$$I = \frac{3}{8}\left(\frac{1}{3}\right)\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right]$$

$$= \frac{1}{8}\left[1 + \frac{9}{4} + \frac{9}{5} + \frac{1}{2}\right] = 0.69375.$$

5.8 METHODS BASED ON UNDETERMINED COEFFICIENTS

The Newton-Cotes methods derived in the previous section can also be obtained using the appr of method of undetermined coefficients. The Newton-Cotes methods are given by

$$\int_a^b f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k .$$

We shall derive the Trapezoidal and Simpson methods using the method of undetermined coeffici

Newton-Cotes Methods

Trapezoidal method

We have $n = 1$, $x_0 = a$, $x_1 = b$ and $h = x_1 - x_0$. We write

$$\int_{x_0}^{x_1} f(x)\, dx = \lambda_0 f(x_0) + \lambda_1 f(x_1).$$

Using (5.61) and definition 5.2, the rule can be made exact for polynomials of degree upto one. F $f(x) = 1$ and x, we get the system of equations

$$f(x) = 1: x_1 - x_0 = \lambda_0 + \lambda_1, \text{ or } h = \lambda_0 + \lambda_1$$

$$f(x) = x: \frac{1}{2}(x_1^2 - x_0^2) = \lambda_0 x_0 + \lambda_1 x_1.$$

We have

$$\frac{1}{2}(x_1 - x_0)(x_1 + x_0) = \lambda_0 x_0 + \lambda_1 x_1$$

or

$$\frac{1}{2} h(2x_0 + h) = \lambda_0 x_0 + \lambda_1(x_0 + h)$$

or

$$\frac{1}{2} h(2x_0 + h) = (\lambda_0 + \lambda_1)x_0 + \lambda_1 h = h x_0 + \lambda_1 h$$

or

$$\lambda_1 h = \frac{h^2}{2}, \text{ or } \lambda_1 = \frac{h}{2}.$$

From the first equation, we get $\lambda_0 = h - \lambda_1 = h/2$. The method becomes

$$\int_{x_0}^{x_1} f(x)\, dx = \frac{h}{2}[f(x_0) + f(x_1)].$$

From (5.69), the error constant is given by

$$C = \int_{x_0}^{x_1} x^2\, dx - \frac{h}{2}[x_0^2 + x_1^2] = \frac{1}{3}[x_1^3 - x_0^3] - \frac{h}{2}[x_0^2 + x_1^2]$$

$$= \frac{1}{6}\,[2(x_0^3 + 3x_0^2 h + 3x_0 h^2 + h^3) - 2x_0^3 - 3x_0^2 h$$
$$-3h(x_0^2 + 2x_0 h + h^2)]$$
$$= -\frac{h^3}{6}.$$

The truncation error becomes

$$R_1 = \frac{C}{2}\,f''(\xi) = -\frac{h^3}{12}\,f''(\xi), \quad x_0 < \xi < x_1.$$

Simpson's method

We have $n = 2$, $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h = b$, $h = (b - a)/2$. We write

$$\int_{x_0}^{x_2} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

The rule can be made exact for polynomials of degree upto two.

For $f(x) = 1, x, x^2$, we get the following system of equations.

$$f(x) = 1: x_2 - x_0 = \lambda_0 + \lambda_1 + \lambda_2, \text{ or } 2h = \lambda_0 + \lambda_1 + \lambda_2 \qquad (5.81\,a)$$

$$f(x) = x: \frac{1}{2}(x_2^2 - x_0^2) = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 \qquad (5.81\,b)$$

$$f(x) = x^2: \frac{1}{3}(x_2^3 - x_0^3) = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2. \qquad (5.81\,c)$$

