Ordinary Differential equations:-

series solution:-

General form of a second order linear Differential equation is $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$ where $a_i(x)$ and $f(x)$ are continuous function of x.

Definition of analytic:-

A function f is said to be analytic at x_0 if its Taylor series about x_0, $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ exists and converges to $f(x)$ for all x in some open interval including itself.

Definition of ordinary points:-

We consider second order linear homogeous Differential equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$

ie $y''(x) + \frac{a_1(x)}{a_2(x)}y'(x) + \frac{a_0(x)}{a_2(x)}y(x) = 0 \quad [a_2(x) \neq 0]$

$\Rightarrow y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$ where

$P_1(x) = \frac{a_1(x)}{a_2(x)}$ and $P_2(x) = \frac{a_0(x)}{a_2(x)}$.

A point x_0 is called ordinary point of a differential equation if both function $P_1(x)$ and $P_2(x)$ are analytic at x_0. If either (or both) of this function is not analytic at x_0 then x_0 is called singular point of the differential equation.

Example:- Consider the differential equation

$$(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{x}y = 0$$

The normality form be

$$\frac{d^2y}{dx^2} + \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x(x-1)}y = 0$$

$\Rightarrow \frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ where

$P_1(x) = \frac{x}{x-1}$ $\quad P_2(x) = \frac{1}{x(x-1)}$

The function $P_1(x)$ is analytic except the point at $x=1$. and the function $P_2(x)$ is analytic except the point at $x=0$ and $x=1$. Those are singular points of the differential equation.

All others points except $x=0$ and $x=1$ are ordinary points.

Problem 1:- Find the power series solution of the differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$ in power of x.

We see that $x=0$ is an ordinary point of the given differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0 \quad \text{--- (1)}$$

Let the solution of (1) be of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{--- (2)}$$ where $c_0, c_1, c_2 \ldots$ are

Constant. Diffⁿ w.r.t. x, we get,

Then $$\frac{dy}{dx} = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{--- (3)}$$

$$\frac{d^2y}{dx^2} = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \quad \text{--- (4)}$$

Substituting (2), (3), (4) in equation of (1) we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + (x^2+2)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} + 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad \text{--- (5)}$$

Equating the coefficient of like power of x from both side,

Coefficient of x^0; $2(2-1)c_2 + 2c_0 = 0 \Rightarrow c_2 = -c_0$ — (6)

Coefficient of x^1; $3(3-1)c_3 + c_1 + 2c_1 = 0 \Rightarrow c_3 = -\frac{1}{2}c_1$ — (7)

coefficient of x^n; $(n+2)(n+1)c_{n+2} + nc_n + c_{n-2} + 2c_n = 0$

$$\Rightarrow c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)} \quad n \geq 2 \quad \text{---} (8)$$

~~The~~

Note:-

(1) The relation (8) is known as recurrence relation.

(2) There is no condition on c_0 & c_1 hence they are arbitrary.

For $n=2$; $c_4 = -\frac{4c_2 + c_0}{12} = \frac{1}{4}c_0$

For $n=3$ $c_5 = -\frac{5c_3 + c_1}{20} = \frac{3}{40}c_1$

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Similarly we ~~can~~ can find even coefficient in term of c_0 and each odd coefficient in term of c_1.

substituting the value of c_0, c_1, c_2 - - - in equation (2) we get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

$$= c_0 + c_1x - c_0x^2 - \frac{1}{2}c_1x^3 + \frac{1}{4}c_0x^4 + \frac{3}{40}c_1x^5 + \cdots$$

$$= c_0\left(1 - x^2 + \frac{1}{4}x^4 + \cdots\right) + c_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots\right) \text{---} (9)$$

where c_0 & c_1 are arbitrary constant

$$y_1(x) = 1 - x^2 + \frac{1}{4}x^4 + \cdots$$

$$y_2(x) = x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots$$

$\therefore$ Equation (9) gives the general solution of equation (1) in power of x where $y_1(x)$ and $y_2(x)$ are power series expension of two linearly independent solution of equation (1).

Problem 2:- Find the power series expand in powers of x of the differential equation

① $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (3x-2)y = 0$

② $(2x^2-3)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0$ Given $y(0)=1, y'(0)=5$.

① We see that $x=0$ is an ~~ord~~ ordinary point of the given differential equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (3x-2)y = 0 \quad \text{—①}$$

Let the solution of the ① be of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{—②}$$ where $c_0, c_1, \ldots$ are constant.

Then $$\frac{dy}{dx} = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad \text{—③}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \quad \text{—④}$$

Substituting ②, ③ & ④ in equation ① we get

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} + (3x-2)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n + 3\sum_{n=0}^{\infty} c_n x^{n+1} - 2\sum_{n=0}^{\infty} c_n x^n = 0 \quad \text{—⑤}$$

Equating the coefficient of like power of x from both side;

Coefficient of x^0; $2(2-1)c_2 - 2c_0 = 0 \Rightarrow c_2 = c_0$ —⑥

Coefficient of x^1; $3(3-1)c_3 - c_1 + 3c_0 - 2c_1 = 0$

$$\Rightarrow c_3 = \frac{1}{2}(c_1 - c_0)$$

Coefficient of x^n; $(n+2)(n+1)c_{n+2} - nc_n + 3c_{n-1} - 2c_n = 0$

$$\Rightarrow c_{n+2} = \frac{(n+2)c_n - 3c_{n-1}}{(n+2)(n+1)}; \quad n \geq 1.$$

For $n=2$; $c_4 = \frac{4c_2 - 3c_1}{12} = \frac{1}{3}c_0 - \frac{1}{4}c_1$

For $n=3$; $c_5 = \frac{5c_3 - 3c_2}{20} = \frac{5/2(c_1-c_0) - 3c_0}{20}$

$= \frac{5c_1 - 11c_0}{40} = \frac{c_1}{8} - \frac{11}{40}c_0$

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Similarly we can find every coefficient in term of c_0 and c_1.

Substituting the value of $c_0, c_1, \ldots$ in equation (2) we get $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots$

$$= c_0 + c_1 x + c_0 x^2 + \frac{1}{2}(c_1 - c_0)x^3 + \left(\frac{1}{3}c_0 - \frac{1}{4}c_1\right)x^4 + \left(\frac{c_1}{8} - \frac{11}{40}c_0\right)x^5 + \ldots$$

$$= \left(1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{11}{40}x^5 + \ldots\right)c_0 + \left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \ldots\right)c_1 \quad \text{—(9)}$$

$\therefore$ equation (9) gives the general solution of equation (1).

(2) We see that $x=0$ be the ordinary point of the given differential equation.

$$(2x-3)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 \quad \text{—(1)}$$

Let the solution of the (1) be of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{—(2)}$$ where $c_0, c_1, \ldots$ are constant

Then

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^n \quad \text{—(3)} \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \quad \text{—(4)}$$

Substituting the value of (2), (3) & (4) in the equation (1) we get

$$(2x^2-3)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2\sum_{n=2}^{\infty} n(n-1)c_n x^n - 3\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n = 0 \quad \text{—(5)}$$

Equating the coefficient of like power of x from both side we get,

Coefficient of x^0; $-3\cdot 2c_2 + c_0 = 0 \Rightarrow c_2 = \frac{1}{6}c_0$ — (6)

Coefficient of x^1; $-3\cdot 3(3-1)c_3 - 2c_1 + c_1 = 0$

$\Rightarrow c_3 = -\frac{1}{18}c_1$ — (7)

Coefficient of x^n;

$$2n(n-1)c_n - 3(n+2)(n+1)c_{n+2} - 2nc_n + c_n = 0$$

$$\Rightarrow c_{n+2} = \frac{2n^2-4n+1}{3(n+2)(n+1)}c_n \qquad n \geq 0 \quad \text{— (8)}$$

For $n=2$; $c_4 = \frac{c_2}{36} = \frac{1}{216}c_0$

For $n=3$; $c_5 = \frac{7}{60}c_3 = -\frac{7}{1080}c_1$

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Similarly we can find every coefficient in term of c_0 and c_1

Substituting the value of $c_0, c_1 \ldots$ in equation (2) we get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$

$$= c_0 + c_1x + \frac{1}{6}c_0x^2 - \frac{1}{18}c_1x^3 + \frac{1}{216}c_0x^4 - \frac{7}{1080}x^5c_1 + \cdots$$

$$= \left(1 + \frac{1}{6}x^2 + \frac{1}{216}x^4 + \cdots\right)c_0 + \left(x - \frac{1}{18}x^3 - \frac{7}{1080}x^5 + \cdots\right)c_1$$

where c_0 and c_1 are arbitrary constant.

Now $y' = \left(\frac{1}{3}x + \frac{4}{216}x^3 + \cdots\right)c_0 + \left(1 - \frac{3}{18}x^2 - \frac{35}{1080}x^4 + \cdots\right)c_1$

now $y(0) = 1$ & $y'(0) = 5$

$\Rightarrow c_0 = 1$ & $c_1 = 5$

$\therefore$ The general solution of equation (1) is given by

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{216}x^4 + \cdots\right) + 5\left(x - \frac{1}{18}x^3 - \frac{7}{1080}x^5 + \cdots\right)$$

Regular Singular point :-

Again we consider a homogeous linear differential equation $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{a_1(x)}{a_2(x)}\frac{dy}{dx} + \frac{a_0(x)}{a_2(x)}y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \text{ where } P_1(x) = \frac{a_1(x)}{a_2(x)}$$

$P_2(x) = \frac{a_0(x)}{a_2(x)}$.

we assume that a point x_0 is singular point i.e. atleast one of the function $P_1(x)$ and $P_2(x)$ is not analytic at x_0.

Now we defined two product $(x-x_0)P_1(x)$ and $(x-x_0)^2P_2(x)$. If the two product are both analytic at x_0 then the point x_0 is called regular singular point of the given differential equation.

If either (or both) of the product functions is not analytic at x_0, then x_0 is called an irregular point of the given differential equation.

Example :- we consider the differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

The normal form be

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{x-5}{2x^2}y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{where}$$

$$P_1(x) = -\frac{1}{2x}, \quad P_2(x) = \frac{x-5}{2x^2}$$

Since both function $P_1(x)$ and $P_2(x)$ are not analytic at the point $x=0$. Then $x=0$ is the singular point of the given differential equation.

Now we define two product $xP_1(x) = \frac{1}{2}$

$$x^2P_2(x) = \frac{x-5}{2}$$

Both of these product of function are analytic at $x=0$.

Then $x=0$ is a regular singular point of the given differential equation.

Example:- we consider the differential equation

$$x^2(x-2)^2\frac{d^2y}{dx^2}+2(x-2)\frac{dy}{dx}+(x+1)y=0.$$

The normal form be

$$\frac{d^2y}{dx^2}+\frac{2}{x^2(x-2)}\frac{dy}{dx}+\frac{x+1}{x^2(x-2)^2}y=0$$

$$\Rightarrow \frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \quad \text{where}$$

$$P_1(x)=\frac{2}{x^2(x-2)} \;\&\; P_2(x)=\frac{x+1}{x^2(x-2)^2}$$

Both function $P_1(x)$ and $P_2(x)$ are not analytic at the point $x=0$ and $x=2$. Then $x=0$ and $x=2$ are singular point of the given differential equation.

Now we defined the product $xP_1(x)=\dfrac{2}{x(x-2)}$

$$x^2P_2(x)=\frac{x+1}{(x-2)^2}$$

~~These product function are not~~ The first product function are not analytic at $x=0$.

$\therefore x_0=0$ is an irregular point of the given differential equation.

we defined the product $(x-2)P_1(x)=\dfrac{2}{x^2}$

$$(x-2)^2P_2(x)=\frac{x+1}{x^2}$$

Both product function are analytic at $x=2$

$\therefore x=2$ is an regular point of the given differential equation.

