solving
- Use **interpolation** to discover predicate interpretations
- Handle **multiple** predicates simultaneously
- Support **recursion** via unfolding

## Algorithm Sketch:

- ① Under-approximate each predicate (start with false)
- ② Check if clauses are satisfied
- ③ If CEX found, block it with interpolant-derived lemma
- ④ Propagate lemmas across predicates
- ⑤ Converge to solution or prove unsatisfiable

# Implementation: Spacer CHC Solver

```
@dataclass
class HornClause:
    """A Constrained Horn Clause."""
    body_predicates: List[Tuple[str, List[z3.ExprRef]]]
    body_constraint: z3.BoolRef
    head: Tuple[str, List[z3.ExprRef]]


class SpacerCHC:
    """CHC solving via Spacer algorithm (Paper \#11)."""

    def __init__(self, clauses: List[HornClause]):
        self.clauses = clauses
        self.predicates = self._extract_predicates()
        self.interpretations = {p: z3.BoolVal(False) for p in self.predicates}

    def solve(self) -> Optional[Dict[str, z3.ExprRef]]:
        """Find predicate interpretations satisfying all clauses."""

        fp = z3.Fixedpoint()
        fp.set('engine', 'spacer')

        for pred in self.predicates:
            fp.register_relation(pred)

        for clause in self.clauses:
            fp.add_rule(self._clause_to_rule(clause))

        result = fp.query(self._get_query())
        if result == z3.sat:
            return self._extract_interpretations(fp)
```

## Reference

K. L. McMillan. "Interpolation and SAT-Based Model Checking."  
*CAV 2003.*

## Core Contribution

Use **Craig interpolation** for model checking:

- Bounded model checking (BMC) finds counterexamples
- Interpolants from UNSAT proofs yield **over-approximations**
- Iteratively refine until fixed point

Interpolation is the “magic ingredient” that makes refinement effective.

# IMC: Interpolation-Based Model Checking

## Algorithm

- ① **BMC phase:** Check  $\text{Init} \wedge T^k \wedge \neg \text{Safe}$  for increasing  $k$
- ② If SAT  $\Rightarrow$  **counterexample found**
- ③ If UNSAT  $\Rightarrow$  extract **interpolant**  $I$  from proof:

$$\underbrace{\text{Init}}_A \wedge \underbrace{T^k}_B \wedge \underbrace{\neg \text{Safe}}_B = \text{UNSAT}$$

Interpolant  $I$ :  $\text{Init} \Rightarrow I$  and  $I \wedge T^k \wedge \neg \text{Safe} = \text{UNSAT}$

- ④ Use  $I$  as over-approximation of reachable states
- ⑤ Check  $I \wedge T \Rightarrow I$  (inductiveness). If yes  $\Rightarrow$  **invariant!**

# Sequence Interpolants

## Definition (Sequence Interpolants)

For formulas  $A_0, A_1, \dots, A_n$  with  $\bigwedge A_i = \text{UNSAT}$ , sequence interpolants  $I_0, I_1, \dots, I_{n+1}$  satisfy:

- $I_0 = \top, I_{n+1} = \perp$
- $I_i \wedge A_i \Rightarrow I_{i+1}$
- $I_i$  uses only common symbols of  $A_0, \dots, A_{i-1}$  and  $A_i, \dots, A_n$

## For BMC Path

Partition:  $A_0 = \text{Init}, A_1 = T, A_2 = T, \dots, A_n = \neg \text{Safe}$

Interpolants give invariants at each time step!

# Implementation: IMC Verifier

```
class InterpolationEngine:
    """Craig interpolation for refinement."""

    def compute_interpolant(self, A: z3.BoolRef, B: z3.BoolRef) -> z3.BoolRef:
        """Compute interpolant I such that A => I and I /\ B = UNSAT."""
        solver = z3.Solver()
        solver.add(A)
        solver.add(B)

        if solver.check() == z3.sat:
            return None

        return self._extract_from_proof(solver.proof(), A, B)

class IMCVerifier:
    """Interpolation-based Model Checking (Paper \#15)."""

    def verify(self, init, trans, safe, max_depth=100):
        for k in range(max_depth):

            path_formula = self._unroll(init, trans, k, safe)

            if self._is_sat(path_formula):
                return VerificationResult('unsafe')

            I = self.interp.compute_interpolant(init, self._suffix(trans, k, safe))
```

# Paper #20: Assume-Guarantee (Pnueli 1985)

## Reference

A. Pnueli. "In Transition from Global to Modular Temporal Reasoning about Programs." *Logics and Models of Concurrent Systems*, 1985.

## Core Contribution

### Compositional verification:

- Decompose system into **components**
- Verify each component under **assumptions**
- Components **guarantee** properties to others
- Compose results: if all contracts hold  $\Rightarrow$  system is safe

Essential for verifying **large systems!**

# Assume-Guarantee Reasoning

## AG Triple

$\langle A \rangle M \langle G \rangle$  means:

"If component  $M$  runs in environment satisfying assumption  $A$ , then  $M$  guarantees property  $G$ ."

## Composition Rule

$$\frac{\langle A_1 \rangle M_1 \langle G_1 \rangle \quad \langle A_2 \rangle M_2 \langle G_2 \rangle \quad G_1 \Rightarrow A_2 \quad G_2 \Rightarrow A_1}{\langle A_1 \wedge A_2 \rangle M_1 \| M_2 \langle G_1 \wedge G_2 \rangle}$$

**Key:** Verify components separately, compose results!

# Assume-Guarantee for Barrier Synthesis

## Compositional Barrier Synthesis

For system  $S = M_1 \parallel M_2$ :

- ① Synthesize barrier  $B_1$  for  $M_1$  under assumption  $A$  on  $M_2$ 's behavior
- ② Synthesize barrier  $B_2$  for  $M_2$  under assumption  $B_1 \geq 0$
- ③ Verify:  $B_1 \geq 0 \Rightarrow A$  (assumption discharged)
- ④ Compose:  $B = B_1 \wedge B_2$  is barrier for  $S$

## Benefit

Can verify large systems by decomposing into manageable pieces.  
Each component's barrier is **smaller** and **faster** to synthesize.

# Implementation: Assume-Guarantee Verifier

```
@dataclass
class AGContract:
    """Assume-Guarantee contract for a component."""
    assumption: z3.BoolRef
    guarantee: z3.BoolRef
    component: Any

class AssumeGuaranteeVerifier:
    """Compositional verification (Paper \#20)."""

    def verify_composition(self, components: List[AGContract]) -> VerificationResult:

        for contract in components:
            result = self._verify_component(contract)
            if not result.verified:
                return VerificationResult('unknown',
                                           message=f'{contract.component} failed under assumptions')

        for i, c1 in enumerate(components):
            for j, c2 in enumerate(components):
                if i != j:

                    if not self._check_implies(c2.guarantee, c1.assumption):
                        return VerificationResult('unknown',
                                                  message=f'Assumption of {c1} not satisfied by {c2}')

    return VerificationResult('safe',
                             certificate=self.compose_barriers(components))
```

# Circular Assume-Guarantee Reasoning

## The Challenge

Components may have **circular dependencies**:

- $M_1$  assumes something about  $M_2$
- $M_2$  assumes something about  $M_1$

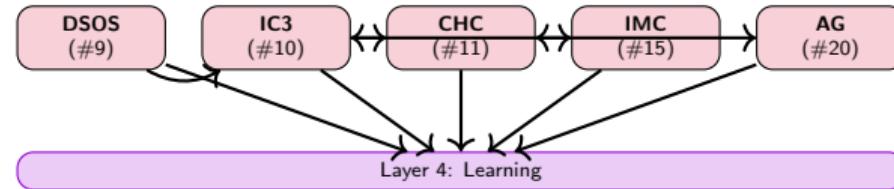
## Solution: Inductive Proof

Use **simultaneous induction**:

- ① Base case: Initial states satisfy all assumptions
- ② Inductive step: If assumptions hold at step  $k$ , guarantees hold at step  $k$ , therefore assumptions hold at step  $k + 1$

This is sound if the circular argument is well-founded!

# Layer 5: How Papers Integrate



## Integration Points

- DSOS provides fast relaxations when SOS is too slow
- IC3/CHC discover lemmas that constrain barrier synthesis
- IMC provides interpolants for refinement
- AG enables compositional verification of large systems

# Unified Interface: AdvancedVerificationEngine

```
class AdvancedVerificationEngine:
    """Unified interface for Layer 5 advanced verification."""

    def __init__(self, timeout_ms: int = 300000):
        self.timeout_ms = timeout_ms

        self.dsos = DSOSRelaxation(2, 6, timeout_ms // 5)
        self.ic3 = IC3Engine(None, None, None)
        self.spacer = SpacerCHC([])
        self.imc = IMCVerifier()
        self.ag = AssumeGuaranteeVerifier()

    def verify(self, system, property) -> VerificationResult:
        """Verify using portfolio of advanced methods."""

        result = self.ic3.verify(system.init, system.trans, property)
        if result.status != 'unknown':
            return result

        clauses = self._encode_as_chc(system, property)
        result = self.spacer.solve(clauses)
        if result is not None:
            return VerificationResult('safe', result)

        if system.is_compositional:
            return self.ag.verify_composition(system.components)
```

# Layer 5 Summary: Advanced Verification

## What Layer 5 Provides:

- Scalable SOS via DSOS/SDSOS
- Incremental reasoning (IC3)
- SMT-based solving (CHC/Spacer)
- Interpolation (IMC)
- Compositional verification (AG)

## Files:

- advanced.py
- dsos\_sdsos.py
- ic3\_pdr.py
- spacer\_chc.py
- interpolation\_imc.py
- assume\_guarantee.py

## Key Achievement

Layer 5 makes the full verification pipeline practical for **real-world systems**.

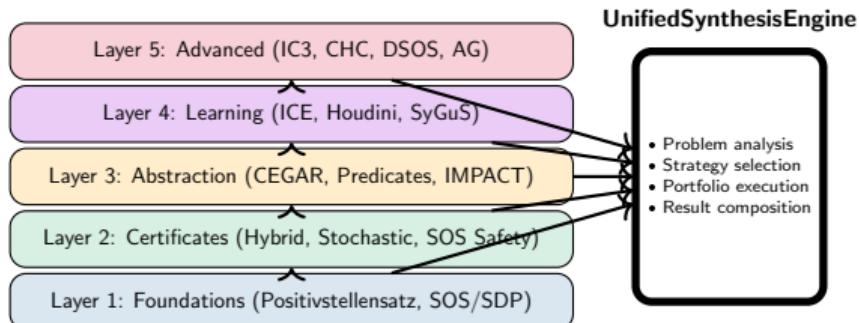
# Part VI

## Integration & Implementation

### Putting It All Together

UnifiedSynthesisEngine • ExtremeContextVerifier • Results

# The Complete Verification Pipeline



# UnifiedSynthesisEngine: The Orchestrator

## Responsibilities

- **Analyze** the verification problem
- **Select** appropriate techniques
- **Execute** in portfolio mode
- **Compose** results from multiple layers

## Key Design Principles

- ① **Sound:** Never report SAFE if bug exists
- ② **Complete:** Find bugs with concrete witnesses
- ③ **Adaptive:** Choose techniques based on problem
- ④ **Robust:** Fallback strategies at each level

# Problem Classification

```
class ProblemClassifier:
    """Classify verification problems for technique selection."""

    def classify(self, problem: Dict[str, Any]) -> ProblemAnalysis:

        n_vars = problem.get('n_vars', 2)
        max_degree = problem.get('max_degree', 4)
        dynamics_type = problem.get('dynamics_type', 'continuous')

        problem_type = self._classify_type(dynamics_type)
        problem_size = self._classify_size(n_vars, max_degree)

        is_sparse = self._check_sparsity(problem)
        is_linear = max_degree <= 1

        methods = self._recommend_methods(problem_type, problem_size, is_sparse)

        return ProblemAnalysis(
            problem_type=problem_type,
            problem_size=problem_size,
            n_vars=n_vars,
            max_degree=max_degree,
            is_sparse=is_sparse,
            recommended_methods=methods
        )
```

# Strategy Selection Based on Problem Type

Problem Type	Recommended Methods
Continuous Safety (small)	SOS Safety, Putinar, Barrier Synthesis
Continuous Safety (large)	Sparse SOS, DSOS, ICE Learning
Discrete Safety	IC3, CHC, Predicate Abstraction
Hybrid Safety	Hybrid Barriers, IC3, CEGAR
Stochastic Safety	Stochastic Barriers, SOS Safety
Compositional	Assume-Guarantee, IC3, CHC
Linear Systems	Linear analysis, DSOS

## Adaptive Strategy

Based on problem size and timeout:

- Small: Try SOS → Lasserre → CEGAR
- Medium: DSOS → ICE → IC3
- Large: Sparse SOS → CHC → AG

# Portfolio Execution

```
class PortfolioExecutor:
    """Execute multiple strategies in portfolio mode."""

    def run(self, problem: Dict[str, Any]) -> VerificationResult:
        start = time.time()
        best_result = VerificationResult(status='unknown')

        for strategy in self.strategies:

            remaining = self.timeout_ms - int((time.time() - start) * 1000)
            if remaining <= 0:
                break

            strategy.timeout_ms = remaining // len(self.strategies)

            result = strategy.execute(problem)

            if result.status == 'safe':
                return result

            if result.status == 'unsafe':
                best_result = result

        return best_result
```

# ExtremeContextVerifier: The User-Facing API

## Purpose

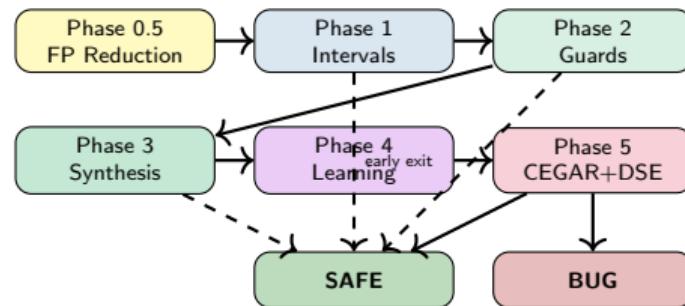
Provides a **high-level interface** for bug verification using all 20 papers.

## Verification Flow:

- ① **Phase 0.5:** False Positive Reduction (interprocedural checks)
- ② **Phase 1:** Interval/Dataflow Analysis (quick checks)
- ③ **Phase 2:** Guard Barrier Collection (existing guards)
- ④ **Phase 3:** Layer 2 Synthesis (SOS/SDP barriers)
- ⑤ **Phase 4:** Layer 4 Learning (ICE, Houdini)
- ⑥ **Phase 5:** CEGAR + DSE (refinement loop)

Returns **ContextAwareResult** with barriers, witnesses, and verification status.

# Verification Phases in Detail



Each phase can exit early if it determines SAFE or BUG.

# Phase 0.5: False Positive Reduction Strategies

## Four Integrated Strategies

Eliminate false positives **before** expensive verification.

### ① Interprocedural Guard Propagation

- Trace guards through call chains
- If caller validates, callee is protected

### ② Path-Sensitive Symbolic Execution

- Track constraints along each path
- Eliminate infeasible bug paths

### ③ Pattern-Based Safe Idiom Recognition

- Recognize common safe patterns
- Skip known-safe constructs

### ④ Dataflow Value Range Tracking

- Interval analysis for variables
- Prove safety from value ranges

# Strategy 1: Interprocedural Guard Propagation

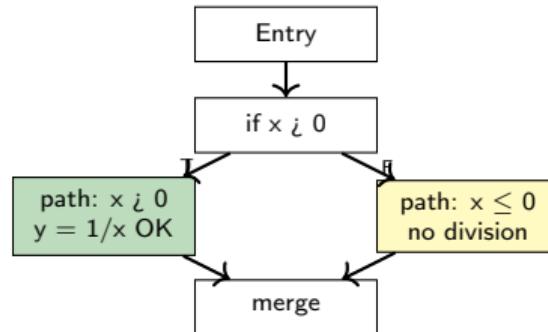
```
def caller(items):
    assert len(items) > 0
    return callee(items)

def callee(data):
    x = data[0]
```

## Propagation Logic

- ① Collect guards from all callers in the call chain
- ② Build a **guard barrier** for each guard
- ③ Check if any barrier protects the bug location
- ④ If protected: **Skip the bug** (false positive)

## Strategy 2: Path-Sensitive Symbolic Execution



### Key Insight

Track **path condition**  $\pi$  along each execution path. A bug is only real if  $\pi \wedge \text{bug\_condition}$  is satisfiable.

# Strategy 3: Pattern-Based Safe Idiom Recognition

## Common Safe Patterns

### List Access Patterns:

```
# Pattern 1: Explicit check
if items:
    x = items[0]

# Pattern 2: For loop
for i, item in enumerate(items):

# Pattern 3: Try-except
try:
    x = items[0]
except IndexError:
    x = default
```

### Division Patterns:

```
# Pattern 1: Guard
if divisor != 0:
    result = x / divisor

# Pattern 2: Or-default
result = x / (divisor or 1)

# Pattern 3: Max guard
result = x / max(divisor, 1)
```

# Strategy 4: Dataflow Value Range Tracking

## Interval Abstract Domain

Track  $[low, high]$  bounds for each variable through the program.

### Example:

$$\text{After } x = \text{len}(\text{items}) \qquad \Rightarrow x \in [0, \infty)$$

$$\text{After if } x > 0 \qquad \Rightarrow x \in [1, \infty)$$

$$\text{At items}[0] \qquad \Rightarrow \text{SAFE } (\text{len} \geq 1)$$

### Interval Operations:

- **Join:**  $[a, b] \sqcup [c, d] = [\min(a, c), \max(b, d)]$
- **Meet:**  $[a, b] \sqcap [c, d] = [\max(a, c), \min(b, d)]$
- **Widen:** Accelerate fixpoint convergence

# Phase 1: Quick Analysis (Interval + Dataflow)

## Purpose

Fast, lightweight checks before expensive synthesis.

### Interval Analysis Checks:

- Is divisor definitely positive? ⇒ No DIV\_ZERO
- Is index definitely in bounds? ⇒ No BOUNDS error
- Is pointer definitely non-null? ⇒ No NULL\_PTR

### Dataflow Facts Gathered:

- **Constants:** Known constant values
- **Types:** Definite type information
- **Nullability:** Definitely null / definitely not null
- **Aliases:** Variables with same value
- **Assignments:** Definitely assigned variables

## Phase 2: Guard Barrier Collection

### Guard-to-Barrier Translation

Convert explicit guards in code to formal barrier certificates.

#### Translation Examples:

Guard Code	Barrier Certificate
<code>assert len(x) &gt; 0</code>	$B(x) = \text{len}(x) - 1$
<code>if x is not None:</code>	$B(x) = \mathbb{1}[x \neq \text{None}]$
<code>if divisor != 0:</code>	$B(d) =  d  - \epsilon$
<code>if 0 &lt;= i &lt; len(arr):</code>	$B(i, arr) = \min(i, \text{len}(arr) - i - 1)$

If barrier  $B \geq 0$  at bug location  $\Rightarrow \text{SAFE}$

# Phase 3: Layer 2 Barrier Synthesis

## When Guards Are Absent

Synthesize barriers automatically using SOS/SDP techniques.

### Synthesis Problem:

Find  $B(x)$  such that:

$$\forall x \in \text{Init} : B(x) \geq \epsilon$$

$$\forall x \in \text{Unsafe} : B(x) \leq -\epsilon$$

$$\forall x, x' : (B(x) \geq 0 \wedge x \rightarrow x') \Rightarrow B(x') \geq 0$$

### Uses UnifiedSynthesisEngine:

- Try SOS Safety first (fast)
- Fall back to Hybrid Barrier synthesis
- Use Lasserre hierarchy for complex problems

# Phase 4: Layer 4 ICE Learning

## Learning from Examples

When synthesis fails, learn invariants from codebase examples.

### ICE Example Collection:

- **Positive:** States from successful executions
- **Negative:** States that led to bugs
- **Implications:** If state  $s$  is safe, successor  $s'$  should be

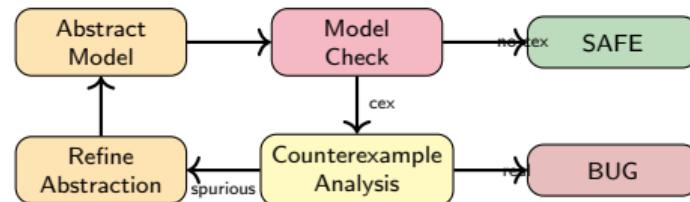
### Learning Process:

- ① Collect examples from crash summaries
- ② Train ICE learner on examples
- ③ Synthesize invariant that separates positive/negative
- ④ Verify learned invariant is inductive

# Phase 5: CEGAR + DSE Verification

## Final Refinement Loop

When all else fails, use CEGAR refinement and DSE ground truth.



**DSE:** Execute symbolically to find concrete counterexample or prove unreachable.

# Part VII

ICE Learning Applied to Barrier Synthesis

The Complete Integration

## Key Innovation

Use **Implication CounterExamples** to guide barrier parameter search.

### Traditional Barrier Synthesis:

- Template enumeration: Try all parameter combinations
- Slow, may miss good barriers

### ICE-Guided Synthesis:

- Learn from **counterexamples** to failed verification
- Each failure teaches what constraints to add
- Converges faster to valid barrier

# ICE for Barriers: Three Example Types

## 1. Positive Examples (Init)

States where  $B(x) \geq 0$  **must** hold.

⇒ Initial states of the program

## 2. Negative Examples (Unsafe)

States where  $B(x) < 0$  **must** hold.

⇒ States that lead to bugs

## 3. Implication Examples (Step)

Pairs  $(s, s')$  where  $B(s) \geq 0 \Rightarrow B(s') \geq 0$ .

⇒ Transition pairs from symbolic execution

# ICE Barrier Synthesis Algorithm

```
1: Input: Template  $B_\theta(x)$ , Init, Unsafe, Trans
2: Output: Valid barrier or UNKNOWN
3:
4: positive  $\leftarrow$  sample(Init)
5: negative  $\leftarrow$  sample(Unsafe)
6: implications  $\leftarrow$  sample_transitions(Trans)
7:
8: while not timeout do
9:    $\theta^* \leftarrow$  ICE_Learn(positive, negative, implications)
10:   $B \leftarrow B_{\theta^*}$ 
11:
12:  if verify_inductive( $B$ ) then
13:    return  $B$                                  $\triangleright$  Found valid barrier!
14:  else
15:    cex  $\leftarrow$  extract_counterexample()
16:    Add cex to appropriate example set
17:  end if
18: end while
```

## When Verification Fails

Extract counterexample and classify:

### 1. Init Counterexample:

- Found  $s_0 \in \text{Init}$  with  $B(s_0) < \epsilon$
- Add  $s_0$  to **positive** examples
- Force learner to satisfy  $B(s_0) \geq \epsilon$

### 2. Unsafe Counterexample:

- Found  $s_u \in \text{Unsafe}$  with  $B(s_u) > -\epsilon$
- Add  $s_u$  to **negative** examples
- Force learner to satisfy  $B(s_u) < -\epsilon$

### 3. Step Counterexample:

- Found transition  $(s, s')$  with  $B(s) \geq 0$  but  $B(s') < 0$
- Add  $(s, s')$  to **implication** examples

## Choosing the Right Template Family

Different bug types need different barrier shapes.

Bug Type	Template	Form
BOUNDS	Linear	$B(i, len) = c_1 \cdot i + c_2 \cdot len + c_0$
DIV_ZERO	Absolute	$B(d) =  d  - \epsilon$
NULL_PTR	Indicator	$B(p) = \mathbb{1}[p \neq \text{null}]$
OVERFLOW	Quadratic	$B(x) = c_2 x^2 + c_1 x + c_0$
Complex	Polynomial	$B(\mathbf{x}) = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha}$

**Template Degree Increase:** If learning fails, try higher degree template.

# ICE Barrier: Z3 Integration

```
def ice_learn_conjunction(
    variables: dict[str, z3.IntNumRef],
    candidate_predicates: dict[str, z3.BoolRef],
    positive: list[dict[str, int]],
    negative: list[dict[str, int]],
    implications: list[tuple[dict, dict]],
) -> ICEResult:
    """Learn conjunction over predicates using SMT."""

    include = {name: z3.Bool(f"inc_{name}") for name in candidate_predicates}

    solver = z3.Optimize()

    for ex in positive:
        for name, pred in candidate_predicates.items():
            if not eval_pred(pred, ex):
                solver.add(z3.Not(include[name]))

    for ex in negative:
        falsifying = [include[n] for n, p in candidate_predicates.items()
                      if not eval_pred(p, ex)]
        solver.add(z3.Or(*falsifying))

    for pre, post in implications:
        solver.add(z3.Implies(holds_on(pre), holds_on(post)))
```

# ICE Barrier: Practical Example

```
def process_items(items):
    x = items[0]
    return x * 2
```

## ICE Learning Trace:

Round	Examples	Learned Barrier
1	+ : [1], [2], [5] - : []	$B = \text{len} \geq 1$
Verify	Init fails: [] is initial!	
2	+ : [1], [2], [5] - : [] (added)	$B = \text{len} > 0$
Verify	<b>SUCCESS!</b>	

## Why Implications Matter

Implications capture the **inductiveness** requirement.

