Volume 2

Feynman Hughes Lectures

Electrostatics
Electrodynamics
Matter-Waves Interacting
Relativity

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John T. Neer

Bookmarks are provided

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INTRODUCTION

This begins a new lecture series and I am confronted with the hard task of talking to you motly crew. I just don't know where to begin. Some of you have been around for a long time; others are new; and some don't belong here. So I am going to try and make the course rather elementary to begin with. Eventually we want to end up in Quantum Electrodynamics (Q.E.D.). But this won't be easy because there are some conceptual and mathematical difficulties in Q.E.D. which we must tackle. To attack these problems we must understand some Quantum Mechanics (QM). I guess this is really a course in physics and we'll see how it goes along.

I really don't know what level to start talking; am I to describe how two pith balls can be charged and made to repel or attract each other as a function of the inverse square of the distance separating them? Some of you are laughing; some not – what am I to conclude? I'll need a lot of questions from you to either slow me down or speed me up. Since I know you're not a timid group, I'm sure we'll be alright.

To begin I want to emphasize that electromagnetic (EM) forces are only a small part of a more remarkable phenomenan – that being the propagation of waves which lets us see the stars and feel the presence of object, etc. It is the influences sensed over great distances which is much more interesting than simple pushes and pulls. You all should know that effects can't proceed faster than the speed of light c. So if I had two objects separated by some distance R, a force would be felt on one of them, say, if I moved the other back and forth. If I moved faster and faster the source object, then inertia will not allow the other object to keep up. But if looked on an atomic level, I could still see the mutual interaction along the radius connecting the two. Let me wiggle the source back and forth so fast that the object doesn't know what I'm doing because the effect takes a finite time to reach it. During this time interval the one object must keep on doing what it had been doing until this effect changes it motion. You see then there is a dynamic effect or aspect to wave propagation which causes a delay in the action. It will turn out the resulting force is not dependent on $1/R^2$ but rather falls off as the inverse of the distance as R becomes large.

I'LL BEGIN, THEN, WITH WHAT YOU ALL KNOW:

$$F = m\frac{d^2R}{dt^2}$$

RIGHT? WRONG! IT'S WRONG FOR TWO REASONS: FIRST, THE MASS IS NOT A CONSTANT BUT RATHER A FUNCTION OF THE VELOCITY v AND, SECONDLY, THERE IS NO such THING AS A WELL DEFINED POSITION, R, SO IT DOESN'T DO ANY GOOD TO KNOW $F = ma$. TO UNDERSTAND THE PROBLEMS WE WANT TO FIRST DISCUSS THE PROBLEM OF MASS AND VELOCITY. THE SECOND ERROR IS MORE DIFFICULT SO WE'LL LET IT GO TIL LATER.

SPECIAL THEORY OF RELATIVITY

REF. CHAPTER 17 VOL I

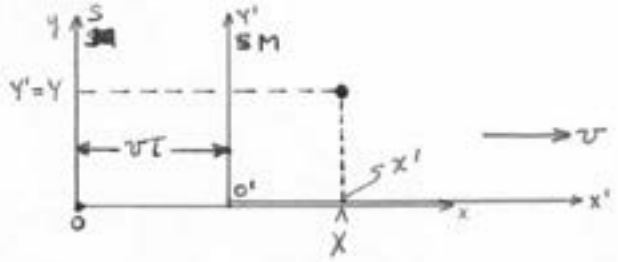
WE MUST BEGIN BY DISCUSSING THE SPECIAL THEORY OF RELATIVITY. THAT IS, FOR MOVING OBJECTS TIME GOES AT DIFFERENT SPEEDS. THE BEST EXAMPLE IS THE TWO CLOCKS, ONE ON EARTH THE OTHER IN A MOVING SPACESHIP. I WILL ASSUME YOU ALL KNOW THIS IN OUR MODERN DAY OF SPACE TRAVEL. SO I WILL NOT DEBATE THIS FACT WITH SOME CRANKS AND ASSUME IT IS TRUE.

I WANT TO DISCUSS THE BEHAVIOR OF THINGS AMONG THEMSELVES ENCLOSED LIKE IN A SPACESHIP AND SHOW HOW THE RESULTS ARE THE SAME AS IF YOU WERE AT REST. STATED ANOTHER WAY THE RESULTS CONDUCTED INSIDE A SPACESHIP COASTING ALONG IN A STRAIGHT LINE AT AN ABSOLUTE SPEED REFERRED TO THE DISTANT STARS ARE NOT INFLUENCED BY THAT MOTION. THE STATEMENT THAT ALL PHENOMENA ARE THE SAME IN A MOVING OR STATIONARY FRAME FORMS THE BASIC HYPOTHESIS OF THE SPECIAL THEORY OF RELATIVITY (STR). FROM THIS HYPOTHESIS A LOT CAN BE DEDUCED.

THE FIRST DESCRIPTION OF THE MOTION OF A BODY IN SPACE WAS GIVEN BY NEWTON. I'LL GIVE HERE A MODERN ANALYTICALLY STATEMENT OF WHAT HE REASONED. WE HAVE TWO COORDINATES FRAMES IN WHICH FRAME M, THE PRIMED SYSTEM, IS MOVING AT A CONSTANT VELOCITY v RELATIVE TO THE STATIONARY, S, FRAME. FOR CONVENIENCE AT TIME $T=0$ THEIR ORIGINS COINCIDED. THUS AN OBJECT IS LOCATED IN THE TWO FRAMES AT:

$$X = X' \quad \text{AND} \quad Y = Y' \quad \text{AT } T=0$$

$$X' = X - vT, \quad Y = Y' \quad \text{AT } t = t'$$



This set of relations show how one man measures position and time as seen from a different point of view. Since the speed of the moving frame is constant and the force is a function of acceleration, the resulting motion is the same as seen by the two observers. That is,

$$F = m\frac{d^2x}{dt^2} \qquad F' = m'\frac{d^2x'}{dt'^2}$$

but $m = m'$, $t = t'$ so we see $F = F'$

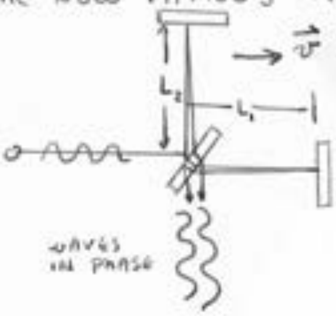
We must keep straight that we are looking at something from two point of views which are different. This makes the situation difficult to analyze. It is made difficult by the human brain that want to follow one line of reasoning. We are often forced to change our point of view in the middle of an argument to figure out what is going to happen. For instance, suppose we had a pool table in a space ship. If we were at rest in the ship, we could write the equations of motion of the ball across the table and predict which spots it would roll over on its way to a predetermined pocket. Now an observer outside looking at the table fly by must get the same results since he knows what should happen if he were fixed inside. Huygen successfully applied similar reasoning to discover the laws of conservation of linear momentum. Instead of a spaceship he had a boat.

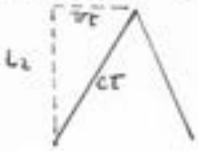
REF: Chapter 15 Vol I

Newton's laws held until the laws governing EM propagation were established in the later 1800's. It was a consequence of Maxwell's equations that waves propagated at the speed C. With a fixed velocity it then seemed possible to detect a moving system. Imagine a spaceship moving at a velocity of 100,000 miles/sec and a light pulse emitted at the rear propagating toward the nose. By the time light had reached the point where the nose was at the time of emission, it was now farther out because it had moved 100,000 m/s times τ. Eventually the light would catch up but the apparent velocity would be $c - v$ = 86,000 m/s as the propagation velocity. If the ship went the other way the apparent velocity was 286,000 m/s. How can it be that light propagates at the velocity C? Obviously it can't. Since Maxwell's laws were newly established, they were the most suspicious.

A experiment was preformed to detect the absolute velocity of the earth. The idea was to measure the time difference in a light going out, striking a mirror, and coming back. Since light oscillates 10^{15} time/sec,

IT WAS REASONED BY MAKING A PHASE COMPARISON OF THE INTERFERING FRINGES ONE PART IN 10^{15} COULD BE MEASURED. THE EXPERIMENT IS THE NOW FAMOUS MICHELSON-MORLEY EXPERIMENT:





TO FIND THE ROUND TRIP TIME ALONG PATH L_1

$$t_1 = \frac{L_1}{c-v} + \frac{L_1}{c+v} = \frac{2L_1 c}{c^2-v^2}$$

TO FIND T_2 WE FIRST NOTE THE APPARATUS IS TRANSLATED THROUGH vt SO BY THE ASSOCIATED TRIANGLE,

$$L_2^2 = c^2 t_2^2 - v^2 t_2^2 = t_2^2 (c^2 - v^2)$$

OR

$$t_2 = \frac{L_2}{\sqrt{c^2-v^2}}$$

SO THE ROUND TRIP TIME ALONG THE ZIG-ZAG PATH IS

$$2t_2 = \frac{2L_2}{\sqrt{c^2-v^2}} = T_2$$

REWRITING THE TWO TIMES

$$T_1 = \frac{2L_1}{c}\left(\frac{1}{1-v^2/c^2}\right)$$

$$T_2 = \frac{2L_2}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

THE DENOMINATORS REPRESENT THE MODIFICATIONS IN THE TIMES CAUSED BY THE MOTION OF THE APPARATUS AND THEY ARE <u>NOT</u> EQUAL. WE MIGHT THINK $L_1 \neq L_2$ BUT THEY NEEDN'T BE EQUAL SINCE ROTATING THE APPARTUS THROUGH 90° NULLS OUT ANY INEQUALITIES WHICH SURELY EXIST. WE THEN LOOK FOR A SHIFT IN THE INTERFERENCE FRINGES WHEN WE ROTATE THE APPARATUS.

SINCE THE EARTH HAS A ORBITAL SPEED OF ABOUT 18 MILES/SEC, $v/c \sim 10^{4}$ SO THEY WERE LOOKING FOR DIFFERENCES ONE PART IN 10^{8} WHICH CORRESPONDS TO A FEW FRINGES. SUCH AN ACCURACY WAS OBTAINABLE BUT NO TIME DIFFERENCE WAS FOUND.

THIS WAS THE BEGINNING OF THE TROUBLE. THIS EXPERIMENT PLUS NEWTON'S LAWS AND MAXWELL'S IDEA OF A FIXED SPEED OF LIGHT LEAD TO INCONSISTENCIES.

AFTER TRYING TO REWORK MAXWELL'S EQUATIONS AND CREATING SOME NEW PHENOMENA WHICH COULD NOT BE OBSERVE, NEWTON'S LAW'S FINALLY CAME UNDER SCRUTINY. THE IDEA OF RELATIVITY WAS ESTABLISHED BUT APPARENTLY HE HAD THE WRONG TRANSFORMATION EQUATIONS. THE IDEA THAT LIGHT TRAVELED AT 186,000 MI/SEC WAS ACCEPTED AND POINCARÉ ESTABLISHED THE IDEA THAT THERE WAS NO WAY TO DETERMINE AN ABSOLUTE VELOCITY.

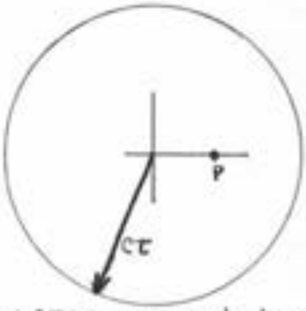
IT WAS LEFT TO LORENTZ TO SOLVE THE APPARENT PARADOX OF THE MICHELSON-MORLEY EXPERIMENT. HIS CONTRIBUTION WAS A NEW SET OF TRANSFORMATION EQUATIONS CONSISTENT WITH BOTH MAXWELL'S EQUATIONS AND THE PRINCIPLE OF RELATIVITY. THEY ARE:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

FURTHERMORE, AT VELOCITIES MUCH LESS THAN C THESE EQUATIONS REDUCE TO THOSE OF NEWTON. WE WANT TO DISCUSS THE CHARACTER OF THE TERMS APPEARING IN THESE EQUATIONS AND SHOW THEY MAKE SENSE.

SUPPOSE WE ARE AT THE ORIGIN OF A REFERENCE FRAME AND WE LET OUR A BURST OF LIGHT; WHERE IS THE WAVE FRONT AT TIME t? WELL, THAT'S EASY; IT'S AT $r = ct$. WHAT'S LITE UP THEN? AGAIN THAT'S EASY; IT'S JUST A SPHERE WHOSE EQUATION IS:

$$[ct]^2 - x^2 - y^2 - z^2 = 0$$



NOW IMAGINE SOME GUY GOING BY AND IS AT POINT P AT TIME t. ASSUME HE ALSO WAS AT THE ORIGIN WHEN THE LIGHT WAS EMITTED. MIRACLES OF MIRACLES THIS JOKER IS GOING TO THINK HE IS IN THE MIDDLE OF THE SAME DAMN SPHERE WHICH SOUNDS INCREDIBLE. LET'S CALCULATE WHAT HE WOULD SEE: THE SPHERE IN <u>HIS</u> COORDINATES WOULD BE GIVEN BY

$$[ct']^2 - x'^2 - y'^2 - z'^2 = 0$$

USING OUR TRANSFORMATIONS:

$$c^2 \frac{(t - \frac{vx}{c^2})^2}{(1 - v^2/c^2)} - \frac{(x - vt)^2}{(1 - v^2/c^2)} - y^2 - z^2 = 0$$

LOOKING AT THE t^2 TERMS

$$\frac{c^2 t^2}{(1 - v^2/c^2)} - \frac{v^2 t^2}{(1 - v^2/c^2)} = \frac{(c^2 - v^2)\, t^2}{(1 - v^2/c^2)} = c^2 t^2 = c^2 t'^2$$

IT IS EASY TO SEE THE CROSS TERMS SUBTRACT OUT

$$-c^2\left(2\frac{tvx}{c^2}\right) + 2xtv = 0$$

NOW THE x^2 TERMS:

$$\frac{c^2\frac{v^2x^2}{c^4}}{1-v^2/c^2} - \frac{x^2}{(1-v^2/c^2} = \frac{-x^2(1-v^2/c^2)}{(1-v^2/c^2)} = -x'^2$$

THE RESULT IS THAT THE TWO MEN SEE THE SAME SPHERE! THAT IS

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

or

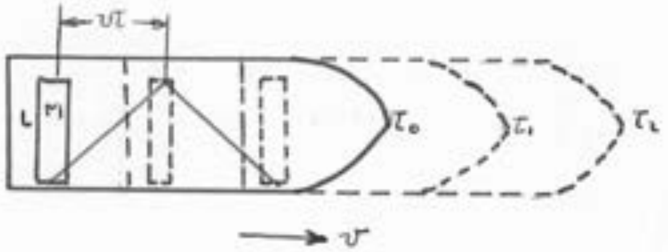
$$ct - r = ct' - r' = 0$$

SUCH A QUANTITY THAT DOES NOT CHANGE ITS LENGTH UNDER A TRANSFORMATION IS AN INVARIANT. STATED ANOTHER WAY: AN INVARIANT DOES NOT CHANGE OR DEPEND ON THE POINT OF VIEW.

WE WOULD LIKE TO UNDERSTAND THIS TRANSFORMATION A LITTLE BETTER SO LET'S LOOK AND SEE, FIRST, WHAT HAPPENS TO TIME. OUR IMMEDIATE OBSERVATION IS THAT $t' \neq t$ SO THAT A MOVING CLOCK CANNOT HAVE THE SAME TIME SCALE AS A STATIONARY CLOCK. HOW CAN THIS BE? WELL FIRST WE MUST DECIDE ON WHAT WE MEAN BY A CLOCK. CAN WE USE A DECAYING μ-MESON OR PERHAPS THE MORE CONVENTIONAL SPRING-MASS CLOCK? NO! WE MUST INVENT A CLOCK WHOSE ULTIMATE MECHANISM IS LIGHT ITSELF. ONE SUCH A CLOCK WOULD BE A HOLLOW ROD, PERHAPS A METER LONG, AT EACH END IS FIXED A MIRROR. WHEN A LIGHT PULSE IS EMITTED AT THE "BOTTOM" IT GOES UP, BOUNCES, AND COMES BACK TO THE BOTTOM - THIS IS ONE CLICK.

NOW WE MAKE TWO OF THESE CLOCKS AND GIVE ONE THE FELLOW IN THE SPACE SHIP AND ONE TO THE FELLOW ON THE GROUND. THEY INITIALLY SYNCHRONIZE THEM AND AGREE THE VERTICAL DISTANCE BETWEEN THE MIRRORS DOES NOT CHANGE. THEY DO THIS WITH LITTLE KNIVES ON BOTH ENDS OF THE ROD WHICH WOULD SCRATCH THE OTHER ROD AS IT GOES BY. BY SYMMETRY THE TWO SETS OF KNIVES MUST RUB BLADES OR A SCRATCH WILL BE LEFT BELOW THE OTHER SET. SINCE THE MOTION IS RELATIVE, THE ARGUMENT COULD BE REVERSED AND CONCLUDED THE MOVING GUY'S KNIVES SCRATCHED OUTSIDE THE "STATIONARY" ONES. THUS $y=y'$ IS A NECESSARY CONSEQUENCE OF THE SYMMETRY OF NATURE.

LET'S LOOK AT THE MOTION OF THE CLOCK AND LIGHT PATH AS VIEWED FROM THE STATIONARY OBSERVER:



The first things S, The stationary guy, discovers is That M's clock is going slow by a factor of $(1-v^2/c^2)^{1/2}$. To see how This could happen let's look at The above drawing from The point of view of M. The round trip time he measures, i.e., a click is given by

$$t = \frac{2L}{c}$$

But as The light propagates The tube is being translated Through $v\tau/2$. Since The light must stay in The tube, it must follow The diagonal shown in The above drawing. That is, light travels farther in The moving system; it Therefore takes longer to make a round trip or a click. To The stationary observer watching his clock he sees his click and has to wait for The moving clock to tick off its "second". We calculate This path length followed by The translating clock just as before.

$$L^2 = c^2t^2 - v^2t^2 = t^2(c^2-v^2)$$

or

$$L = tc\sqrt{1-v^2/c^2}$$

We see The "effective" speed of light is changed by a factor of $\sqrt{1-v^2/c^2}$. Comparing our two results

$$\frac{t_s}{t_M} = \frac{2L/c}{\frac{2L}{c}\frac{1}{\sqrt{1-v^2/c^2}}} = \sqrt{1-v^2/c^2}$$

So we conclude The moving clock is slower by a factor of $(1-v^2/c^2)^{-1/2}$, i.e.,

$$t_m = \frac{t_s}{\sqrt{1-v^2/c^2}}$$

I want to stress of generality of This result; it is independent of The type of clock we use. Be it radioactive decay or wheels and gears The slow-down factor is still $1/\sqrt{1-v^2/c^2}$. Since all clocks go slower in a moving space ship, we must accept That all phenomena such as biological movements are also slowed down – That is, we would live longer in a faste moving space ship!

LORENTZ CONTRACTION

ONE MORE CONSEQUENCE OF THIS EXPERIMENT. If WE TURN THE ROD 90 DEGREES SO IT IS LYING ALONG THE DIRECTION OF MOTION, WE FIND THE LENGTH OF THE ROD HAS CHANGE. IN FACT, IT CONTRACTS BY ~~by~~ THE SQUARE ROOT $\sqrt{1-v^2/c^2}$ OR

$$L_{\parallel} = L_0 \sqrt{1-v^2/c^2}$$

THUS THE MOVING OBSERVER IS MEASURING THE LIGHT PATH OVER A SHORTENED RULER.

SIMULTANEITY

SO FAR WE DON'T UNDERSTAND THE TERM vx/c^2 IN THE TIME TRANSFORMATION BUT THIS IS QUITE INTERESTING AND UNEXPECTED. WE WANT TO TALK ABOUT 2 EVENTS LIKE FLASHES OF LIGHTS OCCURRING AT A DEFINITE TIME. IT TURNS OUT TWO EVENTS OCCURING SIMULTANEOUSLY TO THE MOVING GUY WILL NOT APPEAR SIMULTANEOUS TO THE STATIONARY OBSERVER. LET'S DEVISE AN EXPERIMENT TO SEE HOW THIS COULD BE.

SUPPOSE M HAS TWO CLOCKS LIKE WE DESCRIBED EARLIER ONE AT THE FRONT OF HIS SHIP THE OTHER AT THE REAR. AT THE CENTER OF HIS APPARTUS HE HAS A LIGHT SOURCE. WHEN A BURST OF LIGHT IS RELEASED, IT SPREADS OUT AND TRAVELS BOTH FORWARD AND BACKWARD. WE DEFLECT THE LIGHT UP THE RODS AND IT BOUNCES BACK TO STRIKE A COUNTER OR BELL OR SOMETHING TO TELL YOU WHEN IT HITS THE BOTTOM.

WITH THIS EXPERIMENT M IS CONVINCED THE TWO RAYS STRIKE THE BOTTOM OF THE RODS AT THE SAME TIME. BUT WHAT DOES THE STATIONARY OBSERVER SEE?

WELL, HE WOULD AGREE WHEN THE FLASH WAS EMITTED BUT HE ALSO NOTES THE FRONT CLOCK, F, IS MOVING AWAY FROM THE LIGHT AT SPEED v WHILE THE REAR CLOCK, R IS APPROACHING THE LIGHT AT THE SAME SPEED v. OBVIOUSLY THEN THE REAR CLOCK WOULD REGISTER BEFORE THE FRONT ONE! HE COULD NOT SAY THE EVENTS OCCURRED SIMULTANEOUSLY. THUS WE HAVE A "FAILURE OF SIMULTANEITY AT A DISTANCE" AND SIMULTANEITY IS NO LONGER A UNIQUE MOMENT. THE ONLY CONCLUSION WE CAN MAKE IS EQUAL VALUES OF t' IN M'S SYSTEM CORRESPOND TO DIFFERENT VALUES OF t IN S'S SYSTEM. THE CONNECTION OF THE TWO TIME SCALES IS GIVEN BY

$$t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}}$$

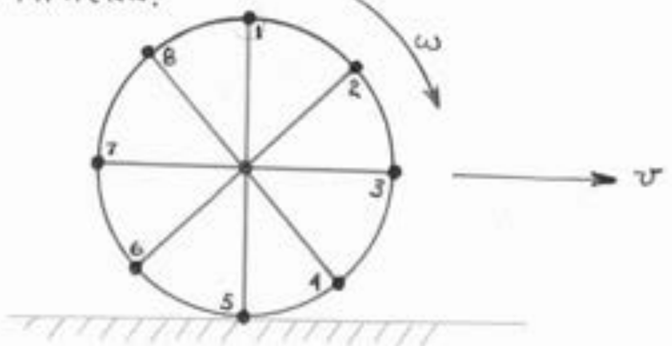
THE WHEEL ANALOGY

I WANT TO GO ON WITH THE DISCUSSION OF RELATIVITY THAT I STARTED TO DEVELOPE LAST TIME. HOWEVER, I WAS TOLD TO LECTURE ON A VERY ELEMENTARY LEVEL TO YOU. THIS DISTURBS ME A LITTLE BECAUSE ALL OF YOU ARE COLLEGE GRADUATES AND I SHOULD BE ABLE TO TALK ON A LITTLE HIGHER LEVEL. SO I'LL CHEAT AND INSTEAD OF REPEATING THE MATERIAL I TAUGHT TO THE FRESHMEN AND SOPHOMORES AT CALTECH, I'LL TRY TO GIVE SOME NEW MATERIAL.

THE TRANSFORMATION EQUATIONS I DEVELOPED WERE THOSE OF LORENTZ AND WHICH I SHALL REWRITE HERE:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

NOW I WOULD LIKE TO GIVE AN ANALOG OF HOW TO INTERPRET THESE EQUATIONS BY DESCRIBING WHAT WE SEE OF A ROLLING, TRANSLATING WHEEL. SUPPOSE WE HAVE A WHEEL WITH SPOKES RADIATING OUT IN THE FOLLOWING PATTERN:

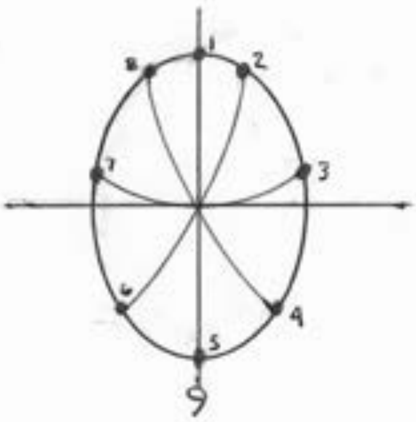


TO AN OBSERVER FIXED IN THE FRAME TRANSLATING WITH VELOCITY v THE EQUATION OF THE MOTION COULD BE WRITTEN

$$x' = a \cos \omega t'$$
$$y' = a \sin \omega t'$$

IN THIS FRAME MASSES 3 AND 7 ARE SAID TO SIMULTANEOUSLY CROSS THE HORIZONTAL AXIS; MASS 3 PASSING DOWNWARD AT THE INSTANT AND MASS 7 MOVING UPWARD.

TO A STATIONARY OBSERVER THE SHAPE OF THE WHEEL IS NOLONGER CIRCULAR BUT TAKES ON AN ELLIPSOIDAL SHAPE WHICH IS SHRUNK EXACTLY AS $\sqrt{1-v^2/c^2}$:



Because the wheel is turning we have to worry about a few things. One there is a contraction at the top which is higher than the mean contraction of the wheel. This moves masses 2 and 8 closer to mass 1 and masses 4 and 6 are farther away from mass 5. Also masses 3 and 7 are no longer simultaneously crossing the horizontal. Mass 7 is already past it and mass 3 has yet to cross it.

As it turned out Feynman was very sorry he brought this analogy up because it is necessary to define the wheel in such a way that general relativity, involving curved space, got into the discussion. In essence, what he wanted to show was given the wheel in stationary motion, i.e., not translating for one observer, the equation of how it should look to someone else could be derived knowing the transformation equations.

Transformation of Velocities

Ref chapter 16 Vol I

Another important problem is the transformation of velocities. We imagine a space ship passing by in which an object is shot forward at velocity u. To the man traveling in the ship he says the bullet has traveled

$$x' = ut'$$

To find out what velocity the guy on the ground sees we must calculate the apparent velocity seen by this stationary observer. The answer is not as simple as $u + v$ if the ship is moving a speed v. We need to then calculate

$$V_{App} = \frac{x}{t}$$

To do this we need the inverse Lorentz transformations which are easily obtained by symmetry as

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \qquad y = y' \qquad z = z' \qquad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

We say easily obtained because all we need do is change $v \rightarrow -v$ since the stationary observers "appears" to be moving in the opposite direction than the space-ship. Thus we have, inserting $x' = ut'$:

$$x = \frac{ut' + vt'}{\sqrt{1 - v^2/c^2}} \qquad y = y' \qquad z = z' \qquad t = \frac{t' + \frac{vut'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

so that

$$V_{App} = \frac{x}{t} = \frac{ut' + vt'}{t' + \frac{vu}{c^2} t'}$$

DIVIDING OUT The t' WE hAVE,

$$V_{APP} = \frac{u+V}{1+\frac{uV}{c^2}}$$

ThIS RELATION TELLS US how TO ADD VELOCITIES IN RELATIVITY. To check This formulA ASSUME $u = c$ Then

$$V_{APP} = \frac{c+V}{1+\frac{V}{c}} = c$$

SO IT DOESN'T APPEAR LIGHT CAN TRAVEL FASTER ThAN C. Suppose $u = \frac{1}{2}c$ And $V = \frac{1}{2}c$,

$$V_{APP} = \frac{\frac{1}{2}c + \frac{1}{2}c}{1+\frac{1}{4}\frac{c^2}{c^2}} = \frac{4}{5}c$$

AND WE DON'T QUITE MAKE IT TO The SPEED of LIGHT.

WE CAN ChANGE ThIS ABOVE EQUATION TO AN AddITION LAW if WE TRANSform The VELOCITIES LIKE

$$u = \tanh W_1 \qquad V = \tanh W_2$$

Then,

$$V_{APP} = \tanh(W_1 + W_2)$$

BUT This MAY be beyond A few of YOU SO I WON'T PURSUE This.

RELATIVISTIC DYNAMICS

There Are OTher QUANTITIES which WE would LIKE TO TRANSform which Are NOT SO EASY TO DO. WE NEED TO ESTABLISh A GUIDE TO USE These TRANSfORMATION LAWS. ThAT IS, WE MUST hAVE SOME PhYSICAL LAWS which "TELL" YOU They MUST be The SAME Under A LORENTZ TRANSformATION. ONE Such SET IS The MAXWELL EQUATIONS which, of course, GOT This whole MESS GOING. BUT These EQUATIONS MUST be INVARIANT Under A LORENTZ TRANSformATION. IT IS This DYNAMIC CASE rATher ThAN The KINEMATIC CASE WE hAVE discussed so far ThAT IMPOSES SOME INTERESTING CRITERIA ON The LAWS of NATURE.

WhAT I WANT TO TELL YOU IS how The Three COMPONENTS of The MOMENTUM AND ENERGY TRANSform. I WANT TO show They TRANSform JUST LIKE x, y, z And t RESPECTIVELY. ThAT IS,

$$P'_x = \frac{P_x - VE/c^2}{\sqrt{1-V^2/c^2}} \qquad P'_y = P_y \quad P'_z = P_z \quad E' = \frac{E - P_x V}{\sqrt{1-V^2/c^2}}$$

where E/c^2 IS ANALOGOUS TO TIME SO WE CALL IT The TIME COMPONENT of The MOMENTUM.

I guess I would like to start over by showing how Newton's laws are changed by relativity. We have a relationship which says

$$\vec{F} = \frac{d}{dt}\left[\frac{m_0}{\sqrt{1-V^2/c^2}}\frac{d\vec{R}}{dt}\right]$$

where $$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

We see that $\vec{F} \neq m\vec{a}$ anymore as we mentioned last time because the relativistic mass is now a function of velocity which must inside the differential since

$$\vec{F} = \frac{d}{dt}\vec{P}$$

And $\vec{P}$ is given by $$\vec{P} = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{d\vec{R}}{dt}$$

This is Einstein's modification of Newton's laws. It is this momentum which must be conserved.

Now what we wanted to show then is that $P_x = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{dx}{dt}$ transforms just like x and so forth. To discuss this relation we get involved with unsymmetrical terms like dx/dt. To get around this difficulty suppose we have a particle moving along some path defined at a given time t by the parameters

$$x(t),\ y(t),\ z(t),\ t$$

Now in another system the equations of the particle are given by $x'(t'), y'(t'), z'(t'), t'$ and the transformations relations get a little messy.

Suppose we define some new parameter α which changes with time so our coordinate becomes:

$$x(\alpha),\ y(\alpha),\ z(\alpha),\ t(\alpha)$$

Then the transformation to the prime coordinate is

$$x'(\alpha) = \frac{x(\alpha) - Vt(\alpha)}{\sqrt{1-V^2/c^2}} \qquad \text{etc.}$$

For the velocities we then have

$$\frac{dx}{d\alpha} = \lambda_x \ , \quad \frac{dy}{d\alpha} = \lambda_y \ , \quad \frac{dz}{d\alpha} = \lambda_z \quad \frac{dt}{d\alpha} = \lambda_t$$

so that $$\lambda_x' = \frac{\lambda_x - V\lambda_t}{\sqrt{1-V^2/c^2}}$$ and the velocities transform the same if we find some parameter α which we all can agree upon. That is, we want to establish some scale which orders the moments along the line which we all agree to. Such a factor is obviously not time.

PROPER TIME

SUPPOSE WE RIDE ALONG WITH THE PARTICLE LORENTZ TRANSFORMING THE TIME AS WE GO. THE INTERVAL BETWEEN TWO MOMENTS CLOSE TO EACHOTHER IS INDEPENDENT OF THE COORDINATE SYSTEM USED AND IS DEFINED TO BE THE PROPER TIME BETWEEN THE EVENTS. THIS TIME MEASURES HOW LONG A CLOCK MOVING WITH THE PARTICLE CLICKS OFF. IT IS DETERMINED BY THE RELATION

$$\Delta S = (\Delta t)\sqrt{1-V^2/c^2}$$

ΔS IS THE TIME INTERVAL IN GOING FROM MOMENT 1 TO 2 MEASURED IN THE REST COORDINATE FRAME.

