①

(2nd Session)

Example :- Solve

$$(1+x)\,dy \cancel{+} y \cdot dx = 0.$$

$$\underline{M(x,y)\,dx} + \underline{N(x,y)\,dy} = 0$$

$$\underline{\underline{F(x)}}\,\underline{\underline{dx}} + \underline{\underline{G(y)}}\,\underline{\underline{dy}} = 0$$

$$(1+x)\,dy = y \cdot dx$$

$$\frac{1}{y} \cdot dy = \frac{1}{1+x}\,dx$$

integrating both sides

$$\int \frac{1}{y} \cdot dy = \int \frac{1}{(1+x)} \cdot dx$$

$$\Rightarrow \ln y = \underline{\ln(1+x) + \ln C}$$

$$\ln m + \ln n = \ln(m \cdot n)$$

$$\underset{e}{\underline{\ln}}(y) = \underset{e}{\underline{\ln}}\left[c(1+x)\right]$$

$$\Rightarrow \boxed{y = c(1+x)}$$

Example :- Solve

$$2xy\cdot y' = 1+y^2,$$

$y(2)=3$.

$x=2,\ y=3$

$$2xy\cdot\frac{dy}{dx} = 1+y^2$$

$m\cdot\ln n = \ln n^m$

$$\frac{y}{1+y^2}\cdot dy = \frac{1}{2x}\cdot dx$$

integrating

$$\frac{1}{2}\int\frac{2y}{(1+y^2)}\cdot dy = \frac{1}{2}\int\frac{1}{x}\cdot dx$$

$$\Rightarrow \frac{1}{2}\ln(1+y^2) = \frac{1}{2}\ln x + \ln c$$

$$\Rightarrow \ln(1+y^2)^{\frac{1}{2}} = \ln x^{\frac{1}{2}} + \ln c$$

$$\Rightarrow \ln_e(1+y^2)^{\frac{1}{2}} = \ln_e(c\,x^{\frac{1}{2}})$$

$$\Rightarrow \boxed{\sqrt{1+y^2} = c\cdot\sqrt{x}} \text{ — } ①$$

$\sqrt{1+y^2} = \sqrt{5x}$

Example :- $y' = \frac{2x}{y+x^2y}$, $y(0)=1$

$\Rightarrow$ $$\frac{d}{dx}y = \frac{2x}{y(1+x^2)}$$

or $$y \cdot dy = \frac{2x}{1+x^2} \cdot dx$$

Integrating.

$$\int y \cdot dy = \int \frac{2x}{1+x^2} \cdot dx$$

$\Rightarrow$ $$\ln y = \ln(1+x^2) + \ln c$$

$\Rightarrow$ $$\ln y = \ln\left[c(1+x^2)\right] \quad \text{— ①}$$

$$\boxed{y = c(1+x^2)}$$

using $y(0)=1$, $x=0, y=1$

① $\Rightarrow$ $$\ln(1) = \ln\left[c(1+0)\right]$$

$$c = 1$$

So, ① $\Rightarrow$ $$\boxed{y = (1+x^2)}$$

Example Solve.

$$(1-x)y' = y^2.$$

$$(1-x)\cdot\frac{d}{dx}y = y^2$$

$$\frac{1}{y^2}\cdot dy = \frac{1}{1-x}\cdot dx$$

$$\int y^{-2}\cdot dy = -\int\frac{-1}{(1-x)^1}\cdot dx$$

$$\Rightarrow \quad \frac{y^{-2+1}}{-2+1} = -\ln(1-x)+C$$

or

$$\frac{y^{-1}}{\cancel{-}1} = \cancel{-}\ln(1-x)+C$$

$$\boxed{\frac{1}{y} = \ln(1-x)+C}$$

Example :- Solve $2\cdot dx + e^{3x}\cdot dy = 0$

$$\Rightarrow \quad 2\cdot dx = -e^{3x}\cdot dy$$

$$\frac{2}{e^{3x}}\cdot dx = -dy$$

$$2\int e^{-3x}\cdot dx = -\int 1\cdot dy$$

$$\Rightarrow \quad 2\cdot\frac{e^{-3x}}{-3} = -y + c$$

$$\boxed{-\frac{2}{3}e^{-3x} = -y + c}$$

Example :- $x^{2}\cdot dx + y(x-1)dy = 0$

$$\frac{x^{2}}{x-1}\cdot dx = -y\cdot dy$$

⑥

$$\frac{x^2}{x-1}dx = - y\cdot dy$$

$$\left[(x+1) + \frac{1}{x-1}\right] dx = -y\,dy$$

$$\int (x+1)\,dx + \int \frac{1}{x-1}\,dx = -\int y\cdot dy$$

$$\Rightarrow \boxed{\frac{(x+1)^2}{2} + \ln(x-1) = -\frac{y^2}{2} + C}$$

$$\begin{array}{r} \boxed{x+1} \\ x-1\,\overline{)\,\cancel{x^2}} \\ \underline{\cancel{x^2} - x} \\ - \quad + \\ +\cancel{x} \\ \underline{\cancel{x} - 1} \\ - \quad + \\ +1 \end{array}$$