Theorem:- The point x_0 is the regular singular point of the differential equation $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$. Then the differential equation has at least one non-trivial solution of the form $|x-x_0|^r \sum_{n=0}^{\infty} c_n (x-x_0)^n$ where r is a defined constant (real or complex) which may be determined & the solution is valid in some open interval $0<|x-x_0|<R$ about x_0, $R>0$. (Method of Frobenius)

Problem 1:- Consider the differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0 \quad \text{--- (1)}$$

$x=0$ is a regular singular point of the given differential equation.

We seek the solution of the differential equation ① has

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad 0<x<R \qquad c_0 \neq 0 \quad \text{--- ②}$$

Then
$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{--- ③}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \quad \text{--- ④}$$

Substituting ②, ③ & ④ in equation ① we get

$$2x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} - x\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + (x-5)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} - 5\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n\{2(n+r)(n+r-1) - (n+r) - 5\}x^{n+r} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r} = 0$$

$\Rightarrow C_0\{2r(r-1)-r-5\}x^r +$

$$\sum_{n=1}^{\infty}\left[C_n\{2(n+r)^r - 3(n+r)-5\} + C_{n-1}\right]x^{n+r} = 0$$

Equating to zero the coefficient of lowest power of x (i.e. x^r) we get quadratic equation

$$2r(r-1) - r - 5 = 0 \quad [C_0 \neq 0]$$

This equation is known as ~~called~~ indicial equation of the differential equation.

$2r^2 - 3r - 5 = 0$. we observed that the roots are $r_1 = \frac{5}{2}$, $r_2 = -1$.

This is called exponents of the differential equation.

Note that the root of the indicial equation are real and unequal and not differ by integer.

Equating to zero the coefficient of x^{n+r} we get the recurrence relation

$$[2(n+r)(n+r-1) - (n+r) - 5]C_n + C_{n-1} = 0,\ n \geq 1$$

when $r = r_1 = \frac{5}{2}$

$$\left[2\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}-1\right) - \left(n+\frac{5}{2}\right) - 5\right]C_n + C_{n-1} = 0$$

$$\Rightarrow \left[\frac{(2n+5)(2n+3)}{2} - \frac{2n+5}{2} - 5\right]C_n + C_{n-1} = 0$$

$$\Rightarrow \left[\frac{2n+5}{2}(2n+2) - 5\right]C_n + C_{n-1} = 0$$

$$\Rightarrow n(2n+7)C_n + C_{n-1} = 0$$

$$\Rightarrow C_n = -\frac{C_{n-1}}{n(2n+7)},\quad n \geq 1.$$

using this we find that

$n=1$, $C_1 = -\frac{C_0}{9}$, $n=2$, $C_2 = -\frac{C_1}{22} = \frac{C_0}{198}$ $n=3$, $C_3 = -\frac{C_2}{39} = -\frac{C_0}{7722}$ .....

using this value of $c_1, c_2 \cdots$ & $r = \frac{5}{2}$ in equation ②
we obtain the solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+5/2} = x^{5/2}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$= x^{5/2}\left(c_0 - \frac{c_0}{9}x + \frac{c_0}{198}x^2 - \frac{c_0}{7722}x^3 + \cdots\right)$$

$$y_1(x) = c_0 x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right) \quad ⑤$$

corresponding to the larger root $r = \frac{5}{2}$.

Now $r = r_2 = -1$. Then we get ~~for~~ from recurrence formula, $[2(n-1)(n-2) - (n-1) - 5]c_n + c_{n-1} = 0, n \geq 1$

$$\Rightarrow n(2n-7)c_n + c_{n-1} = 0 \qquad n \geq 1$$

$$\Rightarrow c_n = -\frac{c_{n-1}}{n(2n-7)}, \quad n \geq 1$$

using this we get

$$c_1 = \frac{1}{5}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, \quad c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \cdots$$

using this coefficient $c_0, c_1 \cdots$ & $r = -1$ in equation ① we obtain the solution

$$y = \sum_{n=0}^{\infty} c_n x^{n-1} = x^{-1}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$= x^{-1}\left(c_0 + \frac{1}{5}c_0 x + \frac{1}{30}c_0 x^2 + \frac{1}{90}c_0 x^3 + \cdots\right)$$

$$y_2(x) = c_0 x^{-1}\left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \cdots\right) \quad ⑥$$

corresponding to the smallest root. $r = -1$.

These two solution ⑤ and ⑥ are linearly independent

$\therefore$ The general solution is given by

$y = Ay_1(x) + By_2(x)$, A, B ~~and~~ are arbitrary constant.

Theorem:- Let the point x_0 be a regular singular point of the differential equation. Let r_2 and r_1 $[Re(r_1) \geq Re(r_2)]$ be the root of the initial equation associated with x_0

Conclusion1:- Suppose $r_1 - r_2 \neq N$, where N is the non-negative integer (i.e. $r_1 - r_2 \neq 0, 1, 2, 3, \ldots$) Then the differential equation has two non-trivial linearly independent solution $y_1(x)$ and $y_2(x)$ given respectively by $y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$ where $c_0 \neq 0$ and

$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n \quad \text{where } c_0^* \neq 0.$$

Conclusion 2:- Suppose $r_1 - r_2 = N$, where N is the positive integer. Then the differential equation has two non-trivial linearly independent solution $y_1(x)$ and $y_2(x)$ given respectively by

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n \quad \text{where } c_0 \neq 0$$

and

$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n + c y_1(x) \ln|x - x_0|,$$

where $c_0^* \neq 0$ and c is a constant.

Conclusion 3:- Suppose $r_1 - r_2 = 0$. Then differential equation has two non-trivial linearly independent solution $y_1(x)$ and $y_2(x)$ given respectively by

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad \text{where } c_0 \neq 0.$$

and

$$y_2(x) = |x - x_0|^{r_1 + 1} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n + y_1(x) \ln|x - x_0|;$$

where $c_0^* \neq 0$.

The solution in conclusion 1, 2 & 3 are valid some interval $0 < |x - x_0| < R$ about x_0.

Problem 1:- Use the method Frobenius to find the solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + (x^2 - 3x)\frac{dy}{dx} + 3y = 0$$ in some interval $0 < x < R$.

We observed that $x=0$ is the regular singular point of the differential equation.

$$x^2 \frac{d^2y}{dx^2} + (x^2 - 3x)\frac{dy}{dx} + 3y = 0 \quad \text{--- (1)}$$

We assume the solution in the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad c_0 \neq 0 \quad \text{--- (2)}$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \quad \text{--- (3)}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \quad \text{--- (4)}$$

Substituting (2), (3), (4) in equation (1) we get

$$x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + (x^2-3x)\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + 3\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r+1} - 3\sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + 3\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+r)(n+r-1) - 3(n+r) + 3] c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+r)(n+r-1) - 3(n+r) + 3] c_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) c_{n-1} x^{n+r} = 0$$

$$\Rightarrow [r(r-1) - 3r + 3] x^r + \sum_{n=1}^{\infty} [\{(n+r)(n+r-1) - 3(n+r) + 3\} c_n + \{(n+r-1) c_{n-1}\}] x^{n+r} = 0$$

Equating to zero the coefficient of lowest power of x i.e. x^r we get

$$r(r-1) - 3r + 3 = 0$$

$$\Rightarrow r^2 - 4r + 3 = 0$$

$$\Rightarrow (r-3)(r-1) = 0$$

$$\Rightarrow r = 3, 1, \quad r_1 = 3, \ r_2 = 1 \ (\text{say})$$ i.e. roots are differ by integer 2.

Equating to zero the coefficient of x^{n+r} we get the recurrence form

$$\{(n+r)(n+r-1) - 3(n+r) + 3\} c_n + (n+r-1) c_{n-1} = 0$$

$$\Rightarrow c_n = -\frac{(n+r-1) c_{n-1}}{(n+r)(n+r-1) - 3(n+r) + 3}$$

Let $r = r_1 = 3$

$$\therefore c_n = -\frac{(n+2) c_{n-1}}{(n+3)(n+2) - 3(n+3) + 3}$$

$$= -\frac{(n+2) c_{n-1}}{n^2 + 2n} = -\frac{c_{n-1}}{n}, \quad n \geq 1$$

For $n = 1$ $\quad c_1 = -c_0$

For $n = 2$ $\quad c_2 = -\frac{c_1}{2} = \frac{c_0}{2}$

For $n = 3$ $\quad c_3 = -\frac{c_2}{3} = -\frac{c_0}{3!}$

For $n = 4$ $\quad c_4 = -\frac{c_3}{4} = \frac{c_0}{4!}$

. . . . . .

For $n = n$ $\quad c_n = (-1)^n \frac{c_0}{n!}$

. . . . . .

Then for $r = 3$ using the value of $c_0, c_1, c_2, \ldots$ in equation (2) we get

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+3} = x^3 (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)$$

$$= c_0 x^3 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)$$

$$\therefore y_1(x) = c_0 x^3 e^{-x}$$ where c_0 is arbitrary constant.

Let $r = r_2 = 1$,

From the recurrence relation

$$c_n = -\frac{n c_{n-1}}{(n+1)n - 3(n+1) + 3} = -\frac{c_{n-1}}{n-2} \quad n \geq 1$$

$\Rightarrow \quad (n-2) c_n = -c_{n-1} \quad n \geq 1.$

For $n=1$ $c_1 = c_0$, For $n = 2$ $c_1 = 0$

$\therefore c_1 = c_0 = 0$

However $c_0 \neq 0$ in our assumption, which is a contradiction, there is no solution when $c_0 = 0$.

Further we observed that

$$c_n = -\frac{c_{n-1}}{n-2}, \quad n \geq 3.$$

For $n = 3$ $c_3 = -c_2$

For $n = 4$ $c_4 = -\frac{c_3}{2} = \frac{c_2}{2!}$

For $n = 5$ $c_5 = -\frac{c_4}{3} = -\frac{c_2}{3!}$

For $n = 6$ $c_6 = -\frac{c_5}{4} = \frac{c_2}{4!}$

- - -

$$c_{n+2} = (-1)^n \frac{c_2}{n!},$$

- - -

For $r = r_2 = 1$, using the above values of c_n in (2)

we get

$$y_2(x) = \sum_{n=0}^{\infty} c_n x^{n+1} = x\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots\right]$$

$$= c_2 x^3 \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right]$$

$$= c_2 x^3 e^{-x}$$

we see that it is essentially.

Now we seek the solution which is linearly independent of the differential equation.

we see in the form given in conclusion (2).

Let $y = f(x)v$ where $f(x)$ is the solution of the differential equation in this case $f(x) = x^3e^{-x}$

$\therefore y = x^3e^{-x}v$

$$\frac{dy}{dx} = x^3e^{-x}\frac{dv}{dx} + (3x^2e^{-x} - x^3e^{-x})v$$

$$\frac{d^2y}{dx^2} = x^3e^{-x}\frac{d^2v}{dx^2} + 2(3x^2e^{-x} - x^3e^{-x})\frac{dv}{dx}$$
$$+ (x^3e^{-x} - 6x^2e^{-x} + 6xe^{-x})v$$

Substituting this in equation ① we get

$$x\frac{d^2v}{dx^2} + (3-x)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = p$

$$\therefore x\frac{dp}{dx} + (3-x)p = 0$$

$$\Rightarrow p = x^{-3}e^{x}$$

$$\Rightarrow v = \int x^{-3}e^{x}dx$$

Hence $y_2(x) = x^3e^{-x}\int x^{-3}e^{x}dx$

$$= x^3e^{-x}\int x^{-3}\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right]dx$$

$$= x^3e^{-x}\int\left[x^{-3} + x^{-2} + \frac{x^{-1}}{2} + \frac{1}{6} + \frac{x}{24} + \cdots\right]dx$$

$$= x^3e^{-x}\left[-\frac{1}{2x^2} - \frac{1}{x} + \frac{1}{2}\ln x + \frac{1}{6}x + \frac{x^2}{48} + \cdots\right]$$

$$= \left[x^3 - x^4 + \frac{x^5}{2} - \frac{x^6}{6} + \cdots\right]\left[-\frac{1}{2x^2} - \frac{1}{x} + \frac{1}{6}x + \frac{x^2}{48} + \cdots\right]$$
$$+ \frac{1}{2}x^3e^{-x}\ln x$$

$$= \left[-\frac{1}{2}x - \frac{1}{2}x^2 + \frac{3}{4}x^3 - \frac{1}{4}x^4 + \cdots\right] + \frac{1}{2}x^3e^{-x}\ln x.$$

which is of the form $\sum_{n=0}^{\infty} c_n^* x^{n+1} + c\,y_1(x)\ln x$

where $y_1(x) = x^3e^{-x}$

$\therefore$ The general solution is $y = Ay_1(x) + By_2(x)$ where A and B are arbitrary constant.