WE NOW UNDERSTAND THAT THE MOMENTUM IS THE RATE OF CHANGE WITH RESPECT TO THE PROPER TIME, I.E.,

$$P_x = m_0 \frac{dx}{dS} \qquad P_Y = m_0 \frac{dY}{dS} \qquad P_z = m_0 \frac{dz}{dS} \qquad P_t = m_0 \frac{dt}{dS}$$

THIS IS EASY TO SEE IF WE SUBSTITUTE FOR dS, $dt\sqrt{1-V^2/c^2}$. THE EQUATIONS WE GET ARE

$$P_x = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{dx}{dt} \qquad P_Y = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{dY}{dt} \qquad P_z = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{dz}{dt} \qquad P_t = \frac{m_0}{\sqrt{1-V^2/c^2}}$$

WHICH IS EXACTLY WHAT WE GET FROM

$$\vec{P} = \frac{m_0}{\sqrt{1-V^2/c^2}}\frac{d\vec{R}}{dt}$$

WE SEE THEN THAT THE SQUARE ROOT $\sqrt{1-V^2/c^2}$ APPEARS IN THE DENOMINATOR OF THE MOMENTUM NOT BECAUSE SOMEBODY WANTED TO PUT IT THERE ARBITRARILY BUT RATHER BECAUSE IT MAKES THE MOMENTUM LORENTZ INVARIANT. THAT IS, BY SHOWING THE PROPER TIME IS THE SAME REGARDLESS OF THE OBSERVER'S FRAME WE THUS SHOW THE MOMENTUM IS THE SAME.

SINCE WE RECALL THE DEFINITION OF THE MOMENTUM TO BE

$$\vec{P} = m\vec{V}$$

WE MUST EQUATE m WITH A FUNCTION OF THE VELOCITY,

$$m = \frac{m_0}{\sqrt{1-V^2/c^2}}$$

A CONSEQUENCE OF THIS RELATION SAYS WHEN YOU HEAT A BOX FULL OF GASS MOLECULES IT WEIGHS MORE BECAUSE THE INCREASED VELOCITY. SUPPOSE $V \ll C$ SO WE CAN WRITE TO A GOOD APPROXIMATION,

$$m \simeq m_0 + \tfrac{1}{2} m_0 V^2 \frac{1}{c^2}$$

THUS, THE INCREASE IN MASS IS PROPORTIONAL TO THE INCREASE IN TEMPERATURE. FURTHER WE CAN WRITE

$$m - m_0 = \Delta m = \frac{\Delta\, K.E.}{c^2}$$

FINALLY, The fourth COMPONENT of The MOMENTUM P_t IS The TIME COMPONENT. WE hAVE The relation

$$P_t = \frac{E}{c^2} = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

OR,

$$E = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = mc^2$$

where m = DYNAMIC MASS

EQUIVALENCE of MASS & ENERGY

Ref. Chapter 15 Vol I

WE would NOW LIKE TO show how ENERGY AND MASS ARE related. If WE ASSUME AS God GIVEN FACTS The following 3 relations:

$$E = mc^2 \qquad \overline{P} = m\vec{V} \qquad \& \qquad \vec{F} = \frac{d}{dt}\vec{P}$$

WE CAN show $m = m_0/\sqrt{1-v^2/c^2}$

TO DO This TAKE The DOT Product of $\overline{V}$ with $\overline{F}$ TO CALCULATE The rATE of change of The ENERGY

$$\frac{dE}{dt} = \overline{F}\cdot\overline{V} = v\frac{d\,mv}{dt}$$

OR

$$dE = v\,d(mv) = v^2 dm + vm\,dv$$

NOW $E = mc^2$ SO

$$c^2 dm = v^2 dm + vm\,dv$$

$$dm\,(c^2 - v^2) = mv\,dv$$

$$\frac{dm}{m} = \frac{v\,dv}{c^2 - v^2}$$

INTEGRATING THIS

$$\ln m = -\tfrac{1}{2}\ln(c^2 - v^2) + \text{CONSTANT}$$

To find The CONSTANT WE ASSUME AT $v = 0$ $m(v) = m_0$ Then

$$\ln m_0 = -\tfrac{1}{2}\ln c^2 + \text{CONSTANT}$$

SUBTRACTING,

$$\ln\frac{m}{m_0} = -\tfrac{1}{2}\ln(1 - v^2/c^2)$$

EXPONENTIATING

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

This EQUIVALENCE of MASS And ENERGY hAS been MEASURED EXPERIMENTALLY. When AN ELECTRON And POSITRON ANNIhILATE EACH-other The DO SO BY EMITTING 2 GAMMA rAYS BOTh WITh ENERGY m_0c^2.

SPACE-TIME INTERVALS

I SHALL BEGIN TODAY'S TALK BY WRITING DOWN, ONCE AGAIN, THE LORENTZ TRANSFORMATION BETWEEN "MOVING" AND "STATIONARY OBSERVERS:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

WE WOULD LIKE TO GET OF THE C'S IN THESE EQUATION AND DO SO BY DEFINING OUR UNITS SO THAT $c = 1$. THIS MEANS TIME IS MEASURED IN TERMS OF DISTANCE LIKE A METER OF TIME. THIS IS SIMPLY THE TIME IT TAKES LIGHT TO TRAVEL A METER, I.E, $1/3 \times 10^{-8}$ SEC. YOU CAN USE DIMENSIONAL ANALYSIS WHEN YOU ARE THROUGH TO SEE IF RESULTS CHECK. THUS WE HAVE

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

THESE EQUATIONS PERMIT US TO MEASURE TIME AND SPACE IN THE SAME UNITS

SINCE WE SHOWED LAST TIME THAT ENERGY-MOMENTUM TRANSFORM JUST LIKE TIME-SPACE WE HAVE IN OUR NEW NOTATION

$$P_x' = \frac{P_x - EV}{\sqrt{1 - V^2}} \qquad P_y' = P_y \quad P_z' = P_z \qquad E' = \frac{E - VP_x}{\sqrt{1 - V^2}}$$

WE MIGHT RECALL AN ANALOGOUS RELATION BETWEEN TWO SYSTEMS - THAT BEING, A ROTATION WHERE THE PRIME AND UNPRIMED COORDINATES ARE RELATED BY:

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -y\cos\theta + x\sin\theta$$
$$z' = z$$

IT IS A SIMPLE MATTER TO RELATE VELOCITIES OF A PARTICLE UNDER SUCH A ROTATION. WE HAVE JUST

$$U_x' = U_x\cos\theta + U_y\sin\theta$$
$$U_y' = -U_y\cos\theta + U_x\sin\theta$$
$$U_z' = U_z$$

IN OTHER WORDS IT IS QUITE EASY TO REPRESENT MANY PHYSICAL QUANTITIES AS VECTORS WHICH WE CAN PROJECT OF DIFFERENT AXES.

NOW WHEN WE ARE DEALING IN FOUR DIMENSIONS, I.E., SPACE-TIME, THE LORENTZ TRANSFORMATIONS ARE NOT EXACTLY THE SAME ~~FY~~ MATHEMATICAL TRANSFORMATIONS AS A ROTATION IN 3 DIMENSIONS. WHILE THE GEOMETRY IS VERY SIMILAR, IT CERTAIN RESPECTS IT IS PECULIAR. WHAT WE WOULD LIKE TO FIND IS A ~~FOURTH~~ FOURTH COMPONENT WHICH, TOGETHER WITH THE KNOWN 3-VECTOR COMPONENTS, ROTATE THE SAME WAY AS POSITION AND TIME IN THE SPACE-TIME TRANSFORMATIONS. IF WE FIND A SET OF 4 SUCH QUANTITIES WHICH DO TRANSFORM LIKE POSITION AND TIME, WE CALL IT A 4 VECTOR

WE HAVE IN FACT FOUND FOUR QUANTITIES AND THEY ARE E, P_x, P_y, P_z. THUS ENERGY AND MOMENTUM FORM A 4 VECTOR WITH THE ENERGY BEING THE TIME PART AND THE MOMENTUM THE POSITION PART.

BUT WHAT ABOUT VELOCITY? WE ALREADY KNOW U_x, U_y, U_z BUT WHAT ABOUT U_T? IS THERE SUCH AN ANIMAL WHICH GIVES US A FOUR VELOCITY? WE ARE TEMPTED TO SAY ~~SR~~ YES SINCE $U_x = \frac{dx}{dt}$, $U_y = \frac{dy}{dt}$, $U_z = \frac{dz}{dt}$, OBVIOUSLY $U_T = \frac{dt}{dt}$. ~~UN~~ BUT WE REMEMBER $\frac{dx}{dt}$ IS UNSYMMETRICAL AND THAT, IN FACT, $U_x' = \frac{U_x - V}{1 - U_x V}$ WHICH IS NOT A LINEAR TRANSFORMATION. WE MUST BE CAREFUL THEN; EVERY VECTOR DOES NOT HAVE SOME UNIQUE THING WHICH IT CAN BE JOINED WITH TO FORM A 4-VECTOR. EVEN THOUGH $dx/dt, dy/dt, dz/dt, dT/dt$, DEE-DAH, DEE-DAH, HAS GOT GODDAMN RHYTHMN TO IT WE MUST CONVERT TO PROPER TIME $ds = dt\sqrt{1-V^2}$.

WHAT WE GET A NEW VELOCITY CALL IT THE PROPER VELOCITY, THE HOKEY-POKEY VELOCITY, THE SCREWED-UP VELOCITY OR ANYTHING YOU GODDAMN WANT WHICH TURNS OUT TO BE

$$\frac{U_x}{\sqrt{1-U^2}}, \quad \frac{U_y}{\sqrt{1-U^2}}, \quad \frac{U_z}{\sqrt{1-U^2}}, \quad \frac{1}{\sqrt{1-U^2}}$$

IT MIGHT BE GOOD TO SHOW THE IDIOT FORMULA $U_x' = \frac{U_x - V}{1 - U_x V}$ DOES, IN FACT, SATISFY THESE EQUATION. SO I HAVE TO DO SOME MATH. TO CHECK TO SEE IF OUR IMPLICATIONS ARE RIGHT. THUS

$$t' = \frac{t - Vx}{\sqrt{1-V^2}} \quad \longrightarrow \quad \frac{1}{\sqrt{1-(U')^2}} = \frac{\frac{1}{\sqrt{1-U^2}} - V\frac{U}{\sqrt{1-U^2}}}{\sqrt{1-V^2}}$$

$$x' = \frac{x - Vt}{\sqrt{1-V^2}} \quad \longrightarrow \quad \frac{U'}{\sqrt{1-U^2}} = \frac{\frac{U}{\sqrt{1-U^2}} - \frac{V}{\sqrt{1-U^2}}\cdot 1}{\sqrt{1-V^2}}$$

IF WE DIVIDE THE ANALOGUE OF x' BY t', WE SEE THAT

$$U_x' = \frac{U_x - V}{(1 - U_x V)},$$

WITH U_x IN THE X DIRECTION ONLY AND V THE VELOCITY OF THE OBJECT PASSING THE STATIONARY OBSERVER. ALSO WE WANT TO SHOW $t' = \frac{1 - UV}{\sqrt{1-U^2}\sqrt{1-V^2}}$. FROM $U_x' = \frac{U_x' - V}{(1 - U_x V)}$,

$$(U')^2 = \frac{U^2 - 2UV + V^2}{1 - 2UV + U^2V^2}, \quad 1 - (U')^2 = \frac{1 - U^2 - V^2 + U^2V^2}{1 - 2UV + U^2V^2}$$

$$\sqrt{1-(U')^2} = \frac{\sqrt{(1-U^2)}\sqrt{1-V^2}}{(1 - UV)} \quad \longrightarrow \quad \frac{1}{\sqrt{1-(U')^2}} = \frac{1 - UV}{\sqrt{1-U^2}\sqrt{1-V^2}} \quad \text{Q.E.D.}$$

MORE ABOUT FOUR-VECTORS

With 3-vectors we had an invariant quantity which was the length of the vector. This means the length does not change under a rotation only, i.e., there is no translation of the origin. Two weeks ago we learned that a similar quantity is preserved in 4-vector algebra, i.e., we showed

$$t'^2 - (x')^2 - (y')^2 - (z')^2 = t^2 - x^2 - y^2 - z^2$$

Another invariant we have determined already is,

$$E^2 - P_x^2 - P_y^2 - P_z^2$$

In general then any four vector a_t, a_x, a_y, a_z has associated with it an invariant quantity $a_t^2 - a_x^2 - a_y^2 - a_z^2$

But what the hell does $E^2 - P_x^2 - P_y^2 - P_z^2$ equal? I remind you last time we learned

$$P_x = E\,U_x = \frac{m_0 U_x}{\sqrt{1-U^2}} \qquad \text{where} \quad E = \frac{m_0}{\sqrt{1-U^2}}$$

likewise for P_y & P_z.

We have then,

$$\frac{m_0^2}{1-U^2} - \frac{m_0^2 U_x^2}{1-U^2} - \frac{m_0^2 U_y^2}{1-U^2} - \frac{m_0^2 U_z^2}{1-U^2} = m_0^2 \frac{(1 - U_x^2 - U_y^2 - U_z^2)}{1-U^2} = m_0^2$$

Thus,

$$E^2 - P_x^2 - P_y^2 - P_z^2 = m_0^2$$

That is, the 4-vector invariant of energy-momentum is the mass squared. When the body is at rest $\vec{P} = 0$ then $E = m_0$ or dimensionally correct $E = m_0 c^2$.

The invariant of the proper velocity is easy; it's just one:

$$\frac{1}{1-U^2} - \frac{U_x^2}{1-U^2} - \frac{U_y^2}{1-U^2} - \frac{U_z^2}{1-U^2} = \frac{1-U^2}{1-U^2} = 1$$

We need a notation for a four vector and one that is most popular is semi-unfortunate because it is unlike the 3-vector. We will write a_μ where it is understood $\mu = t, x, y, z$ in that order. We must form the equivalent of the scalar product for a 3-vector, i.e., the invariant which is written

$$a_\mu a_\mu = a_t^2 - a_x^2 - a_y^2 - a_z^2$$

where we must remember the negative sign on the space components.

There are several rules about 4-vector which we need:

(i). If C_μ is a 4-vector, then αC_μ is a 4 vector if α is invariant like 3, 7 or d/ds.

(ii). If C_μ and d_μ are 4 vectors, then $C_\mu + d_\mu = e_\mu$ another 4 vector

(iii) If a_μ and b_μ are 4 vectors, $a_\mu b_\mu = a_t b_t - a_x b_x - a_y b_y - a_z b_z$

Other useful 4-vector products are

$$P_\mu P_\mu = m_0^2 \qquad P_t = E$$

$$U_\mu U_\mu = 1$$

$$P_\mu = m_0 U_\mu = m_0 \frac{dX_\mu}{dS} \qquad \text{where} \quad U_\mu = \frac{dX_\mu}{dS}$$

$$(dS)^2 = dX_\mu\, dX_\mu$$

WHAT ABOUT FORCE? IS THERE A 4-FORCE? WE KNOW BY DEFINITION THAT, $F_x = \frac{dP_x}{dt}$, $F_Y = \frac{dP_Y}{dt}$, $F_z = \frac{dP_z}{dt}$

BUT AGAIN SINCE d/dt IS A LOPSIDED TERM WE MUST DIFFERENTIATE WITH RESPECT TO PROPER TIME, dS. FURTHERMORE, OUR 4^{th} COMPONENT of FORCE IS THE RATE OF CHANGE OF THE ENERGY $dE/dt = W$ OR MORE PROPERLY $\gamma W = dE/dS$. THUS WE CAN DEFINE A PROPER FORCE AS

$$f_\mu = \frac{dP_\mu}{dS} \quad \text{where } f_t = \frac{W}{\sqrt{1-U^2}} \quad f_x = \frac{F_x}{\sqrt{1-U^2}} \text{ ETC.}$$

SINCE $P_\mu = m_0 \frac{dX_\mu}{dS}$, $\quad f_\mu = m_0 \frac{d^2 X_\mu(S)}{dS^2}$

WE SEE THAT BY CHANGING TO PROPER TIME IN NEWTON'S EQUATION WE ESTABLISH THE CORRECT LORENTZ INVARIANT FORCE. AN INTERESTING INVARIANT QUANTITY WHICH I WILL LEAVE FOR YOU TO WORK OUT IS $f_\mu \cdot \frac{dX_\mu}{dS} = ?$

PRODUCTION OF ANTI-PROTONS VOL. II 25-4

AN INTERESTING ILLUSTRATION OF THE FOUR-VECTOR DOT PRODUCT IS IN THE CALCULATION OF ENERGY NECESSARY TO PRODUCE ANTI-PROTONS FROM PROTONS.

$$\text{PROTON} \longrightarrow \boxed{} \longrightarrow \begin{cases} \text{3 PROTONS} \\ \text{1 ANTI-PROTON} \end{cases}$$

IF WE CALL THE INCIDENT PROTON $P_{1\mu}$ AND THE TARGET PROTON $P_{2\mu}$ WE MUST HAVE

$$P_{1\mu} + P_{2\mu} = \mathbb{P}_\mu \quad \text{IN SOME SYSTEM}$$

IF $P_{1\mu}$ HAS JUST ENOUGH ENERGY TO MAKE THE REACTION GO $\mathbb{P}_\mu = (4M, 0, 0, 0)$

SINCE $\mathbb{P}_\mu \mathbb{P}_\mu$ IS INVARIANT

$$(P_{1\mu} + P_{2\mu})(P_{1\mu} + P_{2\mu}) = \mathbb{P}_\mu \mathbb{P} = (4M)^2$$

or

$$P_{1\mu}P_{1\mu} + P_{2\mu}P_{2\mu} + 2P_{1\mu}P_{2\mu} = 16M^2$$

$$2P_{1\mu}P_{2\mu} = 14M^2$$

SINCE $P_{1\mu}P_{1\mu} = E^2 - P^2 = M^2$

SO

$$P_{1\mu}P_{2\mu} = 7M^2$$

SINCE $P_{2\mu}$ IS AT REST $(M, 0)$ AND $P_{1\mu} = (E_1, \vec{P}_1)$

$$E_1 M = 7M^2$$

OR

$$E_1 = 7M \approx 938 \text{ MeV}$$

STILL MORE ON FOUR-VECTOR

I GAVE YOU A PROBLEM LAST TIME WHICH I WOULD LIKE TO WORK OUT FOR YOU. I ASKED WHAT THE PRODUCT OF THE FOUR-FORCE AND FOUR VELOCITY IS, I.E.,

$$f_\mu \frac{dx_\mu}{ds} = ?$$

THIS EQUATION MEANS

$$f_\mu \frac{dx_\mu}{ds} = f_t \frac{dt}{ds} - f_x \frac{dx}{ds} - f_y \frac{dy}{ds} - f_z \frac{dz}{ds}$$

WHERE WE RECALL THE FOUR FORCE IS

$$f_\mu = f_t(\text{the rate of work}), \frac{F_x}{\sqrt{1-v^2}}, \frac{F_y}{\sqrt{1-v^2}}, \frac{F_z}{\sqrt{1-v^2}} = m_0 \frac{d^2x_\mu}{ds^2}$$

AND

$$\frac{dx_\mu}{ds} = \frac{1}{\sqrt{1-v^2}}, \frac{v_x}{\sqrt{1-v^2}}, \frac{v_y}{\sqrt{1-v^2}}, \frac{v_z}{\sqrt{1-v^2}}$$

NOW IT IS EASY TO SHOW

$$\frac{dx_\mu}{ds}\frac{dx_\mu}{ds} = 1 = \left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 - \left(\frac{dz}{ds}\right)^2$$

BUT $ds^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$

AND THE RATE OF CHANGE OF 1 IS 0 SO

$$2 \frac{dx_\mu}{ds}\frac{d^2x_\mu}{ds^2} = 0$$

OR

$$f_\mu \frac{dx_\mu}{ds} = 0$$

WRITTEN ANOTHER WAY,

$$f_\mu \frac{dx_\mu}{ds} = \frac{\text{rate of work}}{\sqrt{1-v^2}} - \frac{[F_x v_x + F_y v_y + F_z v_z]}{\sqrt{1-v^2}}$$

SINCE THE DIFFERENCE OF THE TWO TERMS ON THE RIGHT MUST BE ZERO, WE CALL THE TERM IN THE BRACKETS THE RATE OF WORK ALSO.

THIS RESULT IS INTERESTING BECAUSE THE FOURTH COMPONENT OF THE FOUR FORCE IS DETERMINED BY THE 3 SPACE COMPONENTS F_x, F_y, F_z. THUS A PROPER FOUR FORCE MAKES THIS INVARIANT ZERO AND WE KNOW HOW TO FIND THE TIME COMPONENT.

APPLICATIONS OF FOUR VECTORS

I WOULD LIKE TO CONTINUE A LITTLE FURTHER AND GIVE YOU A COUPLE OF EXAMPLES OF WHAT YOU CAN DO WITH FOUR VECTORS. WE KNOW THE FOUR MOMENTUM IS GIVEN AS $P_\mu = (E, P_x, P_y, P_z)$ SO THAT THE PRODUCT

$$P_\mu P_\mu = E^2 - \bar{P}\cdot\bar{P} = m_0^2$$

WHERE WE RECALL, $E = \dfrac{m_0}{\sqrt{1-v^2}}$ $\quad \bar{P} = \dfrac{m_0 \bar{v}}{\sqrt{1-v^2}}$

SUPPOSE WE CONSIDER PARTICLES OF ZERO REST MASS, I.E., $m_0 = 0$. There ARE ONLY TWO SUCH PARTICLES KNOWN ~~B~~ EXPERIMENTALLY AND ONE Theorized; They ARE The PHOTON, NEUTRINO AND GRAVITON respectively. The GRAVITON BEING POSTULATE IN The QUANTUM Theory of GRAVITATION. When $m_0 = 0$ IT IS CLEAR $P_\mu P_\mu = 0$ which IMPLIES

$$E^2 = p^2 c^2$$

OR

$$\text{VELOCITY} = \frac{P}{E} = C$$

This follows from OR EQUATIONS for E & P:

$$\frac{P}{E} = \frac{m_0 v / \sqrt{1-v^2}}{m_0 / \sqrt{1-v^2}} = v = c$$

BUT WE NOTE A MEANINGLESS RESULT of This CALCULATION. We have JUST Show That 0 ~~MASS~~ REST MASS PARTICLES ONLY TRAVEL AT The SPEED of LIGHT. SINCE $E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$ WE SEE if $m_0 = 0$ And $v = C$ The ENERGY becomes INDETERMINATE, I.E., $0/0$.

For AN ELECTRON The REST ENERGY, $m_0 c^2$ IS About 0.5 MeV. IN The SYNCHROTRONS WE CAN GET Their ENERGY UP TO The BILLIONS of ELECTRON VOLTS (BeV's) where The SPEED of The ELECTRON Differs by 1 PART IN 10^{10} from C.

THE CHARACTER OF WAVES

I would LIKE TO SAY SOMETHING ABOUT WAVES. I feel STUPID TALKING ABOUT THIS SUBJECT; SURELY YOU KNOW SOMETHING ABOUT WAVES.

The MOST HAPPY WAVE I CAN THINK Of IS A COSINE WAVE PROPAGATING OR MOVING IN The PLUS Z DIRECTION. If WE ARE WATCHING A PRESSURE VARIATION, SAY, The CHARACTER of The WAVE IS DESCRIBED AS

$$P = P_0 \cos(\omega t - kz)$$

If $z=0$ AND The WAVE IS JUST OSCILLATING UP AND DOWN, The PERIOD of OSCILLATION IS

$$T = \frac{2\pi}{\omega}$$

AT A fixed TIME WE HAVE THAT,

$$k\lambda = 2\pi$$

OR

$$k = \frac{2\pi}{\lambda}$$

where k = NUMBER of WAVES PASSING A POINT IN 1 SEC
λ = WAVELENGTH

AS The WAVE MOVES ON IN TIME AND POSITION The SHAPE of The WAVE STAYS The SAME,

$$P = P_0 \cos(\omega t_1 + \omega\Delta t, - kz_1 - k\Delta z)$$

where

$$\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$$ = VELOCITY of A NODE OR The PHASE VELOCITY AS IT IS MORE COMMONLY CALLED

The PHASE VELOCITY SAYS HOW FAST The NODE IS MOVING DOWN The AXIS. IN The CASE of LIGHT

$$v_{phase} = \frac{\omega}{k} = c$$

OR

$$\omega = kc$$ = The ANGULAR FREQUENCY

MORE PROPERLY k, The WAVE NUMBER, IS The NUMBER of CYCLES PER CENTIMETER TIMES 2π. AND THAT'S WAVES!

NOW I CAN GET MORE difficult AND DESCRIBE MOTION IN SOME OTHER DIRECTION. CONSIDER A PRESSURE WAVE WHICH IS INDEPENDENT of The X-Y COORDINATES, i.e., WE HAVE A PLANE WAVE WHERE The PRESSURE IS CONSTANT OVER A PLANE. If The WAVE IS MOVING IN SOME ODD BALL DIRECTION The WAVE MOTION MIGHT BE CHARACTERIZED AS

$$P = P_0 \cos(\omega t - k_x X - k_y Y - k_z Z)$$

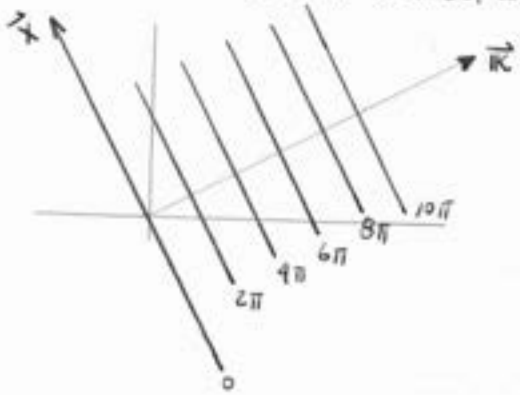
OR MORE SIMPLY

$$P = P_0 \cos(\omega T - \vec{K}\cdot\vec{X})$$

If WE EXAMINE The SHAPE of The PLANE WAVE AT A GIVEN INSTANT, The PHASE IS ZERO AND The fronts SATISFY

$$\vec{K}\cdot\vec{X} = 0$$

THIS EQUATION, $\vec{K}\cdot\vec{x}=0$, SAYS ANY POINT THROUGH THE ORIGIN HAS $\vec{K}\cdot\vec{x}=0$ AND THE SET OF PLANES AT $\pm 2n\pi$ ARE NORMAL TO $\vec{K}$. SO $\vec{K}$ IS A VECTOR DESCRIBING THE PROPAGATION OF THE WAVES. WHERE $|K| = \frac{2\pi}{\lambda}$.



IT TURNS OUT IN QUANTUM MECHANICS THAT PARTICLE AND WAVES HAVE AN INTERESTING RELATIONSHIP. PARTICLES DON'T BEHAVE NORMALLY; THEY BEHAVE IN THE GRAND QUANTUM MECHANICAL WAY. SOMETIMES THEY BEHAVE MUCH LIKE WAVES; OTHER TIMES MORE LIKE PARTICLES. SOMETIMES THEY HAVE TO BE EXPLAINED QUANTUM MECHANICALLY. THE CONNECTION BETWEEN THE WAVE-PARTICLE DUALITY IS OFTEN WRITTEN IN TWO EQUATIONS

$$E = \hbar\omega \qquad \bar{P} = \hbar\bar{K}$$

WHERE $\hbar = \frac{h}{2\pi} = 1.054\times10^{-27}$ erg-sec

THIS IS CALLED THE GRAND CONNECTION BUT IT IS NOT QUITE TRUE. IT WOULD BE BETTER TO WRITE THESE EQUATIONS AS

$$E <?> \hbar\omega \qquad \bar{P} <?> \hbar\bar{K}$$

WHERE WE HAVE INTRODUCED A MYSTERY SYMBOL SINCE WE DON'T REALLY UNDERSTAND THE CONNECTION. I WANT TO EMPHASIZE THAT WE SHOULD REALIZE THAT WE HAVE THIS LACK OF UNDERSTANDING OTHERWISE IT MIGHT CAUSE SOME DISCOMFORT LATER. THIS MIGHT SOUND SILLY BUT IT'S NOT. THE TRUE ~~$~~ PICTURE IS QUITE COMPLICATED AND I MUST POSTPONE THE DISCUSSION UNTIL YOU ARE READY FOR IT. BY NOT CLAIMING TO UNDERSTAND EVERYTHING I THINK YOU ARE BETTER OFF THAN IF YOU CLAIM TO KNOW IT ALL.

THE PROPAGATION VECTOR AS A FOUR-VECTOR

IN RELATIVITY WE TALKED ABOUT 4-VECTORS AND WE MIGHT TRY TO INVENT A POSSIBLE 4-VECTOR

$$k_\mu = (\omega, k_x, k_y, k_z)$$

WE MUST FIND OUT HOW THIS VECTOR TRANSFORMS SO THAT PRESSURE NODES, I.E., WHERE THERE IS NO PRESSURE, REMAINS INVARIANT UNDER A TRANSFORMATION. WE MUST INSURE THAT ONE MAN'S NODE IS ANOTHER MAN'S NODE.

RECALLING OUR TRANSFORMATION EQUATIONS WE CAN WRITE

$$\cos(\omega t - kx) = \cos\left[\omega\frac{(t'-vx')}{\sqrt{1-v^2}} - k\frac{(x'-vt')}{\sqrt{1-v^2}}\right]$$

$$= \cos\left[\frac{(\omega+kv)t'}{\sqrt{1-v^2}} - \frac{(k+\omega v)}{\sqrt{1-v^2}}x'\right]$$

$$= \cos\left[\omega' t' - k' x'\right]$$

THUS OUR TRANSFORMATION RELATIONS ARE

$$\omega' = \frac{\omega + kv}{\sqrt{1-v^2}} \quad \text{AND} \quad k' = \frac{k+\omega v}{\sqrt{1-v^2}}$$

SO, INDEED, k_μ IS A 4-VECTOR. THUS WE CAN WRITE

$$P = P_0 \cos(k_\mu x_\mu)$$

WITH

$$P_\mu = \hbar K_\mu$$

THE GROUP VELOCITY

WE HAVE DISCUSSED THE PHASE VELOCITY AND SAID IS IS $v_p = \frac{\omega}{|k|}$. NOW WE WANT TO CONSIDER THE VELOCITY OF A VARIATION IN THE WAVE. THAT IS, A SECTION OF BUMPS CREATED BY THE SUPERPOSITION OF MANY WAVES OF SLIGHTLY DIFFERENT FREQUENCIES. THE WAVE BECOMES MODULATED WITH BUMPS AND WE WANT TO DETERMINE HOW FAST THESE SIGNITURES MOVE DOWN THE PATH.

IF WE HAVE TWO WAVES OF FREQUENCY ω_1 AND ω_2 SUPERIMPOSED, THEN

$$P_{TOTAL} = P_{01}\cos(\omega_1 t - k_1 x) + P_{02}\cos(\omega_2 t - k_2 x)$$

WHILE $\quad v_{phase,1} = \frac{\omega_1}{k_1}$ AND $v_{phase,2} = \frac{\omega_2}{k_2}$ AND BOTH EQUAL C.

SINCE $E^2 - p^2 = m_0^2$ WE HAVE, $\hbar^2\omega^2 - \hbar^2 K^2 = m_0^2$ AND

$$\omega = \sqrt{\frac{m_0^2}{\hbar^2} + K^2}$$

SO

$$v_{p_1} = \frac{\sqrt{\frac{m_0^2}{\hbar^2} + K_1^2}}{K_1} \quad \text{AND} \quad v_{p_2} = \frac{\sqrt{\frac{m_0^2}{\hbar^2} + K_2^2}}{K_2} \quad \text{AND THESE ARE NOT EQUAL.}$$

IN A DISPERSIVE MEDIA, I.E., DIFFERENT ω'S TRAVEL WITH DIFFERENT ~~k's~~, v_p's, WE HAVE A MUCH MORE ENTERTAINING PROBLEM. WE HAVE TO DO A LITTLE MATHEMATICAL ANALYSIS TO SEE HOW THE BUMPS GO ALONG. IF WE LET $P_{01} = P_{02} = 1$ FOR SIMPLICITY AND ADD THE TWO WAVES WE GET

$$P_T = 2\cos\left[\frac{\omega_1+\omega_2}{2}t - \frac{K_1+K_2}{2}x\right]\cos\left[\frac{\omega_1-\omega_2}{2}t - \frac{K_1-K_2}{2}x\right]$$

When $\omega_1 \approx \omega_2 = \omega$ WE CAN WRITE

$$P_T = 2\cos(\omega t - kz)\cos\left[\frac{\Delta\omega t}{2} - \frac{\Delta K}{2}x\right]$$

The LAST FACTOR ON The RIGHT DETERMINES how The MODULATION MOVES. BY ADDING A CERTAIN AMOUNT AMOUNT Δt WE CAN OFFSET The EFFECT BY ADDING A CERTAIN Δx_0, I.E.,

$$\Delta\omega T_0 = \Delta K x_0$$

OR

$$\frac{x_0}{t_0} = \frac{\Delta\omega}{\Delta K} = v_{group} = \text{VELOCITY of The bumps or ENERGY, or SIGNALS or SOMETHING.}$$

IN The LIMIT

$$v_g = \frac{d\omega}{dK}$$

The GROUP VELOCITY DOES NOT DEPEND ON The choice of ΔK for A SLOWLY VARYING WAVE where $\omega_1 \approx \omega_2 = \omega$ holds.

