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# Riccati's Equation :-

This is an equation of the form:

$$A(x)y' + B(x).y + C(x)y^2 = R(x)$$

if $\boxed{y_1 = y(x)}$ is a particular solution of this equation then by substituting $\boxed{y = \underline{z} + y_1}$ the given equation becomes a Bernoulli's equation.

## Example :- Solve

$$xy' - y + y^2 = x^2 \quad \text{—— (1)}$$

given that $\underline{y_1 = x}$ is a particular solution.

put $y = z + x$

Diff w.r.t $x$

$$\Rightarrow \frac{d}{dx}y = \frac{d}{dx}z + 1$$

using above in Eq. 1.

$$\Rightarrow x\left[\frac{d}{dx}z + 1\right] - [z+x] + [z+x]^2 = x^2$$

or $$x\frac{d}{dx}z + \cancel{x} - z - \cancel{x} + z^2 + \cancel{x^2} + 2zx = \cancel{x^2}$$

or $$x\frac{dz}{dx} \underline{-z} + z^2 + \underline{2zx} = 0$$

or $$x\frac{d}{dx}z + (-1+2x)z + z^2 = 0.$$

or $$1\cdot\frac{d}{dx}z + \left(\frac{-1+2x}{x}\right)z + \frac{1}{x}z^2 = 0$$

or $$\boxed{\frac{d}{dx}z + \left(-\frac{1}{x} + 2\right)z = -\frac{1}{x}z^2} \quad (2)$$

This is the Bernoulli's equation.

Dividing eq (2) by $z^2$

$$\Rightarrow \quad z^{-2}\cdot\frac{d}{dx}z + \left(-\frac{1}{x}+2\right)z^{-1} = -\frac{1}{x} \quad \text{--- (3)}$$

put $\boxed{u = z^{-1}}$

Diff w.r.t "x"

$$\Rightarrow \quad \frac{d}{dx}u = -1\cdot z^{-2}\frac{dz}{dx}$$

or

$$-\frac{d}{dx}u = z^{-2}\cdot\frac{d}{dx}z$$

Eq. (3)

$$\Rightarrow \quad -\frac{d}{dx}u + \left(-\frac{1}{x}+2\right)u = -\frac{1}{x}.$$

$$\Rightarrow \quad \frac{d}{dx}u + \left(\frac{1}{x}-2\right)u = +\frac{1}{x} \quad \text{--- (4)}$$

This is a linear eq.

$$I.F = e^{\int P(x)dx} = e^{\int\left(\frac{1}{x}-2\right)dx}$$

$$= \cancel{e}^{\ln x}\cdot e^{-2x} = \underline{\underline{x\cdot e^{-2x}}}$$

The solution is

$$u(xe^{-2x}) = \int (\cancel{x}e^{-2x}) \cdot \frac{1}{\cancel{x}} \cdot dx + c$$

$$= \int e^{-2x} \cdot dx + c$$

$$\boxed{u x e^{-2x} = -\frac{1}{2} e^{-2x} + c}$$

$$\Rightarrow \frac{1}{z} x e^{-2x} = -\frac{1}{2} e^{-2x} + c$$

$$\Rightarrow \boxed{\left(\frac{1}{y-x}\right) x e^{-2x} = -\frac{1}{2} e^{-2x} + c}$$

$$\boxed{y(I \cdot F) = \int (I \cdot F)\, Q(x) dx + C}$$

Example :- Solve

$$xy' - 2xy + y^2 = x - x^2 \quad \text{--- (1)}$$

given that $y = x$ is a particular solution of eq. (1).

put ~~$y = z + x$~~ $y = z + x$

Diff w.r.t $x$

$$\frac{d}{dx}y = \frac{dz}{dx} + 1$$

using above in Eq. (1)

$$\Rightarrow x\left(\frac{d}{dx}z + 1\right) - 2x(z + x) + (z + x)^2 = x - x^2$$

or

$$x\frac{d}{dx}z + \cancel{x} - \cancel{2xz} - \cancel{2x^2} + z^2 + \cancel{x^2} + \cancel{2zx} = \cancel{x} - \cancel{x^2}$$

or

$$x\frac{d}{dx}z + z^2 = 0 \quad \text{--- (2)}$$

$$x\frac{d}{dx}z = -z^2$$

⑥

$\frac{1}{z^2} . dz = - \frac{dx}{x}$ | $\int x^n . dx = \frac{x^{n+1}}{n+1}$

Integrating.

$$\int z^{-2} dz = - \int \frac{1}{x} dx$$

$$\frac{z^{-2+1}}{-2+1} = - \ln x + c$$

$$\boxed{- \frac{1}{z} = - \ln x + c}$$

$$\Rightarrow \boxed{- \frac{1}{y-x} = - \ln x + c}$$

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Linear D. Eq.