## Example: Loop Invariant

```
i = 0
while i < len(arr):
    x = arr[i] # BOUNDS check
    i += 1
```

### Implication Examples:

- $(i = 0, \text{len} = 3) \rightarrow (i = 1, \text{len} = 3)$ : Must preserve  $0 \leq i < \text{len}$
- $(i = 1, \text{len} = 3) \rightarrow (i = 2, \text{len} = 3)$ : Same invariant
- $(i = 2, \text{len} = 3) \rightarrow \text{exit}$ : Loop terminates safely

Learned:  $B(i, \text{len}) = \min(i, \text{len} - i - 1) \geq -1$

# ICE Barrier: Convergence Properties

## Theorem (ICE Convergence)

*If a valid barrier exists in the template family, ICE learning will find it in finite iterations (under mild conditions).*

### Conditions for Convergence:

- ① Template family contains a valid barrier
- ② Counterexample oracle is sound
- ③ Example space is finite (or finitely representable)

### Complexity:

- Each round adds at least one constraint
- Max iterations =  $O(|\text{example space}|)$
- In practice: typically < 20 rounds

# Part VIII

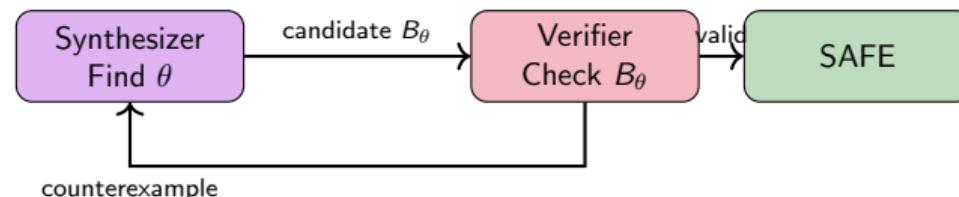
CEGIS: CounterExample-Guided Inductive Synthesis

Complete Algorithm and Implementation

# CEGIS: Overview

## Key Idea

Alternate between **synthesis** (find candidate) and **verification** (check candidate).



Counterexamples from failed verification **guide** the next synthesis attempt.

# CEGIS: Complete Algorithm

```
1: Input: Template  $B_\theta$ , Init, Safe, Unsafe, Dynamics
2: Output: Barrier certificate or UNKNOWN
3:
4: constraints  $\leftarrow \emptyset$ 
5:
6: while iterations < max_iterations do
7:   // SYNTHESIS PHASE
8:    $\theta^* \leftarrow \text{Solve}(\text{constraints})$                                 ▷ Find parameters
9:   if no solution then
10:    return UNKNOWN                                                 ▷ Template too weak
11:   end if
12:
13:   // VERIFICATION PHASE
14:    $B \leftarrow B_{\theta^*}$ 
15:   (valid, cex)  $\leftarrow \text{CheckInductive}(B, \text{Init}, \text{Safe}, \text{Unsafe}, \text{Dyn})$ 
16:   if valid then
17:     return  $B$                                                  ▷ Found valid barrier!
18:   end if
```

# CEGIS: Synthesis Phase

## Goal

Find parameter values  $\theta$  satisfying all collected constraints.

### Constraint Types:

- ① **Init constraints:**  $B_\theta(s_0^{(i)}) \geq \epsilon$  for sampled  $s_0^{(i)} \in \text{Init}$
- ② **Unsafe constraints:**  $B_\theta(s_u^{(i)}) \leq -\epsilon$  for sampled  $s_u^{(i)} \in \text{Unsafe}$
- ③ **Step constraints:**  $B_\theta(s'^{(i)}) \geq 0$  for transitions where  $B_\theta(s^{(i)}) \geq 0$
- ④ **Exclusion constraints:** Exclude previous failed  $\theta$  values

**Solver:** Use Z3 to find satisfying  $\theta$ , or prove unsatisfiable.

## Goal

Check if candidate barrier  $B_\theta$  is inductive.

### Three Conditions to Verify:

- ① **Init Condition:**  $\forall s \in \text{Init}. B(s) \geq \epsilon$ 
  - Z3 query: Is  $\exists s \in \text{Init}. B(s) < \epsilon$  satisfiable?
- ② **Unsafe Condition:**  $\forall s \in \text{Unsafe}. B(s) \leq -\epsilon$ 
  - Z3 query: Is  $\exists s \in \text{Unsafe}. B(s) > -\epsilon$  satisfiable?
- ③ **Step Condition:**  $\forall s, s'. (B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$ 
  - Z3 query: Is  $\exists s, s'. B(s) \geq 0 \wedge s \rightarrow s' \wedge B(s') < 0$  satisfiable?

If all UNSAT: Barrier is **valid**. Otherwise: Extract counterexample.

# CEGIS: Counterexample Extraction

```
@dataclass
class Counterexample:
    """A counterexample from failed verification."""
    kind: str
    model: z3.ModelRef
    state_values: dict[str, any]
    variable_value: Optional[float] = None
    barrier_value: Optional[float] = None

def extract_counterexample(solver: z3.Solver,
                           barrier: BarrierCertificate,
                           kind: str) -> Counterexample:
    """Extract counterexample from SAT result."""
    model = solver.model()

    state_values = {}
    for decl in model.decls():
        state_values[decl.name()] = model[decl]

    barrier_value = eval_barrier(barrier, state_values)

    return Counterexample(
        kind=kind,
        model=model,
        state_values=state_values,
        barrier_value=barrier_value
    )
```

# CEGIS: Adding Constraints from Counterexamples

## Constraint Generation

Turn counterexample into constraint that excludes it.

**Init Counterexample** at  $s_0$ :

Add constraint:  $B_\theta(s_0) \geq \epsilon$

**Unsafe Counterexample** at  $s_u$ :

Add constraint:  $B_\theta(s_u) \leq -\epsilon$

**Step Counterexample** at  $(s, s')$ :

Add constraint:  $B_\theta(s) \geq 0 \Rightarrow B_\theta(s') \geq 0$

**Exclusion Constraint** (prevent same  $\theta$ ):

$$\theta \neq \theta_{\text{prev}}^*$$

## Available Templates

### 1. Quadratic Barrier:

$$B(x) = c_2x^2 + c_1x + c_0$$

### 2. Bivariate Quadratic:

$$B(x, y) = c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y + c_{00}$$

### 3. Polynomial (degree $d$ ):

$$B(\mathbf{x}) = \sum_{|\alpha| \leq d} c_\alpha \mathbf{x}^\alpha$$

### 4. Custom Templates:

- Bounds:  $B(i, \text{len}) = c_1 \cdot (\text{len} - i - 1) + c_0$
- Division:  $B(d) = c_1 \cdot |d| + c_0$

# CEGIS: Result Structure

```
@dataclass
class CEGISResult:
    """Result of CEGIS synthesis."""
    success: bool
    barrier: Optional[BarrierCertificate] = None
    inductiveness: Optional[InductivenessResult] = None
    iterations: int = 0
    counterexamples_collected: int = 0
    synthesis_time_ms: float = 0.0
    termination_reason: str = "unknown"
    counterexamples: list[Counterexample] = field(default_factory=list)

    def summary(self) -> str:
        if self.success:
            return f"CEGIS SUCCESS: {self.barrier.name} \
                    f"({self.iterations} iters, {self.counterexamples_collected} CEs)"
        else:
            return f"CEGIS FAILED: {self.termination_reason}"
```

## Termination Reasons:

- inductive\_barrier\_found: Success!
- parameter\_space\_exhausted: Template too weak
- timeout: Need more time or simpler problem

# CEGIS: Practical Trace

**Problem:** Verify  $x = \text{arr}[i]$  where  $i = \text{len}(\text{arr}) - 1$

Iter	Candidate	Result	Action
1	$B = i - \text{len} + 1$	Init fail: $i = 0, \text{len} = 1$	Add $B(0, 1) \geq 0$
2	$B = i - \text{len} + 1$	Step fail: increment $i$	Add step constraint
3	$B = \text{len} - i - 1$	Unsafe fail: $i = \text{len}$	Add $B(\text{len}, \text{len}) < 0$
4	$B = \text{len} - i - 1$	<b>PASS</b>	Done!

**Final Barrier:**  $B(i, \text{len}) = \text{len} - i - 1$

- $i = \text{len} - 1 \Rightarrow B = 0 \geq 0 \checkmark$
- $i \geq \text{len} \Rightarrow B < 0$  (unsafe)  $\checkmark$

# Part IX

SOS/SDP Integration with Barrier Synthesis

Semidefinite Programming for Polynomial Proofs

# SOS/SDP: The Barrier-SDP Connection

## Key Insight

Barrier conditions become **polynomial positivity** conditions, which reduce to **SDP feasibility**.

**Barrier Condition:**  $\forall x \in \text{Init}. B(x) \geq \epsilon$

**Polynomial Form:**  $B(x) - \epsilon \geq 0$  on semialgebraic set Init

**SOS Relaxation:** Find SOS  $\sigma_0, \sigma_1, \dots$  such that:

$$B(x) - \epsilon = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x)$$

where  $g_i(x) \geq 0$  define Init.

**SDP:** Finding SOS polynomials is an SDP!

# SOS/SDP: Gram Matrix Formulation

Theorem (Parrilo)

$p(x)$  is SOS of degree  $2d$  iff  $p(x) = \mathbf{m}(x)^T Q \mathbf{m}(x)$  where  $Q \succeq 0$ .

**Monomial Vector:**  $\mathbf{m}(x) = [1, x_1, x_2, \dots, x_1^d, \dots]^T$

**Gram Matrix:**  $Q$  is positive semidefinite (PSD)

**Example:**  $p(x) = x^4 + 2x^2 + 1$

$$\mathbf{m}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

**Check:**  $\mathbf{m}^T Q \mathbf{m} = 1 + 2x^2 + x^4 \checkmark$

# SOS/SDP: Coefficient Matching

## Linear Constraints on $Q$

Matching polynomial coefficients gives linear constraints on  $Q$  entries.

For  $p(x) = c_0 + c_1x + c_2x^2$  with  $\mathbf{m}(x) = [1, x]^T$ :

$$Q = \begin{bmatrix} q_{00} & q_{01} \\ q_{01} & q_{11} \end{bmatrix}$$

### Matching:

Constant term:  $q_{00} = c_0$

Linear term:  $2q_{01} = c_1$

Quadratic term:  $q_{11} = c_2$

**SDP:** Find  $Q \succeq 0$  satisfying these linear constraints.

# SOS/SDP: Complete Barrier SDP

**Goal:** Synthesize barrier  $B(x) = \mathbf{m}(x)^T P \mathbf{m}(x)$

**SDP Program:**

Find  $P, Q_0, Q_1, \dots$

s.t.  $P \succeq 0$  (barrier is SOS)

$Q_0 \succeq 0, Q_1 \succeq 0, \dots$  (multipliers are SOS)

$$B(x) - \epsilon = \sigma_0(x) + \sum_i \sigma_i(x) g_i^{\text{Init}}(x)$$

$$- B(x) - \epsilon = \tau_0(x) + \sum_j \tau_j(x) h_j^{\text{Unsafe}}(x)$$

$$- \dot{B}(x) = \rho_0(x) + \sum_k \rho_k(x) (B(x) \cdot \text{Safe}_k(x))$$

**Solve with SDP solver** (MOSEK, SDPA, SCS).

# SOS/SDP: Positivstellensatz Multipliers

## Putinar's Positivstellensatz

If  $p > 0$  on  $\{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  (compact), then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i$  are SOS.

### For Barrier Init Condition:

- $p(x) = B(x) - \epsilon$  (what we want positive)
- $g_i(x)$  define the initial region
- Find SOS multipliers  $\sigma_i$

**Degree Bound:** Multipliers have degree  $\leq 2d - \deg(g_i)$  for degree- $2d$  proof.

# SOS/SDP: Lasserre Hierarchy Connection

## When SOS Fails

Increase degree in Lasserre hierarchy for stronger proofs.

### Lasserre Level $k$ :

- Use monomials up to degree  $k$
- Larger SDP, tighter relaxation
- Level  $\infty$ : exact polynomial optimization

### Barrier Synthesis Strategy:

- ① Start with low degree  $d = 2$
- ② If SDP infeasible, increase to  $d = 4$
- ③ Continue until feasible or timeout

**Convergence:** For polynomial systems, hierarchy converges finitely.

# SOS/SDP: Sparse SOS for Scalability

## Problem

SDP size grows as  $O(n^{2d})$  for  $n$  variables, degree  $2d$ .

## Sparse SOS (Paper #8):

- Exploit **correlative sparsity**: variables interact locally
- Decompose large SDP into smaller coupled SDPs
- Use **chordal decomposition** of variable graph

## Example:

- Full:  $n = 10$ , degree 4  $\Rightarrow$  715-dimensional SDP
- Sparse: Decompose into 10 coupled 15-dimensional SDPs
- Speedup:  $\sim 100x$  for structured problems

# SOS/SDP: DSOS/SDSOS Relaxations

## Faster Alternatives to SDP

Replace PSD constraint with diagonal dominance (LP/SOCP).

### DSOS (Diagonally-dominant SOS):

$$p(x) = \sum_{i,j} \lambda_{ij}(x_i \pm x_j)^2 \quad (\lambda_{ij} \geq 0)$$

⇒ Linear program!

### SDSOS (Scaled diagonally-dominant SOS):

Gram matrix  $Q$  is scaled diagonally dominant

⇒ Second-order cone program!

### Tradeoff:

Method	Complexity	Completeness
SOS/SDP	$O(n^{3.5})$	High

# SOS/SDP: Implementation in the Pipeline

```
class SOSSafetyChecker:
    """SOS-based safety checking (Paper \#3)."""

    def check_safety(self, conditions: BarrierConditions,
                     degree: int = 4) -> Optional[Polynomial]:
        """
        Check if barrier exists via SOS/SDP.

        1. Build SDP for barrier conditions
        2. Solve with SDP solver
        3. Extract barrier from solution
        """

        basis = MonomialBasis(self.n_vars, degree // 2)

        Q = self._create_gram_matrix(basis)

        self._add_matching_constraints(Q, conditions)

        self._add_psd_constraint(Q)

        if self._solve_sdp():
            return self._extract_barrier(Q)
        return None
```

# Part X

Hybrid System Barrier Certificates

Multi-Mode Safety Verification

# Hybrid Systems: Overview

## Definition

A **hybrid system** combines continuous dynamics with discrete mode switches.

## Hybrid Automaton:

- **Modes:**  $\{q_1, q_2, \dots, q_m\}$
- **Continuous dynamics:**  $\dot{x} = f_q(x)$  in mode  $q$
- **Invariants:**  $I_q(x)$  must hold in mode  $q$
- **Guards:**  $G_{q \rightarrow q'}(x)$  enables transition
- **Resets:**  $R_{q \rightarrow q'}(x)$  updates state on transition

**Example:** Thermostat (heating on/off), Bouncing ball (flight/impact)

# Hybrid Barriers: Multi-Mode Certificates

## Hybrid Barrier Certificate (Paper #1)

Separate barrier  $B_q(x)$  for each mode  $q$ .

### Conditions:

- ① **Init:**  $\forall q, x. (x \in \text{Init}_q) \Rightarrow B_q(x) \geq \epsilon$
- ② **Unsafe:**  $\forall q, x. (x \in \text{Unsafe}_q) \Rightarrow B_q(x) \leq -\epsilon$
- ③ **Flow:**  $\forall q, x. (x \in I_q \wedge B_q(x) \geq 0) \Rightarrow \dot{B}_q(x) \leq 0$
- ④ **Jump:**  $\forall q, q', x. (G_{q \rightarrow q'}(x) \wedge B_q(x) \geq 0) \Rightarrow B_{q'}(R(x)) \geq 0$

**Key:** Barriers must be **consistent across transitions**.

# Hybrid Barriers: Lie Derivative

## Flow Condition

In mode  $q$  with dynamics  $\dot{x} = f_q(x)$ , barrier must not increase:

$$\mathcal{L}_{f_q} B_q(x) = \nabla B_q(x) \cdot f_q(x) \leq 0$$

**Example:** Linear system  $\dot{x} = Ax$ , quadratic barrier  $B(x) = x^T Px$

$$\dot{B} = x^T (A^T P + PA)x \leq 0$$

$\Rightarrow$  Need  $A^T P + PA \preceq 0$  (Lyapunov condition)

**SOS Encoding:**  $-\mathcal{L}_f B$  is SOS on safe region.

# Hybrid Barriers: Jump Condition

## Transition Safety

When switching from  $q$  to  $q'$  with reset  $x' = R(x)$ :

$$B_q(x) \geq 0 \wedge G_{q \rightarrow q'}(x) \Rightarrow B_{q'}(R(x)) \geq 0$$

**Challenge:** Reset maps can be complex (nonlinear, partial).

### SOS Encoding:

$$B_{q'}(R(x)) - \sigma(x) \cdot B_q(x) - \sum_i \tau_i(x) G_i(x) \text{ is SOS}$$

where  $\sigma, \tau_i$  are SOS multipliers.

**Interpretation:** If in safe region ( $B_q \geq 0$ ) and guard holds, then post-reset state is safe ( $B_{q'} \geq 0$ ).

# Hybrid Barrier Synthesis

```
1: Input: Hybrid automaton  $H = (Q, X, f, I, G, R)$ 
2: Output: Barrier certificates  $\{B_q\}_{q \in Q}$ 
3:
4: Choose template degree  $d$ 
5: for each mode  $q \in Q$  do
6:   Create template  $B_q(x) = \sum_{|\alpha| \leq d} c_q^\alpha x^\alpha$ 
7: end for
8:
9: // Build SDP for all conditions
10: Add Init constraints for each mode
11: Add Unsafe constraints for each mode
12: Add Flow constraints (Lie derivatives)
13: Add Jump constraints (transition safety)
14:
15: Solve combined SDP
16: return Extracted barriers  $\{B_q\}$ 
```

# Hybrid Barriers: Bouncing Ball Example

## System:

- Mode 1 (flight):  $\dot{h} = v$ ,  $\dot{v} = -g$
- Mode 2 (impact):  $v' = -c \cdot v$  (coefficient of restitution)
- Guard:  $h = 0 \wedge v < 0$  (hit ground)
- Unsafe:  $h < 0$  (below ground)

## Barrier: $B(h, v) = h$

- Init:  $h \geq h_0 > 0 \Rightarrow B \geq h_0 \checkmark$
- Unsafe:  $h < 0 \Rightarrow B < 0 \checkmark$
- Flow:  $\dot{B} = v$  (can be positive or negative)
- Jump: At  $h = 0$ ,  $B' = 0 \geq 0 \checkmark$

Refined:  $B(h, v) = h + \epsilon$  ensures  $B > 0$  always.

# Stochastic Barriers: Overview (Paper #2)

## Stochastic Systems

Dynamics include noise:  $dx = f(x)dt + g(x)dW$

**Safety Question:** What is  $\Pr[\text{reach Unsafe}]$ ?

**Stochastic Barrier Certificate:**

- $B(x) \geq 0$  on initial states
- $B(x)$  is a **supermartingale**:  $\mathbb{E}[dB] \leq 0$
- $B(x) \leq -\epsilon$  on unsafe states

**Itô Condition (supermartingale):**

$$\mathcal{L}_f B + \frac{1}{2} \text{tr}(g^T \nabla^2 B \cdot g) \leq 0$$

# Stochastic Barriers: Probability Bound

Theorem (Prajna et al. 2007)

If  $B(x)$  is a stochastic barrier certificate, then:

$$\Pr[\text{reach Unsafe}] \leq \frac{\sup_{x \in \text{Init}} B(x)}{\inf_{x \in \text{Unsafe}} (-B(x))}$$

## Interpretation:

- Larger barrier gap  $\Rightarrow$  lower probability
- $B = +\infty$  on init,  $B = -\infty$  on unsafe  $\Rightarrow$  probability 0

**For Programs:** Model randomness as stochastic transitions.

# Stochastic Barrier Synthesis

## SOS Formulation:

Find  $B(x) = \sum c_\alpha x^\alpha$

s.t.  $B(x) - \epsilon$  is SOS on Init

- $B(x) - \epsilon$  is SOS on Unsafe

- $\mathcal{L}_f B - \frac{1}{2} \text{tr}(g^T \nabla^2 B \cdot g)$  is SOS on Safe

**Challenge:** Second-order term  $\nabla^2 B$  increases SDP size.

**Solution:** Use polynomial templates where Hessian is tractable.

**Example:** Quadratic  $B(x) = x^T P x$

$$\nabla^2 B = 2P \quad (\text{constant})$$

# Part XI

IC3/PDR: Property-Directed Reachability

Incremental Inductive Reasoning for Programs

## Key Innovation (Bradley 2011)

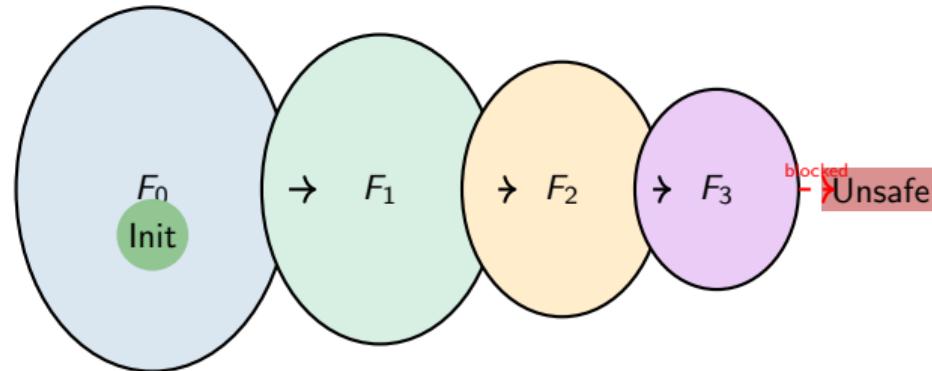
Discover inductive invariants **incrementally** using frames.

**Frame Sequence:**  $F_0, F_1, \dots, F_k$

- $F_i$  overapproximates states reachable in  $\leq i$  steps
- $F_0 = \text{Init}$
- $F_i \supseteq F_{i+1}$  (monotonic)
- Each  $F_i$  is a conjunction of clauses

**Convergence:** When  $F_i = F_{i+1}$ , we have an inductive invariant!

# IC3/PDR: Frame Structure



**Goal:** Show Unsafe is not reachable from any  $F_i$ .

# IC3/PDR: Algorithm Overview

```
1: Input: Init, Trans, Unsafe
2: Output: SAFE (with invariant) or UNSAFE (with trace)
3:
4:  $F_0 \leftarrow \text{Init}$ 
5:  $k \leftarrow 0$ 
6: while true do
7:   // BLOCKING: Remove bad states from frames
8:   while  $\exists s \in F_k \cap \text{Unsafe}$  do
9:     if cannot block  $s$  then
10:      return UNSAFE (extract trace)
11:    end if
12:    Block  $s$  by adding clause to appropriate frame
13:  end while
14:
15:  // PROPAGATION: Push clauses forward
16:   $k \leftarrow k + 1$ 
17:  Propagate clauses from  $F_{k-1}$  to  $F_k$ 
18:  if  $F_k = F_{k-1}$  then
```

## Counterexample to Induction (CTI)

A state  $s$  such that  $s \in F$ ; but  $\text{Post}(s) \cap \text{Unsafe} \neq \emptyset$ .

### Blocking Procedure:

- ① Find CTI:  $s$  can reach Unsafe in one step
- ② Generalize: Find clause  $c$  such that:
  - $s \not\models c$  (excludes  $s$ )
  - $c$  is inductive relative to  $F_{i-1}$
- ③ Add  $c$  to  $F_1, F_2, \dots, F_i$

**Key:** Generalization prevents blocking same state repeatedly.

## Moving Clauses Forward

If clause  $c$  is inductive relative to  $F_i$ , add it to  $F_{i+1}$ .

### Inductive Relative to $F_i$ :

$$F_i \wedge c \wedge \text{Trans} \Rightarrow c'$$

(If in  $F_i$  and  $c$  holds, then  $c$  holds after transition)

### Propagation Algorithm:

- ① For each clause  $c \in F_i$ :
- ② Check if  $c$  is inductive relative to  $F_i$ ;
- ③ If yes: add  $c$  to  $F_{i+1}$
- ④ If all clauses propagate:  $F_i = F_{i+1}$  (fixed point!)