RECALLING THAT $\omega = \sqrt{\frac{m_0^2}{\hbar^2} + k^2}$ AND $v_{phase} = \frac{\omega}{k} = \frac{E}{P}$ for A PARTICLE of O MOMENTUM WE CAN DETERMINE The GROUP VELOCITY.

$$v_{group} = \frac{d\omega}{dk} = \frac{k}{\sqrt{\frac{m_0^2}{\hbar^2} + k}} = \frac{k}{\omega} = \frac{P}{E}$$

WE CAN NOW ESTABLISH A RELATION BETWEEN The GROUP AND PHASE VELOCITIES,

$$v_{phase} = \frac{c^2}{v_{group}}$$

The UNCERTAINTY PRINCIPLE

I our DISCUSSION WE "CREATED" A PARTICLE AS The SUPERPOSITION of A LOT of WAVES SLIGHTLY OUT of PHASE WITH ONE ANOTHER. WE WOULD LIKE TO KNOW OVER how WIDE A RANGE IS This DISTURBANCE, SAY, L IS The DIAMETER of The DISTURBANCE or The "PARTICLE."

If L then is The LENGTH ~~in~~ of The WAVE TRAIN, The EXACT WAVE NUMBER K IS UNCERTAIN OVER THAT LENGTH AND GIVEN BY

$$\Delta K = \frac{2\pi}{L}$$

SINCE The MOMENTUM AND WAVE NUMBER K ARE DIRECTLY RELATED BY $\hbar$, SO WILL Their UNCERTAINTIES.

$$\Delta P = \hbar \Delta K$$

SUBSTITUTING IN for ΔK, WE CAN FIND The ASSOCIATE MOMENTUM UNCERTAINTY,

$$\Delta P = \hbar\frac{2\pi}{L}$$

OR, MORE OFTEN WRITTEN,

$$\Delta P \Delta x \doteq h \qquad \text{where } L = \Delta x$$

This UNCERTAINTY PRINCIPLE IS A NEGATIVE RULE SINCE IT TELLS US JUST how FAR ~~WE CAN GO~~ NOT TO GO IN The PARTICLE VIEW.

MORE ON THE GROUP VELOCITY

I WANT TO POINT OUT SOMETHING INTERESTING THAT CONTINUES FROM LAST TIME. THAT IS, THE GROUP VELOCITY TURNS OUT TO BE THE CLASSICAL PARTICLE VELOCITY. WE RECALL OUR DEFINITION WAS

$$v_{group} = \frac{d\,\omega(k)}{dk} = \frac{d\,E(P)}{dP}$$

which follows from our WAVE-PARTICLE EQUATIONS

$$E = \hbar\omega \quad \text{AND} \quad P = \hbar K$$

If THE ENERGY IS SOME FUNCTION OF THE MOMENTUM, WE WOULD LIKE TO SHOW HOW THIS STATEMENT IS TRUE. FIRST TAKE THE FREE PARTICLE WERE $E = \frac{P^2}{2m}$; CLEARLY THEN $\frac{dE}{dP} = \frac{P}{m} = v$ AND THE CLASSICAL VELOCITY IS THE GROUP VELOCITY ALSO. CONSIDERING THE RELATIVISTIC ENERGY EQUATION,

$$E = \sqrt{m^2 + P^2}$$

WE HAVE

$$\frac{dE}{dP} = \frac{P}{\sqrt{m^2+P^2}} = \frac{P}{E} = \frac{\hbar k}{\hbar\omega} = v$$

MORE GENERALLY, IF A FORCE $F = \frac{dP}{dt}$ IS ACTING SAY ALONG THE LINE OF MOTION, PERHAPS THE X AXIS FOR SIMPLICITY, THEN THE RATE OF DOING WORK IS JUST

$$Fv = \frac{dE}{dt}$$

If THE ENERGY IS SOME COMPLICATED FUNCTION OF THE MOMENTA P_x, P_Y, P_Z, WE WRITE THIS EQUATION AS

$$\bar{v}\cdot\frac{d\bar{P}}{dt} = \frac{d\,E\,(P_x, P_Y, P_Z)}{dt}$$

OR

$$dE = \vec{v}\cdot d\bar{P}$$

This differential EQUATION CAN be expanded.

$$\frac{\partial E}{\partial P_x}\,dP_x + \frac{\partial E}{\partial P_Y}\,dP_Y + \frac{\partial E}{\partial P_Z}\,dP_Z = V_x\,dP_x + V_Y\,dP_Y + V_Z\,dP_Z$$

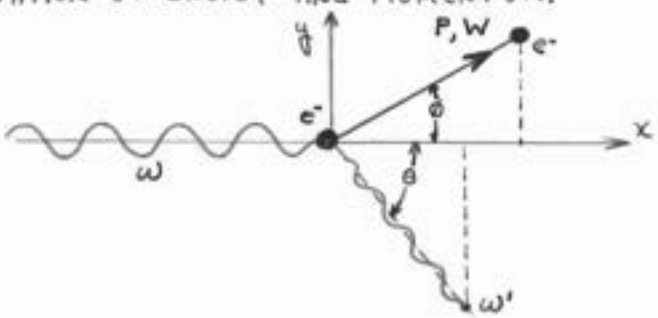
EQUATING THE RESPECTIVE COEFFICIENTS WE HAVE

$$V_x = \frac{\partial E}{\partial P_x}, \quad V_Y = \frac{\partial E}{\partial P_Y}, \quad V_Z = \partial$$

When THE CLASSICAL VELOCITY IS DEFINED IN THIS MANNER WE SEE THE RELATION does NOT depend ON ANY WAVE-PARTICLE DUALITY for IT TO hold. SO WE, indeed have found A USEFUL formula which WE SHALL USE LATTER WE DISCUSSING such THINGS AS PHONON behavior where ITS $\omega = \omega(k)$ And $d\omega/dk$ = CLASSICAL · VELOCITY.

COMPTON SCATTERING

Now I want to show you a few applications of the material I have presented the past few week to see if you learned anything. Suppose we have a photon hitting an electron and being scattered at some angle. We know the outgoing light frequency is less than the incoming because of an energy loss to the electron. This is called the Compton effect. We can determine the resulting frequency from a consideration of the conservation of energy and momentum.



Let ω be the photon's frequency before collision and ω' afterward with θ the photon scattering angle such that $\omega' = \omega'(\theta)$. Let P and W be the momentum and energy of the scattered electron through an angle ϕ. We would like to determine ω', say.

It is helpful to make up a little table of all the important quantities before and after the collision,

		E_x	P_x	P_Y	P_z
BEFORE	PHOTON	$\hbar\omega$	$\hbar\omega/c$	0	0
	ELECTRON	m_0c^2	0	0	0
AFTER	PHOTON	$\hbar\omega'$	$\hbar\omega'\cos\theta$	$-\hbar\omega'\sin\theta$	0
	ELECTRON	W	$P\cos\phi$	$P\sin\phi$	0

And W is related to P by $\quad W^2 - P^2 = m^2 \quad$ or $\quad W = \sqrt{m^2 + P^2}$

Now by conservation of energy

$$\hbar\omega + m = \hbar\omega' + W$$

By conservation of linear momentum:

$$\begin{array}{lll} \text{X component:} & \hbar\omega + 0 & = \hbar\omega'\cos\theta + P\cos\phi \\ \text{Y} \quad " & 0 + 0 & = -\hbar\omega'\sin\theta + P\sin\phi \end{array}$$

Rewriting these 3 equations as

$$\hbar\omega + m - \hbar\omega' = W$$
$$\hbar\omega - \hbar\omega'\cos\theta = P\cos\phi$$
$$+\hbar\omega'\sin\theta = P\sin\phi$$

We can square these and add the last two to get

$$(\hbar\omega)^2 - 2\hbar^2\omega\omega'\cos\theta + \hbar^2\omega'^2 = P^2$$

The first equation squared is

$$(\hbar\omega)^2 + \hbar^2\omega'^2 + m^2 + 2m\hbar\omega - 2m\hbar\omega' - 2\hbar\omega\hbar\omega' = W^2$$

Subtracting these and noting $W^2 - P^2 = M^2$ we get the familiar old law:

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{mc}(1 - \cos\theta)$$

OR REWRITING,

$$\lambda' - \lambda = \frac{2\pi\hbar}{mc}(1-\cos\theta)$$

AND $\frac{2\pi\hbar}{mc} = \lambda_c$ THE COMPTON WAVELENGTH. FOR AN ELECTRON THIS IS ABOUT 0.02 Å AND FOR A PROTON ABOUT 10^{-14} cm. THIS COMPTON EFFECT ORIGINALLY SERVED AS A CHECK OF THE THEORETICAL HYPOTHESIS OF WAVE PARTICLE DUALITY. NOW IT IS JUST AN EXERCISE FOR STUDENTS.

THE COMPTON EFFECT IN 4-VECTOR NOTATION

FOR THE SAKE OF NO GOOD REASON I SHALL WRITE OUT THESE EQUATIONS IN OUR NEWLY ACQUIRED 4-VECTOR NOTATION. TO BEGIN WE DEFINE THE FOLLOWING 4-VECTOR MOMENTA AND FREQUENCIES:

BEFORE	AFTER
k_μ^{IN}	k_μ^{OUT}
p_μ^{IN}	p_μ^{OUT}

BY CONSERVATION OF MOMENTUM

$$p_\mu^{IN} + k_\mu^{IN} = p_\mu^{OUT} + k_\mu^{OUT}$$

IF WE WANT TO OBSERVE THE PHOTON WE SOLVE THE PROBLEM BY FIND THE OUTGOING ELECTRON MOMENTUM AND GETTING RID OF IT. TO DO THIS WE WRITE

$$p_\mu^{OUT} = p_\mu^{IN} + k_\mu^{IN} - k_\mu^{OUT}$$

CALCULATE THE INVARIANT

$$p_\mu^{OUT} p_\mu^{OUT} = (p_\mu^{IN} + k_\mu^{IN} - k_\mu^{OUT})(p_\mu^{IN} + k_\mu^{IN} - k_\mu^{OUT})$$

SINCE THE PRODUCTS $p_\mu p_\mu$ ARE INVARIANTS, I.E., THE EXPERIMENT CAN BE DONE WITH STATIONARY OR MOVING ELECTRON, WE HAVE SINCE $p_\mu p_\mu = m^2$

$$m^2 = m^2 + k_\mu^{IN} k_\mu^{IN} + k_\mu^{OUT} k_\mu^{OUT} + 2p_\mu^{IN} k_\mu^{IN} - 2p_\mu^{IN} k_\mu^{OUT} - 2k_\mu^{IN} k_\mu^{OUT}$$

SUBTRACTING OUT THE m^2 AND REMEMBERING $k_\mu^{IN} k_\mu^{IN}$ AND $k_\mu^{OUT} k_\mu^{OUT}$ BOTH ARE ZERO SINCE THE PHOTON HAS 0 REST MASS, WE HAVE,

$$p_\mu^{IN} k_\mu^{IN} = p_\mu^{IN} k_\mu^{OUT} + k_\mu^{IN} k_\mu^{OUT}$$

IT IS NOW EASY TO DETERMINE EACH OF THESE FACTORS,

$$p_\mu^{IN} k_\mu^{IN} = m\omega$$

$$p_\mu^{IN} k_\mu^{OUT} = m\omega'$$

$$k_\mu^{IN} k_\mu^{OUT} = \omega\omega' - \omega(\omega'\cos\theta) = \omega\omega'(1-\cos\theta)$$

ONCE AGAIN WE GET

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{mc}(1-\cos\theta)$$

BUT THIS PROCEDURE IS A LITTLE MORE ELEGANT AND MUCH MORE USEFUL BECAUSE IT HELPS US SOLVE MORE DIFFICULT SCATTERING PROBLEMS.

ANOTHER PROBLEM I ASSIGN TO YOU CONCERN THE THRESHOLD LEVEL FOR CREATING ANTIPROTONS BY THE REACTION

$$P + P \longrightarrow P + P + P + \bar{P}$$

IF P_μ^a AND P_μ^b REPRESENT THE COLLIDING MOMENTA AND $\mathcal{P}_\mu$ THE MOMENTUM OF THE AGGREGATE PARTICLE AFTER COLLISION, I.E., $\mathcal{P}_\mu = (4M, 0, 0, 0)$, WE HAVE:

$$P_\mu^a + P_\mu^b = \mathcal{P}_\mu$$

THEN CALCULATING THE INVARIANT IN THE CENTER OF MASS SYSTEM,

$$\mathcal{P}_\mu \mathcal{P}_\mu = 16M^2 = (P_\mu^a + P_\mu^b)^2 = M^2 + M^2 + 2P_\mu^a P_\mu^b$$

SO

$$P_\mu^a P_\mu^b = 7M^2$$

IN THE LAB SYSTEM,

$$EM = 7M^2$$

OR

$$E = 7M \doteq 7 \text{ BeV}$$

SINCE THE REST ~~FE~~ MASS OF THE PROTON $M_0c^2 \approx 1$ BeV (938 MeV). THUS IT TAKES AND EXTRA 6 BeV TO GET THE REACTION GOING.

SUPPOSE WE WOULD LIKE TO PREFORM A REAL HIGH ENERGY EXPERIMENT WHERE WE SLAM A 500 BeV PROTON INTO ANOTHER PROTON; HOW WOULD IT LOOK FROM THE CENTER OF MASS SYSTEM. THAT IS, WHAT WOULD THE COLLISION BE LIKE IF TWO BEAMS OF THESE HIGH ENERGY PROTONS COLLIDED HEAD-ON. IN THIS REFERENCE FROM THE PRODUCT

$$P_\mu^a P_\mu^b = ME = W^2 + P^2$$

WHERE W IS THEIR ENERGIES AND THEIR MOMENTA ARE OPPOSITE. THUS

$$P_\mu^a P_\mu^b = 2W^2 - M^2$$

IF WE TAKE A LITTLE MORE CONSERVATIVE ENERGY SAY $W = 100$ BeV, IT IS CLEAR THE APPARANT COLLISION ENERGY IS 20,000 BeV. SO TECHNICALLY WE CAN GO TO MUCH HIGH ENERGIES USING THESE COLLIDING BEAM EXPERIMENTS. THIS MEANS, QUITE SIMPLY, IT IS NECESSARY TO JACK A BEAM UP TO 20,000 BeV AND SLAM IT INTO A STATIONARY PROTON TO CREATE THIS SAME REACTION.

FOR ELECTRONS THIS PHENOMENA IS MUCH MORE INTERESTING BECAUSE THE MASS IS SO MUCH LIGHTER. TWO 100 MeV ELECTRON BEAMS COLLIDING SIMULATE A 40,000 MeV COLLISION. SINCE A BEAM OF ELECTRONS IS VERY TENOUS AND THE PARTICLES PER CUBIC CENTIMETERS IS SO LOW, THE OTHER BEAM NEVER FINDS A PARTICLE TO COLLIDE WITH. THUS THIS IDEA HAS ITS LIMITATIONS.

REF: CHAPTER 2 VOL I
"ELEMENTARY PARTICLES"
by GELL-MANN & ROSENBAUM
SCIENTIFIC AMERICAN
JULY 1957

AN OUTLINE of THE WORLD AS WE KNOW IT

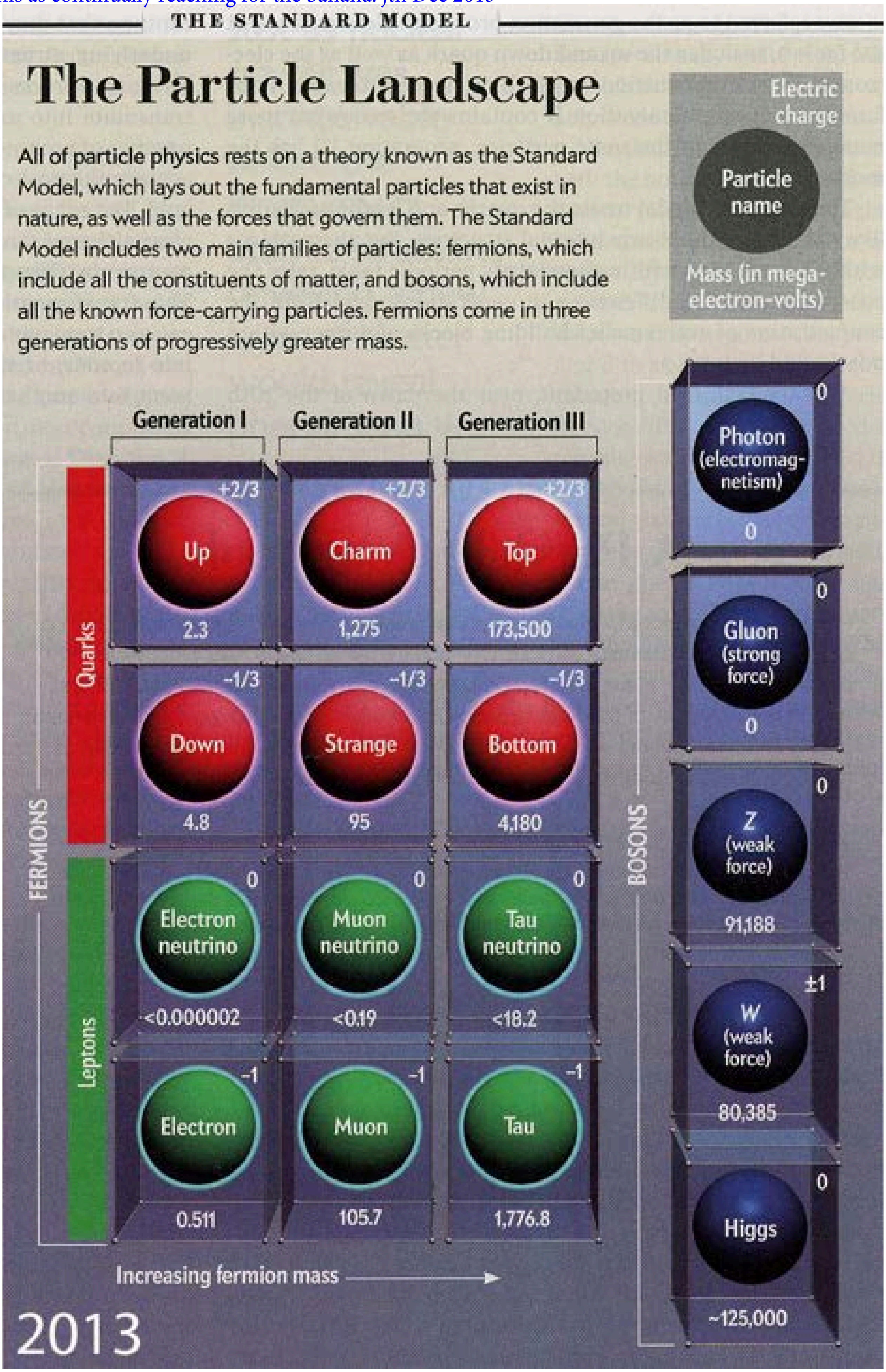
I WANT TO briefly describe what we know of The world by TELLING YOU what IT IS MADE of. TO DO THIS I WANT TO discuss The MAKE-UP of The NUCLEUS AND The PARTICLES EXTERNAL TO The NUCLEUS. WE CAN GENERATE A SMALL TABLE of A few of These PARTICLES

	PARTICLE	CHARGE	SPIN	REST MASS IN MeV	MEAN LIFE (SECONDS)
LEPTONS	ELECTRON, e^-	−	1/2	.511	STABLE
	MU MESON, μ^-	−	1/2	105.6	2.2×10^{-6}
	NEUTRINO				
	ELECTRON, ν_e	0	1/2	0	STABLE
	MUON, ν_μ	0	1/2	0	STABLE
	PHOTON, γ	0	1	0	STABLE
	(GRAVITON)	(0)	(2)	(0)	(STABLE) NOT found
BARYONS	PROTON, P	+	1/2	938	STABLE
	NEUTRON, N	0	1/2	939	STABLE
	LAMBDA Λ^0	0	1/2	1115	2.7×10^{-10}
	SIGMA				
	Σ^+	+	1/2	1189	$.7 \times 10^{-10}$
	Σ^-	−	1/2	1196	1.5×10^{-10}
	Σ^0	0	1/2	1191	NOT MEASURED
	XI				
	$\Xi^{-(-)}$	−	1/2	1319	10^{-10} TO 10^{-9}
	$\Xi^{0(0)}$	0	1/2	1311	
MESONS	K MESONS				
	K^+	+	0	494	1.2×10^{-8}
	K^-	−	0	494	1.2×10^{-8}
	K_1^0	0	0	498	1×10^{-10}
	K_2^0	0	0	498	$3. \times 10^{-8}$ TO 10^{-6}
	PIONS				
	π^+	+	0	139.6	2.6×10^{-8}
	π^-	−	0	139.6	2.6×10^{-8}
	π^0	0	0	135.0	10^{-16} TO 10^{-15}

ELEMENTARY PARTICLES

IN ALL There ARE OVER 300 SUCH PARTICLES which have been EXPERIMENTALLY verified. BUT fortunately IT TURNS OUT THAT THEY ARE EASY TO CATEGORIZE. EVEN MORE IMPORTANT IS The SKILL of GELL-MANN who CAN find A PLACE for EACH NEW DISCOVERY.

As readers today should know the particle world has evolved significantly of the perior my notes were taken. The "standard model" as it is referred to has become far more complex than in the '60's. The graphics below represents our best understanding today. Even hints that there may be subparticles within quarks suggests a further simplification of this model. This only suggests that there is much more to discover and learn about our world and its makeup. Feynman described this as continually reaching for the banana. jtn Dec 2013



THIS LARGE NUMBER OF PARTICLES, AS A WHOLE, SEEMS TO BE THE FUNDAMENTAL CONSTITUENTS OF MATTER. IT TURNS OUT THAT THE NUMBER OF INTERACTIONS BETWEEN ALL THESE PARTICLES IS QUITE SMALL - FOUR, IN FACT. THEY ARE, IN ORDER OF DECREASING STRENGTH: STRONG INTERACTION, WEAK INTERACTIONS, ELECTROMAGNETIC AND GRAVITATIONAL. IF WE ASSIGN A UNIT STRENGTH TO THE STRONG INTERACTIONS, I.E, A MESON TO A BARYON SAY, THEN THE RELATIVE STRENGTHS ARE:

INTERACTION	STRENGTH	EXAMPLE
GRAVITY	$\sim 10^{-42}$	COUPLED TO ALL ENERGY
ELECTROMAGNETIC	$\sim 10^{-2}$	PHOTON TO CHARGED PARTICLES
WEAK INTERACTIONS	$\sim 10^{-5}$	BETA-DECAY
STRONG INTERACTIONS	1	NUCLEAR FORCES

SINCE THE GRAVITATIONAL INTERACTION IS SO SMALL WE CAN DISREGARD IT AS INSIGNIFICANT FOR THE MOMENT.

AS STATED ABOVE AN EXAMPLE OF A WEAK INTERACTION IS BETA-DECAY AND SOME EXAMPLES ARE

$$C^{14} \rightarrow N^{14} + e^- + \bar{\nu}$$
$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$
$$\Lambda \rightarrow p^+ + \pi^-$$

ALL THESE REACTIONS ARE SLOW, I.E., THEY HAVE A MATRIX ELEMENT ON THE ORDER OF 10^{-5} WHICH DEFINES THE PROBABILITY OF DECAY.

OUR CURRENT KNOWLEDGE CONCERNING THE LAWS OF INTERACTION BETWEEN PHOTONS AND CHARGE IS DESCRIBED QUANTUM ELECTRODYNAMICALLY. THE EXACT STRENGTH OF COUPLING BETWEEN A PHOTON AND A CHARGED PARTICLE IS 1/137. THERE ARE SECONDARY INTERACTIONS WHICH OCCUR SAY WHEN A PHOTON INTERACTS WITH THE ELECTRIC FIELD FROM AN ELECTRON OR PROTON. A PHOTON CAN NEVER INTERACT WITH ITSELF.

THE LAWS OF ELECTRODYNAMICS HAVE BEEN CHECKED DOWN TO 10^{-14} CM AND THERE ARE NO ERRORS YET. THIS MEANS THAT WE CAN'T CHECK THE TECHNICAL PROBLEMS AT THAT DIMENSION. WEAK INTERACTIONS ALSO HAVE BEEN CHECKED DOWN TO 10^{-14} CM. FOR STRONG INTERACTIONS WE ARE NOT QUITE SO SURE WHAT HAPPENS AT 10^{-14} CM BECAUSE THE PARTICLE SIZE IMPLIES A COMPTON WAVELENGTH ON THIS ORDER. SO I'M SAYING WE DON'T KNOW THE LAW AT ALL AND IT STANDS INCOMPLETE NOW.

I STILL HAVE ONE MORE EXAMPLE OF A 4-VECTOR WHICH I HAVE FAILED TO GIVE YOU DURING THE PAST TWO LECTURES. THE COMPONENTS OF THE FOUR VECTOR j_μ ARE THE DENSITY OF CHARGE AND THE 3 SPACE COMPONENTS OF THE CURRENT j_x, j_y, j_z.

WE REALIZE FIRST CHARGE IS AN INVARIANT QUANTITY. THAT IS, THE TOTAL CHARGE OF A GLOB OF MATERIAL IN A BOX IS THE SAME WHETHER VIEWED FROM A STATIONARY OR MOVING FRAME. BUT THE TOTAL CHARGE IS SIMPLY THE CHARGE DENSITY TIMES THE VOLUME ENCLOSED BY THE BOX. WE ASSUME THAT ρ_0 = STATIONARY CHARGE DENSITY. I POINT OUT ρ_0 CANNOT REPRESENT A MASS DENSITY BECAUSE MASS IS NOT A LORENTZ INVARIANT. MORE ACCURATELY ρ_0 IS JUST A NUMBER DENSITY LIKE 19 BALLS. SUPPOSE NOW THE CONTAINER IS MOVING WITH SOME VELOCITY v ALONG THE X-AXIS, SAY. IF THE STATIONARY LENGTH OF THE BOX IS L_0, THE MOVING LENGTH IS CONTRACTED BY

$$L = L_0\sqrt{1-v^2/c^2}$$

SINCE THE TOTAL CHARGE $Q = \rho_0 L_0 A_0$ IN THE REST FRAME WE MUST HAVE THIS EQUALING THE CHARGE IN THE MOVING FRAME SO,

$$\rho_0 L_0 A_0 = \rho L A_0$$

WHERE ρ AND L ARE THE MOVING DENSITY AND LENGTH. ALSO WE NOTE THERE IS NOT CONTRACTION IN THE YZ DIRECTION. OUR RESULT IS SIMPLY

$$\rho = \frac{\rho_0}{\sqrt{1-v^2/c^2}}$$

WE SEE THEN THE CHARGE DENSITY OF A MOVING DISTRIBUTION OF CHARGES VARIES IN THE SAME WAY AS THE RELATIVISTIC MASS OF A PARTICLE.

WE ALREADY SAW THE FOUR VELOCITY HAD THE COMPONENTS

$$\frac{1}{\sqrt{1-v^2}},\ \frac{v_x}{\sqrt{1-v^2}},\ \frac{v_y}{\sqrt{1-v^2}},\ \frac{v_z}{\sqrt{1-v^2}}$$

FURTHER THE CURRENT FLOWING IS JUST THE NUMBER OF CHARGE CROSSING A SQUARE CENTIMETER EACH SEC SO IT IS JUST

$$j = \rho v$$

THUS WE HAVE THE 4 COMPONENTS OF THE 4 CURRENT BEING

$$j_t = \frac{\rho_0}{\sqrt{1-v^2}} \qquad j_x = \frac{\rho_0 v_x}{\sqrt{1-v^2}} \qquad j_y = \frac{\rho_0 v_y}{\sqrt{1-v^2}} \qquad j_z = \frac{\rho_0 v_z}{\sqrt{1-v^2}}$$

IF THE TOTAL VOLUME CONTAINS A MIXTURE OF VARIOUS DENSITIES AND CURRENTS, SAY ELECTRONS AND IONS, THEN THE RESULT 4 CURRENT j_μ IS THE RESULTANT SUM OF THE VARIOUS CONSTITUENTS. AN INTERESTING EXAMPLE IS A NEUTRAL WIRE CARRYING A CURRENT OF ELECTRONS CAN HAVE A TIME VARYING COMPONENT IN ANOTHER FRAME. WE'LL COME BACK TO THIS PROBLEM LATER. FOR THE MEANTIME I THINKS THAT'S ALL I WANT TO SAY ON 4-VECTORS. BUT NOT QUITE! A QUESTION AROSE AS TO THE 4 VECTOR ANALOG OF ANGULAR MOMENTUM.

As you must recall the angular momentum is a vector quantity formed from two other vectors, the position and momentum vectors,

$$\vec{M} = \vec{P} \times \vec{R}$$

This cross product gives us terms like $M_z = P_x Y - X P_y$ etc. Thus for motion in three dimensions, quantities like $(v_x Y - X v_y)$ become important. All together there are three such important quantities for motion in three dimension. They are the 3 components of the angular momentum. In order to identify these 3 quantities with a vector, we have to make an artificial right hand rule. Our success in forming this vector resulted in the very unusual property of M_{ij} (where i & j equal $x, y,$ or z) which is

$$M_{ij} = -M_{ji} \qquad M_{ii} = 0$$

This animal is what we call an antisymmetric tensor of the second rank. In tabular form we have

	X	Y	Z
X	0	M_z	$-M_Y$
Y	$-M_z$	0	M_x
Z	M_Y	$-M_x$	0

It turns out of the nine possible quantities only 3 are independent. And M_{ij} can then be associated with a vector since its components transform like a vector. That is, consider a rotation,

$$P_x' = P_x \cos\theta + P_Y \sin\theta \qquad X' = X\cos\theta + Y\sin\theta$$
$$P_Y' = P_Y \cos\theta - P_x \sin\theta \qquad Y' = Y\cos\theta - X\sin\theta$$
$$P_z' = P_z \qquad z' = z$$

And compute $P_x'Y' - X'P_Y'$. We have

$$M_z' = (P_x\cos\theta + P_Y\sin\theta)(Y\cos\theta - X\sin\theta) - (X\cos\theta + Y\sin\theta)(P_Y\cos\theta - P_x\sin\theta)$$

Simplifying we get $M_z' = M_z$.

Now in 4 dimensions we aren't so lucky. Where in 3-dimensions we could uniquely define a surface by a vector ~~nomia~~ normal to it, no such quantity exist in 4-dimensions. The various combinations of M_{ijk} (where i, j, k equal $x, y, z,$ and t) yield 6 independent quantities and you can't represent six things by 4 things. Thus in calculating the 4 vector given by $P_\mu X_\nu - P_\nu X_\mu$ we have a table given by

	X	Y	Z	t
X	0	M_z	$-M_Y$	N_x
Y	$-M_z$	0	M_x	N_Y
Z	M_Y	$-M_x$	0	N_z
t	$-N_x$	$-N_y$	$-N_z$	0

where $N_x = P_x t - X P_t$
$N_Y = P_Y t - Y P_t$
$N_z = P_z t - z P_t$

I trust some of you have heard about such things as the electric and magnetic fields. It will turns out the three components of the magnetic field ($B_x, B_y, \& B_z$) plus the three electric field components (E_x, E_y, E_z) transform like the six components $M_x, M_y, M_z, N_x, N_y, N_z$. So let's first find the general transformation laws between M and M' and N and N'; then see what the M and N's mean physically in terms of the electric and magnetic fields.

