August 30, 2020 (2nd session)

Method of Undetermined Coefficients.

Case (2) :- When RHS is an exponential function. i.e., RHS $= K e^{\alpha x}$; α is a constt.
Assume a solution of the form

$$y_p = A e^{\alpha x} \quad ; \quad A \text{ is a constant to be determined.}$$

Example :- Solve $y'' - y' - 2y = e^{3x}$ —— ①

For y_c $\quad (D^2 - D - 2) y = 0.$

The auxiliary equation is

$$D^2 - D - 2 = 0$$
$$D^2 - 2D + D - 2 = 0$$
$$D(D-2) + 1(D-2) = 0$$
$$(D-2)(D+1) = 0$$
$$D = 2, -1$$

$$\Rightarrow \boxed{y_c = c_1 e^{2x} + c_2 e^{-x}}$$

Suppose $y_p = Ae^{3x}$ —— (2)

$\Rightarrow$ $y_p' = 3Ae^{3x}$

and $y_p'' = 9Ae^{3x}$.

using above values in Eq. (1).

$$y'' - y' - 2y = e^{3x}$$

$$\Rightarrow (9Ae^{3x}) - (3Ae^{3x}) - 2(Ae^{3x}) = e^{3x}$$

$$4Ae^{3x} = e^{3x}$$

On comparing coefficient of e^{3x}

$\Rightarrow$

e^{3x}: $4A = 1$

$$\Rightarrow \boxed{A = \frac{1}{4}}$$

putting in (2)

$$\Rightarrow \boxed{y_p = \frac{1}{4}e^{3x}}$$

$\therefore$ $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4}e^{3x}}$$

Example :- $y'' + y' + y = e^{-x}$ —— (1)

For y_c $\quad (D^2 + D + 1)y = 0.$

The auxiliary equation is

$$D^2 + D + 1 = 0$$

$$\Rightarrow \quad D = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$D = \frac{-1 \pm i\sqrt{3}}{2}$$

$$D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \boxed{y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]}$$

For y_p Let $y_p = Ae^{-x}$ —— (2)

$$\Rightarrow \quad y_p' = -Ae^{-x}, \quad y_p'' = +Ae^{-x}.$$

using above in Eq. (1)

$$\Rightarrow \quad \cancel{(Ae^{-x})} + \cancel{(-Ae^{-x})} + Ae^{-x} = e^{-x}$$

On comparing $\boxed{A = 1}$

Using value of $A=1$ in Eq ②

$$\Rightarrow \boxed{y_p = 1 \cdot e^{-x}}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{-x}}$$

Example :- $y'' - 3y' + 2y = e^{5x}$ —— ①

For y_c: $(D^2 - 3D + 2)y = 0$.

A.E. $\Rightarrow$ $D^2 - 2D - D + 2 = 0$

$$D(D-2) - 1(D-2) = 0$$

$$(D-2)(D-1) = 0$$

$$D = 2, 1$$

$$\Rightarrow \boxed{y_c = c_1 e^{2x} + c_2 e^{x}}$$

For y_p

Let $y_p = Ae^{5x}$ —— (2)

$\Rightarrow$ $y_p' = 5Ae^{5x}$, $y_p'' = 25Ae^{5x}$.

using above in Eq. (1)

$\Rightarrow (25Ae^{5x}) - 3(5Ae^{5x}) + 2(Ae^{5x}) = e^{5x}$

$\underline{25Ae^{5x}} \underline{-15Ae^{5x}} + 2Ae^{5x} = e^{5x}$

$12Ae^{5x} = e^{5x}$

On comparing the coefficients e^{5x}.

e^{5x}: $12A = 1$

$\Rightarrow \boxed{A = \frac{1}{12}}$

putting in (2)

$\Rightarrow \boxed{y_p = \frac{1}{12}e^{5x}}$

$\therefore$ $y = y_c + y_p$ —— $+$ ——

Example :- Solve $4y'' + 4y' - 3y = e^{2x}$. —— (1)

$$(4D^2 + 4D - 3)y = 0$$

A.E $\Rightarrow$ $4D^2 + 4D - 3 = 0$

$$4D^2 + 6D - 2D - 3 = 0$$

$$2D(2D+3) - 1(2D+3) = 0$$

$$(2D+3)(2D-1) = 0$$

$$D = +\frac{1}{2}, -\frac{3}{2}$$

$$\Rightarrow \boxed{y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{3}{2}x}}$$

Let (2) — $y_p = Ae^{2x}$, $y_p' = 2Ae^{2x}$

$$y_p'' = 4Ae^{2x}$$

(1) $\Rightarrow$ $4(4Ae^{2x}) + 4(2Ae^{2x}) - 3(Ae^{2x}) = e^{2x}$

$$21Ae^{2x} = e^{2x}$$

$$\Rightarrow A = \frac{1}{21}$$

$$\therefore \boxed{y_p = \frac{1}{21}e^{2x}}$$

$\therefore y = y_c + y_p =$ —— + ——

Example :- $y'' - 4y' + 4y = e^{2x}$ —— (1)

For $y_c \Rightarrow (D^2 - 4D + 4)y = 0$

The auxiliary equation is

$$D^2 - 4D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D(D-2) - 2(D-2) = 0$$

$$(D-2)(D-2) = 0$$

$$D = +2, +2$$

$$\Rightarrow \boxed{y_c = (c_1 + c_2 x)e^{2x}}$$

For y_p:

Let $y_p = Ae^{2x}$

$\Rightarrow$

$$y_p' = 2Ae^{2x}, \quad y_p'' = 4Ae^{2x}.$$

using above in Eq. (1)

$$\Rightarrow (4Ae^{2x}) - 4(2Ae^{2x}) + 4(Ae^{2x}) = e^{2x}$$