PARTIAL DIFFERENTIAL EQUATIONS-I

Classification of second order partial differential equations. Three Fundamental equations: Laplace, Wave and Diffusion. Their solution.

REFERENCE BOOKS

1. Introduction to Partial Differential Equations – K. Sankara Rao (Prentice Hall of India, Private Limited)
2. Differential Equations – S. L. Ross (John Wiley & Sons.)

✓ PARTIAL DERIVATIVES: Basic Concepts

Let $u(x, y)$ be a function of the independent variables x and y. The **first derivative of u with respect to x** is defined by

$$u_x(x, y) = \frac{\partial u}{\partial x}(x, y) = \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h}$$

provided that the limit exists.

Likewise, the **first derivative of u with respect to y** is defined by

$$u_y(x, y) = \frac{\partial u}{\partial y}(x, y) = \lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{h}$$

provided that the limit exists.

We can define higher order derivatives such as

$$u_{xx}(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) = \lim_{h \to 0} \frac{u_x(x+h, y) - u_x(x, y)}{h}$$

$$u_{yy}(x, y) = \frac{\partial^2 u}{\partial y^2}(x, y) = \lim_{h \to 0} \frac{u_y(x, y+h) - u_y(x, y)}{h}$$

$$u_{xy}(x, y) = \frac{\partial^2 u}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \lim_{h \to 0} \frac{u_y(x+h, y) - u_y(x, y)}{h}$$

and

$$u_{yx}(x, y) = \frac{\partial^2 u}{\partial y \partial x}(x, y) = \lim_{h \to 0} \frac{u_x(x, y+h) - u_x(x, y)}{h}$$

provided that the limits exist.[1]

[1] If u_{xy} and u_{yx} are continuous then $u_{xy}(x, y) = u_{yx}(x, y)$.

✓ PARTIAL DERIVATIVES: Basic Concepts

An important formula of differentiation is the so-called **chain rule.** If $u = u(x, y)$, where $x = x(s,t)$ and $y = y(s,t)$, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

✓ Partial Differential Equation (PDE)

A partial differential equation (PDE) is an equation for some quantity u (dependent variable) which depends on the independent variables $x_1, x_2, x_3, \ldots, x_n$, $n \geq 2$, and involves derivatives of u with respect to at least some of the independent variables.

$$F(x_1, \ldots, x_n, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial^2_{x_1} u, \partial^2_{x_1 x_2} u, \ldots, \partial^n_{x_1 \ldots x_n} u) = 0.$$

1. In applications x_i are often space variables (e.g. x, y, z) and a solution may be required in some region Ω of space. In this case there will be some conditions to be satisfied on the boundary $\partial\Omega$; these are called boundary conditions (BCs).

2. Also in applications, one of the independent variables can be time (t say), then there will be some initial conditions (ICs) to be satisfied (i.e., u is given at $t = 0$ everywhere in Ω)

- The order of PDE is the order of the highest order partial derivative

✓ 1st Order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

✓ 2nd Order PDE

$$u_{xx} + u_{yy} + u_{zz} = 0 \ (Laplace\ \ Equation)$$

$$u_t = \alpha\left(u_{xx} + u_{yy} + u_{zz}\right)(Heat\ \ Equation)$$

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right)(Wave\ Equation)$$

✓ Formation of PDEs

- PDE can be formed either by elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

✓ Examples

1. Eliminate two arbitrary constants a and b from $(x-a)^2+(y-b)^2+z^2=R^2$, R is known constant

Solution: $(x-a)^2+(y-b)^2+z^2=R^2 \ldots\ldots(1)$

Differentiating both sides with respect to x and y

$$\left.\begin{aligned} &2z\frac{\partial z}{\partial x}=-2(x-a), \quad 2z\frac{\partial z}{\partial y}=-2(y-b) \\ &\frac{\partial z}{\partial x}=p, \quad \frac{\partial z}{\partial y}=q \end{aligned}\right\}$$

$$\Rightarrow \quad x-a=-pz, \quad y-b=-qz$$

By substituting all these values in (1)

$$p^2 z^2 + q^2 z^2 + z^2 = R^2 \Rightarrow z^2 = \frac{R^2}{p^2 + q^2 + 1}$$

$$\Rightarrow \quad z^2 = \frac{R^2}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

2. Find the PDE by eliminating arbitrary functions from $z = f(x^2 - y^2)$

Solution: $z = f(x^2 - y^2)$.........(1)

Partial derivative w.r.to x and y gives

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2) \times 2x \text{......}(2)$$

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2) \times -2y \text{......}(3)$$

using $\frac{(2)}{(3)}$

$$\frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} = \frac{-x}{y} \Rightarrow \frac{p}{q} = \frac{-x}{y} \Rightarrow py + qx = 0$$

3. Find PDE by eliminating two arbitrary functions from $z = yf(x) + xg(y)$

Solution: $$z = yf(x) + xg(y)......(1)$$

Differentiating both sides with respect to x and y

$$\frac{\partial z}{\partial x} = yf'(x) + g(y)........(2)$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y)........(3)$$

Again differentiating equations (2) ((3)) w .r. to y (x)

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

Finally $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xg(y) + yf(x) + xy(f'(x) + g'(y)) = z + xy\left(f' + g'\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

✓ LINEAR PDE

- A PDE is linear if the PDE and any boundary or initial conditions do not include any product of the dependent variables or their derivatives

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Phi(x, y), \qquad 2u_{xx} + u_{xt} + 3u_{tt} + 4u_x + \cos(2t) = 0,$$

$$2u_{xx} - 3u_t + 4u_x = 0$$

✓ NONLINEAR PDE

- A PDE which is not linear, is called nonlinear PDE

$$2u_{xx} + (u_{xt})^2 + 3u_{tt} = 0, \quad \sqrt{u_{xx}} + 2u_{xt} + 3u_t = 0, \quad u_t + uu_x + u_{xxx} = 0,$$ (Korteweg de Vries Equation)

$$2u_{xx} + 2u_{xt}u_t + 3u_t = 0 \qquad u_t + uu_x = u_{xx}$$ (Burgers' Equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \qquad \frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = 0$$

✓ A nonlinear PDE is semilinear if the coefficients of the highest derivative are functions of the independent variables only

$$(x+3)\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} = u^3, \quad x\frac{\partial^2 u}{\partial x^2} + (xy+y^2)\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = u^4 \quad y\frac{\partial u}{\partial x} + (x^3+y)\frac{\partial u}{\partial y} = u^3$$

✓ A nonlinear PDE of order m is quasilinear if it is linear in the derivatives of order m with coefficients depending only on x, y,…. and derivatives order <m

$$\left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial y^2} = 0 \qquad x^2u\frac{\partial u}{\partial x} + (y+u)\frac{\partial u}{\partial y} = u^3$$

✓ Superposition Principle of linear PDE

Principle of superposition: A linear equation has the useful property that if u_1 and u_2 both satisfy the equation then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in \mathbb{R}$. This is often used in constructing solutions to linear equations (for example, so as to satisfy boundary or initial conditions; c.f. Fourier series methods). This is not true for nonlinear equations, which helps to make this sort of equations more interesting, but much more difficult to deal with.

✓ Homogeneous Linear PDE

- Equations in which partial derivatives occurring are all of same order (with degree one) are called homogeneous linear PDE.

✓ 2nd Order Linear PDE in two variables

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y, but do not depend on the unknown function u. The classification of second order equations depends on the form of the leading part of the equations consisting of the second order terms. So, for simplicity of notation, we combine the lower order terms and rewrite the above equation in the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0. \qquad (1)$$

Facts:

- The expression $Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}$ is called the Principal part of the equation.
- Classification of such PDEs is based on this principal part.

✓ Classifications of 2nd order Linear PDE

As we will see, the type of the above equation depends on the sign of the quantity

$$\Delta(x,y) = B^2(x,y) - 4A(x,y)C(x,y), \longrightarrow \textit{discriminant} \tag{2}$$

At the point (x_0, y_0) the second order linear PDE (1) is called

i) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

ii) *elliptic*, if $\Delta(x_0, y_0) < 0$

Notice that in general a second order equation may be of one type at a specific point, and of another type at some other point.

