August 17, 2020

The linear equation of order one :-

The equation

$$\frac{d}{dx}y + P(x)\cdot y = Q(x) \quad -(1)$$

is said to be linear differential equation.

For solution of Eq (1)

$$I.F = e^{\int P(x)\,dx}$$

Then the solution is

$$y\cdot(I.F) = \int (I.F)\,Q(x)\,dx + C$$

Example :- Solve

$$\frac{d}{dx}y + \frac{1}{x}\cdot y = x^2 \quad \text{———} \; (1)$$

On comparing with

$$\frac{d}{dx}y + P(x)\cdot y = Q(x)$$

$$\Rightarrow \quad P(x) = \frac{1}{x}, \qquad Q(x) = x^2$$

$$\Rightarrow \quad I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx}$$

$$= e^{\cancel{\int}\ln x \, dx} = \cancel{\log x}\, e^{\ln x} = x$$

So, the solution of Eq. (1) is

$$y(x) = \int x \cdot x^2 \, dx + C$$

$$\boxed{xy = \frac{x^4}{4} + C}$$

Example :- Solve

$$\frac{d}{dx}y + 3x^2 y = x^2 \quad (1)$$

On comparing with

$$\frac{d}{dx}y + P(x)y = Q(x)$$

$\Rightarrow$

$$P(x) = 3x^2, \quad Q(x) = x^2.$$

So,

$$I.F = e^{\int 3x^2 dx} = e^{3\int x^2 \cdot dx}$$

$$= e^{\cancel{3} \cdot \frac{x^3}{\cancel{3}}} = e^{x^3}$$

Thus, the solution of eq (1) is

$$y \cdot (e^{(x^3)}) = \frac{1}{3}\int e^{x^3} \cdot 3x^2 \cdot dx + C$$

$$\boxed{y e^{x^3} = \frac{1}{3} e^{x^3} + C}$$

Example :- Solve

$$(x^4+2y)dx - x\,dy = 0$$

$$(x^4+2y)\,dx = x.dy$$

or $$\frac{x^4+2y}{x} = \frac{dy}{dx}$$

or $$\frac{d}{dx}y = x^3 + \frac{2}{x}y$$

or $$\frac{d}{dx}y + \left(-\frac{2}{x}\right)y = x^3 \quad \text{—}(1)$$

On comparing with

$$\frac{d}{dx}y + P(x)y = Q(x)$$

$$\Rightarrow \quad P(x) = -\frac{2}{x}, \quad Q(x) = x^3$$

$$I.F = e^{\int -\frac{2}{x}.dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

The solution is

$$\underbrace{y\left(\frac{1}{x^2}\right) = \int \frac{1}{\cancel{x^2}}.\cancel{x^3}^{\,x}.dx + c = \frac{x^2}{2} + c}$$

<u>Example :-</u> Solve

$$\frac{d}{dx}y + 2xy = x \text{ ——— (1)}, \quad \underline{y(0) = -3}$$

Here, $P(x) = 2x$, $Q(x) = x$.

$$I.F = e^{\int 2x\,dx} = e^{x^2}.$$

The solution is

$$y.(e^{x^2}) = \underline{\int e^{x^2}.x.dx} + C$$

$$\boxed{ye^{x^2} = \frac{1}{2}e^{x^2} + C} \text{ ——— (2)}$$

using $y(0) = -3 \Rightarrow x = 0$
$y = -3$

(2) $\Rightarrow$

$$(-3)e^0 = \frac{1}{2}e^0 + C$$

$$-3 - \frac{1}{2} = C = -\frac{7}{2}$$

Eq.(2) $\Rightarrow$ $\boxed{ye^{x^2} = \frac{1}{2}e^{x^2} - \frac{7}{2}}$

The Bernoulli's equation:

The equation

$$\frac{d}{dx}y + P(x)y = Q(x)\cdot y^n \quad \text{—} (1)$$

is called the Bernoulli's equation.

Example: Solve

$$\frac{d}{dx}y - y = -xe^{-2x}\cdot y^3$$

$$\Rightarrow \frac{d}{dx}y + (-1)y = (-xe^{-2x})y^3 \quad \text{—} (1)$$

Dividing Eq. (1) by y^3

$$\Rightarrow y^{-3}\cdot\frac{d}{dx}y + (-1)y^{-2} = (-xe^{-2x})\cdot 1 \quad \text{—} (2)$$

let $z = y^{-2}$

Diff w.r.t x

$$\frac{d}{dx}z = -2\cdot y^{-3}\cdot\frac{dy}{dx}$$

$$-\frac{1}{2}\cdot\frac{dz}{dx} = y^{-3}\cdot\frac{dy}{dx}$$

using above substitution in Eq. (2)

$$\Rightarrow \cancel{y^{-3}\cdot\frac{dy}{dx} + (-1)z = -xe^{-2x}}$$

$$-\frac{1}{2}\cdot\frac{dz}{dx} + (-1)z = -xe^{-2x}$$

$$\boxed{\frac{dz}{dx} + \underbrace{(2)}_{P}z = \underbrace{+2xe^{-2x}}_{Q}}$$

This is a linear differential eq.

$$\Rightarrow \quad I.F = e^{2\int 1\,dx} = e^{2x}.$$

The solution is

$$z\left(e^{2x}\right) = \int \underbrace{e^{2x}\cdot 2xe^{-2x}}\cdot dx + C$$

$$\boxed{\frac{1}{y^2}e^{2x} = x^2 + C}$$

Example :- Solve

$$xy\,dx + (x^2 - 3y)\,dy = 0$$

$$xy\,dx = -(x^2 - 3y)\,dy$$

$$\frac{xy}{x^2 - 3y} = -\frac{d}{dx}y$$

$$\frac{dx}{dy} = -\frac{x^2 - 3y}{xy}$$

$$= -\frac{x^2}{xy} + \frac{3y}{xy}$$

$$\frac{dx}{dy} = -\frac{x}{y} + \frac{3}{x}$$

$$\frac{d}{dy}x = \left(-\frac{1}{y}\right)x + \frac{3}{x}$$

or

$$\frac{d}{dy} + \left(\frac{1}{y}\right)x = \frac{3}{x}.$$

$$\frac{d}{dx}y + P(x)y = Q(x)$$

$$\frac{dx}{dy} + P(y)x = Q(y)$$

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$$\frac{d}{dy}x + \left(\frac{1}{y}\right)x = 3 \cdot x^{-1}.$$

Dividing by x^{-1}.

$$x \cdot \frac{d}{dy}x + \frac{1}{y}x^2 = 3 \quad \text{——— (1)}$$

let $z = x^2$

$$\Rightarrow \frac{d}{dy}z = 2x \cdot \frac{dx}{dy}$$

$$\frac{1}{2}\frac{dz}{dy} = x\frac{d}{dy}x$$

using in (1)

$$\Rightarrow \frac{1}{2}\frac{dz}{dy} + \frac{1}{y} \cdot z = 3$$

$$\frac{dz}{dy} + \left(\frac{2}{y}\right)z = 6 \quad \text{——— (2)}$$

$$I.F = e^{\int \frac{2}{y} dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

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$\therefore$ The solution is

$$z \cdot y^2 = \int y^2 \cdot 6 \cdot dy + C$$

$$= \cancel{6}\frac{y^3}{\cancel{3}} + C$$

$$= 2y^3 + C$$

$$\boxed{zy^2 = 2y^3 + C}$$

$$\Rightarrow \boxed{x^2y^2 = 2y^3 + C}$$