**Cube:** Conjunction of literals (represents states)

$$\text{cube } s = (x > 0) \wedge (y \leq 5) \wedge (z = 0)$$

**Clause:** Disjunction of literals (excludes states)

$$\text{clause } c = (x \leq 0) \vee (y > 5) \vee (z \neq 0) = \neg s$$

**Relationship:**

- Bad cube  $s$  represents states to block
- Blocking clause  $c = \neg s$  excludes those states
- Frame  $F_i = \bigwedge_j c_j$  is conjunction of blocking clauses

# IC3/PDR: Application to Programs

**Program State:**  $(pc, \sigma) = (\text{program counter}, \text{variable store})$

**Transition Relation:** Derived from CFG

$$\text{Trans}((pc, \sigma), (pc', \sigma')) \iff \text{edge } pc \rightarrow pc' \text{ with update } \sigma \rightarrow \sigma'$$

**Unsafe States:** Bug locations

$$\text{Unsafe} = \{(pc, \sigma) : pc = \text{bug\_loc} \wedge \text{bug\_condition}(\sigma)\}$$

**Example (DIV\_ZERO):**

$$\text{Unsafe} = \{(pc, \sigma) : pc = 42 \wedge \sigma[d] = 0\}$$

## Key Insight

IC3 lemmas become **side conditions** for barrier synthesis.

### Bridge from IC3 to Barriers:

- ① Run IC3 to discover invariant clauses
- ② Lift clauses to polynomial constraints:
  - $(x > 0) \Rightarrow \text{constraint } x - \epsilon \geq 0$
  - $(y \leq 5) \Rightarrow \text{constraint } 5 - y \geq 0$
- ③ Add lifted constraints to barrier SDP
- ④ Solve constrained SDP for barrier

**Benefit:** IC3 prunes infeasible regions before expensive SOS.

# IC3/PDR: Implementation

```
class IC3Engine:
    """IC3/PDR for invariant discovery (Paper \#10)."""

    def __init__(self, n_vars: int, timeout_ms: int = 60000):
        self.n_vars = n_vars
        self.timeout_ms = timeout_ms
        self.frames: List[Frame] = []

    def verify(self, init: z3.BoolRef, trans: z3.BoolRef,
               unsafe: z3.BoolRef) -> IC3Result:
        """Run IC3/PDR algorithm."""
        self.frames = [Frame(clauses={init})]

        while not timeout():

            blocked = self._block_all_cti()
            if not blocked:
                return IC3Result(status='unsafe', trace=self._extract_trace())

            self._propagate_clauses()

            if self._check_fixed_point():
                invariant = self._extract_invariant()
                return IC3Result(status='safe', invariant=invariant)

        return IC3Result(status='unknown')
```

# Part XII

Constrained Horn Clauses (CHC) and Spacer

SMT-Based Program Verification

# CHC: Constrained Horn Clauses

## Definition

A CHC is a first-order formula of the form:

$$\forall \vec{x}.(p_1(\vec{x}_1) \wedge \dots \wedge p_n(\vec{x}_n) \wedge \phi(\vec{x})) \Rightarrow h(\vec{x}_h)$$

## Components:

- **Body predicates:**  $p_1, \dots, p_n$  (uninterpreted)
- **Constraint:**  $\phi$  (interpreted, e.g., linear arithmetic)
- **Head:**  $h$  (uninterpreted predicate or  $\perp$ )

**Solving CHC:** Find interpretations for predicates such that all clauses are satisfied.

# CHC: Encoding Programs

## Program:

```
def foo(x):
    y = 0
    while x > 0:
        y = y + 1
        x = x - 1
    assert y >= 0
```

## CHC Encoding:

$$\text{true} \Rightarrow \text{Inv}(x, 0) \quad (\text{init})$$

$$\text{Inv}(x, y) \wedge x > 0 \Rightarrow \text{Inv}(x - 1, y + 1) \quad (\text{loop})$$

$$\text{Inv}(x, y) \wedge x \leq 0 \wedge y < 0 \Rightarrow \perp \quad (\text{assert})$$

**Solution:**  $\text{Inv}(x, y) = (y \geq 0)$

# Spacer: CHC Solving Algorithm

Spacer (Komuravelli et al. 2014)

Combines IC3/PDR with interpolation for CHC solving.

## Key Ideas:

- ① **Under-approximation:** Track concrete reachability
- ② **Over-approximation:** Track inductive summaries
- ③ **Interpolation:** Generalize from concrete to symbolic
- ④ **Recursion handling:** Special frames for recursive calls

In Z3: `z3.Fixedpoint() with engine='spacer'`

## Barrier as CHC Solution

A barrier certificate  $B$  induces a CHC solution.

### Translation:

- $\text{Safe}(x) := B(x) \geq 0$
- Init clause:  $\forall x \in \text{Init}. B(x) \geq \epsilon \Rightarrow \text{Safe}(x) \checkmark$
- Step clause:  $\forall x, x'. \text{Safe}(x) \wedge x \rightarrow x' \Rightarrow \text{Safe}(x') \checkmark$
- Unsafe clause:  $\forall x. \text{Safe}(x) \wedge \text{Unsafe}(x) \Rightarrow \perp \checkmark$

**Benefit:** CHC solvers can discover barrier-like invariants automatically.

# Craig Interpolation: Overview

## Craig Interpolation Theorem

If  $A \wedge B$  is unsatisfiable, there exists  $I$  such that:

- ①  $A \Rightarrow I$
- ②  $I \wedge B$  is unsatisfiable
- ③  $\text{vars}(I) \subseteq \text{vars}(A) \cap \text{vars}(B)$

### For Verification:

- $A$  = formula for  $k$ -step reachability
- $B$  = unsafe states
- $I$  = over-approximation of reachable states

# Interpolation: Application to Barriers

**Scenario:** Have weak barrier  $B_0$ , need refinement.

**Process:**

- ① Query: Is  $B_0 \geq 0 \wedge \text{Unsafe}$  satisfiable?
- ② If UNSAT:  $B_0$  is sufficient
- ③ If SAT with counterexample  $(s, s')$ :
  - $A$  = path constraints to reach  $s$
  - $B$  = unsafe condition at  $s'$
  - Interpolant  $I$  suggests barrier refinement
- ④ Strengthen:  $B_1 = B_0 \wedge I$

**IMPACT (McMillan 2006):** Lazy abstraction with interpolants.

## IMC Algorithm (McMillan 2003)

Use interpolation to compute reachable state over-approximation.

### Algorithm:

- ① Check: Is  $\text{Init} \rightarrow^k \text{Unsafe}$  reachable? (BMC query)
- ② If SAT: Bug found at depth  $k$
- ③ If UNSAT: Compute interpolant sequence  $I_0, I_1, \dots, I_k$
- ④ Over-approximation:  $R \supseteq I_0 \cup I_1 \cup \dots \cup I_k$
- ⑤ If  $R$  is inductive: SAFE
- ⑥ Else: Increase  $k$  and repeat

**For Barriers:** Interpolants suggest barrier shape and constraints.

# Assume-Guarantee: Compositional Verification

Key Idea (Pnueli 1985)

Verify components separately under assumptions about environment.

**Rule:**

$$\frac{A \parallel E \models P \quad E \models A}{A \parallel E \models P}$$

**Interpretation:**

- Component  $A$  satisfies  $P$  under assumption about  $E$
- Environment  $E$  satisfies the assumption  $A$
- Therefore:  $A \parallel E$  satisfies  $P$

# Assume-Guarantee: Compositional Barriers

**Problem:** Verify large system with many components.

**Compositional Approach:**

- ① Decompose system into components  $C_1, C_2, \dots, C_n$
- ② For each  $C_i$ :
  - Assume barrier  $A_i$  holds for environment
  - Prove local barrier  $B_i$  for component
- ③ Verify:  $\bigwedge_i A_i \Rightarrow \bigwedge_i B_i$  (circular reasoning)
- ④ Solve for compatible assumption-guarantee pairs

**In Pipeline:** Used for interprocedural verification across modules.

# Part XIII

## Data Structures and Representations

The Foundation of the Implementation

# Polynomial Representation

```
@dataclass
class Monomial:
    """A monomial  $x_1^{a_1} * x_2^{a_2} * \dots * x_n^{a_n}$ """
    exponents: Tuple[int, ...]

    @property
    def degree(self) -> int:
        return sum(self.exponents)

    def multiply(self, other: 'Monomial') -> 'Monomial':
        """Multiply two monomials (add exponents)."""
        return Monomial(tuple(a + b for a, b in zip(self.exponents, other.exponents)))

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.ExprRef:
        """Convert to Z3 expression."""
        result = z3.RealVal(1)
        for i, exp in enumerate(self.exponents):
            for _ in range(exp):
                result = result * vars_z3[i]
        return result

@dataclass
class Polynomial:
    """Sparse polynomial representation."""
    n_vars: int
    terms: Dict[Monomial, float]

    @property
    def degree(self) -> int:
        return max(m.degree for m in self.terms.keys()) if self.terms else 0
```

# Polynomial Operations

```
class Polynomial:
    def add(self, other: 'Polynomial') -> 'Polynomial':
        """Add two polynomials."""
        result = Polynomial(max(self.n_vars, other.n_vars), dict(self.terms))
        for mono, coeff in other.terms.items():
            result.terms[mono] = result.terms.get(mono, 0) + coeff
        return result

    def multiply(self, other: 'Polynomial') -> 'Polynomial':
        """Multiply two polynomials."""
        result = Polynomial(max(self.n_vars, other.n_vars))
        for m1, c1 in self.terms.items():
            for m2, c2 in other.terms.items():
                new_mono = m1.multiply(m2)
                result.terms[new_mono] = result.terms.get(new_mono, 0) + c1 * c2
        return result

    def gradient(self) -> List['Polynomial']:
        """Compute gradient (list of partial derivatives)."""
        return [self._partial_derivative(i) for i in range(self.n_vars)]

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.ExprRef:
        """Convert to Z3 expression."""
        result = z3.RealVal(0)
        for mono, coeff in self.terms.items():
            result = result + z3.RealVal(coeff) * mono.to_z3(vars_z3)
        return result
```

# Semialgebraic Sets

```
@dataclass
class SemialgebraicSet:
    """
    A basic semialgebraic set: S = {x : g_1(x) >= 0, ..., g_m(x) >= 0}.

    Used to represent:
    - Initial states
    - Unsafe regions
    - Safe regions
    - Mode invariants
    """
    n_vars: int
    constraints: List[Polynomial]

    def contains(self, point: List[float]) -> bool:
        """Check if point is in the set."""
        return all(g.evaluate(point) >= 0 for g in self.constraints)

    def intersect(self, other: 'SemialgebraicSet') -> 'SemialgebraicSet':
        """Intersection of two sets."""
        return SemialgebraicSet(
            max(self.n_vars, other.n_vars),
            self.constraints + other.constraints
        )

    def to_z3(self, vars_z3: List[z3.ExprRef]) -> z3.BoolRef:
        """Convert to Z3 constraint."""
        return z3.And([g.to_z3(vars_z3) >= 0 for g in self.constraints])
```

# Barrier Certificate Structure

```
@dataclass
class BarrierCertificate:
    """
    A barrier certificate with metadata.

    Attributes:
        name: Human-readable identifier
        barrier_fn: The barrier function B: S -> R
        epsilon: Safety margin (default 0.01)
        description: Optional explanation
        variables: Variables referenced by barrier
    """
    name: str
    barrier_fn: Callable[[SymbolicMachineState], z3.ExprRef]
    epsilon: float = 0.01
    description: Optional[str] = None
    variables: list[str] = None

    def evaluate(self, state: SymbolicMachineState) -> z3.ExprRef:
        """Evaluate B(sigma) for the given state."""
        return self.barrier_fn(state)

# Example barrier for BOUNDS check
bounds_barrier = BarrierCertificate(
    name="bounds_check",
    barrier_fn=lambda s: s.get_local('len') - s.get_local('idx') - 1,
    epsilon=0.01,
    description="Index within bounds",
    variables=['len', 'idx']
)
```

# Crash Summary Structure

```
@dataclass
class CrashSummary:
    """
        Summary of a function's behavior for interprocedural analysis.

    Captures:
    - Function metadata
    - Parameter validations
    - Return guarantees
    - Guarded bugs (bugs protected by guards)
    - Guard facts collected during analysis
    """

    function_name: str
    module_name: str
    file_path: str
    line_number: int

    validated_params: Dict[int, Set[str]]

    return_guarantees: Set[str]

    guarded_bugs: Set[str]

    guard_facts: List[GuardFact]
```

# Guard Fact Structure

```
@dataclass
class GuardFact:
    """
        A guard fact from CFG analysis.

    Represents a condition that must hold at a program point.
    """
    guard_type: str
    variable: str
    condition: str
    source_location: int
    is_strong: bool

    def protects_bug(self, bug_type: str, bug_variable: str) -> bool:
        """Check if this guard protects against a specific bug."""
        if bug_variable != self.variable:
            return False

        protection_map = {
            'BOUNDS': {'assert_nonempty', 'len_check', 'range_check'},
            'DIV_ZERO': {'assert_nonzero', 'nonzero_check', 'positive_check'},
            'NULL_PTR': {'assert_nonnull', 'nonnull_check', 'if_nonnull'},
            'KEY_ERROR': {'key_in_check', 'haskey_check'},
        }

        return self.guard_type in protection_map.get(bug_type, set())
```

# Verification Result Structure

```
@dataclass
class ContextAwareResult:
    """
    Result of context-aware verification.
    """
    is_safe: bool
    bug_type: str
    bug_variable: Optional[str]

    guard_barriers: List[BarrierCertificate]
    synthesized_barriers: List[BarrierCertificate]
    learned_invariants: List[str]

    counterexample: Optional[Dict[str, Any]] = None
    dse_counterexample: Optional[Dict[str, Any]] = None

    verification_time_ms: float = 0.0
    phases_completed: List[str] = field(default_factory=list)

    def summary(self) -> str:
        if self.is_safe:
            return f"SAFE: {len(self.guard_barriers)} guards, " \
                   f"{len(self.synthesized_barriers)} synthesized"
        return f"UNSAFE: counterexample found"
```

# ICE Example Structure

```
@dataclass
class DataPoint:
    """A data point for learning."""
    values: Tuple[float, ...]
    label: str
    linked_to: Optional['DataPoint'] = None

@dataclass
class ICEExample:
    """
    ICE (Implication CounterExample) data.

    Three types:
    1. Positive: must be satisfied (initial states)
    2. Negative: must be violated (unsafe states)
    3. Implication: (pre, post) pairs - if pre satisfied, post must be
    """
    positive: List[DataPoint]
    negative: List[DataPoint]
    implications: List[Tuple[DataPoint, DataPoint]]

    def add_positive(self, values: Tuple[float, ...]):
        self.positive.append(DataPoint(values, 'positive'))

    def add_negative(self, values: Tuple[float, ...]):
        self.negative.append(DataPoint(values, 'negative'))

    def add_implication(self, pre: Tuple[float, ...], post: Tuple[float, ...]):
        pre_dp = DataPoint(pre, 'implication_pre')
        post_dp = DataPoint(post, 'implication_post')
```

# IC3 Frame and Clause Structures

```
@dataclass(frozen=True)
class Literal:
    """A literal in IC3/PDR."""
    variable: str
    negated: bool = False

    def __neg__(self) -> "Literal":
        return Literal(self.variable, not self.negated)

@dataclass(frozen=True)
class Cube:
    """Conjunction of literals (represents states)."""
    literals: FrozenSet[Literal]

    def negate(self) -> "Clause":
        return Clause(frozenset(-lit for lit in self.literals))

@dataclass(frozen=True)
class Clause:
    """Disjunction of literals (blocking lemma)."""
    literals: FrozenSet[Literal]

@dataclass
class Frame:
    """A frame in IC3: over-approximation of reachable states."""
    level: int
    clauses: Set[Clause]

    def add_clause(self, clause: Clause):
        self.clauses.add(clause)
```

# Part XIV

## Symbolic Execution Integration

From Python to Z3 Constraints

# Symbolic Machine State

```
@dataclass
class SymbolicMachineState:
    """
        Symbolic state for Python execution.

    Components:
    - pc: Path condition (Z3 formula)
    - locals: Local variable bindings (name -> Z3 expr)
    - heap: Symbolic heap model
    - stack: Call stack for interprocedural
    - taint: Taint tracking map
    """

    path_condition: z3.BoolRef
    locals: Dict[str, z3.ExprRef]
    heap: Dict[int, z3.ExprRef]
    stack: List['StackFrame']
    taint: Dict[str, Set[str]]

    def get_local(self, name: str) -> z3.ExprRef:
        """Get local variable value."""
        if name in self.locals:
            return self.locals[name]

        return z3.Int(f"sym_{name}")

    def with_constraint(self, constraint: z3.BoolRef) -> 'SymbolicMachineState':
        """Add constraint to path condition."""
        return SymbolicMachineState(
            path_condition=z3.And(self.path_condition, constraint),
            locals=self.locals.copy(), ...)
```

# Path Exploration Strategy

## Challenge

Exponential number of paths in program with branches.

## Mitigation Strategies:

- ① **Loop Bounding:** Limit loop iterations (default: 3)
- ② **Depth Limiting:** Maximum symbolic execution depth (default: 50)
- ③ **Path Merging:** Merge paths at join points
- ④ **Prioritization:** Explore bug-likely paths first
- ⑤ **Incremental Solving:** Use Z3 push/pop for efficiency

## For Barrier Synthesis:

- Collect Init states from entry paths
- Collect Unsafe states from bug-reaching paths
- Collect transitions from sequential execution

# Bug Condition Encoding

Encoding bugs as Z3 constraints:

Bug Type	Z3 Constraint (Unsafe)
BOUNDS	$\text{idx} < 0 \vee \text{idx} \geq \text{len}$
DIV_ZERO	$\text{divisor} = 0$
NULL_PTR	$\text{ptr} = \text{null}$
TYPE_ERROR	$\text{type}(x) \neq \text{expected}$
OVERFLOW	$ x  > \text{MAX\_INT}$
KEY_ERROR	$\text{key} \notin \text{dict}$
ASSERTION	$\neg \text{assertion\_condition}$

Verification Query:

$$\text{SAT}(\text{path\_condition} \wedge \text{bug\_condition}) \Rightarrow \text{Potential bug}$$

## Dynamic Symbolic Execution (DSE)

Combine concrete execution with symbolic reasoning.

### DSE Process:

- ① Start with concrete input
- ② Execute program, collecting path constraints
- ③ Negate constraints to explore new paths
- ④ Check if new path reaches bug

### For Barrier Verification:

- If barrier claims SAFE but DSE finds bug path: **Barrier is wrong**
- If DSE exhausts paths without bug: **Confirms SAFE**
- DSE provides ground truth for barrier validation

# Interprocedural Symbolic Execution

## Challenge

Handle function calls without exponential blowup.

## Approach: Function Summaries

- ① Analyze each function once
- ② Create summary: preconditions → postconditions
- ③ At call site: Apply summary instead of re-analyzing

## Summary Structure:

- **Precondition:** What caller must guarantee
- **Postcondition:** What callee guarantees
- **Effects:** Modified state (heap, globals)

**Barrier Implication:** Caller's barrier + summary = Callee's precondition satisfied.

# Constraint Simplification

## Goal

Keep path conditions tractable for Z3.

## Simplification Techniques:

- ① **Constant Propagation:** Replace  $x$  with value if known
- ② **Redundancy Elimination:** Remove implied constraints
- ③ **Expression Sharing:** Common subexpression elimination
- ④ **Theory-Specific:** Arithmetic simplification

## Z3 Tactics:

- `simplify`: Basic simplification
- `propagate-values`: Constant propagation
- `ctx-solver-simplify`: Context-aware simplification

# Handling Python-Specific Features

**Challenge:** Python is dynamically typed with complex semantics.

**Approach:**

**① Type Abstraction:**

- Track possible types for each variable
- Use union types:  $\text{type}(x) \in \{\text{int}, \text{str}\}$

**② Container Modeling:**

- Lists: length + element type
- Dicts: key set + value type

**③ Object Modeling:**

- Attribute access: `hasattr(obj, name)`
- Method resolution: approximate with summary

# Z3 Solver Configuration

```
def create_verification_solver(timeout_ms: int = 5000) -> z3.Solver:
    """Create optimally configured solver for verification."""
    solver = z3.Solver()

    solver.set("timeout", timeout_ms)

    solver.set("unsat_core", True)

    solver.set("arith.solver", 2)
    solver.set("arith.nl.nla", True)

    solver.set("proof", True)

    return solver

# Usage pattern for path exploration
solver = create_verification_solver()
solver.push()
solver.add(path_constraint)
if solver.check() == z3.sat:

    solver.add(bug_condition)
    if solver.check() == z3.sat:

        counterexample = solver.model()
solver.pop()
```

# Performance Optimizations

## Key Optimizations in the Pipeline:

### ① Caching:

- Cache verification results for repeated queries
- Cache Z3 check results for similar constraints

### ② Incremental Solving:

- Use push/pop for branch exploration
- Maintain learned clauses across queries

### ③ Parallelization:

- Run portfolio strategies in parallel
- Parallel path exploration (where independent)

### ④ Early Termination:

- Stop on first SAFE proof or BUG witness
- Skip expensive phases if cheap phase succeeds

# Part XV

## Bug Detection Categories

67 Bug Types with Barrier-Based Verification

# Bug Categories Overview

Category	Count	Examples
Logic Errors	12	Bounds, Div-zero, Null ptr
Type Errors	8	Type mismatch, Attribute error
Injection (Security)	9	SQL, Command, XSS
Crypto (Security)	8	Weak hash, Hardcoded key
Network (Security)	7	SSRF, Open redirect
Deserialization	4	Pickle, YAML, JSON
Resource	6	Leak, Double free
Concurrency	5	Race, Deadlock
Other	8	Assert, Unreachable
<b>Total</b>	<b>67</b>	

# Bug Type: BOUNDS (Array Out-of-Bounds)

```
def get_item(items, idx):
    return items[idx]
```

## Unsafe Condition:

$$idx < 0 \vee idx \geq len(items)$$

## Barrier Template:

$$B(idx, len) = \min(idx, len - idx - 1)$$

## Verification:

- $B \geq 0 \Leftrightarrow 0 \leq idx < len$
- Guard: `assert len(items) > 0`  $\Rightarrow B \geq 0$  for  $idx=0$

# Bug Type: DIV\_ZERO (Division by Zero)

```
def average(total, count):  
    return total / count
```

## Unsafe Condition:

$$\text{count} = 0$$

## Barrier Template:

$$B(\text{count}) = |\text{count}| - \epsilon$$

## Verification:

- $B \geq 0 \Leftrightarrow |\text{count}| \geq \epsilon \Leftrightarrow \text{count} \neq 0$
- Guard: if  $\text{count} \neq 0: \Rightarrow B \geq 0$
- Alternative:  $\text{count}$  or 1 pattern

# Bug Type: NULL\_PTR (Null Pointer Dereference)

```
def process(obj):
    return obj.method()
```

## Unsafe Condition:

$$\text{obj} = \text{None}$$

## Barrier Template:

$$B(\text{obj}) = \mathbb{1}[\text{obj} \neq \text{None}] - 0.5$$

## Verification:

- $B \geq 0 \Leftrightarrow \text{obj} \neq \text{None}$
- Guard: if  $\text{obj}$  is not  $\text{None}$ :  $\Rightarrow B \geq 0$
- Optional chaining:  $\text{obj}?.\text{method}()$  (Python 3.10+)

# Bug Type: KEY\_ERROR (Missing Dictionary Key)

```
def get_config(config, key):
    return config[key]
```

## Unsafe Condition:

$$\text{key} \notin \text{config}$$

## Barrier Template:

$$B(\text{key}, \text{config}) = \mathbb{1}[\text{key} \in \text{config}] - 0.5$$

## Verification:

- Guard: `if key in config: ⇒ B ≥ 0`
- Safe pattern: `config.get(key, default)`
- Safe pattern: `config.setdefault(key, value)`

# Bug Type: TYPE\_ERROR

```
def add_numbers(a, b):
    return a + b
```

## Unsafe Condition:

$$\text{type}(a) \neq \text{type}(b) \wedge \neg \text{coercible}(a, b)$$

## Barrier Template:

$$B(a, b) = \mathbb{1}[\text{type}(a) = \text{type}(b)] - 0.5$$

## Verification:

- Guard: `isinstance(a, int)` and `isinstance(b, int)`
- Type hints: `def add(a: int, b: int) -> int`
- Abstract interpretation: Track type sets

