August 31, 2020 (2nd Session)

Method of Undetermined coefficients

Modifications :- If any term of the assumed solution, disregarding the multiplicative constts, is also a term of y_c, then the assumed solution must be modified by multiplying it by x^m; m is the smallest positive integer such the product of x^m with the assumed solution has no term in common with y_c.

<u>Example</u>: Solve $y'' = 9x^2 + 2x - 1$ ——— (1)

For $y_c \Rightarrow$ $D^2 y = 0$

The auxiliary equation is

$$D^2 = 0$$

$$\Rightarrow D = 0, 0$$

$$\Rightarrow y_c = (c_1 + c_2 x) e^{0x} = c_1 + c_2 x$$

$$\boxed{y_c = c_1 + c_2 x}$$

For y_p let $y_p = A_2 x^2 + A_1 x + A_0$.

$$\Rightarrow y_p = x^1 \cdot (A_2 x^2 + A_1 x + A_0)$$

$$y_p = A_2 x^3 + A_1 x^2 + A_0 x$$

$$\Rightarrow y_p = x(A_2 x^3 + A_1 x^2 + A_0 x)$$

$$\Rightarrow \boxed{y_p = A_2 x^4 + A_1 x^3 + A_0 x^2} \text{ ——— (2)}$$

$\Rightarrow \quad y_p = A_2 x^4 + A_1 x^3 + A_0 x^2$

$\Rightarrow$

$$y_p' = 4A_2 x^3 + 3A_1 x^2 + 2A_0 x$$

and $y_p'' = 12A_2 x^2 + 6A_1 x + 2A_0.$

Using above in Eq. 1

$\Rightarrow$

$$12A_2 x^2 + 6A_1 x + 2A_0 = 9x^2 + 2x - 1$$

On comparing the coefficients of $x^2, x, 1$

$\Rightarrow x^2: \quad 12A_2 = 9 \Rightarrow A_2 = \frac{3}{4}$

$x: \quad 6A_1 = 2 \Rightarrow A_1 = \frac{1}{3}.$

$1: \quad 2A_0 = -1 \Rightarrow A_0 = -\frac{1}{2}$

using values of A_2, A_1 and A_0 in Eq. 2

$$\Rightarrow \boxed{y_p = \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2}$$

So, $\quad y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 + c_2 x + \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2}$$

Example :- Solve $y' - 5y = 2e^{5x}$ — (1)

For y_c $\quad (D-5)y = 0$

$A \cdot E : \Rightarrow \quad D - 5 = 0$

$D = 5.$

$\therefore \quad \boxed{y_c = c_1 e^{5x}}$

For $y_p \Rightarrow \quad y_p = Ae^{5x}$

$\Rightarrow \quad y_p = \cancel{A}\,\underline{x}\,\underline{e^{5x}}$ — (2)

$\Rightarrow$

$y_p' = A\left[x(5e^{5x}) + e^{5x}\cdot(1)\right]$

$= \cancel{Ax\,e^{5x} + 5Ae^{5x}} = \underline{5Axe^{5x} + Ae^{5x}}$

$= \cancel{(Ax+5)}e$

using in (1)

$\Rightarrow$

$(5Axe^{5x} + Ae^{5x}) - 5(Axe^{5x}) = 2e^{5x}$

$$5A\cancel{xe^{5x}} + Ae^{5x} - 5A\cancel{xe^{5x}} = 2e^{5x}$$

$$Ae^{5x} = 2e^{5x}$$

On comparing the coefficients of e^{5x}

$$\Rightarrow \boxed{A = 2}$$

using value of $A=2$ in Eq. 2

$$\Rightarrow \boxed{y_p = 2xe^{5x}}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 e^{5x} + 2xe^{5x}}$$

Example :- Solve

$$y'' - 3y' = 6e^{3x} - 5\sin x \quad \text{--- (1)}$$

For y_c $\quad (D^2 - 3D)y = 0$

The auxiliary equation is.