From (5.81 b), we get

$$\frac{1}{2}(x_2 - x_0)(x_2 + x_0) = \lambda_0 x_0 + \lambda_1(x_0 + h) + \lambda_2(x_0 + 2h)$$

or
$$\frac{1}{2}(2h)(2x_0 + 2h) = (\lambda_0 + \lambda_1 + \lambda_2)x_0 + (\lambda_1 + 2\lambda_2)h$$
$$= 2h\,x_0 + (\lambda_1 + 2\lambda_2)h \quad \text{using (5.81 } a)$$

or
$$2h = \lambda_1 + 2\lambda_2. \qquad (5.81\,d)$$

From (5.81 c), we get

$$\frac{1}{3}\,[(x_0^3 + 6x_0^2 h + 12x_0 h^2 + 8h^3) - x_0^3] = \lambda_0 x_0^2 + \lambda_1(x_0^2 + 2x_0 h + h^2)$$
$$+ \lambda_2(x_0^2 + 4x_0 h + 4h^2)$$

or
$$2x_0^2 h + 4x_0 h^2 + \frac{8}{3}h^3 = (\lambda_0 + \lambda_1 + \lambda_2)x_0^2 + 2(\lambda_1 + 2\lambda_2)x_0 h + (\lambda_1 + 4\lambda_2)h^2$$
$$= 2h\,x_0^2 + 4x_0 h^2 + (\lambda_1 + 4\lambda_2)h^2$$

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or
$$\frac{8}{3}h = \lambda_1 + 4\lambda_2.$$

Solving (5.81*d*), (5.81*e*) and using (5.81 *a*), we obtain $\lambda_0 = h/3$, $\lambda_1 = 4h/3$, $\lambda_2 = h/3$.
The method is given by

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\,[f(x_0) + 4f(x_1) + f(x_2)].$$

From (5.76), the error constant is given by

$$C = -\frac{(b-a)^5}{120} = -\frac{4}{15}h^5$$

and
$$R_2 = \frac{C}{4!}f^{iv}(\eta) = -\frac{h^5}{90}f^{iv}(\eta), \quad x_0 < \eta < x_2.$$

The method of undetermined coefficients can be used to derive quadrature formulas of a given type. We illustrate such derivations through the following examples.

Example 5.14 Determine *a*, *b* and *c* such that the formula

$$\int_0^h f(x)\,dx = h\left\{af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right\}$$

is exact for polynomials of as high order as possible, and determine the order of the truncation error.

(Uppsala Univ., Sweden, BIT 13(1973), ...)

Making the method exact for polynomials of degree upto 2, we obtain

for $f(x) = 1$: $h = h(a + b + c)$, or $a + b + c = 1$.

for $f(x) = x$: $\frac{h^2}{2} = h\left(\frac{bh}{3} + ch\right)$, or $\frac{1}{3}b + c = \frac{1}{2}$.

for $f(x) = x^2$: $\frac{h^3}{3} = h\left(\frac{bh^2}{9} + ch^2\right)$, or $\frac{1}{9}b + c = \frac{1}{3}$.

Solving the above equations, we get

$$a = 0, \quad b = 3/4, \text{ and } \quad c = 1/4.$$

The truncation error of the formula is given by

$$TE = \frac{C}{3!}f'''(\xi), \quad 0 < \xi < h$$

and
$$C = \int_0^h x^3\,dx - h\left[\frac{bh^3}{27} + ch^3\right] = -\frac{h^4}{36}.$$

Hence, we have

$$TE = -\frac{h^4}{216} f'''(\xi) = O(h^4).$$

Example 5.15 Find the quadrature formula

$$\int_0^1 f(x) \frac{dx}{\sqrt{x(1-x)}} = \alpha_1 f(0) + \alpha_2 f\left(\frac{1}{2}\right) + \alpha_3 f(1)$$

which is exact for polynomials of highest possible degree. Then use the formula on

$$\int_0^1 \frac{dx}{\sqrt{x - x^3}}$$

and compare with the exact value.

(Oslo Univ., Norway, BIT 7(1967), 170)

Making the method exact for polynomials of degree upto 2, we obtain

$$\text{for } f(x) = 1: \quad I_1 = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \alpha_1 + \alpha_2 + \alpha_3$$

$$\text{for } f(x) = x: \quad I_2 = \int_0^1 \frac{x\,dx}{\sqrt{x(1-x)}} = \frac{1}{2}\alpha_2 + \alpha_3$$

$$\text{for } f(x) = x^2: \quad I_3 = \int_0^1 \frac{x^2\,dx}{\sqrt{x(1-x)}} = \frac{1}{4}\alpha_2 + \alpha_3$$

where

$$I_1 = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^1 \frac{dx}{\sqrt{1-(2x-1)^2}} = \int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{-1}^1 = \pi$$