- Each category relates to specific problems

1. Laplace's Equation

$$u_{xx} + u_{yy} = 0.$$

2. Wave Equation

$$u_{tt} - u_{xx} = 0.$$

3. Heat Equation

$$u_t = u_{xx}.$$

✓ Examples

1. (Laplace Equation)

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0;\ \ A=1, B=0, C=1 \Rightarrow B^2-4AC<0 \Rightarrow \text{Laplace Equation } \textit{is Elliptic}$$

2. (1D Heat Equation)

$$\alpha\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial u(x,t)}{\partial t}=0;\ A=\alpha,\ \ B=0,\ \ C=0 \Rightarrow B^2-4AC=0 \ \Rightarrow \text{Heat Equation } \textit{is Parabolic}$$

3. (1D Wave Equation)

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^2 u(x,t)}{\partial t^2}=0;\ A=c^2>0,\ \ B=0,\ \ C=-1 \Rightarrow B^2-4AC>0 \ \Rightarrow \text{Wave Equation } \textit{is Hyperbolic}$$

4. Determine the regions in the xy plane where the following equation is hyperbolic, parabolic, or elliptic.

$$u_{xx}+yu_{yy}+\frac{1}{2}u_y=0.$$

The coefficients of the leading terms in this equation are

$$A=1, B=0, C=y.$$

The discriminant is then $\Delta=B^2-4AC=-4y$. Hence the equation is hyperbolic when $y<0$, parabolic when $y=0$, and elliptic when $y>0$. □

Problem Set - I

1. What is the type of each of the following PDEs?

   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

2. Find the regions in the xy plane where the PDE

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

   is elliptic, hyperbolic, or parabolic. Sketch them.

3. What is the type of the PDE

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

4. Classify each of the following as hyperbolic, parabolic or elliptic at every point (x, y) of domain

   a. $x\,u_{xx} + u_{yy} = x^2$

   b. $x^2\,u_{xx} - 2xy\,u_{xy} + y^2u_{yy} = e^x$

   c. $e^xu_{xx} + e^yu_{yy} = u$

   d. $u_{xx} + u_{xy} - xu_{yy} = 0$ in the left half plane $(x \leq 0)$

   e. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + xyu_x + y^2u_y = 0$

   f. $u_{xx} + xu_{yy} = 0$ (Tricomi equation)

Problem Set - I

5. Classify each of the following constant coefficient equations

a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
c. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

6. For each of the differential equations in Exercises 1–10 find a solution which contains two arbitrary functions. In each case determine whether the equation is hyperbolic, parabolic, or elliptic.

1. $\dfrac{\partial^2 u}{\partial x^2} - 7\dfrac{\partial^2 u}{\partial x\,\partial y} + 6\dfrac{\partial^2 u}{\partial y^2} = 0.$

2. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} - 6\dfrac{\partial^2 u}{\partial y^2} = 0.$

3. $\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x\,\partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0.$

4. $4\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

6. $2\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 0.$

7. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} = 0.$

8. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial^2 u}{\partial y^2} = 0.$

9. $8\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0.$

10. $2\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial^2 u}{\partial y^2} = 0.$

Problem Set - I

7. Consider the equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0.$$

where A, B, C, D, E, and F are functions of x and y. Extending the definition of the text, we say that this equation is

(i) hyperbolic at all points at which $B^2 - 4AC > 0$;

(ii) parabolic at all points at which $B^2 - 4AC = 0$; and

(iii) elliptic at all points at which $B^2 - 4AC < 0$.

(a) Show that the equation

$$(x^2 - 1)\frac{\partial^2 u}{\partial x^2} + 2y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic for all (x, y) outside the region bounded by the circle $x^2 + y^2 = 1$, parabolic on the boundary of this region, and elliptic for all (x, y) inside this region.

(b) Determine all points (x, y) at which the equation

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} - xy\frac{\partial u}{\partial x} = 0$$

(i) is hyperbolic;
(ii) is parabolic;
(iii) is elliptic.

Methods and Techniques for Solving PDEs

- Change of coordinates: A PDE can be changed to an ODE or to an easier PDE by changing the coordinates of the problem.
- Separation of variables: A PDE in n independent variables is reduced to n ODEs.
- Integral transforms: A PDE in n independent variables is reduced to one in $(n-1)$ independent variables. Hence, a PDE in two variables could be changed to an ODE.
- Numerical Methods

✓ Canonical / Standard form of 2nd Order Linear PDE

Change of coordinates: Canonical Transformations

Define the new variables as

$$\begin{cases} \xi = \xi(x,y), \\ \eta = \eta(x,y), \end{cases} \qquad \text{with} \qquad J = \det\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0. \tag{3}$$

We then use the chain rule to compute the terms

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}.$$

The equation becomes [(1)]

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + I^*(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{4}$$

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y, \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned} \tag{5}$$

✓ Canonical / Standard form of 2nd Order Linear PDE

One can form the discriminant for the equation in the new variables via the new coefficients

$$\Delta^* = (B^*)^2 - 4A^*C^* \longrightarrow \Delta^* = J^2\Delta \qquad (6)$$

- Thus the type of the PDE is invariant under nondegenerate co-ordinate transformations since the signs of Δ *and* Δ^* coincide (same)

Type	Canonical form
Hyperbolic	$\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots = 0$
Parabolic	$\frac{\partial^2 u}{\partial \eta^2} + \ldots = 0$
Elliptic	$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + \ldots = 0$

$$\alpha = \xi + \eta$$
$$\beta = i(\xi - \eta)$$

✓ Characteristics of 2nd Order Linear PDE

Notice that the expressions for A^* and C^* in (5) have the same form, with the only difference being in that the first equation contains the variable ξ, while the second one has η. Due to this, we can try to chose a transformation, which will make both A^* and C^* vanish.

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0. \qquad (7)$$

We use ζ $(zeta)$ in place of both ξ and η.

- The solutions of equation (7) are called characteristics / characteristic curves of PDE (1).

We divide both sides of the above equation by ζ_y^2 to get

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \qquad (8)$$

Without loss of generality we can assume that $A \neq 0$. Indeed, if $A = 0$, but $C \neq 0$, one can proceed in a similar way, by considering the ratio ζ_y/ζ_x instead of ζ_x/ζ_y. Otherwise, if both $A = 0$, and $C = 0$, then the equation is already in the reduced form, and there is nothing to do.

Along such curves $\zeta(x, y) = \text{const}$

$$d\zeta = \zeta_x dx + \zeta_y dy = 0.$$

✓ Slope of characteristic curves: $\dfrac{dy}{dx} = -\dfrac{\zeta_x}{\zeta_y}$ $\Longrightarrow$ $A\left(\dfrac{dy}{dx}\right)^2 - B\left(\dfrac{dy}{dx}\right) + C = 0.$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \qquad (9)$$

Canonical Transformations: Hyperbolic PDE

✓ Hyperbolic PDE $\Delta > 0$

we can apply the change of variable $(x, y) \to (\eta, \xi)$ to transform the original PDE to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + (\text{lower order terms}) = 0. \tag{10}$$

Characteristics: From (9), we get two distinct families of characteristics

$$y = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x + c, \qquad \text{or} \qquad \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x - y = c.$$

These equations give the following change of variables $\begin{cases} \xi = \frac{B+\sqrt{B^2-4AC}}{2A} x - y \\ \eta = \frac{B-\sqrt{B^2-4AC}}{2A} x - y \end{cases}$

Then the equation (1) reduces to the form (10), the first canonical form of the Hyperbolic equation.

Under the orthogonal transformations $\begin{cases} x' = \xi + \eta \\ y' = \xi - \eta \end{cases}$

The equation (10) reduces to the second canonical form of the Hyperbolic equation;

$$u_{x'x'} - u_{y'y'} + \cdots = 0$$

the same form in its leading terms as the wave equation (with $c = 1$).

Canonical Transformations: Parabolic PDE

✓ Parabolic PDE $\Delta = B^2 - 4AC = 0$,

we can apply the change of variable $(x, y) \to (\eta, \xi)$ to transform the original PDE to

$$\frac{\partial^2 u}{\partial \eta^2} + (\text{lower order terms}) = 0. \qquad (11)$$

Characteristics: From (9), we get only one family of characteristics

$$\frac{B}{2A}x - y = c.$$

Let us choose ξ to be the unique solution of (9). Then, we can choose η arbitrarily as long as the change of co-ordinates formulas (3) define a non-degenerate transformations (i.e. ξ and η are independent)

change of variables $\begin{cases} \xi = \frac{B}{2A}x - y \\ \eta = x \end{cases}$

The Jacobian determinant of this transformation is

$$J = \begin{vmatrix} B/(2A) & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0. \quad \text{(non-degenerate)}$$

Then the equation (1) reduces to the form (11), the canonical form of the Parabolic equation.