What we are after is the Lorentz transformation of an anti-symmetric tensor of the second order or more concisely a 6vector. The transformation laws are straight forward but involve a little algebra. For simplicity let, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

Then the momentum and position 4 vectors transform as,

$$P_t' = \gamma(P_t - vP_x) \quad P_z' = \gamma(P_z - vP_t) \quad P_x' = P_x \quad P_y' = P_y$$

$$t' = \gamma(t - vz) \quad z' = \gamma(z - vt) \quad x' = x \quad y' = y$$

For motion along the z axis. First let's calculate M_z'

$$M_z' = P_x' y' - P_y' x' = P_x y - P_y x = M_z$$

Thus the angular momentum of a photon, say, as you run forward. This also means $B_z' = B_z$ which transforms differently from $z' = \gamma(z - vt)$ in the direction of motion. Now let's find M_x',

$$M_x' = P_y' z' - P_z' y' = P_y \gamma(z - vt) - \gamma(P_z - vP_t)y$$
$$= \gamma(P_y z - P_z y) - \gamma v(P_y t - P_t y) = \gamma[M_x - vN_y]$$

Substituting in B and E we get the Lorentz transformation

$$B_x' = \frac{B_x - vE_y}{\sqrt{1-v^2}}$$

Similarly we find,

$$M_y' = \gamma[M_y + vN_x]$$

so that

$$B_y' = \frac{B_y + vE_x}{\sqrt{1-v^2}}$$

Now we want to find N_x',

$$N_x' = P_x' t' - P_t' x'$$
$$= P_x \gamma(t - vz) - \gamma(P_t - vP_z)x$$
$$= \gamma(P_x t - P_t x) - \gamma v(P_x z - P_z x) = \gamma N_x + \gamma v M_y$$
$$N_x' = \gamma[N_x + vM_y]$$

Similarly $\qquad N_y' = \gamma[N_y - vM_x]$

And $\qquad N_z' = N_z$

WE CAN NOW TABULATE OUR RESULTS AND HAVE THE LORENTZ TRANSFORMATION FOR THE ELECTRIC AND MAGNETIC FIELDS

$$E'_z = E_z \qquad B'_z = B_z$$

$$E'_Y = \frac{E_Y - V B_X}{\sqrt{1-V^2}} \qquad B'_Y = \frac{B_Y + V E_X}{\sqrt{1-V^2}}$$

NOTE : C = 1

$$E'_X = \frac{E_X + V B_Y}{\sqrt{1-V^2}} \qquad B'_X = \frac{B_X - V E_Y}{\sqrt{1-V^2}}$$

AS A MORE GENERAL TRANSFORMATION LAW

$$E'_z = E_z \qquad B'_z = B_z$$

$$E'_Y = \frac{(\bar{E} + \bar{v} \times \bar{B})_y}{\sqrt{1-V^2}} \qquad B'_Y = \frac{(\bar{B} - \frac{\bar{v}}{c^2} \times \bar{E})_y}{\sqrt{1-V^2}}$$

$$E'_z = \frac{(\bar{E} + \bar{v} \times \bar{B})_z}{\sqrt{1-V^2}} \qquad B'_X = \frac{(\bar{B} - \frac{\bar{v}}{c^2} \times \bar{E})_z}{\sqrt{1-V^2}}$$

I WANT TO POINT OUT THAT IT IS NOT OBVIOUS THAT $\bar{B}$ AND $\bar{E}$ FORM THE COMPONENTS OF A 6 VECTOR FROM ANY STATEMENT I HAVE MADE. I WILL DISCUSS LATER WHY WE ARE JUSTIFIED TO MAKE THIS CLAIM.

THE PHYSICAL SIGNIFICANCE OF THE N'S CAN BE INTERPERTED FROM ONE OF THE COMPONENTS SAY $N_x = P_x t - \mathcal{E} X = \mathcal{E}\left(\frac{P_x}{\mathcal{E}} t - X\right)$

OR I PREFER TO WRITE

$$N_X = -\mathcal{E}\,(X - V_x t)$$

NOW WHEN $t=0$

$$N_X = -\mathcal{E} X_0$$

WHERE X_0 IS SOME GIVEN INITIAL POSITION. BUT AS t PROGRESSES THE DIFFERENCE $X - V_x t$ REMAINS CONSTANT AND EQUAL TO X_0. N_X BECOMES A CONSTANT OF THE MOTION. QUITE LITERALLY ONE MAN'S N IS ANOTHER MAN'S M. OR MORE FAMILARLY ONE MAN'S ELECTRIC FIELD IS ANOTHER MAN'S MAGNETIC FIELD. SINCE ENERGY IS MASS THE TERM $\mathcal{E} X_0$ CAN BE INTERPERTED AS THE MOMENT OF MASS OF THE TOTAL MOVING SYSTEM.

IF YOU RECALL OUR WHEEL ANALOGY THAT WE DISCUSSED EARLIER, YOU WILL REMEMBER THAT THE MASS OF THE PARTICLES IN THE UPPER PART OF THE WHEEL ARE MORE MASSIVE BECAUSE THEY GO FASTER THAN THE BOTTOM ONES. WHERE THE STATIONARY CENTER OF MASS WAS COINCIDENT WITH THE GEOMETRIC CENTER, IN THE MOVING WHEEL THE CENTER OF MASS IS SHIFTED VERTICALLY UPWARD A DISTANCE X_0. THUS THE VECTOR $\bar{N}$ IS THE MASS MOMENT CREATED BY THE MOVEMENT.

IN GENERAL THEN THE 6 COMPONENTS OF THE ELECTRIC AND MAGNETIC FIELD COMBINED ARE THE ELEMENTS OF AN ANTISYMMETRIC TENSOR OF THE SECOND ORDER $F_{\mu\nu}$ WHERE,

$F_{XY} = -B_z \quad F_{YZ} = -B_X \quad F_{ZX} = -B_Y \quad F_{Xt} = E_X \quad F_{Yt} = E_Y \quad F_{Zt} = E_Z$

WE SEE A 6 VECTOR IS THEN COMPRISED OF AN AXIAL AND POLAR VECTOR.

CONSERVATIVE FORCES AND THE PRINCIPLE OF LEAST ACTION

REF: CHAPTER 19 VOL. II

NOW I WANT TO START TALKING ABOUT THE LAWS OF MECHANICS. BUT IN ORDER TO TEACH YOU SOMETHING YOU ALREADY KNOW I SHALL TEACH THE MATERIAL IN A DIFFERENT WAY. I WANT TO DISCUSS THE CHARACTER OF THE LAWS OF FORCE IN A WAY DIFFERING FROM NEWTON. ALSO, I WANT TO DISCUSS THE NATURE OF THE KINDS OF THE FORMS OF FORCES LIKE ELECTRICITY, MAGNITISM, AND GRAVITY.

THE IDEA I WANT TO DISUSS IS CONSERVATIVE FORCES. NOW FORGET, OR RATHER PUT ASIDE, ALL THE RELATIVITY I HAVE TOUGHT YOU AND GO DIRECTLY TO GOOD OLD NEWTONIAN MECHANICS -

$$F = m\ddot{x}$$

THIS TIME - NO TRICKS. THE FORCE IS THE SECOND TIME DERIVATIVE OF THE POSITION. REMEMBER NEWTON ALSO INCLUDED FRICTIONAL FORCES WHICH DEPEND ON A PARTICLE'S VELOCITY. TO NEWTON FRICTION WAS APPARENTLY AN EXAMPLE OF A NON CONSERVATIVE FORCE. BUT FRICTION IS REALLY A CONSERVATIVE FORCE WHEN EXAMINED ON THE ATOMIC LEVEL FOR ENERGY IS CONSERVED BY HEATING UP THE PARTICLES AND MAKING THEM GO FASTER. THUS THE TOTAL ENERGY, POTENTIAL PLUS KINETIC, IN THE UNIVERSE CAN BE SAID TO BE CONSTANT.

THERE APPEARS THEN CLASSIFICATION OF FORCES WHICH WE SHALL CALL FUNDAMENTAL OR PRIMARY WHICH ARE CONSERVATIVE. THIS STATEMENT IS NOT OBVIOUS FROM NEWTON'S LAWS. WE MUST SEEK AN EXPLANATION AS TO THE VALIDITY OF THE CLAIM: ALL PRIMARY FORCES ARE CONSERVATIVE.

WHAT DO WE MEAN BY A CONSERVATIVE FORCE? SUCH A FORCE IS DERIVABLE FROM A POTENTIAL OR SCALAR FUNCTION. THIS FUNCTION V, CALL IT, CAN ALSO BE A FUNCTION OF TIME, $V = V(x, y, z, t)$

ASSUMING $V \neq V(t)$ FOR THE TIME BEING WE CAN WRITE OUR FORCE AS

$$\bar{F} = -\bar{\nabla} V$$

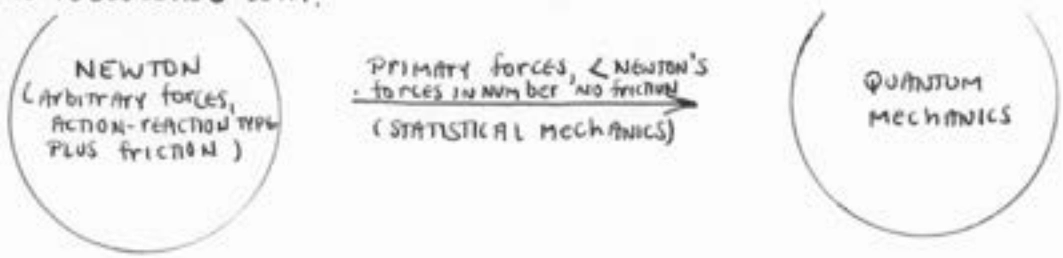
THE x COMPONENT BEING,

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

THUS PRIMARY FORCES HAVE A SPECIAL MATHEMATICAL FORM. THIS IMPLIES THE NUMBER OF SUCH FORCES IS GOING TO BE SMALLER THAN NEWTON ALLOWS. IN THE ABOVE FORM THE FORCE IS INDEPENDENT OF VELOCITY. IF THE FORCE VARIES AS THE FIRST POWER OF THE VELOCITY, THEN WE CAN DEDUCE THE LAWS OF ELECTRODYNAMICS. IF THE FORCE VARIES AS THE SQUARE OF THE VELOCITY, WE CAN DEDUCE THE GRAVITATIONAL LAWS OF EINSTEIN.

MORE SPECIFICALLY I SHALL CALL CONSERVATIVE FORCES, THOSE FORCES WHICH CAN BE DEDUCED FROM QUANTUM MECHANICS IN THE CLASSICAL LIMIT. AS YOU KNOW Q.M. IS THE UNDERPINNING OF ~~of~~ NATURE AND Q.M IS RATHER PECULIAR. WHEN WE EXAMINE A PARTICLE IN DETAIL IT DOES UNUSUAL THINGS. BUT IN BIG GLOBS THESE PARTICLES BEHAVE AS STATED BY CLASSICAL MECHANICS. THERE IS A SUBBRANCH OF FORCES, THE PRIMARY FORCE, WHICH LIE BETWEEN Q.M. AND CLASSICAL MECHANICS.

Historically, we can compare man's attempt to try to understand nature in the following way.



I said these ~~m~~ primary forces have special mathematical forms and thus called the laws of natural mechanics rather than Newtonian ~~b~~ mechanics. This special way of writing the laws is called the principle of least action. From this principle it is easy to get to quantum mechanics. I know want to show the form of the least action and then I'll explain the idea next to show it is right.

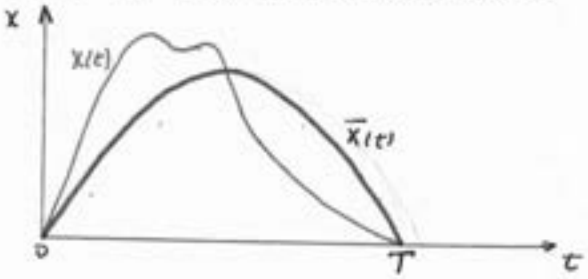
Let's consider first motion in a gravitational field. Since the potential energy of a particle moving under its own weight is given by $V = mgx$, then the equation of motion is

$$m\ddot{x} = -\frac{\partial}{\partial x}(mgx)$$

or

$$\ddot{x} + g = 0$$

This motion is described by a parabola. Suppose a particle starts at the origin and at some later time T comes back down; its position as a function of time then looks like:



The real motion or path is thus $\bar{x}(t)$. Now imagine some other path $x(t)$ also represents the particle's motion from 0 to T as show above. There is a quantity called the action which is least when the actual path $\bar{x}(t)$ is followed. The action is given by

$$\text{Action} = \int_0^T \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x(t)) \right] dt = S$$

Thus when the kinetic energy minus the ~~potent~~ potential energy is integrated with respect to time from the initial to final time, this quantity is a minimum for the actual dynamical path. We must require, however, the end points to be fixed, i.e, $x(0) = 0$, $x(T) = 0$.

HOW DOES THIS RELATE TO QUANTUM MECHANICS? WELL, Q.M. has TO do with MATTER-WAVES AND The MOTION IS LIKE A SPREADING WAVES ARRIVING AT A CERTAIN POINT IN different PHASES. IN order for A PARTICLE TO be detected There MUST be AN INTERFERENCE of WAVES which Add IN PHASE TO GIVE A MEASURABLE "AMPLITUDE." WHAT IS The AMPLITUDE for EACH PATH ALLOWABLE TO The PARTICLE? IT TURNS OUT The AMPLITUDE IS PROPORTIONAL TO $\exp(iS/\hbar)$ where S IS The ACTION for THAT PATH. THUS The PHASE ANGLE IS $S/\hbar$ where $\hbar$ is, of course, PLANCK'S CONSTANT. SINCE S IS Expressed IN UNITS of ENERGY-SEC AS IS $\hbar$, $S/\hbar$ IS A CONSTANT, AND THIS CONSTANT determines when Q.M. IS IMPORTANT.

IN GENERAL S IS MUCH GREATER THAN $\hbar$. THIS MEANS NEARBY PATHS HAVE very different PHASES AND Thus CANCEL EACH other's effect WHEN TAKING Their SUM. BuT for ONE SMALL REGION ABOUT ONE SPECIAL PATH, The PATH which MAKES The ACTION A MINIMUM, WILL The NEIGHBORING PHASES ~~Add~~ be The SAME. THIS MEANS ALL These NEARBY PATHS HAVE The SAME ACTION WITH $\hbar$. THIS THEN IS The CONNECTION BETWEEN QUANTUM MECHANICS AND The PRINCIPLE of LEAST ACTION (or MORE PROPERLY, LEAST PHASE). FURTHERMORE, The PRINCIPLE of LEAST ACTION IS AN INVARIANT QUANTITY.

LAST TIME WE WERE DISCUSSING THE PRINCIPLE OF LEAST ACTION AND WE EXPRESSED THE ACTION AS,

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x(t),t) \right] dt$$

WHAT I WOULD LIKE TO DO IS TO deduce THE LAWS of MECHANICS WHICH MAKES THIS NUMBER A MINIMUM. NOW THE ACTION depends ON THE choice of THE CURVE WE CHOOSE AND THE idea of CALCULATING AN EXTREMUM by differentiATING IS NOT THE THING TO DO. THE APPROACH IS A LITTLE MORE HIGH CLASS. I MIGHT POINT OUT THAT WHEN WE HAVE A QUANTITY LIKE S WHICH IS A FUNCTION of THE CURVE, WE SAY S IS A FUNCTIONAL of $x(t)$.

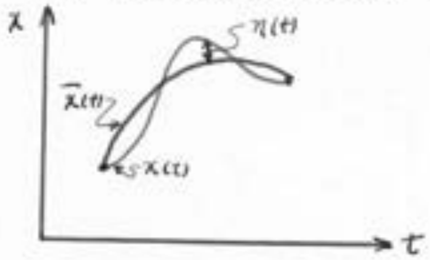
OUR PLAN IS TO VARY THE ACTUAL PATH $\bar{x}(t)$ by A SLIGHT AMOUNT $\eta(t)$ SO THAT OUR NEW CURVE IS

$$x(t) = \bar{x}(t) + \eta(t)$$

WE WANT TO CALCULATE THE ACTION ALONG THIS NEW PATH AND THEN COMPUTE THE difference between $S(\bar{x}(t)+\eta(t))$ AND $S(\bar{x}(t))$. WHEN THIS difference YIELDS first order VARIATIONS IN $\eta(t)$ of ZERO, WE ARE AT AN EXTREMUM. THAT IS, THE CHANGE IN SLOPE AROUND THE EXTREMUM IS INFINITESIMAL AND ONLY SECOND order AND higher TERMS REMAIN. THAT IS,

$$S[\bar{x}(t)+\eta(t)] = S[\bar{x}(t)] + 0\cdot\eta(t) + \text{order } \eta^2 + \text{higher terms}$$

SINCE THE COEFFICIENT of THE first order TERM IN $\eta(t)$ IS 0, WE KNOW THE SLOPE of THE FUNCTIONAL IS AN EXTREMUM. AND WE'LL WORRY ABOUT IT BEING A MAX OR MIN LATER. ONE FURTHER IMPORTANT COMMENT AND THAT IS THE LEAST ACTION PRINCIPLE holds for ALL CHOICES of $\eta(t)$ provided THEY ARE SUFFICIENTLY SMALL AND $\eta(t_1) = \eta(t_2) = 0$



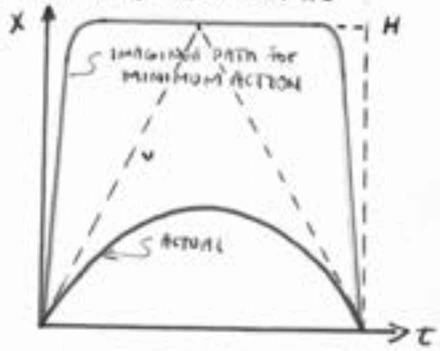
$\eta(t)$ IS EXAGGERATED here

TO GET A LITTLE BETTER feel for THIS idea recall LAST TIME WE WERE CONSIDERING A PARTICLE MOVING IN A POTENTIAL $V = mgx$ AND WE HAD THE ACTION EXPRESSED AS

$$S = \int_{t_1}^{t_2} \frac{m}{2}\dot{x}^2\, dt - \int mgx\, dt$$

If WE divide THE ACTION by THE PERIOD of FLIGHT T WE HAVE THE AVERAGE KINETIC ENERGY MINUS THE AVERAGE POTENTIAL. THE LAST TERM IS REALLY JUST THE AVERAGE height. THUS if WE IMAGINED A MOTION WHERE THE PARTICLE NEVER GOT of THE GROUND THE WE AREN'T TAKING AWAY from THE first INTEGRAL AND THE ACTION becomes A MAXIMUM.

It appears obvious then we want a bigger average height so that we are subtracting a very large number and making the action very small. So we can imagine a path where the particle zooms way up, coasts along, and comes back down so the average height is very high. The drawing compares the actual path with this hypothetical one. In this motion the action is made small but the time of ~~***~~ travel is longer than the actual path.



Let's approximate this path with the dotted path where the velocity of the particle $\dot{x} = \frac{2H}{T}$. The action can be written as

$$S = \left[\frac{m}{2}\left(\frac{2H}{T}\right)^2 - \frac{m}{2} g H\right] T = \frac{2mH^2}{T} - \frac{mgHT}{2}$$

Since $S = S(H)$, differentiating with respect to H and finding the least H,

$$\frac{\partial S}{\partial H} = 0 \Rightarrow \frac{4m\bar{H}}{T} = \frac{mgT}{2} \quad \text{OR} \quad \bar{H} = \frac{gT^2}{8}$$

This is how high you can climb in time T. This gives an action of

$$S = -\frac{mg^2T^3}{32}$$

In essence, the particle tries to get the potential energy as big as possible in the minimum time.

Let's return to our original problem and compute our new action along the path $x(t) = \bar{x}(t) + \eta(t)$. We have upon substitution

$$S_{x(t)} = \int_{t_1}^{t_2} \left\{\frac{m}{2}\left[\frac{d\bar{x}(t)}{dt} + \frac{d\eta(t)}{dt}\right]^2 - V(\bar{x}(t) + \eta(t), t)\right\} dt$$

We require that $\eta(t)$ is very small so terms like $\eta(t)^2$, $\dot{\eta}(t)^2$, and higher order terms are neglected. Then the squared term is

$$\left(\frac{d\bar{x}}{dt}\right)^2 + 2\frac{d\bar{x}}{dt}\frac{d\eta}{dt}$$

Also we expand the potential in a Taylor's series

$$V(\bar{x}(t) + \eta(t), t) = V(\bar{x}(t), t) + \eta(t)\, V'(\bar{x}(t), t) + \frac{\eta^2}{2} V''(\bar{x}, t) + \cdots$$

Combining these two results

$$S_{x(t)} = \int \left[\frac{m}{2}\left(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{\eta}\right) - V(\bar{x}, t) - \eta V'(\bar{x}, t) + (\text{higher order terms})\right] dt$$

rearranging

$$S_{x(t)} = \int\left[\frac{m}{2}\dot{x}^2 - V(\bar{x}, t)\right] dt + \int\left[m\dot{\bar{x}}(t)\,\dot{\eta}(t) - V'(\bar{x}, t)\right] dt$$

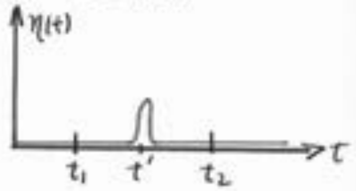
$$= S_{\bar{x}(t)} + 1^{\text{st}} \text{ order change}$$

We want this last integral to be zero for first order changes to be zero. To continue on we need a lemma:

When we have an integral like

$$\int_{t_1}^{t_2} f(t)\,\eta(t)\,dt = 0$$

If $\eta(t)$ is an function, then $f(t)$ is necessary 0 at all point except at the ends where, perhaps, it is not 0 since $\eta(t_1) = \eta(t_2) = 0$. As a simple ~~poor~~ proof let $\eta(t)$ be the simple curve as shown here. Since $\eta(t) \neq 0$ at $t' \pm \epsilon$, say, then $f(t)$ must equal 0 where ~~this~~ this pip is. But since the pip can be anywhere $f(t)$ must be zero everywhere.



Now we make use of the fact that the slope of a curve is not independent of that curve so we can relate $\dot{\eta}(t)$ to $\eta(t)$. In fact, we can integrate, $\int m\dot{x}\dot{\eta}\,dt$ by parts. Remember the rule

$$\int u\,dv = uv - \int v\,du \qquad \text{or} \qquad \int U\frac{dv}{dt}\,dt = uv - \int v\frac{du}{dt}\,dt$$

let $\quad U = m\dot{\bar{x}} \qquad \dfrac{dV}{dt} = \dfrac{d\eta}{dt} \quad$ so we have

$$\int_{t_1}^{t_2} m\dot{\bar{x}}\,\dot{\eta}\,dt = m\dot{\bar{x}}\,\eta(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta(t)\,\frac{d}{dt}\left[m\frac{d\bar{x}}{dt}\right]dt$$

recall our limits on $\eta(t)$, i.e., $\eta(t_1) = \eta(t_2) = 0$ so the first term on the right is zero. How clever! Maybe that was to tricky for you. Suppose I have a times b and a is identically zero then the product is zero for all values of b!

Collecting our results from the previous page, we have

$$S(x(t)) = S(\bar{x}(t)) + \int_{t_1}^{t_2}\left[-m\frac{d^2\bar{x}}{dt^2} - V'(\bar{x},t)\right]\eta(t)\,dt$$

By our happy theorem or lemma we immediately have oure result

$$f(t) = -m\frac{d^2\bar{x}}{dt^2} - V'(\bar{x},t) = 0$$

more familiarly,

$$m\frac{d^2\bar{x}}{dt^2} = -V'(\bar{x},t)$$

And we have proved that classical mechanics is derivable from the principle of least action. That is, for a conservative system, the path that has the minimum action is the one satisfying Newton's law.

There are a few situations where this extremum is not a minimum, i.e., correspond to a least time. There could be a saddle point at the extremum which is not a minimum or a maximum. So what our principle really says is that the first order change in S is zero. It is not necessarily a minimum.

A quick word on extending the idea to three dimension. The math is just the same where now we vary three parameters $x(t)$, $y(t)$ and $z(t)$ by $\eta(t)$, $\xi(t)$, $\zeta(t)$ say. The kinetic energy is simply $\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ it is a simple matter to transform to some screwball coordinate system, say, r, θ, and φ in the same way varying each parameter separately

I WANT TO NOW TELL YOU WHAT THE ACTION IS FOR RELATIVISTIC MOTION. REMEMBER A WHILE AGO WE SAID THE ACTION WAS THE INTEGRAL OF THE KINETIC MINUS THE POTENTIAL ENERGY. THIS IS TRUE ONLY IN THE NON-RELATIVISTIC CASE. FOR THE RELATIVISTIC CASE WE MUST HAVE AN ACTION WHICH YIELD THE PROPER EQUATIONS OF MOTION,

$$\frac{d}{dt}\left[\frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}\right] = -\frac{\partial V}{\partial x}$$

INSTEAD OF WORKING BACKWARD TO FIND THE ACTION, I'LL SIMPLY GIVE IT TO.

$$S = \int_{t_1}^{t_2}\left[-m_0c^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - V(x,t)\right]dt$$

NOW WHAT I'M DOIN' AIN'T A BIG DEAL; IT'S JUST ENTERTAINING - A SMALL DEAL. I WANT TO SHOW THIS ACTION DOES GIVE THE RIGHT EQUATION OF MOTION. FOLLOWING THE SAME ANALYSIS AS BEFORE, WE START WITH A SMALL PATH VARIATION,

$$x = \bar{x} + \eta \rightarrow \dot{x} = \dot{\bar{x}} + \dot{\eta} \rightarrow \dot{x}^2 \simeq \dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{\eta}$$

WE MUST EVALUATE THE SQUARE ROOT TERM

$$\sqrt{1-\dot{x}^2} = \sqrt{1-(\dot{\bar{x}}^2+2\dot{\bar{x}}\dot{\eta})} = (1-\dot{\bar{x}}^2)^{1/2}\left(1-\frac{2\dot{\bar{x}}\dot{\eta}}{1-\dot{\bar{x}}^2}\right)^{1/2}$$

EXPANDING THE LAST TERM TO FIRST POWERS IN $\dot{\eta}$ WE HAVE

$$\sqrt{1-\dot{x}^2} \doteq \sqrt{1-\dot{\bar{x}}^2} - \frac{\dot{\bar{x}}\dot{\eta}}{\sqrt{1-\dot{\bar{x}}^2}} + \text{higher order terms}$$

OUR VARIATION IN THE RELATIVISTIC ACTION YIELDS

$$S_x = \int\left[-m_0c^2\left(\sqrt{1-\dot{\bar{x}}^2} - \frac{\dot{\bar{x}}\dot{\eta}}{\sqrt{1-\dot{\bar{x}}^2}}\right) - V(\bar{x},t) - V'\eta\right]dt$$

$$S_x = S_{\bar{x}} + \int_{t_1}^{t_2}\left[\frac{m_0\dot{\bar{x}}\dot{\eta}}{\sqrt{1-\dot{\bar{x}}^2}} - V'\eta\right]dt$$

INTEGRATING BY PARTS AGAIN

$$S_x = S_{\bar{x}} + \int_{t_1}^{t_2}\left[-\frac{d}{dt}\frac{m_0\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2}} - V'(x,t)\right]\eta(t)\,dt$$

THUS

$$\frac{d}{dt_0}\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-\dot{\bar{x}}^2}}\right] = -\frac{\partial V}{\partial x}$$

SO, IN FACT, WE DO GET THE PROPER EQUATION OF MOTION.

WE CAN GO FARTHER AND DISCUSS THE RELATIVISTIC INVARIANCE OF THIS ACTION PRINCIPLE IN THE FOLLOWING MANNER. IN THREE DIMENSIONS WE HAVE

$$S = -m_0\int_{t_1}^{t_2}\sqrt{1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)}\,dt - \int_{t_1}^{t_2}V(x,t)\,dt$$

SINCE $\frac{ds}{dt} = \sqrt{1-v^2}$ OR $ds^2 = dt^2(1-v^2)$ WE HAVE FOR THE FIRST INTEGRAL,

$$S = -m_0\int\sqrt{dt^2-dx^2-dy^2-dz^2} = -m_0\int_{t_1}^{t_2}ds$$

THUS WE WANT THE PATH WHICH MAKES THE PROPER TIME AN EXTREMUM.

THE WHOLE RELATIVISTIC INVARIANT ACTION IS THUS GIVEN BY

$$S = \int_{t_1}^{t_2} \left[-m_0 ds - V(x,y,z,t)\, dt \right]$$

THE INVARIANCE MEANS THAT IF S IS LEAST FOR ME, IT IS LEAST FOR YOU. THAT IS, IN ANY COORDINATE SYSTEM S IS AN EXTREMUM FOR THE ACTUAL RELATIVISTIC PATH. BUT WE MUST INVENT A MODIFICATION OF THE POTENTIAL TERM SO WE GET THE RIGHT FORCES ON THE EQUATIONS OF MOTIONS, I.E., $\partial V/\partial x$ IS NOT SYMMETRICAL ANYMORE.

ONE POSSIBLE POTENTIAL IS A SCALAR POTENTIAL $\mathcal{X}(x,y,z,t)$ SO

$$S = \int_{t_1}^{t_2} \left[-m_0 ds - \mathcal{X}(x,y,z,t)\, ds \right]$$

SO FAR THERE ARE NO KNOWN LAWS WHICH ARE DERIVABLE FROM THIS ACTION. BUT IT IS FUN TO PREDICT THE BEHAVIOR OF PARTICLES UNDER SUCH A POTENTIAL IF IT WAS A VALID LAW.

ANOTHER POSSIBILITY IS TO HAVE A 4-POTENTIAL A_μ SO THAT

$$S = \int_{t_1}^{t_2} \left[-m_0 ds - A_t(x,y,z,t)\, dt + A_x\, dx + A_y\, dy + A_z\, dz \right]$$

FROM THIS ACTION WE CAN DERIVE THE LAWS OF ELECTRICITY.

ONE MORE POSSIBILITY IS AN ACTION GIVEN BY

$$S = \int -m_0 ds - g_{tt} \frac{dt}{ds}\frac{dt}{ds} ds - g_{tx} \frac{dt}{ds}\frac{dx}{ds} ds + g_{ty} \frac{dt}{ds}\frac{dy}{ds} ds + \dots$$

THIS IS A 10-POTENTIAL SINCE THERE ARE TEN POSSIBLE COMBINATIONS OF x,y,z,t. AN EXAMPLE IS OF A FORCE DERIVABLE FROM THIS ACTION IS GRAVITY

WHAT WE ARE TRYING TO DO BY WRITING OUR CUSTOMARY EQUATIONS OF MOTION IN THIS FUNNY FORM IS TO TRY TO FIND A FRAMEWORK FOR MECHANICS THAT IS SIMPLE. WE ARE NOT TRYING TO DEDUCE ELECTRODYNAMICS. IN SOME FORM, IT SEEMS THAT AN EQUATION WILL TAKE ON A VERY NATURAL SIMPLE APPEARANCE.

MOTION OF A PARTICLE IN AN ELECTROMAGNETIC FIELD

LAST TIME I GAVE YOU A PROBLEM TO WORK OUT AND NOW I WANT TO DO IT. I GAVE YOU THE ACTION FOR A PARTICLE MOVING IN AN ELECTROMAGNETIC FIELD,

$$S = -\int m_0\, ds - e\int A_\mu\, dx_\mu$$

EXPANDING OUT WE HAVE

$$S = -\int m_0 \sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}\, dt - \int e\,\varphi(x,y,z,t)\,dt + e\int A_x(x,y,z,t)\,\frac{dx}{dt}\,dt$$
$$+ e\int A_y(x,y,z,t)\,\frac{dy}{dt}\,dt + e\int A_z(x,y,z,t)\,\frac{dz}{dt}\,dt$$

WE HAVE CHOSEN TO DEFINE THE PATH OF THE PARTICLE BY THE THREE COMPONENTS, $X(t)$, $Y(t)$, AND $Z(t)$. UNFORTUNATELY THIS IS A HIGHLY UNRELATIVISTIC (NOTE, THIS IS DIFFERENTIATED FROM A NON-RELATIVISTIC QUANTITY) APPROACH BECAUSE THE EQUATIONS BECOME LOPSIDED. THAT IS, THEY DEPEND ON THE CHOICE OF COORDINATES. BUT THE ACTION IS INVARIANT AND I WILL PROVE THIS AT THE END. I HAVE CHOSEN TO MAKE THE ANALYSIS AS EASY AS POSSIBLE SO I DON'T DESTROY YOU ALONG THE WAY.

