August 24, 2020 (2^{nd} session)

Case(4):

when $\frac{1}{f(D)}\left(e^{ax} \cdot V\right)$

; V is any function of x.

$$\Rightarrow \frac{1}{f(D)}\left(e^{ax} V\right) = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

Example: Solve $y'' - 2y' + 4y = e^{x} \cdot \cos x$

For $y_c \Rightarrow$ $(D^2 - 2D + 4)y = 0$.

The auxiliary equation is

$$D^2 - 2D + 4 = 0$$

$$D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$D = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$$

$$= 1 \pm i\sqrt{3}$$

(2)

$$\Rightarrow y_c = e^{x}\cdot\left[c_1\cos(\sqrt{3}\,x) + c_2\sin(\sqrt{3}\,x)\right]$$

For y_p

$$\Rightarrow y_p = \frac{1}{D^2-2D+4}\cdot e^{x}\cdot\cos x$$

$$= e^{x}\cdot\frac{1}{(D+1)^2-2(D+1)+4}\cdot\cos x$$

$$= e^{x}\cdot\frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-2+4}\cdot\cos x$$

$$= e^{x}\cdot\left[\frac{1}{D^2+3}\cdot\cos x\right]$$

$D^2 = -a$

$$= e^{x}\cdot\frac{1}{-(1)^2+3}\cdot\cos x$$

$$y_p = \frac{e^{x}\cdot\cos x}{2}$$

So,

$$y = y_c + y_p$$

$$\Rightarrow y = \_\_\_\_\_\_ + \_\_\_\_\_\_$$

Example :- Solve $y'' - 5y' + 6y = e^{4x} \cdot x$

For y_c

$$\Rightarrow (D^2 - 5D + 6)\, y = 0$$

The auxiliary equation is:

$$D^2 - 5D + 6 = 0$$

$$(D-2)(D-3) = 0$$

$$D = 2, 3.$$

$$\Rightarrow \boxed{y_c = c_1 e^{2x} + c_2 e^{3x}}$$

For y_p

$$\Rightarrow y_p = \frac{1}{D^2 - 5D + 6} \cdot e^{4x} \cdot x$$

$$= e^{4x} \cdot \frac{1}{(D+4)^2 - 5(D+4) + 6} \cdot x$$

$$= e^{4x} \cdot \frac{1}{D^2 + 8D + 16 - 5D - 20 + 6} \cdot x$$

$$= e^{4x} \left(\frac{1}{D^2 + 3D + 2} \cdot x \right)$$

$$= e^{4x} \cdot \frac{1}{2\left[1 + \frac{D^2 + 3D}{2}\right]} \cdot x$$

(4)

$$y_p = e^{4x} \cdot \frac{1}{2\left[1 + \frac{D^2+3D}{2}\right]} \cdot x$$

$$= e^{4x} \cdot \frac{1}{2} \cdot \left[1 + \frac{D^2+3D}{2}\right]^{-1} \cdot x$$

$$= \frac{e^{4x}}{2} \cdot \left[1 - \left(\frac{D^2+3D}{2}\right)^1\right] \cdot x$$

$$= \frac{e^{4x}}{2} \cdot \left\{1 - \frac{1}{2}(D^2+3D)\right\} x$$

$$= \frac{e^{4x}}{2} \left[x - \frac{1}{2}(\underline{D^2}+3D)\underline{x}\right]$$

$$= \frac{e^{4x}}{2} \left[x - \frac{1}{2}(0+3(1))\right]$$

$$= \frac{e^{4x}}{2} \left[x - \frac{3}{2}\right]$$

$$\boxed{y_p = e^{4x}\left[\frac{x}{2} - \frac{3}{4}\right]}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow$$

$$y = \_\_\_\_\_\_ + \_\_\_\_\_\_$$

Example :- $y'' + y = x e^{2x}$.

