Inverse CDF transformation

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For any cdf $F(x)$, one can use its right continuity property to define the pseudo-inverse

$$F^{[-1]}(u) = \inf\{x : F(x) \geq u\}, \qquad u \in (0,1).$$

If you take a uniform random variable Y (i.e., the pdf of Y is given by $f_Y(y) = 1$ if $0 < y < 1$, and 0 otherwise), then $X = F^{[-1]}(Y)$ is a random variables with cdf $F_X = F$.

If F corresponds to a discrete random variable X that takes values $x_1, x_2, \cdots$ with $P(X = x_i) = p_i$, then the pseudo-inverse can be written as follows.

$$\begin{array}{rclcc}
F^{[-1]}(u) & = & x_1 & \text{for} & 0 < u \leq p_1 \\
F^{[-1]}(u) & = & x_2 & \text{for} & p_1 < u \leq p_1 + p_2 \\
\vdots & & \vdots & & \vdots \\
F^{[-1]}(u) & = & x_k & \text{for} & \sum_{i=1}^{k-1} p_i < u \leq \sum_{i=1}^{k} p_i \\
\vdots & & \vdots & & \vdots
\end{array}$$

If F corresponds to a continuous random variable X whose pdf $f(x)$ is strictly positive over an interval (a, b) and is zero outside (in other words $\{x : f(x) > 0\} = (a, b)$), then F is actually invertible (i.e, for every $u \in (0, 1)$ there is a unique $x \in (a, b)$ for which $F(x) = u$) and in this case $F^{[-1]}(u) = F^{-1}(x)$.