IN THE ABOVE ACTION I HAVE DEFINED THE COMPONENTS IN THE 4-POTENTIAL A_μ IN FOLLOWING WAY

$$A_t = \varphi(x,y,z,t) \quad \text{THE SCALAR POTENTIAL}$$
$$A_{x,y,z} = \text{VECTOR POTENTIAL}$$

FURTHER I HAVE USED THE FACT THAT

$$ds = \sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}\,dt = \sqrt{1-v^2}\,dt$$

NOW LET'S VARY THE PATH AS BEFORE SO WE HAVE A NEW PATH DEFINED BY A VARIATION ~~ARE~~ ON THE ACTUALLY DYNAMICAL PATH

$$X = \bar{X}(t) + \xi(t) \qquad Y(t) = \bar{Y}(t) + \eta(t) \qquad Z(t) = \bar{Z}(t) + \zeta(t)$$

WE NOW WRITE THE ACTION IN TERMS OF FIRST ORDER VARIATIONS ON THE ACTUAL PATH. THAT IS, WE NEGLECT SMALL TERMS LIKE $\xi\eta$, $\dot{\xi}^2$, $\dot{\xi}\dot{\eta}$, ETC. BUT THAT IS AWFULLY DIFFICULT TO DO BECAUSE OF SO MANY TERMS. SO LET'S SIMPLIFY OUR CALCULATION BY ONLY VARYING THE X COMPONENT AND HOLDING $Y(t)$ AND $Z(t)$ FIXED AT $\bar{Y}(t)$ AND $\bar{Z}(t)$ RESPECTIVELY. WHAT WE NOW HAVE FOR THE ACTION IS

$$S = -m_0\int \sqrt{1-(\dot{\bar{x}}+\dot{\xi})^2-\dot{\bar{y}}^2-\dot{\bar{z}}^2}\,dt - e\int \varphi(\bar{x}+\xi,\bar{y},\bar{z},t)\,dt + e\int A_x(\bar{x}+\xi,\bar{y},\bar{z})(\dot{\bar{x}}+\dot{\xi})\,dt$$
$$+ e\int A_y(\bar{x}+\xi,\bar{y},\bar{z},t)\,\dot{\bar{y}}\,dt + e\int A_z(\bar{x}+\xi,\bar{y},\bar{z},t)\,\dot{\bar{z}}\,dt$$

NOW WE HAVE TO DO SOME EXPANDING AS BEFORE. BEGINNING WITH THE FIRST TERM WE MUST APPROXIMATE THE SQUARE ROOT TO FIRST ORDER IN $\dot{\xi}$,

$$\sqrt{1-(\dot{\bar{x}}+\dot{\xi})^2-\dot{\bar{y}}^2-\dot{\bar{z}}^2} = \sqrt{1-v^2-2\dot{\bar{x}}\dot{\xi}} \doteq \sqrt{1-v^2} - \frac{\dot{\bar{x}}\dot{\xi}}{\sqrt{1-v^2}} \quad \text{BY THE BINOMIAL THEOREM}$$

The second term can be written as

$$e\int \varphi(\bar{x}+\xi,\bar{y},\bar{z})\,dt = e\int \varphi(\bar{x},\bar{y},\bar{z},t) + \xi\frac{\partial\varphi(x,y,z,t)}{\partial x}\,dt$$

The same type of expansion into a Taylor series goes for the vector potential so we can write altogether to first order

$$S = -m_0\int\sqrt{1-v^2}\,dt + \int\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}} + eA_x\right]\dot{\xi}\,dt + \int\mathcal{E}(t)\left[-\frac{\partial\varphi}{\partial x}e + e\left(\frac{\partial A_x}{\partial x}\dot{\bar{x}} + \frac{\partial A_y}{\partial y}\dot{\bar{y}} + \frac{\partial A_z}{\partial z}\dot{\bar{z}}\right)\right]dt$$

Transforming the second integral by parts we get,

$$\mathcal{E}\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}} + eA_x\right]_{\text{inital}}^{\text{final}} - \int\mathcal{E}(t)\frac{d}{dt}\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}} + eA_x\right]$$

A side note here the terms in the brackets are the generalized momenta and can be written in the form $\frac{\hbar}{i}\frac{\partial}{\partial x}$ but I won't go into that now. But the first term is zero since we don't vary the end points. If we did vary them, it is like changing the momentum a small amount.

Putting all this dribble together,

$$S = -\int m_0\sqrt{1-v^2}\,dt - e\int\bar{\varphi}\,dt + e\int\bar{A}_x\,dt\,\dot{\bar{x}} + e\int\bar{A}_y\,\dot{y}\,dt + e\int\bar{A}_z\,\dot{\bar{z}}\,dt$$
$$+\int\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}} + eA_x\right]dt + \int\mathcal{E}(t)\left[\frac{\partial\varphi}{\partial x} + e\left(\frac{\partial A_x}{\partial x}\dot{\bar{x}} + \frac{\partial A_y}{\partial y}\dot{\bar{y}} + \frac{\partial A_z}{\partial z}\dot{\bar{z}}\right)\right]dt$$

Rewriting the change in the action

$$S = -m_0\int ds - \int eA_\mu\,dx_\mu + \int\left[-\frac{d}{dt}\left(\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}}\right) - \frac{d}{dt}eA_x - e\frac{\partial\phi}{\partial x} + e\frac{\partial A_x}{\partial x}\dot{\bar{x}} + e\frac{\partial A_y}{\partial x}\dot{\bar{y}} + e\frac{\partial A_z}{\partial z}\dot{\bar{z}}\right]\mathcal{E}\,dt$$

We can now equate the coefficient of $\mathcal{E}(t)$

$$-\frac{d}{dt}\left(\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}}\right) = -F_x = +\frac{d}{dt}eA_x + e\frac{\partial\phi}{\partial x} - \left(e\frac{\partial A_x}{\partial x}\dot{\bar{x}} + e\frac{\partial A_y}{\partial x}\dot{\bar{y}} + e\frac{\partial A_z}{\partial z}\dot{\bar{z}}\right)$$

Now it is not correct to take the time derivative of A_x because A_x is constantly changing with. The real time derivative is by expanding the following differential to first order

$$A_x[\bar{x}(t+\Delta t),\bar{y}(t+\Delta t),\bar{z}(t+\Delta t),t+\Delta t)] - A_x(\bar{x}(t),\bar{y}(t),\bar{z}(t),t)$$

$$A_x[\bar{x}+\Delta t\,\dot{\bar{x}},\ \bar{y}+\Delta t\,\dot{\bar{y}},\ \bar{z}+\Delta t\,\dot{\bar{z}},\ t+\Delta t) - A_x[\bar{x},\bar{y},\bar{z},t] =$$

$$\frac{\partial A_x}{\partial x}\dot{\bar{x}}\,\Delta t + \frac{\partial A_x}{\partial y}\dot{\bar{y}}\,\Delta t + \frac{\partial A_x}{\partial z}\dot{\bar{z}}\,dt + \frac{\partial A_x}{\partial t}\Delta t = \text{first order change}$$

Finally then,

$$-\frac{d}{dt}\left[\frac{m_0\dot{\bar{x}}}{\sqrt{1-v^2}}\right] = -F_x = e\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + e\dot{\bar{y}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + e\dot{\bar{z}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

This is the way a particle moves in one dimension in a electromagnetic field. We can expend this law to three dimensions quite easily, The form of F_x could have been expressed in the following way

$$F_x = e[\overline{E}(\overline{x},\overline{y},\overline{z}) + \vec{v} \times \overline{B}]_x$$

So generally we have

$$\overline{F} = e\vec{E} + e\overline{v} \times \overline{B}$$

As we see by comparing the above F_x with our previous result, the electric and magnetic field vectors $\bar{E}$, and $\bar{B}$, respectively, are related to ϕ and A the scalar and vector potentials,

$$\overline{E} = -\overline{\nabla}\phi - \frac{\partial}{\partial t}\overline{A} \qquad \overline{B} = \overline{\nabla} \times \overline{A}$$

It is more customary to give the fields than the potentials when expressing the force. If we write these two equation all out, we find that the six field components are related to the 4 components of the four vector A_μ. But the relationships are not independent and we have two limitations or conditions we must satisfy,

$$\overline{\nabla} \cdot \overline{B} = 0 \qquad \text{And} \qquad \overline{\nabla} \times \overline{E} = \frac{\partial \overline{B}}{\partial t}$$

$$\boxed{\begin{aligned} E_x &= -\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t} \\ E_y &= -\frac{\partial\phi}{\partial y} - \frac{\partial A_y}{\partial t} \\ E_z &= -\frac{\partial\phi}{\partial z} - \frac{\partial A_z}{\partial t} \\ B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{aligned}}$$

Taking the first condition we have

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

So

$$\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0$$

A similar mathematical proof will show $\overline{\nabla} \times \overline{E} = \frac{\partial \overline{B}}{\partial t}$.

I might point out that historically, the vector potential, A, was discovered before the conditions $\vec{\nabla} \times \vec{E} = \vec{B}$ and $\vec{\nabla} \cdot \vec{B} = 0$. Maxwell made the discovery. He was working with the phenomena described by Faraday. When he put all the known laws together, he found them to be mutually inconsistent. He found he ~~de~~ needed a displacement current to untangle the problem. The vector potential was called the electrotonic state vector and to be absolutely fair it was so named by Von Neumann.

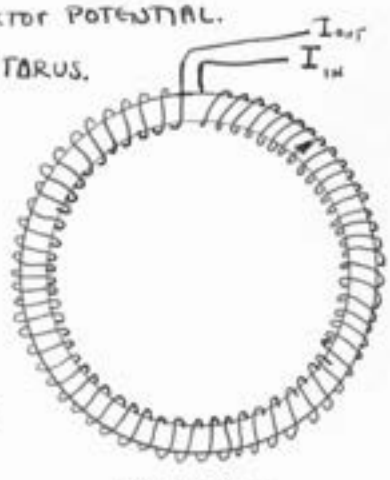
There appears to be some arbitrariness in where you start this whole discussion. Whether I start with this action crap and deduce the force laws or claim to know the laws and discover the electric and magnetic field connection is all immaterial. But I choose to write the action to begin with because it is a more eloquent and simple equation from which I can deduce a lot. You can't force me or nature in telling you which are the assumptions and which are the deductions. I like to start from the obscure and deduce the common place so the other way, i.e., giving the force laws to start with, is the crazy man's assumption.

HOW TO DETECT THE VECTOR POTENTIAL

DURING THE EARLY YEARS OF WORK IN ELECTROMAGNETICS THERE WASN'T A WAY TO MEASURE THE PRESENCE OF THE VECTOR POTENTIAL. THE ONLY MEASURABLE QUANTITIES WERE THE ELECTRIC AND MAGNETIC FIELDS. BUT NOW THAT WE HAVE QUANTUM MECHANICS WE KNOW THAT THE ACTION IS RELATED TO THE PHASE OF A WAVE WHICH IN TURN IS A FUNCTION OF THE INTEGRAL OF THE VECTOR POTENTIAL AROUND A CLOSED PATH. THAT IS,

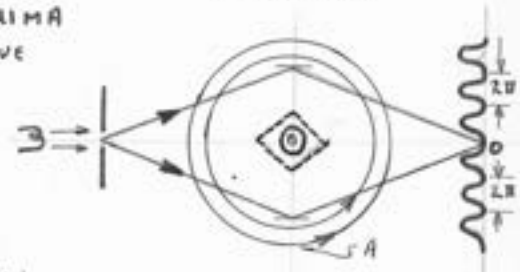
$$\text{ACTION} \propto \oint A_\mu ds$$

SINCE WE CAN'T WRITE QUANTUM MECHANICS IN TERMS OF E AND B, WE BECOME AWFULLY SUSPICIOUS THAT MAYBE THERE DOES EXIST AN EXPERIMENT WHICH WILL ALLOW US TO DETECT THE PRESENCE OF THE VECTOR POTENTIAL.



LARGE TORUS

LET'S CONSIDER A VERY LARGE COIL MADE IN THE SHAPE OF TORUS. NOW CONSIDER A CROSS SECTION OF THIS COIL WHERE THERE IS A B-FIELD PRESENT, I.E., GET CLOSE TO THE COIL. AROUND THIS B-FIELD CIRCULATES A VECTOR POTENTIAL; EVEN IN REGION SUFFICIENTLY FAR FROM THE COIL WHERE B = 0. SAY, OUTSIDE THE LITTLE DIAMOND B=0. NOW LET ELECTRONS MOVE INTO THE REGION OF ZERO ELECTRIC AND MAGNETIC FIELD BY EMITTING THEM THROUGH A CONTROLLED SLIT WHICH SPLITS THE BEAM AROUND THE COIL, REFLECTS OFF SOME MIRRORS AND COMES TOGETHER ON A SCREEN WHICH RECORDS THE NUMBER AND LOCATION OF THE IMPINGING ELECTRONS. WE WILL FIND TO OUR ABSOLUTE AMAZEMENT THAT A TYPICAL DIFFRACTION PATTERN COMPLETE WITH MAXIMA AND MINIMA IS OBSERVED. IF THE B-FIELD IS INCREASED WE OBSERVE THE ZERO POINT TRANSLATE UP AND DOWN BUT THE SPACING BETWEEN MINIMA REMAINS CONSTANT AND EQUAL TO 2π. FOR THE ELECTRONS TO BE DIFFRACTED LIKE THIS THEY MUST FEEL THE PRESENCE OF A NEW KIND OF FIELD, VIZ., THE VECTOR POTENTIAL. IT IS POSSIBLE TO DETERMINE THE LINE INTEGRAL OF "A" AROUND A CLOSED PATH EVEN IN REGIONS OF B=0, I.E.,



$$\oint A \cdot ds = \int B_n \, d(\text{area}) \qquad \text{where } \bar{B} = \bar{\nabla} \times \bar{A}$$

THE PRESENCE OR RATHER EXISTENCE OF THE VECTOR POTENTIAL IS A CONSEQUENCE OF REQUIRING THE ACTION TO BE A LOCAL PHENOMENA. THAT IS, THERE IS A CONDITION IN SPACE WHICH WE CHOOSE TO CALL THE VECTOR POTENTIAL WHICH PROPERLY EXALAINS THE BEHAVIOR OF PARTICLES MOVING IN REGIONS OF ZERO B-FIELD. ON REQUIREMENT ON LOCAL PHENOMENA RIDS OUR THEORY OF INSTANTANEOUS ACTION AT A DISTANCE WHICH WE WOULD HAVE TO ALLOW IN ORDER TO EXPLAIN THE BEHAVIOR OF THE PARTICLE IN REGIONS OF, THEORETICALLY ZERO. IN CLASSICAL PHYSICS WE WOULD NOT BE ABLE TO DIRECTLY MEASURE THE A-POTENTIAL'S EFFECT AS SEEN BY THE EXPERIMENT.

One consequence of this argument is that the vetor potential cannot be determined at a single point in space. Rather it is evaluated over a closed path. Furthermore if we multiply this line integral by the charge over Planck's constant, we discover this quantity is determinable only to an integral multiple of 2π. That is, we say it is modulo integer 2π. The reason for this is utterly unknown. If any changes occurred in this quantity it would have to be related to a change in the charge, q, but so far all charges are multiples of a fundamental unit. If we recall our classical approximation of the phase is is given by the action over Planck's constant,

$$\phi = \frac{S}{\hbar} = -\frac{m_0}{\hbar}\int ds + \frac{e}{\hbar}\int A\cdot ds + \frac{1}{\hbar}\int V dt$$

With the second integral equaling $2\pi n$ we begin to understand how the various waves could add in phase. All the difficulty we have in determining A is due to the fact that we must interfer with the electron to deflect it and measure it which might make it appear a little less real to us. But it is analogous to the observation of an E and B field where to some moving coordinate system B might be zero.

Four-Dimensional Gradient Operator

ref. Vol II Chapter 25

I want to go back an piddle around with w the electric and magnetic field components which I gave you last time; for instance,

$$E_x = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t} \qquad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad \text{etc}$$

I choose to with the scalar potential ϕ as A_t so I can recognize the relativistic character of these equations. What I come up with is another example of a 6 vector if I write:

$$\begin{array}{llll} F_{xt} = E_x & F_{yt} = E_y & F_{zt} = E_z & \text{and} \quad F_{\mu\nu} = -F_{\nu\mu} \\ F_{xy} = -B_z & F_{yz} = -B_x & F_{zx} = -B_y & \qquad F_{\mu\mu} = 0 \end{array}$$

I have to realize the following. If I have some vector potential $\mathcal{X}(x,y,z,t)$ which is invariant, then the quantity

$$\nabla_\mu \mathcal{X} = \frac{\partial \mathcal{X}_t}{\partial t} - \frac{\partial \mathcal{X}}{\partial x} - \frac{\partial \mathcal{X}}{\partial y} - \frac{\partial \mathcal{X}}{\partial z}$$

is also invariant. Here I have introduced the four dimension gradient operator

$$\nabla_\mu = \left(\frac{\partial}{\partial t}, -\bar{\nabla}\right) = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

The invariance of the operator implies it behaves like a 4-vector. I won't bother proving the invariance but rather let myself believe it is.

INSTEAD LET ME SHOW HOW THIS OPERATOR WORKS, CONSIDER A WAVE FUNCTION SUCH THAT

$$\chi = e^{i(\omega t - k_x X - k_y Y - k_z Z)}$$

Where $k_\mu = (\omega, k_x, k_y, k_z)$

Then by our rule for differentiation

$$\frac{\partial \chi}{\partial t} = i\omega\chi \,, \quad \frac{\partial \chi}{\partial x} = -ik_x\chi \,, \quad \frac{\partial \chi}{\partial Y} = -ik_y\chi \quad \frac{\partial \chi}{\partial z} = -ik_z\chi$$

IN order to obtain the desired result,

$$\nabla_\mu \chi = i k_\mu \chi$$

WE SEE THE LAST THREE PARTIAL DERIVATIVES MUST BE NEGATIVE SO WE GET $-\frac{\partial \chi}{\partial x} = ik_x\chi$, ETC. THUS THE REASON FOR THE NEGATIVE SIGNS.

RETURNING NOW TO OUR FORCE TENSOR $F_{\mu\nu}$, I CAN CHANGE MY NOTATION INTO A NEW FORM USING THE 4-GRADIENT,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

where $A_{\mu\nu} = (\varphi, A_x, A_Y, A_z)$

TO SHOW THIS IS RIGHT CONSIDER THE TERM F_{xt},

$$F_{xt} = \nabla_x A_t - \nabla_t A_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_t}{\partial t}$$

NOW TRY F_{XY},

$$F_{XY} = \nabla_x A_Y - \nabla_Y A_x = -\frac{\partial A_Y}{\partial x} + \frac{\partial A_x}{\partial Y} = -B_z$$

AND THE OTHER COMPONENTS ARE JUST AS EASY. WHAT I HAVE DONE HERE IS TO PROVE HOW $F_{\mu\nu}$ CAN BE RELATED TO THE THREE ELECTRIC AND THREE MAGNETIC FIELD COMPONENTS WHICH I GAVE YOU EARLY AS AN EXAMPLE OF A 6-VECTOR.

THE EQUATIONS OF MOTION IN RELATIVISTIC NOTATION

Ref. VOL II Chapter 26.

WE NOW WANT TO RETURN TO OUR RELATIVISTIC FORM OF THE FORCE,

$$\frac{d}{dt}\left[\frac{m_0 \vec{V}}{\sqrt{1-v^2}}\right] = q\left[\vec{E} + \vec{V}\times\vec{B}\right] = \vec{F}$$

BUT REMEMBER THE TIME-DERIVATIVE OF A 4-VECTOR IS NOT A 4-VECTOR AS IT STANDS. THE EXPRESSION $\partial/\partial t$ IS A SPECIAL THING WITH IS NOT INVARIANT. WE MUST CHANGE TO PROPER, I.E., $dS = \sqrt{1-v^2}\,dt$. SO IF WE DIVIDE BOTH SIDES OF OUR EQUATION BY $\frac{1}{\sqrt{1-v^2}}$ WE WILL HAVE BY RECALLING ALSO OUR DEFINITION OF THE FOUR-VELOCITY

$$\frac{dX_\mu}{dS} = \frac{\vec{V}}{\sqrt{1-v^2}}$$

SINCE

$$\frac{v}{\sqrt{1-v^2}} = \frac{dx/dt}{\sqrt{1-v^2}} = \frac{dX}{dS}$$

SO THAT WE HAVE AS OUR FORCE EQUATION,

$$\frac{d}{ds}\left[m_0 \frac{dx_\mu}{ds} \right] = q\left[\frac{\vec{E}}{\sqrt{1-v^2}} + \frac{\vec{V}}{\sqrt{1-v^2}} \times \vec{B} \right]$$

WE ARE NOT READY TO WRITE THIS EQUATION RELATIVISTICALLY INVARIANTLY. TO DO THAT LET'S TAKE FOR SIMPLICITY'S SAKE JUST THE X-COMPONENT,

$$m_0 \frac{d^2 x_\mu}{ds^2} = q\left[\frac{E_x}{\sqrt{1-v^2}} + \frac{(\vec{V}\times\vec{B})_x}{\sqrt{1-v^2}} \right] = q\left[\frac{dt}{ds} E_x + \frac{dy}{ds} B_z - \frac{dz}{ds} B_y \right]$$

REMEMBERING THAT $F_{yx} = B_z$ AND $F_{xz} = B_y$

$$m_0 \frac{d^2 x_\mu}{dS^2} = q\left[F_{xt} \frac{dt}{ds} - F_{xx}\frac{dx}{ds} - F_{xy}\frac{dy}{ds} - F_{xz}\frac{dz}{ds} \right]$$

SINCE $F_{xx} = 0$, WE HAVE STUCK IT IN FOR THE HECK OF IT. THE SECOND INDEX IS NOW INVARIANT WHILE THE FIRST INDEX APPEARS ON EITHER SIDE.

THUS WE WRITE THE EQUATION OF MOTION OF A PARTICLE OF REST MASS m_0 AS

$$f_\mu = m_0 \frac{d^2 x_\mu}{dS^2} = q \sum_\nu F_{\mu\nu} \frac{dx_\nu}{dS}$$

CURIOSITY DEMANDS US TO FIND THE 4th COMPONENT SO LET μ EQUAL THE TIME VARIABLE t. IN OUR EQUATION WE HAVE,

$$\frac{d}{ds}\left[m_0 \frac{1}{\sqrt{1-v^2}} \right] = q\left[F_{tt} \frac{dt}{ds} - f_{tx}\frac{dx}{ds} - f_{ty}\frac{dy}{ds} - f_{tz}\frac{dz}{ds} \right]$$

NOTING $f_{tt} = 0$ AND $E_x = F_{tx}$, $E_y = f_{ty}$, $E_z = f_{tz}$. EXPANDING OUT,

$$\frac{1}{\sqrt{1-v^2}} \frac{d}{dt}\left(m_0 \frac{1}{\sqrt{1-v^2}} \right) = q \frac{\left[E_x V_x + E_y V_y + E_z V_z \right]}{\sqrt{1-v^2}}$$

THE FINAL BRILLIANT EQUATION IS

$$\frac{d}{dt}\left[\frac{m_0}{\sqrt{1-v^2}} \right] = q\, \vec{V}\cdot\vec{E}$$

THE TERM INSIDE THE BRACKET WE RECOGNIZE AS ENERGY SO WE HAVE THE TIME RATE OF CHANGE OF ENERGY OR THE POWER WHICH IS EQUAL TO $\vec{V}\cdot\vec{F}$. WE COULD HAVE MADE THIS DISCOVERY EARLY IF WE HAD CALCULATED $\vec{V}\cdot\vec{F}$,

$$\vec{F}\cdot\vec{V} = \vec{V}\cdot q\left[\vec{E} + \vec{V}\times\vec{B} \right] = q\vec{V}\cdot\vec{E} + q\underbrace{\vec{V}\cdot(\vec{V}\times\vec{B})}_{=0}$$

THUS WE HAVE SHOWN IT IS POSSIBLE TO FIND THE FORTH COMPONENT OF THE FOUR FORCE GIVEN THE OTHER THREE.

ONE FURTHER COMMENT, I GAVE YOU THE PROBLEM OF SHOWING $\frac{dx_\mu}{ds} f_\mu = 0$. AND SHOWED YOU SINCE $\frac{dx_\mu}{ds}\frac{d}{ds}\left(m_0 \frac{dx_\mu}{ds} \right) = \frac{m}{2}\frac{d}{ds}\left[\frac{dx_\mu}{ds}\frac{dx_\mu}{ds} \right] = 0$ SINCE $\frac{dx_\mu}{ds}\frac{dx_\mu}{ds} = 1$ AND $\frac{d1}{ds} = 0$. A MORE GENERAL FORM OF THIS LAW IS

$$\sum_\mu f_\mu \frac{dx_\mu}{dS} = q \sum_\mu \sum_\nu F_{\mu\nu} \frac{dx_\nu}{dS}\frac{dx_\mu}{dS} = 0$$

BY EXPANDING THE PRODUCT OF THE ANTISYMMETRICAL 4 VECTOR $F_{\mu\nu}$ WITH THE SYMMETRICAL ONES $\frac{dx_\nu}{dS}$ YOU CAN SHOW ALL THE TERMS ADD UP TO ZERO. WHAT THIS MEANS IS THAT THERE IS NO VELOCITY DEPENDEND FORCE COMPONENT IN THE DIRECTION OF THE VELOCITY. ~~For~~ SUCH A TERM IS ANALOGOUS TO A RESISTANCE TERM WHICH WOULD IMPLY A NON-CONSERVATIVE ENERGY IN VIOLATION OF OUR ACTION PRINCIPLE.

MOTION OF A PARTICLE IN A CONSTANT ELECTRIC FIELD

LET'S GO BACK AN SOLVE THE FORCE EQUATION

$$\frac{d}{dt}\left[\frac{m_0 \bar{v}}{\sqrt{1-v^2}}\right] = q\left[\bar{E} + \bar{v}\times\bar{B}\right]$$

FOR SOME SPECIAL CASES. THIS EQUATION IS OUR STARTING PLACE TOGETHER WITH OUR CONDITIONS THAT,

$$\bar{\nabla}\cdot\bar{B} = 0 \quad \text{AND} \quad \bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

AS OUR FIRST CASE LET'S CONSIDER THE MOTION OF A PARTICLE WHICH STARTS AT REST AND MOVES IN A CONSTANT ELECTRIC FIELD. AS YOU MIGHT ALREADY KNOW THE PARTICLE IS ACCELERATED FASTER AND FASTER UNTIL IT BECOMES RELATIVISTIC AND ITS SPEED APPROACHES THAT OF LIGHT. AS WE HAVE LEARNED SO FAR THE MASS WILL CONTINUALLY INCREASE DURING THIS MOTION BUT LET'S SEE A LITTLE BETTER WHAT HAPPENS.

FOR SIMPLICITY LET $\bar{E} = E_0\vec{e}_x$ SO WE ONLY HAVE TO CONSIDER MOTION IN A SINGLE DIMENSION. THUS WE HAVE,

$$\frac{d}{dt}\left[\frac{m_0 \dot{x}}{\sqrt{1-\dot{x}^2}}\right] = qE$$

INTEGRATING ONCE

$$\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = \frac{qE}{m_0}t$$

BY MULTIPLYING BY $\sqrt{1-\dot{x}^2}$, SQUARING, MULTIPLYING THROUGH, SUBTRACTING, AND TURNING EVERYTHING UPSIDE DOWN IT IS QUITE EASY TO SHOW

$$\dot{x} = \frac{qEt/m}{\sqrt{1+(\frac{q}{m}Et)^2}}$$

IF IT BOTHERS YOU HOW I DID THIS SO FAST, I DID IT AT HOME AND I HAPPENED TO REMEMBER HOW IT CAME OUT. NOW THIS CAN BE INTEGRATED AGAIN TO GIVE

$$x = \frac{m}{qE}\left[\sqrt{1+(\tfrac{qE}{m}t)^2} - 1\right]$$

WHERE THE -1 IS INSERTED TO CORRECT FOR $t=0$.

NOW LET'S EXAMINE THE BEHAVIOR OF THIS MOTION AND SEE IF IT MAKES SOME COMMON SENSE. CONSIDER FIRST MOTION VERY EARLY WHERE WE CAN EXPAND THE SQUARE ROOT TO FIRST ORDER AND WE GET

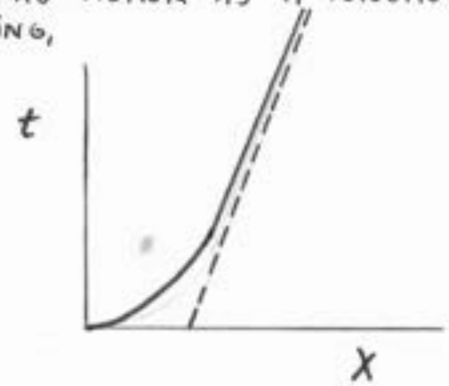
$$x = \frac{1}{2}\frac{qE}{m}t^2 \qquad \text{FOR} \quad t < 1$$

THIS EXPRESSION IS JUST LIKE A FORCE WHERE THE TERM $\frac{qE}{m}$ CORRESPONDS TO AN ACCELERATION. THIS IS WHAT WE EXPECT FOR THE NON-RELATIVISTIC FORCE WHEN $\dot{x} \ll c$. FOR TIMES MUCH LATER THE 1 IN THE SQUARE ROOT IS NEGLIGIBLE SO WE FIND

$$x = t - \frac{m}{qE}$$

SINCE $c = 1$ IN OUR UNITS, WE SEE THE ~~POSITION~~ PARTICLE IS TRAVELING AT THE SPEED OF LIGHT MINUS SOME SMALL AMOUNT.

A plot of this motion as a function of time would look like the following,



We can also examine how the mass changes since we know

$$m = \frac{m_0}{\sqrt{1-\dot{x}^2}}$$

When we are just getting started we can approximate $\dot{x}$ as

$$\dot{x} \approx \frac{qE}{m} t \qquad (\text{early})$$

Since $\dot{x}$ is small, also expand the square root

$$m \approx m_0 \left(1 + \tfrac{1}{2}\dot{x}^2\right) = m_0 \left[1 + \tfrac{1}{2}\left(\frac{qEt}{m}\right)^2\right]$$

Or since

$$\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = \frac{qEt}{m_0} \rightarrow \frac{m_0}{\sqrt{1-\dot{x}^2}} = \frac{qEt}{\dot{x}} = \frac{qEt}{\frac{qEt}{m_0}\sqrt{1+\left(\frac{q}{m}Et\right)^2}}$$

$$m \approx m_0 \sqrt{1+\left(\frac{q}{m}Et\right)^2} \approx m_0 \left[1 + \tfrac{1}{2}\left(\frac{q}{m}Et\right)^2\right]$$

From this form we have again

$$\frac{m_0}{\sqrt{1-\dot{x}^2}} = m_0 \sqrt{1+\left(\frac{q}{m}Et\right)^2}$$

but $\quad x q E + m_0 = m_0 \sqrt{1+\left(\frac{qEt}{m}\right)^2}$ from the previous page

Thus we see that

$$\frac{m_0}{\sqrt{1-\dot{x}^2}} = qEx + m_0$$

This simply says that the energy of the particle is equal to the rest energy plus the work done on it, i.e., the energy per foot imparted to it. We can see this by remembering $F = qE$ and $Fx = \text{work}$. This is the basic idea behind the linear accelerator.

Another way to express this result is in proper time where we let $\frac{qEt}{m} = \sinh \frac{eEs}{m}$ so that $x = \frac{m}{qE}\left(\cosh \frac{eEs}{m} - 1\right)$ and $\dot{x} = \tanh\left(\frac{qEs}{m}\right)$. Recalling a while back I talked about addition of velocities in terms of what called rapidity, w, i.e.,

$$V = \tanh(w_1 + w_2) = V_1 + V_2$$

where $w = \frac{eEs}{m}$ the rapidity here.

MOTION OF A PARTICLE IN A CONSTANT MAGNETIC FIELD

LET'S CONSIDER THE CASE WHERE WE HAVE ONLY A MAGNETIC FIELD ACTING ON THE PARTICLE, i.e.,

$$\frac{d}{dt}\left[\frac{m_0\vec{v}}{\sqrt{1-v^2}}\right] = q\vec{v}\times\vec{B}$$

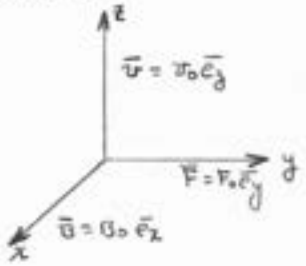
SINCE THE TIME COMPONENT OF THE FOUR FORCE IS ZERO

$$\frac{d}{dt}\left[\frac{1}{\sqrt{1-v^2}}\right] = q\vec{v}\cdot\vec{E} = 0$$

WE CAN TAKE THE SQUARE ROOT OUTSIDE THE DIFFERENTIAL AND GET THE FOLLOWING FORCE LAW

$$\frac{1}{\sqrt{1-v^2}}\frac{d}{dt}m_0\vec{v} = q\vec{v}\times\vec{B}$$

NOW IF WE CONSIDER THE VELOCITY AND MAGNETIC FIELD ARE NORMAL TO EACH OTHER THEN WE HAVE SIMPLY



$$\frac{1}{\sqrt{1-v^2}}m_0 a = qvB$$

WHERE a IS THE ACCELERATION OF THE PARTICLE.

IF THE MAGNETIC FIELD IS CONSTANT, THE PARTICLE WILL SIMPLY GO IN A CIRCULAR PATH WITH ACCELERATION

$$a = \frac{v^2}{R}$$

WHERE R IS THE RADIUS OF THE CIRCLE.