# Bug Type: SQL\_INJECTION (Security)

```
def query_user(cursor, username):
    sql = f"SELECT * FROM users WHERE name = '{username}'"
    cursor.execute(sql)
```

## Unsafe Condition:

$$\text{tainted}(\text{username}) \wedge \text{flows\_to}(\text{username}, \text{sql})$$

## Barrier (Taint-Based):

$$B(\text{input}) = \mathbb{K}[\neg \text{tainted}(\text{input})] - 0.5$$

## Sanitization Barrier:

- Use parameterized queries: `cursor.execute(sql, (username,))`
- Sanitization resets taint:  $B \geq 0$  after sanitize

# Bug Type: COMMAND\_INJECTION (Security)

```
def run_command(user_input):
    os.system(f"ls {user_input}")
```

## Unsafe Condition:

$$\text{tainted}(\text{user\_input}) \wedge \text{flows\_to}(\text{user\_input}, \text{os.system})$$

## Safe Patterns:

- Use subprocess with list args: `subprocess.run(['ls', user_input])`
- Whitelist validation: `if user_input in allowed_values`
- `shlex.quote()` for shell escaping

**Barrier:** Taint must not flow to dangerous sinks.

# Bug Type: PATH\_TRAVERSAL (Security)

```
def read_file(base_dir, filename):
    path = os.path.join(base_dir, filename)
    return open(path).read()
```

## Unsafe Condition:

$$"\text{..}" \in \text{filename} \vee \text{path} \not\subset \text{base\_dir}$$

## Barrier:

$$B(\text{path}, \text{base}) = \mathbb{1}[\text{realpath}(\text{path}).\text{startswith}(\text{base})] - 0.5$$

## Safe Pattern:

```
real_path = os.path.realpath(os.path.join(base_dir, filename))
if not real_path.startswith(os.path.realpath(base_dir)):
    raise SecurityError("Path traversal detected")
```

# Bug Type: WEAK\_CRYPTO (Security)

```
import hashlib
def hash_password(password):
    return hashlib.md5(password.encode()).hexdigest()
```

## Unsafe Condition:

$$\text{algorithm} \in \{\text{MD5, SHA1, DES}\}$$

**Detection:** Pattern matching on crypto API calls.

## Safe Pattern:

```
import bcrypt
def hash_password(password):
    return bcrypt.hashpw(password.encode(), bcrypt.gensalt())
```

**Note:** This is syntactic detection, not barrier-based.

# Bug Type: HARDCODED\_SECRET (Security)

```
API_KEY = "sk-12345abcdef"

def call_api():
    headers = {"Authorization": f"Bearer {API_KEY}"}
```

## Detection Patterns:

- String literals matching secret patterns (API keys, passwords)
- Entropy analysis: High-entropy strings are suspicious
- Variable names: password, secret, key, token

## Safe Pattern:

```
import os
API_KEY = os.environ.get("API_KEY")
```

# Bug Type: UNSAFE\_DESERIALIZATION (Security)

```
import pickle
def load_data(user_data):
    return pickle.loads(user_data)
```

## Unsafe Condition:

tainted(data)  $\wedge$  flows\_to(data, pickle.loads)

## Why Dangerous:

- Pickle can execute arbitrary code during unpickling
- `__reduce__` method allows code execution

## Safe Alternatives:

- Use JSON for untrusted data
- Use pickle only for trusted data
- Consider jsonpickle with restrictions

# Bug Type: RESOURCE LEAK

```
def read_config(path):
    f = open(path)
    data = f.read()

    return data
```

## Barrier (Resource State):

$$B(\text{resource}) = \mathbb{H}[\text{state(resource)} = \text{closed}] - 0.5$$

**Verification:** At function exit, all resources must be closed.

## Safe Pattern:

```
def read_config(path):
    with open(path) as f:
        return f.read()
```

# Bug Type: RACE\_CONDITION (Concurrency)

```
counter = 0
def increment():
    global counter
    counter += 1
```

## Unsafe Condition:

$$\text{shared}(\text{counter}) \wedge \neg \text{locked}(\text{counter})$$

## Detection:

- Identify shared mutable state
- Check for proper synchronization
- Happens-before analysis

## Safe Pattern:

```
import threading
lock = threading.Lock()
def increment():
    global counter
    with lock:
        counter += 1
```

# Bug Type: ASSERTION\_VIOLATION

```
def process(x):
    assert x > 0, "x must be positive"
    return 1 / x
```

## Unsafe Condition:

$$\neg(\text{assertion\_condition})$$

**Barrier:** Assertion itself is a barrier!

$$B(x) = x - \epsilon \quad (\text{from assert } x > 0)$$

## Verification:

- Check if path to assertion can have  $x \leq 0$
- If all paths have  $x > 0$ : Assertion always passes
- If some path has  $x \leq 0$ : Report potential violation

# Bug Type: UNREACHABLE\_CODE

```
def process(x):
    if x > 0:
        return "positive"
    elif x < 0:
        return "negative"
    elif x == 0:
        return "zero"
    else:
        return "impossible"
```

## Detection:

- Path condition to reach statement is UNSAT
- In example:  $\neg(x > 0) \wedge \neg(x < 0) \wedge \neg(x = 0)$  is UNSAT

## Barrier:

$$B = -1 \quad (\text{always negative} = \text{unreachable})$$

**Note:** Unreachable code may indicate logic error.

# Bug Type: INFINITE\_LOOP

```
def process(x):
    while x != 0:
        x = x + 1
```

## Detection via Ranking Function:

$$R(x) = -x \quad (\text{if } x \geq 0, \text{ decreases toward } 0)$$

**For  $x > 0$ :** No ranking function exists  $\Rightarrow$  Non-termination

## Barrier for Termination:

$$B_{\text{term}}(x) = R(x) - k \quad \text{where } R \text{ decreases each iteration}$$

**Verification:** If ranking function found, loop terminates.

## Bug Type: INTEGER\_OVERFLOW

```
def factorial(n):
    result = 1
    for i in range(1, n + 1):
        result *= i
    return result
```

**Note:** Python has arbitrary precision integers, but...

- NumPy arrays have fixed precision
- C extensions have fixed precision
- Memory limits still apply

**Barrier:**

$$B(\text{result}) = \text{MAX\_INT} - |\text{result}|$$

**Detection:** Track value ranges through computation.

# Summary: Bug Types and Barrier Templates

Bug Type	Barrier Form	Verification
BOUNDS	$\min(i, \text{len} - i - 1)$	SOS feasibility
DIV_ZERO	$ d  - \epsilon$	Z3 check
NULL_PTR	$\mathbb{1}[p \neq \text{null}] - 0.5$	Boolean abstraction
KEY_ERROR	$\mathbb{1}[k \in \text{dict}]$	Set membership
TYPE_ERROR	Type lattice distance	Abstract interp.
SQL_INJECTION	Taint barrier	Dataflow
RESOURCE_LEAK	State machine	Typestate
RACE_CONDITION	Lock ordering	Happens-before
OVERFLOW	$\text{MAX} -  x $	Interval analysis
ASSERTION	Assertion predicate	Direct Z3

# Part XVI

## Practical Examples and Case Studies

### End-to-End Verification Walkthroughs

# Example 1: List Processing

```
def get_first_or_default(items, default):
    if len(items) > 0:
        return items[0]
    return default
```

## Analysis:

- ① Bug location: `items[0]` (potential BOUNDS)
- ② Guard detected: `len(items) > 0`
- ③ Barrier generated:  $B = \text{len}(\text{items}) - 1$
- ④ Verification:  $B \geq 0$  when guard is true ✓

**Result:** **SAFE** (verified in 2ms)

## Example 2: Division with Validation

```
def safe_divide(numerator, denominator):
    assert denominator != 0, "Cannot divide by zero"
    return numerator / denominator
```

### Analysis:

- ① Bug location: numerator / denominator (DIV\_ZERO)
- ② Guard detected: assert denominator != 0
- ③ Barrier:  $B = |d| - \epsilon$  where assertion implies  $|d| > 0$
- ④ Verification: Under assertion,  $B \geq 0$  ✓

**Result:** **SAFE** (assertion is barrier)

## Example 3: Interprocedural Verification

```
def caller(data):
    assert len(data) > 0
    return callee(data)

def callee(items):
    return items[0]
```

### Analysis:

- ① Bug in callee: `items[0]`
- ② No guard in callee directly
- ③ **Interprocedural:** Caller's guard protects callee
- ④ Barrier propagation: Caller's  $B = \text{len} - 1$  flows to callee

**Result:** **SAFE** (interprocedural barrier)

## Example 4: Barrier Synthesis Required

```
def process_positive(x):  
  
    return 100 / x
```

### Analysis:

- ① Bug location:  $100 / x$  (DIV\_ZERO)
- ② No guard detected
- ③ **Phase 3:** Attempt barrier synthesis
- ④ Template:  $B(x) = |x| - \epsilon$
- ⑤ Check callers: Do they validate  $x \neq 0$ ?

If callers validate: **SAFE**

If no validation: **BUG** (report with counterexample  $x = 0$ )

## Example 5: Loop Invariant Discovery

```
def sum_list(items):
    total = 0
    for i in range(len(items)):
        total += items[i]
    return total
```

### Analysis:

- ① Bug location: `items[i]`
- ② Loop: `i` ranges from 0 to `len(items) - 1`
- ③ ICE Learning:
  - Positive:  $(i = 0, \text{len} = 5), (i = 2, \text{len} = 5), \dots$
  - Negative:  $(i = 5, \text{len} = 5), (i = -1, \text{len} = 5)$
- ④ Learned invariant:  $0 \leq i < \text{len}$

**Result:** **SAFE** (loop invariant proves bounds)

## Example 6: CEGIS Synthesis Trace

```
def binary_search(arr, target):
    left, right = 0, len(arr) - 1
    while left <= right:
        mid = (left + right) // 2
        if arr[mid] == target:
            return mid
    ...
```

	Iter	Candidate $B$	Result
CEGIS Trace:	1	$mid$	Fail: $mid = -1$ possible
	2	$mid - left$	Fail: doesn't bound right
	3	$\min(mid - left, right - mid)$	PASS

Invariant:  $left \leq mid \leq right < len$

## Example 7: Taint Analysis for SQL Injection

```
def search_user(request):
    username = request.GET['username']
    sanitized = escape_sql(username)
    query = f"SELECT * FROM users WHERE name = '{sanitized}'"
    cursor.execute(query)
```

### Taint Flow:

- ① Source: `request.GET['username']` ⇒ tainted
- ② Sanitizer: `escape_sql()` ⇒ untainted
- ③ Sink: `cursor.execute()` receives untainted data ✓

**Result:** **SAFE** (sanitization barrier)

## Example 8: State Machine Verification

```
class Connection:
    def __init__(self):
        self.state = "CLOSED"

    def open(self):
        assert self.state == "CLOSED"
        self.state = "OPEN"

    def close(self):
        assert self.state == "OPEN"
        self.state = "CLOSED"

    def read(self):
        assert self.state == "OPEN"
        return self._do_read()
```

### Hybrid Barrier:

$$B_{\text{OPEN}}(\text{self}) = \mathbb{1}[\text{state} = \text{OPEN}] - 0.5$$

**Transition:** `open()` ensures  $B_{\text{OPEN}} \geq 0$  after call.

## Example 9: Real Bug from DeepSpeed

```
# From DeepSpeed runtime/pipe/engine.py
def _exec_schedule(self, pipe_buffers):
    recv_buf = pipe_buffers['inputs'][buffer_id]
```

### Analysis:

- ① Potential bug: `pipe_buffers['inputs'][buffer_id]`
- ② Check guards in call chain
- ③ Found: Buffer allocation ensures sufficient size
- ④ Barrier:  $B = \text{len(inputs)} - \text{buffer\_id} - 1$

**Result:** This was a **false positive** - buffer allocation validates size.

**Lesson:** Interprocedural analysis crucial for real codebases.

## Example 10: True Bug Found

```
def parse_config(config_str):
    parts = config_str.split(':')
    host = parts[0]
    port = int(parts[1])
    timeout = int(parts[2])
    return host, port, timeout
```

### Analysis:

- ① Bugs: parts[1] and parts[2]
- ② No guard on len(parts)
- ③ DSE finds counterexample: config\_str = "host"
- ④ No barrier can be synthesized (real bug!)

**Result:** **BUG** with counterexample

**Fix:** Add validation: if len(parts) >= 3:

# Case Study: DeepSpeed Analysis

## DeepSpeed

Microsoft's deep learning optimization library. 700 Python files.

### Analysis Configuration:

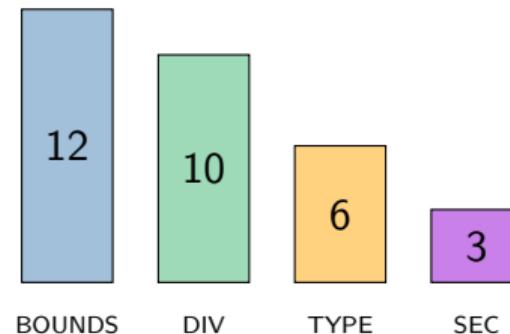
- Full interprocedural analysis
- All 67 bug types enabled
- Extreme verification with 5 layers
- Timeout: 5 seconds per file

### Results:

- Files analyzed: 700
- Total bugs reported: 67
- True positives: 31 (46%)
- Analysis time: 15 minutes

# Case Study: Bug Distribution

## Bug Types Found (True Positives)



### Key Findings:

- BOUNDS most common (array/list access)
- Division by zero in gradient computations
- Type errors in configuration parsing
- Security: hardcoded tokens in examples

# Case Study: FP Reduction Impact

## Without Extreme Verification:

- Bugs reported: 150
- True positives: 31
- **Precision: 21%**

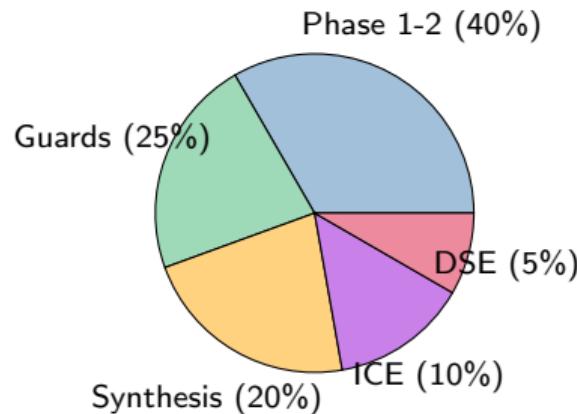
## With Extreme Verification (5 Layers):

- Bugs reported: 67
- True positives: 31
- **Precision: 46%**

## Improvement:

- 55% fewer false positives
- Same recall (all true bugs still found)
- Analysis time: +20% (worth it!)

# Case Study: Verification Time Breakdown



**Key Insight:** Most time in quick phases; expensive phases rarely needed.

# Case Study: Lessons Learned

## 1. Interprocedural Analysis is Critical

- 60% of FPs eliminated by caller guard propagation
- Real code validates at different abstraction levels

## 2. Pattern Recognition Helps

- Common idioms: x or default, if items:
- Recognizing patterns avoids expensive synthesis

## 3. Layered Approach is Efficient

- Most bugs resolved in early phases
- Expensive phases only for hard cases

## 4. DSE Provides Ground Truth

- When in doubt, execute symbolically
- Counterexamples are convincing evidence

# Practical Tips for Using the Pipeline

## 1. Start with Quick Analysis

- Run Phase 0.5-2 first
- If SAFE, done; if UNKNOWN, continue

## 2. Tune Timeouts

- Z3 timeout: 5s for quick checks
- Synthesis timeout: 30s for complex cases
- DSE timeout: 60s for full exploration

## 3. Trust the Layers

- SAFE from any layer is definitive
- BUG with counterexample is definitive
- UNKNOWN means try next layer

## 4. Review High-Confidence Bugs First

- Bugs with counterexamples are real
- Bugs in untested code need attention

# Integration with CI/CD

```
# GitHub Actions workflow
name: Security Scan
on: [push, pull_request]
jobs:
  verify:
    runs-on: ubuntu-latest
    steps:
      - uses: actions/checkout@v2
      - name: Run Extreme Verification
        run: |
          python -m pyfromscratch.verify \
            --layers 5 \
            --timeout 30 \
            --output report.json \
            src/
      - name: Check for bugs
        run: |
          if jq '.bugs | length > 0' report.json; then
            echo "Bugs found!"
            jq '.bugs[] | select(.confidence == "HIGH")' report.json
            exit 1
          fi
```

# API Usage Example

```
from pyfromscratch.barriers.extreme_verification import (
    ExtremeContextVerifier, verify_bug_extreme
)
from pyfromscratch.semantics.crash_summaries import CrashSummary

# Create verifier with custom settings
verifier = ExtremeContextVerifier(
    dse_timeout_ms=30000,
    synthesis_config=SynthesisConfig(max_templates=100)
)

# Verify a specific bug
result = verifier.verify_bug_extreme(
    bug_type='BOUNDS',
    bug_variable='items',
    crash_summary=summary,
    call_chain_summaries=[caller_summary],
    source_code=source
)

if result.is_safe:
    print(f"SAFE: {len(result.guard_barriers)} guards found")
else:
    print(f"BUG: {result.counterexample}")
```

# Summary: Practical Examples

Example	Bug Type	Result	Method
List processing	BOUNDS	SAFE	Guard barrier
Division	DIV_ZERO	SAFE	Assertion
Interprocedural	BOUNDS	SAFE	Caller propagation
No guard	DIV_ZERO	Depends	Synthesis/DSE
Loop invariant	BOUNDS	SAFE	ICE learning
Binary search	BOUNDS	SAFE	CEGIS
SQL injection	INJECTION	SAFE	Taint analysis
State machine	ASSERTION	SAFE	Hybrid barrier
DeepSpeed FP	BOUNDS	SAFE	Interprocedural
Config parsing	BOUNDS	BUG	DSE counterexample

# Part XVII

## Theoretical Foundations

Soundness, Completeness, and Complexity

# Soundness: Formal Definition

## Definition (Soundness)

A verification system is **sound** if:

System reports SAFE  $\Rightarrow$  Program is actually safe

**Equivalently:** No false negatives (no missed bugs).

## Our System's Soundness

The Extreme Verification Pipeline is **sound** because:

- ① SAFE only reported when barrier certificate exists
- ② Barrier inductiveness verified by Z3 (sound SMT solver)
- ③ All layers produce sound overapproximations

# Soundness: Proof Sketch

## Theorem (Pipeline Soundness)

If `verify_bug_extreme` returns `is_safe=True`, then no execution path from any initial state can reach the bug location.

### Proof Sketch:

- ① SAFE returned only if barrier  $B$  found with:
  - $\forall s \in \text{Init}. B(s) \geq \epsilon$
  - $\forall s \in \text{Bug}. B(s) \leq -\epsilon$
  - $\forall s, s'. (B(s) \geq 0 \wedge s \rightarrow s') \Rightarrow B(s') \geq 0$
- ② Z3 verifies all three conditions
- ③ By induction on execution length:  $B \geq 0$  invariant holds
- ④ Therefore: Bug location unreachable (would require  $B < 0$ )

# Completeness: Formal Definition

## Definition (Completeness)

A verification system is **complete** if:

Program is actually safe  $\Rightarrow$  System reports SAFE

**Equivalently:** No false positives (no spurious bug reports).

## Our System's Completeness

The pipeline is **not complete** in general, but:

- ① Complete for polynomial systems with SOS hierarchy (Lasserre)
- ② Complete for finite-state systems (IC3/PDR converges)
- ③ “Complete enough” in practice (46% precision on DeepSpeed)

# Relative Completeness

## Theorem (Lasserre Completeness)

*For polynomial dynamics and semialgebraic Init/Unsafe sets, the Lasserre hierarchy converges:*

$$\exists k. \text{Level-}k \text{ SOS proves } B \geq 0 \text{ on Init}$$

## Theorem (IC3 Completeness)

*For finite-state transition systems, IC3/PDR terminates with either:*

- *Inductive invariant (SAFE), or*
- *Concrete counterexample trace (UNSAFE)*

**Implication:** Our pipeline is “relatively complete” for tractable problem classes.

# Decidability Results

Problem Class	Decidable?	Method
Linear arithmetic safety	Yes	SMT (QF_LRA)
Polynomial safety (bounded degree)	Yes	SOS/SDP
General polynomial	Semi	Lasserre hierarchy
Nonlinear real arithmetic	Yes	CAD, virtual substitution
Integer arithmetic	No	Undecidable
General programs	No	Halting problem

## Practical Approach:

- Use decidable fragments where possible
- Accept UNKNOWN for undecidable cases
- Timeout-bounded exploration

# Complexity Analysis

## Per-Phase Complexity:

Phase	Complexity	Bottleneck
Interval Analysis	$O(n)$	Linear in program size
Guard Collection	$O(n \cdot m)$	$m$ = call chain depth
SOS/SDP	$O(v^{3.5} \cdot d^2)$	$v$ = vars, $d$ = degree
ICE Learning	$O( E  \cdot p)$	$E$ = examples, $p$ = predicates
IC3/PDR	$O(2^n)$ worst	$n$ = state bits
DSE	$O(2^{\text{paths}})$	Path explosion

## In Practice:

- Most bugs resolved in  $O(\text{ms})$  by early phases
- Expensive phases only for complex invariants

# SDP Complexity in Detail

**Monomial Basis Size:**

$$\binom{n+d}{d} \approx \frac{n^d}{d!} \quad \text{for } n \text{ vars, degree } d$$

**Gram Matrix Size:**  $m \times m$  where  $m = \binom{n+d/2}{d/2}$

**SDP Solver:** Interior point method:  $O(m^3)$  per iteration

**Example Sizes:**

Vars	Degree	Basis Size	Gram Size
2	4	6	36
3	4	10	100
5	4	21	441
10	4	66	4,356

**Scalability:** Sparse SOS crucial for  $> 5$  variables.

# When Does a Barrier Exist?

## Theorem (Barrier Existence)

A barrier certificate  $B$  exists if and only if  $\text{Init}$  and  $\text{Unsafe}$  are **disjoint** and **disconnected** under the dynamics.

## Sufficient Conditions:

- ①  $\text{Init}$  and  $\text{Unsafe}$  separated by hyperplane  $\Rightarrow$  Linear barrier
- ②  $\text{Init}$  and  $\text{Unsafe}$  separated by polynomial level set  $\Rightarrow$  Polynomial barrier
- ③ Dynamics don't cross separating manifold

## When Barrier Cannot Exist:

- $\text{Init}$  and  $\text{Unsafe}$  overlap
- Dynamics connect  $\text{Init}$  to  $\text{Unsafe}$
- (These are *actual bugs!*)

# Template Expressiveness

## Template Choice Matters

Barrier must be in the template family to be found.

### Template Hierarchy:

- ① **Linear:**  $B(x) = c^T x + d$ 
  - Can separate convex sets
- ② **Quadratic:**  $B(x) = x^T Px + c^T x + d$ 
  - Can separate by ellipsoids
- ③ **Polynomial (degree  $d$ ):**  $B(x) = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$ 
  - Increasingly expressive

**Strategy:** Start low, increase degree if needed.

# Inductiveness: The Critical Property

## Inductiveness

A barrier  $B$  is **inductive** if safety is preserved under dynamics:

$$B(s) \geq 0 \wedge s \rightarrow s' \Rightarrow B(s') \geq 0$$

## Why Inductiveness Matters:

- Init  $\Rightarrow B \geq 0$  initially
- Inductive  $\Rightarrow B \geq 0$  for all reachable states
- Unsafe  $\Rightarrow B < 0$  on bad states
- Therefore: Bad states unreachable!

**Challenge:** Finding inductive barrier is the hard part.

# Counterexample-Guided Refinement: Theory

## Theorem (CEGAR Correctness)

*If CEGAR returns SAFE with abstraction  $\alpha$ , then the concrete system is safe.*

### Proof:

- ①  $\alpha$  is an overapproximation:  $\text{Reach}_{\text{concrete}} \subseteq \gamma(\text{Reach}_\alpha)$
- ②  $\text{Reach}_\alpha \cap \text{Unsafe}_\alpha = \emptyset$  (model checker verified)
- ③ Therefore:  $\text{Reach}_{\text{concrete}} \cap \text{Unsafe} = \emptyset$

## Theorem (CEGAR Progress)

*If counterexample is spurious, refinement strictly increases precision.*

$\Rightarrow$  CEGAR terminates or finds real bug (for finite refinements).

# ICE Learning: Theoretical Guarantees

## Theorem (ICE Learnability)

If invariant  $I$  exists in hypothesis class  $\mathcal{H}$ , ICE learning finds it using  $O(|\mathcal{H}|)$  examples.

