August 29, 2020 (1st Session)

Inverse Operator method :-

Cases of failure :-

1) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

2) $\frac{1}{f(D^2)}$ Cos ax or Sin ax

$= \frac{1}{f(-a^2)}$; $f(-a^2) \neq 0$

if $f(a) = 0$ or $f(-a^2) = 0$. Then above results fail and therefore we have the following theorems:

Theorem :- i) $\frac{1}{f(D)} e^{ax} = x . \frac{1}{f'(D)} . e^{ax}$

ii) $\frac{1}{f(D^2)}$ cos ax or Sin ax

$= x . \frac{1}{(f(D^2))'}$ Cos ax or Sin ax

Example: Solve $y'' - 3y' + 2y = e^x$.

For y_c $\quad (D^2 - 3D + 2)y = 0$

The auxiliary equation

$$D^2 - 3D + 2 = 0$$

$$(D-1)(D-2) = 0$$

$$D = 1, 2$$

$$\therefore \Rightarrow \boxed{y_c = c_1 e^x + c_2 e^{2x}}$$

For $y_p \Rightarrow$ $\quad y_p = \frac{1}{D^2 - 3D + 2} \cdot e^x$

$$= \frac{1}{\cancel{(1)^2} - \cancel{3(1)} + \cancel{2}} e^x = \frac{1}{0} \quad \Big/ \text{Case Failure}$$

$$\Rightarrow y_p = x \cdot \frac{1}{2D - 3} e^x$$

$$\Rightarrow y_p = x \cdot \frac{1}{2(1) - 3} e^x = -x e^x \quad \Big/ a = 1$$

$$\therefore \boxed{y_p = -x e^x}$$

$$\therefore \boxed{y = y_c + y_p = c_1 e^x + c_2 e^{2x} - x e^x}$$

Example :- Solve $y''' - y'' - y' + y = e^{x}$

For $y_c \Rightarrow$ $(D^3 - D^2 - D + 1)y = 0$

The auxiliary equation is

$D^3 - D^2 - D + 1 = 0.$

$D^2(D-1) - 1(D-1) = 0$

$(D-1)(D^2-1)$

$(D-1)(D-1)(D+1) = 0$

$D = 1, 1, -1$

$\Rightarrow \boxed{y_c = (c_1 + c_2 x)e^{x} + c_3 e^{-x}}$

For $y_p \Rightarrow y_p = \frac{1}{D^3 - D^2 - D + 1} \cdot e^{x}$

$\Rightarrow \quad y_p = \frac{1}{\cancel{(1)^3} - \cancel{(1)^2} - \cancel{1} + \cancel{1}} e^{x} = \frac{1}{0} e^{x}$ / case failure

$\Rightarrow$

$y_p = x \cdot \frac{1}{3D^2 - 2D - 1} e^{x}$

$$y_p = x \cdot \frac{1}{\cancel{3(1)^2} - \cancel{2(1)} - \cancel{1}} e^x = \frac{1}{0} e^x$$

Case failure.

$$\Rightarrow$$

$$y_p = x^2 \cdot \frac{1}{6D-2} e^x$$

$$\Rightarrow \quad y_p = x^2 \cdot \frac{1}{6(1)-2} e^x = \frac{x^2 e^x}{4}.$$

So,

$$y = y_c + y_p$$

$$\boxed{y = (c_1 + c_2 x)e^{+x} + c_3 e^{-x} + \frac{x^2 e^x}{4}}$$

Example :- Solve

$$y'' + a^2 y = \sin ax$$

For y_c $\quad (D^2 + a^2) y = 0$

The auxiliary equation

$\Rightarrow \quad D^2 + a^2 = 0$

$D^2 = -a^2$

$\Rightarrow \quad D = \pm ia$

$\therefore$

$$\boxed{y_c = 1 \cdot [c_1 \cos ax + c_2 \sin ax]}$$

$$\boxed{y_p = -\frac{x}{2a} \cos ax}$$

$\therefore y = y_c + y_p$

$= \_\_ + \_\_$

For $y_p \Rightarrow y_p = \frac{1}{D^2 + a^2} \cdot \sin ax$ $\quad$ $D^2 = -a^2$

$\Rightarrow \quad y_p = \frac{1}{-(a)^2 + a^2} \cdot \sin ax$ $\quad$ case failure

$\Rightarrow \quad y_p = x \cdot \frac{1 \cdot D}{2D \cdot D} \cdot \sin ax$

$= x \cdot \frac{D}{2D^2} \cdot \sin ax = x \cdot \frac{D}{2 \cdot [-(a)^2]} \sin ax$