THUS, WE HAVE

$$\frac{1}{\sqrt{1-v^2}}m_0\frac{v^2}{R} = qvB$$

OR

$$qBR = \frac{m_0 v}{\sqrt{1-v^2}} = |P| \text{, THE MOMENTUM}$$

THE MOMENTUM OF A PARTICLE CAN BE DETERMINED FROM THE SIZE OF THE CIRCLE IN MOVES AROUND WHEN SUBJECTED TO A CONSTANT B-FIELD.

FOR THE NON-RELATIVISTIC CASE WE IGNORE $\sqrt{1-v^2}$ ~~AS NO~~ SINCE IT DOESN'T DIFFER MUCH FROM 1. THEN WE HAVE

$$qBR = m_0 v$$

AND FROM THIS WE GET THE CYCLOTRON FREQUENCY, AS IT IS CALLED,

$$\omega = \frac{v}{R} = \frac{2\pi}{T} = \frac{qB}{m_0}$$

WHERE T IS THE PERIOD OF THE MOTION

THE CYCLOTRON FREQUENCY IS CONSTANT FOR VELOCITIES MUCH LESS THAN THE SPEED OF LIGHT. AS THE VELOCITY INCREASES, IT IS POSSIBLE TO MAINTAIN A CONSTANT FREQUENCY BY INCREASING THE B-FIELD PROPORTIONATELY. BUT THIS DOES NOT WORK BEYOND A CERTAIN ENERGY LEVEL. ONE PROBLEM IS MAINTAINING A HIGH AVERAGE MAGNETIC FIELD INSIDE THE ORBIT. ONE WAY AROUND THIS IS TO DESIGN THE COIL SO THAT THE FIELD IS HIGHER IN THE CENTER. ANOTHER DIFFICULTY IS THE ENERGY LOSSES DUE TO THE PARTICLES RADIATION OF ENERGY, SYNCHROTRON RADIATION. WHAT WE HAVE BEEN DISCUSSING IS THE IDEA OF THE BETATRON AND CYCLOTRON, MACHINES FOR ACCELERATING ELECTRONS TO HIGH ENERGIES.

MOTION OF A PARTICLE IN A CONSTANT ELECTRIC AND MAGNETIC FIELD

When last seen we found the equation of force on a particle moving in an electric and magnetic field. The equation was given as

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

where we recall two other equations necessary to complete this equation,

$$\mathbb{E} = -\nabla\phi - \frac{\partial \mathbb{A}}{\partial t} \qquad \mathbb{B} = \nabla \times \mathbb{A}$$

Now we have lots of directions to go from here and we consider a couple of those directions last time.

I should do another problem — that is, the case for motion of a particle in a constant electric and magnetic field. This case is very amusing; so I'll describe it. Consider the B-field into the paper and the E-field directed vertically upward. As a particle starts out in the direction of E it is deflected to the left by the B-field. The path follows a curve until the particle is curled back and is now being deaccelerated as it headed against the E-field. Soon a reversal point is reached and the particle starts to accelerate again. The net path is a cycloid translating along the direction of B so actually we have a helix or corkscrew motion.

Depending on the direction of the initial velocity you can get different motions but they are always cycloidal.

I will leave it up to you to figure out the differential equations of motion. But I want to point out there is another way to understand this motion and that way is with relativity. Whenever a particle is moving at some velocity v we can recall our electric and magnetic field transformations that we developed a while back. Those equations went something like

$$\mathbb{B}' = \frac{\mathbb{B} - \frac{\mathbb{V} \times \mathbb{E}}{c^2}}{\sqrt{1 - v^2/c^2}} \qquad \mathbb{E}' = \frac{\mathbb{E} + \mathbb{V} \times \mathbb{B}}{\sqrt{1 - v^2/c^2}}$$

where $B_z' = B_z$ and $E_z' = E_z$

If another coordinate system we are able to describe the ~~motion~~ motion of the particle as if it were only acted upon by a magnetic field instead of both an electric and magnetic field. But since the new coordinate is translating at velocity v, it is easier to understand how we get a helical motion. If the magnitude of E is greater than B the particle is never turned around but is just deflected.

I don't know if its worth talking about more complicated motions. So I think I'll go on to something new and let you work out some more cases.

The Gauge (or "no good") Transformation ref. Vol II Chapter 18

I want to tell you how to calculate the vector and scalar potentials given a certain charge distribution. Then you can find the electric and magnetic fields right away. First I want to consider a few special cases and later tell you how to work with any kind of charge distribution. However, let me point out that it is not possible to uniquely define $\bar{A}$ and ϕ for a given $\bar{E}$ and $\bar{B}$ and quantum mechanical probability. I'd better explain the reason for this.

We have talked about quantum mechanics in terms of the principle of least action and expressed it in the following way

$$S_m = m_0\int_1^2 ds + \int_1^2 (A_t\,dt - \bar{A}\cdot\bar{v}\,dt)$$

calculated along some path from point 1 to point 2. Suppose we call the above action S_m and say it is the action observed by Moe. Now if Joe has a different vector potential than Moe because he is moving or something, his action could be expressed as:

$$S_J = m_0\int_1^2 ds + \int_1^2 (A_t'\,dt - \bar{A}'\cdot\bar{v}\,dt)$$

determined along the same path. What we want to know is whether both Joe and Moe experience the same minimum action for the actual dynamical path? The answer to this question is yes if the difference between the actions does not depend on the path of integration. Under such a condition both Joe and Moe experience the same physics, i.e., the force laws are the same.

What is the difference? Well, it can be written as

$$S_m - S_J = \int_1^2 (a_t - \bar{a}\cdot\bar{v})\,dt$$

where $a_\mu = A_\mu - A_\mu'$. Again the two motions will be equivalent if this integral does not depend on the path taken. Such a condition would occur if the integrand was a perfect differential, i.e.,

$$\int_1^2 \frac{d\mathcal{X}(x,y,z,t)}{dt}\,dt = \mathcal{X}(x_2 y_2, z_2 t) - \mathcal{X}(x_1, y_1, z_1, t)$$

This is equivalent to the following

$$\int_1^2 \left(\frac{\partial \mathcal{X}}{\partial t} + \dot{x}\frac{\partial \mathcal{X}}{\partial x} + \dot{y}\frac{\partial \mathcal{X}}{\partial y} + \dot{z}\frac{\partial \mathcal{X}}{\partial z} \right) dt = \mathcal{X}(2) - \mathcal{X}(1)$$

Now if we have

$$a_t = \frac{\partial \mathcal{X}}{\partial t} \quad \text{And} \quad \vec{a} = -\vec{\nabla}\mathcal{X}$$

or more elegantly

$$a_\mu = \nabla_\mu \mathcal{X}$$

Then all would be well; The two actions would produce the same effects. Thus it does not depend on whether you start with A or A'. Where A and A' are related by the above restriction, namely,

$$\bar{A}' = \bar{A} - \bar{\nabla}\mathcal{X} \qquad A_t' = A_t + \frac{\partial \mathcal{X}}{\partial t}$$

The laws of classical physics, i.e., $\underline{F} = q(\bar{E} + \bar{v} \times \bar{B})$, hold for the above transformation. We can see that $\bar{A}$ can be changed by an amount $\bar{\nabla}\mathcal{X}$ in the following way:

Since $\bar{B} = \bar{\nabla} \times \bar{A}$ we can determine a new magnetic field B' using the transformation to A' or

$$\vec{B}' = \bar{\nabla} \times \bar{A}'$$

Upon substituting for A' $A - \nabla\mathcal{X}$, we have:

$$\bar{B}' = \bar{\nabla} \times \bar{A} - \bar{\nabla} \times \bar{\nabla}\mathcal{X} = \bar{\nabla} \times \bar{A} - 0$$

The last term on the right is zero and can also be shown to be true. Take the z-component alone of $(\bar{\nabla} \times \bar{A})$

$$(\bar{\nabla} \times \bar{A})_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}$$

Since $\bar{A}$ is derivable from a gradient of a scalar $\mathcal{X}$, we have

$$A_x = \frac{\partial \mathcal{X}}{\partial x} \qquad A_y = \frac{\partial \mathcal{X}}{\partial y}$$

Substituting back into the previous equation we find

$$\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = \frac{\partial^2 \mathcal{X}}{\partial y\, \partial x} - \frac{\partial^2 \mathcal{X}}{\partial x\, \partial y} = 0$$

Now we all know the second partials such as the previous ones are equal so they must subtract to give zero. This is the only place in all of physics where I get a little nervous because partial derivatives were given to me in math. All the other math I have learned has been through physics. For instance, I had calculus for the practical man and other such nonsense. So I set it equal to zero cause I have always gotten away with it.

Also you remember how to transform the electric field since $\bar{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}$. Determining the E-field in the primed system I have

$$\bar{E}' = -\nabla\phi' - \frac{\partial A'}{\partial t}$$

REMEMBERING THAT $\varphi' = \varphi + \frac{\partial \chi}{\partial t}$ AND $\bar{A}' = \bar{A} - \bar{\nabla}\chi$, WE HAVE

$$\bar{E}' = -\bar{\nabla}\varphi' + \nabla\frac{\partial \chi}{\partial t} - \frac{\partial A}{\partial t} + \frac{\partial}{\partial t}\bar{\nabla}\chi$$

NOW IT BOTHERS ME TO INTERCHANGE THE GRADIENT AND TIME DERIVATIVE OPERATORS BUT I'LL DO IT ANYWAY SO I'LL GET THE RIGHT ANSWER, I.E.,

$$\bar{E}' = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t} = \bar{E}$$

WE THAT THAT EVERYTHING WORKS SO FAR.

NOW FOR QUANNUM MECHANICS THINGS ARE NOT QUITE THE SAME. REMEMBER THAT THE AMPLITUDE OF A THING TO HAPPEN IS AN EXPONENTIAL OF THE ACTION DIVIDED BY $\hbar$ OR THE PHASE AS IT IS CALLED. THE DIFFERENCE OF THE TWO ACTIONS WOULD GIVE US A PROBABILITY OF

$$e^{\frac{i}{\hbar}\chi(2)} - e^{-\frac{i}{\hbar}\chi(1)}$$

UNFORTUNATELY THE PHASE DEPENDS WHERE YOU ARE BUT WE HAVE TO REMEMBER THE PROBABILITY DEPENDS ON THE ABSOLUTE VALUE OF THE AMPLITUDE AND EVERYTHING WILL STILL BE ALRIGHT. WE MUST CHANGE THE WAVE FUNCTION ~~B~~ ψ TO $\psi' = \psi e^{\frac{i}{\hbar}\chi}$ FOR THE CHANGE TO A NEW A.

I CAN SUMMARIZE THEN WHAT I HAVE DONE SO FAR BY EXPRESSING THE VECTOR POTENTIAL, SCALAR POTENTIAL TRANSFORMATION IN FOUR-VECTOR NOTATION

$$A'_\mu = A_\mu + \nabla_\mu \chi$$

THIS IF FOR NO GOOD REASON CALLED A GAUGE TRANSFORMATION. THAT IS, IT IS NO GOOD BECAUSE THE TERM GAUGE IS SILLY OR, PERHAPS, IRRATIONAL. ONCE I GAVE A LECTURE AND SPELLED GAUGE GUAGE. SOME CHARACTER WROTE ME CORRECTING MY SPELLING AND TELLING ME GUA IS PRONOUNCED LIKE GWA IN GUATEMALA. BUT GUAGE HAS ALWAYS LOOKED RIGHT TO ME SO I WROTE BACK TO THE GUY ONE SENTENCE — CAN YOU GUARANTEE THAT! I THINK ITS OUR CRAZY ENGLISH.

WE MIGHT WANT TO MAKE A PARTICULAR CHOICE OF THE VECTOR POTENTIAL SO THAT $\bar{\nabla}\cdot\bar{A} = 0$. SOMEONE ELSE MIGHT CHOOSE $\bar{\nabla}\cdot\bar{A} = -\frac{\partial A_t}{\partial t}$. THE FORMER IS CALLED A COULOMB GAUGE; THE LATTER IS CALLED A LORENTZ GAUGE. BUT THESE GAUGES ARE NOT RELATIVISTICALLY INVARIANT. ANOTHER GAUGE WHICH IS INVARIANT IS $\nabla_\mu \cdot A_\mu = 0$. IT IS A GREAT GAME GIVING NAMES TO EQUATIONS AND WHENEVER THERE IS A FREEDOM IN PICKING A PARTICULAR EQUATION AS ABOVE, YOU ALWAYS GET A LOT OF DIFFERENT NAMES.

FIELDS PRODUCED BY STATIC CHARGE DISTRIBUTIONS

NOW I WANT TO START A NEW SUBJECT AND TALK ABOUT FIELDS THAT ARE PRODUCE BY DIFFERENT CHARGE DISTRIBUTIONS. I WILL GIVE THEM TO YOU WITHOUT KNOWING HOW TO DERIVE THEM. THE SIMPLEST CASE I CAN THINK OF IS A SINGLE, INFINITESIMALLY SMALL CHARGE SOME DISTANCE R FROM A POINT AT WHICH I WOULD LIKE TO KNOW THE POTENTIAL.

THE POTENTIAL IS GIVEN BY THE FOLLOWING RELATION

$$\phi = k \frac{Q}{R}$$

WHERE k IS A KONSTANT OF PROPORTIONALITY

IN THE MKS UNIT THIS CONSTANT IS DEFINED TO BE

$$k = \frac{1}{4\pi\epsilon_0} = 10^{-7} c^2 \qquad c^2 \doteq 9 \times 10^{16} \left(\frac{m}{sec}\right)^2$$

THE MKS UNIT GIVES Q IN COULOMBS, R IN METERS, AND ϕ IN VOLTS. THE FACTOR 10^{-7} IS FOUND IN THIS CONSTANT IN ORDER TO LUMP ALL THE CRAZY FACTORS OF TEN RUNNING AROUND IN ALL THE OTHER EQUATIONSIN ELECTRICITY AND MAGNETISM. WE ARE ASSUMING THE ELECTROSTATIC CASE WHERE THE CHARGE Q IS NOT MOVING; THEN THE VECTOR POTENTIAL IS ZERO

CONDITION: $\vec{A} = 0$

IF WE WERE SMART AND Q WAS MOVING IN A STRAIGHT LINE, THEN WE COULD FIND ϕ AND $\vec{A}$ BY TRANSFORMING EQUATIONS FOR A MOVING CHARGE.

I'D LIKE TO SEE HOW THE ELECTRIC FIELD, $\vec{E}$, LOOKS FOR THIS CHARGE. I'LL BET YOU'VE NEVER SEEN IT, EH? WE'LL WE ARE PRETTY SMART AND, THEREFORE, WE KNOW THAT THE ELECTRIC FIELD IS GIVEN BY

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = -\nabla\phi$$

SINCE $\vec{A} = 0$

SO WE MUST DO SOME MATH TO FIND THE GRADIENT OF THE POTENTIAL,

$$\vec{E} = -\nabla \frac{1}{r} \frac{Q}{4\pi\epsilon_0}$$

AND THAT, OF COURSE IS JUST

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

Proof: CONDER THE X-COMPONENT ALONE,

$$E_x = -\frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

WHERE THE CHARGE IS AT THE ORIGIN AND THE DISTANCE TO THE FIELD POINT IS $r = \sqrt{x^2+y^2+z^2}$

WORKING OUT THE DIFFERENTIATION

$$E_x = \frac{x}{\sqrt{x^2+y^2+z^2}^{\,3}} \frac{Q}{4\pi\epsilon_0} = \frac{Q}{4\pi\epsilon_0} \frac{x}{r^3}$$

SUMMING THE Y AND Z COMPONENTS

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} \qquad Q.E.d$$

You realize that this equation says the electric field points in the direction of increasing scalar potential, ϕ, i.e., radially. The magnitude of the electric field is given by

$$|E| = \vec{E}\cdot\vec{E} = \frac{Q}{r^2 4\pi\epsilon_0}$$

This is the key to the law of electrostatics.

Law of Superposition

I want to talk about a general theorem in electricity that is true absolutely. That means as far as we know there are no violations. The law has to do with the field and potential arising from an aggregate of not one or two but many charges. The theorem, quite simply says the to potential ϕ_T due to ϕ_i potentials is just a linear superposition of all these individual potentials ϕ_i

$$\phi_T = \sum_{i=1}^{N} \phi_i$$

or expressed in terms of the charge and separation distance r_i

$$\phi_{T\,\text{at}\,P} = \sum \frac{Q_{\text{charge}}}{r_{\text{charge to }P}}$$

where P is the point in question.

For instance consider two potentials ϕ_1 and ϕ_2 where the total potential and vector potentials are

$$\phi = \phi_1 + \phi_2 \qquad \vec{A} = \vec{A}_1 + \vec{A}_2$$

Since the magnetic field is given by $\vec{B} = \vec{\nabla}\times\vec{A}$, we see immediately by our differentiation rules

$$\vec{B} = \vec{\nabla}\times\vec{A}_1 + \vec{\nabla}\times\vec{A}_2 = \vec{B}_1 + \vec{B}_2$$

The magnetic field add linearly also. The same is, of course true for the electric field,

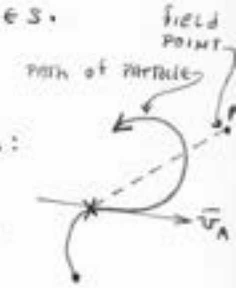
$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = -\nabla(\phi_1+\phi_2) - \frac{\partial}{\partial t}(\vec{A}_1+\vec{A}_2)$$
$$= -\vec{\nabla}\phi_1 - \frac{\partial\vec{A}_1}{\partial t} - \vec{\nabla}\phi_2 - \frac{\partial\vec{A}_2}{\partial t} = +\vec{E}_1 + \vec{E}_2$$

Thus we see that the superposition rule is, indeed, a general statement.

Exceptions to the Rule ?

There are a couple of cases which appear to contradict the superposition principle that I just stated. One of them has to do with the phenomena of dielectricity. Suppose we have a piece of matter which is fairly close to some charge Q. The field produced by Q distorts the charge distribution over the region of the surface closest to the charge. This distortion is due to a rearrangement of the electrons and protons in the matter. Now add another charge Q_2 nearby which also distorts the charge distribution of the matter.

The effective field at point P also in the area is not just the sum of the distortions created by the two charges individually. The field at point P is the sum of the effects of all the charges in the region. Because, then, of non-linear responses of matter to different charges. This is all related to the dielectric properties of matter.



RETARDED TIME

Now the other case I want to talk about is the following: what is the field due to a charge moving any which way. In what wbay does the superposition law give us the potential at some point P? It works this way —

If you want ϕ and $\vec{A}$ at P, there is some point ~~in~~ along the path of the charge where an imaginary flash of light would be emitted and arrive at P at the precise time you want to determine the potentiial. For instance, the light we see from Andromeda takes four years to reach us so it is the light emitted in the past that we see now. It looks, then, like ϕ is produced ~~by~~ from some retarded position where the particle is moving with ~~a~~ some apparent velocity, v_a, in a straight line. We call the time at which we are determine ϕ now the retarded time. When adding the effects of many charges, it is necessary to consider the retardation effect when calculaténg the total potential.

With the time delay considered in calculating ϕ we have the keystone to a complete theory of electrodynamics. I have given you, now, all the laws of electrodynamics,

$$\text{Coulomb potential } \left(\sim \tfrac{1}{R}\right) + \text{Relativity} + \underset{\text{(information travels at } c)}{\text{Time delay}} = \text{Electrodynamics}$$

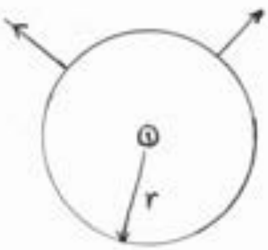
A Philosophical note on the way physical laws are written

It may seem odd that the same physical law can be expressed in many different ways. Some seem utterly different others less utterly different. I don't why this is. I guess it depends on what you consider simple. For instance, on a macroscopic level friction is just an opposition to motion and apparently proportional to the velocity of the motion. But friction is not such a simple phenomena as we care to make it. The explanation of molecular forces involves a very complicated quantum electrodynamical consideration of field interactions further material properties, wear, fatguing, etc all complicate our understanding of this very basic phenomena. Such an explanation of the laws of physics take on special forms. They are unique to the particular case in question. The reason for this is due to the special mathematical form the law takes on. There isn't another form or expression which we can choose a-priori. In order words all other formulations become impossible by the very nature of the problem.

So much for the exceptionally hard to explain phenomena. Let me go back and give another form for the potential due to a stationary charge, i.e., $\phi = \frac{Q}{4\pi\epsilon_0 r}$. Remember we said the electric field had the form $\overline{E} = \frac{Q}{4\pi\epsilon_0}\frac{\overline{r}}{r^3}$ where $\overline{E}$ is radial and the factor $\frac{1}{r^2}$ permits us to understand the electric field by a model or analog which is not relativistically invariant.

In the liquid world an interesting problem is to consider the flow of fluid out of a long thin tube inside another fluid. If you watch the outward flow by coloring the injected fluid, it will move radially with some ~~uniform~~ some velocity, v. How big is the velocity flow? Well, it's proportional to one over r squared, i.e., $v \propto 1/r^2$. If the amount of water flowing out is S and measured in m^3/sec, then we have the following relation

$$v\, 4\pi r^2 = S \qquad \text{OR} \qquad v = \frac{S}{4\pi r^2}$$



We compare this to the strength of the electric field,

$$\mathcal{E} = \frac{Q/\epsilon_0}{4\pi r^2}$$

We see that we do have a good analog between a velocity flow and electric field is we make the identity $Q/\epsilon_0 = S$. Since S is a measure of flux outward, then Q/ϵ_0 must be a measure of the electric field flux.

The results are easy to analyze with the fluid analogy. The bigger the ball; the more crap flows out and the lesser the velocity so the same amount of crap goes out per unit time.

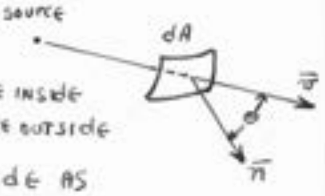
Let's take another closed surface, for instance a goat's liver, and put a charge inside. Now the amount of water by analogy through the liver must be the same over each little area of surface. If the water source is outside the closed surface, then the flux is zero since the same amount of crap leaves the surface each seconds as enters.

source inside surface

source outside

Returning to the electric field problem we should calculate the flux through some surface, A, and this should equal Q/ϵ_0. Consider the velocity vector directed out of the closed region over a small area dA. In order to reference the flux to a known quantity characteristic of the surface we choose the normal to the area element. We then have the projected velocity $v_{out}\cos\theta\, dA$ or $\vec{v}\cdot\vec{n}$ normal to dA. The integral over the whole region is equal to the amount of flux out, i.e.,

$$\int_{\substack{\text{closed}\\ \text{surface}}} \vec{v}\cdot\vec{n}\, dA = \text{flux of soup} = \begin{cases} S & \text{if source inside} \\ 0 & \text{if source outside} \end{cases}$$



This integral is identically zero if the source is ~~ze~~ outside as discussed above.

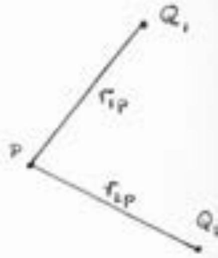
BY ANALOGY THEN

$$\text{ELECTRIC FLUX} = \int_{\text{ANY SURFACE } A} \vec{E}\cdot\vec{n}\, dA = \begin{cases} Q/\varepsilon_0 & \text{IF CHARGE }\cancel{\text{OUT}}^{\text{IN}}\text{SIDE } A \\ 0 & \text{IF CHARGE OUTSIDE } A \end{cases}$$

GAUSS' THEOREM

NOW LET'S GET TRICKY AND CONSIDER MORE THAN ONE CHARGE, SAY TWO. THE TOTAL FIELD DUE TO THIS CHARGE DISTRIBUTION IS BY OUR SUPERPOSITION LAW,

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{r_{1P}} + \frac{Q_2}{r_{2P}}\right]$$



THUS WE SEE RIGHT AWAY THAT THE ELECTRIC FIELD AT P IS JUST

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

TO ASSURE THIS CONSIDER

$$\int \bar{E}\cdot\bar{n}\, dA = \text{TOTAL FLUX}$$

IF THE SURFACE ENCLOSES BOTH CHARGES, THEN

$$\int_{\text{CLOSED SURFACE}} (\bar{E}_1 + \bar{E}_2)\cdot\bar{n}\, dA = \frac{Q_1}{\varepsilon_0} + \frac{Q_2}{\varepsilon_0} = \frac{1}{\varepsilon_0}(Q_1 + Q_2) = \frac{Q_{TOTAL}}{\varepsilon_0}$$

THE FINAL GREAT ANSWER IS THEN

$$\int \bar{E}\cdot\bar{n}\, dA = \frac{1}{\varepsilon_0}(\text{TOTAL CHARGE INSIDE THE SURFACE})$$

THIS IS GAUSS' THEOREM.

ANOTHER WAY TO EXPRESS THIS RESULT IF INSTEAD OF A CHARGE HERE AND A CHARGE THERE WE HAVE SOME CHARGE DISTRIBUTION REPRESENTED BY $\rho = \rho(x,y,z)$ = TOTAL CHARGE/UNIT VOLUME, THEN GAUSS' THEOREM TAKES THE FORM

$$\int_{\text{SURFACE}} \vec{E}\cdot\vec{N}\, dA = \frac{1}{\varepsilon_0}\int_{\text{VOLUME}} \rho(x,y,z)\, d(\text{VOLUME})$$

EVEN IF $\rho = \rho(x,y,z,t)$ THIS THEOREM IS STILL TRUE SO IT IS INDEED A 'MASTERFUL' THEOREM.

THE POTENTIAL AT SOME POINT (x_1, y_1, z_1) DUE TO A CHARGE DISTRIBUTION $\rho(x_2, y_2, z_2)$ IS GIVEN MORE ELEGANTLY AS

$$\phi(x_1, y_1, z_1) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x_2, y_2, z_2)\, dx_2\, dy_2\, dz_2}{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}}$$

THAT IS, BY MAKING LITTLE CUBES AROUND EACH LITTLE REGION OF CHARGE THE RESULTANT EFFECT OVER AT POINT 1 CAN BE DETERMINED. WITH THIS INTEGRAL AT HAND MANY INTERESTING PROBLEMS CAN BE WORKED OUT. SUCH AS DETERMINING THE POTENTIAL FROM A SHEET OF CHARGE. OR THE POTENTIAL DUE TO A CHARGED SPHERE. BUT DURING THE TEACHING OF PHYSICS WE HAVE INVENTED TRICKS SUCH AS GAUSS' THEOREM TO HELP US AVOID DOING THE INTEGRAL. THIS, I BELIEVE, IS A MISTAKE IN OUR ELEMENTARY COURSES ON ELECTROMAGNETISM.

Field Due to a Spherical Charge Distribution

I'd like to work out some simple electrostatics problems, that is, I want to work out some special problems in which the integral,

$$\phi(x_1, y_1, z_1) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x_2, y_2, z_2)\; dx_2\, dy_2\, dz_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

gives some interesting results. Remember we gave another expression for the above and called it Gauss' Theorem:

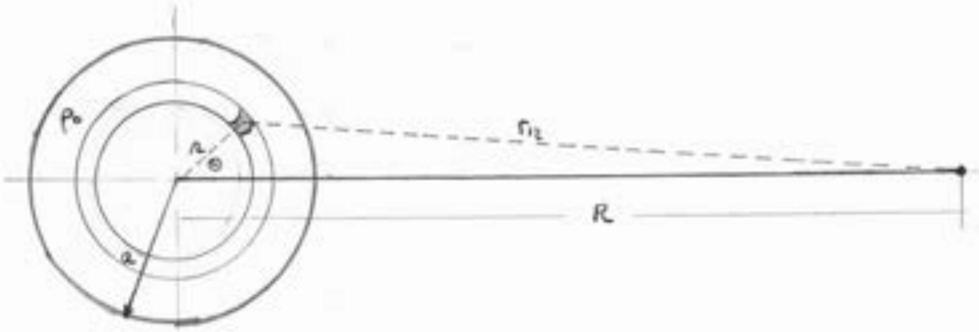
$$\int \vec{E}\cdot\vec{n}\, dA = \frac{Q_{inside}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho\; d\,\text{vol}.$$

Now we want to find ϕ due to different charge distributions.

For the fun of it lets find the potential due to a spherical charge distribution. First let's adopt a simplifying notion:

$$\rho(1) = \rho(x_1, y_1, z_1) \qquad r_{12} = \left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{1/2}$$

$$\phi(1) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(2)\, dV_2}{r_{12}}$$



It is easier to work in polar coordinates so we'll do it. The integral now has the form

$$\Phi(R) = \int \frac{\rho_0\, 2\pi r^2\, dr\; d(\cos\theta)}{\sqrt{r^2+R^2-2rR\cos\theta}}$$

where I slipped in a little geometry on you, i.e.,

$$r_{12}^2 = r^2+R^2-2rR\cos\theta$$

and I used the identity

$$d(\cos\theta) = \sin\theta\, d\theta$$

Integrating $d(\cos\theta)$ we have

$$I = \int_{+1}^{-1}\frac{dx}{\sqrt{a+bx}} \qquad \text{where} \quad \begin{aligned} x &= \cos\theta \\ a &= r^2+R^2 \\ b &= -2rR \end{aligned}$$

using the tables

$$I = \frac{2\sqrt{a+bx}}{b}\Bigg|_{+1}^{-1} = -\frac{2}{2rR}\sqrt{r^2+R^2-2rRx}\;\Bigg|_{+1}^{-1}$$

$$I = -\frac{1}{rR}\left[\sqrt{r^2+R^2+2rR} - \sqrt{r^2+R^2-2rR}\right]$$

$$I = \frac{1}{rR}\left[\sqrt{r^2+R^2-2rR} - \sqrt{r^2+R^2+2rR}\right]$$

But we have the wrong sign since we made a mistake. If we go back you will see $d\cos\theta \neq \sin\theta\, d\theta$ but rather $d\cos\theta = -\sin\theta\, d\theta$. So we have

$$I = \frac{1}{rR}\Big(|r+R| - |r-R| \Big)$$

Thus we have so far

$$\phi(R) = \frac{1}{4\pi\varepsilon_0}\int_0^a \rho_0\, 2\pi r^2 \frac{1}{rR}\Big(|r+R| - |r-R| \Big)\, dr$$

Now we have two cases we must consider. First when the field point is outside the little shell of charge, $R > r$. In this case our integral simplifies to

$$\phi(R) = \frac{1}{4\pi\varepsilon_0}\,\frac{\int_0^a \rho_0\, 4\pi r^2\, dr}{R}$$

But the integral is just the total charge on the shell. Then miracles of miracle we learn that the potential outside a charged sphere behaves just like a point charge at the center. Thus the electric field is just
$$E = \frac{Q}{4\pi\varepsilon_0 R^2}$$
This miracle is true even if $\rho = \rho(r)$.

Now to find the potential inside the shell we have the case where $r > R$. In this case the integral becomes

$$\phi(R) = \frac{1}{4\pi\varepsilon_0}\int_0^R \rho_0\, 4\pi \frac{r^2}{r}\, dr$$

This time we eat up the $1/R$ in the integral. Now the potential is no longer constant because it depends on where you are. That is, all the shells outside of r do not exert a force on the inner point and therefore do not contribute to the field.

We can now write the combined potential as,

$$\Phi = \frac{1}{4\pi\varepsilon_0 R}\int_0^R \rho_0 4\pi r^2 dr + \frac{1}{4\pi\varepsilon_0}\int_R^\infty 4\pi r \rho_0\, dr$$

Since the electric field is given by $\vec{E} = -\nabla\Phi = -\partial\Phi/\partial R$, we can determine the E-field

$$E = \frac{1}{4\pi R^2}\underbrace{\int_0^R 4\pi r^2 dr\, \rho(r)}_{\Phi\ Q_{inside}} + \cancel{\frac{4\pi}{4\pi\varepsilon_0}\frac{R^2\rho}{R}} - \cancel{\frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2}{R}\rho_0}$$

For uniform density we get

$$E_{inside} = \frac{1}{R^2}\,\frac{4\pi R^3 \rho_0}{4\pi\varepsilon_0\, 3} = \frac{R\rho_0}{3\varepsilon_0}$$

Since the force on a charge is proportional to its displacement, the charge will oscillate at the plasma frequency. A plot of the electric field as a function of distance is the following

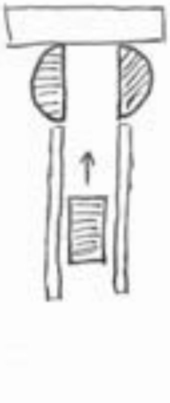
FISSION AND The Atomic Bomb

There are a number of things we can determine once we understand how to calculate static fields. We could find the radius of a bromine atom if you like. We know that uranium splits into two separate atoms. The nuclei are little spheres of charge determined by the amount of energy liberated. I want to point out that the amount of energy radiated during the fission process is not $E = mc^2$ as we are led to believe. Given the radius of any atom is approximately $r = 1.1 \times 10^{-13} A^{1/3}$ cm and the coulomb forces binding the atom, we can find the electric field as a function of the radius. It is more correct to say that Coulomb was the father of the atomic bomb rather than Einstein. Most of the energy in an atom is pure electrostatic. If you built a bomb and got more, then Einstein would tell you how much more you will get.