$$I_2 = \int_0^1 \frac{x\,dx}{\sqrt{x(1-x)}} = 2\int_0^1 \frac{x\,dx}{\sqrt{1-(2x-1)^2}} = \int_{-1}^1 \frac{t+1}{2\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\int_{-1}^1 \frac{t\,dt}{\sqrt{1-t^2}} + \frac{1}{2}\int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}.$$

$$I_3 = \int_0^1 \frac{x^2\,dx}{\sqrt{x(1-x)}} = 2\int_0^1 \frac{x^2\,dx}{\sqrt{1-(2x-1)^2}} = \frac{1}{4}\int_{-1}^1 \frac{(t+1)^2}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{4}\int_{-1}^1 \frac{t^2}{\sqrt{1-t^2}}\,dt + \frac{1}{2}\int_{-1}^1 \frac{t}{\sqrt{1-t^2}}\,dt + \frac{1}{4}\int_{-1}^1 \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{3\pi}{8}.$$

Hence, we have the equations

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

$$\frac{1}{2}\alpha_2 + \alpha_3 = \frac{\pi}{2}$$

$$\frac{1}{4}\alpha_2 + \alpha_3 = \frac{3\pi}{8}$$

which gives

$$\alpha_1 = \pi/4, \quad \alpha_2 = \pi/2, \quad \alpha_3 = \pi/4.$$

The quadrature formula is given by

$$\int_0^1 \frac{f(x)\,dx}{\sqrt{x(1-x)}} = \frac{\pi}{4}\left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right]$$

We now use this formula to evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{x - x^3}} = \int_0^1 \frac{dx}{\sqrt{1+x}\,\sqrt{x(1-x)}} = \int_0^1 \frac{f(x)\,dx}{\sqrt{x(1-x)}}$$

where $f(x) = 1/\sqrt{1+x}$.

We obtain

$$I = \frac{\pi}{4}\left[1 + \frac{2\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{2}\right] \approx 2.62331.$$

The exact value is $I = 2.62205755$.

Gauss Quadrature Methods

In the integration method (5.70), the nodes x_k's and the weights λ_k's, $k = 0(1)n$ can also be obtained by making the formula exact for polynomials of degree upto m. When the nodes are known, that is, $m = n$, the corresponding methods are called Newton-Cotes methods. When the nodes are also to be determined, we have $m = 2n + 1$ and the methods are called **Gaussian integration methods.** Since any finite interval $[a, b]$ can always be transformed to $[-1, 1]$, using the transformation

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

we consider the integral in the form

$$\int_{-1}^{1} w(x)\, f(x)\, dx = \sum_{k=0}^{n} \lambda_k f_k \tag{5.82}$$

where $w(x) > 0$, $-1 \le x \le 1$, is the weight function.

Gauss-Legendre Integration Methods

Let the weight function be $w(x) = 1$. Then, the method (5.82) reduces to

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=0}^{n} \lambda_k f(x_k). \tag{5.83}$$

In this case, all the nodes x_k and weights λ_k are unknown. Consider the following cases.

One-point formula $n = 0$. The formula is given by

$$\int_{-1}^{1} f(x)\, dx = \lambda_0 f(x_0). \tag{5.84}$$

The method has two unknowns λ_0, x_0. Making the method exact for $f(x) = 1, x$, we get

$$f(x) = 1: \quad 2 = \lambda_0$$

$$f(x) = x: \quad 0 = \lambda_0 x_0 \text{ or } x_0 = 0.$$

Hence, the method is given by

$$\int_{-1}^{1} f(x)\, dx = 2f(0) \tag{5.85}$$

which is same as the mid-point formula. The error constant is given by

$$C = \int_{-1}^{1} x^2\, dx - 2[0] = \frac{2}{3}.$$

Hence,

$$R_1 = \frac{C}{2!} f''(\xi) = \frac{1}{3} f''(\xi), \quad -1 < \xi < 1.$$

Two-point formula $n = 1$. The formula is given by

$$\int_{-1}^{1} f(x)\, dx = \lambda_0 f(x_0) + \lambda_1 f(x_1). \tag{5.86}$$

The method has four unknowns, x_0, x_1, λ_0 and λ_1. Making the method exact for $f(x) = 1, x, x^2, x^3$, we get

$$f(x) = 1 \quad : \quad 2 = \lambda_0 + \lambda_1 \tag{5.87 a}$$

$$f(x) = x \quad : \quad 0 = \lambda_0 x_0 + \lambda_1 x_1 \tag{5.87 b}$$

$$f(x) = x^2 \quad : \quad \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 \tag{5.87 c}$$

$$f(x) = x^3 \quad : \quad 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3. \tag{5.87 d}$$

