October 13, 2020

$$-\frac{1}{3xy}+\frac{2}{3}\ln x-\frac{1}{3}\ln y=c$$

(2) $(x^2y^2+xy+1)y\,dx+(x^2y^2-xy+1)x\,dy=0$

$$(x^2y^3+xy^2+y)dx+(x^3y^2-x^2y+x)dy=0 \quad \text{---(i)}$$

Here, $M=x^2y^3+xy^2+y$, $N=x^3y^2-x^2y+x$.

$$\text{I.F.}=\frac{1}{xM-yN}$$

$$=\frac{1}{x[x^2y^3+xy^2+y]-y[x^3y^2-x^2y+x]}$$

$$=\frac{1}{\cancel{x^3y^3}+x^2y^2+\cancel{xy}-\cancel{x^3y^3}+x^2y^2-\cancel{xy}}=\frac{1}{2x^2y^2}$$

multiplying eq. 1 by $\frac{1}{2x^2y^2} = $ I.F.

$$\Rightarrow \frac{1}{2}\left[xy + \frac{1}{xy} + \frac{1}{x^2y^2}\right]y\,dx + \frac{1}{2}\left[xy - \frac{1}{xy} + \frac{1}{x^2y^2}\right]x\,dy = 0$$

or

$$\left(xy^2 + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x^2y - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0 \quad \text{— (2)}$$

$$\Rightarrow M = xy^2 + \frac{1}{x} + \frac{1}{x^2y}, \quad N = x^2y - \frac{1}{y} + \frac{1}{xy^2}$$

$$\Rightarrow M_y = 2xy + 0 - \frac{1}{x^2y^2}, \quad N_x = 2xy + 0 - \frac{1}{x^2y^2}$$

$$M_y = N_x$$

$\Rightarrow$ Eq. (2) is exact D.Eq.

Eq. 1 is not exact.

Now,
$$I.F. = \frac{1}{xM - yN}$$

$$= \frac{1}{x[y+xy^2] - y[x - x^2y]}$$

$$= \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2}$$

$$\boxed{I.F = \frac{1}{2x^2y^2}}$$

multiplying eq. 1 by $I.F. = \frac{1}{2x^2y^2}$.

$$\frac{1}{2x^2y^2}(y + xy^2)dx + \frac{1}{2x^2y^2}(x - x^2y)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \quad \text{—(2)}$$

The solution of eq. (2) is

$$\int_{y\ \text{const}} \left(xy^2+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\int\left(-\frac{1}{y}\right)dy=c$$

$$\frac{x^2}{2}y^2+\ln x-\frac{1}{xy}-\ln y=c$$

$$\boxed{\frac{x^2y^2}{2}-\frac{1}{xy}+\ln\left(\frac{x}{y}\right)=c}$$

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(3) (a) $(1+xy)y\,dx+(1-xy)x\,dy=0$

$$\underbrace{(y+xy^2)}_{M}dx+\underbrace{(x-x^2y)}_{N}dy=0 \quad \text{——} (1)$$

$$\Rightarrow M_y=1+2xy, \quad N_x=1-2xy$$

$$M_y\neq N_x$$

Here,

$$M = \frac{1}{2x^2y} + \frac{1}{2x}, \quad N = \frac{1}{2xy^2} - \frac{1}{2y}$$

$\Rightarrow$

$$M_y = -\frac{1}{2x^2y^2} + 0, \quad N_x = \frac{1}{2x^2y^2} - 0$$

$\therefore M_y = N_x$

$\Rightarrow$ Eq. 2 is exact.

The solution of eq 2 is

$$\int_{y\text{-const}} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \int \left(-\frac{1}{2y}\right) dy = c$$

$$\boxed{-\frac{1}{2xy} + \frac{1}{2}\ln x - \frac{1}{2}\ln y = c}$$