Further it is incorrect to say that mass is converted to energy. They are one in the same. As an example, any chemical process such as gasoline burning does not expend mass or energy since it is theoretically possible to capture all the products of the combustion and compare with the mass before combustion. The difference is zero.

As an aside I might mention some of the characteristic problems of spontaneous fission, i.e., how to make a safe bomb so you aren't blown up by it before you release it. When two blobs of uranium are put together, fission results. That is, the uranium atom is split into two smaller atoms by high energy neutrons which bombard the uranium atom. The mean free path of a neutron is large but there is a small probability of it leaking out and starting the reaction. Since the uranium must be maintained at a supercritical temperature (about 1/1000 above critical temperature), the fission process will get going. One way to contain the reaction is to surround the material with a lot of inertia to absorb these fast neutrons. Often this is just dead uranium. But since we must be in a supercritical state, the ever present bombardment of cosmic rays will eventually kick off a neutron and start the reaction so the bomb explodes before you get to the target.

What you have to do is put the pieces of uranium together fast enough when you want it to go-boom. On the Hiroshima this was done by screwing a gun barrel up against to hemispheres of uranium. Then we can fire another uranium slug between them and not miss the target.



FLUX FROM A VOLUME

SINCE I HAVE GOTTEN PERMISSION TO TALK ABOUT ANYTHING (I WAS TOLD YOU JUST NEED AN EDUCATION IT DOESN'T MATTER WHAT YOU LEARN), I WANT TO DO TWO THINGS. FIRST, I WOULD LIKE TO TAKE THE EQUATION $\int \bar{E}\cdot\bar{n}\,dA = \frac{1}{\epsilon_0}\int \rho\,dV$ AND TURN IT INTO A DIFFERENTIAL EQUATION OF THE FORM $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$. TO DO THIS I WILL PROVE A RELATION I NEVER WANT YOU TO FORGET, VIZ.

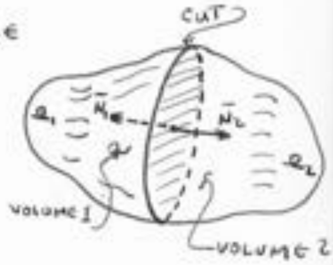
$$\int_{surface} \bar{E}\cdot\bar{N}\,dA = \int_{volume} \bar{\nabla}\cdot\bar{E}\,dV$$

THIS IS PURELY A MATHEMATICAL PROPOSITION WHICH REPRESENTS A PROPERTY OF ALL VECTORS. IT IS MORE IMPORTANT HERE TO UNDERSTAND WHAT IT IS WE ARE PROVING RATHER THAN HOW TO PROVE IT; THIS IS WHY I ALSO HATE MATHEMATICIANS (AT THE BEGINNING OF THE LECTURE DR. FEYNMAN WAS ON A DISCOURSE EXPLAINING WHY HE THOUGHT PHYLOSOPHERS WERE, IN ESSENCE, BIG WIND BAGS).

OUR STARTING POINT IS OUR IDEA OF WHAT THE SURFACE INTEGRAL IS,

$$\int \bar{E}\cdot\bar{N}\,d\,\text{surface} = \text{flux through the surface}$$

NOW LET'S CONSIDER A NEW PROBLEM IN WHICH WE DIVIDE SOME ARBITRARY VOLUME INTO TWO REGIONS (SEE FIGURE).



THE CUT GIVES US TWO SUBSURFACES Ω_1 AND Ω_2. NOW WE WOULD LIKE TO CALCULATE THE FLUX THROUGH THE TWO SURFACES. I CAN SHOW THAT THE FLUX THROUGH THE SURFACE MADE BY THE CUT IS ZERO. THIS IS BECAUSE THE NORMALS TO THE CUT ARE OPPOSITE FOR THE TWO SURFACES Ω_1 AND Ω_2. THE NET CONTRIBUTION IS THEREFORE ZERO OVER THIS AREA. IT IS PERFECTLY CLEAR I CAN DIVIDE THE VOLUME UP INTO SMALLER AND SMALLER CUBES AND IT WILL TURN OUT THAT THE FLUX THROUGH Ω_{TOTAL} EQUALS THE SUM OF THE FLUX OUT OF THE ITSY-BITSY PIECES I HAVE CUT. I WILL IGNORE THOSE IDIOT SHAPED VOLUMES AROUND THE EDGE BUT YOU CAN PROVE THAT THEY WON'T CHANGE THE RESULT. FOR PROOF OF THE RELATION

$$\Sigma\,\text{flux}_{\Omega_i} = \int \bar{\nabla}\cdot\bar{E}\,dV$$

SEE THE REFERENCE (3-9).

SINCE WE HAVE SHOWN $\vec{E} = -\nabla\phi$ THE EQUATION $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$ CAN BE EXPRESSED IN TERMS OF THE POTENTIAL ϕ AS

$$\nabla^2\phi = -\rho/\epsilon_0$$

WHERE $\bar{\nabla}\cdot\bar{E} = \bar{\nabla}\cdot(-\bar{\nabla}\phi) = -(\bar{\nabla}\cdot\bar{\nabla})\phi = -\nabla^2\phi$

AND $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ = LAPLACIAN OPERATOR

THUS THE LAWS OF ELECTROSTATICS CAN BE WRITTEN IN TERMS OF THE POISSON EQUATION

$$\nabla^2\phi = -\rho/\epsilon_0$$

I WANT TO DEVELOPE A NUMBER OF MATHEMATICAL TOOLS WHICH WE CAN APPLY TO A LARGE NUMBER OF PROBLEMS RELATED TO CHARGE DISTRIBUTIONS, FIELDS, AND THE WHOLE BIT. BUT BEFORE I GO INTO THAT THERE IS ONE LITTLE THING I WANT TO DO.

LAST YOU RECALL I GAVE YOU THE POTENTIAL ϕ DUE TO SOME CHARGE DISTRIBUTION, I.E.,
$$\phi(1) = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0 r_{12}}$$
I WENT ON TO SAY THAT ASSOCIATED WITH THIS PARTICULAR POTENTIAL IS A DIFFERENTIAL EQUATIONS,
$$\nabla^2\phi = \frac{\rho(1)}{\varepsilon_0}$$
THAT IS, $\phi(1)$ AS GIVEN BY THE INTEGRAL FORMULA IS THE SOLUTION TO THIS SECOND ORDER DIFFERENTIAL EQUATION. NOW I WOULD LIKE TO PLUG ϕ INTO THE EQUATION AND SEE IF THIS IS AN HONEST CLAIM. IN THE PROCESS I ENCOUNTER SOME DIFFICULTIES IN THE MATH SO I'D LIKE TO GO THROUGH IT.

IF YOU WILL LET ME, I'LL CHANGE THE FORM OF THE POTENTIAL INTO THE FOLLOWING
$$\phi(1) = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0\sqrt{R_{12}^2 + a^2}}$$
NOTE IF $a = 0$, WE HAVE OUR ORIGINAL POTENTIAL. SO WE REALLY SHOULD TAKE THE LIMIT OF THE INTEGRAL AS $a \to 0$. WHAT I WANT TO DO IS MODIFY R_{12} A LITTLE BIT AND I COULD HAVE CHOSEN
$$\frac{1}{R_{12}} \to \frac{1}{R_{12}}\left(1 - e^{-\alpha R_{12}}\right) \quad \text{where } \alpha << 0$$
THE PROBLEM I AM AVOIDING BY MAKING THIS CHANGE NOW HAS TO DO WITH DIFFERENTING $1/R_{12}$ TWICE AND EVALUATING AT THE LIMITS. WE GET CRAZY RESULTS LIKE INFINITIES AND ZEROES. WELL LET'S START BY TAKING, SAY, THE FIRST DERIVATIVE OF ϕ WITH RESPECT TO X_1 AND THEN BY SYMMETRY ADD UP OUR PARTS IN THE END.
$$\frac{\partial\phi}{\partial X_1} = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{2(X_1 - X_2)}{\sqrt{R_{12}^2 + a^2}^{\,3}}\right] = E_x$$
TAKING THE DERIVATIVE OF THIS
$$\frac{\partial^2\phi}{\partial X_1^2} = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0}\left[-\frac{1}{\sqrt{R_{12}^2 + a^2}^{\,3}} + \frac{3(X_1 - X_2)^2}{\sqrt{R_{12}^2 + a^2}^{\,5}}\right]$$
SINCE WE WILL HAVE THE SAME FORM OF RESULT FOR Y_1 AND Z_1, WE CAN SUM THE PARTS TO GET
$$\nabla^2\phi = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0}\left[-\frac{3}{\sqrt{R_{12}^2 + a^2}^{\,3}} + \frac{3R_{12}^2}{\sqrt{R_{12}^2 + a^2}^{\,5}}\right]$$

I CAN PUT THE BRACKETED TERM OVER THE COMMON DENOMINATOR AND WRITE,

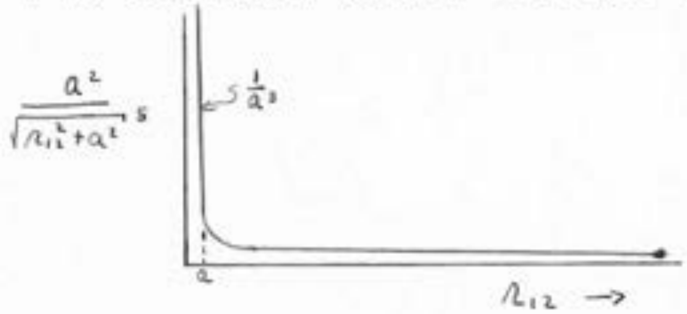
$$\nabla^2\phi = \int \frac{\rho(2)\, dV_2}{4\pi\varepsilon_0}\left[-\frac{3a^2}{\sqrt{r_{12}^2+a^2}^{\,5}}\right]$$

NOW IF a IS ZERO, THE RIGHT SIDE IS ALSO ZERO BUT SUPPOSEDLY IT EQUAL ρ/ε_0. SO WE MUST HAVE MADE A MISTAKE IN CALCULATING THE INTEGRAL BECAUSE IT DOES NOT CONVERGE. LET'S LOOK AT THE FUNCTION INSIDE THE BRACKET AND SEE WHAT WE HAVE.

IF r_{12} IS ANYTHING BUT A VERY SMALL QUANTITY, THEN THE BRACKET IS A VERY SMALL, INFINITESIMAL NUMBER FOR ANY CHOICE OF $a \ll 0$. WHEN r_{12} CONVERGES TO ZERO AND BECOMES LESS THAN a ITSELF, THE BRACKET GOES AS $1/a^3$ AND THE INTEGRAL PEAKS LIKE MAD. IN FACT, IF WE ASSUME THE CHARGE DISTRIBUTION IS A SMOOTH FUNCTION OVER DISTANCES OF a THEN OUR POINT 2 CAN'T BE FAR FROM 1 AND WE CAN APPROXIMATE THE INTEGRAL AS

$$\nabla^2\phi \cong -\frac{\rho(1)}{4\pi\varepsilon_0}\int \frac{dV_2\, 3a^2}{\left(\sqrt{r_{12}^2+a^2}\right)^5}$$

THE FUNCTION I AM DESCRIBING WOULD LOOK LIKE THE FOLLOWING -



THE INTEGRAL IS VERY UNUSUAL. IT EXIST, REALLY, ONLY OUT TO A RADIUS a. HOW BIG IS THE INTEGRAL. WELL, WE INTEGRATE OVER A SPHERE WHOSE VOLUME IS PROPORTIONAL TO a^3 AND THE FUNCTION WHICH GOES AS $1/a^3$. SO THE VOLUME INTEGRAL COMES OUT TO BE FINITE AND, SPECIFICALLY, ONE IF WE NORMALIZE.

WE MIGHT TRY TO DO THE INTEGRAL BY LETTING $r_{12} = r$

$$\int_0^\infty \frac{4\pi r^2 dr\; 3a^2}{\sqrt{r^2+a^2}^{\,5}} = 4\pi$$

WHERE YOU MIGHT RECALL DERIVATIVE OF $\frac{r^3}{\sqrt{r^2+a^2}^{\,3}}\left[\frac{3}{r} - \frac{3}{2}\frac{2r}{(r^2+a^2)}\right] = \frac{r^2}{\sqrt{r^2+a^2}^{\,3}}$

OUR RESULT, ONCE DOING THE INTEGRAL IS

$$\nabla^2\phi = -\frac{\rho(1)}{\varepsilon_0}$$

WHICH IS THE RIGHT ANSWER.

For those of you who have heard of these things I am talking about, we call these functions delta functions. In three dimensions we can express this as

$$\nabla^2 \frac{1}{r_{12}} = -4\pi\, \delta^3 (\mathbb{R}_1 - \mathbb{R}_2)$$

So this has been a little mathematical summary which is elementary but that is what we are teaching, right? I have just shown you how to fix up this problem so you don't get into hot water. That hot water, I might remind you, comes from the difficulty in evaluating $\left(-\frac{3}{r_{12}} + \frac{3 r_{12}^2}{r_{12}^5}\right)$ when you have no "a" present. At the limits of 0 and ∞ you see there are some problems. I like to throw in this extra gear so I can screw down and get the right answer.

I'll give you a couple little theorems you can practise on to see how well you understand this. Suppose we have to solve the equation,

$$\nabla^2 \phi - m^2 \phi = S$$

Then I claim the solution is

$$\varphi = -\int \frac{dv_2\, e^{-m r_{12}}\, S(2)}{4\pi\epsilon_0\, r_{12}}$$

This problem has the same disease as the last one. For those of you who are extremely ambitious, try to solve

$$\nabla^2 \phi(R,t) - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = S(\mathbb{R},t)$$

A solution of the above equation is

$$\phi(\mathbb{R}_1, t) = -\int \frac{dv_2\, S(\mathbb{R}_2, t - \frac{r_{12}}{c})}{4\pi\epsilon_0\, r_{12}}$$

This solution is not unique but it is one solution. I will assume these solutions when I need them and the last equation is important from a relativistic point of view.

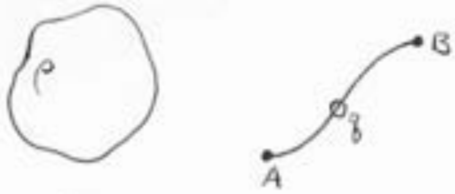
This has been an exercise to satisfy a need for completeness not to learn anything new. These little equations will come up later and I want you to get a feel for them now.

ref. Vol II chapter 8

ELECTROSTATIC ENERGY

I want to develope something else now. A handy gadget is the law of conservation of energy. This law is handy because it is often easier to calculate energies than forces. So want to develope some formulas for electrostatic energy. It must be appreciated that when you have this, you have all of electrostatics.

Let's take some arbitrary shaped charge distribution in space which has some field φ associated with it. Away from the charge distribution is some other point charge q. This charge we will move from point A to point B and during this motion assume q does not act back on the distribution ρ.



We must calculate the work done on the charge in moving it from A to B. This is given by the integral

$$W = \int_A^B (F_x dx + F_y dy + F_z dz)$$

The integral is calculated along the path joining the points and there is proportional to the path chosen. This integral can also be written, for those who like vector analysis, as

$$W = \int_A^B \vec{F} \cdot d\bar{S}$$

In our case $\bar{F} = q\bar{E}$ so that we can write

$$W = q\int \bar{E} \cdot d\bar{S}$$

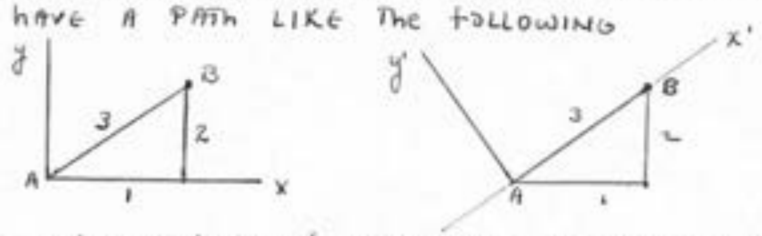
Substituting $-\bar{\nabla}\varphi$ for $\bar{E}$ we have

$$W = -q\int \bar{\nabla}\varphi \cdot d\bar{S}$$

Now we need a theorem from mathematics which is true in general that

$$\int_A^B \bar{\nabla}\phi \cdot d\bar{S} = \varphi(B) - \phi(A)$$

That is the path integral is independent of the path and only depends on your starting and stopping points. One way to see this is to have a path like the following



Since we have an understanding of invariance we can rotate the axis so that getting from A to B is indeed independent of the path chosen.

Thus when q is at point A its electric energy is equal to $q\,\phi(A)$. And the work done in getting the charge to B is the difference in energy between these points

$$W = -q\left(\varphi(B) - \varphi(A)\right)$$

The energy is somewhat arbitrary in that you can change the zero point by adding a constant but we won't worry about this.

I want to point out that whenever the path is a closed loop, the work done in carrying the charge around the loop is zero. Therefore, in electrostatic no work is done in traversing such a path.

The next thing we'd like to know is given a charge distribution how much work did it take to put it together. This is child's play. The charge per unit volume is just $\rho(1)\,dV_1$ so the electrostatic energy is

$$U = \int \varphi(1)\,\rho(1)\,dV_1$$

Right? Wrong! Suppose the charge consists of two patch dq_1 and dq_2 separated by some distance r_{12}. The energy at the little charge element dq_2 is due to two parts

$$dq_2\,\frac{dq_1}{4\pi\varepsilon_0 r_{12}} + dq_2\,\frac{dq_1}{4\pi\varepsilon_0 r_{12}}$$

So we get twice as much as we thought we would. We can generalize this result and state

$$\text{Mutual energy between point charges} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}}$$

This theorem could serve as a basis of electrostatic and all other force laws subsequently derived. If there are a lot of charges around, we can express the total electrostatic energy of the system as the sum of all the pairs of charges:

$$U = \sum_{\text{pairs}} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

When considering a continous charge distribution as we did above we must divide the integral by 1/2 to account for integrating the pairs twice, i.e.,

$$U = \frac{1}{2}\int \varphi(1)\,\rho(1)\,dV_1$$

If we substitute for $\varphi(1)$ into this equation, we have

$$U = \frac{1}{2}\iint \frac{\rho(2)\,\rho(1)\,dV_1\,dV_2}{4\pi\varepsilon_0 r_{12}}$$

We can see a little better the double integration here. dV_1 can be around dq_1 and dq_2 at a latter time; likewise for dV_2.

I can rewrite the electrostatic energy in still another way this time using the potential. Remember $\nabla^2\varphi = -\frac{\rho(1)}{\varepsilon_0}$; so substituting for $\rho(1)$ into the above equation

$$U = -\frac{\varepsilon_0}{2}\int (\nabla^2\phi)\,\phi\,dV_1$$

Now remember also the little differential identity

$$\bar\nabla\cdot(\varphi\bar\nabla\varphi) = \bar\nabla\varphi\cdot\bar\nabla\varphi + \varphi\nabla^2\varphi$$

so I can substitute for the last term on the right $\bar\nabla\cdot\varphi\bar\nabla\varphi - \bar\nabla\varphi\cdot\bar\nabla\varphi$.

$$U = \frac{\varepsilon_0}{2}\int \bar\nabla\varphi\cdot\bar\nabla\varphi\,dV + \frac{\varepsilon_0}{2}\int \bar\nabla\cdot(\varphi\bar\nabla\varphi)\,dV.$$

BY GAUSS' THEOREM I CAN CHANGE THE LAST INTEGRAL TO A SURFACE INTEGRAL, I.E.,

$$\int \bar{\nabla}\cdot(\varphi\bar{\nabla}\varphi)\, dV = \int \varphi\bar{\nabla}\varphi \cdot \bar{N}\ d\,\text{Surface}$$

SUPPOSE ALL THE CHARGE IS FINITE AND BOUNDED BY A VERY LARGE REGION. IF WE LET THE SURFACE EXTEND OVER ALL SPACE THE SECOND INTEGRAL TENDS TO ZERO SINCE $d\,\text{surface} \propto r^2$, $\nabla\varphi \propto 1/r^2$ AND $\varphi \propto 1/r$ SO THE INTEGRAND GOES AS $1/r$. NOW IT IS EASY TO WRITE OUR FINAL RESULT IN TERMS OF THE ELECTRIC FIELD SINCE $\bar{E} = -\nabla\varphi$

$$U = \frac{\varepsilon_0}{2}\int \bar{\nabla}\varphi\cdot\bar{\nabla}\varphi\, dV$$

or

$$U = \frac{\varepsilon_0}{2}\int \bar{E}\cdot\bar{E}\, dV = \frac{\varepsilon_0}{2}\int |E|^2 dV.$$

THESE FORMULAS ARE VALID FOR ELECTRODYNAMICS AND THE LAST FORMULA FOR U IS CORRECT IF WE JUST ADD THE MAGNETIC ENERGY. THE OTHER FORMULAS FOR U IN TERMS OF THE SUMMATION AND THE TWO VOLUME INTEGRALS ARE NOT SO GOOD.

I MUST MAKE SOME REMARKS ABOUT TWO PARTICULAR QUESTIONS: WHAT IS THE ENERGY WHEN YOU HAVE POINT CHARGES PRESENT AND JUST WHERE IS THE ENERGY. I'LL DEAL WITH THE FIRST QUESTION FIRST.

LET'S FIND U FOR POINT CHARGES USING THE LAST EQUATION WE DEVELOPED,

$$U_{pc} = \frac{\varepsilon_0}{2}\int_0^\infty \frac{Q^2}{(4\pi\varepsilon_0)^2}\frac{4\pi r^2 dr}{r^4} \qquad \text{SINCE } E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

INTEGRATING BETWEEN OUR LIMITS

$$U_{pc} = \frac{\varepsilon_0}{2}\frac{Q^2\, 4\pi}{(4\pi\varepsilon_0)^2}\left[\frac{1}{r}\right]_0^\infty = \frac{Q^2}{8\pi\varepsilon_0}\left.\frac{1}{r}\right|_0^\infty$$

THE UPPER LIMIT GIVES US NO DIFFICULTY; THE LOWER LIMIT GIVES US INFINITE DIFFICULTY. WHICH IS WRONG? WHAT IS φ AT THE CENTER OF A CHARGE DISTRIBUTED OVER ITS SURFACE, SAY A BALL? WELL, IT IS JUST $\varphi = \frac{Q}{4\pi\varepsilon_0 a}$ SO WE MUST CHANGE OUR LOWER LIMIT TO a THE RADIUS INSTEAD OF ZERO

$$U = \frac{\varepsilon_0}{2}\int_a^\infty \frac{Q^2}{(4\pi\varepsilon_0)^2}\frac{4\pi\, dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 a}$$

OUR DIFFICULTY IS THAT A POINT CHARGE SEEMS TO HAVE INFINITE ENERGY. BUT WHEN WE CALCULATE U AS THE SUM OF THE PAIRS OF CHARGE INTERACTING THE ANSWER IS NOT INFINITE. THIS IS AN OLD PROBLEM TO CLASSICAL PHYSICS. JUST WHAT IS THE ENERGY OF A POINT CHARGE LIKE AN ELECTRON? ONE WAY TO AVOID THE PROBLEM IS TO DO AS WE DID AND ONLY TAKE PAIRS BUT THIS IS NOT A SOLUTION TO THE PROBLEM. IT WAS NEVER RESOLVED BY CLASSICAL PHYSICS.

One day Quantum Mechanics came along and hope ran high that the problem would be resolved. But low and behold infinities still persisted even with this new understanding of the dynamic interaction of matter. Even today there is no theory which does not represent charges as points when in the classical limit. One attempt was made to explain how an electron produces a force on itself but this is ridiculous. If nature would present us with two objects of equal mass but one charged and the other not, we could measure the energy in the two and compare them. And in fact nature was kind to us and gave us the proton and neutron. The measurable forces, electrical that is, were slightly different. And the electrostatic energy was also slightly different. Together with the positive and neutral pi meson we must accept the fact that charges act on themselves yet we can't have a quantum theory of point particles. It appears that either the laws of electrodynamics fails at short distances or there are such things as non-point charges in nature. Right now we don't know which is right. Nature being what it is, the solution is probably in a third direction. Maybe some little jerk in some university has got it all worked out and we aren't paying any attention to him

PROBLEM: Here is a problem you might try to work out. It is not very useful but it is entertaining: If a sphere is empty of charge so inside $\nabla^2 \varphi = 0$, show that φ at the center is the average value of φ on the surface.

Where is the Energy?

ref. vol. II chapter 8

Let's go on to the second question and see if we can answer it. First, just what does it mean – where is the energy. If we have two charges that are near eachother and could somehow tack them down inside a box, we would say there is electrostatic energy in the box. Well-ll — it's sort of in the box. Shouldn't we refine this idea of energy so it is a little more palitable? Since energy is represented by just a number maybe we can't find the answer by looking for a formula or some way to express where it is located. So the whole issue might not mean anything as you all hope. But there are some troubling phenomena which we must consider.

As we have learned from relativity mass and energy are one in the same thing. Since mass is the source of gravity, it must be true that energy is, likewise, the source of gravitational force. If we don't know where the energy is we don't know where the mass is. For this physical reason it is important to specify where the energy is or else any theory of gravitation would be incomplete.

LET'S CONSIDER ANOTHER PROBLEM. OUR IDEA OF ~~of~~ ENERGY IS THE ABILITY TO DO WORK. BUT DO WORK WHERE? OUT THERE; AWAY FROM THE CHARGE. TO CLEAR THIS UP A LITTLE CONSIDER OUR TWO CHARGE SYSTEM WHICH HAS AN ELECTROSTATIC ENERGY ASSOCIATED WITH IT. AS ANOTHER CHARGE IS PASSED BY THIS SYSTEM, WE MUST DO WORK TO OVERCOME THE FORCES OR ENERGY SURROUNDING THE TWO. SO THE TEST CHARGE IS DEFLECTED. NOW IF OUR DOUBLE CHARGE SYSTEM IS OSCILLATING OR MOVING OR DOING SOMETHING SO FAST THAT THE TEST CHARGE DOES NOT RECEIVE A SIGNAL FROM THE TWO UNTIL A TIME r/c LATTER, IT MUST NECESSARY REACT TO THE ENERGY AT THE TIME IT PASSES A PARTICULAR POINT. THIS IS A VERY INTERESTING PROBLEM.

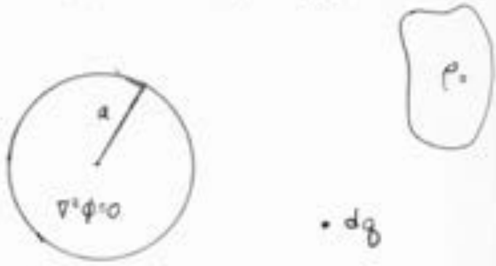
BUT TO LOOK FOR SOME SORT OF ANSWER WE MUST CONSIDER TWO KINDS OF CONSERVATION LAWS: OVERALL OR LOCAL CONSERVATION OF ENERGY. LOCAL CONSERVATION IS THE ONLY KIND WHICH IS CONSISTENT WITH OUR IDEAS OF RELATIVITY; OVERALL IS NOT. THE REASON IS BECAUSE SIMULTANEITY IS NOT DEFINED. CONSIDER BY ANALOGY CONSERVATION OF MATTER. SUPPOSE WE HAVE A PIECE OF CHALK 2/3 OF AN INCH LONG IN ONE BOX AND ANOTHER PIECE 1/3 OF AN INCH IN ANOTHER BOX. THE TOTAL AMOUNT OF CHALK IS ONE INCH. A GUY WHO IS MOVING BY THE BOXES AT SOME VELOCITY V WOULD SEE MORE CHALK IN BOTH BOXES. WE MUST SAY THE CHALK IS CONSERVED LOCALLY OR ELSE WE ARE IN TROUBLE.

FURTHER IT IS UTTERLY USELESS TO SAY THE CONSERVATION OF ENERGY THROUGHT THE UNIVERSE IS CONSTANT. WE MUST HAVE KNOWLEDGE OF ONE REGION OF SPACE WHERE CHANGES ARE TAKING PLACE AND STUFF IS FLOWING IN AND OUT OF REGIONS. WE MUST LOCATE ENERGY SOMEWHERE BUT WE CAN'T DO IT WITH ELECTROSTATICS ALONE. LATTER WE WILL FIND THE FORMULAS FOR FINDING OUT WHERE THE ENERGY IS AND HOW IT MOVES FROM PLACE TO PLACE.

PROOF OF A CUTE THEOREM

Today I'd like to start out by solving that little theorem I gave you last time. It is cute but not very useful. I would like to use the energy theorems I have derived to show the average of the surface potential is the value of the potential at the center of the spherical shell.

Let's imagine some charge distribution outside the shell to hold it at the potential ϕ. Inside the shell there are no charges so $\nabla^2\phi = 0$. Now let's put some small piece of charge dq in the space nearby and if we multiply by the potential at the point where dq is, the integral of ϕdq will give us the energy of interaction between dq and the sphere, i.e.,



$$\text{Energy}, U = \int \phi_c \, dq$$ where ϕ_c is the potential at the charge

Now we have two cases to consider when calculating this integral:

(1). When dq is outside of the sphere the potential appears to be at the center of the sphere, i.e., it acts as a point charge so the integral becomes

$$U = Q\,\phi_{center}$$

(2). When inside the sphere there appears to be a uniform surface distribution of charge and the integral comes out

$$U = Q\,\phi_{\text{average over surface}}$$

Now it would suffice to show the interactions energies are the same and that is easy to do. From ρ_0 all we see are point charges and the energy is

$$U = \int \phi \rho \, dV$$

which is equated to the above two energies and thus all are equal.

This is a very beautiful theorem and a good application of the usefulness of the energy theorems. But as I have said I have never found a use for it. So there you are.

EQUILIBRIUM IN AN ELECTROSTATIC FIELD   ref Vol II chapter 5

You have all the tools to work out a number of things and right now I'd like to talk about charge equilibrium. Suppose I have three positive charges placed at the corners of an equilateral triangle. At the center of the triangle I put a minus charge of equal amount to each of the positive ones. The minus charge will be in equilibrium at that point but it is an unstable one. Any disturbance from the point will cause the charge to move further away.

If the minus charge is at the center of a positive tetrahedron, this point is also an unstable equilibrium. If the minus is displaced, it senses a restoring force in the direction of change. But the electric field is not like this. To see how it looks consider looking down on the tetrahedron as shown on the right →

The black minus charge in the center is shown in its equilibrium position 0. When displaced to position 1, the electric field is radially inward by Gauss' theorem $\nabla \cdot E = 0$. Thus the force felt by the minus charge is outward away from the equilibrium position. Thus Gauss' theorem is violated since there is no charge at 0.

However, a charged placed inside a uniformly charged ball will be restored harmonically when displaced. This we showed before

$$\text{Force} = \underbrace{\left(\frac{4\pi\rho R^3}{3}\right)}_{Q_{\text{inside}}R}\frac{1}{4\pi\varepsilon_0 R^2} = \cancel{\text{field}} = \frac{\rho R}{3\varepsilon_0}$$

So a charge is stable in a sphere but not in empty space.

Conductors

Ref. Vol II Chapter 5

Now let the charges outside of the test charge be rigid, i.e., they may reside in a conductor. The minus charge will now induce a charge in the conductor so the positive charges will respond to the test charge. This is not a stable condition either. If you move the minus charge up, you change the field. This fluctuating field causes a reaction in the conductor which changes the charge distribution there. Thus the field at the new position is different than it was with the test charge at the center or equilibrium position. You just can't hold that charge electrostatically stable

If the charges are fixed, as we have in the electrostatic case so far, the electrostatic energy, U, rises when you move away from the equilibrium point. But we argued this was impossible for fixed charges and we, in fact, move to lower U. This motion represents positive work done. So we have an unstable equilibrium condition which could be made stable if we moved the charges dynamically. That is, by sensing and compensating for the motion of the minus charge we constantly maintain equilibrium. This is like balancing a stick on your finger

Let's see how the charges inside a conductor move. One way to describe a conductor is simply in terms of time. For an ideal conductor the charges have plenty of time to respond to the field being applied. The time period here is on the order of microsecond. Within that period of time the charges redistribute themselves so that there is zero field inside the conductor.

