August 16, 2020 (2nd Session)

Non-Exact Differential equations:

Type 1:

$$M\,dx + N\,dy = 0.$$

$$\cancel{I.F} \quad \frac{M_y - N_x}{N} = f(x)$$

Then

$$\cancel{I.F = e^{\int (I.F)\,dx}}$$

$$I.F = e^{\int f(x)\,dx}$$

x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Example: Solve

$$(x^2+y^2+2x)\,dx + (2y)\,dy = 0 \quad —(1)$$

$$\Rightarrow M = x^2+y^2+2x, \quad N = 2y$$

$$\Rightarrow M_y = 2y, \quad N_x = 0$$

$$\because M_y \neq N_x$$

$\therefore$ Eq.(1) is not exact D.Eq.

$$\frac{M_y - N_x}{N} = \frac{2y - 0}{2y} = 1 = f(x).$$

$$\therefore \quad I.F = e^{\int f(x)dx} = e^{\int 1.dx} = \underline{e^{x}}$$

multiplying eq① by $I.F = e^{x}$.

$$\Rightarrow \quad \underline{e^{x}(x^2+y^2+2x)dx} + \underline{e^{x}.2y\,dy} = 0 \quad \text{— ②}$$

$$\Rightarrow \quad M = \underline{e^{x}x^2} + e^{x}y^2 + \underline{2xe^{x}}, \quad N = \underline{2ye^{x}}.$$

$$\Rightarrow \quad M_y = \underline{\underline{e^{x}.(2y)}}, \quad N_x = \underline{\underline{2y\,(e^{x})}}$$

$\Rightarrow$ Eq.② is Exact.

The Solution is

$$\int_{y\text{-const}} (\underline{x^2e^{x}} + e^{x}y^2 + 2\underline{e^{x}x})\,dx + 0 = c$$

$$\boxed{\begin{aligned}&(x^2e^{x} - 2xe^{x} + 2e^{x}) + y^2\underline{\underline{e^{x}}}\\ &+2(xe^{x} - e^{x}) = C\end{aligned}}$$

x^2	$+$	e^{x}
$2x$	$-$	e^{x}
2	$+$	e^{x}
0		e^{x}

x		e^{x}
1	$+$	e^{x}
0		e^{x}

$$M\,dx + N\,dy = 0$$

Example :- Solve

$$(xy^2 - e^{\frac{1}{x^3}})\,dx - x^2y\,dy = 0 \quad \text{--- (1)}$$

$\Rightarrow$

$$M = xy^2 - e^{\frac{1}{x^3}}, \quad N = -x^2y.$$

$$M_y = 2xy - 0, \quad N_x = -2xy$$

$$\because \quad M_y \neq N_x$$

$\Rightarrow$ Eq. (1) is not exact.

$$\frac{M_y - N_x}{N} = \frac{2xy - (-2xy)}{-x^2y}$$

$$= \frac{4\cancel{x}\cancel{y}}{-x^{\cancel{2}}\cancel{y}} = -\frac{4}{x} = f(x)$$

$\Rightarrow$

$$I.F = e^{\int f(x)dx} = e^{-\int \frac{4}{x}dx} = e^{-4\ln x}$$

$$\boxed{I.F} = e^{\ln x^{-4}} = x^{-4} = \boxed{\frac{1}{x^4}}$$

multiplying eq. ① by $I.F = \frac{1}{x^4}$.

$$\frac{1}{x^4}\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - \frac{1}{x^4}\left(x^2 y\right)dy = 0$$

$$\left(\frac{1}{x^3}y^2 - \frac{1}{x^4}.e^{\frac{1}{x^3}}\right)dx - \left(\frac{1}{x^2}y\right)dy = 0 \quad \text{—②}$$

$$\Rightarrow M = \frac{y^2}{x^3} - \frac{1}{x^4}.e^{\frac{1}{x^3}}, \quad N = -\frac{y}{x^2}.$$

$$\Rightarrow$$

$$M_y = \frac{2y}{x^3} - 0, \quad N_x = +\frac{2y}{x^3}$$

$$\because \quad M_y = N_x.$$

So, Eq. ② is exact D. Eq.

$x^{-4}.e^{x^{-3}}$

x^{-3}

$= -3x^{-4}$

The solution is

$$\int\left(\frac{1}{x^3}y^2 - \frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx + 0 = C$$

y const

$$\boxed{-\frac{y^2}{2}x^{-2} + \frac{1}{3}e^{\frac{1}{x^3}} = C}$$

$$\int 2x.e^{x^2}dx = e^{x^2}$$

Rule 2 :-

$$\frac{M_y - N_x}{M} = f(y)$$

$$I.F = e^{-\int f(y)\, dy}.$$

Example:- Solve

$$(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0 \quad —(1)$$

$\checkmark M = y^4+2y, \qquad \checkmark N = xy^3+2y^4-4x.$

$\Rightarrow M_y = 4y^3+2, \qquad N_x = y^3+0-4$

$\because M_y \neq N_x$

$\Rightarrow$ Eq.(1) is not exact.