Notice that this equation has the same leading terms as the heat equation $u_{xx} - u_t = 0$.

Canonical Transformations: Elliptic PDE

✓ Hyperbolic PDE $\Delta = B^2 - 4AC < 0$,

we can apply the change of variable $(x, y) \rightarrow (\eta, \xi)$ to transform the original PDE to

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + \text{(lower order terms)} = 0. \tag{12}$$

Characteristics: From (9), we get pair of complex conjugate solutions. The PDE has no real characteristics.

change of variables

$$\begin{cases} \xi = \frac{B+\sqrt{B^2-4AC}}{2A} x - y \\ \eta = \frac{B-\sqrt{B^2-4AC}}{2A} x - y \end{cases}$$

To keep the transformation real, we apply a further change of variables $(\xi, \eta) \rightarrow (\alpha, \beta)$

$$\begin{aligned} \alpha &= \xi + \eta \\ \beta &= i(\xi - \eta) \end{aligned} \quad \Longrightarrow \quad \begin{cases} \alpha = \frac{B}{2A} x - y \\ \beta = \frac{\sqrt{B^2-4AC}}{2A} x \end{cases} \quad \Longrightarrow \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2}$$

Then the equation (1) reduces to the form (12), the canonical form of the Elliptic equation.

✓ Examples

1. Reduce to the canonical form

$$y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{xy}\left(y^3\frac{\partial u}{\partial x}+x^3\frac{\partial u}{\partial y}\right).$$

$$A=y^2,\ B=-2xy,\ C=x^2 \Rightarrow B^2-4AC=4\left(x^2y^2-x^2y^2\right)=0 \Rightarrow Parabolic\quad Equation$$

Characteristics: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{x}{y}$ On $\xi=$ constant, $\xi=x^2+y^2$

We can choose η arbitrarily provided ξ and η are independent. We choose $\eta=y$. (Exercise, try it with $\eta=x$.) Then

$$\frac{\partial u}{\partial x}=2x\frac{\partial u}{\partial \xi},\qquad \frac{\partial u}{\partial y}=2y\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta},\qquad \frac{\partial^2 u}{\partial x^2}=2\frac{\partial u}{\partial \xi}+4x^2\frac{\partial^2 u}{\partial \xi^2},$$

$$\frac{\partial^2 u}{\partial x\partial y}=4xy\frac{\partial^2 u}{\partial \xi^2}+2x\frac{\partial^2 u}{\partial \xi\partial \eta},\qquad \frac{\partial^2 u}{\partial y^2}=2\frac{\partial u}{\partial \xi}+4y^2\frac{\partial^2 u}{\partial \xi^2}+4y\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2},$$

and the equation becomes

$$2y^2\frac{\partial u}{\partial \xi}+4x^2y^2\frac{\partial^2 u}{\partial \xi^2}-8x^2y^2\frac{\partial^2 u}{\partial \xi^2}-4x^2y\frac{\partial^2 u}{\partial \xi\partial \eta}+2x^2\frac{\partial u}{\partial \xi}+4x^2y^2\frac{\partial^2 u}{\partial \xi^2}$$
$$+4x^2y\frac{\partial^2 u}{\partial \xi\partial \eta}+x^2\frac{\partial^2 u}{\partial \eta^2}=\frac{1}{xy}\left(2xy^3\frac{\partial u}{\partial \xi}+2x^3y\frac{\partial u}{\partial \xi}+x^3\frac{\partial u}{\partial \eta}\right)$$

i.e. $\dfrac{\partial^2 u}{\partial \eta^2}-\dfrac{1}{\eta}\dfrac{\partial u}{\partial \eta}=0.$ (canonical form)

This has solution

$$u=f(\xi)+\eta^2\,g(\xi),\qquad\qquad u=f(x^2+y^2)+y^2\,g(x^2+y^2)$$

where f and g are arbitrary functions (via integrating factor method),

We need to impose two conditions on u or its partial derivatives to determine the functions f and g i.e. to find a particular solution.

2. Reduce to canonical form and then solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}+1=0 \quad \text{in } 0\le x\le 1, y>0, \text{ with } u=\frac{\partial u}{\partial y}=x \text{ on } y=0.$$

$$A=1,\ B=1,\ C=-2 \Rightarrow B^2-4AC=1+8=9>0 \Rightarrow \textit{Hyperbolic}\quad \textit{Equation}$$

Characteristics:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\pm\frac{3}{2}=-1 \text{ or } 2 \qquad \left(=-\frac{\partial\xi/\partial x}{\partial\xi/\partial y} \text{ or } -\frac{\partial\eta/\partial x}{\partial\eta/\partial y}\right).$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\Rightarrow x-\frac{1}{2}y=\text{constant} \quad\text{and}\quad \frac{\mathrm{d}y}{\mathrm{d}x}=-1\Rightarrow x+y=\text{constant}. \quad\Rightarrow \left.\begin{array}{l}\xi=x-\frac{y}{2}\\ \eta=x+y\end{array}\right\}\Leftrightarrow \left.\begin{array}{l}x=\frac{1}{3}(\eta+2\xi)\\ y=\frac{2}{3}(\eta-\xi)\end{array}\right\}.$$

So,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta},\qquad \frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta},\qquad \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial u}{\partial \xi\partial\eta}+\frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{2}\frac{\partial^2 u}{\partial \xi\partial\eta}+\frac{\partial^2 u}{\partial \eta^2},\qquad \frac{\partial^2 u}{\partial y^2}=\frac{1}{4}\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \xi\partial\eta}+\frac{\partial^2 u}{\partial \eta^2},$$

and the equation becomes

(canonical form)

$$\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial u}{\partial \xi\partial\eta}+\frac{\partial^2 u}{\partial \eta^2}-\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{2}\frac{\partial^2 u}{\partial \xi\partial\eta}+\frac{\partial^2 u}{\partial \eta^2}-\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}+2\frac{\partial^2 u}{\partial \xi\partial\eta}-2\frac{\partial^2 u}{\partial \eta^2}+1=0 \Rightarrow \frac{9}{2}\frac{\partial^2 u}{\partial \xi\partial\eta}+1=0.$$

So $\partial^2 u/\partial\xi\partial\eta=-2/9$ and general solution $\Rightarrow u(\xi,\eta)=-\frac{2}{9}\xi\,\eta+f(\xi)+g(\eta)$,

where f and g are arbitrary functions; now, we need to apply two conditions to determine these functions.

When, $y=0$, $\xi=\eta=x$ so the condition $u=x$ at $y=0$ gives

$$u(\xi=x,\eta=x)=-\frac{2}{9}x^2+f(x)+g(x)=x\Longleftrightarrow f(x)+g(x)=x+\frac{2}{9}x^2. \qquad (2.1)$$

Also, using the relation

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = \frac{1}{9}\eta - \frac{1}{2}f'(\xi) - \frac{2}{9}\xi + g'(\eta),$$

the condition $\partial u/\partial y = x$ at $y = 0$ gives

$$\frac{\partial u}{\partial y}(\xi = x, \eta = x) = \frac{1}{9}x - \frac{1}{2}f'(x) - \frac{2}{9}x + g'(x) = x \Longleftrightarrow g'(x) - \frac{1}{2}f'(x) = \frac{10}{9}x,$$

and after integration, $g(x) - \frac{1}{2}f(x) = \frac{5}{9}x^2 + k$, where k is a constant. (2.2)

Solving (2.1) and (2.2)

$$f(x) = \frac{2}{3}x - \frac{2}{9}x^2 - \frac{2}{3}k \quad \text{and} \quad g(x) = \frac{1}{3}x + \frac{4}{9}x^2 + \frac{2}{3}k,$$

or, in terms of ξ and η $f(\xi) = \frac{2}{3}\xi - \frac{2}{9}\xi^2 - \frac{2}{3}k$ and $g(\eta) = \frac{1}{3}\eta + \frac{4}{9}\eta^2 + \frac{2}{3}k.$