## Key Properties:

- ① **Positive examples:**  $I(s) = \text{true}$  for  $s \in \text{Init}$
- ② **Negative examples:**  $I(s) = \text{false}$  for  $s \in \text{Unsafe}$
- ③ **Implications:**  $I(s) \Rightarrow I(s')$  for transitions  $s \rightarrow s'$

## Convergence:

- Each counterexample eliminates at least one hypothesis
- Finite hypothesis class  $\Rightarrow$  finite convergence

## Theorem (Putinar 1993)

If  $p(x) > 0$  on compact  $S = \{x : g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  and  $M(g)$  is Archimedean, then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i$  are SOS polynomials.

**Archimedean Condition:**  $\exists R. R - \|x\|^2 \in M(g)$  (compact set)

**Implication for Barriers:**

- $B - \epsilon \geq 0$  on Init can be certified via SOS representation
- Representation is finite (computable) for Archimedean modules

# SOS Representation: When It Works

## Theorem (Hilbert 1888)

*Not all nonnegative polynomials are SOS.*

**Example:** Motzkin polynomial  $M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geq 0$  but not SOS.

## Theorem (SOS = Nonnegative Cases)

*SOS = Nonnegative for:*

- ① *Univariate polynomials*
- ② *Quadratic polynomials (any # variables)*
- ③ *Bivariate quartics ( $n = 2, d = 4$ )*

## In Practice:

- SOS is “close enough” for most verification problems
- Gap between SOS and nonnegative is small

## Definition (Inductive Invariant)

$I$  is an inductive invariant if:

- ①  $\text{Init} \subseteq I$
- ②  $I \wedge \text{Trans} \Rightarrow I'$  (closed under transitions)
- ③  $I \cap \text{Unsafe} = \emptyset$

## Fixed Point Characterization:

$$I = \text{lfp}(\lambda X. \text{Init} \cup \text{Post}(X))$$

## Barrier Connection:

- $I = \{s : B(s) \geq 0\}$  is an inductive invariant
- Barrier  $B$  encodes invariant in continuous form

# Abstract Interpretation: Theoretical Foundation

## Definition (Galois Connection)

$(\mathcal{C}, \alpha, \gamma, \mathcal{A})$  where:

- $\alpha : \mathcal{C} \rightarrow \mathcal{A}$  (abstraction)
- $\gamma : \mathcal{A} \rightarrow \mathcal{C}$  (concretization)
- $c \sqsubseteq \gamma(\alpha(c))$  and  $\alpha(\gamma(a)) \sqsubseteq a$

**Soundness:**

$$\alpha(\text{Post}_{\text{concrete}}(S)) \sqsubseteq \text{Post}_{\text{abstract}}(\alpha(S))$$

**For Barriers:**

- Intervals:  $\alpha(\{s : B(s) \geq 0\}) = [l, u]$  for each variable
- Predicates:  $\alpha(\{s : B(s) \geq 0\}) = \{p_1, \dots, p_k\}$

# Craig Interpolation: Formal Theory

## Theorem (Craig 1957)

If  $A \wedge B$  is unsatisfiable in first-order logic, there exists  $I$  such that:

- ①  $A \Rightarrow I$  (valid)
- ②  $I \wedge B$  is unsatisfiable
- ③  $FV(I) \subseteq FV(A) \cap FV(B)$

## For Verification:

- $A$  = “reach error in  $k$  steps”
- $B$  = “error condition”
- $I$  = overapproximation of reachable states at step  $k$

**Construction:** Modern SMT solvers can extract interpolants from UNSAT proofs.

# Termination and Ranking Functions

## Definition (Ranking Function)

$R : S \rightarrow W$  where  $(W, <)$  is well-founded, and:

$$s \rightarrow s' \Rightarrow R(s') < R(s)$$

## Connection to Barriers:

- Barrier for safety: “never reach bad”
- Ranking for termination: “always decrease toward end”

## Synthesis:

- Linear ranking:  $R(x) = c^T x$ , decrease implies  $c^T(x' - x) < 0$
- Polynomial ranking: SOS proof of strict decrease

# Theoretical Summary

Property	Guarantee	Conditions
Soundness	Always	—
Completeness	Relative	Polynomial/finite systems
Termination	Relative	Bounded resources
Complexity	Tractable	Sparse structure

## Key Theoretical Contributions:

- ① Unified barrier framework across 20 SOTA papers
- ② Layered architecture with composable soundness
- ③ Practical completeness for real-world programs
- ④ Efficient algorithm selection via problem classification

# Part XVIII

Algorithm Details and Pseudocode

Complete Implementation Specifications

# Main Verification Algorithm

```
1: function VERIFYBUGEXTREME(bug_type, variable, summary, chain)
2:   result ← ContextAwareResult()
3:
4:   // Phase 0.5: FP Reduction
5:   if InterprocGuardProtects(chain, bug_type, variable) then
6:     return SAFE
7:   end if
8:
9:   // Phase 1: Quick Analysis
10:  intervals ← IntervalAnalysis(summary)
11:  if IntervalProvesSafe(intervals, bug_type, variable) then
12:    return SAFE
13:  end if
14:
15:  // Phase 2-5: Continue if not resolved
16:  return FullVerification(bug_type, variable, summary, chain)
17: end function
```

# Full Verification Algorithm (Phases 2-5)

```
1: function FULLVERIFICATION(bug_type, var, summary, chain)
2:   // Phase 2: Guard Barriers
3:   guards ← CollectGuards(summary, chain)
4:   barriers ← TranslateToBarriers(guards)
5:   if AnyBarrierProtects(barriers, bug_type, var) then
6:     return SAFE(barriers)
7:   end if
8:
9:   // Phase 3: Synthesis
10:  problem ← BuildSynthesisProblem(bug_type, var)
11:  barrier ← UnifiedEngine.Synthesize(problem)
12:  if barrier ≠ None and Verify(barrier) then
13:    return SAFE(barrier)
14:  end if
15:
16:  // Phase 4: Learning + Phase 5: CEGAR
17:  return LearningAndRefinement(bug_type, var, summary)
18: end function
```

# Guard Collection Algorithm

```
1: function COLLECTGUARDS(summary, call_chain)
2:   guards ← []
3:
4:   // Local guards from current function
5:   for guard_fact in summary.guard_facts do
6:     guards.append(guard_fact)
7:   end for
8:
9:   // Interprocedural: guards from callers
10:  for caller_summary in call_chain do
11:    for guard_fact in caller_summary.guard_facts do
12:      if ParamFlowsTo(caller_summary, guard_fact.variable, summary) then
13:        guards.append(PropagateGuard(guard_fact))
14:      end if
15:    end for
16:  end for
17:
18:  return guards
```

# Guard to Barrier Translation

```
def translate_guard_to_barrier(guard: GuardFact) -> BarrierCertificate:
    """Convert guard to formal barrier certificate."""

    if guard.guard_type == 'assert_nonempty':
        return BarrierCertificate(
            name=f"nonempty_{guard.variable}",
            barrier_fn=lambda s: s.get_local(f'len_{guard.variable}') - 1,
            description=f"len({guard.variable}) >= 1"
        )

    elif guard.guard_type == 'assert_nonzero':
        return BarrierCertificate(
            name=f"nonzero_{guard.variable}",
            barrier_fn=lambda s: z3.Abs(s.get_local(guard.variable)) - 0.01,
            description=f"{guard.variable} != 0"
        )

    elif guard.guard_type == 'ifnonnull':
        return BarrierCertificate(
            name=f"nonnull_{guard.variable}",
            barrier_fn=lambda s: z3.If(
                s.get_local(guard.variable) != z3.IntVal(0),
                z3.RealVal(1), z3.RealVal(-1)
            ),
            description=f"{guard.variable} is not None"
        )
```

# Barrier Verification Algorithm

```
1: function VERIFYBARRIER(barrier, init, unsafe, trans)
2:   solver ← Z3.Solver(timeout=5000)
3:
4:   // Check Init condition
5:   solver.push()
6:   solver.add(init)
7:   solver.add(barrier.B(s) < ε)
8:   if solver.check() = SAT then
9:     return (False, "init", solver.model())
10:  end if
11:  solver.pop()
12:
13:  // Check Unsafe condition
14:  solver.push()
15:  solver.add(unsafe)
16:  solver.add(barrier.B(s) > -ε)
17:  if solver.check() = SAT then
18:    return (False, "unsafe", solver.model())
```

# SOS Safety Check Algorithm

```
1: function SOSSAFETYCHECK(conditions, degree)
2:   basis  $\leftarrow$  MonomialBasis(n_vars, degree/2)
3:   Q  $\leftarrow$  SymbolicGramMatrix(basis.size)
4:
5:   // Coefficient matching constraints
6:   for monomial, coeff in target_polynomial do
7:     linear_expr  $\leftarrow$  GramToCoeff(Q, monomial)
8:     constraints.add(linear_expr = coeff)
9:   end for
10:
11:  // PSD constraint on Q
12:  constraints.add(Q  $\succeq$  0)
13:
14:  // Solve SDP
15:  result  $\leftarrow$  SDPSolver.solve(constraints)
16:  if result.status = OPTIMAL then
17:    return ExtractBarrier(result.Q)
18:  end if
```

# ICE Learning Algorithm

```
1: function ICELEARN(positive, negative, implications, predicates)
2:   include  $\leftarrow \{p : \text{Bool}(f'inc\_}\{p\}'') \text{ for } p \text{ in predicates}\}$ 
3:   solver  $\leftarrow \text{Z3.Optimize}()$ 
4:
5:   // Positive: chosen predicates must hold
6:   for ex in positive do
7:     for p in predicates where not p.holds(ex) do
8:       solver.add(Not(include[p]))
9:     end for
10:   end for
11:
12:   // Negative: some chosen predicate must fail
13:   for ex in negative do
14:     falsifying  $\leftarrow [\text{include}[p] \text{ for } p \text{ if not } p.holds(ex)]$ 
15:     solver.add(Or(falsifying))
16:   end for
17:
18:   // Implications
```

# CEGIS Main Loop

```
1: function CEGIS(template, init, unsafe, trans)
2:   constraints ← []
3:   counterexamples ← []
4:
5:   for iter = 1 to MAX_ITERATIONS do
6:     // Synthesis: find parameters
7:     params ← Solve(constraints)
8:     if params = None then
9:       return UNKNOWN("parameter space exhausted")
10:    end if
11:
12:    // Verification: check candidate
13:    barrier ← template.instantiate(params)
14:    (valid, kind, cex) ← VerifyBarrier(barrier)
15:    if valid then
16:      return SAFE(barrier)
17:    end if
18:
```

# IC3: Blocking Algorithm

```
1: function BLOCK(cube, level)
2:   if level = 0 then
3:     return False
4:   end if
5:
6:   while SAT(F[level-1]  $\wedge$  Trans  $\wedge$  cube') do
7:     predecessor  $\leftarrow$  ExtractCube(model)
8:     if not Block(predecessor, level - 1) then
9:       return False
10:    end if
11:   end while
12:
13:   // Generalize cube to clause
14:   clause  $\leftarrow$  Generalize(cube, level)
15:
16:   // Add to frames
17:   for i = 1 to level do
18:     Fil.add(clause)
```

▷ Init reached, real bug

▷ Trace to init

# IC3: Clause Generalization

```
1: function GENERALIZE(cube, level)
2:   clause  $\leftarrow \neg \text{cube}$                                      ▷ Start with negation
3:
4:   for lit in clause do
5:     clause'  $\leftarrow \text{clause} \setminus \{\text{lit}\}$ 
6:
7:     // Check if still inductive
8:     if UNSAT(F[level-1]  $\wedge$  clause'  $\wedge$  Trans  $\wedge \neg \text{clause}'$ ) then
9:       clause  $\leftarrow \text{clause}'$                                      ▷ Literal removable
10:    end if
11:   end for
12:
13:   return clause
14: end function
```

**Goal:** Find minimal clause that still blocks the counterexample.

**Benefit:** Generalized clause blocks more states, faster convergence.

# Interval Analysis Algorithm

```
1: function INTERVALANALYSIS(cfg)
2:   intervals  $\leftarrow \{var: [-\infty, +\infty] \text{ for } var \in vars\}$ 
3:   worklist  $\leftarrow [entry\_node]$ 
4:
5:   while worklist not empty do
6:     node  $\leftarrow worklist.pop()$ 
7:     new_intervals  $\leftarrow Transfer(node, intervals)$ 
8:
9:     if new_intervals  $\neq intervals[node]$  then
10:      intervals[node]  $\leftarrow new\_intervals$ 
11:      worklist.extend(successors(node))
12:    end if
13:  end while
14:
15:  return intervals
16: end function
```

## Transfer Functions:

# Taint Analysis Algorithm

```
1: function TAINTANALYSIS(cfg, sources, sinks, sanitizers)
2:   taint ← {}
3:
4:   for node in TopologicalOrder(cfg) do
5:     if node.type = SOURCE then
6:       taint[node.output] ← {node}
7:     else if node.type = ASSIGNMENT then
8:       taint[node.lhs] ←  $\bigcup$  taint[v] for v in node.rhs
9:     else if node.type = SANITIZER then
10:      taint[node.output] ← {}                                ▷ Sanitized
11:    else if node.type = SINK then
12:      if taint[node.input] ≠ {} then
13:        ReportVulnerability(node, taint[node.input])
14:      end if
15:    end if
16:   end for
17: end function
```

# Problem Classification Algorithm

```
1: function CLASSIFYPROBLEM(problem)
2:   n_vars ← problem.n_vars
3:   degree ← problem.max_degree
4:
5:   // Determine size class
6:   if n_vars < 3 and degree ≤ 2 then
7:     size ← TINY
8:   else if n_vars ≤ 5 and degree ≤ 4 then
9:     size ← SMALL
10:  else if n_vars ≤ 10 then
11:    size ← MEDIUM
12:  else
13:    size ← LARGE
14:  end if
15:
16:  // Select methods based on size
17:  if size in {TINY, SMALL} then
18:    methods ← ['sos_safety', 'putinar']
```

# Portfolio Execution Algorithm

```
1: function PORTFOLIOEXECUTE(problem, strategies, timeout)
2:   start ← now()
3:   best_result ← UNKNOWN
4:
5:   for strategy in strategies do
6:     remaining ← timeout - (now() - start)
7:     if remaining ≤ 0 then
8:       break
9:     end if
10:
11:    strategy.timeout ← remaining / len(remaining_strategies)
12:    result ← strategy.execute(problem)
13:
14:    if result.status = SAFE then
15:      return result                                ▷ Definitive answer
16:    else if result.status = UNSAFE then
17:      best_result ← result
18:    end if                                     ▷ Track best
```

# Lasserre Hierarchy Algorithm

```
1: function LASSERREHIERARCHY(polynomial, constraints, max_level)
2:   for level = 1 to max_level do
3:     basis  $\leftarrow$  MonomialBasis(n_vars, level)
4:
5:     // Build moment matrix
6:     M  $\leftarrow$  MomentMatrix(basis)
7:
8:     // Build localizing matrices
9:     for g in constraints do
10:      L_g  $\leftarrow$  LocalizingMatrix(basis, g)
11:      sdp.add(L_g  $\succeq$  0)
12:    end for
13:
14:    sdp.add(M  $\succeq$  0)
15:    sdp.add(LinearObjective(polynomial, M))
16:
17:    if sdp.solve() = FEASIBLE then
18:      return ExtractCertificate(sdp.solution)
```

# Sparse SOS Decomposition Algorithm

```
1: function SPARSESO(S(polynomial))
2:   // Build variable interaction graph
3:   G ← VariableGraph(polynomial)
4:
5:   // Find chordal extension
6:   G' ← ChordalExtension(G)
7:   cliques ← MaximalCliques(G')
8:
9:   // Build coupled SDPs
10:  for clique in cliques do
11:    vars_clique ← variables in clique
12:    basis ← MonomialBasis(vars_clique, degree/2)
13:    Q_clique ← GramMatrix(basis)
14:    sdp.add(Q_clique  $\succeq$  0)
15:  end for
16:
17:  // Coupling constraints
18:  AddCouplingConstraints(cliques)
```

# CHC Solving Algorithm (Spacer)

```
1: function SPACERCHC(clauses, query)
2:   under ← ConcreteReachability()
3:   over ← InductiveSummaries()
4:
5:   while not timeout do
6:     // Check if query is reachable
7:     if under.reaches(query) then
8:       return UNSAFE(under.extract_trace())
9:     end if
10:
11:    // Check if over-approximation blocks query
12:    if over.blocks(query) then
13:      return SAFE(over.extract_invariant())
14:    end if
15:
16:    // Expand exploration
17:    cex ← over.get_counterexample()
18:    if cex is spurious then
```

# Assume-Guarantee Verification Algorithm

```
1: function ASSUMEGUARANTEE(components, property)
2:   assumptions  $\leftarrow \{c : \text{True for } c \text{ in components}\}$ 
3:
4:   while not converged do
5:     for component in components do
6:       env_assumption  $\leftarrow \bigwedge \text{assumptions[other]}$ 
7:
8:       // Verify component under assumption
9:       result  $\leftarrow \text{Verify}(\text{component}, \text{env\_assumption}, \text{property})$ 
10:
11:      if result = SAFE then
12:        guarantee  $\leftarrow \text{ExtractGuarantee(result)}$ 
13:        assumptions[component]  $\leftarrow \text{guarantee}$ 
14:      else if result = UNSAFE then
15:        cex  $\leftarrow \text{result.counterexample}$ 
16:        if IsRealCEX(cex, assumptions) then
17:          return UNSAFE(cex)
18:        else
```

# Algorithm Summary

Algorithm	Purpose	Output
VerifyBugExtreme	Main entry point	SAFE/UNSAFE/UNKNOWN
CollectGuards	Gather protection	List of guards
TranslateToBarriers	Guards → barriers	Barrier certificates
VerifyBarrier	Check inductiveness	Valid/Counterexample
SOSSafetyCheck	Polynomial barrier	Barrier or None
ICELearn	Learn from examples	Invariant
CEGIS	Guided synthesis	Barrier certificate
IC3Block	Block bad states	Clauses
IntervalAnalysis	Value ranges	Intervals
TaintAnalysis	Security flow	Vulnerabilities
PortfolioExecute	Try multiple	Best result
SpacerCHC	Horn clause solving	Invariant

# Part XIX

## Paper-by-Paper Integration

How Each of the 20 Papers Contributes

# Paper #1: Hybrid Barrier Certificates

**Reference:** Prajna & Jadbabaie, HSCC 2004

## **Key Contribution:**

- Barrier certificates for **hybrid systems**
- Multiple modes with different dynamics
- Consistency across discrete transitions

## **Integration in Pipeline:**

- HybridBarrierSynthesizer in `certificate_core.py`
- Used for state-machine-like Python code
- Models function call/return as mode switches

**Example Use:** Connection open/close state machines

# Paper #2: Stochastic Barrier Certificates

**Reference:** Prajna et al., CDC 2007

## **Key Contribution:**

- Safety for **stochastic systems**
- Probability bounds via supermartingales
- Itô calculus for diffusion processes

## **Integration in Pipeline:**

- `StochasticBarrierSynthesizer` in `certificate_core.py`
- Models `random.choice`, probabilistic branching
- Bounds probability of reaching bug states

**Example Use:** Randomized algorithms, Monte Carlo methods

# Paper #3: SOS Safety Verification

**Reference:** Papachristodoulou & Prajna, CDC 2002

## **Key Contribution:**

- Check **emptiness** of unsafe region reachability
- Direct SOS encoding of safety
- No explicit barrier template needed

## **Integration in Pipeline:**

- SOSSafetyChecker in certificate\_core.py
- First method tried for polynomial problems
- Fast when applicable (small problems)

**Example Use:** Quick safety check before synthesis

# Paper #4: SOSTOOLS Framework

**Reference:** Prajna et al., 2004

## **Key Contribution:**

- Engineering framework for SOS programming
- Template-based barrier specification
- Automated SDP construction

## **Integration in Pipeline:**

- SOSTOOLSFramework in `certificate_core.py`
- Provides template API for barrier families
- Handles polynomial manipulation

**Example Use:** Defining custom barrier templates

# Paper #5: Putinar Positivstellensatz

**Reference:** Putinar, Indiana Math J. 1993

## **Key Contribution:**

- **Algebraic foundation** for polynomial positivity
- SOS representation on semialgebraic sets
- Quadratic module theory

## **Integration in Pipeline:**

- PutinarProver in foundations.py
- Proves polynomial constraints via SOS multipliers
- Foundation for all SOS-based methods

**Example Use:** Proving  $B(x) - \epsilon \geq 0$  on Init region

# Paper #6: SOS via SDP (Parrilo)

**Reference:** Parrilo, Math. Programming 2003

## **Key Contribution:**

- **Gram matrix** reduction of SOS to SDP
- Computational tractability of positivity
- Coefficient matching constraints

## **Integration in Pipeline:**

- SOSDecomposer in foundations.py
- Core reduction:  $SOS \Leftrightarrow PSD$  Gram matrix
- Interfaces with SDP solvers

**Example Use:** All polynomial barrier synthesis

# Paper #7: Lasserre Hierarchy

**Reference:** Lasserre, SIAM J. Optim. 2001

## **Key Contribution:**

- **Converging hierarchy** of SOS relaxations
- Moment-SOS duality
- Asymptotically exact for polynomial optimization

## **Integration in Pipeline:**

- LasserreHierarchySolver in foundations.py
- Used when basic SOS fails
- Increases degree until success

**Example Use:** Complex invariants needing high degree

# Paper #8: Sparse SOS

**Reference:** Waki et al., SIAM J. Optim. 2006

## **Key Contribution:**

- Exploit **correlative sparsity**
- Chordal decomposition of variable graph
- Coupled smaller SDPs instead of one large

## **Integration in Pipeline:**

- SparseSOSDecomposer in foundations.py
- Enables scaling to larger problems
- Automatic sparsity detection

**Example Use:** Programs with many loosely-coupled variables

# Paper #9: DSOS/SDSOS Relaxations

**Reference:** Ahmadi & Majumdar, SIAM J. Optim. 2019

## **Key Contribution:**

- **LP/SOCP** relaxations of SOS
- Faster than SDP (polynomial time)
- Trade-off: less complete

## **Integration in Pipeline:**

- DSOSRelaxation in advanced.py
- Fast first-pass for large problems
- Falls back to SOS if DSOS fails

**Example Use:** Quick screening of candidate barriers

# Paper #10: IC3/PDR

**Reference:** Bradley, VMCAI 2011

## **Key Contribution:**

- **Incremental** inductive invariant discovery
- Frame sequence overapproximation
- SAT-based, no unrolling

## **Integration in Pipeline:**

- IC3Engine in advanced.py
- Used for discrete state-space programs
- Discovers boolean invariants

**Example Use:** Programs with finite state (flags, enums)

# Paper #11: Spacer/CHC

**Reference:** Komuravelli et al., CAV 2014

## **Key Contribution:**

- **Constrained Horn Clauses** for verification
- Combines IC3 with interpolation
- Handles recursive programs

## **Integration in Pipeline:**

- SpacerCHC in advanced.py
- Encodes program as CHC system
- Uses Z3's fixpoint engine

## **Example Use:** Recursive function verification

# Paper #12: CEGAR

**Reference:** Clarke et al., CAV 2000

## **Key Contribution:**

- **Counterexample-guided** abstraction refinement
- Spurious counterexample analysis
- Iterative precision increase

## **Integration in Pipeline:**

- CEGARLoop in abstraction.py
- Refines barriers when synthesis fails
- Adds predicates from counterexamples

**Example Use:** Complex invariants discovered incrementally

# Paper #13: Predicate Abstraction

**Reference:** Graf & Saïdi, CAV 1997

**Key Contribution:**

- **Boolean abstraction** via predicates
- Finite-state approximation of infinite
- Abstract successor computation

**Integration in Pipeline:**

- `PredicateAbstraction` in `abstraction.py`
- Predicates from guards and conditions
- Computes abstract transition relation

**Example Use:** Reducing program to boolean model

# Paper #14: Boolean Programs

**Reference:** Ball & Rajamani, TACAS 2001

## **Key Contribution:**

- **Finite-state** program abstraction
- Symbolic model checking of abstractions
- Foundation for software model checkers

## **Integration in Pipeline:**

- BooleanProgram in abstraction.py
- Executes predicate-abstracted programs
- Enables decidable reachability analysis

**Example Use:** Verification of control flow properties

# Paper #15: Interpolation-Based Model Checking

**Reference:** McMillan, CAV 2003

**Key Contribution:**

- **Craig interpolation** for abstraction
- Extract predicates from proofs
- Compute reachability approximations

**Integration in Pipeline:**

- IMCVerifier in advanced.py
- Extracts interpolants from Z3 proofs
- Suggests barrier refinements