Note:- * *

Here we consider the case where the components are either equal or differ by an integer.

i) When exponents are equal, $r_1 = r_2$ in this case two linear independent solution of the differential equation $y = y_1(x)$ and $y = y_2(x)$ are given by $y_1(x) = Y(x, r_1)$ and $y_2(x) = \frac{\partial}{\partial r} Y(x, r)\Big|_{r=r_1}$.

ii) When the exponents are differ by an integer $r_1 = r_2 + N$, where N be the positive integer.

In this case linearly independent solution $y = y_1(x)$ and $y = y_2(x)$ are given by $y_1(x) = \lim_{r \to r_1} Y(x, r)$ and

$$y_2(x) = \lim_{r \to r_2} \frac{\partial}{\partial r} Y(x, r) \text{ where}$$

$$Y(x, r) = (r - r_2) y(x, r).$$

Problem 1:-

Solve the differential equation (2018)

$x(1-x)y'' - 3xy' - y = 0$ near $x = 0$. (2013)

The normal form be

$$y'' - \frac{3}{1-x} y' - \frac{1}{x(1-x)} y = 0$$

$\Rightarrow y'' + P_1(x) y' + P_2(x) y = 0$ where $P_1 = -\frac{3}{1-x}$ and $P_2 = -\frac{1}{x(1-x)}$

The function P_1 is not analytic at $x = 1$ & the function P_2 is not analytic at $x = 0$ and $x = 1$

$\therefore$ $x = 0$ and $x = 1$ be the singular point of the given differential equation.

We defined the product $x P_1(x) = \frac{x}{1-x}$

$$x P_2(x) = \frac{1}{1-x}$$

$\therefore$ The both product are analytic at $x = 0$

Hence $x = 0$ is an regular singular point of the given differential equation, $x(1-x)y'' - 3xy' - y = 0$ —①

Now we seek the solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad c_0 \neq 0 \quad —②$$

Diffⁿ w.r.t. x we get

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \quad —③$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \quad —④$$

Substituting ②, ③, ④ in equation ① we get

$$x(1-x)\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-2} - 3x\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$
$$- \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r}$$
$$-3\sum_{n=0}^{\infty} c_n (n+r)x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty}\Big[\{(n+r)(n+r-1)$$
$$3(n+r)+1\}c_n x^{n+r}\Big] = 0$$

Equating to zero, the coefficient of the lowest power of x i.e. x^{r-1} we get the indicial equation

$$r(r-1) = 0 \qquad [c_0 \neq 0]$$

$$\Rightarrow \quad r = 1, 0,$$

Set $r_1 = 1$, $r_2 = 0$ in this case $r_1 - r_2 = 1$ i.e. The root of the differential equation is differed by an integer 1.

Equating to zero, the coefficient of x^{n+r-1} we get

$$c_n (n+r)(n+r-1) - \{(n+r-1)(n+r-2) + 3(n+r-1) + 1\}c_{n-1} = 0$$

$$\Rightarrow c_n (n+r)(n+r-1) - (n+r)^2 c_{n-1} = 0$$

$$\Rightarrow c_n = \frac{(n+r)}{(n+r-1)} c_{n-1} \qquad n \geq 1.$$

Putting $n=1$ $\quad c_1 = \dfrac{r+1}{r} c_0$

$n=2$ $\quad c_2 = \dfrac{r+2}{(r+1)} c_1 = \dfrac{r+2}{r} c_0$

$n=3$ $\quad c_3 = \dfrac{r+3}{r+2} c_2 = \dfrac{r+3}{r} c_0$

- - - - - - -

putting the values of c_1, c_2 --- in equation (2) we get

$$y = x^r (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$= c_0 x^r \left(1 + \frac{r+1}{r} x + \frac{r+2}{r} x^2 + \frac{r+3}{r} x^3 + \cdots \right)$$

$$y(x,r) = c_0 \frac{x^r}{r} \left(r + (r+1)x + (r+2)x^2 + (r+3)x^3 + \cdots \right)$$

Now $Y(x,r) = (r-0)\, y(x,r)$

$$= c_0 x^r (r + (r+1)x + (r+2)x^2 + (r+3)x^3 + \cdots)$$

$\therefore$ The one solution corresponding to $r = r_1 = 1$ is given by $y_1(x) = \lim\limits_{r\to 1} Y(x,r) = \lim\limits_{r \to 1} c_0 x^r (r + (r+1)x + \cdots)$

$$= c_0 (x + 2x^2 + 3x^3 + \cdots).$$

The another solution corresponding to $r = r_2 = 0$ is given by

$$Y_2(x) = \lim_{r \to r_2} \frac{\partial Y(x,r)}{\partial r}$$

$$= \lim_{r \to 0} c_0 \Big\{ x^r \log x \,[r + (r+1)x + (r+2)x^2 + (r+3)x^3 + \cdots] + x^r [1 + x + x^2 + x^3 + \cdots] \Big\}$$

$$= c_0 \Big\{ \log x \,.\, y_1(x) + [1 + x + x^2 + x^3 + \cdots] \Big\}$$

$\therefore$ The required solution is given by

$$y(x) = a[x + 2x^2 + 3x^3 + \cdots] + b[\log x ; (x + 2x^2 + 3x^3 + \cdots) + (1 + x + x^2 + \cdots)].$$

where a and b are arbitrary constant.

Extra: (Solution about ordinary point).

Problem:-

Find the general power series solution near $x=0$ of the Legendere's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + P(P+1)y = 0,$$ where P is an arbitrary constant.

The given differential equation be

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + P(P+1)y = 0 \quad \text{--- (1)}$$

The normal form be

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{P(P+1)}{1-x^2}y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{where } P_1(x) = -\frac{2x}{1-x^2}$$

$$P_2(x) = \frac{P(P+1)}{1-x}$$

now at $x=0$ both function $P_1(x)$ and $P_2(x)$ are analytic. so $x=0$ is an ordinary point of the given differential equation.

let the solution of (1) ~~of~~ be of the form

$$y = \sum_{n=0}^{\infty} C_n x^n \text{ --- (2)}$$ where $c_0, c_1, \dots$ are arbitrary constant

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} C_n n x^{n-1} = \sum_{n=1}^{\infty} C_n n x^{n-1} \quad \text{--- (3)}$$

$$\frac{d^2y}{dx^2} = \sum_{n=1}^{\infty} C_n n(n-1)x^{n-2} = \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} \quad \text{--- (4)}$$

substituting (2), (3) and (4) in equation (1) we get

$$(1-x^2)\sum_{n=2}^{\infty} C_n(n-1)n x^{n-2} - 2x\sum_{n=1}^{\infty} C_n n x^{n-1} + P(P+1)\sum_{n=0}^{\infty} C_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} C_n n(n-1)x^n - 2\sum_{n=1}^{\infty} C_n n x^n + P(P+1)\sum_{n=0}^{\infty} C_n x^n = 0$$

Equating the coefficient of like power of x from both side, we get.

Coefficient x^0, $C_2\, 2\cdot(2-1) + C_0 P(P+1) = 0 \Rightarrow C_2 = -\frac{P(P+1)}{2!} C_0$

Coefficient of x^1, $C_3\, 3(3-1) - 2C_1 + P(P+1)C_1 = 0$

$$\Rightarrow C_3 = \frac{1}{3!}(2 - P(P+1))C_1$$

$$\Rightarrow C_3 = -\frac{1}{3!}(P-1)(P+2)C_1$$

- - - -

Coefficient of x^n, $C_{n+2}(n+2)(n+1) - C_n n(n-1) - 2C_n n + P(P+1)C_n = 0$

$$\Rightarrow C_{n+2} = \frac{n(n-1) + 2n - P(P+1)}{(n+2)(n+1)} C_n$$

$$\Rightarrow C_{n+2} = \frac{n^2 + n - P(P+1)}{(n+2)(n+1)} C_n$$

$$= -\frac{P^2 - n^2 + P - n}{(n+2)(n+1)} C_n = -\frac{(P-n)(P+n+1)}{(n+2)(n+1)} C_n, \quad n \geq 2$$

For $n = 2$

$$C_4 = -\frac{(P-2)(P+3)}{4\cdot 3} \cdot C_2 = \frac{P(P-2)(P+1)(P+3)}{4!} C_0$$

For $n = 3$

$$C_5 = -\frac{(P-3)(P+4)}{5\cdot 4} C_3 = \frac{(P-1)(P-3)(P+2)(P+4)}{5!} C_1$$

For $n = 4$

$$C_6 = -\frac{(P-4)(P+5)}{6\cdot 5} C_4 = -\frac{(P-4)(P-2)P(P+1)(P+3)(P+5)}{6!} C_0$$

- -

Similarly we can find every coefficient in term of C_0 and C_1.

Substituting the value of C_2 C_3 - - - in equation ②

we get

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots$$

$$= C_0 + C_1 x - \frac{P(P+1)}{2!} C_0 x^2 - \frac{1}{3!}(P-1)(P+2)C_1 x^3 + \frac{(P-2)P(P+1)(P+3)}{4!} C_0 \cdot x^4 + \frac{(P-1)(P-3)(P+2)(P+4)}{5!} C_1 x^5 - \frac{(P-4)(P-2)P(P+1)(P+3)(P+5)}{6!} \cdot C_0 x^6 + \cdots$$

$$\therefore y = \left(1 - \frac{P(P+1)}{2!}x^2 + \frac{(P-2)P(P+1)(P+3)}{4!}x^4 - \frac{(P-4)(P-2)P(P+1)(P+3)(P+5)}{6!}x^6 + \cdots\right)c_0 +$$

$$\left(x - \frac{(P-1)(P+2)}{3!}x^3 + \frac{(P-1)(P-3)(P+2)(P+4)}{5!}x^5\right)c_1$$

where c_0 and c_1 be arbitrary constant.

This is the general solution of equation ①.

Problem:

Find the general solution of $y'' + (x-3)y' + y = 0$ near $x = 2$.

The differential equation be

$$y'' + (x-3)y' + y = 0 \quad \text{— ①}$$

$$\Rightarrow y'' + P_1(x)y' + P_2(x)y = 0$$ where $P_1(x) = x-3$ and $P_2(x) = 1$.

Since both function $P_1(x)$ and $P_2(x)$ are analytic at $x=2$. So $x = 2$ be an ordinary point of the differential equation.

Let the solution of ① be of the form

$$y = \sum_{n=0}^{\infty} c_n(x-2)^n \quad \text{— ②}$$ where $c_0, c_1, c_2 \cdots$ arbitrary Constant.