So, full solution is

$$\begin{aligned} u(\xi,\eta) &= -\frac{2}{9}\xi\eta + \frac{2}{3}\xi - \frac{2}{9}\xi^2 + \frac{1}{3}\eta + \frac{4}{9}\eta^2, \\ &= \frac{1}{3}(2\xi + \eta) + \frac{2}{9}(\eta - \xi)(2\eta + \xi). \end{aligned}$$

$$\Rightarrow u(x,y) = x + xy + \frac{y^2}{2}. \qquad \text{(check this solution.)}$$

3. Reduce to canonical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$A = 1,\ B = 1,\ C = 1 \Rightarrow B^2 - 4AC = 1 - 4 = -3 < 0 \Rightarrow \mathit{Elliptic} \quad \mathit{Equation}$$

Find ξ and η via

$$\left.\begin{array}{l} \xi = \text{constant on} \quad \mathrm{d}y/\mathrm{d}x = (1 + i\sqrt{3})/2 \\ \eta = \text{constant on} \quad \mathrm{d}y/\mathrm{d}x = (1 - i\sqrt{3})/2 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{l} \xi = y - \frac{1}{2}(1 + i\sqrt{3})x \\ \eta = y - \frac{1}{2}(1 - i\sqrt{3})x \end{array}\right\}.$$

To obtain a real transformation, put

$$\alpha = \eta + \xi = 2y - x \quad \text{and} \quad \beta = i(\xi - \eta) = x\sqrt{3}.$$

So,

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \alpha} + \sqrt{3}\,\frac{\partial u}{\partial \beta}, \quad \frac{\partial u}{\partial y} = 2\,\frac{\partial u}{\partial \alpha}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} - 2\sqrt{3}\,\frac{\partial^2 u}{\partial \alpha \partial \beta} + 3\,\frac{\partial^2 u}{\partial \beta^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = -2\,\frac{\partial^2 u}{\partial \alpha^2} + 2\sqrt{3}\,\frac{\partial^2 u}{\partial \alpha \partial \beta}, \quad \frac{\partial^2 u}{\partial y^2} = 4\,\frac{\partial^2 u}{\partial \alpha^2},$$

and the equation transforms to

$$\frac{\partial^2 u}{\partial \alpha^2} - 2\sqrt{3}\,\frac{\partial^2 u}{\partial \alpha \partial \beta} + 3\,\frac{\partial^2 u}{\partial \beta^2} - 2\,\frac{\partial^2 u}{\partial \alpha^2} + 2\sqrt{3}\,\frac{\partial^2 u}{\partial \alpha \partial \beta} + 4\,\frac{\partial^2 u}{\partial \alpha^2} = 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = 0 \quad \text{(canonical form)}$$

4. Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} - 4u_{xy} + 4u_{yy} = \cos(2x + y).$$

- The given equation is a parabolic one.
- The characteristics are:

$$\xi = x, \; \eta = x - \frac{B}{2C}y = x + (1/2)y$$

- The canonical form will be $u_{\xi\xi} = \cos\eta$.
- Integrating partially with respect to ξ:

$$u_\xi = \xi\cos\eta + f(\eta).$$

- Again integrating w.r.t. ξ

$$u(\xi,\eta) = \frac{\xi^2}{2}\cos\eta + \xi f(\eta) + g(\eta).$$

5. $$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$$

Show that the canonical form is

$$u_{\eta\eta} = 0 \qquad \text{for } y \neq 0$$

$$u_{xx} = 0 \qquad \text{for } y = 0.$$

To solve (2.6.2) we integrate with respect to η twice (ξ is fixed) to get

$$u(\xi, \eta) = \eta F(\xi) + G(\xi).$$

Since the transformation to canonical form is

$$\xi = \frac{y}{x} \qquad \eta = y \qquad \text{(arbitrary choice for } \eta)$$

then

$$u(x, y) = yF\left(\frac{y}{x}\right) + G\left(\frac{y}{x}\right).$$

6. Obtain the general solution for

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$$

The transformation

$$\xi = y - x,$$

$$\eta = y - \frac{x}{4},$$

leads to the canonical form

$$u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}.$$

Let $v = u_\eta$ then

$$v_\xi = \frac{1}{3}v - \frac{8}{9}$$

which is a first order linear ODE (assuming η is fixed.) Therefore

$$v = \frac{8}{3} + e^{\xi/3}\phi(\eta).$$

Now integrating with respect to η yields

$$u(\xi, \eta) = \frac{8}{3}\eta + G(\eta)e^{\xi/3} + F(\xi).$$

In terms of x, y the solution is

$$u(x, y) = \frac{8}{3}\left(y - \frac{x}{4}\right) + G\left(y - \frac{x}{4}\right)e^{(y-x)/3} + F(y - x).$$

Problem Set - II

1. Find the general solution of the equation $u_{xx} - 2u_{xy} - 3u_{yy} = 0$.
2. Classify the following linear second order partial differential equation and find its general solution .

$$xyu_{xx} + x^2 u_{xy} - yu_x = 0.$$

Ans: $u(\xi, \eta) = \int f(\xi) d\xi + G(\eta) = F(\xi) + G(\eta)$ i.e. $u(x, y) = F(x^2 - y^2) + G(y)$

3. Classify, reduce to normal form and obtain the general solution of the partial differential equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 4x^2$$

Ans: $u(\xi, \eta) = 2\xi^2 + \xi f(\eta) + g(\eta)$ i.e. $u(x, y) = 2x^2 + xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$

4. Reduce to canonical form and then solve $u_{xx} + 5u_{xy} + 6u_{yy} = 0$.

Ans: $u_{\mu\eta} = 0$ $u(x, y) = F(3x - y) + G(2x - y)$

5. Reduce to canonical form and then solve $y^2 u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y$.

Ans: $u_{\eta\eta} = 6\eta$. $u(x, y) = y^3 + y \cdot f\left(\frac{y^2}{2} + x\right) + g\left(\frac{y^2}{2} + x\right)$

6. *Find the characteristics of the partial differential equation* $xu_{xx} + (x - y)u_{xy} - yu_{yy} = 0$, $x > 0,\ y > 0$, *and then show that it can be transformed into the canonical form* $(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$ *whence* ξ *and* η *are suitably chosen canonical coordinates. Use this to obtain the general solution in the form*

$$u(\xi, \eta) = f(\xi) + \int^{\eta} \frac{g(\eta')\, d\eta'}{(\xi^2 + 4\eta')^{\frac{1}{2}}}$$

where f *and* g *are arbitrary functions of* ξ *and* η.

Problem Set - II

7. Reduce to canonical form (a) $x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} + u_y = 0.$

(b) $u_{xx} + (1+y^2)^2 u_{yy} - 2y(1+y^2)u_y = 0.$

8. Find the general solution of the PDE by first reducing it to a canonical form.

$$x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} = 0$$

9. Find the characteristic equation, characteristic curves and obtain a canonical form for ch

a. $x\, u_{xx} + u_{yy} = x^2$
b. $u_{xx} + u_{xy} - x u_{yy} = 0 \qquad (x \le 0, \text{ all } y)$
c. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + xy u_x + y^2 u_y = 0$
d. $u_{xx} + x u_{yy} = 0$
e. $u_{xx} + y^2 u_{yy} = y$
f. $\sin^2 x u_{xx} + \sin 2x u_{xy} + \cos^2 x u_{yy} = x$

10. Determine the general solution of

a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0 \qquad c = \text{constant}$
b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
c. $u_{xx} + u_{xy} = 0$
d. $u_{xx} + 10 u_{xy} + 9 u_{yy} = y$

Problem Set - II

11. Transform the following equations to

$$U_{\xi\eta} = cU$$

by introducing the new variables

$$U = ue^{-(\alpha\xi+\beta\eta)}$$

where α, β to be determined

a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$
b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

(Hint: First obtain a canonical form. If you take the right transformation to canonical for then you can transform the equation in (a) to

$$U_{\xi\eta} = cU_\eta$$

and this can be integrated to get the exact solution. Is it possible to do that for part b?)

