May 5, 2020

Example :- Solve

$$x^2y'' - xy' + 3y = x\ln x \quad \text{--- (1)}$$

Let $x = e^t \Rightarrow t = \ln x$.

$$\Rightarrow \quad xy' = \frac{d}{dt}y$$

and $$x^2y'' = \frac{d^2}{dt^2}y - \frac{d}{dt}y$$

using above transformations in eq. (1).

$$\Rightarrow \left(\frac{d^2}{dt^2}y - \frac{d}{dt}y\right) - \left(\frac{d}{dt}y\right) + 3y = e^t . t$$

or

$$\frac{d^2}{dt^2}y - 2 . \frac{d}{dt}y + 3y = te^t \quad \text{--- (2)}$$

For y_c

$$(D^2 - 2D + 3)y = 0$$

A.E. $\Rightarrow$ $$D^2 - 2D + 3 = 0$$

(2)

$$D^2 - 2D + 3 = 0$$

$$\Rightarrow D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$D = \frac{+2 \pm \sqrt{4-12}}{2}$$

$$D = \frac{2 \pm 2\sqrt{2}\, i}{2}$$

$$D = 1 \pm i\sqrt{2} = \alpha \pm i\beta$$

$$\Rightarrow \alpha = 1, \ \beta = \sqrt{2}$$

$$\therefore \boxed{y_c = e^{t}\left[c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)\right]}$$

Now, for $y_p = \dfrac{1}{D^2 - 2D + 3} e^{t} \cdot t$

$$y_p = \frac{1}{D^2 - 2D + 3} \cdot e^t t$$

$$y_p = e^t \frac{1}{(D+1)^2 - 2(D+1) + 3} \cdot t$$

$$y_p = e^t \cdot \frac{1}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} \cancel{-2} \cancel{+3}} \cdot t$$

$$y_p = e^t \cdot \frac{1}{D^2 + 2} \cdot t$$

$$y_p = e^t \cdot \frac{1}{2\left(1 + \frac{D^2}{2}\right)} \cdot t$$

$$= e^t \cdot \frac{1}{2} \cdot \left(1 + \frac{D^2}{2}\right)^{-1} \cdot t$$

$$= \frac{e^t}{2} \cdot \left[1 - \left(\frac{D^2}{2}\right)^{1}\right] \cdot t$$

$$y_p = \frac{e^t}{2} \cdot \left[t - 0\right] = \frac{e^t \cdot t}{2}$$

$$\boxed{y_p = \frac{e^t \cdot t}{2}}$$

The complete solution is

$$y = y_c + y_p$$

$\Rightarrow$

$$y = e^t \left[c_1 \cos(\sqrt{2}\, t) + c_2 \sin(\sqrt{2}\, t) \right] + \frac{e^t\, t}{2}$$

using back substitution. $\Big/ \; x = e^t, \; t = \ln x$

$\Rightarrow$

$$\boxed{y = x \cdot \left[c_1 \cos(\sqrt{2} \cdot \ln x) + c_2 \sin(\sqrt{2} \cdot \ln x) \right] + \frac{x \cdot \ln x}{2}}$$

Example :- Solve

$$x^2 y'' + 7x y' + 5y = x^5$$

$$\boxed{y = c_1 x^{-1} + c_2 x^{-5} + \frac{x^5}{60}}$$

Power Series Solutions :-

There are many differential equations whose solutions can't be found explicitly in terms of the elementary functions by the methods that we have already discussed. However, their solutions can be obtained in the form of power series. Such a solution is called Power Series solution of the differential equation.

We begin our study with examples of 1st order differential equations.

$$y = c_1 e^{x} + c_2 e^{-x}$$ // Explicit function

$$x^2y^2 + y = c_1 e^{x} + c_2 e^{4x},$$ / implicit function

Example :- Find a power series solution of the differential equation

$$y' = 2xy \quad \text{--- (1)}$$

Solution :-

We assume a power series solution of the given differential equation (1) as

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots +$$

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n.} \quad \text{--- (2)}$$

$$\Rightarrow \quad y' = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\boxed{y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}} \quad \text{--- (3)}$$

using above in Eq. (1)

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = 2x \cdot \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = \sum_{n=0}^{\infty} 2a_n \cdot x^{n+1}$$

replace n by $n+2$

$$\sum_{n+2=1}^{\infty} a_{n+2} \cdot (n+2) \cdot x^{(n+2)-1} = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

or $$\sum_{n=-1}^{\infty} a_{n+2} \cdot (n+2) \cdot x^{n+1} = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

or

$$a_1 \cdot (1) \cdot x^0 + \sum_{n=0}^{\infty} a_{n+2}(n+2)x^{n+1} = \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

or

$$a_1 + \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2)x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

or

$$a_1 + \sum_{n=0}^{\infty} \underbrace{\left[a_{n+2} \cdot (n+2) - 2a_n\right]}_{\text{Recurrence relation}} x^{n+1} = 0$$

$$\Rightarrow \boxed{a_1 = 0}, \quad a_{n+2}(n+2) - 2a_n = 0 \; ; \; n \geq 0$$

$$\boxed{a_1 = 0}$$

$$a_{n+2} \cdot (n+2) - 2a_n = 0 \qquad ; n \geq 0$$

$$\boxed{a_{n+2} = \frac{2a_n}{n+2}} \quad ; n \geq 0$$

For

$n = 0$

$$a_2 = \frac{\cancel{2}a_0}{\cancel{2}} = a_0$$

$$\boxed{a_2 = a_0}$$

$n=1 \Rightarrow \quad a_3 = \frac{2a_1}{3} = 0 \qquad \because a_1 = 0$

$$\boxed{a_3 = 0}$$

$n=2 \Rightarrow \quad a_4 = \frac{2a_2}{4} = \frac{a_2}{2} = \frac{a_0}{2}$

$$\boxed{a_4 = \frac{1}{2}a_0} \qquad \because a_0 = a_2$$

For $n=3$ $\Rightarrow$ $a_5 = \frac{2a_3}{5} = 0$

$$\boxed{a_5 = 0}$$

$n=4$ $\Rightarrow$ $a_6 = \frac{2a_4}{6} = \frac{1}{3}a_4$

$$a_6 = \frac{1}{3}\cdot\left(\frac{1}{2}a_0\right) = \frac{1}{3\cdot 2}\cdot a_0$$

$$\boxed{a_6 = \frac{1}{3!}a_0}$$

$\therefore$ Eq. 2

$$y = a_0 + \cancel{0x} + a_0\cdot x^2 + \cancel{0\cdot x^3} + \frac{1}{2!}a_0 x^4 + \cancel{0x^5} + \frac{1}{3!}a_0 x^6 + \cdots$$

or

$$y = a_0 + a_0 x^2 + \frac{1}{2!}a_0 x^4 + \frac{1}{3!}a_0 x^6$$

$$= a_0\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right] = \underline{a_0 e^{x^2}}$$

(10)

$$e^{x} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots$$

$$\boxed{y = a_0 e^{x^2}}$$

$$y' = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = 2\int x\,dx$$

$$\ln y = x^2 + C$$

$$y = e^{x^2+C} = e^{x^2} \cdot e^{C}$$

$$\boxed{y = a_0 e^{x^2}}$$