Eliminating λ_0 from (5.87 *b*), (5.87 *d*), we get

$$\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0, \text{ or } \lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0.$$

Since $\lambda_1 \neq 0$, $x_0 \neq x_1$, we get $x_1 + x_0 = 0$ or $x_1 = -x_0$. Note that if $x_1 = 0$, then from (5.87 *b*), we get $x_0 = 0$ since $\lambda_0 \neq 0$. Therefore, $x_1 \neq 0$.

Substituting in (5.87 *b*), we get $\lambda_0 - \lambda_1 = 0$, or $\lambda_0 = \lambda_1$.

Substituting in (5.87 *a*), we get $\lambda_0 = \lambda_1 = 1$.

Using (5.87 *c*), we get $x_0^2 = 1/3$, or $x_0 = \pm 1/\sqrt{3}$, and $x_1 = \mp 1/\sqrt{3}$. Therefore, the two-point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)\, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \tag{5.88}$$

The error constant is given by

$$C = \int_{-1}^{1} x^4\, dx - \left[\frac{1}{9} + \frac{1}{9}\right] = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}.$$

The error term R_4 becomes

$$R_4 = \frac{C}{4!} f^{(4)}(\xi) = \frac{1}{135} f^{(4)}(\xi), \quad -1 < \xi < 1. \tag{5.89}$$

Three-point formula $n = 2$. The method is given by

$$\int_{-1}^{1} f(x)\, dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

There are six unknowns in the method and it can be made exact for polynomials of degree upto five. For $f(x) = x^i$, $i = 0(1)5$, we get the system of equations

$$f(x) = 1 \;:\; \lambda_0 + \lambda_1 + \lambda_2 = 2 \tag{5.90 a}$$

$$f(x) = x \;:\; \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0 \tag{5.90 b}$$

$$f(x) = x^2 \;:\; \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = \frac{2}{3} \tag{5.90 c}$$

$$f(x) = x^3 \;:\; \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0 \tag{5.90 d}$$

$$f(x) = x^4 \;:\; \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = \frac{2}{5} \tag{5.90 e}$$

$$f(x) = x^5 \;:\; \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 = 0. \tag{5.90 f}$$

Eliminating λ_0 from (5.90 *b*), (5.90 *d*) and (5.90 *d*), (5.90 *f*), we get

$$\lambda_1 x_1(x_1^2 - x_0^2) + \lambda_2 x_2(x_2^2 - x_0^2) = 0$$

$$\lambda_1 x_1^3(x_1^2 - x_0^2) + \lambda_2 x_2^3(x_2^2 - x_0^2) = 0.$$

Eliminating the first term from these two equations, we get

$$\lambda_2 x_2^3(x_2^2 - x_0^2) - \lambda_2 x_2 x_1^2(x_2^2 - x_0^2) = 0$$

or

$$\lambda_2 x_2(x_2^2 - x_0^2)(x_2^2 - x_1^2) = 0.$$

Since x_0, x_1, x_2 are distinct, we get on cancelling the terms $(x_2 - x_0)$ and $(x_2 - x_1)$

$$\lambda_2\, x_2(x_2 + x_0)\,(x_2 + x_1) = 0.$$

We have $\lambda_2 \neq 0$ and let $x_2 \neq 0$. Then, we have either $x_2 = -\,x_0$ or $x_2 = -\,x_1$. Let $x_2 = -\,x_0$. Then, from (5.90 *b*), (5.90 *d*), we get

$$(\lambda_0 - \lambda_2)x_0 + \lambda_1\, x_1 = 0$$

$$(\lambda_0 - \lambda_2)x_0^3 + \lambda_1\, x_1^3 = 0.$$

Eliminating the first term, we get $\lambda_1 x_1\,(x_1^2 - x_0^2) = 0$. Since, $\lambda_1 \neq 0$, $x_1 \neq x_0$, $x_1 \neq -\,x_0$ (otherwise $x_1 = x_2$), we get $x_1 = 0$.