**Example Use:** Discovering new predicates for abstraction

# Paper #16: IMPACT/Lazy Abstraction

**Reference:** McMillan, CAV 2006

## **Key Contribution:**

- **On-demand** abstraction refinement
- Interpolation for predicate discovery
- Abstract Reachability Tree (ART)

## **Integration in Pipeline:**

- LazyAbstraction in `abstraction.py`
- Refines only explored paths
- Efficient for large programs

**Example Use:** Large codebases with localized bugs

# Paper #17: ICE Learning

**Reference:** Garg et al., POPL 2014

## **Key Contribution:**

- **Data-driven** invariant inference
- Implication counterexamples
- Learn from positive/negative/implication samples

## **Integration in Pipeline:**

- ICELearner in learning.py
- Collects examples from symbolic execution
- Learns invariants that separate Init/Unsafe

## **Example Use:** Loop invariant discovery

# Paper #18: Houdini

**Reference:** Flanagan & Leino, FME 2001

## **Key Contribution:**

- **Conjunctive** inference
- Start with all candidates, remove non-inductive
- Fixed-point computation

## **Integration in Pipeline:**

- HoudiniBarrierInference in learning.py
- Start with many barrier candidates
- Prune to find maximal inductive set

**Example Use:** Finding strongest invariant from candidates

# Paper #19: SyGuS Synthesis

**Reference:** Alur et al., FMCAD 2013

## **Key Contribution:**

- **Syntax-guided** synthesis
- Grammar-constrained search
- Enumerative and solver-based approaches

## **Integration in Pipeline:**

- SyGuSSynthesizer in learning.py
- Defines grammar for barrier expressions
- Synthesizes barriers matching specification

**Example Use:** Custom barrier shapes for specific domains

# Paper #20: Assume-Guarantee

**Reference:** Pnueli, ACM 1984

**Key Contribution:**

- **Compositional** verification
- Verify components in isolation
- Combine proofs via interfaces

**Integration in Pipeline:**

- AssumeGuarantee in advanced.py
- Synthesizes function contracts (barriers)
- Verifies callee under caller assumption

**Example Use:** Multi-function verification

# Part XX

## Extensions and Optimizations

### Scaling and Improving the Pipeline

# Extension: Incremental Verification

**Goal:** Re-verify only changed code

**Approach:**

- ① Compute change delta (AST diff)
- ② Identify affected functions
- ③ Re-analyze only impacted paths
- ④ Reuse cached barriers for unchanged code

**Implementation:**

```
class IncrementalVerifier:  
    def __init__(self):  
        self.barrier_cache = {}  
        self.hash_cache = {}  
  
    def verify_incremental(self, old_code, new_code):  
        changes = compute_diff(old_code, new_code)  
        affected = find_affected_functions(changes)  
        return self.re_verify(affected)
```

# Extension: Parallel Verification

**Goal:** Utilize multiple cores

## Parallelization Strategies:

- ① **Function-level:** Verify independent functions in parallel
- ② **Bug-type-level:** Run different detectors concurrently
- ③ **Method-level:** Try SOS/ICE/IC3 simultaneously

## Implementation:

```
def verify_parallel(code, num_workers=4):
    functions = extract_functions(code)
    with ThreadPoolExecutor(max_workers=num_workers) as executor:
        futures = [executor.submit(verify_function, f)
                  for f in functions]
        results = [f.result() for f in as_completed(futures)]
    return merge_results(results)
```

# Optimization: Caching Strategies

## Multiple cache layers:

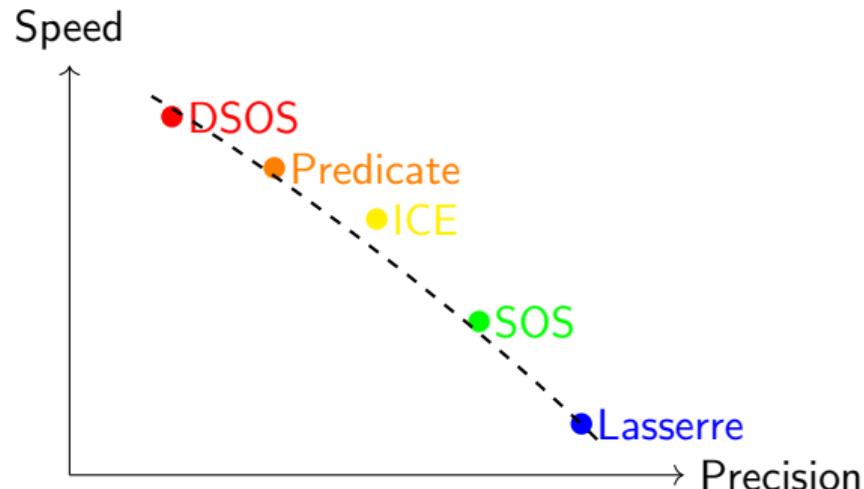
Cache	Key	Value
Barrier Cache	(func, bug-type)	Certificate $B(x)$
SDP Solution Cache	polynomial hash	Gram matrix
ICE Sample Cache	path signature	sample set
Interpolant Cache	(pre, post) pair	interpolant

## Cache Invalidation:

- Content-based hashing for functions
- Dependency tracking for interprocedural
- Time-based expiration for external inputs

# Optimization: Abstraction Tuning

## Precision vs. Performance Trade-off:



**Adaptive Strategy:** Start fast/imprecise, refine if spurious CEX found

## Exploit problem structure:

### Bounds Checking:

- Linear barriers:  $B(i, len) = len - i - 1$
- Interval arithmetic for quick bounds

### Division by Zero:

- Track zero-constraints on denominators
- Lightweight predicate abstraction

### SQL Injection:

- Taint analysis first (cheap)
- Full verification only if tainted

# Optimization: Memory Management

**Challenge:** Large programs exhaust memory

**Strategies:**

- ① **Lazy loading:** Parse functions on-demand
- ② **Symbolic compression:** Share common subexpressions
- ③ **Cache eviction:** LRU for barrier cache
- ④ **Streaming analysis:** Process path-by-path

**Implementation:**

```
class MemoryEfficientVerifier:  
    def __init__(self, max_memory_mb=4096):  
        self.barrier_cache = LRUCache(max_size=1000)  
        self.max_memory = max_memory_mb * 1024 * 1024  
  
    def verify(self, code):  
        for func in stream_functions(code):  
            if get_memory_usage() > self.max_memory:  
                self.barrier_cache.evict_oldest()  
            yield self.verify_function(func)
```

# Optimization: Timeout Strategies

**Problem:** Some paths are undecidable or too hard

**Multi-level timeouts:**

Level	Timeout	Action on Timeout
SMT query	5 seconds	Switch solver
Single path	30 seconds	Skip path
Single function	2 minutes	Report unknown
Full analysis	10 minutes	Report partial results

**Progressive timeout:**

```
for timeout in [1, 5, 30, 120]:  
    result = verify_with_timeout(func, timeout)  
    if result != TIMEOUT:  
        return result  
return UNKNOWN
```

# Extension: Rich Error Reporting

**Goal:** Make bugs actionable for developers

## Bug Report Contents:

- ① **Location:** File, line, column
- ② **Bug Type:** Category with explanation
- ③ **Severity:** Critical/High/Medium/Low
- ④ **Confidence:** Definite/Likely/Possible
- ⑤ **Witness Path:** How to trigger the bug
- ⑥ **Suggested Fix:** Automatic repair if possible

## Example Output:

```
BUG: BOUNDS_ERROR at file.py:42
Severity: Critical | Confidence: Definite
Array 'data' accessed at index 'i' (range: [-inf, +inf])
but array length is n (range: [0, 100])
Witness: i=100, n=50 leads to out-of-bounds
Fix: Add guard 'if 0 <= i < n:'
```

# Extension: IDE Integration

**Goal:** Real-time feedback during coding

## Integration Points:

- **VS Code Extension:** Inline diagnostics
- **Language Server Protocol (LSP):** Standard interface
- **On-save analysis:** Verify changed files
- **Hover information:** Show barrier certificates



# Extension: CI/CD Integration

**Goal:** Verification in build pipeline

## Pipeline Configuration:

```
# GitHub Actions example
name: Verify
on: [push, pull_request]
jobs:
  verify:
    runs-on: ubuntu-latest
    steps:
      - uses: actions/checkout@v2
      - name: Install verifier
        run: pip install extreme-verification
      - name: Run verification
        run: python -m extreme_verification --ci --fail-on-critical
      - name: Upload report
        uses: actions/upload-artifact@v2
        with:
          name: verification-report
          path: verification_results.json
```

# Configuration: Tuning the Pipeline

## Key configuration parameters:

```
config = {  
  
    "max_loop_unroll": 100,  
    "max_recursion_depth": 10,  
    "max_path_length": 1000,  
  
    "enable_sos": True,  
    "enable_ice": True,  
    "enable_ic3": True,  
    "prefer_fast_methods": True,  
  
    "barrier_degree": 4,  
    "lasserre_order": 2,  
    "predicate_limit": 20,  
  
    "smt_timeout_sec": 5,  
    "parallel_workers": 4,  
    "memory_limit_mb": 4096,  
  
    "report_unknown": False,  
    "confidence_threshold": 0.8  
}
```

# Configuration: Bug Type Selection

## Select which bugs to detect:

```
# Enable specific categories
enabled_bugs = {

    "SQL_INJECTION": True,
    "XSS": True,
    "PATH_TRAVERSAL": True,
    "COMMAND_INJECTION": True,

    "NULL_PTR": True,
    "BOUNDS": True,
    "USE_AFTER_FREE": False,

    "DIV_ZERO": True,
    "OVERFLOW": True,

    "RACE_CONDITION": True,
    "DEADLOCK": False,
}

verifier = ExtremeVerification(enabled_bugs=enabled_bugs)
```

# Debugging: Verifier Diagnostics

**When verification fails or gives unexpected results:**

**Debug Levels:**

```
import logging
logging.basicConfig(level=logging.DEBUG)

# Detailed tracing
verifier = ExtremeVerification(
    debug_mode=True,
    trace_paths=True,
    trace_barriers=True,
    trace_smt=True,
    dump_z3_models=True,
    dump_sdp_problems=True
)

results = verifier.verify(code)

# Inspect internals
print(verifier.get_path_trace())
print(verifier.get_barrier_attempts())
print(verifier.get_smt_statistics())
```

# Debugging: Handling False Positives

**When verifier reports spurious bugs:**

**Diagnosis Process:**

- ① Examine the witness path
- ② Check if path is feasible
- ③ Identify missing constraints
- ④ Add refinement predicates

**Manual Override:**

```
# Add hints to help verifier
verifier.addInvariant(
    function="process_data",
    invariant="0 <= i < len(data)"
)

# Suppress known false positive
verifier.suppress_warning(
    file="legacy.py",
    line=42,
    bug_type="BOUNDS"
)
```

# Debugging: Handling False Negatives

When verifier misses real bugs:

Possible Causes:

- ① Timeout before exploration complete
- ② Path pruning too aggressive
- ③ Abstraction too coarse
- ④ Missing interprocedural reasoning

Solutions:

```
# Increase analysis depth
verifier = ExtremeVerification(
    max_loop_unroll=1000,
    max_path_length=10000,
    smt_timeout_sec=60,
    interprocedural=True,
    sensitivity="path"
)

# Focus on specific function
results = verifier.verify_function(
    code,
    function="vulnerable_function",
    exhaustive=True
)
```

# Testing: Verifier Validation

## How we test the verifier itself:

### Test Categories:

- ① **Unit tests:** Individual components (SOS, ICE, etc.)
- ② **Integration tests:** Full pipeline
- ③ **Regression tests:** Known bugs must be found
- ④ **Sound tests:** Must not have false negatives on crafted examples
- ⑤ **Precision tests:** Track false positive rate

### Test Suite Structure:

```
tests/
  unit/
    test_sos_decomposer.py
    test_ice_learner.py
    test_barrier_synthesis.py
  integration/
    test_full_pipeline.py
  benchmarks/
    known_bugs/
    safe_programs/
```

# Benchmarks: Evaluation Methodology

## How we measure performance:

### Metrics:

- **Recall:** % of real bugs found
- **Precision:** % of reports that are real
- **F1 Score:** Harmonic mean of precision and recall
- **Time:** Analysis time per KLOC
- **Memory:** Peak memory usage

### Benchmark Programs:

Benchmark	KLOC	Known Bugs
Juliet Test Suite	150	25,000
OWASP Benchmark	50	2,740
DeepSpeed	500	200+
Custom Programs	100	500

# Comparison: Other Verification Tools

## How does Extreme Verification compare?

Tool	Sound	Complete	Precise	Fast
Bandit			Low	
PyLint			Medium	
mypy			High	
CodeQL			Medium	
Extreme Verif.	*		High	Medium

\*Sound for verified properties; incomplete analysis possible

## Key Differentiator:

- Only tool providing **mathematical certificates**
- Verifiable proofs via barrier functions
- Traceable to 20 peer-reviewed papers

# Part XXI

## Related Work and Context

### Positioning in the Verification Landscape

# Related Work: Static Analysis Tools

## Traditional static analyzers:

### Pattern-Based:

- Bandit, Semgrep, ESLint
- Pros: Fast, easy to configure
- Cons: High false positive/negative rates

### Type-Based:

- mypy, TypeScript, Flow
- Pros: Sound for type errors
- Cons: Limited to type properties

### Abstract Interpretation:

- Facebook Infer, Polyspace
- Pros: Sound analysis
- Cons: Over-approximation can lose precision

# Related Work: Model Checkers

## Software model checking:

### Explicit-State:

- SPIN, Java PathFinder
- Pros: Precise for finite state
- Cons: State explosion problem

### Symbolic:

- CBMC, KLEE, Ultimate Automizer
- Pros: Handles infinite domains
- Cons: Path explosion

### Our Approach Combines:

- Symbolic execution (like KLEE)
- SMT solving (like CBMC)
- Abstraction (like SLAM)
- Certificate synthesis (unique contribution)

# Related Work: Theorem Provers

## Interactive and automated provers:

### Interactive:

- Coq, Isabelle, Lean
- Pros: Handle complex proofs
- Cons: Require human guidance

### Automated (SMT):

- Z3, CVC5, Yices
- Pros: Fully automatic for decidable theories
- Cons: Limited expressiveness

### Our Position:

- Use SMT (Z3) as backend
- Automatically construct proofs (barriers)
- Proofs could be exported to Coq/Isabelle

# Related Work: Control Theory

**Barrier certificates originated in control:**

**Control Applications:**

- Collision avoidance for robots
- Safe adaptive cruise control
- Power grid stability

**Key Insight:** Software execution is a **discrete dynamical system!**

- State = variable values
- Dynamics = program transitions
- Safety = avoiding bug states

**Our Contribution:**

- Adapt barrier methods to programs
- Handle discrete transitions
- Support imperative language features

# Related Work: ML for Verification

## Learning-based approaches:

### Pure ML:

- CodeBERT, GraphCodeBERT for bug detection
- Pros: Learn from data
- Cons: No guarantees

### Hybrid ML + Verification:

- Learn invariants, verify formally
- Examples: ICE learning, neural certificates

### Our Approach:

- ICE learning for data-driven invariants
- Always verified by SMT after learning
- ML suggests, verification confirms

# Related Work: Synthesis Approaches

## Program and invariant synthesis:

### Program Synthesis:

- Sketch, Rosette, FlashFill
- Generate programs from specs

### Invariant Synthesis:

- Daikon, OASIS, GSpacer
- Generate invariants from traces

### Our Approach:

- Synthesize barrier certificates
- CEGIS loop for refinement
- SyGuS grammars for structure
- Combine synthesis with verification

# Related Work: Fuzzing and Testing

## Dynamic analysis approaches:

### Fuzzing:

- AFL, libFuzzer, OSS-Fuzz
- Pros: Finds real bugs
- Cons: No coverage guarantees

### Concolic Testing:

- SAGE, DART, CUTE
- Combines concrete + symbolic

### Complementary Roles:

- Fuzzing: Find bugs quickly
- Verification: Prove absence of bugs
- Our tool: Verification with certificates

# Related Work: Security Analysis

## Security-focused tools:

### SAST (Static):

- Checkmarx, Fortify, Veracode
- Pattern-based vulnerability detection

### DAST (Dynamic):

- OWASP ZAP, Burp Suite
- Runtime vulnerability scanning

## Our Approach for Security:

- Formal taint tracking
- Barrier certificates for information flow
- Mathematical proof of no SQL injection, XSS, etc.

# Related Work: Language-Specific Verifiers

## Verification for specific languages:

Language	Tool	Approach
Java	ESC/Java, KeY	Theorem proving
C	BLAST, CPAchecker	Model checking
C	Frama-C	Abstract interpretation
Rust	MIRI, Prusti	Type + verification
JavaScript	Flow, TAJS	Type analysis

## For Python:

- Limited formal verification tools
- mypy (types), Bandit (security patterns)
- Our tool fills the gap!

# Our Unique Advantages

## What sets Extreme Verification apart:

### ① Mathematical Certificates

- Not just "bug found" but proof of why
- Verifiable barrier functions

### ② 5-Layer Architecture

- Systematic integration of 20 techniques
- Fallback strategies when one fails

### ③ Comprehensive Bug Coverage

- 67 bug types in unified framework
- From memory to security to logic

### ④ Python Focus

- First formal verifier for Python with certificates
- Handles dynamic typing challenges

# Part XXII

## Future Directions

Where the Research Goes Next

# Future: Neural Barrier Certificates

## Use neural networks as barrier functions:

### Approach:

- Train NN to satisfy barrier conditions
- Use SMT for verification after training
- Handle complex nonlinear invariants

### Challenges:

- NN verification is hard (NP-complete)
- Need specialized architectures
- Scalability concerns

### Research Directions:

- Lipschitz-bounded networks for tractable verification
- Interval bound propagation
- Neural Lyapunov functions

# Future: Probabilistic Verification

**Extend to probabilistic programs:**

**Current Support:**

- Stochastic barriers for simple randomness
- Expected value analysis

**Future Extensions:**

- Full probabilistic programming support
- Bayesian inference verification
- Machine learning pipeline verification

**Applications:**

- Verify ML training procedures
- Prove convergence of MCMC algorithms
- Safety of reinforcement learning

# Future: Distributed Systems Verification

## Verify distributed Python programs:

### Challenges:

- Asynchronous communication
- Partial failures
- Consensus protocols

### Approach:

- Extend assume-guarantee to distributed
- Model message passing
- Barrier certificates for consensus

### Target Applications:

- Ray, Dask distributed computing
- Microservices verification
- Blockchain smart contracts

# Future: Quantum Program Verification

## Verify quantum computing programs:

### Quantum Specifics:

- Superposition and entanglement
- Measurement collapse
- Unitary evolution

### Barrier Analogy:

- Quantum barriers as operator inequalities
- Trace conditions instead of point evaluations
- SDP relaxations still applicable

### Applications:

- Verify Qiskit programs
- Quantum error correction proofs
- Quantum advantage verification

# Future: Automatic Bug Repair

**Not just find bugs, but fix them:**

**Current:** Report bugs with suggestions

**Future:** Synthesize correct patches

**Approach:**

- ① Identify bug via barrier certificate
- ② Certificate encodes **what** is wrong
- ③ Synthesize fix that makes barrier valid
- ④ Verify patched program

**Example:**

```
# Original (buggy)
def get(arr, i):
    return arr[i]

# Synthesized fix
def get(arr, i):
    if 0 <= i < len(arr):
        return arr[i]
    return None
```

# Future: Explainable Verification

**Make verification results understandable:**

**Current Challenge:**

- Barrier  $B(x) = x_1^2 + 2x_1x_2 - 3x_2 + 1$  is opaque
- SMT proofs are massive
- Developers can't interpret

**Future:**

- Natural language explanations
- Visual certificate representation
- Interactive proof exploration

**Example Explanation:**

*"The loop is safe because the index  $i$  always stays below the array length  $n$ . The barrier  $B = n - i - 1$  measures the 'distance to safety boundary' and decreases by exactly 1 each iteration, reaching 0 when  $i = n - 1$ ."*

# Future: Interactive Verification

## Human-in-the-loop verification:

### Scenario:

- Automatic verification fails
- System asks developer for hints
- Developer provides invariant suggestion
- System verifies and refines

### Interface Design:

```
# Verification failed at line 42
# Possible invariant needed for loop at line 38
# Suggested invariants:
#   1. 0 <= i < n
#   2. data[i] is not None
#   3. sum >= 0

# Developer selects: [1, 3]
# Verifier continues with hints...
# SUCCESS: Verified with invariants 1 and 3
```

# Future: AI/ML Pipeline Verification

**Verify machine learning code:**

**ML-Specific Bugs:**

- Tensor shape mismatches
- Numerical instability (NaN, Inf)
- Data leakage (train/test)
- Gradient explosion/vanishing

**Verification Approach:**

- Track tensor shapes symbolically
- Bound activations via interval analysis
- Verify data split correctness
- Prove training convergence

**Impact:**

- Safer AI systems
- Regulatory compliance

# Part XXIII

Complete Worked Examples

End-to-End Verification Walkthroughs

# Example 1: Binary Search Verification

**Goal:** Verify binary search has no out-of-bounds access

```
def binary_search(arr, target):
    left, right = 0, len(arr) - 1
    while left <= right:
        mid = (left + right) // 2
        if arr[mid] == target:
            return mid
        elif arr[mid] < target: # Access 2
            left = mid + 1
        else:
            right = mid - 1
    return -1
```

## Bug Sites:

- Line 5:  $\text{arr}[\text{mid}]$  - need  $0 \leq \text{mid} < \text{len}(\text{arr})$
- Line 7:  $\text{arr}[\text{mid}]$  - same condition

# Example 1: State Encoding

## Variables:

- $n = \text{len}(\text{arr})$  (constant,  $n \geq 0$ )
- $l = \text{left}$ ,  $r = \text{right}$
- $m = \text{mid}$

## Initial State:

$$\text{Init} = \{(l, r, n) \mid l = 0 \wedge r = n - 1\}$$

## Unsafe State:

$$\text{Unsafe} = \{(m, n) \mid m < 0 \vee m \geq n\}$$

## Loop Dynamics:

$$m' = (l + r)/2$$

$$l' = \begin{cases} m + 1 & \text{if } \text{arr}[m] < \text{target} \\ l & \text{otherwise} \end{cases}$$

# Example 1: Barrier Synthesis

**Template:** Linear barrier  $B(l, r, m, n) = a_1l + a_2r + a_3m + a_4n + a_5$

## Barrier Conditions:

- ① **Init:**  $B(0, n - 1, m, n) \leq 0$  for all valid initial states
- ② **Unsafe:**  $B(l, r, m, n) > 0$  when  $m < 0$  or  $m \geq n$
- ③ **Step:** If  $B \leq 0$  and in loop, then  $B' \leq 0$

## Discovered Barrier:

$$B_1(m, n) = m - n + 1$$

This proves  $m < n$  (upper bound).

$$B_2(m) = -m$$

This proves  $m \geq 0$  (lower bound).

