09.10.2020

## Simple Harmonic Oscillator. (Strogatz).

$$\left.\begin{aligned} \frac{dx}{dt} &= \dot{x} = v \\ \frac{dv}{dt} &= \dot{v} = -\omega^2 x \end{aligned}\right\} \Rightarrow \frac{dv}{dx} = \frac{\frac{dv}{dt}}{\frac{dx}{dt}} = \frac{\dot{v}}{\dot{x}} = -\frac{\omega^2 x}{v}$$

$$\Rightarrow v\dot{v} + \omega^2 x \dot{x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2\right) = 0.$$

$$\Rightarrow \frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2 = \text{Constant} = C \text{ (say)}.$$

Let initially $x = x_0$ and $v = v_0$ then

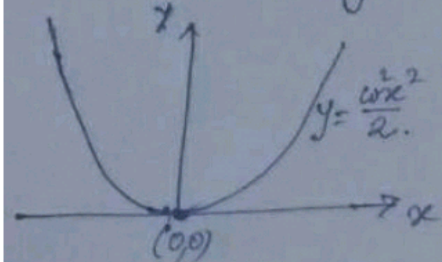
$$C = \frac{1}{2}v_0^2 + \frac{1}{2}\omega^2 x_0^2 \longrightarrow (1)$$

Therefore, $\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2 = \frac{1}{2}v_0^2 + \frac{1}{2}\omega^2 x_0^2$ gives us the orbit in the phase space starting from $(x_0, v_0)$ point.

Clearly $\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2$ = total energy of a simple harmonic oscillator.

If we choose a ~~pt~~ point $(x_0', v_0')$ in the phase space such that $\frac{1}{2}v_0^2 + \frac{1}{2}\omega^2 x_0^2 \neq \frac{1}{2}v_0'^2 + \frac{1}{2}\omega^2 x_0'^2$ then we definitely get another elliptic orbit which is different from (1).

According to our choice of axes the total energy can take any real value in $[0,\infty)$.




There are uncountable ~~many~~ infinite number of real numbers in $[0,\infty)$. Therefore there exist uncountable many periodic orbits in the phase plane. Moreover, if we consider the interval $E_1, E_2$, $0 < E_1 < E_2 < \infty$ then in $[E_1, E_2]$ interval there are uncoutable many periodic orbits. (Since the system is conservative the total energy of the system remains conserved along a phase trajectory.). Hence between any two closed orbits with total energy $E_1$ and $E_2$ $(E_1 \neq E_2)$ there is a closed orbit with total energy $E_1 < \frac{E_1 + E_2}{2} < E_2$. Hence there are dense set of periodic orbits in the phase space of SHO.

**Definition**

A nonzero vector $V_0$ is called an *eigenvector* of $A$ if $AV_0 = \lambda V_0$ for some $\lambda$. The constant $\lambda$ is called an *eigenvalue* of $A$.

As we observed, there is an important relationship between eigenvalues, eigenvectors, and solutions of systems of differential equations:

**Theorem.** *Suppose that $V_0$ is an eigenvector for the matrix $A$ with associated eigenvalue $\lambda$. Then the function $X(t) = e^{\lambda t} V_0$ is a solution of the system $X' = AX$.* ∎

Now we turn to the question of finding nonequilibrium solutions of the linear system $X' = AX$. The key observation here is this: Suppose $V_0$ is a nonzero vector for which we have $AV_0 = \lambda V_0$ where $\lambda \in \mathbb{R}$. Then the function

$$X(t) = e^{\lambda t} V_0$$

is a solution of the system. To see this, we compute

$$\begin{aligned} X'(t) &= \lambda e^{\lambda t} V_0 \\ &= e^{\lambda t}(\lambda V_0) \\ &= e^{\lambda t}(AV_0) \\ &= A(e^{\lambda t} V_0) \\ &= AX(t) \end{aligned}$$

so $X(t)$ does indeed solve the system of equations. Such a vector $V_0$ and its associated scalar have names:

**Theorem 8.1** *All solutions of the regular linear system* $\dot{x} = A(t)x + f(t)$ *have the same Liapunov stability property (unstable, stable, uniformly stable, asymptotically stable, uniformly and asymptotically stable). This is the same as that of the zero (or any other) solution of the homogeneous equation* $\dot{\xi} = A(t)\xi$. ■

$\dot{X} = AX$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. have two eigenvalues $\lambda_1, \lambda_2$ and $\lambda_1 \neq \lambda_2$.

Let $V_1$ and $V_2$ are eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ respectively. then $V_1, V_2$ are linearly independent, hence forms a basis of $R^2$. Let $X_0 \in R^2$

then $X_0 = c_1 V_1 + c_2 V_2$ $\quad$ ($c_1, c_2$ are scalars)

Now $\dot{X}_0$ = velocity components at $X_0$

$$= AX_0 = A(c_1 V_1 + c_2 V_2)$$

According to the time evolution rule

$$= c_1 AV_1 + c_2 AV_2$$

$$= c_1 \lambda_1 V_1 + c_2 \lambda_2 V_2$$

ie it is a linear combination of $V_1$ and $V_2$ (for linear system).

Hence at an arbitrary point in the phase space velocity componants ~~are~~ are linear combination of $V_1$ and $V_2$.

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Consider the nonlinear system

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\} \quad \ldots \quad ①$$

(See Strogatz's Book)

Linearising the system ① near the fixed point $(x^*, y^*)$ we get

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}_{(x^*,y^*)} \begin{pmatrix}u\\ v\end{pmatrix} \quad \ldots \quad ②$$

Example 1.

$$\left.\begin{aligned}\dot{x} &= x - x^3 \; [= f(x,y)]\\ \dot{y} &= y - y^3 \; [= g(x,y)]\end{aligned}\right\} \longrightarrow ⓐ$$

ie
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}-x^3\\ -y^3\end{pmatrix}$$

= Linear ~~part~~ term + Nonlinear term

Clearly $(0,0)$ is a fixed point of system ⓐ, and linearizing the system near $(0,0)$ we obtain

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}1-3x^2 & 0\\ 0 & 1-3y^2\end{pmatrix}_{(0,0)} \begin{pmatrix}u\\ v\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\begin{pmatrix}u\\ v\end{pmatrix}$$

This is clearly equal to considering the linear terms only??

Example 2.

$$\left.\begin{aligned}\dot{x} &= \sin y \; [= f(x,y)]\\ \dot{y} &= \cos x \; [= g(x,y)]\end{aligned}\right\} \longrightarrow ⓑ$$

ie
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\sin y\\ \cos x\end{pmatrix}$$

= Linear term + nonlinear term.

Q: Linearize the system near $(0,0)$ fixed point

Example 2.

Linearizing near (0,0) we get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & \cos y \\ -\sin x & 0 \end{pmatrix}_{(0,0)} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} u \\ v \end{pmatrix}$$

Clearly this is different from linear terms of system ⑤. Therefore contribution of nonlinear terms is also important even in case of linearization near (0,0) fixed point. Note that Sin y and Cos y are not polynomial functions. Hence we can conclude that ~~in cas~~ when $f(x,y)$ and $g(x,y)$ are polynomial functions then only no contribution of nonlinear terms will be there in the linearized system near (0,0) fixed point.

If we agree with the linearized eqn ② then we should follow it there is no point of questioning it.