Hence, $(\lambda_0 - \lambda_2)\,x_0 = 0$, or $\lambda_0 = \lambda_2$ since $x_0 \neq 0$.

Now, (5.90 *c*), (5.90 *e*) give

$$2\lambda_0\, x_0^2 = \frac{2}{3},\quad 2\lambda_0\, x_0^4 = \frac{2}{5}.$$

Dividing, we get $x_0^2 = 3/5$, or $x_0 = \pm\sqrt{3/5}$. Then $x_2 = \mp\sqrt{3/5}$.

Now, $\lambda_0\, x_0^2 = 1/3$ gives $\lambda_0 = 5/9$ and $\lambda_2 = \lambda_0 = 5/9$. From (5.90*a*), we get $\lambda_1 = 2 - 2\lambda_2 = 8/9$.

Therefore, the three-point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)\, dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{5.91}$$

If we take $x_2 = -\,x_1$, then we get $x_0 = 0$ and $x_2 = \pm\sqrt{3/5}$ giving the same method. The nodes are symmetrically placed about $x = 0$.

The error constant is given by

$$C = \int_{-1}^{1} x^6\, dx - \frac{1}{9}\left[5\left(-\sqrt{\frac{3}{5}}\right)^6 + 0 + 5\left(\sqrt{\frac{3}{5}}\right)^6\right]$$

$$= \frac{2}{7} - \frac{6}{25} = \frac{8}{175}.$$

The error in the method becomes

$$R_6 = \frac{C}{6!}\, f^{(6)}(\xi) = \frac{8}{(6!)\,175}\, f^{(6)}(\xi) = \frac{1}{15750}\, f^{(6)}(\xi),\; -1 < \xi < 1.$$

In the later part of this section, we shall prove that the abscissas of the above formulas are the zeros of the Legendre polynomials of the corresponding order. Hence, they are called the Gauss-Legendre quadrature methods.

The nodes and the corresponding weights for the **Gauss-Legendre integration method** (5.83) for $n = 1(1)5$ are given in Table 5.3.

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Table 5.3 Nodes and Weights for Gauss-Legendre Integration Method (5.83).

n	nodes x_k	weights λ_k
1	±0.5773502692	1.0000000000
	0.0000000000	0.8888888889
2	±0.7745966692	0.5555555556
	±0.3399810436	0.6521451549
3	±0.8611363116	0.3478548451
	0.0000000000	0.5688888889
4	±0.5384693101	0.4786286705
	±0.9061798459	0.2369268851
	±0.2386191861	0.4679139346
5	±0.6612093865	0.3607615730
	±0.9324695142	0.1713244924

Example 5.16 Evaluate the integral

$$I = \int_0^1 \frac{dx}{1+x}$$

using Gauss-Legendre three-point formula.

First we transform the interval [0, 1] to the interval [–1, 1]. Let $t = ax + b$. We have

$$-1 = b, \quad 1 = a + b$$

or

$$a = 2, \quad b = -1, \text{ and } t = 2x - 1.$$

$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{dt}{t+3}.$$

Using Gauss-Legendre three-point rule (corresponding to $n = 2$), we get

$$I = \frac{1}{9}\left[8\left(\frac{1}{0+3}\right) + 5\left(\frac{1}{3+\sqrt{3/5}}\right) + 5\left(\frac{1}{3-\sqrt{3/5}}\right)\right]$$

$$= \frac{131}{189} = 0.693122.$$

The exact solution is $I = \ln 2 = 0.693147$.

Example 5.17 Evaluate the integral $I = \int_1^2 \frac{2x\,dx}{1+x^4}$, using the Gauss-Legendre 1-point, 2-point and 3-point quadrature rules. Compare with the exact solution

$$I = \tan^{-1}(4) - (\pi/4).$$

To use the Gauss-Legendre rules, the interval [1, 2] is to be reduced to [–1, 1]. Writing $x = at + b$, we get

$$1 = -a + b, \quad 2 = a + b$$

whose solution is $b = 3/2$, $a = 1/2$. Therefore, $x = (t + 3)/2$, $dx = dt/2$ and

$$I = \int_{-1}^{1} \frac{8(t+3)\,dt}{[16 + (t+3)^4]} = \int_{-1}^{1} f(t)dt.$$

Using the 1-point rule, we get

$$I = 2\,f(0) = 2\left[\frac{24}{16+81}\right] = 0.4948.$$

Using the 2-point rule, we get

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.3842 + 0.1592 = 0.5434.$$

Using the 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\,[5(0.4393) + 8(0.2474) + 5(0.1379)] = 0.5406.$$

The exact solution is $I = 0.5404$.