$$D^2 - 3D = 0$$

$$D(D-3) = 0$$

$$\Rightarrow D = 0, 3.$$

So, $\boxed{y_c = c_1 + c_2 e^{3x}}$

For y_p. Let $y_p = Ae^{3x} + B\cos x + C\sin x$

$\Rightarrow$

$$y_p = Axe^{3x} + B\cos x + C\sin x$$

$\Rightarrow$

$$y_p' = A\left[x(3e^{3x}) + e^{3x}.1\right] + \left[-B\sin x + C\cos x\right]$$

$\Rightarrow$

$$y_p'' = A\left[\left(x(9e^{3x}) + (3e^{3x}).1\right) + 3e^{3x}\right] + \left[-B\cos x - C\sin x\right]$$

7

or

$$y_p'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x} + (-B\cos x - C\sin x)$$

or

$$y_p'' = 9Axe^{3x} + 6Ae^{3x} + (-B\cos x - C\sin x)$$

using above in Eq. 1

$$\Rightarrow \left[9Axe^{3x} + 6Ae^{3x} + (-B\cos x - C\sin x)\right] - 3\left[\left(A(3xe^{3x}) + Ae^{3x}\right) + (-B\sin x + C\cos x)\right] = 6e^{3x} - 5\sin x$$

or

$$\left(\cancel{9Axe^{3x}} + 6Ae^{3x} - \cancel{9Axe^{3x}} - 3Ae^{3x}\right) + \left((-B\cos x - C\sin x) + (-3B\sin x + 3C\cos x)\right) = 6e^{3x} - 5\sin x$$

or

$$(3Ae^{3x}) + \left[(-B+3C)\cos x + (-C-3B)\sin x\right] = 6e^{3x} - 5\sin x$$

On comparing.

e^{3x} : $3A = 6 \Rightarrow \boxed{A = 2}$

$\sin x$: $-3B - C = -5$

$\cos x$: $-B + 3C = 0$

or

$$3B + C = 5 \quad \text{--- (i)}$$

$$-B + 3C = 0 \quad \text{--- (ii)}$$

multiplying Eq (ii) by 3 and adding to (i)

$$\Rightarrow \quad \cancel{3B} + C = 5$$

$$-\cancel{3B} + 9C = 0 \qquad +$$

$$10C = 5$$

$$\boxed{C = \frac{1}{2}}$$

using $C = \frac{1}{2}$ in Eq (ii)

$$\Rightarrow \quad -B + 3\left(\frac{1}{2}\right) = 0$$

$$-B = -\frac{3}{2} \Rightarrow \boxed{B = \frac{3}{2}}$$

∴ Eq. (2)

$$\Rightarrow \boxed{y_p = 2x e^{3x} + \frac{3}{2}\cos x + \frac{1}{2}\sin x}$$

So,

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 + c_2 e^{3x} + 2x e^{3x} + \frac{3}{2}\cos x + \frac{1}{2}\sin x}$$

$$y'' + y' - 2y = 2x - 40\cos 2x.$$

$$y_p = \frac{1}{D^2+D-2} \cdot (2x - 40 \cdot \cos 2x)$$

$$= \frac{1}{-2\left[1 - \frac{D^2+D}{2}\right]} 2x - \frac{40}{D^2-D-2} \cdot \cos 2x$$

$$y_p = -\frac{1}{\cancel{2}}\left[1 - \frac{D^2+D}{2}\right]^{-1} (\cancel{2}x) - \frac{40 \cdot}{-(\cancel{2})^2 - D - 2} \cdot \cos 2x$$

$$= -1 \cdot \left[1 + (D^2+D)\right] x - \frac{40}{-4-D-2} \cdot \cos 2x$$

$$= -\left[x + (0+1)\right] - \frac{40}{-D-6} \cdot \cos 2x$$

$$= (-x-1) + \frac{40}{D+6} \cdot \frac{D-6}{D-6} \cdot \cos 2x$$

$$= (-x-1) + \frac{40 \cdot}{D^2-36} \cdot (D-6)\cos 2x$$

$$= (-x-1) + \frac{40}{-(2)^2-36} \cdot (D-6)\cos 2x$$

$$= (-x-1) + \frac{\cancel{40}}{-\cancel{40}} (D-6)\cos 2x$$

$$y_p = (-x-1) - 1\cdot(D-6)(\cos 2x)$$

$$= -x-1-(-2\sin 2x - 6\cos 2x)$$

$$\boxed{y_p = -x-1+2\sin 2x+6\cos 2x}$$