August 23, 2020 (1st session)

Higher order differential equations with constant coefficients :-

$$a_0 \cdot \frac{d^n y}{dx^n} + a_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = X(x) \quad \text{--- (1)}$$

$a_0, a_1, a_2, \dots, a_n$ constt. coefficients.

nth order differential equation.

$$\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} + 3y = 0$$

3rd order differential equation

Operators :-

$$\frac{d}{dx} = D^1, \quad \frac{d^2}{dx^2} = D^2,$$

$$\dots, \quad \frac{d^n}{dx^n} = D^n$$

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = X$$

or $$\underline{\underline{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)}}\, y = X$$

operator is being operated on y

Auxiliary Equation :—

e.g. $$(\underline{D^3 + 2D^2 + D + 1})\, y = x^2$$

The equation obtained by equating to zero the symbolic coeficient of y is called the auxiliary equation, provided D is taken as an algebraic quantity.

$$\underline{D^3 + 2D^2 + D + 1} = 0$$

Example :- Solve.

$$\frac{d^2}{dx^2}y - 3\frac{d}{dx}y - 4y = 0$$

In operator form

$$D^2y - 3Dy - 4y = 0$$

or

$$(D^2 - 3D - 4)y = 0$$

The auxiliary equation is:

$$D^2 - 3D - 4 = 0$$

$$D^2 - 4D + D - 4 = 0$$

$$D(D-4) + 1(D-4) = 0$$

$$(D-4)(D+1) = 0$$

$$D = \underline{\underline{4}}, \underline{\underline{-1}}$$

Therefore, its complete solution is

$$\boxed{y_c = c_1 e^{4x} + c_2 e^{-x}}$$

To solve the auxiliary equation, we have three cases.

Case (1) :- When all the roots of A.E are real and different

Let $m_1, m_2, \ldots, m_n$ are n real and different roots of the A.E. Then the solution is written as:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

e.g. $D = 1, 2, 3$

$$\Rightarrow \boxed{y_c = c_1 e^{1x} + c_2 e^{2x} + c_3 e^{3x}}$$

Case (2) :- When roots of the A.E are real and equal (repeated).

Let $m_1, m_1, m_3, m_4, \ldots, m_n$ are the roots of A.E. Then

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

e.g $D = 2, 2$.

$$\Rightarrow y_c = (c_1 + c_2 x)e^{2x}$$

Example :- $\frac{d^2}{dx^2}y + 2\frac{d}{dx}y + y = 0$

In operator form

$$D^2 y + 2Dy + y = 0$$

A.E $\Rightarrow$ $D^2 + 2D + 1 = 0$

$$D^2 + D + D + 1 = 0$$

$$D(D+1) + 1(D+1) = 0$$

$$(D+1)(D+1)=0$$

$$\Rightarrow D=-1,-1$$

So, the complete solution is

$$y_c=(c_1+c_2x)e^{-x}$$

Case (3)

: When roots of the A.E. are imaginary of the type $\alpha+i\beta$

$$y_c=e^{\alpha x}\left[c_1\cos\beta x+c_2\sin\beta x\right]$$

$$D=2\pm 3i$$

$$\Rightarrow y_c=\cancel{\quad}\, e^{2x}\left[c_1\cos 3x+c_2\sin 3x\right]$$

Example :- Solve

$$\frac{d^2}{dx^2}y - 4\frac{d}{dx}y + y = 0$$

In operator form

$$D^2y - 4Dy + y = 0$$

or

$$(D^2 - 4D + 1)y = 0$$

The auxiliary equation is:

$$D^2 - 4D + 1 = 0$$

$$D = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$D = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2} = \frac{\cancel{2}[2 \pm \sqrt{3}]}{\cancel{2}}$$

$$= \underline{\underline{2+\sqrt{3}}},\ \underline{\underline{2-\sqrt{3}}}$$

The solution is

$$y_c = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$$

Example :- Solve $\frac{d^2}{dx^2}y + \frac{d}{dx}y + y = 0$

$$\Rightarrow \quad D^2y + Dy + y = 0$$

$$(D^2 + D + 1)\,y = 0$$

The A.E is

$$D^2 + D + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$= \alpha + i\beta$$

The complete solution is

$$y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Example:- Solve $\frac{d^3}{dx^3}y - 7\frac{d}{dx}y - 6y = 0$

In operator form

$$D^3 y - 7Dy - 6y = 0$$

or $$(D^3 - 7D - 6)y = 0$$

The A-E is

$$1D^3 - 7D - 6 = 0$$

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}}$$

Synthetic Division

$$\begin{array}{c|cccc} -1 & 1 & 0 & -7 & -6 \\ & \downarrow & -1 & +1 & +6 \\ \hline -2 & 1 & -1 & -6 & \underline{|0} \\ & \downarrow & -2 & +6 & \\ \hline 3 & 1 & -3 & \underline{|0} & \\ & \downarrow & 3 & & \end{array}$$

$$\boxed{D = -1, -2, +3}$$

Example :- Solve

$$\frac{d^3}{dx^3}y - 2\cdot\frac{d^2}{dx^2}y + 4\frac{d}{dx}y - 8y = 0$$

In operator form

$$(D^3 - 2D^2 + 4D - 8)y = 0$$

The A.E.

$$\Rightarrow D^3 - 2D^2 + 4D - 8 = 0$$

$$D^2(D-2) + 4(D-2) = 0$$

$$(D-2)(D^2+4) = 0$$

$$D - 2 = 0, \Rightarrow \boxed{D = +2}$$

and

$$D^2 + 4 = 0$$

$$\Rightarrow D^2 = -4$$

$$D = \pm 2i$$

$$\Rightarrow D = 2, 0 \pm 2i$$

$$\Rightarrow \boxed{y_c = c_1 e^{2x} + [c_1 \cos 2x + c_2 \sin 2x]}$$

Example :- $\frac{d^4}{dx^4}y - 5\cdot\frac{d^2}{dx^2}y + 4y = 0$

In operator form

$$D^4 y - 5D^2 y + 4y = 0$$

or

$$(D^4 - 5D^2 + 4)y = 0.$$

The auxiliary equation is

$$D^4 - 5D^2 + 4 = 0$$

$$D^4 - 4D^2 - D^2 + 4 = 0$$

$$D^2(D^2 - 4) - 1(D^2 - 4) = 0$$

$$(D^2 - 4)(D^2 - 1) = 0$$

$$D^2 = 4 \Rightarrow D = \pm 2$$

$$D^2 - 1 = 0 \Rightarrow D^2 = 1 \Rightarrow D = \pm 1$$

So, the solution is

$$\boxed{y_c = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}}$$

Example: Solve

$$\frac{d^2}{dx^2}y - 4\frac{d}{dx}y + y = 0$$

$$\Rightarrow D^2y - 4Dy + y = 0$$

$$(D^2 - 4D + 1)y = 0$$

A.E $\Rightarrow$ $D^2 - 4D + 1 = 0$

$$D = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{+4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{\cancel{2}(2 \pm 2\sqrt{3})}{\cancel{2}} = 2 \pm 2\sqrt{3}$$

$$\boxed{y_c = c_1 e^{(2+2\sqrt{3})x} + c_2 e^{(2-2\sqrt{3})x}}$$