Gauss-Chebyshev Integration Methods

Let the weight function be $w(x) = 1/\sqrt{1-x^2}$. Then, the method (5.82) reduces to

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \sum_{k=0}^{n} \lambda_k\, f(x_k). \tag{5.92}$$

The abscissas x_k and weights λ_k are unknown. Consider the following cases.

One-point formula $n = 0$. The formula is given by

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \lambda_0\, f(x_0). \tag{5.93}$$

The method has two unknowns λ_0, x_0. Making it exact for $f(x) = 1, x$, we get

$$f(x) = 1: \quad \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lambda_0, \text{ or } \left[\sin^{-1}(x)\right]_{-1}^{1} = \lambda_0, \text{ or } \lambda_0 = \pi$$

$$f(x) = x: \quad \int_{-1}^{1} \frac{x\,dx}{\sqrt{1-x^2}} = \lambda_0\, x_0, \text{ or } \lambda_0\, x_0 = 0, \text{ or } x_0 = 0.$$

Hence, the method is given by

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \pi\, f(0). \tag{5.94}$$

The error constant is given by

$$C = \int_{-1}^{1} \frac{x^2\, dx}{\sqrt{1-x^2}} = 2\int_{0}^{1} \frac{x^2\, dx}{\sqrt{1-x^2}} = 2\int_{0}^{\pi/2} \sin^2\theta\, d\theta = \frac{\pi}{2}.$$

Hence,

$$R_1 = \frac{C}{2!} f''(\xi) = \frac{\pi}{4} f''(\xi), \quad -1 < \xi < 1.$$

Two-point formula $n = 1$. The formula is given by

$$\int_{-1}^{1} \frac{f(x)\, dx}{\sqrt{1-x^2}}\, dx = \lambda_0 f(x_0) + \lambda_1 f(x_1). \tag{5.95}$$

The method has four unknowns. Making the method exact for $f(x) = 1, x, x^2, x^3$, we get

$$f(x) = 1: \quad \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lambda_0 + \lambda_1, \text{ or } \lambda_0 + \lambda_1 = \pi. \tag{5.96a}$$

$$f(x) = x: \quad \int_{-1}^{1} \frac{x\, dx}{\sqrt{1-x^2}} = \lambda_0 x_0 + \lambda_1 x_1, \text{ or } \lambda_0 x_0 + \lambda_1 x_1 = 0. \tag{5.96b}$$

$$f(x) = x^2: \quad \int_{-1}^{1} \frac{x^2\, dx}{\sqrt{1-x^2}} = \lambda_0 x_0^2 + \lambda_1 x_1^2, \text{ or } \lambda_0 x_0^2 + \lambda_1 x_1^2 = \frac{\pi}{2}. \tag{5.96c}$$

$$f(x) = x^3: \quad \int_{-1}^{1} \frac{x^3\, dx}{\sqrt{1-x^2}} = \lambda_0 x_0^3 + \lambda_1 x_1^3, \text{ or } \lambda_0 x_0^3 + \lambda_1 x_1^3 = 0. \tag{5.96d}$$

Eliminating λ_0 from (5.96 *b*), (5.96 *d*), we get

$$\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0, \text{ or } \lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0.$$

Since $\lambda_1 \neq 0$, $x_0 \neq x_1$, we get $x_1 + x_0 = 0$ or $x_1 = -x_0$. If $x_1 = 0$, then from (5.96 *b*), we get $x_0 = 0$ since $\lambda_0 \neq 0$. Therefore, $x_1 \neq 0$.

Substituting in (5.96*b*), we get $\lambda_0 = \lambda_1$.

Substituting in (5.96*a*), we get $\lambda_0 = \lambda_1 = \pi/2$.

Using (5.96*c*), we get $2x_0^2 = 1$, or $x_0 = \pm 1/\sqrt{2}$, and $x_1 = \mp 1/\sqrt{2}$.