Differentiating with respect to x we get

$$y' = \sum_{n=0}^{\infty} c_n n (x-2)^{n-1} = \sum_{n=1}^{\infty} c_n n (x-2)^{n-1} \quad \text{--- (3)}$$

$$y'' = \sum_{n=1}^{\infty} c_n n(n-1)(x-2)^{n-2} = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \quad \text{--- (4)}$$

Substituting (2), (3), (4) in equation (1) we get,

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + (x-3) \sum_{n=1}^{\infty} c_n n (x-2)^{n-1} + \sum_{n=0}^{\infty} c_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n (n-1) n x^{n-2} + \sum_{n=1}^{\infty} c_n n (x-2)^n - \sum_{n=1}^{\infty} c_n n (x-2)^{n-1}$$

$$+ \sum_{n=0}^{\infty} c_n (x-2)^n = 0$$

Equating the coefficient of like power of $(x-2)$ from both side we get

Coefficient of $(x-2)^0$; $c_2 \, 2(2-1) - c_1 + c_0 = 0$

$$\Rightarrow c_2 = \frac{1}{2}(c_1 - c_0)$$

Coefficient of $(x-2)^1$; $c_3 \, 3(3-1) + c_1 - c_2 \cdot 2 + c_1 = 0$

$$\Rightarrow c_3 = \frac{1}{6}[(c_1 - c_0) - 2c_1]$$

$$\Rightarrow c_3 = -\frac{1}{6}(c_1 + c_0).$$

...

Coefficient of $(x-2)^n$; $c_{n+2}(n+2)(n+1) + c_n n - c_{n+1}(n+1)$

$$+ c_n = 0$$

$$\Rightarrow c_{n+2} = \frac{c_{n+1} - c_n}{(n+2)}$$

For $n = 2$, $c_4 = \dfrac{c_3 - c_2}{4} = -\dfrac{1}{4}\left[\dfrac{1}{6}(c_1 + c_0) + \dfrac{1}{2}(c_1 - c_0)\right]$

$$= -\frac{1}{4}\left[\frac{4c_1 - 2c_0}{6}\right] = \frac{1}{12}c_0 - \frac{1}{6}c_1$$

Similarly we can find every coefficient in term of c_1

and C_0.

Substituting the value of C_2, C_3, C_4 --- ~~wegger~~ in equation (2) we get

$$y = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3 + C_4(x-2)^4 + \cdots$$

$$= C_0 + C_1(x-2) + \frac{1}{2}(C_1 - C_0)(x-2)^2 - \frac{1}{6}(C_1 + C_0)(x-2)^3 + \left(\frac{1}{12}C_0 - \frac{1}{6}C_1\right)(x-2)^4 + \cdots$$

$$= \left(1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\right)C_0 + \left((x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \cdots\right)C_1$$

where C_0 and C_1 are arbitrary constant, This is the general solution of the given differential equation.

Problems

i) Solve $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$ in powers of x.

ii) Find the power series solution of the differential equation $y' = 2xy$.

Legendre's equation and Legendre's Polynominals:-

Let us Consider the function $u \equiv u(r, \theta, \phi)$

Equation be $\nabla^2 u = 0$

The spherical Co-ordinate System

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{--- (1)}$$

$$u \equiv u(r, \theta, \phi) = F(r)\, G(\theta, \phi) \quad \text{--- (2)}$$

Substituting this in equation (1) we get after same Simplification

$$\frac{r^2}{F}\left(F'' + \frac{2}{r}F'\right) = -\frac{1}{G}\left(\frac{\partial^2 G}{\partial \phi^2} + \cot\phi \frac{\partial G}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 G}{\partial \theta^2}\right) = K \text{ (say)}$$

Left hand side is a function of r and right hand Side is a function of ϕ, and θ.

$\therefore$ Both Side must be Constant then we get

$$F'' + \frac{2}{r}F' = \frac{KF}{r^2}$$

$$\Rightarrow r^2 F'' + 2rF' - KF = 0.$$

The other equation is $\dfrac{\partial^2 G}{\partial \phi^2} + \cot\phi \dfrac{\partial G}{\partial \phi} + \dfrac{1}{\sin^2\phi}\dfrac{\partial^2 G}{\partial \theta^2} + KG = 0$

Next we specialised our approach, axes symmetric i.e. the problem is no θ dependend. G is only function of ϕ.

Then we get $\dfrac{d^2 G}{d\phi^2} + \cot\phi \dfrac{dG}{d\phi} + KG = 0$

put $W = \cos\phi$

$$\frac{dG}{d\phi} = \frac{dG}{dW}\cdot\frac{dW}{d\phi} = -\sin\phi \frac{dG}{dW}$$

$$\frac{d^2 G}{d\phi^2} = \frac{d}{d\phi}\left(-\sin\phi \frac{dG}{dW}\right) = -\cos\phi \frac{dG}{dW} - \sin\phi \frac{d}{dW}\left(\frac{dG}{dW}\right)\frac{dW}{d\phi}$$

$$= -\cos\phi \frac{dG}{dW} + \sin^2\phi \frac{d^2 G}{dW^2}$$

Substituting this we get

$(1-w^2)\, G''(w) - 2w\, G'(w) + K\, G(w) = 0$ This equation is known as Legendre's equation.

Legendre's equation and Legendre's polynomial:

Differential equation is of a form

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ — ① is called the Legendre's equation of degree n.

we seek the solution of ① in the form

$$y = \sum_{m=0}^{\infty} c_m x^{k-m} \quad \text{— ②} \qquad c_0 \neq 0$$

Differentiating w.r.t. x we get,

$$y' = \sum_{m=0}^{\infty} c_m (k-m)\, x^{k-m-1} \quad \text{— ③}$$

$$y'' = \sum_{m=0}^{\infty} c_m (k-m)(k-m-1)\, x^{k-m-2} \quad \text{— ④}$$

Substituting ②, ③, ④ in equation ① we get

$$(1-x^2)\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)\, x^{k-m-2} - 2x \sum_{m=0}^{\infty} c_m (k-m)\, x^{k-m-1}$$

$$+ n(n+1) \sum_{m=0}^{\infty} c_m x^{k-m} = 0.$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m (k-m)(k-m-1)\, x^{k-m-2} - \sum_{m=0}^{\infty} c_m (k-m)(k-m-1)\, x^{k-m}$$

$$-2\sum_{m=0}^{\infty} c_m (k-m)\, x^{k-m} + n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0.$$

$$\Rightarrow \sum_{m=2}^{\infty} c_{m-2}(k-m+2)(k+m+1)\, x^{k-m} - \sum_{m=0}^{\infty} \{(k-m)(k-m-1)$$

$$+ 2(k-m) - n(n+1)\} c_m x^{k-m} = 0.$$

$$\Rightarrow -c_0\{k(k-1) + 2k - n(n+1)\}x^k - \{(k-1)(k-2) + 2(k-1)$$

$$- n(n+1)\} c_1 x^{k-1} + \sum_{m=2}^{\infty} \{c_{m-2}(k-m+2)(k+m+1) + c_m (k-m-n)$$

$$\cdot (k-m+n-1)\} x^{k-m} = 0$$

Equating to zero the lowest power of x i.e. x^k we get

$C_0(k+n)(k+n+1) = 0 \Rightarrow k = n, -(n+1) \quad C_0 \neq 0.$

equating to zero the coefficient of x^{k-1} we get

$C_1(k-1-n)(k+n) = 0$

$\Rightarrow C_1 = 0 \quad$ Since $(k-1-n)(k+n) \neq 0$

Also equating the coefficient of x^{k-m} we get the recurrence formula.

$$C_m = -\frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n-1)} C_{m-2} \quad m \geq 2.$$

Since $C_1 = 0$ we get $C_1 = C_3 = C_5 = \cdots = 0,$

when $k = n.$

$$C_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} C_{m-2}$$

putting $m = 2, 4, 6 \cdots$ we get

$$C_2 = -\frac{n(n-1)}{2(2n-1)} C_0, \quad C_4 = -\frac{(n-2)(n-3)}{4(2n-3)} C_2$$

$$= \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} C_0,$$

and so on.

we have for $k = n$ in equation (2) and substituting the values of $C_2, C_4 \cdots$ and $C_{2n+1} = 0$

$$y = \sum_{m=0}^{\infty} C_m x^{k-m} = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + C_3 x^{n-3} + \cdots$$

$$\therefore y_1(x) = C_0\left(x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} + \cdots\right) \quad -(5)$$

when $k = -(n+1)$ then

$$C_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} C_{m-2}$$

putting $m = 2, 4, \cdots$ we get

$$c_2 = \frac{(n+1)(n+2)}{2(2n+3)} c_0, \quad c_4 = \frac{(n+3)(n+4)}{4(2n+5)} c_2$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)} c_0$$

and so on.

putting this values $k = -(n+1)$ and using the values of c_2, c_4 -- in equation ② we get another solution

$$y_2(x) = c_0 \Big[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)} x^{-n-5} + \cdots \Big) \quad -⑥$$

The two solution ⑤ and ⑥ are linearly independent solution of equation ①

If we take $c_0 = \frac{1\cdot 3 \cdots (2n-1)}{n!}$ in the solution ⑤ then we get

$$y_1(x) = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} \Big[x^n - \frac{n(n-1)}{2\cdot(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} - \cdots \Big]$$ and

This solution is denoted by $P_n(x)$. $P_n(x)$ is called Legendre function of first kind or Legendre's polynominal. of degree n.

$$\therefore P_n(x) = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} \Big[x^n - \frac{n(n-1)}{2\cdot(2n-1)} x^{n-2} + \cdots \Big]$$

$$\& \; P_n(x) = \sum_{r=0}^{[\frac{1}{2}n]} (-1)^r \frac{(2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

where $[\frac{1}{2}n] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$

Again if we take $c_0 = \frac{n!}{1\cdot 3\cdot 5 \cdots (2n+1)}$ in equation ⑥ we get the legendre's function of second kind and its denoted by $Q_n(x)$ where

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5 \cdots (2n+1)} \Big[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \cdots \Big]$$

The general solution of ① is

$Y(x) = A P_n(x) + B Q_n(x)$ where A and B are arbitrary constant.

Some properties of Legendre polynomial:-

1) Show that $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

$|x| \leq 1$ $|z| \leq 1$. (2010) (2014) (2017) (2016)

Since $|x| \leq 1$, $|z| \leq 1$ we have

$$(1-2xz+z^2)^{-1/2} = [1-z(2x-z)]^{-1/2}$$

$$= 1 + \frac{1}{2} z(2x-z) + \frac{1}{2}\cdot\frac{3}{4}\cdot z^2(2x-z)^2 + \cdots$$

$$+ \frac{1\cdot 3 \cdots (2n-3)}{2\cdot 4 \cdots (2n-2)} z^{n-1}(2x-z)^{n-1} + \cdots$$

$$\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} z^n (2x-z)^n + \cdots$$

Now the coefficient of z^n in

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4 \cdots 2n} z^n (2x-z)^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4 \cdots 2n} 2^n x^n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{(2\cdot 1)(2\cdot 2)\cdots(2\cdot n)} 2^n x^n$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!} x^n.$$

Again the coefficient of z^n in

$$\frac{1\cdot 3 \cdots (2n-3)}{2\cdot 4 \cdots (2n-2)} z^{n-1}(2x-z)^{n-1}$$

$$= \frac{1\cdot 3 \cdots (2n-3)}{(2\cdot 1)(2\cdot 2)\cdots(2(n-1))} \{-(n-1)(2x)^{n-2}\}$$

$$= \frac{1\cdot 3 \cdots (2n-3)(2n-1)}{2^{n-1}\, n!} \frac{n}{(2n-1)} \{-(n-1)2^{n-2}x^{n-2}\}$$

$$= -\frac{1\cdot 3 \cdots (2n-1)}{n!} \cdot \frac{n(n-1)}{2(2n-1)} x^{n-2}$$ and so on.