$\underline{\underline{1}} \cdot \frac{d}{dx} y + \underline{\underline{P(x)}} . \underline{\underline{y}} = \underline{\underline{Q(x)}}$

e.g. $\cancel{x} . \frac{dy}{dx} + \underline{\underline{(2x+1)}} y = \underline{\underline{x^2}}$

Example :- Solve

$$y' + (2x-1)y - xy^2 = x-1, \quad (1)$$

given that $y_1 = 1$ is a particular solution.

put $\quad y = z + 1$

$$\Rightarrow \quad \frac{d}{dx}y = \frac{d}{dx}z + 0$$

using in (1)

$$\Rightarrow \quad \frac{dz}{dx} + (2x-1)(z+1) - x(z+1)^2 = x-1$$

or

$$\frac{d}{dx}z + 2xz + 2x - z - 1 - x(z^2+2z+1) = x-1$$

or

$$\frac{d}{dx}z + \cancel{2xz} + \cancel{2x} - z - \cancel{1} - xz^2 - \cancel{2xz} - \cancel{x} = \cancel{x} - \cancel{1}$$

or

$$\frac{d}{dx}z - z - xz^2 = 0$$

$$\frac{d}{dx}z + (-1)z = xz^2 \quad (2)$$

Bernoulli's eq.

Dividing eq. (2) by $z^2$

$$z^{-2}\cdot\frac{d}{dx}z+(-1)z^{-1}=x \quad \text{--- (3)}$$

put $u=z^{-1}$

$$\Rightarrow \quad \frac{d}{dx}u=-z^{-2}\cdot\frac{d}{dx}z$$

or

$$-\frac{d}{dx}u=z^{-2}\frac{dz}{dx}$$

using in (3)

$$\Rightarrow \quad -\frac{d}{dx}u+(-1)u=x$$

or

$$\frac{d}{dx}u+(+1)u=-x \quad \text{--- (4)}$$

This is a linear diff eq.

$$I.F=e^{\int 1\cdot dx}=e^{x}.$$

The sol. is

$$u(e^{x})=-\int e^{x}\cdot x\cdot dx+c$$

$$ue^x = -\int xe^x\,dx + c$$

| | |
|---|---|
| $x$ | $e^x$ |
| $1$ | $+\ e^x$ |
| $0$ | $-\ e^x$ |

$$ue^x = -(xe^x - x) + c$$

$$\boxed{ue^x = -xe^x + x + c}$$

$$\Rightarrow \quad \frac{1}{z}\cdot e^x = -xe^x + x + c$$

$$\Rightarrow \quad \boxed{\frac{1}{y-1}\cdot e^x = -xe^x + x + c}$$

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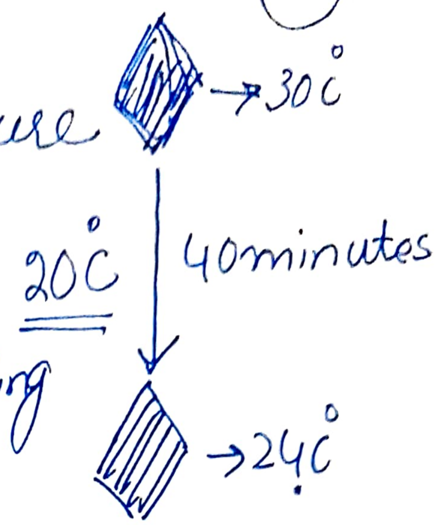
$$\frac{d(y)}{dt} \propto (y - y_1)$$

object — surrounding medium

$$\boxed{\frac{d}{dt}y = K(y - y_1)}$$




Solution: let the temperature of the object be $y^\circ C$.

According to Newton's law of Cooling

$$\frac{dy}{dt} \propto (y-20)$$

or $$\frac{d}{dt}y = K(y-20) \quad \text{——} (1)$$

or

$$\int \frac{1 \cdot dy}{(y-20)^1} = K\int dt$$

$$\Rightarrow \cancel{\log_e} \ln(\cancel{y}\, y-20) = Kt + C_1$$

$$\Rightarrow y-20 = e^{Kt} \cdot e^{C_1} \quad ; e^{C_1} = A$$

$$\Rightarrow y-20 = Ae^{Kt}$$

or $$y = 20 + Ae^{Kt} \quad \text{——} (2)$$

at $t=0$, $y=30$

using in (2) $y = 20 + Ae^{kt}$ — (2)

$\Rightarrow$ $30 = 20 + A \cdot 1$

$\boxed{A = 10}$

Eq (2) $\Rightarrow$ $y = 20 + 10e^{kt}$ — (3)

and at $t=40$, $y=24$

using in (3)

$\Rightarrow$ $24 = 20 + 10e^{k(40)}$

$\frac{24-20}{10} = 0.4 = e^{k(40)}$

$\Rightarrow$ $\ln(0.4) = 40k \cdot (\ln e)$

$k = \frac{\ln(0.4)}{40} =$

$k = -0.0229$

(12)

using in (3)

$\Rightarrow$

a) $y = 20 + 10 \cdot e^{(-0.0229)t}$ —— (4)

(i) at $t = 15$

Eq (4) $\Rightarrow$ $y = 20 + 10\, e^{(-0.0229)(15)}$

$$\boxed{y = 27.1^{\circ}C}$$

b) at $y = 21$ $t = ?$

Eq (4) $\Rightarrow$ $21 = 20 + 10 e^{(-0.0229)t}$

$\frac{21-20}{10} = e^{(-0.0229)t}$

$0.1 = e^{(-0.0229)t}$

$t = \frac{\ln(0.1)}{-0.2229} = 100.5$ minutes