Therefore, the two-point Gauss-Chebyshev method is given by

$$\int_{-1}^{1} \frac{f(x)\, dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right]. \tag{5.97}$$

The error constant is given by

$$C = \int_{-1}^{1} \frac{x^4\, dx}{\sqrt{1-x^2}} - \frac{\pi}{2}\left[\frac{1}{4} + \frac{1}{4}\right].$$

Setting $x = \sin\theta$, we get

$$C = \int_{-\pi/2}^{\pi/2} \sin^4 \theta \, d\theta - \frac{\pi}{4} = 2\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) - \frac{\pi}{4} = \frac{\pi}{8}.$$

The error term R_4 becomes

$$R_4 = \frac{C}{4!} f^{(4)}(\eta) = \frac{\pi}{192} f^{(4)}(\eta), \quad -1 < \eta < 1. \tag{5.98}$$

Three-point formula $n = 2$. The method is given by

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2). \tag{5.99}$$

There are six unknowns in the method and it can be made exact for polynomials of degree upto five. For $f(x) = x^i$, $i = 0(1)5$, we get the system of equations

$f(x) = 1$: $\quad \lambda_0 + \lambda_1 + \lambda_2 = \pi.$ (5.100 *a*)

$f(x) = x$: $\quad \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0.$ (5.100 *b*)

$f(x) = x^2$: $\quad \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = \dfrac{\pi}{2}.$ (5.100 *c*)

$f(x) = x^3$: $\quad \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0.$ (5.100 *d*)

$f(x) = x^4$: $\quad \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = \dfrac{3\pi}{8}.$ (5.100 *e*)

$f(x) = x^5$: $\quad \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 = 0.$ (5.100 *f*)

Eliminating λ_0 from (5.100 *d*) and (5.100 *f*), we get

$$\lambda_1 x_1^3 (x_1^2 - x_0^2) + \lambda_2 x_2^3 (x_2^2 - x_0^2) = 0 \tag{5.100 g}$$

Eliminating λ_0, x_0 from (5.100 *b*) and (5.100 *d*), we get

$$\lambda_1 x_1 (x_1^2 - x_0^2) + \lambda_2 x_2 (x_2^2 - x_0^2) = 0. \tag{5.100 h}$$

Eliminating λ_2, x_2 from (5.100 *g*) and (5.100 *h*), we get

$$\lambda_1 x_1 (x_1^2 - x_0^2)(x_2^2 - x_1^2) = 0.$$

Since x_i are distinct, we get

$$\lambda_1 x_1 (x_1 + x_0)(x_2 + x_1) = 0. \tag{5.100 i}$$

Let $x_1 = 0$. Then, (5.100 *h*) gives $\lambda_2 x_2 (x_2 - x_0)(x_2 + x_0) = 0$. Again, since x_i are distinct, we get $x_2 + x_0 = 0$ or $x_2 = -x_0$.

Substituting in (5.100 *b*), we get $(\lambda_0 - \lambda_2)x_0 = 0$, or $\lambda_0 = \lambda_2$. (5.100 *j*)

Substituting in (5.100 *c*), we get $2\lambda_0 x_0^2 = \pi/2$.

Substituting in (5.100 *e*), we get $2\lambda_0 x_0^4 = 3\pi/8$.

Hence,

$$x_0^2 = 3/4 \text{ or } x_0 = \pm \sqrt{3}/2, \text{ and } x_2 = -x_0 = \mp \sqrt{3}/2.$$

Using (5.100 *j*), we get $\lambda_0 = \pi/3$. Hence, $\lambda_2 = \pi/3$. Using (5.100 *a*), we get $\lambda_1 = \pi/3$.

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If in (5.100 *i*), we choose $x_2 + x_1 = 0$, then we get $x_0 = 0$ and

$$\lambda_0 = \lambda_1 = \lambda_2 = \pi/3,\ x_2 = -x_1 = \pm\sqrt{3}/2.$$

If in (5.100 *i*), we choose $x_1 + x_0 = 0$, then we get $x_2 = 0$ and

$$\lambda_0 = \lambda_1 = \lambda_2 = \pi/3,\quad x_1 = -x_0 = \pm\sqrt{3}/2.$$

In all the cases, the same method is obtained.