Using this expression we see that coefficient of x^n in the expression of $(1-2xz+z^2)^{-1/2}$ is given by

$$\frac{1\cdot 3\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)}x^{n-4} - \cdots\cdots\right]$$

$= P_n(x)$

we find that $P_1(x), P_2(x)$ ... will be the Coefficient of $z, z^2 \cdots$ in the expression of $(1-2xz+z^2)^{-1/2}$ thus we may write.

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + \cdots + z^nP_n(x) + \cdots$$

$$= \sum_{n=0}^{\infty} z^n P_n(x).$$

② Show that $\dfrac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^nP_n(x)$ (2010)

using property ① differentiating w.r.t. z and multiply both side by $2z$ and adding with ① we get the result.

③ Laplace's definite integral. prove that

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}\cos\phi\right]^n d\phi \quad (2014)\ (2017)\ (2016)$$

From the integral Calculus we know that

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad a^2 > b^2.$$

Let $a = 1 - zx$ and $b = z\sqrt{x^2-1}$

$\therefore a^2 - b^2 = (1-zx)^2 - z^2(x^2-1) = 1 - 2xz + z^2$

using this value in the above integral we get

$$\pi(1-2xz+z^2)^{-1/2} = \int_0^{\pi}\left[1 - zx \pm z\sqrt{x^2-1}\cos\phi\right]^{-1}d\phi$$

$$= \pi\sum_{n=0}^{\infty} z^n P_n(x) = \int_0^{\pi}(1-zt)^{-1}d\phi, \left[t = x \pm \sqrt{x^2-1}\cos\phi\right]$$

$$= \int_0^{\pi} (1 + zt + z^2t^2 + \cdots)\, d\phi$$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} z^n t^n \, d\phi = \sum_{n=0}^{\infty} z^n \int_0^{\pi} (x \pm \sqrt{x^2-1}\cos\phi)^n \, d\phi$$

Equating both side the coefficient of z^n we get

$$\pi P_n(x) = \int_0^{\pi} (x \pm \sqrt{x^2-1}\cos\phi)^n \, d\phi$$

i.e. $P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x \pm \sqrt{x^2-1}\cos\phi)^n \, d\phi$

④ Show that $P_n(\cos\theta) = \frac{1}{\pi} \int_0^{\pi} [\cos\theta + i \sin\theta \cos\phi]^n \, d\phi$

[Hints: put $x = \cos\theta$ in problem ③]

⑤ Show that

i) $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}, \; n \geq 2$ (2015)

ii) $nP_n = xP'_n - P'_{n-1}$ (2015)

iii) $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

iv) $(1-x^2)P'_n = n(P_{n-1} - xP_n)$.

① We have $(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$

Differentiating w.r.t. z we get

$$-\frac{1}{2}(1 - 2xz + z^2)^{-3/2} \cdot (-2x + 2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

Multipling both side by $(1 - 2xz + z^2)$

$$(x - z)(1 - 2xz + z^2)^{-1/2} = (1 - 2xz + z^2) \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$\Rightarrow (x - z) \sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2) \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

Equating both side coefficient of z^n we get

$$xP_n(x) - P_{n-1} = (n+1)P_{n+1} - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

Replace n by $n-1$ we get

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2} \qquad n \geqslant 2.$$

Ex. 4. *Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.*

Sol. We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$,

and $P_4(x) = (35x^4 - 30x^2 + 3)/8$.

These $\Rightarrow$ $x^4 = (8/35)P_4(x) + (6/7)x^2 - (3/35)$, ...(1)

$x^3 = (2/5)P_3(x) + (3/5)x$, $x^2 = (2/3)P_2(x) + (1/3)$, ...(2)

$x = P_1(x)$ and $1 = P_0(x)$ ...(3)

Now, $x^4 + 2x^3 + 2x^2 - x - 3 = (8/35)P_4(x) + (6/7)x^2 - (3/35) + 2[(2/5)P_3(x) + (3/5)x] + 2x^2 - x - 3$

[Putting values of x^4 and x^3 with help of (1) and (2)]

$= (8/35)P_4(x) + (4/5)P_3(x) + (20/7)x^2 + (1/5)x - (108/35)$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{20}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + \frac{1}{5}P_1(x) - \frac{108}{35}, \text{ using (2) and (3)}$$

$= (8/35)P_4(x) + (4/5)P_3(x) + (40/21)P_2(x) + (1/5)P_1(x) - (224/105)P_0(x)$, using (3)

Ex. 5. *Prove that* $P_n(-\frac{1}{2}) = P_0(-\frac{1}{2})P_{2n}(\frac{1}{2}) + P_1(-\frac{1}{2})P_{2n-1}(\frac{1}{2}) + \dots P_{2n}(-\frac{1}{2})P_0(\frac{1}{2})$.

Sol. We have $$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \quad \text{...(1)}$$

Replacing x by $1/2$ and $-1/2$ successively, (1) gives

$$(1 - z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(\tfrac{1}{2}) \quad \text{...(2)}$$

and $$(1 + z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-\tfrac{1}{2}). \quad \text{...(3)}$$

Next, replacing z by z^2 in (3), $$(1 + z^2 + z^4)^{-1/2} = \sum_{n=0}^{\infty} z^{2n} P_n(-\tfrac{1}{2}). \quad \text{...(4)}$$

But $1 + z^2 + z^4 = (1 + z^2)^2 - z^2 = (1 + z^2 + z)(1 + z^2 - z)$

$\therefore$ $(1 + z^2 + z^4)^{-1/2} = (1 + z + z^2)^{-1/2} . (1 - z + z^2)^{-1/2}$

or $$\sum_{n=0}^{\infty} z^{2n} P_n(-\tfrac{1}{2}) = \sum_{n=0}^{\infty} z^n P_n(-\tfrac{1}{2}) . \sum_{n=0}^{\infty} z^n P_n(\tfrac{1}{2}), \text{ by (2), (3) and (4)}$$

or $$\sum_{n=0}^{\infty} z^{2n} P_n(-\tfrac{1}{2}) = \left[P_0(-\tfrac{1}{2}) + z P_1(-\tfrac{1}{2}) + \dots + z^{2n-1} P_{2n-1}(-\tfrac{1}{2}) + z^{2n} P_{2n}(-\tfrac{1}{2}) + \dots\right] \times \left[P_0(\tfrac{1}{2}) + z P_1(\tfrac{1}{2}) + \dots\right.$$
$$\left. + z^{2n-1} P_{2n-1}(\tfrac{1}{2}) + z^{2n} P_{2n}(\tfrac{1}{2}) + \dots\right]$$

Equating the coefficients of z^{2n} from both sides of the above equation, we get the desired result.

Ex. 6. *Prove that* $$\frac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n.$$

[Agra 2003; Kanpur 2005; Purvanchal 2007; Meerut 2010]

Sol. We have $$(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'z', $-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z)=\sum_{n=0}^{\infty} n z^{n-1} P_n$

or $$(x-z)(1-2xz+z^2)^{-3/2}=\sum_{n=0}^{\infty} n z^{n-1} P_n. \qquad ...(2)$$

Multiplying both sides of (2) by $2z$, $$2z(x-z)(1-2xz+z^2)^{-3/2}=2\sum_{n=0}^{\infty} n z^n P_n. \qquad ...(3)$$

Adding (1) and (3), $$\frac{1}{(1-2xz+z^2)^{1/2}}+\frac{2z(x-z)}{(1-2xz+z^2)^{3/2}}=\sum_{n=0}^{\infty} z^n P_n+\sum_{n=0}^{\infty} 2nz^n P_n$$

or $$\frac{1-2xz+z^2+2z(x-z)}{(1-2xz+z^2)^{3/2}}=\sum_{n=0}^{\infty}(2n+1)z^n P_n \quad \text{or} \quad \frac{1-z^2}{(1-2xz+z^2)^{3/2}}=\sum_{n=0}^{\infty}(2n+1)z^n P_n.$$

Ex. 7. *Prove that* $$\frac{1+z}{z\sqrt{(1-2xz+z^2)}}-\frac{1}{z}=\sum_{n=0}^{\infty}(P_n+P_{n+1})z^n$$

Sol. We have, $$(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

$\therefore$ L.H.S. of the required result $=(1/z)\times(1-2xz+z^2)^{-1/2}+(1-2xz+z^2)^{-1/2}-(1/z)$

$$=\frac{1}{z}\sum_{n=0}^{\infty} z^n P_n+\sum_{n=0}^{\infty} z^n P_n-\frac{1}{z}, \text{ by (1)} \qquad ...(2)$$

But $\sum_{n=0}^{\infty} z^n P_n=P_0+zP_1+z^2P_2+....+z^nP_n+z^{n+1}P_{n+1}+...=1+z(P_1+zP_2+...+z^nP_{n+1}+...)$, as $P_0=1$

Thus, $$\sum_{n=0}^{\infty} z^n P_n=1+z\sum_{n=0}^{\infty} z^n P_{n+1}. \qquad ...(3)$$

Using (3) in (2), the L.H.S. of the required result

$$=\frac{1}{z}\left[1+z\sum_{n=0}^{\infty} z^n P_{n+1}\right]+\sum_{n=0}^{\infty} z^n P_n-\frac{1}{z}=\sum_{n=0}^{\infty} z^n P_{n+1}+\sum_{n=0}^{\infty} z^n P_n=\sum_{n=0}^{\infty}(P_n+P_{n+1})z^n$$

= R.H.S. of the required result.

Ex. 8. *Prove that* $(1-2xz+z^2)^{-1/2}$ *is a solution of the equation* $z\frac{\partial^2(zv)}{\partial z^2}+\frac{\partial}{\partial x}\left[(1-x^2)\frac{dv}{dx}\right]=0.$

Sol. Let $$v=(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

$\therefore$ $$zv=z\sum_{n=0}^{\infty} z^n P_n=\sum_{n=0}^{\infty} z^{n+1} P_n.$$

9.9. Recurrence relations (formulae). To show that

I. $nP_n = (2n-1)\,xP_{n-1} - (n-1)P_{n-2}$, $n \geq 2$ **[Delhi Physics (H) 2000; Agra 2005; Bilaspur 1998, Purvanchal 2005, Kanpur 2012]**

or $(n+1)P_{n+1} = (2n+1)\,xP_n - nP_{n-1}$, $n \geq 1$ **[Rohilkhand 2006; Agra 2005, 06; Kanpur 2014; Lucknow 2011; Meerut 2012; Ravishankar 2004; Vikram 2004]**

or $$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x).$$ **[Utkar 2003; Agra 1998]**

II. $nP_n = xP'_n - P'_{n-1}$. **[Avadh 2007; Agra 2013; Purvanchal 2004; Kanpur 2011; Lucknow 2006; Vikram 2000; Bangalore 1995; Meerut 1996]**

III. $(2n+1)P_n = P'_{n+1} - P'_{n-1}$. **[Lucknow 2008]**

IV. $(n+1)P_n = P'_{n+1} - xP'_n$. or $P_n' - xP'_{n-1} = nP_{n-1}$. [Rohilkhand 2011; Kanpur 2011]

V. $(1-x^2)P_n' = n(P_{n-1} - xP_n)$ or $(x^2-1)P'_n = nxP_n - nP_{n-1}$. [Kanpur 2010; Rohilkhand 2010] [Agra 2011; Meerut 1997]

VI. $(1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$.

Proof I. From generating function, we have $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(1)

Differentiating both sides of (1) w.r.t. 'z', we get

$$-\tfrac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x). \quad ...(2)$$

Multiplying both sides by $1-2xz+z^2$, (2) gives

$$(x-z)(1-2xz+z^2)^{-1/2} = (1-2xz+z^2)\sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

or $$(x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} n z^{n-1} P_n(x), \text{ by (1)}$$

or $$x\sum_{n=0}^{\infty} z^n P_n - \sum_{n=0}^{\infty} z^{n+1} P_n(x) = \sum_{n=0}^{\infty} n z^{n-1} P_n - 2x\sum_{n=0}^{\infty} n z^n P_n + \sum_{n=0}^{\infty} n z^{n+1} P_n.$$

Equating coefficients of z^n from both sides, we get

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} \quad ...(3)$$

or $$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}.$$

Replacing n by $n-1$ in (3), we get $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$. ...(4)

Again (3) can be re-arranged to give another form $xP_n = \dfrac{n+1}{2n+1}P_{n+1} + \dfrac{n}{2n+1}P_{n-1}$.