$$(D^2+1) = 0$$

$$D = \pm i$$

$$\Rightarrow y_c = 1 \cdot [c_1 \cos x + c_2 \sin x].$$

For $y_p \Rightarrow$ $y_p = \frac{1}{D^2+1} \cdot e^{2x} \cdot x$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+1} x$$

$$= e^{2x} \cdot \frac{1}{D^2+4D+4+1} \cdot x$$

$$= e^{2x} \cdot \left(\frac{1}{D^2+4D+5} \cdot x \right)$$

$$= e^{2x} \cdot \frac{1}{5\left[1+\frac{D^2+4D}{5}\right]} x$$

$$y_p = \frac{e^{2x}}{5} \cdot \left[1+\frac{D^2+4D}{5}\right]^{-1} x.$$

$$y_p = \frac{e^{2x}}{5} \cdot \left[1 - \frac{D^2+4D}{5}\right] x$$

$$= \frac{e^{2x}}{5} \cdot \left[x - \frac{1}{5} \cdot (D^2+4D)x\right]$$

$$= \frac{e^{2x}}{5} \left[x - \frac{1}{5} (0 + 4(1))\right]$$

$$y_p = \frac{e^{2x}}{5} \cdot \left[x - \frac{4}{5}\right]$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = (c_1 \cos x + c_2 \sin x) + \frac{e^{2x}}{5}\left(x - \frac{4}{5}\right)}$$

Cases :- $\frac{1}{f(D)}(xV)$

; V is any function of x

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Example :- Solve $y'' + 4y = x\sin x$

For y_c $\quad (D^2+4)y = 0$

A.E $\Rightarrow$ $\quad D^2+4 = 0$

$D = \pm 2i$

$$\Rightarrow \boxed{y_c = c_1\cos 2x + c_2\sin 2x}$$

For $y_p \Rightarrow y_p = \frac{1}{D^2+4} \cdot x\sin x$

$$\Rightarrow y_p = x \cdot \frac{1}{D^2+4} \cdot \sin x - \frac{2D}{(D^2+4)^2} \cdot \sin x$$

$$y_p = x \cdot \frac{1}{D^2+4} \sin x - \frac{2D}{(D^2+4)^2} \cdot \sin x$$ (8)

$$= x \cdot \frac{1}{-(1)^2+4} \cdot \sin x - \frac{2D}{\left(-(1)^2+4\right)^2} \cdot \sin x$$

$$= x \frac{1}{3} \cdot \sin x - \frac{2D}{9} \sin x$$

$$\boxed{y_p = \frac{x}{3} \sin x - \frac{2}{9} \cos x}$$

So, $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cos x}$$

Example :- Solve $y'' - 4y = x\cos 2x$

For $y_c \Rightarrow (D^2-4)y = 0$

The auxiliary equation:

$$D^2 - 4 = 0$$
$$D^2 = 4$$
$$D = \pm 2$$

$$\Rightarrow \boxed{y_c = c_1 e^{2x} + c_2 e^{-2x}}$$

For y_p

$$\Rightarrow y_p = \frac{1}{D^2-4}\, x\cos 2x$$

Case (ii)
sin ax or cos ax
$D^2 = -a^2$

$$= x \cdot \frac{1}{D^2-4}\cos 2x - \frac{2D}{(D^2-4)^2}\cdot \cos 2x$$

$$= x \cdot \frac{1}{-(2)^2-4}\cos 2x - \frac{2D}{\left(-(2)^2-4\right)^2}\cos 2x$$

$$= x \cdot \frac{-1}{8}\cos 2x - \frac{2D}{64}\cdot \cos 2x.$$

$$\boxed{y_p = -\frac{x}{8}\cos 2x - \frac{1}{32}\cdot(-2\sin 2x)}$$

Example: Solve $(D^2-1)y = x.\sin x$.

For $y_c \Rightarrow$ $D^2-1=0$

$D=\pm 1$

$$\Rightarrow \boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$$

For $y_p \Rightarrow$ $y_p = \frac{1}{D^2-1} . x\sin x$

$\Rightarrow$

$$y_p = x . \frac{1}{D^2-1} \sin x - \frac{2D}{(D^2-1)^2} . \sin x$$

$$= x . \frac{1}{-(1)^2-1} \sin x - \frac{2D}{(-(1)^2-1)^2} \sin x$$

$$= x . \frac{1}{-2} \sin x - \frac{\cancel{2}D}{\cancel{4}} \sin x$$

$$y_p = -\frac{x}{2}\sin x - \frac{1}{2}(\cos x)$$

$\therefore$ $y = y_c + y_p$

$\Rightarrow$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - \frac{x}{2}\sin x - \frac{\cos x}{2}}$$