12. Show that

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

is parabolic for a, b, d constants. Show that the substitution

$$u(x,t) = v(x,t)e^{\frac{b}{2}x}$$

transforms the equation to

$$v_{xx} = av_t + de^{-\frac{b}{2}x}$$

Problem Set - II

13. Transform each of the partial differential equations in Exercises 1–10 into canonical form.

1. $\dfrac{\partial^2 u}{\partial x^2} - 5\dfrac{\partial^2 u}{\partial x\,\partial y} + 6\dfrac{\partial^2 u}{\partial y^2} = 0.$

2. $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 4\dfrac{\partial^2 u}{\partial y^2} = 0.$

3. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} - 8\dfrac{\partial^2 u}{\partial y^2} + 9\dfrac{\partial u}{\partial x} = 0.$

4. $2\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial x\,\partial y} - 9\dfrac{\partial^2 u}{\partial y^2} + 4\dfrac{\partial u}{\partial x} = 0.$

5. $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 13\dfrac{\partial^2 u}{\partial y^2} - 9\dfrac{\partial u}{\partial y} = 0.$

6. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} + 3\dfrac{\partial u}{\partial x} + 9u = 0.$

7. $6\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0.$

8. $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial u}{\partial x} + 5\dfrac{\partial u}{\partial y} = 0.$

9. $2\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial^2 u}{\partial y^2} + u = 0.$

10. $\dfrac{\partial^2 u}{\partial x^2} - 5\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} + 3u = 0.$

Problem Set - II

14. Show that the transformation

$$\xi = y - \frac{x^2}{2},$$

$$\eta = x,$$

reduces the equation

$$\frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \, \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

to

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial u}{\partial \xi}.$$

15. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + (2x + 3) \frac{\partial^2 u}{\partial x \, \partial y} + 6x \frac{\partial^2 u}{\partial y^2} = 0. \tag{A}$$

(a) Show that for $x = \frac{3}{2}$, Equation (A) reduces to a parabolic equation, and reduce this parabolic equation to canonical form.

(b) Show that for $x \neq \frac{3}{2}$, the transformation

$$\xi = y - 3x,$$

$$\eta = y - x^2$$

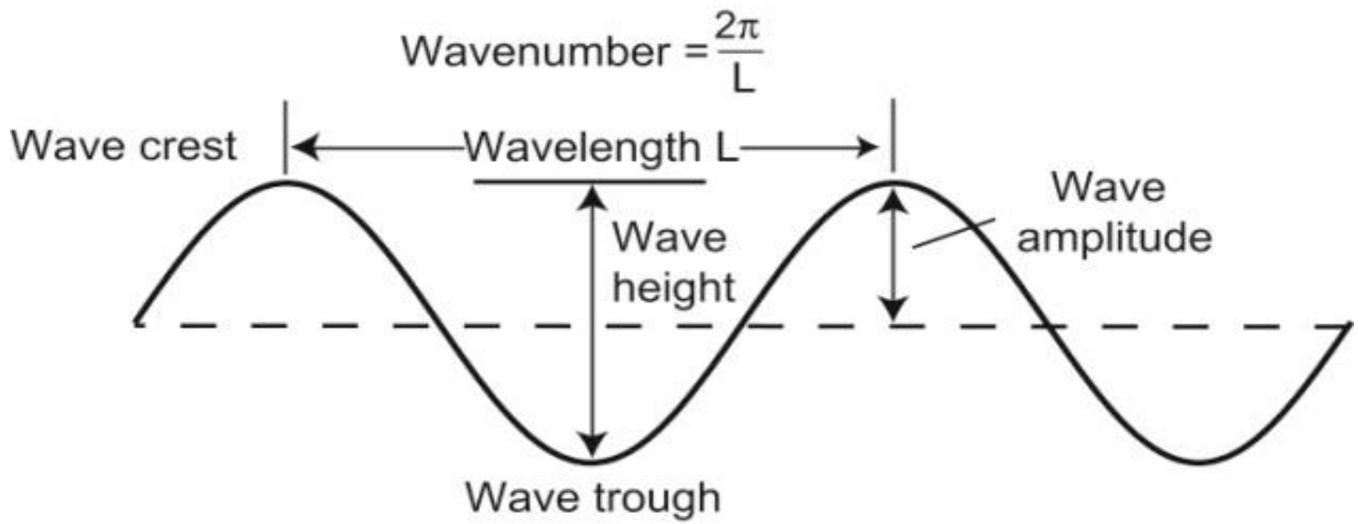
reduces Equation (A) to

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = \frac{2 \dfrac{\partial u}{\partial \eta}}{4(\eta - \xi) - 9}.$$

PARTIAL DIFFERENTIAL EQUATIONS-II

Concept of nonlinearity and wave breaking. Solution of Burger and KdV equations.

✓ One Dimensional LINEAR WAVE: Basic Terminology



✓ One Dimensional LINEAR WAVE: Basic Concepts

Let us consider the simplest model one-dimensional linear equation:

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

where $u(x,t)$ is the amplitude of the wave and c is the positive constant. The general solution of this equation for smooth functions f and g is given by

$$u(x,t) = f(x - ct) + g(x + ct)$$

- $x \pm ct =$ Constant is the phase of the wave.
- $c = \frac{dx}{dt}$

The 1D wave equation can be rewritten as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0.$$

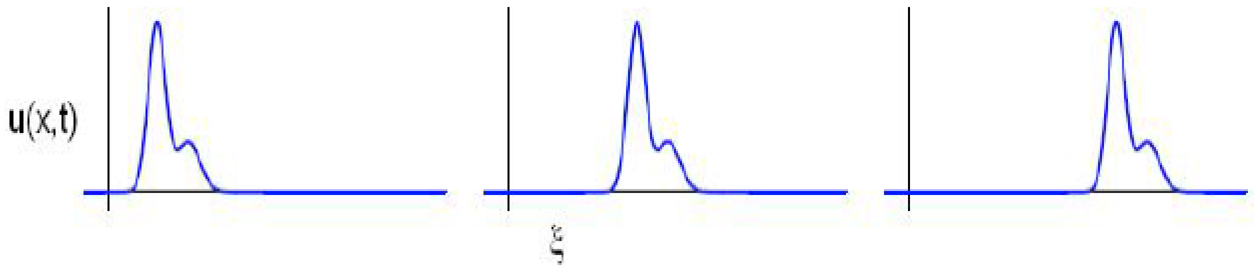
Consider the homogeneous equation

$$u_t + cu_x = 0. \tag{2}$$

As a consequence of the above, the solutions are traveling waves of the form

$$u(x,t) = h(\xi), \qquad \xi = x - ct = \text{phase}.$$

✓ One Dimensional LINEAR WAVE: Basic Concepts



1. To a stationary observer the solution appears a wave of unchanging for moving with velocity c.
2. When $c > 0$ the wave translate to the right as shown in the figure.
3. When $c < 0$, the wave moves to the left.

✓ One Dimensional LINEAR WAVE: Basic Concepts

❑ Dispersion Relation

The Harmonic wave solution $u(x,t) \sim \exp(i(kx - \omega t))$ in (ω, k) space provides the dispersion relation:

$$\omega = \omega(k). \tag{3}$$

As an example, the dispersion relation of (1)/(2) is

$$\omega^2 = c^2 k^2 \ / \ \omega = ck.$$

- A natural question: is the dispersion function $\omega(k)$ [equation (3)] always real?
- The answer is no (we will discuss it later).

✓ One Dimensional LINEAR WAVE: Basic Concepts

❑ Wave Dispersion: Phase Velocity

The phase velocity of this wave is defined as

$$c_p = \frac{\omega(k)}{k}.$$

- The wavefront of a wave propagate at the phase velocity.
- If $c_p = c_p(k)$, the different wave number components will have different speeds: a phenomenon known as **wave dispersion**. Therefore the way they interfere with one another will change with time – so the shape of the disturbance will change.
- Clearly, c_p is independent of k for all k only if $\omega(k) = \text{constant} \times k-$ this is the nondispersive case: All disturbances, including localized ones, propagate without change of shape.