II. We have $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(1)

Differentiating (1) w.r.t. 'z', we get $-\tfrac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n$

or $$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} n z^{n-1} P_n. \quad ...(2)$$

Again, differentating (1) w.r.t. 'x' and simplifying, we get

$z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'$. or $z(x-z)(1-2xz+z^2)^{-3/2} = (x-z)\sum_{n=0}^{\infty} z^n P_n'$

or $z\sum_{n=0}^{\infty} n z^{n-1} P_n = (x-z)\sum_{n=0}^{\infty} z^n P'_n$, by (2) or $\sum_{n=0}^{\infty} n z^n P_n = x\sum_{n=0}^{\infty} z^n P_n' - \sum_{n=0}^{\infty} z^{n+1} P_n'$.

Equating coefficient of z^n on both sides we get $nP_n = xP'_n - P'_{n-1}$.

Particular Case. Putting $n = 9$, we get $9P_9 = xP_9' - P_8'$

Thus, $xP_9' = P_8' + 9P_9$ [Meerut 2008]

III. From recurrence relation I, $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$.

Differentiating it w.r.t. 'x', $(2n+1)xP'_n + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$

or $$(2n+1)(nP_n + P'_{n-1}) + (2n+1)P_n = (n+1)P'_{n+1} - nP'_{n-1}$$

[$\because$ from recurrence II, $xP'_n = nP_n + P'_{n-1}$]

or $$(2n+1)(n+1)P_n = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

or $$(2n+1)P_n = P'_{n+1} - P'_{n-1}. \qquad ...(1)$$

Replacing n by $n-1$ in (1), we have $$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

or $$\frac{dP_n(x)}{dx} = \frac{dP_{n-2}(x)}{dx} + (2n-1)P_{n-1}(x) \qquad ...(2)$$

(1) and (2) are the required forms of the results.

IV. From recurrence relations II and III, we get

$$nP_n = xP'_n - P'_{n-1} \qquad ...(1)$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad ...(2)$$

Subtracting (1) from (2), $$(n+1)P_n = P'_{n+1} - xP'_n$$

V. From recurrence relations II and IV, we get

$$nP_n = xP'_n - P'_{n-1} \qquad ...(1)$$

and $$(n+1)P_n = P'_{n+1} - xP'_n. \qquad ...(2)$$

Replacing n by $n-1$ in (2), $$nP_{n-1} = P'_n - xP'_{n-1}. \qquad ...(3)$$

Multiplying both sides of (1) by x, $$xnP_n = x^2P'_n - xP'_{n-1}. \qquad ...(4)$$

Subtracting (4) from (3), we have

$$n(P_{n-1} - xP_n) = (1-x^2)P'_n \quad \text{or} \quad (x^2-1)P'_n = nxP_n - nP_{n-1}.$$

VI. From recurrence relations I and V, we have

$$(2x+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \qquad ...(1)$$

and $$(1-x^2)P'_n = n(P_{n-1} - xP_n). \qquad ...(2)$$

Re–writting (1), $$[(n+1)+n]xP_n = (n+1)P_{n+1} + nP_{n-1}$$

or $$(n+1)(xP_n - P_{n+1}) = n(P_{n-1} - xP_n). \qquad ...(3)$$

Now, from (2) and (3), $$(1-x^2)P'_n = (n+1)(xP_n - P_{n+1}).$$

Bessel's Equations and Bessel's functions :-

Derive the Laplace's Equation in cylindrical Co-ordinate.

Laplace's Equation be

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{--- (1)}$$

Let (x, y, z) be the Cartesian Co-ordinate of the point whose cylindrical co-ordinate be (r, θ, z).

Then we have $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$\therefore$ $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiating partially w.r.t. x and y we get

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{\cos\theta}{r}$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \quad \text{--- (2)}$$

Similarly,

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \quad \text{--- (3)}$$

Substituting ② and ③ in equation ① we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

This is the required Laplace's equation in cylindrical Co-ordinates.

Bessel's Equation:-

In cylindrical Co-ordinate system (r, θ, z) Laplace's equation is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{---(i)}$$

u is a function of r, θ, z

$$\therefore u(r,\theta,z) = R(r)\,\phi(\theta)\,Z(z)$$

Substituting in equation (i) we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\phi Z\frac{dR}{dr}\right) + \frac{1}{r^2}RZ\frac{d^2\phi}{d\theta^2} + R\phi\frac{d^2 Z}{dz^2} = 0.$$

$$\Rightarrow \frac{\phi Z}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2}RZ\frac{d^2\phi}{d\theta^2} + R\phi\frac{d^2 Z}{dz^2} = 0$$

$$\Rightarrow \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\phi}\frac{d^2\phi}{d\theta^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2 \text{ (Say)}$$

Left Side of the equation is a function of r and θ and right hand Side is a function of z.

$\therefore$ Both Side must be Constant.

Then $\frac{d^2 Z}{dz^2} - k^2 Z = 0$ also

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\phi}\frac{d^2\phi}{d\theta^2} = -k^2$$

$$\Rightarrow \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\phi}\frac{d^2\phi}{d\theta^2} = -k^2$$

$$\Rightarrow \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + k^2r^2 = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \gamma^2 \text{ (Say)}$$

Left hand Side of the above equation is a function of r and right hand side in a function of θ

$\therefore$ so the both side must be Constant.

Then we get $\frac{d^2\varphi}{d\theta^2} + \gamma^2\varphi = 0$ also

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\gamma^2}{\rho^2}\right)R = 0$$

Put $x = k\rho$ we get

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\gamma^2}{x^2}\right)R = 0$$

$\Rightarrow x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - \gamma^2)R = 0$. This equation is known as standard Bessel's equation.

Solution of Bessel's Equation:-

The Bessel's equation of order k is defined as $x^2y'' + xy' + (x^2 - k^2)y = 0$ —(1) where k is any constant.

Here we restrict our study for the real $k \geq 0$.

The normal form be

$$y'' + \frac{1}{x}y' + \left(1 - \frac{k^2}{x^2}\right)y = 0$$

$\Rightarrow y'' + P_1(x)y' + P_2(x)y = 0$ where $P_1(x) = \frac{1}{x}$

$P_2(x) = 1 - \frac{k^2}{x^2}$

Here at $x = 0$ both function are not analytic, so $x = 0$ is an singular point of the differential equation.

Consider the product $xP_1(x) = 1$

$x^2P_2(x) = x^2 - k^2$

Now both function are analytic at $x = 0$. So $x = 0$ is an regular singular point.

We seek the solution $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ $c_0 \neq 0$ —(2)

Differentiating w.r.t. x we get

$$y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \quad \text{—(3)}$$

$$y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} \quad \text{—(4)}$$

Substituting the value (2), (3), (4) in equation (1)

$$x^2\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + x\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$$
$$+ (x^2-k^2)\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r}$$
$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} - k^2\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(n+r)-k^2\right]c_n x^{n+r}$$
$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+r)^2-k^2\right]c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 0$$

$$\Rightarrow (r^2-k^2)c_0x^r + \left[(r+1)^2-k^2\right]c_1 x^{r+1}$$
$$+ \sum_{n=2}^{\infty}\left[\{(n+r)^2-k^2\}c_n + c_{n-2}\right]x^{n+r} = 0$$

Equating to zero the coefficient of lowest power of x i.e x^r

$$(r^2-k^2)c_0 = 0 \Rightarrow r = k, -k \quad [c_0 \neq 0]$$

Say $r_1 = k$ $\quad r_2 = -k$ [r_1 and r_2 are not differ by integer].

Equating to zero the coefficient of x^{r+1} we get $\left[(r+1)^2-k^2\right]c_1 = 0 \Rightarrow c_1 = 0$

$$[\because (r+1)^2-k^2 \neq 0].$$

Equating to zero the coefficient of x^{n+r} we get the recurrence relation.

$$c_n = -\frac{c_{n-2}}{(n+r)^2-k^2} \qquad n = 2, 3, \ldots$$

First we find the solution corresponding to $r = p = k$. Then the recurrence formula becomes

$$c_n = -\frac{c_{n-2}}{n^2 + 2nk}$$

Since $c_1 = 0$, then $c_3 = c_5 = c_7 = \cdots = c_{2n+1} = 0$.

For $n = 2$, $\quad c_2 = -\frac{c_0}{2(2+2k)} = -\frac{c_0}{4(k+1)}$

$n = 4$, $\quad c_4 = -\frac{c_2}{4(4+2k)} = \frac{c_0}{2!\,4^2 (k+2)(k+1)}$

$n = 6$; $\quad c_6 = -\frac{c_4}{6(6+2k)} = -\frac{c_0}{3!\,4^3 (k+3)(k+2)(k+1)}$

$\cdots$

$n = 2n$, $\quad c_{2n} = -\frac{c_n}{2n(2n+2k)} = (-1)^n \frac{c_0}{n!\,4^n (k+n)(k+n-1)\cdots(k+2)\cdot(k+1)}$

Thus one solution of the Bessel's equation is given by substituting the values of $c_0, c_1, \cdots$ in equation ②

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = x^r \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots \right.$$

$$= c_0 x^k \left[1 - \frac{1}{4(k+1)} x^2 + \frac{1}{2!\,4^2 (k+2)(k+1)} x^4 - \frac{1}{3!\,4^3 (k+3)(k+2)(k+1)} x^6 + \cdots \right]$$

$$\therefore y_1(x) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+k}}{n!\,4^n (k+n)(k+n-1)\cdots(k+1)}$$

A particular choice of arbitrary parameter c_0 result in a from of y_1 i.e. use in application of physical problem.

Let $c_0 = (2^k\, k!)^{-1}$

$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+k}}{n!\, 2^{2n+k} (k+n)!}$$

$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(k+n)!} \left(\frac{x}{2}\right)^{2n+k} = J_k(x).$$

$J_k(x)$ is called Bessel's function of 1st kind of order k and $J_k(x)$ is convergent for all x and $k \geq 0$.

The second linearly independent solution of $J_k(x)$ of the Bessel's equation may be obtained by replacing k by $-k$ [Note that here k is not an integer]

Then $y_2(x) = J_{-k}(x)$

$\therefore$ The general solution is $y = A\, J_k(x) + B\, J_{-k}(x)$ where A and B are arbitrary constant.

If $r_1 - r_2$ is differ by integer the one solution of the Bessel's equation be

$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(k+n)!} \left(\frac{x}{2}\right)^{2n+k} = J_k(x)$$

and the second solution be

$$y_2(x) = x^{-k} \sum_{n=0}^{\infty} C_n^{*} x^n + C\, J_k(x) \ln(x)$$

The constant C_n^{*} and C may be obtained by substituting in the Bessel's equation.

The solution of Bessle's equation called a Bessel's function of second kind of order k i.e. linearly independent of $J_k(x)$ is denoted by $Y_k(x)$ and defined by

$$Y_k(x) = \frac{2}{\pi}\left\{\left(\ln\frac{x}{2} + \gamma\right) J_k(x) - \frac{1}{2}\sum_{n=0}^{k-1} \frac{(k-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-k} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^{n+1}\left(\sum_{p=1}^{n} \frac{1}{p} + \sum_{p=1}^{n+k} \frac{1}{p}\right)\left[\frac{1}{n!\,(n+k)!}\left(\frac{x}{2}\right)^{2n+k}\right]\right\}$$

where $\gamma = \lim_{p\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p} - \ln p\right) = 0.5772157.$

Bessel's function of first kind of order n.