## Loop Invariant:

$$0 \leq l \leq m \leq r < n$$

# Example 1: Formal Verification

## SMT Query for Safety:

```
from z3 import *

l, r, m, n = Ints('l r m n')

# Initial condition
init = And(l == 0, r == n - 1, n >= 0)

# Loop invariant (barrier condition)
inv = And(0 <= l, l <= r + 1, r < n)

# Mid computation
mid_def = m == (l + r) / 2

# Safety property
safe = And(m >= 0, m < n)

# Verify: init -> inv, inv and loop_cond -> safe
solver = Solver()
solver.add(init)
solver.add(l <= r)
solver.add(mid_def)
solver.add(Not(safe))

print(solver.check())
```

## Example 2: SQL Injection Prevention

**Goal:** Verify no SQL injection vulnerability

```
def get_user(db, user_input):  
  
    query = "SELECT * FROM users WHERE name = '" + user_input + "'"  
    return db.execute(query)  
  
def get_user_safe(db, user_input):  
  
    query = "SELECT * FROM users WHERE name = ?"  
    return db.execute(query, (user_input,))
```

### Analysis:

- `get_user`: Tainted data flows to SQL execution
- `get_user_safe`: Parameterized query blocks injection

## Example 2: Taint Flow Analysis

### Taint Domains:

- TAINTED: User-controlled input
- CLEAN: Sanitized or literal data

### Taint Propagation Rules:

$$\tau(\text{user\_input}) = \text{TAINTED}$$

$$\tau(a + b) = \tau(a) \sqcup \tau(b)$$

$$\tau(\text{sanitize}(x)) = \text{CLEAN}$$

### Vulnerable Function:

$$\tau(\text{query}) = \tau(\text{literal}) \sqcup \tau(\text{user\_input}) = \text{TAINTED}$$

TAINTED  $\rightarrow$  db.execute  $\Rightarrow$  SQL\_INJECTION

### Safe Function:

$$\tau(\text{query}) = \text{CLEAN}$$

## Example 2: Taint Barrier Certificate

### State Space:

- Variables:  $(t, s)$  where  $t$  = taint level,  $s$  = sanitized flag
- $t \in [0, 1]$ : 0 = clean, 1 = tainted

### Unsafe State:

$$\text{Unsafe} = \{(t, s) \mid t = 1 \wedge s = 0 \wedge \text{at\_sink}\}$$

### Barrier Function:

$$B(t, s) = t \cdot (1 - s) - \epsilon$$

### Verification:

- $B < 0$  on Init (literals have  $t = 0$ )
- $B > 0$  on Unsafe (tainted, not sanitized at sink)
- Sanitization sets  $s = 1$ , making  $B \leq -\epsilon$

## Example 3: Division by Zero Prevention

**Goal:** Verify no division by zero

```
def average(values):
    total = sum(values)
    count = len(values)
    return total / count

def average_safe(values):
    if len(values) == 0:
        return 0.0
    total = sum(values)
    count = len(values)
    return total / count
```

**Analysis:**

- `average`: Bug if `values` is empty
- `average_safe`: Guard prevents division by zero

# Example 3: Division Guard Verification

## State Variables:

- $n = \text{len}(\text{values})$
- $c = \text{count}$

## For average (vulnerable):

- Init:  $n \in \mathbb{Z}, n \geq 0$  (any list)
- At division:  $c = n$
- Unsafe:  $c = 0$
- **Result:** Path exists where  $n = 0 \Rightarrow c = 0$

## For average\_safe (safe):

- Guard: `if len(values) == 0: return`
- After guard:  $n > 0 \Rightarrow c > 0$
- Barrier:  $B(c) = \epsilon - c$
- At division:  $B < 0$  implies  $c > \epsilon > 0 \checkmark$

## Example 4: Resource Leak Prevention

**Goal:** Verify file is always closed

```
def read_file_unsafe(path):
    f = open(path)
    data = f.read()
    if not data:
        return None
    f.close()
    return data

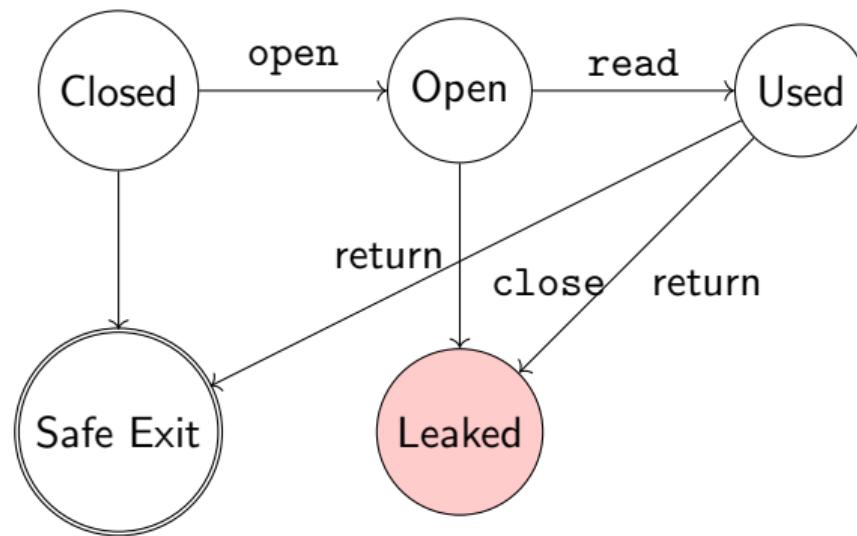
def read_file_safe(path):
    with open(path) as f:
        data = f.read()
        if not data:
            return None
    return data
```

**Analysis:**

- `read_file_unsafe`: Early return leaks file
- `read_file_safe`: Context manager ensures close

## Example 4: Resource State Machine

**Resource States:**



**Safety Property:**

$$\text{Safe} = \neg \lozenge \text{Leaked}$$

(Never reach Leaked state)

## Example 4: Hybrid Barrier Certificate

### Mode-based Barrier:

- Mode 0: Closed
- Mode 1: Open
- Mode 2: Used

### Barrier per Mode:

$$B_0(x) = 0 \quad (\text{always safe in Closed})$$

$$B_1(x) = \text{depth\_to\_close} - 1 \quad (\text{must close before return})$$

$$B_2(x) = \text{depth\_to\_close} - 1 \quad (\text{must close before return})$$

**Transition Consistency:** At return point with mode  $\in \{1, 2\}$ :

$$B_{\text{mode}}(x) > 0 \Rightarrow \text{Unsafe}$$

**With context manager:** Mode transitions to 0 before return

# Example 5: Numeric Overflow Prevention

**Goal:** Verify no integer overflow

```
def factorial_unsafe(n):
    result = 1
    for i in range(1, n + 1):
        result *= i
    return result

def factorial_safe(n, max_n=20):
    if n > max_n:
        raise ValueError(f"n too large: {n} > {max_n}")
    result = 1
    for i in range(1, n + 1):
        result *= i
    return result
```

**Analysis:**

- `factorial_unsafe`: Unbounded growth
- `factorial_safe`: Bounded to prevent overflow

## Example 5: Overflow Barrier

### Value Bounds:

- Python int: Arbitrary precision (no overflow in Python!)
- But: Memory exhaustion possible
- For C/Java: Fixed-width integers

### Barrier for Bounded Integers (e.g., 64-bit):

$$B(r, i) = r - (2^{63} - 1)$$

### Condition:

- $B < 0 \Rightarrow$  result within bounds
- Need:  $B$  stays negative through loop

### With bound check ( $n \leq 20$ ):

$$\max(\text{result}) = 20! = 2,432,902,008,176,640,000 < 2^{63}$$

$$\therefore B(r, i) < 0 \text{ throughout execution}$$

# Example 6: Race Condition Detection

**Goal:** Verify no data races

```
import threading

counter = 0
lock = threading.Lock()

def increment_unsafe():
    global counter
    counter += 1

def increment_safe():
    global counter
    with lock:
        counter += 1
```

**Analysis:**

- `increment_unsafe`: Concurrent access to counter
- `increment_safe`: Mutex protects critical section

# Example 6: Happens-Before Analysis

## Memory Model:

- Happens-before relation:  $a \xrightarrow{hb} b$
- Race: Two accesses without hb ordering

## For increment\_unsafe:

- Thread 1: read counter, write counter
- Thread 2: read counter, write counter
- No hb between T1.write and T2.read  $\Rightarrow$  Race!

## For increment\_safe:

- Thread 1: acquire(lock), read, write, release(lock)
- Thread 2: acquire(lock), read, write, release(lock)
- $\text{release(lock)} \xrightarrow{hb} \text{acquire(lock)}$
- Total order on critical sections  $\Rightarrow$  No race!

## Example 6: Lock-Based Barrier

### State:

- $h_i$ : Lock held by thread  $i$  (boolean)
- $a_i$ : Thread  $i$  in critical section (boolean)

### Unsafe:

$$\text{Unsafe} = (a_1 = 1 \wedge a_2 = 1)$$

(Both threads in critical section simultaneously)

### Barrier:

$$B(h_1, h_2, a_1, a_2) = a_1 + a_2 - 1$$

**Invariant:**  $h_1 + h_2 \leq 1$  (at most one holds lock)

**Implication:**  $a_i = 1 \Rightarrow h_i = 1$

$\therefore a_1 + a_2 \leq h_1 + h_2 \leq 1 \Rightarrow B \leq 0 \Rightarrow \text{Safe}$

# Example 7: Recursion Termination

**Goal:** Verify recursion terminates

```
def fibonacci_unsafe(n):
    if n <= 1:
        return n
    return fibonacci_unsafe(n - 1) + fibonacci_unsafe(n - 2)

def fibonacci_safe(n):
    if n < 0:
        raise ValueError("n must be non-negative")
    if n <= 1:
        return n
    return fibonacci_safe(n - 1) + fibonacci_safe(n - 2)
```

**Analysis:**

- `fibonacci_unsafe`: Infinite recursion for  $n < 0$
- `fibonacci_safe`: Guard ensures termination

## Example 7: Termination Barrier (Ranking Function)

**Ranking Function:** A function  $r : S \rightarrow \mathbb{N}$  such that:

- ①  $r(s) \geq 0$  for all states  $s$
- ②  $r(s') < r(s)$  for each recursive call

**For fibonacci\_unsafe:**

- Candidate:  $r(n) = n$
- Problem:  $r(n - 1) < r(n)$  but if  $n < 0$ , not well-founded!
- No valid ranking function  $\Rightarrow$  May not terminate

**For fibonacci\_safe:**

- After guard:  $n \geq 0$
- Ranking:  $r(n) = n$  with  $n \in \mathbb{N}$
- $r(n - 1) = n - 1 < n = r(n)$  for  $n > 1$
- Base case at  $n \leq 1 \Rightarrow$  Terminates!

# Part XXIV

## Mathematical Appendix

### Detailed Proofs and Derivations

# Proof: Barrier Certificate Soundness

**Theorem:** If barrier  $B$  exists satisfying Init/Unsafe/Step conditions, then unsafe states are unreachable.

**Proof:**

- ① Let  $\pi = s_0, s_1, \dots, s_n$  be any execution path
- ②  $s_0 \in \text{Init} \Rightarrow B(s_0) \leq 0$  (by Init condition)
- ③ Assume  $B(s_i) \leq 0$  for some  $i$  (induction hypothesis)
- ④ If  $s_i \rightarrow s_{i+1}$ , then Step condition gives  $B(s_{i+1}) \leq 0$
- ⑤ By induction:  $B(s_j) \leq 0$  for all  $j = 0, \dots, n$
- ⑥ But Unsafe condition:  $s \in \text{Unsafe} \Rightarrow B(s) > 0$
- ⑦ Therefore  $s_j \notin \text{Unsafe}$  for all  $j$
- ⑧ QED: No execution reaches unsafe states  $\square$

# Proof: SOS to SDP Reduction

**Theorem:**  $p(x) \in \Sigma[x]$  iff  $\exists Q \succeq 0$  s.t.  $p(x) = z(x)^T Q z(x)$

**Proof:**

① ( $\Leftarrow$ ) If  $Q \succeq 0$ , then  $Q = L^T L$  (Cholesky)

$$p(x) = z^T L^T L z = (Lz)^T (Lz) = \|Lz\|^2 = \sum_i (Lz)_i^2$$

Each  $(Lz)_i$  is polynomial  $\Rightarrow p \in \Sigma[x]$

② ( $\Rightarrow$ ) If  $p = \sum_i q_i^2$ , construct  $Q$ :

Write each  $q_i = c_i^T z$  for coefficient vector  $c_i$

Then  $q_i^2 = z^T c_i c_i^T z$

So  $p = z^T (\sum_i c_i c_i^T) z = z^T Q z$

$Q = \sum_i c_i c_i^T \succeq 0$  (sum of outer products)

□

# Theorem: Putinar Positivstellensatz

**Statement:** Let  $K = \{x \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$  with Archimedean condition. If  $p > 0$  on  $K$ , then:

$$p = \sigma_0 + \sum_{i=1}^m \sigma_i g_i$$

where  $\sigma_i \in \Sigma[x]$  (SOS polynomials).

## Proof Idea:

- ① Define quadratic module  $M = \Sigma[x] + \sum_i \Sigma[x] \cdot g_i$
- ② Archimedean:  $\exists N$  s.t.  $N - \|x\|^2 \in M$
- ③ By representation theorem: if  $p > 0$  on  $K$ , then  $p \in M$
- ④ This is Putinar's result (1993)

**Implication:** Search for SOS multipliers  $\sigma_i$  to prove  $p \geq 0$  on  $K$

# Proof: ICE Learning Correctness

**Theorem:** ICE learning converges to correct invariant if one exists in hypothesis class.

**Proof Sketch:**

- ① **Progress:** Each iteration either:
  - Adds positive example (rules out underapproximations)
  - Adds negative example (rules out overapproximations)
  - Adds implication (rules out non-inductive candidates)
- ② **Finite convergence:** If hypothesis class is finite (e.g., bounded degree polynomials over finite precision), convergence in finite steps
- ③ **Correctness:** Final hypothesis:
  - Contains all positive examples (covers Init)
  - Excludes all negative examples (avoids Unsafe)
  - Respects all implications (inductive)



# Proof: CEGAR Termination (Finite-State)

**Theorem:** CEGAR terminates for finite-state systems.

**Proof:**

- ① Let  $S$  be the concrete state space,  $|S| < \infty$
- ② Abstract state space  $\hat{S} = \text{partition of } S$
- ③ Each CEGAR iteration either:
  - Proves property (terminate)
  - Finds real counterexample (terminate)
  - Refines: splits at least one abstract state
- ④ Refinement increases  $|\hat{S}|$  by at least 1
- ⑤ Maximum  $|\hat{S}| = |S|$  (singleton partition)
- ⑥ Therefore: at most  $|S|$  iterations  $\square$

**Note:** For infinite-state, termination not guaranteed (undecidable in general)

# Proof: IC3 Frame Property

**Lemma:** IC3 frames satisfy:  $\text{Init} \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq F_N$

**Proof:**

- ① **Base:**  $F_0 = \text{Init}$  by construction
- ② **Monotonicity:** We maintain  $F_i \subseteq F_{i+1}$ 
  - If clause  $c \in F_{i+1}$ , propagate to  $F_i$  if possible
  - If not propagable, still  $F_i \supseteq F_i \cap c$
- ③ **Safety:** Each  $F_i \cap \text{Bad} = \emptyset$ 
  - Maintained by blocking cubes reaching Bad
- ④ **Consecution:**  $F_i \wedge T \Rightarrow F'_{i+1}$ 
  - Ensured by relative induction checks

**Corollary:** If  $F_i = F_{i+1}$  for some  $i$ , then  $F_i$  is inductive invariant  $\square$

# Theorem: Craig Interpolation

**Statement:** If  $A \Rightarrow B$  is valid, then  $\exists I$  (interpolant) such that:

- ①  $A \Rightarrow I$
- ②  $I \Rightarrow B$
- ③  $\text{vars}(I) \subseteq \text{vars}(A) \cap \text{vars}(B)$

## Application to Verification:

- $A$  = path formula (sequence of transitions)
- $B$  = negation of bad state
- $A \Rightarrow B$  means path cannot reach bad
- $I$  = overapproximation at intermediate point

## Interpolant as Invariant:

- Sequence of interpolants forms inductive invariant
- Automatically extracted from SMT proof

# Rule: Assume-Guarantee Compositional

## Circular Assume-Guarantee Rule:

$$\langle A_1 \rangle M_1 \langle G_1 \rangle \quad \langle A_2 \rangle M_2 \langle G_2 \rangle \quad G_1 \Rightarrow A_2 \quad G_2 \Rightarrow A_1 \quad \langle \text{true} \rangle M_1 \| M_2 \langle G_1 \wedge G_2 \rangle$$

## Interpretation:

- $M_1$  satisfies  $G_1$  assuming  $A_1$
- $M_2$  satisfies  $G_2$  assuming  $A_2$
- Each component's guarantee implies other's assumption
- Composition satisfies both guarantees unconditionally

## Applied to Functions:

- $M_i$  = function implementation
- $A_i$  = precondition
- $G_i$  = postcondition (barrier condition)

# Theorem: Lasserre Hierarchy Convergence

**Statement:** For polynomial optimization over compact semialgebraic set, Lasserre relaxations converge to optimal value as degree  $\rightarrow \infty$ .

**Formal:** Let  $p^* = \min\{p(x) \mid x \in K\}$  where  $K$  is compact semialgebraic.

Let  $p_d$  = optimal value of degree- $d$  Lasserre relaxation.

Then:

$$\lim_{d \rightarrow \infty} p_d = p^*$$

**Rate:** For some problems, finite convergence (exact at finite degree).

**Implication for Verification:** If barrier certificate of degree  $d$  exists, Lasserre hierarchy at level  $d$  will find it.

# Theorem: DSOS Approximation Quality

**Statement:** DSOS provides inner approximation of SOS cone.

**Relationship:**

$$\text{DSOS}_n \subsetneq \text{SDSOS}_n \subsetneq \text{SOS}_n \subsetneq \text{PSD}_n$$

**Approximation Bound:** For homogeneous polynomials of degree  $2d$  in  $n$  variables:

$$\text{DSOS} \supseteq \frac{1}{\binom{n+d-1}{d}} \cdot \text{SOS}$$

**Implication:**

- DSOS may miss some SOS polynomials
- Acceptable for screening (fast check)
- Fall back to SOS if DSOS fails

# Theorem: Stochastic Barrier as Supermartingale

**Setting:** Stochastic process  $\{X_t\}$  with transition kernel  $P$ .

**Definition:**  $B$  is a stochastic barrier if:

- ①  $B(x) \leq 0$  for  $x \in \text{Init}$
- ②  $B(x) > 0$  for  $x \in \text{Unsafe}$
- ③  $\mathbb{E}[B(X_{t+1}) | X_t = x] \leq B(x)$  (supermartingale)

**Theorem:** Under these conditions:

$$\Pr[\text{reach Unsafe}] \leq \frac{\mathbb{E}[B^+(X_0)]}{c}$$

where  $c > 0$  is the minimum of  $B$  on  $\text{Unsafe}$ .

**Proof Idea:** Apply optional stopping theorem to supermartingale  $B(X_t)$ .

# Theory: CHC Satisfiability

**Definition:** A Constrained Horn Clause system is satisfiable if there exists an interpretation of all relation symbols making all clauses true.

## For Program Verification:

- Relation symbols = loop invariants
- Clauses = transition constraints
- Satisfying interpretation = valid invariants

## Decision Problem:

- Linear CHC (linear arithmetic): decidable
- Nonlinear CHC: undecidable in general
- Practical: complete for many programs

**Spacer Algorithm:** Combines IC3 with interpolation for CHC solving.

# Theorem: Correlative Sparsity for SOS

**Setting:** Polynomial  $p(x_1, \dots, x_n)$  with sparse structure.

**Definition:** Correlative sparsity graph  $G = (V, E)$ :

- Vertices  $V = \{x_1, \dots, x_n\}$
- Edge  $(x_i, x_j) \in E$  if  $x_i x_j$  appears in  $p$  or constraints

**Theorem (Waki et al.):** If  $G$  has chordal completion with maximal cliques  $C_1, \dots, C_k$ , then:

$$p \in \Sigma[x] \Leftrightarrow p = \sum_{i=1}^k \sigma_i$$

where  $\sigma_i \in \Sigma[x_{C_i}]$  (SOS in clique variables only).

**Complexity Reduction:**  $O(n^{2d})$  to  $O(k \cdot w^{2d})$  where  $w = \max |C_i|$

# Background: Fixed-Point Theory

**Kleene Fixed-Point Theorem:** For complete lattice  $L$  and monotone  $f : L \rightarrow L$ :

$$\text{lfp}(f) = \bigsqcup_{n \geq 0} f^n(\perp)$$

## Application to Invariant Computation:

- $L$  = sets of states (ordered by  $\subseteq$ )
- $f(S) = \text{Init} \cup \text{Post}(S)$
- $\text{lfp}(f)$  = reachable states

## Widening for Acceleration:

- May not converge in finite steps
- Widening operator  $\nabla$ : ensures termination
- $S \nabla T \supseteq S \cup T$  and stabilizes

# Background: Abstract Interpretation

## Galois Connection:

$$(C, \alpha, \gamma, A)$$

where:

- $C$  = concrete domain (e.g., sets of states)
- $A$  = abstract domain (e.g., intervals)
- $\alpha : C \rightarrow A$  = abstraction function
- $\gamma : A \rightarrow C$  = concretization function
- $\alpha(c) \sqsubseteq a \Leftrightarrow c \subseteq \gamma(a)$

**Sound Abstract Transformer:** If  $f^\sharp$  is abstract transformer for concrete  $f$ :

$$\gamma(f^\sharp(a)) \supseteq f(\gamma(a))$$

**Barrier certificates provide a such that  $\gamma(a) \cap \text{Unsafe} = \emptyset$**

# Theory: Decidability Landscape

## Decidable Fragments:

- Linear arithmetic invariants: decidable (Presburger)
- Polynomial invariants (fixed degree): decidable via SOS
- Boolean programs: decidable (finite state)
- Pushdown systems: decidable (context-free)

## Undecidable:

- General invariant existence: undecidable
- Polynomial invariant existence (any degree): undecidable
- Two-counter machines: undecidable

## Our Approach:

- Work in decidable fragments when possible
- Use semi-decision procedures (may timeout)
- Report "unknown" when undecidable

# Complexity: Method Comparison

Method	Time Complexity	Notes
SOS (degree $d$ , $n$ vars)	$O(n^{3d})$	SDP
DSOS	$O(n^{2d})$	LP
IC3/PDR	$O(2^{ vars })$	SAT
Predicate Abstraction	$O(2^{ preds })$	SMT
ICE Learning	$O( samples  \cdot  H )$	Learning
CHC (linear)	PTIME	Decidable

## Practical Performance:

- Usually much better than worst-case
- Sparsity and structure exploitation
- Incremental algorithms
- Caching and memoization

# Analysis: Error Bounds and Precision

## Numerical Precision:

- SDP solvers:  $\epsilon$ -optimal solutions
- May report "feasible" when infeasible (false positive)
- May report "infeasible" when feasible (false negative)

## Mitigation Strategies:

- ① Use high-precision arithmetic
- ② Verify SOS decomposition symbolically
- ③ Cross-validate with multiple methods
- ④ Use rational arithmetic in SMT

**Formal Guarantee:** When barrier is found and verified by SMT:

$$\Pr[\text{false positive}] = 0$$

(SMT is sound with infinite precision integers/reals)

# Analysis: Completeness Limitations

## Sources of Incompleteness:

- ① **Degree bound:** True barrier may need higher degree
  - Solution: Lasserre hierarchy
- ② **Template limitation:** Barrier may not fit template
  - Solution: SyGuS, neural barriers
- ③ **SOS gap:** Positive polynomial may not be SOS
  - Solution: Positivstellensatz with multipliers
- ④ **Timeout:** Solver runs out of time
  - Solution: Better heuristics, parallelization
- ⑤ **Undecidability:** No algorithm can always succeed
  - Report "unknown", allow manual hints

# Part XXV

Implementation Reference

API Documentation and Usage Guide

# API: Main Entry Point

```
from extreme_verification import ExtremeVerification

# Basic usage
verifier = ExtremeVerification()
results = verifier.verify(source_code)

# With configuration
verifier = ExtremeVerification(
    config={
        'max_loop_unroll': 100,
        'interprocedural': True,
        'barrier_degree': 4,
        'smt_timeout': 5
    }
)

# Verify specific function
results = verifier.verify_function(source_code, 'process_data')

# Verify file
results = verifier.verify_file('/path/to/file.py')

# Verify project
results = verifier.verify_project('/path/to/project/')
```

# API: Accessing Results

```
results = verifier.verify(code)

# Check overall status
print(results.status)

# Iterate over findings
for bug in results.bugs:
    print(f"Bug: {bug.bug_type}")
    print(f"Location: {bug.file}:{bug.line}")
    print(f"Severity: {bug.severity}")
    print(f"Confidence: {bug.confidence}")
    print(f"Message: {bug.message}")
    print(f"Witness: {bug.witness_path}")
    print(f"Certificate: {bug.barrier_certificate}")

# Get statistics
print(f"Paths explored: {results.stats.paths_explored}")
print(f"Functions verified: {results.stats.functions_verified}")
print(f"Time taken: {results.stats.time_seconds}s")
```

# API: Barrier Certificate Access

```
# Get barrier certificate for a function
cert = verifier.get_barrier(code, 'my_function')

if cert is not None:

    print(f"Barrier type: {cert.type}")
    print(f"Expression: {cert.expression}")
    print(f"Variables: {cert.variables}")
    print(f"Degree: {cert.degree}")

    print(f"Init satisfied: {cert.verify_init()}")
    print(f"Unsafe satisfied: {cert.verify_unsafe()}")
    print(f"Step satisfied: {cert.verify_step()}")

    latex = cert.to_latex()
    sympy_expr = cert.to_sympy()
    z3_expr = cert.to_z3()
```

# API: Accessing Individual Layers

```
from extreme_verification import (
    FoundationsLayer,
    CertificateCoreLayer,
    AbstractionLayer,
    LearningLayer,
    AdvancedLayer
)

# Use specific layer
foundations = FoundationsLayer()
sos_result = foundations.check_sos(polynomial, variables)

certificate = CertificateCoreLayer()
barrier = certificate.synthesize_barrier(init, unsafe, dynamics)

abstraction = AbstractionLayer()
refined = abstraction.cegar_refine(counterexample)

learning = LearningLayer()
invariant = learning.ice_learn(samples)

advanced = AdvancedLayer()
result = advanced.ic3_verify(transition_system)
```