✓ One Dimensional LINEAR WAVE: Basic Concepts

❑ Wave Dispersion: Group Velocity

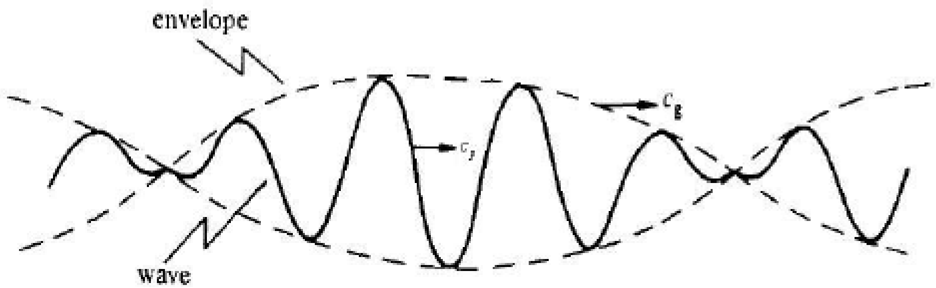
When the various frequency components of a waveform have different phase velocities, the phase velocity of the waveform is an average of these velocities (the phase velocity of the carrier wave), but the waveform itself moves at a different speed than the underlying carrier wave called the *group velocity* and is defined as

$$c_g = \frac{d\omega}{dk} = c_p + k\frac{dc_p}{dk} \neq c_p \Longrightarrow c_g \ll c_p.$$

- The carrier wave propagates at the phase velocity, the modulation envelope propagates at the group velocity.
- This is an important concept, as it is the velocity that governs the propagation of information.

✓ One Dimensional LINEAR WAVE: Basic Concepts

❑ Phase Velocity and Group Velocity



1. The speed at which a given phase propagates does not coincide with the speed of the wave packet (envelope).
2. The phase velocity is greater than the group velocity.

‡ A wave is called *dispersive* if $c_g = f(k)$.

- In this case the phase velocity becomes a function of k (wave number) and thereby wave of different wave number propagate at different velocities: characteristic of dispersive wave.

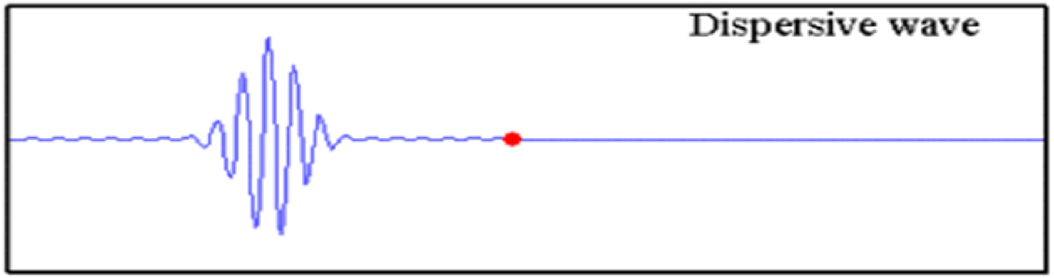
✓ One Dimensional LINEAR WAVE: Basic Concepts

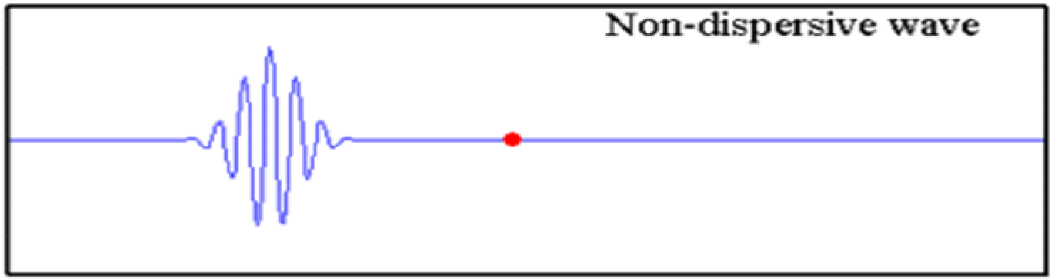
- Example of Wave Dispersion:

$$u_t + cu_x + u_{xxx} = 0 \tag{4}$$

  - Dispersion Relation: $\omega = ck - k^3$
  - Phase Velocity: $c_p = c - k^2$
  - Group Velocity: $c_g = c - 3k^2$

▷ So the wave is dispersive.





✓ One Dimensional LINEAR WAVE: Basic Concepts

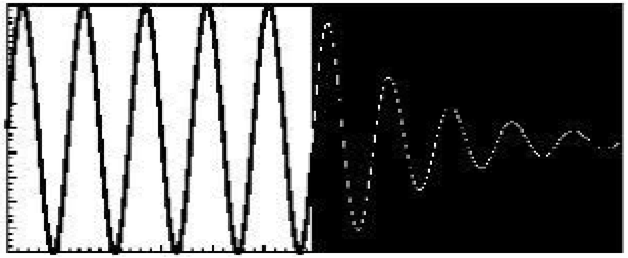
- Example of Wave Dissipation:

$$u_t + cu_x - u_{xx} = 0 \tag{5}$$

  - Dispersion Relation: $\omega = ck - ik^2$

$$u(x,t) \sim \left[e^{-k^2 t}\right] \exp ik\,(x - ct)$$

▷ So the wave decays exponentially. It is dissipative in nature.



- Odd powers in spatial derivatives $\rightarrow$ *Dispersion*.
- Even powers in spatial derivatives $\rightarrow$ *Dissipation*.

✓ NONLINEAR WAVE: Basic Concepts of Breaking

❑ Solution of a Quasilinear Equation

Consider the first order quasilinear PDE

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \qquad \text{(ql-1)}$$

where the functions a, b and c can involve u but not its derivatives.

The method of characteristics is the only method applicable for quasilinear PDEs.

Characteristic equations $(\mathrm{d}\{x,y\}/\mathrm{d}s)$ and compatibility equation $(\mathrm{d}u/\mathrm{d}s)$ are simultaneous first order ODEs in terms of a dummy variable s (curvilinear coordinate along the characteristics); we cannot solve the characteristic equations and compatibility equation independently

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x,y,u),$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = b(x,y,u),$$
$$\frac{\mathrm{d}u}{\mathrm{d}s} = c(x,y,u).$$

Solving characteristic and compatibility equations, we get two independent first integrals

$$\phi(x,y,u) = c_1 \quad \text{and} \quad \psi(x,y,u) = c_2$$

Then the solution of equation (ql-1) satisfies $F(\phi,\psi) = 0$ for some arbitrary function F (equivalently, $\phi = G(\psi)$ for some arbitrary G), where the form of F (or G) depends on the initial conditions.

✓ NONLINEAR WAVE: Basic Concepts of Breaking

A partial differential equation (PDE) of the form (hyperbolic type)

$$T_t + X_x = 0,$$

is called a *conservation law* with T representing the density/velocity (etc.) of some quantity and X is the associated rightward flux.

- *Conservation Laws* arise in plasma dynamics, fluid dynamics and many other fields.

The simplest example of nonlinear one-dimensional *conservation law* is the following *inviscid Burgers equation*

- $T = u$
- $X = \frac{u^2}{2}$

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \Longrightarrow u_t + uu_x = 0.$$

✓ NONLINEAR WAVE: Basic Concepts of Breaking

Instead of equation (2), let us consider the following equation (the simplest form of 1D nonlinear equation)

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \Longrightarrow u_t + uu_x = 0. \qquad (6)$$

This equation is also known as inviscid Burgers' equation.

- Linearizing this equation against constant background $u = u_0$ and looking for the solution in the form $\sim \exp(i(kx - \omega t)$, we obtain the dispersion relation $\omega = u_0 k$.
- The phase velocity $c_p = u_0$.
- The wave is non-dispersive as group velocity $c_g = u_0(\text{constant})$.

We shall try to solve (3) with the initial condition $u(0, x) = f(x)$. (7)

In trying to solve this equation, we find two independent integrals of

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0} \qquad \begin{aligned} \phi &= u(x,t) = C_1 \\ \psi &= x - u(x,t)t = C_2 \end{aligned}$$

So the general solution is the following traveling wave form according to the initial condition (7).

$$u(x,t) = f[x - u(x,t)t] \qquad (8)$$

✓ NONLINEAR WAVE: Basic Concepts of Breaking

➢ This solution indicates that the phase $\xi = x - u(x,t)t$ moves faster where $u(x,t)$ is longer.

On the characteristics both $\phi[=u(x,t)]$ and $\psi[=x-u(x,t)t]$ are constants. So the characteristics equations are

$$\left.\begin{aligned} \frac{du}{dt} &= 0 \Rightarrow u = C_3 \xrightarrow{t=0,u=f(x),x=\xi} u = f(x) \\ \frac{dx}{dt} &= u \Rightarrow x = u\,t + C_4 \xrightarrow{t=0,u=f(x),x=\xi} x = u + \xi \end{aligned}\right\} \Rightarrow x = f(\xi)t + \xi \qquad \text{(straight lines)}$$

The slope of the characteristics $1/f(\xi)$ varies from one line to another and so two curves can intersect.