Bessel's function of first kind of order n is denoted by $J_n(x)$ and defined by

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{2r+n}$$ where

n is non-negative constant.

Note:

$\Gamma(m) = \infty$ when m is zero or negative integer.

Properties:-

(1)

i) When n is positive integers $J_{-n}(x) = (-1)^n J_n(x)$

ii) When n is any integers $J_{-n}(x) = (-1)^n J_n(x)$.

i)

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2r+n}$$

$$\therefore\ J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(r-n+1)} \left(\frac{x}{2}\right)^{2r-n}.$$

Since $n>0$, $\Gamma(r-(n-1))$ is infinite for $r = 0, 1, \ldots (n-1)$.

Then $$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(r-n+1)} \left(\frac{x}{2}\right)^{2r-n}$$

put $r-n = m$

$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \frac{1}{(m+n)!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

$$= (-1)^n J_n(x)$$

$$\therefore\ J_{-n}(x) = (-1)^n J_n(x).$$

ii) When $n<0$ Let $n=-p$ $p>0$, Then we have

$$J_{-p}(x) = (-1)^p J_p(x)$$

$$\Rightarrow J_p(x) = (-1)^{-p} J_{-p}(x)$$

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

Hence the required result holds for any integers.

- Note:-

When n is an integer $J_{-n}(x)$ is not independent of $J_n(x)$ because $J_{-n}(x)$ is an constant multiple of $J_n(x)$. Hence $y = AJ_n(x) + BJ_{-n}(x)$ is not the general solution of Bessel's equation when n is ~~not~~ is an integer. Of course, when n is not an integer, the most general solution of Bessel's equation is given by $y = AJ_n(x) + BJ_{-n}(x)$.

- Bessel's function of second kind of order n:-

Bessel's function of second kind of order n is denoted by $Y_n(x)$ and defined by

$$Y_n(x) = \frac{\cos n\pi \, J_n(x) - J_{-n}(x)}{\sin n\pi} \quad \text{when } n \text{ is not integer}$$

Since n is not integer $\sin n\pi \neq 0$. clearly $Y_n(x)$ is a linear combination of $J_n(x)$ and $J_{-n}(x)$. We know that $J_{-n}(x)$ and $J_n(x)$ are linearly independent solution if n is not integer. Hence $J_n(x)$ and $Y_n(x)$ are two linearly independent solution of Bessel's equation.

and

$$Y_n(x) = \lim_{\gamma \to n} \frac{J_\gamma(x)\cos\gamma\pi - J_{-\gamma}(x)}{\sin\gamma\pi} \quad \text{when } n \text{ is integer}$$

- Note:-

General solution of Bessel's equation when n is integer is $y = AJ_n(x) + BY_n(x)$.

(2)

i) Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

ii) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

iii) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$.

i) From definition of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2\cdot 2(n+1)} + \frac{x^4}{2\cdot 4\cdot 2^2 (n+1)(n+2)} - \cdots \right] \quad \text{— (A)}$$

Replace n by $-\frac{1}{2}$ we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma(1/2)} \left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

ii) Again putting $n = \frac{1}{2}$ in (A) we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)} \left[1 - \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots \right]$$

$$= \frac{1}{2^{1/2} x^{1/2} \frac{1}{2}\sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

By (i) and (ii) we get

$$[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}.$$

(3) Show that

i) $\frac{d}{dx} J_0(x) = -J_1(x)$

ii) $\frac{d}{dx}[x^k J_k(x)] = x^k J_{k-1}(x)$

iii) $\frac{d}{dx}[x^{-k} J_k(x)] = -x^{-k} J_{k+1}(x)$.

i) From (A) we get

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{1}{2!}\right)^2\left(\frac{x}{2}\right)^4 - \cdots + (-1)^{n+1}\left(\frac{1}{(n+1)!}\right)^2\left(\frac{x}{2}\right)^{2n+2} + \cdots$$

$$\frac{d}{dx} J_0(x) = -\left(\frac{x}{2}\right) + \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 - \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 + \cdots + (-1)^{n+1}\frac{1}{n!\,(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$$

$$= -\sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\,(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$$

$$= -J_1(x).$$

ii) we have

(2018)
(2015) $$x^K J_K(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{2n+k}\cdot n!\,(n+k)!}\, x^{2n+2k}$$

Differentiating both side w.r.t. x we get

$$\frac{d}{dx}\left(x^k J_k(x)\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n+2k)}{2^{2n+k}\cdot n!\,(n+k)!}\, x^{2n+2k-1}$$

$$= x^k \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\,(n+k-1)!}\left(\frac{x}{2}\right)^{2n+k-1}$$

$$= x^k J_{k-1}(x).$$

iii) we have

(2018) $$x^{-k} J_k(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{2n+k}\, n!\,(n+k)!}\, x^{2n}$$

Differentiating w.r.t. x we get

$$\frac{d}{dx}\left[x^{-k} J_k(x)\right] = \sum_{n=0}^{\infty}(-1)^n \frac{2n}{2^{2n+k}\cdot n!\cdot(n+k)!}\, x^{2n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{1}{(n-1)!\,(n+k)!}\,\frac{1}{2^{2n+k-1}}\cdot x^{2n-1}$$

$[\because (n-1)! = \infty \text{ if } n=0]$

put $n-1 = m$

$$= \sum_{m=0}^{\infty} (-1)^{m+1} \frac{1}{m!\,(m+k+1)!} \cdot \frac{x^{2m+1}}{2^{2m+k+1}}$$

$$= - x^{-k} \sum_{m=0}^{\infty} (-1)^{m} \frac{1}{m!\,(m+k+1)} \left(\frac{x}{2}\right)^{2m+(k+1)}$$

$$= - x^{-k} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{n!\,(n+k+1)!} \left(\frac{x}{2}\right)^{2n+(k+1)}$$

$$= - x^{-k} J_{k+1}(x).$$

(4) Show that (2013)

i) $J_{k-1}(x) - J_{k+1}(x) = 2 \frac{d}{dx} (J_k(x))$

ii) $J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x).$

we have

$$\frac{d}{dx}\left(x^k J_k(x)\right) = x^k J_{k-1}(x)$$

$$\Rightarrow x^k \frac{d}{dx} J_k(x) + k x^{k-1} J_k(x) = x^k J_{k-1}(x)$$

$$\Rightarrow \frac{d}{dx} J_k(x) = J_{k-1}(x) - \frac{k}{x} J_k(x) \quad \text{— (A)} \quad (2018)\ (2015)$$

Also

$$\frac{d}{dx}\left(x^{-k} J_k(x)\right) = -x^{-k} J_{k+1}(x)$$

$$\Rightarrow x^{-k} \frac{d}{dx} J_k(x) - k x^{-k-1} J_k(x) = -x^{-k} J_{k+1}(x)$$

$$\Rightarrow \frac{d}{dx} J_k(x) = \frac{k}{x} J_k(x) - J_{k+1}(x) \quad \text{— (B)} \quad (2018)$$

2015.

Adding (A) and (B) we get

$$J_{k-1}(x) - J_{k+1}(x) = 2 \frac{d}{dx} J_k(x).$$

substracting (B) from (A) we get

$$J_{k+1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x).$$

⑤ The orthogonality relation satisfied by the Bessel's equation.

The Bessel's function $J_p(x)$ satisfy the differential equation (Bessel's equation) be

$$x^2 \frac{d^2 J_p(x)}{dx^2} + x \frac{d J_p(x)}{dx} + (x^2 - p^2) J_p(x) = 0 \quad \text{--- ①}$$

This equation can be written in the form

$$x \frac{d}{dx}\left(x \frac{d J_p}{dx} \right) + (x^2 - p^2) J_p(x) = 0. \quad \text{--- ②}$$

Now define a new variable t such that $x = at$, where a is a constant which will take to be 0 of $J_p(x)$ (i.e $J_p(a) = 0$)

Let us define

$$u(t) = J_p(at)$$

$$\text{so } u(1) = 0.$$

putting $x = at$ in equation ② we get

$$t \frac{d}{dt}\left(t \frac{du}{dt}\right) + (a^2 t^2 - p^2) u = 0 \quad \text{--- ③}$$

We can also write down another equation similarly by taking $J_p(b) = 0$, and

$v(t) = J_p(bt)$; $v(1) = 0$.

and we get

$$t \frac{d}{dt}\left(t \frac{dv}{dt}\right) + (b^2 t^2 - p^2) v = 0. \quad \text{--- ④}$$

multiply ③ by v and ④ by u and subtracting and dividing by t we get

$$v \frac{d}{dt}\left(t \frac{du}{dt}\right) - u \frac{d}{dt}\left(t \frac{dv}{dt}\right) + (a^2 - b^2) tuv = 0$$

$$\Rightarrow \frac{d}{dt}\left(vt \frac{du}{dt} - ut \frac{dv}{dt}\right) + (a^2 - b^2) tuv = 0$$

Integrating both side w.r.t. t between 0 to 1 we get,

$$\left[vt\frac{du}{dt} - ut\frac{dv}{dt}\right]_0^1 + (a^2-b^2)\int_0^1 tuv\,dt = 0.$$

Then we have,

$$\int_0^1 tuv\,dt = 0 \quad [a \neq b]$$

$$\Rightarrow \int_0^1 t\, J_p(at)\, J_p(bt)\, dt = 0$$

This is the desired orthogonal equation.

Prove that if $a=b$ then $\int_0^1 t\, J_p^2(at)\, dt = \frac{1}{2} J_p'^2(a)$.

we have $\frac{d}{dt}\left[vt\frac{du}{dt} - ut\frac{dv}{dt}\right] + (a^2-b^2)tuv = 0$

Integrating, w.r.t. t from 0 to 1 we get

$$\left[vt\frac{du}{dt} - ut\frac{dv}{dt}\right]_0^1 + (a^2-b^2)\int_0^1 tuv\,dt = 0.$$

$$\Rightarrow (a^2-b^2)\int_0^1 t\,J_p(at)\,J_p(bt)\,dt + \Big[J_p(bt)\,t.\,a J_p'(at) - J_p(at).t.\,bJ_p'(bt)\Big]_0^1 = 0.$$

$$\Rightarrow (a^2-b^2)\int_0^1 t\,J_p(at)\,J_p(bt)\,dt + \Big[a J_p(b)\,J_p'(a) - b\,J_p(a).J_p'(b)\Big] = 0.$$

when $a=b$. then

$$\int_0^1 t\,J_p(at)\,J_p(at)\,dt = \lim_{b\to a}\left[-\frac{a J_p(b)\,J_p'(a) - b\,J_p(a)\,J_p'(b)}{(a^2-b^2)}\right]$$

$$\Rightarrow \int_0^1 t\,J_p^2(at)\,dt = \lim_{b\to a} \frac{a J_p'(a)\,J_p'(b)}{2b} \quad [\because J_p(a)=0]$$

$$= \frac{a J_p'(a)\,J_p'(a)}{2a}$$

$$\therefore \int_0^1 t\,J_p^2(at)\,dt = \frac{1}{2}\left[J_p'(a)\right]^2$$

⑥ $e^{x/2(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$ i.e. the function $e^{\frac{x}{2}(z-z^{-1})}$ is the generating function of the first kind Bessel's equation. $[G(a_n, x) = \sum_{n=0}^{\infty} a_n x^n]$

A generating function of another function is the function whose power series has a^n as coefficient of x^n i.e. the generating function of a_n is the function $G(a_n, x)$. where,

$$G(a_n, x) = \sum_{n=0}^{\infty} a_n x^n.$$

we know that $e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}$

now $e^{\left(\frac{x}{2}z - \frac{x}{2}\frac{1}{z}\right)} = e^{\frac{x}{2}z} \cdot e^{-\frac{x}{2}\cdot\frac{1}{z}}$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}z\right)^m}{m!} \cdot \sum_{k=0}^{\infty} \frac{\left(-\frac{x}{2}\frac{1}{z}\right)^k}{k!}$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m}{m!} z^m \cdot \sum_{k=0}^{\infty} \frac{\left(-\frac{x}{2}\right)^k}{k!} z^{-k}$$