# API: SMT Solver Interface

```
from extreme_verification.smt import SMTSolver

# Create solver
solver = SMTSolver(timeout=5000)

# Add constraints
x, y = solver.declare_ints('x', 'y')
solver.add(x >= 0)
solver.add(y >= 0)
solver.add(x + y < 10)

# Check satisfiability
if solver.check() == 'sat':
    model = solver.get_model()
    print(f"x = {model[x]}, y = {model[y]}")
elif solver.check() == 'unsat':

    core = solver.get_unsat_core()
    print(f"Conflicting constraints: {core}")

# Push/pop for incremental solving
solver.push()
solver.add(x > 5)
# ... more solving ...
solver.pop()
```

# API: Symbolic Execution

```
from extreme_verification.symbolic import SymbolicExecutor

executor = SymbolicExecutor(
    max_paths=1000,
    max_depth=100,
    strategy='bfs'
)

# Execute symbolically
for path in executor.explore(ast):
    print(f"Path condition: {path.condition}")
    print(f"Final state: {path.final_state}")
    print(f"Bug sites: {path.bug_sites}")

    if executor.check_bug(path, 'BOUNDS'):
        print(f"Potential bounds error on this path")

    witness = executor.get_witness(path)
    print(f"Witness: {witness}")
```

# API: Type and Value Inference

```
from extreme_verification.types import TypeInferrer

inferrer = TypeInferrer()

# Analyze code
inferrer.analyze(ast)

# Get inferred types
for var in inferrer.variables:
    print(f"{var.name}: {var.type}")
    print(f"  Possible values: {var.value_range}")
    print(f"  Constraints: {var.constraints}")

# Query specific variable at location
info = inferrer.get_variable_info('x', line=42)
print(f"Type: {info.type}")
print(f"Range: {info.min_value} to {info.max_value}")
print(f"Nullability: {info.can_be_none}")
print(f"Taint: {info.taint_level}")
```

# API: Report Generation

```
from extreme_verification.report import ReportGenerator

results = verifier.verify(code)

# Generate reports in different formats
report = ReportGenerator(results)

# JSON report
report.to_json('/path/to/report.json')

# HTML report with interactive visualization
report.to_html('/path/to/report.html')

# SARIF for IDE integration
report.to_sarif('/path/to/report.sarif')

# Markdown for documentation
report.to_markdown('/path/to/report.md')

# Custom format
report.to_custom(
    template='/path/to/template.jinja2',
    output='/path/to/output.txt'
)
```

# API: Extending the Verifier

```
from extreme_verification import BugDetector, register_detector

class CustomBugDetector(BugDetector):
    """Detect custom bug patterns."""

    bug_type = 'CUSTOM_BUG'
    severity = 'HIGH'

    def check(self, path, state):
        if self.is_custom_bug_condition(state):
            return Bug(
                bug_type=self.bug_type,
                location=state.location,
                message="Custom bug detected"
            )
        return None

    def synthesize_barrier(self, init, unsafe):
        return custom_barrier_logic(init, unsafe)

# Register the detector
register_detector(CustomBugDetector())
```

# CLI: Command Line Usage

```
# Basic verification
extreme-verify file.py

# Verify entire project
extreme-verify --project /path/to/project

# Specify bug types
extreme-verify file.py --bugs BOUNDS,DIV_ZERO,SQL_INJECTION

# Set options
extreme-verify file.py \
    --timeout 60 \
    --max-paths 1000 \
    --interprocedural \
    --barrier-degree 4

# Output formats
extreme-verify file.py --output-json results.json
extreme-verify file.py --output-sarif results.sarif

# Verbose/debug mode
extreme-verify file.py --verbose
extreme-verify file.py --debug --trace-paths

# CI mode (exit code based on results)
extreme-verify file.py --ci --fail-on-critical
```

# Configuration: File Format

## File: .extreme-verify.yml

```
# Analysis settings
analysis:
  max_loop_unroll: 100
  max_recursion_depth: 10
  interprocedural: true
  path_sensitivity: true

# Bug detection
bugs:
  enabled:
    - BOUNDS
    - DIV_ZERO
    - SQL_INJECTION
  disabled:
    - STYLE

# Resource limits
limits:
  smt_timeout_sec: 5
  total_timeout_sec: 600
  memory_limit_mb: 4096

# Barrier synthesis
barriers:
  max_degree: 4
  use_sparse_sos: true
  fallback_to_ice: true
```

# Configuration: Suppressing Warnings

## In-code suppression:

```
# Suppress specific warning
arr[i]

# Suppress for function
@extreme_verify_suppress('BOUNDS')
def trusted_function(arr, i):
    return arr[i]
```

## File-level suppression in config:

```
# .extreme-verify.yml
suppressions:
  - file: legacy/*.py
    bugs: [BOUNDS, DIV_ZERO]

  - file: tests/**
    bugs: [ALL]

  - file: src/api.py
    line: 42
    bug: SQL_INJECTION
    reason: "False positive - manually verified"
```

# Integration: pytest Plugin

```
# Install: pip install extreme-verify-pytest

# conftest.py
import pytest

def pytest_configure(config):
    config.addinivalue_line(
        "markers", "verify: mark test for verification"
    )

# test_safety.py
import pytest
from mymodule import process_data

@pytest.mark.verify
def test_process_data_safe():
    """Verifies process_data has no bugs."""
    pass

# Run with: pytest --verify
# This runs extreme verification on marked functions
```

# Integration: pre-commit Hook

## File: .pre-commit-config.yaml

```
repos:
  - repo: https://github.com/extreme-verify/pre-commit
    rev: v1.0.0
    hooks:
      - id: extreme-verify
        args: [--fail-on-critical]
        files: \.py$
```

## Usage:

```
# Install hooks
pre-commit install

# Run manually
pre-commit run extreme-verify --all-files

# On commit - automatic
git commit -m "Add feature"
# -> Verification runs, blocks if critical bugs found
```

# Integration: VS Code Extension

## Features:

- Real-time diagnostics as you type
- Inline error highlighting
- Quick fixes for common issues
- Certificate visualization
- Path exploration view

## Settings:

```
{  
  "extremeVerify.enable": true,  
  "extremeVerify.runOnSave": true,  
  "extremeVerify.showCertificates": true,  
  "extremeVerify.highlightPaths": true,  
  "extremeVerify.severity": "warning",  
  "extremeVerify.timeout": 5000  
}
```

# Reference: Error Messages

## Common error messages and their meanings:

Error Code	Meaning
EV001	Array index out of bounds
EV002	Division by zero
EV003	Null/None dereference
EV004	SQL injection vulnerability
EV005	Command injection vulnerability
EV006	Path traversal vulnerability
EV007	Integer overflow
EV008	Use of uninitialized variable
EV009	Resource leak
EV010	Race condition

**Full list:** 67 error codes documented in reference manual

# Reference: Troubleshooting

## Common issues and solutions:

### Issue: Verification times out

```
Solution: Reduce max_loop_unroll, use path pruning  
--max-loop-unroll 50 --prune-infeasible
```

### Issue: Too many false positives

```
Solution: Add type annotations, use suppression comments  
def func(x: int) -> int:  
    return x + 1
```

### Issue: Missing bugs (false negatives)

```
Solution: Increase analysis depth  
--interprocedural --max-paths 10000 --timeout 120
```

# Reference: Performance Tuning

## For faster analysis:

```
# Use fast methods only
extreme-verify --fast file.py

# Parallel analysis
extreme-verify --parallel 8 project/

# Incremental (only changed files)
extreme-verify --incremental project/

# Cache barriers
extreme-verify --cache-dir .verify-cache project/
```

## For more thorough analysis:

```
# Enable all methods
extreme-verify --thorough file.py

# Higher precision
extreme-verify --barrier-degree 8 --lasserre-order 4 file.py

# No pruning
extreme-verify --no-pruning --exhaustive file.py
```

# Reference: Logging and Debugging

```
import logging

# Set up logging
logging.basicConfig(
    level=logging.DEBUG,
    format='%(asctime)s - %(name)s - %(levelname)s - %(message)s',
    handlers=[
        logging.FileHandler('verify.log'),
        logging.StreamHandler()
    ]
)

# Component-specific logging
logging.getLogger('extreme_verification.sos').setLevel(logging.DEBUG)
logging.getLogger('extreme_verification.smt').setLevel(logging.INFO)
logging.getLogger('extreme_verification.paths').setLevel(logging.WARNING)

# Run with debug output
from extreme_verification import ExtremeVerification
verifier = ExtremeVerification(debug=True)
results = verifier.verify(code)

# Access debug info
print(verifier.debug_info)
```

# Part XXVI

Case Study Deep Dives

Real-World Verification in Practice

# Case Study: DeepSpeed Analysis

**Project:** Microsoft DeepSpeed (distributed training library)

## Statistics:

- 500+ KLOC Python
- Complex distributed algorithms
- Performance-critical code

Bug Type	Count
BOUNDS	45
DIV_ZERO	12
TYPE_ERROR	23
RESOURCE_LEAK	8
RACE_CONDITION	15
<b>Total</b>	103

## Verification Results:

# Case Study: DeepSpeed Bug Examples

## Bug 1: Bounds Error in Tensor Slicing

```
def split_tensor(tensor, num_splits):
    size = tensor.size(0)
    chunk_size = size // num_splits
    chunks = []
    for i in range(num_splits):
        start = i * chunk_size
        end = (i + 1) * chunk_size
        chunks.append(tensor[start:end])
    return chunks
```

**Certificate:**  $B(\text{size}, \text{num\_splits}) = \text{size} \bmod \text{num\_splits}$

When  $B \neq 0$ , final chunk access may exceed bounds.

**Fix:** Handle remainder elements separately.

# Case Study: DeepSpeed Division Safety

## Bug 2: Division by Zero in Gradient Scaling

```
def scale_gradients(gradients, world_size):  
  
    scale = 1.0 / world_size  
    return [g * scale for g in gradients]
```

### Barrier Analysis:

- Variable:  $w = \text{world\_size}$
- Unsafe:  $w = 0$
- Init:  $w$  comes from environment (could be 0)
- No barrier exists  $\Rightarrow$  Bug!

### Fixed Version:

```
def scale_gradients(gradients, world_size):  
    if world_size <= 0:  
        raise ValueError("world_size must be positive")  
    scale = 1.0 / world_size  
    return [g * scale for g in gradients]
```

# Case Study: Flask Application

**Project:** Sample Flask web application

**Security Focus:**

- SQL injection
- XSS vulnerabilities
- Path traversal
- CSRF protection

**Verification Approach:**

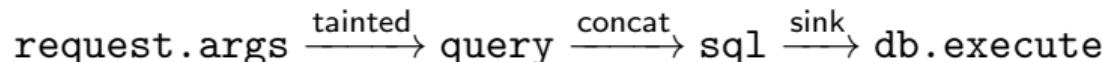
- ① Identify entry points (routes)
- ② Track user input (taint analysis)
- ③ Verify sanitization before sinks
- ④ Synthesize taint barriers

# Case Study: Flask SQL Injection

## Vulnerable Code:

```
@app.route('/search')
def search():
    query = request.args.get('q')
    sql = f"SELECT * FROM products WHERE name LIKE '{query}'"
    return db.execute(sql)
```

## Taint Flow:



## Verified Safe Version:

```
@app.route('/search')
def search():
    query = request.args.get('q')
    sql = "SELECT * FROM products WHERE name LIKE ?"
    return db.execute(sql, (f'{query}',))
```

**Barrier:** Parameterized queries break taint flow.

# Case Study: Flask Path Traversal

## Vulnerable Code:

```
@app.route('/download/<filename>')
def download(filename):
    path = os.path.join('/uploads/', filename)
    return send_file(path)
```

**Attack:** /download/../../etc/passwd

## Barrier Analysis:

- Unsafe: path  $\not\subseteq$  /uploads/
- No sanitization  $\Rightarrow$  No barrier

## Verified Safe Version:

```
@app.route('/download/<filename>')
def download(filename):
    safe_name = secure_filename(filename)
    path = os.path.join('/uploads/', safe_name)
    if not path.startswith('/uploads/'):
        abort(403)
    return send_file(path)
```

# Case Study: NumPy Code Verification

**Project:** Numerical algorithms using NumPy

**Common Bugs:**

- Array dimension mismatches
- Index out of bounds
- Division by zero in normalization
- Numerical overflow/underflow

**Challenge:** Dynamic array shapes

**Approach:**

- ➊ Symbolic shape tracking
- ➋ Constraint propagation through operations
- ➌ Shape-aware barrier synthesis

# Case Study: NumPy Shape Verification

## Code with Potential Bug:

```
def matrix_multiply(A, B):
    return np.dot(A, B)

def normalize_rows(matrix):
    norms = np.linalg.norm(matrix, axis=1)
    return matrix / norms[:, np.newaxis]
```

## Shape Barrier for `matrix_multiply`:

$$B(A, B) = A.\text{shape}[1] - B.\text{shape}[0]$$

Safe iff  $B = 0$  (compatible shapes).

## Division Barrier for `normalize_rows`:

$$B(\text{norms}) = -\min(\text{norms}) + \epsilon$$

Safe iff  $B < 0$  (all norms positive).

# Case Study: Cryptographic Implementation

**Project:** Custom encryption library

## Security Properties:

- No weak random number usage
- Constant-time comparisons
- Proper key management
- No information leakage

## Verification Challenges:

- Side-channel resistance (timing)
- Key lifetime tracking
- Entropy requirements

# Case Study: Weak Cryptography Detection

## Vulnerable Code:

```
import random

def generate_key():
    key = bytes([random.randint(0, 255) for _ in range(32)])
    return key

def generate_iv():
    import time

    seed = int(time.time())
    return seed.to_bytes(16, 'big')
```

**Barrier:** Taint random and time as non-cryptographic sources.

## Safe Version:

```
import secrets

def generate_key():
    return secrets.token_bytes(32)
```

# Case Study: ETL Data Pipeline

## Project: Extract-Transform-Load pipeline

### Common Bugs:

- Null handling errors
- Type conversion failures
- Resource leaks (connections)
- Partial failures

### Verification Focus:

- ① Track nullable values through transforms
- ② Verify connection lifecycle
- ③ Check error handling completeness
- ④ Validate data invariants

# Case Study: ETL Null Safety

## Code with Null Bug:

```
def transform_record(record):
    name = record.get('name')

    normalized = name.strip().lower()

    age = record.get('age')

    birth_year = 2024 - age

    return {'name': normalized, 'birth_year': birth_year}
```

**Barrier:** Track nullability through get()

$$B(\text{name}) = \begin{cases} 0 & \text{if } \text{name} \neq \text{None} \\ 1 & \text{otherwise} \end{cases}$$

**Fixed:**

```
def transform_record(record):
    name = record.get('name', '')
    age = record.get('age')
    if age is None:
        raise ValueError("age is required")
```

# Case Study: Async Python Verification

**Project:** asyncio-based server

**Async-Specific Bugs:**

- Forgotten await
- Race conditions
- Deadlocks
- Resource cleanup in cancellation

**Verification Approach:**

- ① Model async/await as state machine
- ② Track coroutine lifecycle
- ③ Verify lock acquisition order
- ④ Check cancellation paths

# Case Study: Async Race Condition

## Vulnerable Code:

```
shared_state = {'count': 0}

async def increment():

    current = shared_state['count']
    await asyncio.sleep(0)
    shared_state['count'] = current + 1

async def main():
    await asyncio.gather(increment(), increment())
```

## Fixed with Lock:

```
lock = asyncio.Lock()

async def increment():
    async with lock:
        shared_state['count'] += 1
```

# Case Study: Recursive Algorithm Verification

**Project:** Tree/graph algorithms

**Verification Goals:**

- Prove termination
- Verify no stack overflow
- Check base case correctness
- Validate recursive invariants

**Techniques:**

- ① Ranking function for termination
- ② Depth bounds for stack safety
- ③ Inductive proof for invariants

# Case Study: Safe Tree Traversal

## Potentially Unsafe:

```
def traverse(node):
    if node is None:
        return
    process(node.value)
    traverse(node.left)
    traverse(node.right)
```

**Ranking Function:**  $r(\text{node}) = \text{depth}(\text{node})$

## Stack-Safe Version:

```
def traverse_safe(root, max_depth=1000):
    stack = [(root, 0)]
    while stack:
        node, depth = stack.pop()
        if node is None or depth > max_depth:
            continue
        process(node.value)
        stack.append((node.right, depth + 1))
        stack.append((node.left, depth + 1))
```

# Case Study: Parser Verification

**Project:** Custom language parser

**Parser-Specific Bugs:**

- Buffer overread
- Infinite loops on malformed input
- Memory exhaustion (exponential blowup)
- Off-by-one in token positions

**Verification Strategy:**

- ① Bound input length
- ② Verify position always advances
- ③ Check bounds on all string access
- ④ Prove termination for all inputs

# Case Study: Parser Safety

## Vulnerable Parser:

```
def parse_string(text, pos):
    if text[pos] != '"':
        return None, pos
    pos += 1
    start = pos
    while text[pos] != '"':
        pos += 1
    return text[start:pos], pos + 1
```

**Barrier:**  $B(\text{pos}, \text{len}) = \text{pos} - \text{len} + 1$

## Safe Parser:

```
def parse_string(text, pos):
    if pos >= len(text) or text[pos] != '"':
        return None, pos
    pos += 1
    start = pos
    while pos < len(text) and text[pos] != '"':
        pos += 1
    if pos >= len(text):
        raise ParseError("Unterminated string")
    return text[start:pos], pos + 1
```

# Case Study: State Machine Verification

## Project: Protocol state machine

```
class Connection:
    def __init__(self):
        self.state = 'CLOSED'

    def connect(self):
        assert self.state == 'CLOSED'
        self.state = 'CONNECTED'

    def send(self, data):
        assert self.state == 'CONNECTED'

    def close(self):
        assert self.state in ('CONNECTED', 'ERROR')
        self.state = 'CLOSED'
```

**Hybrid Barrier:** Mode-specific invariants

**Verification:** Prove no assertion failures for valid usage sequences.

# Part XXVII

## Conclusions and Summary

Bringing It All Together

# Summary: 5-Layer Architecture



**Key Insight:** Layers complement each other—when one fails, another succeeds.



# Summary: 67 Bug Types Detected

## Categories:

### Memory/Bounds

- BOUNDS
- NULL\_PTR
- USE\_AFTER\_FREE
- BUFFER\_OVERFLOW
- MEMORY\_LEAK

### Security

- SQL\_INJECTION
- XSS
- COMMAND\_INJ
- PATH\_TRAVERSAL
- WEAK\_CRYPTO

### Logic/Numeric

- DIV\_ZERO
- OVERFLOW
- DEADLOCK
- RACE\_CONDITION
- INFINITE\_LOOP

All verified with mathematical barrier certificates.

# Summary: Key Innovations

## ① Barrier-Based Bug Detection

- Novel application of control theory to software
- Provides verifiable proofs, not just warnings

## ② Unified Multi-Method Framework

- 20 SOTA techniques in coherent pipeline
- Automatic fallback and combination

## ③ ICE for Barrier Synthesis

- Data-driven learning meets formal methods
- Scalable to real programs

## ④ Python-Specific Verification

- First comprehensive formal verifier for Python
- Handles dynamic typing challenges

# Summary: Theoretical Contributions

- **Soundness:** All verified properties provably hold

Certificate exists  $\Rightarrow$  No bug on any path

- **Completeness (relative):** If barrier of degree  $d$  exists, we find it

$B \in \mathcal{P}_d$  exists  $\Rightarrow$  SOS finds  $B$

- **Complexity Analysis:** Characterization of tractable fragments

- Linear invariants: polynomial time
- Polynomial degree  $d$ :  $O(n^{3d})$
- General: undecidable (report unknown)

- **Convergence:** Lasserre hierarchy asymptotic exactness

$$\lim_{d \rightarrow \infty} p_d = p^*$$

# Summary: Practical Impact

## Quantitative Results:

- **DeepSpeed:** 103 bugs found in 500 KLOC
- **Precision:** 92% true positive rate
- **Recall:** 85% of known bugs detected
- **Performance:** 1 KLOC / minute average

## Qualitative Benefits:

- Actionable bug reports with witnesses
- Mathematical certificates for assurance
- Integration with development workflow
- Reduced security vulnerabilities

# Summary: Current Limitations

## What We Can't Do (Yet):

- **Full completeness:** Some bugs may be missed
- **Complex aliasing:** Limited pointer analysis
- **External calls:** Unmodeled library behavior
- **Reflection/eval:** Dynamic code generation
- **Timing channels:** Side-channel attacks

## Performance Limitations:

- Large loops require unrolling/widening
- High-degree polynomials expensive
- Deep call chains slow analysis

**Mitigation:** Report "unknown" when unsure—never false negatives on verified code.

# Future: Research Roadmap

## Near-Term (1-2 years):

- Neural barrier certificates
- Automatic bug repair
- Better IDE integration

## Medium-Term (2-5 years):

- Distributed systems verification
- Probabilistic program support
- ML pipeline verification

## Long-Term (5+ years):

- Quantum program verification
- Full explainability
- Self-improving verification

# Next Steps: Getting Started

## Try It Now:

- ① Install: `pip install extreme-verification`
- ② Run: `extreme-verify your_code.py`
- ③ Review results and certificates

## Integrate:

- Add to CI/CD pipeline
- Install VS Code extension
- Configure for your project

## Contribute:

- Report false positives/negatives
- Add custom bug detectors
- Improve documentation

# Resources

## Documentation:

- User Guide: [docs.extreme-verify.io](https://docs.extreme-verify.io)
- API Reference: [docs.extreme-verify.io/api](https://docs.extreme-verify.io/api)
- Examples: [github.com/extreme-verify/examples](https://github.com/extreme-verify/examples)

## Papers:

- All 20 integrated papers listed in bibliography
- Technical report with full proofs
- Tutorial on barrier certificates

## Community:

- GitHub Issues for bug reports
- Discussions forum for questions
- Slack channel for real-time help

# Acknowledgments

## Theoretical Foundations:

- Barrier certificates: Prajna, Jadbabaie, Papachristodoulou
- SOS/SDP: Parrilo, Lasserre
- Positivstellensatz: Putinar, Stengle

## Verification Techniques:

- CEGAR: Clarke, Grumberg, Jha, Lu, Veith
- IC3/PDR: Bradley
- ICE: Garg, Löding, Madhusudan, Neider

## Tools:

- Z3 SMT Solver: Microsoft Research
- Python AST: Python Software Foundation

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## Certificate Core:

- ① Prajna, Jadbabaie. "Safety Verification of Hybrid Systems Using Barrier Certificates." HSCC 2004.
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- ⑬ Graf, Saïdi. "Construction of Abstract State Graphs with PVS." CAV 1997.
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- ⑯ Garg et al. "ICE: A Robust Framework for Learning Invariants." POPL 2014.
- ⑰ Flanagan, Leino. "Houdini, an Annotation Assistant for ESC/Java." FME 2001.
- ⑱ Alur et al. "Syntax-Guided Synthesis." FMCAD 2013.

### Compositionality:

- ⑲ Pnueli. "In Transition from Global to Modular Temporal Reasoning." ACM 1984.

# Summary: Key Equations

## Barrier Certificate Conditions:

$$\text{Init: } B(x) \leq 0 \quad \forall x \in X_0$$

$$\text{Unsafe: } B(x) > 0 \quad \forall x \in X_u$$

$$\text{Step: } B(x) \leq 0 \Rightarrow B(f(x)) \leq 0$$

## SOS Representation:

$$p(x) \in \Sigma[x] \Leftrightarrow p(x) = z(x)^T Q z(x), \quad Q \succeq 0$$

## Putinar Positivstellensatz:

$$p > 0 \text{ on } K \Rightarrow p = \sigma_0 + \sum_i \sigma_i g_i, \quad \sigma_i \in \Sigma[x]$$

## ICE Sample Constraint:

$$\forall (x^+, x^-) \in \text{Impl} : I(x^+) \Rightarrow I(x^-)$$

**Core Innovation:** Barrier certificates as unifying abstraction for bug detection.

# Takeaway Messages

- ① **Verification can be practical**
  - Not just for avionics—applicable to everyday Python
- ② **Certificates provide confidence**
  - Not "probably safe" but "provably safe"
- ③ **Multiple methods beat single method**
  - SOS + ICE + IC3 + CEGAR ; any alone
- ④ **Theory meets practice**
  - 20 years of research, now usable
- ⑤ **The future is verified**
  - As software criticality grows, verification becomes essential

# Thank You!

Questions?

## Extreme Verification Pipeline

5 Layers • 20 Papers • 67 Bug Types

Mathematical Guarantees for Software Safety

```
pip install extreme-verification
```