Now, $\dfrac{M_y-N_x}{M} = \dfrac{(4y^3+2)-(y^3-4)}{} = 3y^3+6$

$$= \frac{3(\cancel{y^3+2})}{y(\cancel{y^3+2})} = \frac{3}{y} = f(y)$$

$$I.F = e^{-\int f(y)dy} = e^{-\int \frac{3}{y} dy}$$

$$\underline{\underline{I.F}} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3} = \underline{\underline{\frac{1}{y^3}}}$$

multiplying eq. 1 by $I.F = \frac{1}{y^3}$

$$\frac{1}{y^3}(y^4+2y)dx + \frac{1}{y^3}\cdot(xy^3+2y^4-4x)dy = 0$$

$$\underline{\underline{\left(y+\frac{2}{y^2}\right)}}dx + \underline{\underline{\left(x+2y-\frac{4x}{y^3}\right)}}dy = 0 \quad \text{--- (2)}$$

$\Rightarrow$

$$M = y + \frac{2}{y^2} \quad , \quad N = x + 2y - \frac{4x}{y^3}.$$

$$M_y = 1 - \frac{4}{y^3} , \quad N_x = 1 + 0 - \frac{4}{y^3}$$

$$\because \quad M_y = N_x$$

So, Eq. (2) is exact.

The solution is

$$\int_{y\text{-const}} \left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = C$$

$$xy + \frac{2}{y^2}x + \cancel{2}\cdot\frac{y^2}{\cancel{2}} = C$$

$$\boxed{xy + \frac{2x}{y^2} + y^2 = C}$$

Rule 3 :- If $M\,dx + N\,dy = 0$ is homogeneous. Then

$$I.F = \frac{1}{xM+yN}.$$

Example :- Solve

$$(x^2y)dx - (x^3+y^3)dy = 0 \quad \text{—(1)}$$

$\Rightarrow$ $M = x^2y$, $\quad N = -x^3 - y^3$

$\Rightarrow$ $M_y = 2xy$, $\quad N_x = -3x^2$.

Eq.(1) is not exact.

So,

$$I.F = \frac{1}{xM + yN}$$

$$= \frac{1}{x(x^2y) + y(-x^3-y^3)}$$

$$= \frac{1}{\cancel{x^3y} - \cancel{x^3y} - y^4} = -\frac{1}{y^4}.$$

multiplying eq.(1) by $I.F = -\frac{1}{y^4}$.

$$-\frac{1}{y^4}(x^2y)\,dx+\frac{1}{y^4}(x^3+y^3)\,dy=0$$

$$\left(-\frac{x^2}{y^3}\right)dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0 \quad \text{—(2)}$$

$$\Rightarrow \quad M=-\frac{x^2}{y^3}, \quad N=\frac{x^3}{y^4}+\frac{1}{y}.$$

$$\Rightarrow \quad M_y=+\frac{3x^2}{y^4}, \quad N_x=\frac{3x^2}{y^4}+0$$

Eq. (2) is exact.

so, the solution of (2) is

$$\int_{y\text{ const}}\left(-\frac{x^2}{y^3}\right)dx+\int\frac{1}{y}\,dy=C$$

$$\boxed{-\frac{x^3}{3y^3}+\ln y=c}$$

Example :- Solve

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad \text{—(1)}$$

$$M = x^2y - 2xy^2, \quad N = -x^3 + 3x^2y$$

$$\Rightarrow M_y = x^2 - 4xy, \quad N_x = -3x^2 + 6xy$$

$$\because \quad M_y \neq N_x$$

$\therefore$ Eq. (1) is not exact.

$$I.F = \frac{1}{xM + yN} = \frac{1}{x(x^2y - 2xy^2) + y(-x^3 + 3x^2y)}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2}$$

$$\boxed{I.F = \frac{1}{x^2y^2}}$$

Rule 4 :-

If the eq. $M \cdot dx + N dy = 0$ is of the form

$$y \cdot f_1(xy)\,dx + x \cdot f_2(xy)\,dy = 0.$$

Then
$$I.F = \frac{1}{xM - yN}$$

Example :- Solve

$$(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0 \quad (1)$$

$$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$$

$$I.F = \frac{1}{xM - yN}$$

$$I.F = \frac{1}{x(xy^2 + 2x^2y^3) - y(x^2y - x^3y^2)} = \boxed{\frac{1}{3x^3y^3}}$$