August 23, 2020 (2^{nd} session)

Case (i) :- if the RHS is e^{ax}

then Particular sol. is $y_p = \frac{1}{f(D)} e^{ax}$

$$y_p = \frac{1}{f(a)} \cdot e^{ax}. \qquad \boxed{D = a}$$

Example :- Solve $y'' + y' + y = e^{-x}$

$$y = \underline{\underline{y_c}} + y_p$$

For y_c

$$y'' + y' + y = 0$$

In operator form

$$D^2 y + Dy + y = 0$$

$$(D^2 + D + 1) y = 0$$

A.E. $\Rightarrow$ $D^2 + D + 1 = 0$

$$D^2+D+1=0$$

$$D=\frac{(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2}=\underbrace{\frac{1}{2}}_{\alpha}\pm i\underbrace{\frac{\sqrt{3}}{2}}_{\beta}$$

$$\Rightarrow \boxed{y_c=e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}$$

For $y_p \Rightarrow$

$$y_p=\frac{1}{D^2+D+1}\cdot e^{-x} \qquad \because a=-1$$

$$D=a$$

$$=\frac{1}{(-1)^2+\cancel{(-1)}+\cancel{1}}\,e^{-x}$$

$$\boxed{y_p=e^{-x}}$$

So

$$y=y_c+y_p$$

$$\Rightarrow \boxed{y=e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]+e^{-x}}$$

Example :- Solve $y'' - 3y' + 2y = e^{5x}$

For y_c

$$(D^2 - 3D + 2)y = 0$$

A.E. $\Rightarrow$ $$D^2 - 3D + 2 = 0$$

$$(D-1)(D-2) = 0$$

$$D = 1, 2$$

$\therefore \Rightarrow$ $$\boxed{y_c = c_1 e^{x} + c_2 e^{2x}}$$

For $y_p \Rightarrow$ $$y_p = \frac{1}{D^2 - 3D + 2} \cdot e^{5x}$$

$$= \frac{1}{(5)^2 - 3(5) + 2} \cdot e^{5x}$$

$$y_p = \frac{1}{25 - 15 + 2} e^{5x} = \frac{1}{12} e^{5x}$$

So, $y = y_c + y_p$

$\Rightarrow$ $$\boxed{y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{12} e^{5x}}$$

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Example :- Solve $\frac{d^2}{dx^2}y - 5\frac{d}{dx}y + 6y = e^{4x}$

For y_c

$$\Rightarrow \quad D^2y - 5Dy + 6y = 0$$

$$(D^2 - 5D + 6)y = 0.$$

The auxiliary equation is

$$D^2 - 5D + 6 = 0$$

$$(D-2)(D-3) = 0$$

$$D = 2, 3.$$

So,
$$\boxed{y_c = c_1 e^{2x} + c_2 e^{3x}}$$

For
$$y_p = \frac{1}{D^2 - 5D + 6} \cdot e^{4x}$$

$$= \frac{1}{(4)^2 - 5(4) + 6} e^{4x}$$

$$y_p = \frac{1}{2} e^{4x}$$

So,
$$\boxed{y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x}}$$

Example :- Solve

$$4y'' + 4y' - 3y = e^{2x}$$

For y_c

$\Rightarrow$ $(4D^2 + 4D - 3)y = 0$

A.E $\Rightarrow$ $4D^2 + 4D - 3 = 0$

$$4D^2 + 6D - 2D - 3 = 0$$

$$2D(2D+3) - 1(2D+3) = 0$$

$$(2D+3)(2D-1) = 0$$

$$D = -\frac{3}{2}, +\frac{1}{2}$$

So, $y_c = c_1 e^{-\frac{3}{2}x} + c_2 e^{\frac{1}{2}x}$

For y_p $\Rightarrow$ $y_p = \frac{1}{4D^2 + 4D - 3} \cdot e^{2x}$

$$= \frac{1}{4(2)^2 + 4(2) - 3} e^{2x} = \boxed{\frac{1}{21} e^{2x}}$$

So, $\boxed{y = c_1 e^{-3/2x} + c_2 e^{1/2x} + 1/21 \cdot e^{2x}}$

Case (ii) :- $\frac{1}{f(D^2)} \cdot \sin ax$ or $\cos ax$

$= \frac{1}{f(-a^2)} \cdot \sin ax$ or $\cos ax$

$D = -(a)^2$

Example: Solve

$$y'' + y' + y = \sin 2x$$

For $y_c \Rightarrow (D^2 + D + 1)y = 0$.

A.E $\Rightarrow D^2 + D + 1 = 0$

$$D = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

For y_p:

$$\Rightarrow \quad y_p = \frac{1}{D^2+D+1} \cdot \sin 2x \qquad \left| \; D^2 = -a^2 \right.$$

$$y_p = \frac{1}{-(2)^2+D+1} \cdot \sin 2x$$

$$= \frac{1}{D-3} \cdot \sin 2x$$

$$= \frac{D+3}{(D-3)(D+3)} \cdot \sin 2x$$

$$= \frac{D+3}{D^2-9} \cdot \sin 2x$$

$$= \frac{D+3}{-(2)^2-9} \sin 2x$$

$$= -\frac{1}{13} \cdot (D+3) \cdot \sin 2x$$

$$= -\frac{1}{13} \cdot \left[D(\sin 2x) + 3 \sin 2x \right]$$

$$\boxed{y_p = -\frac{1}{13}\left[2\cos 2x + 3 \sin 2x \right]}$$

$$\Rightarrow \quad y = y_c + y_p = \underline{\quad\quad} + \underline{\quad\quad}$$

Example: Solve $y'' - 5y' + 6y = \sin 2x$

For $y_c \Rightarrow (D^2 - 5D + 6)y = 0$

A E $\Rightarrow$

$$D^2 - 5D + 6 = 0$$

$$(D-2)(D-3) = 0$$

$$\Rightarrow \quad D = 2, 3$$

$$\boxed{y_c = c_1 e^{2x} + c_2 e^{3x}}$$

For y_p

$$y_p = \frac{1}{D^2 - 5D + 6} \cdot \sin 2x$$

$$= \frac{1}{-(2)^2 - 5D + 6} \cdot \sin 2x$$

$$= \frac{1}{-5D + 2} \sin 2x$$

$$= \frac{2 + 5D}{(2 - 5D)(2 + 5D)} \sin 2x$$

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$$y_p = \frac{2+5D}{(2)^2-(5D)^2} \cdot \sin 2x$$

$$= \frac{2+5D}{4-25D^2} \sin 2x$$

$$= \frac{2+5D}{4-25[-(2)^2]} \cdot \sin 2x$$

$$= \frac{2+5D}{4-25[-4]} \sin 2x$$

$$= \frac{2+5D}{104} \sin 2x$$

$$= \frac{1}{104}\left[2\sin 2x + 5D(\sin 2x)\right]$$

$$= \frac{1}{104}\left[2\sin 2x + 5(2\cos 2x)\right]$$

$$\boxed{y_p = \frac{1}{52}\left[\sin 2x + 5\cos 2x\right]}$$

$y = y_c + y_p =$ —— + ——