$$\frac{7}{3}$$

$$\begin{array}{r} ① \\ 3\,\overline{)\,7} \\ \underline{3} \\ ④ \end{array}$$

$$1\frac{4}{3}$$

Homogeneous Differential equations:-

$$\boxed{f(\lambda x, \lambda y) = \lambda^n \cdot f(x, y)}$$

example:- $f(x,y) = 2x^4 - x^2y^2 + 5y^3x$

$$\Rightarrow f(\lambda x, \lambda y) = 2(\lambda x)^4 - (\lambda x)^2(\lambda y)^2 + 5(\lambda x)(\lambda y)^3$$

$$= 2\lambda^4 x^4 - \lambda^2 x^2 \cdot \lambda^2 y^2 + 5\lambda^4 x y^3$$

$$= \lambda^4 \{2x^4 - x^2y^2 + 5xy^3\}$$

$$\boxed{f(\lambda x, \lambda y) = \lambda^4 \cdot f(x, y)}$$

This is a homogeneous function.

$$f(x,y) = x^2 - xy + y^2$$

Example: $f(x,y) = \frac{1}{x^2+y^2}\cdot e^{\frac{x}{y}}$ //

$$\Rightarrow f(\lambda x,\lambda y) = \frac{1}{(\lambda x)^2+(\lambda y)^2}\cdot e^{\frac{\cancel{\lambda} x}{\cancel{\lambda} y}}$$

$$= \frac{1}{\lambda^2(x^2+y^2)}\cdot e^{\frac{x}{y}}$$

$$= \lambda^{\boxed{-2}}\left[\frac{1}{x^2+y^2}\cdot e^{\frac{x}{y}}\right]$$

This is a homogeneous function of degree -2.

Homogeneous Differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

with M, N are both homogeneous functions of the same degree.

Example:- Solve $\underline{\underline{(x^2+y^2)}}\,dx + \underline{\underline{(x^2-xy)}}\,dy = 0$

$$(x^2+y^2)\,dx = -(x^2-xy)\,dy$$

$$\frac{d}{dx}y = -\frac{x^2+y^2}{x^2-yx} \quad \text{——} ①$$

put $\boxed{y = \underline{\underline{vx}}}$ ✓ $\Rightarrow \boxed{v = \frac{y}{x}}$

$$\Rightarrow \frac{d}{dx}y = v.1 + x.\frac{d}{dx}v$$

or $\boxed{\frac{d}{dx}y = v + x.\frac{d}{dx}v}$ ✓

putting above in Eq. ①

$$\Rightarrow \quad v + x\cdot\frac{d}{dx}v = -\frac{x^2+(vx)^2}{x^2+(vx)x}$$

$$= -\frac{x^2+x^2v^2}{x^2+vx^2}$$

$$= -\frac{\cancel{x^2}(1+v^2)}{\cancel{x^2}(1+v)}$$

$$v + x\cdot\frac{d}{dx}v = -\frac{1+v^2}{1+v}$$

$$x\cdot\frac{d}{dx}v = -\frac{1+v^2}{1+v} - \frac{v}{1}$$

$$= -\left[\frac{1+v^2}{1+v^2}+\frac{v}{1}\right]$$

$$= -\left[\frac{1+v^2+v(1+v)}{1+v^2}\right]$$

$$= -\left[\frac{1+\cancel{v^2}+v+\cancel{v^2}}{1+v^2}\right]$$

$$x \cdot \frac{d}{dx} v = -\left[\frac{1+v}{1+v^2}\right]$$

$$\int \frac{1+v^2}{1+v} \cdot dv = -\int \frac{dx}{x}$$

$$\int \left(v-1+\frac{2}{v+1}\right) dv = -\int \frac{dx}{x}$$

$$\begin{array}{r|l} & v-1 \\ \hline v+1 & v^2+1 \\ & \cancel{v^2} + v \\ & - \quad - \\ \hline & -v+1 \\ & \cancel{-v}-1 \\ & + \quad + \\ \hline & +2 \end{array}$$

$$\int (v-1)\, dv + 2\int \frac{1}{v+1} \cdot dv = -\int \frac{dx}{x}$$

$$\ln(v-1) + 2\ln(v+1) = -\ln x + \ln C$$

$$\underline{\ln(v-1) + \ln(v+1)^2} = \ln\left(\frac{C}{x}\right)$$

$$\ln\left[(v-1)(v+1)^2\right] = \ln\left(\frac{C}{x}\right)$$

$$\Rightarrow \boxed{(v-1)(v+1)^2 = \frac{C}{x}}$$

$$\boxed{\left(\frac{y}{x}-1\right)\left(\frac{y}{x}+1\right)^2 = \frac{C}{x}} \quad ✓$$