[If two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ each have a radius of convergent $R>0$ then their product also be expressed in power series in the disc $|z|<R$ $(fg)(z) = \sum_{n=0}^{\infty} c_n z^n$ where

$$c_n = \sum_{k=0}^{\infty} a_k b_{n-k} \text{ (Cauchy product)}]$$

Apply this property we have

$$\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m}{m!} z^m \cdot \sum_{k=0}^{\infty} \frac{\left(-\frac{x}{2}\right)^k}{k!} z^{-k}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m-k=n \\ m,k\geq 0}} \frac{(-1)^k \left(\frac{x}{2}\right)^{m+k}}{m!\,k!} z^{m-k} \right)$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{k+n+k}}{(n+k)!\,k!} \right) z^n$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!\,k!} \left(\frac{x}{2}\right)^{2k} \left(\frac{x}{2}\right)^n \right) z^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, z^n.$$

Integral representation of Bessel's function:
(2017)

From the Complex analysis a recidue theorem state that if a function $f(z)$ define on the Complex plane has the Laurent series representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and if c is any positively oriented closed curve in a Complex plane Containing $f(0)$ then $\oint_c f(z)\,dz = 2\pi i\, a_{-1}$

Again we know that from property no. ⑤

$$e^{x/2(t - \frac{1}{t})} = \sum_{m=-\infty}^{\infty} J_m(x)\, t^m.$$

multipling both side by $\frac{1}{t^{n+1}}$

$$\frac{e^{\frac{x}{2}(t - \frac{1}{t})}}{t^{n+1}} = \sum_{m=-\infty}^{\infty} J_m(x)\, t^{m-n-1}$$

now integrating both side on the closed Contour c.

$$\oint_c \frac{e^{x/2(t - \frac{1}{t})}}{t^{n+1}}\,dt = \oint_c \sum_{m=-\infty}^{\infty} J_m(x)\, t^{m-n-1}\,dt$$

where c any positively oriented closed curve in the Complex plane Containing origin.

$$\oint_c \frac{e^{x/2(t - \frac{1}{t})}}{t^{n+1}}\,dt = \oint_c \sum_{m=-\infty}^{\infty} J_m(x)\, t^{m-n-1}\,dt = 2\pi i\, J_n(x).$$

[$m = n$]

$$\Rightarrow 2\pi i\, J_n(x) = \oint_c \frac{e^{x/2(t - 1/t)}}{t^{n+1}}\,dt$$

put $t = e^{i\theta}$, $dt = ie^{i\theta}\, d\theta$

$$2\pi i J_n(x) = i\int_0^{2\pi} \frac{e^{x/2\,(e^{i\theta} - e^{-i\theta})}}{e^{i(n+1)\theta}}\, e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} e^{(ix\sin\theta - in\theta)}\, d\theta$$

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)]\, d\theta$$

Equating both side the real or imaginary part we get

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta - n\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\, d\theta$$

(2017)

Due to symmetry

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(x\sin\theta - n\theta)\, d\theta = 0.$$

A particular case when $n=0$ then

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\, d\theta.$$

Problem:- (Extra)/(Bessel's function).

① prove that $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$

$$\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x).$$

we know that

$$e^{x/2(z - z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

$$= J_0 + (z - z^{-1})J_1 + (z^2 + z^{-2})J_2 + \cdots$$

put $z = e^{i\theta}$

$$e^{x/2(e^{i\theta} - e^{-i\theta})} = J_0 + (e^{i\theta} - e^{-i\theta})J_1 + (e^{i2\theta} + e^{-i2\theta})J_2 + (e^{i3\theta} - e^{-i3\theta})J_3 + \cdots$$

$\Rightarrow e^{ix\sin\theta} = J_0 + 2i\sin\theta\, J_1 + 2\cos 2\theta\, J_2 + 2i\sin 3\theta\, J_3 + \cdots$

$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = (J_0 + 2\cos 2\theta\, J_2 + 2\cos 4\theta \cdot J_4 + \cdots) + 2i(\sin\theta \cdot J_1 + \sin 3\theta \cdot J_3 + \sin 5\theta \cdot J_5 + \cdots)$

Equating real or imaginary part we get

$$\cos(x\sin\theta) = J_0 + 2\cos 2\theta \cdot J_2 + 2\cos 4\theta \cdot J_4 + \cdots$$

$$\sin(x\sin\theta) = 2(\sin\theta \cdot J_1 + \sin 3\theta \cdot J_3 + \sin 5\theta \cdot J_5 + \cdots).$$

put $\theta = \frac{\pi}{2} - \phi$ we get

$$\cos(x\cos\phi) = J_0 - 2\cos 2\phi \cdot J_2 + 2\cos 4\phi \cdot J_4 + \cdots$$

$$\sin(x\cos\phi) = 2(\cos\phi \cdot J_1 - \cos 3\phi \cdot J_3 + 2\cos 5\phi \cdot J_5 - \cdots)$$

put $\phi = 0$ we get

$$\cos x = J_0 - 2J_2 + 2J_4 - \cdots = J_0 + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(x) \quad \text{--- (1)}$$

$$\sin x = 2(J_1 - J_3 + J_5 - \cdots) = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x) \quad \text{--- (2)}$$

Also put $\phi = \pi/2$ we get

$$\cos(x\cos\pi/2) = J_0 + 2J_2 + 2J_4 + \cdots$$

$$\Rightarrow 1 = J_0 + 2\sum_{n=1}^{\infty} J_{2n}(x).$$

(2) Prove that $\frac{d}{dx}(x^n J_n(ax)) = a x^n J_{n-1}(ax)$ and hence deduce that $\frac{d}{dx}(x J_1(x)) = x J_0(x)$.

Similarly properties (6).

(3) Show that $\int_0^1 \frac{u J_0(xu)}{(1-u^2)^{1/2}}\, du = \frac{\sin x}{x}$.

we know that

$$J_k(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+k)!} \left(\frac{x}{2}\right)^{2n+k}$$

$$= \frac{x^k}{2^k \cdot k!}\left[1 - \frac{x^2}{2\cdot 2(k+1)} + \frac{x^4}{2\cdot 4\cdot 2^2 (k+1)(k+2)} - \cdots\right]$$

$\therefore J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$

$$\therefore \int_0^1 \frac{u\, J_0(xu)}{(1-u^2)^{1/2}}\, du = \int_0^1 \frac{u}{(1-u^2)^{1/2}} \left[1 - \frac{x^2u^2}{4} + \frac{x^4u^4}{64} - \frac{x^6u^6}{2304} \cdots \right] du$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \left[1 - \frac{x^2}{4}\sin^2\theta + \frac{x^4}{64}\sin^4\theta - \frac{x^2}{2304}\sin^6\theta + \cdots\right] \cos\theta\, d\theta.$$

put $u = \sin\theta$
i.e. $du = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \left[\sin\theta - x^2/4 \sin^3\theta + x^2/64 \sin^5\theta - \cdots\right] d\theta$$

$$= \left[-\cos\theta\right]_0^{\pi/2} - \frac{x^2}{4}\cdot\frac{2}{3} + \frac{x^2}{64}\cdot\frac{4}{5}\cdot\frac{2}{3} - \cdots$$ [using Wallis's Formula of integral Calculus]

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$= \frac{1}{x}\left(x - \frac{x^3}{3} + \frac{x^5}{5!} - \cdots\right) = \frac{\sin x}{x}.$$

Note that: (Wallis's formula of integral Calculus).

If n is even, $\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x\, dx = \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots n}\cdot\frac{\pi}{2}$

If n is odd, $\int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \sin^n x\, dx = \frac{2\cdot 4\cdot 6 \cdots (n-1)}{1\cdot 3\cdot 5 \cdots n}.$

(4) Show that $\int_0^{\pi/2} J_1(z\cos\theta)\, d\theta = \frac{1-\cos z}{z}.$

We know that $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(r+n)!}\left(\frac{x}{2}\right)^{2r+n}.$

$$\therefore J_1(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(r+1)!}\left(\frac{x}{2}\right)^{2r+1}.$$

So $J_1(z\cos\theta) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(r+1)!}\left(\frac{z\cos\theta}{2}\right)^{2r+1}.$

$$\therefore \int_0^{\pi/2} J_1(z\cos\theta)\, d\theta = \sum_{r=0}^{\infty} (-1)^r \frac{z^{2r+1}}{r!\,(r+1)!}\cdot\frac{1}{2^{2r+1}} \int_0^{\pi/2} \cos^{2r+1}\theta\, d\theta$$

$$= \frac{z}{2}\int_0^{\pi/2}\cos\theta\, d\theta - \frac{1}{2!}\left(\frac{z}{2}\right)^3\int_0^{\pi/2}\cos^3\theta\, d\theta +$$

$$\frac{1}{2!\cdot 3!}\left(\frac{z}{2}\right)^5\int_0^{\pi/2}\cos^5\theta\, d\theta - \cdots$$

$$= \frac{z}{2} - \frac{1}{2!}\cdot\frac{z^3}{2^2}\cdot\frac{2}{3} + \frac{1}{2!\cdot 3!}\cdot\frac{z^5}{2^5}\cdot\frac{2}{3}\cdot\frac{4}{5} - \cdots$$

$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$$ [using walli's ~~~~ formula]

$$= \frac{1}{z} - \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z}\cos\theta = \frac{1-\cos z}{z}.$$

(5) Prove that $J_n(x) = \frac{x^n}{2^{n-1}\Gamma(n)}\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\, J_0(x\sin\theta)\, d\theta$, where $n > -\frac{1}{2}$.

We know that

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{2r+n}$$

$$\therefore J_0(x\sin\theta) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(r+1)}\left(\frac{x\sin\theta}{2}\right)^{2r}$$

Now $$\frac{x^n}{2^{n-1}\Gamma(n)}\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\, J_0(x\sin\theta)\,d\theta$$

$$= \frac{x^n}{2^{n-1}\Gamma(n)}\cdot\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\cdot\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(r+1)}\left(\frac{x\sin\theta}{2}\right)^{2r}d\theta$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{x^n\cdot x^{2r}}{2^{n-1}\Gamma(n)\, r!\,\Gamma(r+1)}\cdot\frac{1}{2^{2r}}\int_0^{\pi/2}\sin^{2r+1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{x^n\cdot x^{2r}}{2^{2r+n}\cdot\Gamma(n)(r!)^2}\cdot\frac{\Gamma(n)\cdot\Gamma(r+1)}{\Gamma(n+r+1)}.$$

$$= \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r+n}\cdot\frac{1}{(r!)^2}\cdot\frac{r!}{(n+r)!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{2r+n} = J_n(x)$$

$$\therefore J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(n)} \int_0^{\pi/2} \sin\theta \cos^{2n-1}\theta \, J_0(x\sin\theta)\,d\theta.$$

⑥ Prove that if $n > m-1$, then $J_n(x) = \frac{2(x/2)^{n-m}}{\Gamma(n-m)} \cdot \int_0^1 (1-t^2)^{n-m-1} t^{m+1} J_m(xt)\,dt.$

⑦ Prove that $J_n(x) = \frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+1)} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} e^{ixt}\,dt$ if $n \geq -\frac{1}{2}$.

⑧ Prove that $J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n-1)} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt\,dt$

[Hints $e^{ixt} = \cos xt + i\sin xt$]

⑨ Obtain the differential equation satisfied by $y = \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}$ where $J_n(x)$ is a Bessel function of order n. Hence find the series form of y. (2010)