Considering two crossing characteristics expressed in terms of ξ_1 and ξ_2,

$$\begin{aligned} x &= f(\xi_1)t + \xi_1, \\ x &= f(\xi_2)t + \xi_2, \end{aligned} \quad \text{intersect at time} \quad t^* = -\frac{\xi_1 - \xi_2}{f(\xi_1) - f(\xi_2)} = -\frac{\Delta\xi}{\Delta f}. \qquad (9)$$

By letting $\Delta\xi \to 0$ the characteristics intersect at

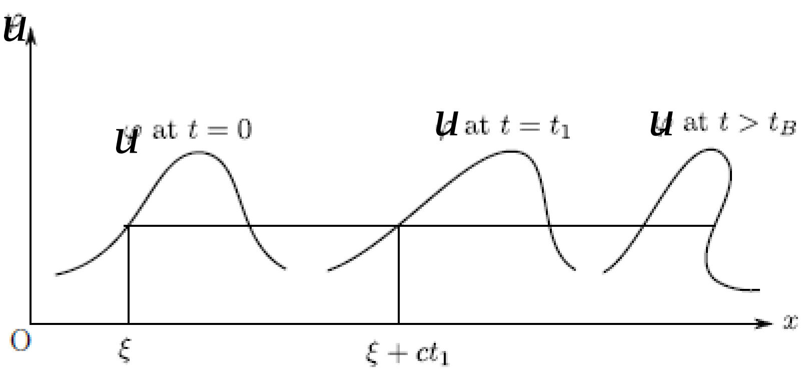
$$t^* = -\frac{1}{f'(\xi)}. \qquad (10)$$

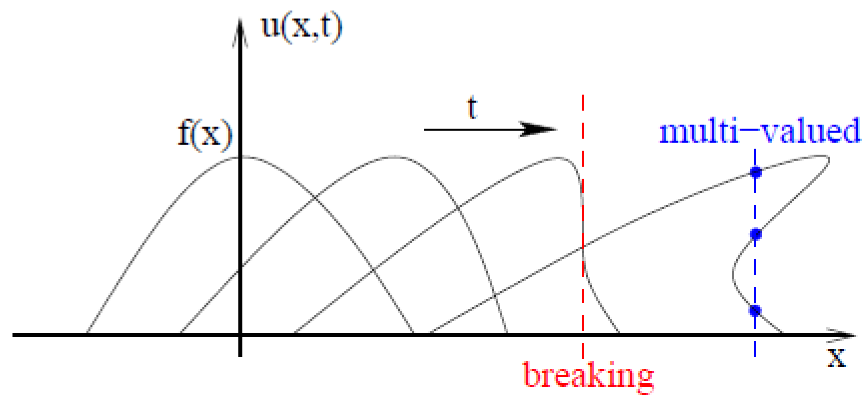
✓ NONLINEAR WAVE: Basic Concepts of Breaking

At this point u will not be single valued and the solution breaks down (wave breaking occurs). The minimum time for which the solution become multi-valued is

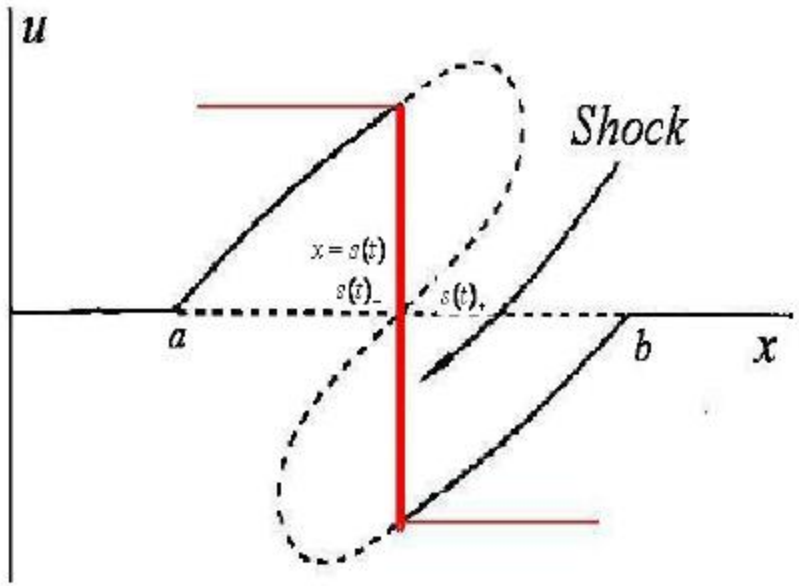
$$t_B = \max(t^*) = -\frac{1}{\max\left[f'(\xi)\right]}. \qquad (11)$$

- ✓ The solution is single valued for $0 \le t < t_B$.
- ✓ $f'(\xi) < 0$, we expect the solution to exist only for a finite time.





1. At some time $t = t_B$, the continuous wave profile may become multiple valued function. This phenomenon is called *wave breaking*.
2. In this case a continuous solution will break down and a discontinuity will occur.



- By definition, a shock is a jump discontinuity in the solution $u(t, x)$.

ARRESTING OF WAVE BREAKING

To arrest the wave breaking, we consider the following physical phenomena:

- Dissipation.
- Dispersion.
- Dissipation and Dispersion.

The effects of dissipation and dispersion on the wave breaking are drastically different. It is therefore, instructive first to consider them separately.

EFFECT OF DISSIPATION: NONLINEAR BURGERS' EQUATION

To introduce the dispersive effect, we put

- $T = u$
- $X = \frac{u^2}{2} - \nu u_x$

in equation (2) and obtain

$$u_t + uu_x = \nu u_{xx}, \qquad \nu \ll 1 \tag{1}$$

This is the well-known nonlinear Burgers equation.
Proceeding as before we obtain the dispersion relation:

$$\omega = ku_0 - i\nu k^2$$

- The wave is dissipative.
- The term νu_{xx} (usually arising due to the viscosity of the medium) mediates the wave dissipation:

$$u \sim e^{-\nu k^2 t} \to 0 \text{ as } t \to \infty \text{ for all } x.$$

SOLUTION: NONLINEAR BURGERS' EQUATION

We consider the Burgers equation (1)

$$u_t + uu_x = \nu u_{xx} \quad (2)$$

with the constant boundary conditions at infinity:

$$u \to \begin{cases} u_2 & x \to -\infty \\ u_1 & x \to \infty \end{cases} \quad (3)$$

Let $u_2 > u_1$. We will look for a steady profile moving with constant velocity U, i.e. a traveling wave, $u = u(\xi)$, where $\xi = x - Ut$. Then Burger equation becomes the following ordinary differential equation (ODE):

$$-Ud_\xi u + ud_\xi u = \nu d_\xi^2 u$$

Integrating once we obtain

$$\nu d_\xi u = \frac{1}{2}u^2 - Uu + C, \text{ where } C \text{ is an arbitrary constant.} \quad (4)$$

Using boundary conditions (3) and assuming that $du/d\xi \to 0$ as $|\xi| \to \infty$, we obtain the following 1st order ODE:

$$\nu \frac{du}{d\xi} = -\frac{1}{2}(u - u_1)(u - u_2) \quad (5)$$

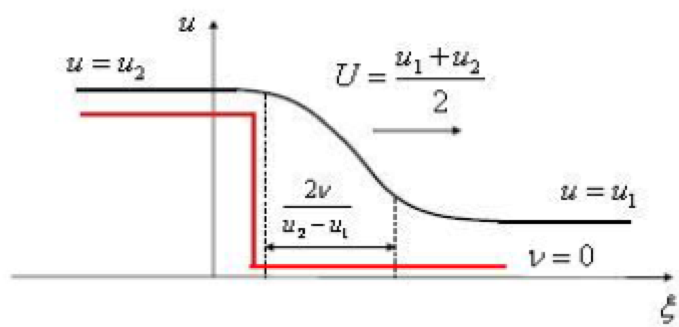
SOLUTION: NONLINEAR BURGERS' EQUATION

with

$$U = \frac{(u_1 + u_2)}{2} \text{ and } C = \frac{u_1 u_2}{2}.$$

Integrating (15), we obtain the following so-called Taylor's shock profile:

$$u(x,t) = u_1 + (u_2 - u_1)\frac{\exp\{-(u_2 - u_1)\xi/(2\nu)\}}{1 + \exp\{-(u_2 - u_1)\xi/(2\nu)\}}$$



- The shock propagates with the shock velocity $U = (u_1 + u_2)/2$.
- The shock width $\Delta = 2\nu/(u_2 - u_1)$.
- So for fixed $u_{1,2}$ and $u_1 \neq u_2$, $\Delta \to 0$ as $\nu \to 0$ and recover the discontinuous shock. Also for fixed ν, $\Delta \to 0$ as $u_2 - u_1 \to 0$.

