August 16, 2020 (1^{st} Session)

Example :- Solve

$$(x^2 - xy + y^2)dx - (xy)dy = 0$$

$$(xy)dy = (x^2 - xy + y^2)dx$$

$$\frac{d}{dx}y = \frac{x^2 - xy + y^2}{xy} \quad \text{--- (1)}$$

let $\quad y = vx$

$$\Rightarrow \quad \frac{d}{dx}y = v + x.\frac{d}{dx}v$$

using above in Eq. (1)

$$\Rightarrow \quad v + x.\frac{d}{dx}v = \frac{x^2 - x(vx) + (vx)^2}{x(vx)}$$

$$= \frac{x^2 - x^2v + x^2v^2}{x^2v}$$

$$= \frac{\cancel{x^2}[1 - v + v^2]}{\cancel{x^2}v}$$

$$v + x.\frac{d}{dx}v = \frac{1 - v + v^2}{v}$$

$$x \cdot \frac{d}{dx} v = \frac{1 - v + v^2}{v} - \frac{v}{1}$$

$$x \cdot \frac{d}{dx} v = \frac{1 - v + \cancel{v^2} - \cancel{v^2}}{v}$$

$$x \cdot \frac{d}{dx} v = \frac{1 - v}{v}$$

$$\int \frac{v}{1 - v} \cdot dv = \int \frac{1}{x} \cdot dx$$

$$\begin{array}{r} -1 \\ -v+1 \,\big)\, \overline{v \quad\;\;} \\ \underline{v - 1} \\ +1 \end{array}$$

$$\int \left(-1 + \frac{1}{1 - v}\right) dv = \int \frac{1}{x} \cdot dx$$

$$\Rightarrow -v - \ln(1 - v) = \ln x + c$$

$$\Rightarrow \boxed{-\frac{y}{x} - \ln\left(1 - \frac{y}{x}\right) = \ln x + c}$$

$y = vx$

$v = \frac{y}{x}$

Example:- Solve $y^2 . dx + x(x+y)dy = 0$

$$\frac{-v}{1+v}$$

$$\begin{array}{r|l} v+1 & -v \quad (-1 \\ & \cancel{-v} - 1 \\ & + \quad + \\ \hline & +1 \end{array}$$

$$-1 + \frac{1}{v+1}$$

$$y^2 . dx + (x^2 + xy)dy = 0$$

or
$$y^2 . dx = -(x^2 + xy)dy$$

or
$$\frac{y^2}{x^2 + xy} = -\frac{dy}{dx}$$

or
$$\frac{d}{dx}y = -\frac{y^2}{x^2 + xy} \quad \text{---} \ (1)$$

put $y = vx$

$$\Rightarrow \quad \frac{d}{dx}y = v + x \cdot \frac{d}{dx}v$$

using above in Eq. 1

$$\Rightarrow \quad v + x \cdot \frac{d}{dx}v = -\frac{(vx)^2}{x^2 + x(vx)}$$

$$v + x \cdot \frac{d}{dx}v = -\frac{v^2x^2}{x^2 + vx^2}$$

$$= -\frac{v^2\cancel{x^2}}{\cancel{x^2}(1+v)}$$

$$v + x \cdot \frac{d}{dx}v = -\frac{v^2}{1+v}$$

$$\Rightarrow \quad x \cdot \frac{d}{dx}v = -\frac{v^2}{1+v} - \frac{v}{1}$$

$$= \frac{-v^2 - v(1+v)}{1+v}$$

$$= \frac{-v^2 - v - v^2}{1+v}$$

$$x \cdot \frac{d}{dx}v = \frac{-2v^2 - v}{1+v}$$

$$\frac{1+v}{-2v^2-v}\,dv = \frac{1}{x}.dx$$

$$-\frac{1+v}{2v^2+v}\,dv = \frac{1}{x}.dx$$

$$\frac{1+v}{v(2v+1)}\,dv = -\frac{1}{x}.dx \quad — ②$$

solving $\boxed{\frac{1+v}{v(2v+1)} = \frac{A}{v} + \frac{B}{2v+1}} = \frac{1}{v} - \frac{1}{2v+1}$

$$\frac{1+v}{\cancel{v(2v+1)}} = \frac{A(2v+1)+Bv}{\cancel{v(2v+1)}}$$

$$1+v = A(2v+1)+Bv$$

$$1+v = 2v.A + A + Bv$$

$$1+v = (2A+B)v + A$$

On comparing.

v: $2A+B=1$

1: $\boxed{A=1}$

$2(1)+B=1$

$\boxed{B=-1}$

using in 2

$$\int \left(\frac{1}{v} - \frac{1}{2v+1}\right) dv = -\int \frac{1}{x}\, dx$$

$$\ln v - \frac{1}{2}\ln(2v+1) = -\ln x + c$$

$$v = \frac{y}{x}.$$

$$\Rightarrow \boxed{\ln\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{2y}{x}+1\right) = -\ln x + c}$$

Partial Fraction :-

Case(1) when denomenator has ~~t~~ linear and different factors.

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Case(2) :- when denomenator has linear and repeated factor

$$\frac{1}{(x+1)^2} = \frac{A}{(x+1)^1} + \frac{B}{(x+1)^2}$$

Case (3) :- When denomenator has ir-reducible quardatic factor

$$\frac{1}{x^2+1} = \frac{Ax+B}{x^2+1}$$

Exact Differential equations :-

The 1st order Differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if

$$\boxed{\frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N}$$

Example :- Solve.

$$(2xy - 3x^2)dx + (x^2 + 2y)dy = 0 \quad \text{--- } (1)$$

Here,

$$M = 2xy - 3x^2$$

$$N = x^2 + 2y$$

$$\Rightarrow \quad \frac{\partial}{\partial y} M = 2x(1) - 0 = 2x.$$

$$\frac{\partial}{\partial x} N = 2x + 0 = 2x$$

$$\because \frac{\partial}{\partial y} M = \frac{\partial}{\partial x} N \Rightarrow \text{Eq. (1) is Exact. D. Eq.}$$

The solution of the exact D. Eq. is

$$\int_{y\ \text{const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\Rightarrow \int_{y\ \text{const}} (2xy - 3x^2)\,dx + \int 2y\,dy = C$$

$$\cancel{2}y \cdot \frac{x^2}{\cancel{2}} - \cancel{3} \cdot \frac{x^3}{\cancel{3}} + \cancel{2} \cdot \frac{y^2}{\cancel{2}} = C$$

or $\boxed{x^2y - x^3 + y^2 = C}$