$y_p = -\frac{x}{2a^2} \cdot D(\sin ax) = -\frac{x}{2a} \cdot [\cancel{a} \cdot \cos ax]$

Example :- Solve $y'' + 4y = e^x + \sin 2x$

For $y_c \Rightarrow$ $(D^2+4)y = 0$

A.E $\Rightarrow$ $D^2 + 4 = 0$

$D^2 = -4$

$\Rightarrow$ $D = \pm i(2) = \pm 2i$

$\Rightarrow$ $\boxed{y_c = \cancel{C_0 e}\,[C_1 \cos 2x + C_2 \sin 2x]}$

For $y_p \Rightarrow$ $y_p = \frac{1}{D^2+4}(e^x + \sin 2x)$

$\Rightarrow$

$y_p = \frac{1}{D^2+4}.e^x + \frac{1}{D^2+4}.\sin 2x$

$\Rightarrow$

$y_p = \frac{1}{(1)^2+4}e^x + \frac{1}{\cancel{-(2)^2+4}}\sin 2x$

$= \frac{1}{5}e^x + x.\frac{1.D}{2D.D}.\sin 2x$

$y_p = \frac{\cancel{Ce}e^x}{5} + x.\frac{D}{2D^2}.\sin 2x$

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$$y_p = \frac{e^x}{5} + x \cdot \frac{D}{2[-(2)^2]} \cdot \sin 2x$$

$$y_p = \frac{e^x}{5} - \frac{x}{8} \cdot D(\sin 2x)$$

$$= \frac{e^x}{5} - \frac{x}{\cancel{8}\,4}(\cancel{2}\cos 2x)$$

$$\Rightarrow \boxed{y_p = \frac{e^x}{5} - \frac{x \cdot \cos 2x}{4}}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} - \frac{x \cos 2x}{4}}$$

Example :- $y'' + y = e^{-x} + \cos x + x^3 + e^x \cdot \sin x$

For $y_c \Rightarrow$ $D^2 + 1 = 0$

$D = \pm i$

$$\Rightarrow \boxed{y_c = c_1 \cos x + c_2 \sin x}$$

For $y_p \Rightarrow$

$$y_p = \frac{1}{D^2+1}\left[e^{-x} + \cos x + x^3 + e^x \sin x\right]$$

$$\Rightarrow y_p = \frac{1}{D^2+1} e^{-x} + \frac{1}{D^2+1} \cos x + \frac{1}{D^2+1} x^3 + \frac{1}{D^2+1} e^x \cdot \sin x$$

$$= \frac{1}{(-1)^2+1} e^{-x} + \frac{1}{-(1)^2+1} \cos x + (1+D^2)^{-1} x^3.$$

Case failure

$$+ e^x \cdot \frac{1}{(D+1)^2+1} \cdot \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x \cdot D}{2D \cdot D} \cos x + \left[1 - (D^2)^1 + (D^2)^2\right] x^3$$

$$+ e^x \cdot \frac{1}{D^2+2D+1+1} \cdot \sin x$$

$$= \frac{1}{2} e^{-x} + x \cdot \frac{D}{2D^2} \cos x + \left[1 - D^2 + D^4\right] x^3$$

$$+ e^x \cdot \frac{1}{D^2+2D+2} \sin x$$

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$$y_p = \frac{e^{-x}}{2} + \frac{xD}{2[-(1)^2]}\sin x + \left[x^3 - D^2(x^3) + D^4(x^3)\right] + e^x \cdot \frac{1}{-(1)^2+2D+2} \cdot \sin x$$

or

$$y_p = \frac{e^{-x}}{2} + \frac{x}{2}(-\sin x) + (x^3 - 6x + 0) + e^x \cdot \frac{1}{2D+1} \cdot \sin x$$

or

$$y_p = \frac{e^{-x}}{2} + \frac{x\sin x}{2} + (x^3 - 6x) + e^x \cdot \frac{2D-1}{(2D+1)(2D-1)} \cdot \sin x$$

$$= \text{\_\_\_\_\_\_} + e^x \cdot \frac{2D-1}{4D^2-1} \cdot \sin x$$

$$= \text{\_\_\_\_\_\_} + e^x \cdot \frac{2D-1}{4[-(1)^2]-1} \sin x$$

$$= \text{\_\_\_\_\_\_} + e^x \cdot \frac{2D-1}{-5} \cdot \sin x$$

$$= \text{\_\_\_\_\_\_} - \frac{e^x}{5}\left[2\cos x - \sin x\right]$$

$$\Rightarrow \boxed{y_p = \frac{e^{-x}}{2} + \frac{x\sin x}{2} + x^3 - 6x - \frac{e^x}{5}(2\cos x - \sin x)}$$