EFFECT OF DISPERSION : CELEBRATED Korteweg –de Vries (KdV) Equation (1895)
Arresting of wave breaking and formation of stable structure called 'SOLITON'

Diederik Johannes Korteweg
(1848-1941)

Gustav de Vries
(1866-1934)

$$u_t \pm 6 \underbrace{u u_x}_{\text{Nonlinearity}} + \underbrace{u_{xxx}}_{\text{Dispersion}} = 0.$$

SOLITON

- Soliton: This is a solution of a nonlinear partial differential equation which represent a solitary traveling wave possessing the following properties:
  1. A permanent wave form: Unchanging in shape.
  2. It is Localized within a bounded region.
  3. It Does not obey the superposition principle.
  4. It does not disperse.

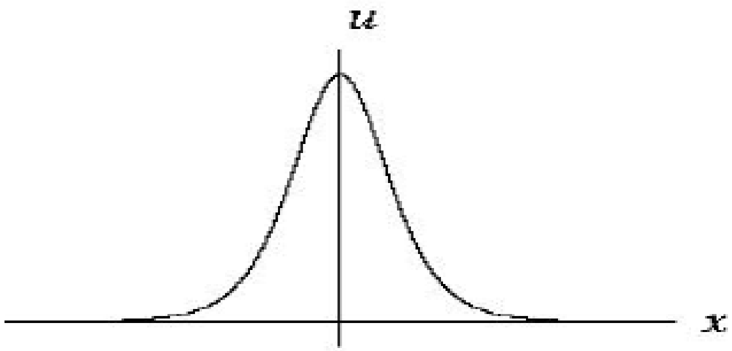


Figure: Steepen + flatten=Stable

Solitary wave solution: Single Soliton

We look for a (traveling wave) solution of the equation

- (Traveling Wave) $u(x,t) = u(\theta), \;\; \theta = x - ct.$

Then KdV equation [with +Ve sign] becomes (after twice integration)

$$\frac{(u')^2}{2} + u^3 - \frac{c}{2}u^2 + B_1 u = B_2$$

It looks like an equation of motion of a "particle" in the "potential":

$$U_{eff} = u^3 - \frac{c}{2}u^2 + B_1 u.$$

- Localized Boundary Conditions: $u, u', u'' \to 0$ as $|\theta| \to \infty$.

We finally obtain

$$u(\theta) = \left(\frac{c}{2}\right) \, sech^2\left(\frac{\sqrt{c}}{2}\theta\right).$$

Outline of the solution

The boundary conditions ensures that $B_1 = 0,\ \ B_2 = 0$ $\quad (u')^2 = u^2(c-2u)$

Rearrange to $\displaystyle\int \frac{du}{u\sqrt{c-2u}} = \pm\int d\theta$ $\quad$ Make Substitution $\quad u = \left(\frac{c}{2}\right)\sec h^2\phi$

$$\int \frac{\mathrm{du}}{\mathrm{u}\sqrt{(\mathrm{c}-2\mathrm{u})}} = \int \mathrm{d}\phi \frac{-\mathrm{c}\,\mathrm{sech}^2\phi\tanh\phi}{\frac{\mathrm{c}}{2}\mathrm{sech}^2\phi\sqrt{\mathrm{c}\tanh^2\phi}} = -\frac{2}{\sqrt{\mathrm{c}}}\int \mathrm{d}\phi$$

$$\mathrm{sech}\,\phi = \frac{2}{\mathrm{e}^{\phi} - \mathrm{e}^{-\phi}},$$

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{sech}\,\phi = -\mathrm{sech}\,\phi\tanh\phi$$

$$-\frac{2}{\sqrt{\mathrm{c}}}\int \mathrm{d}\phi = \pm\int \mathrm{d}\theta \Rightarrow -\frac{2\phi}{\sqrt{\mathrm{c}}} = \pm\theta + \frac{\mathrm{x_o}}{\sqrt{\mathrm{c}}}$$

X_0 is the constant of integration.

Rearrange to $\quad \phi = \dfrac{\sqrt{\mathrm{c}}}{2}\left(\mp\theta - \mathrm{x_o}\right)$

Make Back Substitution $\quad \mathrm{u} = \left(\frac{\mathrm{c}}{2}\right)\mathrm{sech}^2\left[\frac{\sqrt{\mathrm{c}}}{2}\left(\mp(\mathrm{x}-\mathrm{ct}-\mathrm{x_o}\right)\right]$

Solitary wave solution: Single Soliton

- Observations:
  1. $A = c/2$ is the amplitude which is proportional to the velocity (c).
  2. $\Delta = 2/\sqrt{c}$ is the spatial width which is inversely proportional to the square root of the velocity $(\sqrt{c})$.
  3. $$\text{Amplitude} \times (\text{Width})^2 = A\Delta^2 = 2 = \text{Constant}.$$
  4. Taller solitary waves are thinner and moves faster.

- Notes:
  1. More general solutions can be found for other choices of B_1 and B_2.
  2. KdV equation has multisoliton solutions.
  3. There is *anti-soliton* solution of the KdV equation obtained by replacing $u \to -u$: $u_t - 6uu_x + u_{xxx}$.

Solitary wave solution: Single Soliton

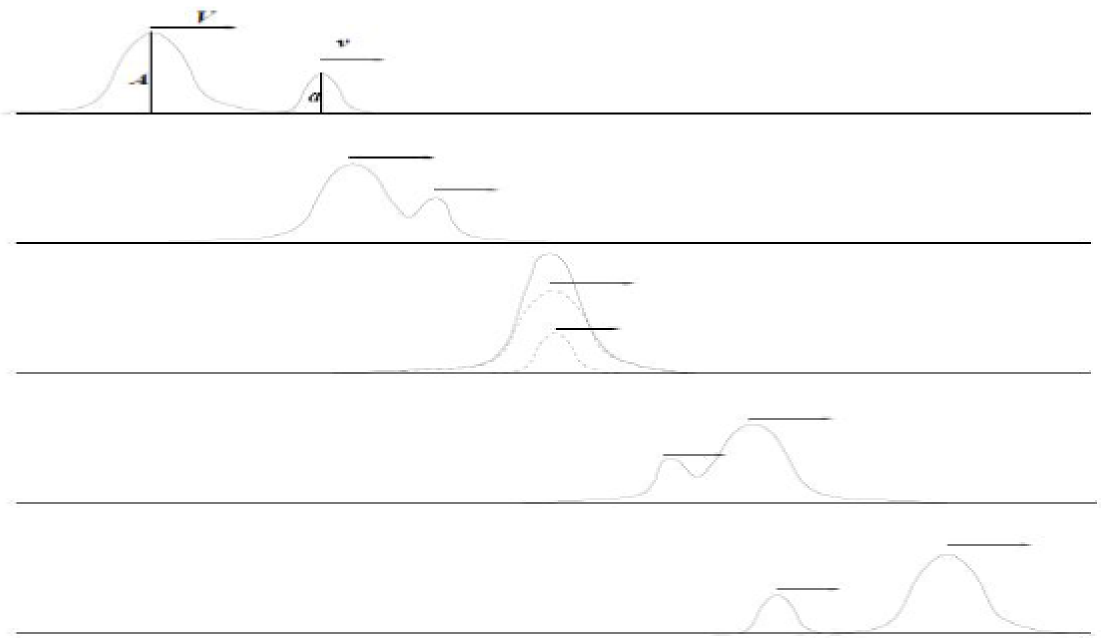


Figure: Collision of two solitons (like perfect elastic collision): Signature of phase-shift

- It satisfies infinitely many conservation Laws.