August 29, 2020 (2^{nd} session)

Variation of parameters :-

let $f(D)y = (a_0 D^2 + a_1 D + a_2) y = Q(x)$ — (1)

be a non-homogeneous differential equation of order two. If y_c, the complementary function of (1) is given by

$$y_c = c_1 y_1 + c_2 y_2$$

then the particular integral of (1) is given by

$$y_p = A y_1 + B y_2$$

; $$A = -\int \frac{y_2 \cdot Q(x)}{W(x)} dx$$

$$B = +\int \frac{y_1 \cdot Q(x)}{W(x)} \cdot dx$$

with

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is the Wronskian.

Two important formulae (short cuts)

① $$\int e^{ax} \cdot \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \cdot \left[a\sin bx - b\cos bx \right] + c$$

② $$\int e^{ax} \cdot \cos bx = \frac{e^{ax}}{a^2+b^2} \cdot \left[a\cos bx - b\sin bx \right]$$

Example: Solve $y'' + y = \tan x$

For y_c $\Rightarrow$ $(D^2+1)y = 0$

The auxiliary equation is

$$D^2 + 1 = 0$$

$$D^2 = -1$$

$$\Rightarrow D = \pm i$$

$$\Rightarrow \boxed{y_c = c_1 \cos x + c_2 \sin x}$$

Here, $y_1 = \cos x$, $y_2 = \sin x$

let $y_p = A\cos x + B\sin x$ — (1)

Here, $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$\Rightarrow W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$$= 1$$

$$A = -\int \frac{y_2 \cdot Q(x)}{W(x)} \cdot dx$$

$\Rightarrow$

$$A = -\int \frac{\sin x \cdot \tan x}{1} \cdot dx$$

$$= -\int \sin x \cdot \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} \cdot dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= -\int \left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\right) dx$$

$$= -\int (\sec x - \cos x) dx$$

$$\boxed{A = -(\ln|\sec x + \tan x|) + \sin x}$$

also,

$$B = +\int \frac{y_1 \cdot Q(x)}{W(x)}\, dx$$

$\Rightarrow$

$$B = \int \frac{\cos x \cdot \tan x}{1} \cdot dx$$

$$= \int \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} \cdot dx$$

$$= \int \sin x\, dx = -\cos x$$

$$\boxed{B = -\cos x}$$

Using values of A and B in eq (1)

$$\Rightarrow y_p = \left[-\ln|\sec x + \tan x| + \sin x\right]\cos x + (-\cos x) \cdot \sin x$$

or

$$y_p = -\ln|\sec x + \tan x| \cdot \cos x + \cancel{\sin x \cdot \cos x} - \cancel{\sin x \cos x}$$

$$\boxed{y_p = -\ln|\sec x + \tan x|}$$

So, $y = y_c + y_p$

$\Rightarrow$ $$\boxed{y = c_1 \cos x + c_2 \sin x - \ln|\sec x + \tan x|}$$

Example :– $(D^2+1)y = \sec^2 x$

For $y_c \Rightarrow$ $(D^2+1)y = 0$

A·E $\Rightarrow$ $D^2+1 = 0$

$D = \pm i$

$\Rightarrow$ $$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

let

$y_p = A \cos x + B \sin x$ —— (1)

$\Rightarrow$ $W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$\Rightarrow$

$$A = -\int \frac{y_2 \cdot Q(x)}{W(x)} \, dx$$

$$A = -\int \frac{\sin x \cdot \sec^2 x}{1} \, dx$$

$$A = -\int \sin x \cdot \frac{1}{\cos^2 x} \, dx$$

$$= -\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \, dx$$

$$= -\int \tan x \cdot \sec x \, dx = -\sec x.$$

$$\boxed{A = -\sec x} \checkmark$$

and

$$B = +\int \frac{y_1 \cdot Q(x)}{W(x)} \, dx$$

$$= \int \frac{\cos x \cdot \sec^2 x}{1} \, dx$$

$$= \int \cos x \cdot \frac{1}{\cos^2 x} \, dx = \int \frac{1}{\cos x} \, dx$$

$$B = \int \sec x \, dx = \ln|\sec x + \tan x|$$

$$\boxed{B = \ln|\sec x + \tan x|} \checkmark$$

using values of A and B in eq (1)

$$\Rightarrow y_p = (-\sec x)\cos x + \ln|\sec x + \tan x|\sin x$$

$$\boxed{y_p = -1 + \ln|\sec x + \tan x| \cdot \sin x}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x - 1 + \ln|\sec x + \tan x| \cdot \sin x}$$

Example : Solve $(D^2+1)y = \sec x \cdot \tan x$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

let

$$y_p = A \cos x + B \sin x$$

$$W(x) = 1$$

$$A = x - \tan x$$

$$B = \ln|\sec x|$$

$$\boxed{y_p = (x - \tan x)\cos x + \sin x \cdot \ln|\sec x|}$$