The three-point Gauss-Chebyshev quadrature formula is given by

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right].$$

The error constant is given by

$$C = \int_{-1}^{1}\frac{x^6\,dx}{\sqrt{1-x^2}} - \frac{\pi}{3}\left[\frac{27}{64}+0+\frac{27}{64}\right].$$

Setting $x = \sin\theta$, we obtain

$$C = \int_{-\pi/2}^{\pi/2}\sin^6\theta\,d\theta - \frac{9\pi}{32} = 2\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) - \frac{9\pi}{32} = \frac{\pi}{32}.$$

The error term R_6 becomes

$$R_6 = \frac{C}{6!}f^{(6)}(\eta) = \frac{\pi}{23040}f^{(6)}(\eta),\quad -1 < \eta < 1.$$

Example 5.18 Evaluate the integral

$$I = \int_{-1}^{1}(1-x^2)^{3/2}\cos x\,dx$$

using the Gauss-Chebyshev 1-point, 2-point and 3-point quadrature rules. Evaluate it also usin Gauss-Legendre 3-point formula.

We write the integral as $I = \displaystyle\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx$

where $f(x) = (1-x^2)^2\cos x$.

Using the 1-point Gauss-Chebyshev formula, we get

$$I = \pi f(0) = \pi = 3.14159.$$

Using the 2-point Gauss-Chebyshev formula, we get

$$I = \frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{\pi}{2}\left[2\left(\frac{1}{4}\right)\cos\left(\frac{1}{\sqrt{2}}\right)\right] = 0.59709.$$

Using the 3-point Gauss-Chebyshev formula, we get

$$I = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\pi}{3}\left[2\left(\frac{1}{16}\right)\cos\left(\frac{\sqrt{3}}{2}\right)+1\right] = 1.13$$

Using the 3-point Gauss-Legendre formula, we get (with $f(x) = (1 - x^2)^{3/2} \cos x$)

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[10\left(\frac{2}{5}\right)^{3/2} \cos\left(\sqrt{\frac{3}{5}}\right) + 8\right] = 1.08979.$$

Determination of Nodes and Weights through Orthogonal Polynomials

Consider again the integration formula (5.60)

$$I = \int_a^b w(x)\, f(x)\, dx \approx \sum_{k=0}^{n} \lambda_k f_k. \tag{5.103}$$

We shall now prove the following theorem.

Theorem 5.1 *If x_k's are selected as zeros of an orthogonal polynomial, orthogonal with respect to the weight function $w(x)$ over $[a, b]$, then the formula (5.103) has precision $2n + 1$ (or exact for polynomials of degree $\le 2n + 1$). Further $\lambda_k > 0$.*

Let $f(x)$ be a polynomial of degree less than or equal to $2n + 1$. Let $q_n(x)$ be the Lagrange interpolating polynomial of degree $\le n$, interpolating the data (x_i, f_i), $i = 0, 1, ..., n$

$$q_n(x) = \sum_{k=0}^{n} l_k(x)\, f(x_k)$$

with

$$l_k(x) = \frac{\pi(x)}{(x - x_k)\pi'(x_k)}.$$

The polynomial $[f(x) - q_n(x)]$ has zeros at $x_0, x_1, ..., x_n$. Hence, it can be written as

$$f(x) - q_n(x) = p_{n+1}(x)\, r_n(x) \tag{5.104}$$

where $r_n(x)$ is a polynomial of degree atmost n and $p_{n+1}(x_i) = 0$, $i = 0, 1, 2, ..., n$. Integrating (5.104), we get

$$\int_a^b w(x)\,[f(x) - q_n(x)]\, dx = \int_a^b w(x)\, p_{n+1}(x)\, r_n(x)\, dx$$

or

$$\int_a^b w(x)\, f(x)\, dx = \int_a^b w(x)\, q_n(x)\, dx + \int_a^b w(x)\, p_{n+1}(x)\, r_n(x)\, dx.$$

The second integral on the right hand side is zero, if $p_{n+1}(x)$ is an orthogonal polynomial, orthogonal with respect to the weight function $w(x)$, to all polynomials of degree less than or equal to n. We then have

$$\int_a^b w(x)\, f(x)\, dx = \int_a^b w(x)\, q_n(x)\, dx = \sum_{k=0}^{n} \lambda_k f(x_k) \tag{5.105}$$

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