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Cauchy-Euler Equation:-

An equation of the form

$$a_0 x^n \cdot \frac{d^n y}{dx^n} + a_1 x^{n-1} \cdot \frac{d^{n-1}}{dx^{n-1}} y + \cdots + a_{n-1} x \cdot \frac{d}{dx} y + a_n y = Q(x) \quad \text{--- (1)}$$

with $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy-Euler equation of order n. We shall discuss only 2^{nd} order Euler's equation which is of the form

$$a_0 x^2 y'' + a_1 x y' + a_2 y = Q(x).$$

This equation can be reduced to a linear differential equation with constant coefficients by substituting

$$x = e^t \quad \text{or} \quad t = \ln x$$

$$x = e^t \Rightarrow t = \ln x$$

$$\boxed{a_0 \underline{x^2} y'' + a_1 \underline{x} y' + a_2 y = Q(x)}$$

$$xy' = x \cdot \frac{d}{dx} y$$

$$= x \cdot \frac{d}{dt} y \cdot \frac{dt}{dx}$$

$$t = \ln x$$

$$\frac{dt}{dx} = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

$$xy' = \cancel{x} \cdot \frac{d}{dt} y \cdot \frac{1}{\cancel{x}}$$

$$\boxed{xy' = 1 \cdot \frac{d}{dt} y}$$

$$\because \quad xy' = \frac{d}{dt} y$$

$$y' = \frac{1}{x} \cdot \frac{d}{dt} y$$

$$\Rightarrow \quad y'' = \frac{d}{dx} \left[\underbrace{\frac{1}{x}}_{II} \cdot \underbrace{\frac{d}{dt} y}_{I} \right]$$

$$= \frac{d}{dt} y \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x} \cdot \underline{\underline{\frac{d}{dx}\left(\frac{d}{dt} y\right)}}$$

$$y'' = -\frac{1}{x^2}\cdot\frac{d}{dt}y + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$y'' = -\frac{1}{x^2}\cdot\frac{d}{dt}y + \frac{1}{x}\left[\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}\right]$$

$$y'' = -\frac{1}{x^2}\cdot\frac{d}{dt}y + \frac{1}{x}\left[\frac{d^2}{dt^2}y\cdot\frac{1}{x}\right]$$

$$y'' = -\frac{1}{x^2}\cdot\frac{d}{dt}y + \frac{1}{x^2}\cdot\frac{d^2}{dt^2}y$$

$$y'' = \frac{1}{x^2}\left[\frac{d^2}{dt^2}y - \frac{dy}{dt}\right]$$

$$\boxed{x^2y'' = \frac{d^2}{dt^2}y - \frac{d}{dt}y}$$

Solve: $x^2y'' - 2xy' + 2y = 0$ — (1)

This is a Cauchy-Euler equation of order 2. To solve let

$$\boxed{\begin{aligned} &x = e^t, \Rightarrow t = \ln x. \\ \Rightarrow\ &xy' = \frac{d}{dt}y \\ \text{and}\ &x^2y'' = \frac{d^2}{dt^2}y - \frac{d}{dt}y \end{aligned}}$$

using above in Eq. (1)

$$\Rightarrow \left(\frac{d^2}{dt^2}y - \frac{d}{dt}y\right) - 2\left(\frac{d}{dt}y\right) + 2y = 0$$

or $$\frac{d^2}{dt^2}y - 3\cdot\frac{d}{dt}y + 2y = 0$$

In operator form

$$\Rightarrow (D^2 - 3D + 2)y = 0$$

A.E $\Rightarrow$ $D^2 - 3D + 2 = 0$

$$(D-2)(D-1) = 0$$

$$D = 2, 1$$

$$\Rightarrow y = c_1 e^{2t} + c_2 e^{t}$$

$$y = c_1 (e^{t})^2 + c_2 e^{t}$$

$$\boxed{y = c_1 x^2 + c_2 x}$$

Example :- Solve

$$x^2 y'' + x y' + y = 1 + x \quad \text{--- (1)}$$

put $x = e^{t} \Rightarrow t = \ln x$

$$\Rightarrow x y' = \frac{d}{dt} y$$

and $x^2 y'' = \frac{d^2}{dt^2} y - \frac{d}{dt} y$

using above in Eq. 1

$$\Rightarrow \left(\frac{d^2}{dt^2}y - \cancel{\frac{d}{dt}y}\right) + \left(\cancel{\frac{d}{dt}y}\right) + y = 1 + e^t$$

$$\frac{d^2}{dt^2}y + y = 1 + e^t \quad \text{--- } (2)$$

For y_c $\quad (D^2+1)y = 0$

A.E $\Rightarrow$ $\quad D^2 + 1 = 0$

$$D = \pm i$$

$$\Rightarrow y_c = [c_1 \cos t + c_2 \sin t]$$

For $y_p = \frac{1}{D^2+1} \cdot (1 + e^t)$

$$= \frac{1}{D^2+1} \cdot e^{0t} + \frac{1}{D^2+1} \cdot e^t$$

$$= \frac{1}{0^2+1} e^{0t} + \frac{1}{1^2+1} e^t$$

$$y_p = 1 + \frac{1}{2}e^t$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 \cos t + c_2 \sin t + 1 + \frac{1}{2} \cdot e^t}$$

back substitution.

$$\Rightarrow y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + 1 + \frac{1}{2} \cdot (x)$$

or

$$\boxed{y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + 1 + \frac{x}{2}}$$