SUPPOSE WE TURN A PIECE OF METAL ON BY BRING A POSITIVE CHARGE CLOSE TO IT. THE ELECTRONS WILL BE ATTRACKED TOWARDS THE OUTSIDE CHARGE. BUT THE ELECTRONS CAN'T GET BEYOND THE SURFACE SO THEY PILE UP THERE. WE WILL ASSUME NO THERMIONIC EMISSION OR OTHER UNUSUAL PHENOMENA TO SCREW UP OUR THINKING. BUT THIS IS THE DYNAMICS OF ELECTROSTATICS. IT IS THE STUFF YOU DON'T HAVE A FEELING FOR WHEN YOU FIRST LEARNED IT. THE LACK OF UNDERSTANDING REALITY IS ONE OF OUR GREAT EDUCATIONAL ACCOMPLISHMENTS. BUT NATURE IS MORE WONDERFUL THAN AN IDEAL CONDUCTOR AND

WHAT IS THE POTENTIAL INSIDE THE CONDUCTOR? SUPPOSE WE HAVE A CONDUCTOR WHICH SUPPLIES AN INFINITY NUMBER OF CHARGES. THEN WE WILL BUILT UP A CHARGE SO BIG THAT THE ELECTRON WHICH DO DRAIN OFF DO NOT CHANGE THE TOTAL CHARGE AT ALL.

JUST FOR FUN LETS COMPUTE THE FORCE BETWEEN TWO MOLES OF WATER 1 KILOMETER APART IF WE DRAIN OFF 1 MILLIONTH OF THE ELECTRONS.

1 MOLE H_2O ≈ 96000 COULOMBS OF ELECTRIC CHARGE

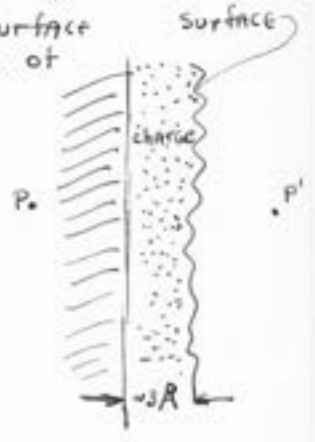
1 PART IN 10^6 ELECTRONS TAKEN OUT

$$F = \frac{Q^2}{4\pi\varepsilon_0 r^2} = \frac{(96000 \times 10^{-6}\,\text{coul})^2 \times 9\times 10^9}{(10^3\,\text{m})^2}\ \frac{\text{NT.}\,\text{m}^2}{\text{coul}^2}$$

FORCE ≈ 80 NEWTONS

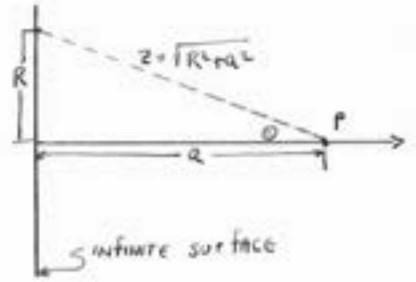
WHICH IS A VERY LARGE FORCE SO YOU WON'T HAVE MUCH SUCCESS IN TAKING THOSE ELECTRONS OFF. IT TAKES A HELLUVA DENSITY OF ELECTRONS BEFORE THE ELECTRIC FIELD IS ZERO INSIDE THE CONDUCTOR BUT ONCE IT IS IT IS ZERO FOR GOOD. BECAUSE IT IS ZERO IMPLIES THERE IS NO CHARGE DISTRIBUTION, ρ INSIDE; THIS IS PROVEN BY GAUSS' THEOREM. BUT THIS IS A HORRIBLE PARADOX TO HAVE A POTENTIAL FROM NO CHARGES.

WELL, AS YOU MAY ALREADY KNOW THERE REALLY ISN'T A PARADOX AT ALL BECAUSE ALL THE CHARGES MOVE TO THE SURFACE AND GET STUCK THERE. THERE IS A PROBLEM THOUGH BECAUSE THERE IS NO PLACE FOR ALL THE ELECTRONS RIGHT ON THE SURFACE SINCE IT IS BUMPY ON THE ATOMIC LEVEL. SO THE ELECTRONS DO SIT THERE; THEY TRY TO GET AWAY AND BALANCE THE ELECTRIC FIELD. IT IS THIS GET AWAY FORCE THAT DETERMINES THE SURFACE CHARGE DENSITY. AT THE SURFACE THEN WITHIN A FEW ANGSTROMS OF THE TOP BOUNDARY THE CHARGE ACCUMULATES.



BECAUSE $\bar{E} = 0$ INSIDE THE POTENTIAL MUST BE CONSTANT THROUGHOUT THE MATERIAL. IF WE HAVE THE POTENTIAL φ AT POINT P INSIDE THE CONDUCTOR, WHAT IS THE POTENTIAL φ AT P' JUST OUTSIDE THE CONDUCTOR. TO FIND THE POTENTIAL WE MUST DETERMINE THE φ FOR A SHEET OF CHARGE. WE WILL DISREGARD THE SURFACE SHAPE AND CONSIDER THE SHEET INFINITE IN EXTENT AS VIEWED FROM P'.

Using the diagram on the right we can find the potential at some point P a distance "a" from the sheet of charge. The sheet has a uniform charge distribution σ over the surface. The integral becomes



$$E\ \Phi = \frac{1}{4\pi\epsilon_0}\int_0^{\infty} \frac{\sigma\, 2\pi R\, dR}{(R^2+a^2)} \cos\theta$$

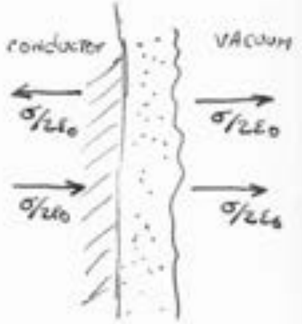
$$= \frac{1}{4\pi\epsilon_0}\int_0^{R} \frac{\sigma\, 2\pi R\, dR\, a}{(R^2+a^2)^{3/2}} \qquad \text{where } \cos\theta = \frac{a}{\sqrt{R^2+a^2}}$$

Upon integrating

$$E\ \varphi = \left.\frac{2\pi a \sigma}{4\pi\epsilon_0 (R^2+a^2)^{3/2}}\right|_0^{\infty} = \frac{\sigma}{2\epsilon_0}$$

Now this is the electric field for a uniformly charged sheet. In a conductor we have two such sheets at the surface. Further the electric field is due to all the rest of the charges in the conductor not just those in the vicinity of the point P we are interested in. But the field at point P and P' are the same as the value we just calculated because all the other charges are so far away as compared with a few angstroms that the differences in the two fields is not detectable. The net field produced by all the other charges is, miracle of miracles, exactly $\sigma/2\epsilon_0$. The total field just outside the conductor is thus a sum of two fields both contributing $\sigma/2\epsilon_0$.

$$E_{outside} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}$$



Inside we see the field is zero since the two contributions are in opposite direction. This implies the ~~the~~ potential does not vary inside the conductor. The surface becomes equipotential since E is normal to the surface. If E was not normal electrons would flow and, of course, the don't. The charge on the surface sloshes and slurps around until it adjusts itself so it no longer feels a force. The dynamics of this balancing phenomena is very difficult to analyze and we don't understand physically how it works.

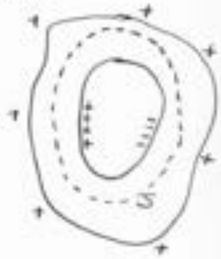
Nature is full of mysteries like this. And one that puzzles me is how a gyroscope works. I guess it is really not too difficult to figure out if you take some time to do it. And considering the gyroscope takes some time itself to figure out what it is going to do I think we might explain it. Inertia in the wheel has a dynamic reaction causing the wheel to turn as it falls. Gravity acts to pull the wheel ~~done~~ down after you let it go, this is very pleasing to us, but as it falls the friction at the contact point causes the wheel to turn and eventually presesses smoothly after it settles down.

How do we solve the problem of a conductor charged to a certain potential V_o? From outside the conductor we have the condition

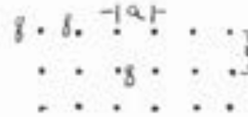
$$\nabla^2 \phi = \frac{\rho_{\text{known outside conductor}}}{\varepsilon_0} + \frac{\rho_{\text{on surface conductor}}}{\varepsilon_0}$$

Know we know $\rho_{outside}$ but we do not know the density of charge on the surface. We can replace the unknown with a constant, the potential $\phi = V_o$ on the conductor. There are many problems which can be approximately solved using this boundary condition technique. But there are complications in determining the exact charge distribution which make it often difficult to find ϕ unless you ~~to~~ have very simple shaped conductors like spheres, plates, cylinders, and wires.

HOLLOW CONDUCTOR



If we have a conductor with a cavity inside, what is the field inside the cavity? Around a Gaussian surface S the field must be zero. So either no charges are inside S or there are equal positive and negative charges piled up on the cavity wall. But if there were charges on the inside wall, they would ~~wo~~ move about until they neutralized themselves. Otherwise $\oint \bar{E} \cdot d\bar{s} \neq 0$ and it must be zero for electrostatics. Thus static charge distributions outside the cavity cannot produce a field inside the cavity. This is called electrostatic shielding.



Electrostatic shielding can be achieved by a grid work of charges with a total surface distribution of $\sigma = q/ab$. You might want to work out how far away from the grid you have to get before the field is no longer varying. It is easy to show that it is not very far from the grid. So a grid of wires like chicken wire makes a pretty good electrostatic shield. Just about as good as a gold plated aluminum shell used commonly today in our aerospace industry.

I often amuse at the stupidity of industry in its hard-headedness when it does things like I just said. I first came across this phenomena a long time ago when a colleague of mine, a guy named Fox, set out from the academic community to pursue work in industry. He went to work for a company which made thermos bottles. Fox's job was to design a gigantic thermos bottle to survive the environment of Antartica. As the worked on the structure-heat characteristic of big bottles, he learned by putting cork plugs periodically spaced around the inside of the vacuum chamber he could keep the contents warm and at the same time prevent breakage. In fact, he determined the whole bottle could be made out of cork and still meet the

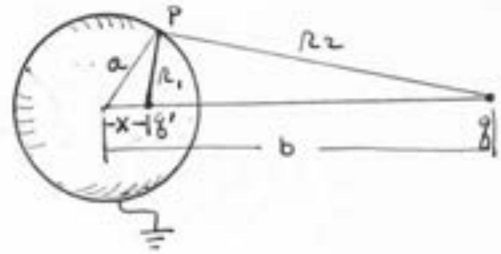
The requirements of the environments. When he went to tell the company's management what he had discovered, he was informed they were not interested in his idea. They were in the Thermos bottle business! So Fox came back to the academic community and left industry to its madness.

The Method of Images

I'd like to talk about a helpful little tool for solving some of the problems that arise in electrostatic. The problem of computing the field due to charge distributions is generally quite complicated. However, for a few simple geometrical shapes we can replace a whole conductor by some imaginary charge Q' at a particular location in space that makes the conductor surface an equipotential. Using this technique many weird shaped conductors have been analyzed and their field given by some nice mathematical relation. What the guy did was to work the problem backward by finding an odd-ball shaped conductor that had a nice image charge representation.

As an example of this idea, let's find the field due to a point charge outside a spherical conductor. Assume first the surface is grounded.

The potential at P from q and q' is proportional to



$$\frac{q}{R_2} + \frac{q'}{R_1}$$

In fact this quantity is a constant for each sphere of a given radius

$$\frac{q}{R_2} + \frac{q'}{R_1} = C$$

Now for $C=0$, where we have ground the surface

$$q = -\frac{R_2}{R_1} q' \quad \text{or} \quad q' = -\frac{R_1}{R_2} q$$

Since we only know $a, b,$ and q we have to find q' in terms of the quantities. So let β be a factor relating b to a and x to a in the following reciprocal way

$$b = a\beta \qquad x = a/\beta$$

Along the radius to q we know

$$\frac{q/\alpha}{a-x} = \frac{q}{b-a}$$

which yield $\quad \frac{1}{\alpha}\,\frac{1}{1-\frac{1}{\beta}} = \frac{1}{\beta-1} \quad$ or $\quad \alpha = \beta = \frac{b}{a}$

Thus $\quad q' = -\frac{a}{b}\, q$

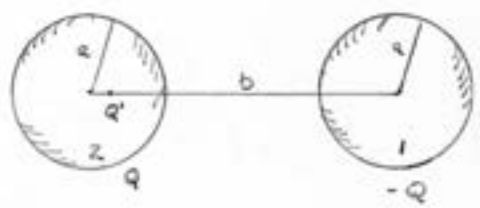
At The surface of The conductor we have a potential due to The image charge inside The sphere and given by

$$V_{surface} = \frac{Q'}{4\pi\epsilon_0 a}$$

For a neutral sphere we know $Q' = \frac{Q}{\alpha} = \frac{a}{b}Q$ so we have as a result, miracle of miracle

$$V = \frac{Q}{4\pi\epsilon_0 b}$$

There is an amusing problem for The more advanced individual where now instead a point charge we have two spheres say of equal but opposite charge and radius. What is The resulting potential in The space between The sphere? Well, to first approximation consider sphere 1 is a point charge a distance b from sphere 2. Then using The above agrument we can find an image charge Q' inside #2. But now let sphere 2 be represented by a charge Q at The center and Q' a distance a^2/b from The center. These two charges have two imagines in sphere 1. And This goes on and on with a rapidly converging sum of charges. There is probably some weird elliptic function or something That represents This sum, The two sphere function) but we haven't found it yet.



Condensers

The previous problem considered two spheres separated by some distance b. We can find The potential difference between The sphere and it turns out being proportional to The charge on The sphere. The constant of proportionsality, for reasons of insanity, is $1/C$ where C is called The capacity so we have

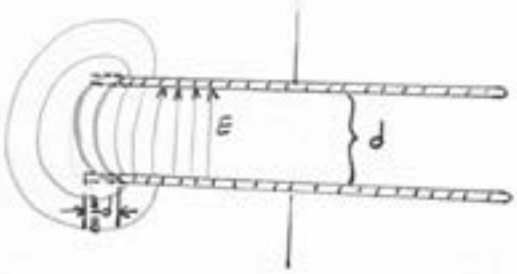
$$V = \frac{1}{C}Q \quad \text{OR} \quad Q = CV$$

The unit of capacity is called The farad and This is a very big number. Most condensers are in The range of micro-micro farad to milli farad.

$$\epsilon_0 = \frac{1}{36\pi 10^9} \frac{\text{farad}}{\text{meter}}$$

The capacity of an object depends on its geometry not its physical length. If we have two condensers of some weird shape with a capacitance $C = Q/V$ and shrink Them by some factor α so There geometric shape is unchanced, The capacitance is unchanged. The electric field increase but The separation distance decreases.

The capacitance between two parallel plates is easy to work out.



The potential difference is the work per unit charge needed to carry a small charge from plate to plate.

$$V = Ed$$

but $E = \frac{\sigma}{\varepsilon_0}$ as we worked out a while back so that

$$V = \frac{\sigma}{\varepsilon_0} d = \frac{Qd}{A\varepsilon_0}$$

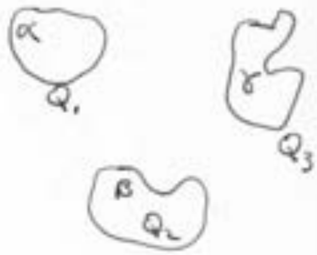
Since

$Q = CV$ we have for the capacity of two parallel plates

$$C = \frac{\varepsilon_0 A}{d}$$

Now this computation is not correct because we have neglected the end effects shown in the figure. As I recall you get a better value for the capacity if you artificially extend the plates a distance 3/8 of the separation distance, d.

If you have more conductors, say three α, β, γ, the potential diff. and capacitance become harder to compute. If we put a little charge dQ on α and measure the new charge on β and then reversing the process by put dQ on β and measuring α, you can go around and compute potential differences between the various combinations. The result is a matrix like representation of the charges Q_1, Q_2, and Q_3



$$Q_1 = C_{11}V_1 + C_{12}V_2 + C_{13}V_3$$
$$Q_2 = C_{21}V_1 + C_{22}V_2 + C_{23}V_3$$
$$Q_3 = C_{31}V_1 + C_{32}V_2 + C_{33}V_3$$

The process I just described is called the reciprocity theorem and has good application here.

The energy between condensers is determined by:

$$dE = V\,dQ = \frac{1}{C} Q\,dQ$$

The electrostatic energy due to all charges is then

$$\mathcal{E} = \frac{1}{2C} Q^2$$

or

$$\mathcal{E} = \frac{1}{2} C V^2$$

I would like to go on and try to find the field due to other configurations. As I have mentioned before it is quite difficult, in general, to find the field surrounding different shaped objects. Such things as cones, polygons, cans present difficult mathematical solutions.

There are many problems which interest us in which the variations of the physical field in one direction are zero. Often the shapes of the objects are surfaces of revolution where there is no z dependence. In such cases the problem reduces to one of two dimensions. The solutions to these problems then appear in a particular mathematical form - functions of a complex variable.

Let $z = x + iy$ the complex variable of x and y. We can express any function of z, $f(z)$ as the following

$$f(z) = u(x,y) + i\,v(x,y)$$

Thus any function can be written as the sum of a real and complex function.

For example
$$f(z) = z^2 = x^2 - y^2 + i\,2xy$$
$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$

Off hand it looks like $f(z)$ has derivatives for a lot of functions such as z^2. According to the definition of a derivative

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{df(z)}{dz}$$

But there is no reason to think the derivative is independent of direction. For a very small class of functions the above limit does exist and we call these analytic functions. Suppose we first consider $\Delta z = \Delta x$ then the derivative becomes

$$\frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - (u+iv)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

Now let $\Delta z = i\Delta y$

$$\frac{u(x, y+i\Delta y) + i\,v(x, y+i\Delta y) - (u+iv)}{i\Delta y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

Equating real and imaginary parts we have the special condition which must be satisfied by analytic functions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the Cauchy-Riemann relations. Note we have implied something else by this condition if we take second derivatives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

which is just like Laplace's equation, for empty space with no z dependence.

So what you do is hunt around for some analytic functions and just how do you do that? One way is through power series expansions of functions of z like

$$e^{z} = 1 + z + \frac{z^2}{2} + \frac{1}{3} z^3 + \cdots$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

To show these are analytic let's take the second one:

$$\frac{1}{1+z} = \frac{1}{1+x+iy} = \frac{1+x-iy}{(1+x)^2+y^2}$$

so

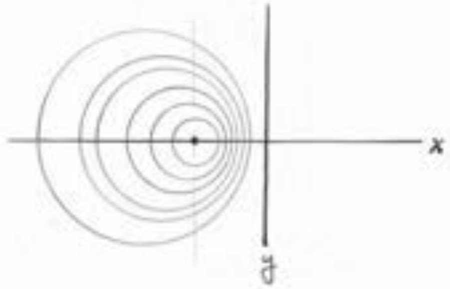
$$U(x,y) = \frac{1+x}{(1+x)^2+y^2} \qquad V(x,y) = \frac{-y}{(1+x)^2+y^2}$$

$$\frac{\partial U}{\partial x} = \frac{1+x}{(1+x)^2+y^2}\left[\frac{1}{1+x} - \frac{2(1+x)}{(1+x)^2+y^2}\right] = \frac{-(1+x)^2+y^2}{[(1+x)^2+y^2]^2}$$

$$\frac{\partial V}{\partial y} = \frac{-y}{(1+x)^2+y^2}\left[\frac{1}{y} - \frac{2y}{(1+x)^2+y^2}\right] = -\frac{(1+x)^2-y^2}{[(1+x)^2+y^2]^2}$$

So it does work.

But what problem have we solved? We have to give a physical meaning to the functions $U(x,y)$ and $V(x,y)$. Both functions are solutions to Laplace's equation in two dimensions; thus each represents a ~~a~~ possible electrostatic potential. Suppose in the above case $U(x,y) = \text{constant}$. The equation $\frac{1+x}{(1+x)^2+y^2} = \text{constant}$ represents ~~a family~~ of circles which correspond to equipotential lines. There exists a a zero at $x=-1$ so we have a set of equipotentials describing a unit charge from an infinite plane.



Given the conductor it is virtually impossible to find the right analytic function so the use of complex variables is not all that practical. But there are a number of analytic functions which are interesting and they correspond to the function,

$$f(z) = z^{\alpha}$$

where $\alpha = \pm\ 1/2,\ 1/3,\ 1,\ 3/2,\ 5/4$, etc

What are the functions U and V for various choices of α and what problem are we solving. I might point out the functions $U(x,y)$ and $V(x,y)$ form mutally orthogonal sets of lines. If we call the equation $V(x,y) = \text{constant}$ the field lines, the $U(x,y) = \text{another constant}$ corresponds to the potential lines. But the choice was arbitrary and we could have inverted our naming.

So let's study the function $f(z) = z^{\alpha}$. First we'll switch to polar coordinates where $z = x + iy = re^{i\theta}$ Then

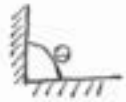
$$f(z) = r^{\alpha} e^{i\alpha\theta} = r^{\alpha}(\cos\alpha\theta + i\sin\alpha\theta)$$

The two functions U and V are now functions of r and θ

$$U(r,\theta) = r^{\alpha}\cos\alpha\theta$$
$$V(r,\theta) = r^{\alpha}\sin\alpha\theta$$

Assume $V(r,\theta)$ corresponds to the potential and let's consider some simple cases:

- when $\theta = 0$, we have $V = 0$ everywhere. This is not too unbelievable
- But when $\theta = \frac{\pi}{\alpha}$ we also have $V = 0$ and this corresponds to a V-shaped groove running along the z axis.
- when $\alpha = 1$, the groove is flat and the potential is $V = r\sin\theta = y$ and we have found the field due to a flat plate. The potential is proportional to the y distance and the electric field is constant
- When $\alpha = 2$ we have the case of a right angle. Then $V = r^2$. The electric field is given by



$$E = -\frac{\partial V}{r\partial\theta} = \frac{r^{\alpha}}{r}\cos\alpha\theta = r^{\alpha-1}$$

  The surface charge density is proportional to r for $\alpha = 2$
- For outside corners $\alpha = 2/3$ and then $V \propto r^{2/3}$ and $\sigma \propto r^{-1/3}$

- As the edge gets sharper and approaches a knife edge, $\alpha \to 1/2$ and the surface charge goes as $1/\sqrt{r}$. So we see the charge accumulating rapidly as the radius shrinks down.

Other values for α can be studied but this is a technique limited to two dimensions and therefore doesn't have much practicality. But it is an amusing little tool.

Capacitance by the Least Energy Technique

We have been talking about condensers and solving various problems relating to the fields produced by the various configurations. We noticed that various geometrical shaped objects were easy too analyze. These included such things as ellipsoids, cylinders, spheres, and other surfaces of revolution such as parabolic hyperboloids. Now if we know the fields from these basic shapes we can get approximations for other more complicated objects. For instances if we let the ellipsoid become a prolate ellipsoid in the limit, we get a needle. If we let the ellipsoid grow oblate, we can approximate a disk. So by combining known fields we are able to find the fields for more complicated objects

I'd like to describe a technical way to get the capacitance for any shaped object such as a cube in a sphere or a cylinder in a sphere. Mathematically there are very few problems we can work out exactly in terms of such things as Legendre polynomials. The simple objects such as a sphere inside a sphere are really quite easy to work out. If the inner sphere is of radius "a" and the other of "b", then the potential is given by

$$V = \int_a^b E\,dr = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

or the capacitance between the two spheres is

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{1/a - 1/b}$$

Now a more powerful technique consider any shaped conductor inside the other conductor. Suppose we want to find the energy from some arbitrary potential ψ inside the system. This is given by the equation

$$\mathcal{E} = \frac{\varepsilon_0}{2}\int (\nabla\psi)^2\, d\,\text{vol}$$

This number varies with ψ and we only know the potentials V_1 and V_2 on the surfaces. By specifying ψ on the boundaries to be a particular value, namely $V_1 = 0$ and $V_2 = V$, we can apply a very nice tool to solving the problem. That is, the integral for the energy is least for the right distribution of charges between the two condensers. That is, the distribution nature assumes is least in energy and satisfies $\nabla^2\psi_{min} = 0$.

Consider that we do have a minimum ψ and call it φ which makes $\mathcal{E}$ least. Let's vary ψ by adding to it another little potential η, i.e., $\psi = \varphi + \eta$. We want to show the first order difference between $\mathcal{E}_{(\varphi+\eta)}$ and $\mathcal{E}_\varphi$ is zero which implies only second order terms remain.

$$\mathcal{E}_{(\varphi+\eta)} = \mathcal{E}_\varphi + 0_{\text{in }\eta} + \text{order } \eta^2$$

so we plug into the integral and get

$$\mathcal{E}_{(\varphi+\eta)} = \frac{\varepsilon_0}{2}\int \left[\nabla^2\varphi + 2(\bar\nabla\varphi\cdot\bar\nabla\eta) + (\nabla\eta)^2\right] d\,\text{vol}$$

since

$$\mathcal{E}_\varphi = \frac{\varepsilon_0}{2}\int \nabla^2\varphi\; d\,\text{vol}$$

The term we are interested in is

$$\epsilon_0 \int [\bar{\nabla}\varphi \cdot \bar{\nabla}\eta]\, d\,\mathrm{vol}$$

Now we want to isolate η from the gradient operator and we use the fact that

$$\int \bar{\nabla}\cdot(\eta\bar{\nabla}\varphi)\,d\mathrm{vol} = \int(\bar{\nabla}\eta\cdot\bar{\nabla}\varphi)\,d\mathrm{vol} + \int \eta\,\nabla^2\varphi\, d\mathrm{vol}$$

The left hand side can be written in terms of a surface integral, i.e.,

$$\int \bar{\nabla}\cdot\bar{F}\,d\mathrm{vol} = \int \bar{F}\cdot\bar{N}\,d\mathrm{surf}$$

so that,

$$\int \bar{\nabla}\cdot(\eta\bar{\nabla}\varphi)\,d\mathrm{vol} = \int \eta\bar{\nabla}\varphi\cdot\bar{N}\,d\mathrm{surf}$$

We can now write

$$\mathcal{E}_{(\varphi+\eta)} = \mathcal{E}_\varphi - \int \eta\,\nabla^2\varphi\,d\mathrm{vol} - \int \eta(\bar{\nabla}\varphi)\cdot\bar{N}\,d\mathrm{surf} + 2^{\mathrm{nd}}\text{ order term}$$

We must now apply our boundary conditions on the electrodes where we made $\eta = 0$ on the inside and outside conductor. In this case the third term on the right is zero. We are left with the condition that

$$\nabla^2\varphi = 0$$

For the first order variation to be zero. for $\psi = 0$ at inside and $\psi = V$ at the outside surface. But we also recall

$$\mathcal{E}_{\mathrm{least}} = \tfrac{1}{2}CV^2$$

So we have a way to guess the capacity of these odd ball objects. The accuracy we get is quite good because the error is only in the second order. So if we have φ within 10% we have the capacity to within 1%.

As an example we might take our two sphere problem and find the capacitance for a straight line approximation of the potential. As a first trial potential consider

$$\psi = \frac{V(r-a)}{(b-a)}$$

which satisfies the conditions that $\psi = 0$ at $r = a$ and $\psi = V$ at $r = b$. Then $\nabla\psi = \dfrac{V}{b-a}$ which gives $(\nabla\psi)^2 = \dfrac{V^2}{(b-a)^2}$ and the integral is easy

$$\mathcal{E} = \frac{\epsilon_0}{2}\,\frac{4\pi}{3}(b^3-a^3)\frac{V^2}{(b-a)^2} = \frac{1}{2}CV^2$$

or

$$\frac{C_{\mathrm{app}}}{4\pi\epsilon_0} = \frac{[b^3-a^3]}{3(b-a)^2}$$

Versus the true result

$$\frac{C_{\mathrm{true}}}{4\pi\epsilon_0} = \frac{1}{1/a - 1/b}$$

If $b=2$ and $a=1$, then numerically we have $\dfrac{C_{\mathrm{true}}}{C_{\mathrm{app}}} = \dfrac{2}{2.33}$ which is not too bad. If I had chosen a worse case where $b=100$ and $a=1$ we are off much worse $\dfrac{C_{\mathrm{true}}}{C_{\mathrm{app}}} = \dfrac{30}{1.01}$. In this case the gradient varies by 100 not as a constant.

As you see by the example, you can't go into the problem totally blind and it takes some degree of intelligence to choose a good representation for the potential. I might take another ψ to show you what I mean. This time let's assume a parabolic potential,

$$\psi = \frac{V\left[(r-a) + \alpha(r-a)^2\right]}{(b-a) + \alpha(b-a)^2}$$

This potential satisfies the boundary condition at a and b, viz. $\psi = 0$ @ $r=a$ and $\psi = V$ @ $r=b$. For simplification let $K = \dfrac{V}{(b-a)+\alpha(b-a)^2}$ so we can write

$$\psi = K\left[(r-a) + \alpha(r-a)^2\right]$$

The gradient of ψ is then

$$\nabla\psi = K + 2\alpha K(r-a)$$

Then

$$(\nabla\psi)^2 = K^2 + 4\alpha K^2(r-a) + 4\alpha^2 K^2(r-a)^2$$

The integral then becomes

$$\mathcal{E} = \frac{\varepsilon_0}{2}K^2\int_a^b 4\pi r^2 dr\left[1 + 4\alpha(r-a) + 4\alpha^2(r-a)^2\right]$$

Upon integrating

$$\mathcal{E} = \tfrac{1}{2}CV^2 = \frac{\varepsilon_0}{2}K^2 4\pi\left[(1-4\alpha a+4\alpha^2a^2)\frac{(b^3-a^3)}{3} + (4\alpha-8\alpha^2 a)\frac{(b^4-a^4)}{4} + 4\alpha^2\frac{(b^5-a^5)}{5}\right]$$

Putting K back in

$$\frac{CV^2}{4\pi\varepsilon_0} = \frac{V^2}{\left[(b-a)+\alpha(b-a)^2\right]^2}\left[(1-2\alpha a)^2\frac{(b^3-a^3)}{3} + 4\alpha(1-2\alpha a)\frac{(b^4-a^4)}{4} + 4\alpha\frac{(b^5-a^5)}{5}\right]$$

or

$$\frac{C}{4\pi\varepsilon_0} = \frac{(1-2\alpha a)^2(b^3-a^3)/3 + \alpha(1-2\alpha a)(b^4-a^4) + (4/5)\alpha(b^5-a^5)}{\left[(b-a)+\alpha(b-a)^2\right]^2}$$

Consider $b=2$ and $a=1$ then our approximate capacitance as a function of α is

$$\frac{C_{Approx}}{4\pi\varepsilon_0} = \frac{(1-2\alpha)^2\, 7/3 + \alpha(1-2\alpha)15 + \frac{124}{5}\alpha}{\left[1+\alpha\right]^2}$$

To find the least capacitance we minimize for α by differentiating with respect to α and set equal to zero then use that value for α_{MIN}.

$$\frac{d\,C_{APP}}{d\alpha} = \frac{(1+\alpha)^2\left[{}^{14}\!/_3 (1-2\alpha)(-2) + (1-2\alpha)15 - 30\alpha + {}^{124}\!/_5 \right] - 2(1+\alpha)\left[(1-2\alpha)^2(7/3) + \alpha(1-2\alpha)15 + \frac{124}{5}\alpha \right]}{[1+\alpha]^4}$$

Therefore

$$(1+\alpha)\left[-\frac{28}{3}(1-2\alpha) + 15 - 60\alpha + \frac{124}{5} \right] = 2\left[\frac{7}{3}(1-2\alpha)^2 + 15\alpha - 30\alpha^2 + \frac{124}{5}\alpha \right]$$

AND ALL THIS SHOULD REDUCE TO A LINEAR EQUATION IN α AND, PERHAPS, IT IS

$$409\alpha - 129 = 0$$

SO THAT

$$\alpha = \frac{129}{409} = .316$$

USING THIS α WE GET FOR THE CAPACITANCE

$$\frac{C_{Approx}}{4\pi\varepsilon_0} = 5.73$$

WHICH COMPARES WITH

$$\frac{C_{True}}{4\pi\varepsilon_0} = 2$$

THE APPROXIMATION IS NOT TOO GOOD SO THE RATIO a/b MUST BE OFF.

IF I TRIED TO USE THIS TECHNIQUE FOR FINDING THE CAPACITANCE FOR THE CUBE IN A SPHERE, I WOULD BE HARD PRESSED TO COME UP WITH THE ANSWER. IT IS A VERY DIFFICULT PROBLEM BECAUSE IT IS HARD TO FIND A FUNCTION SATISFYING OUR BOUNDARY CONDITIONS. AS A CLEVER IDEA I MIGHT WRITE THE ~~E~~ EQUATION FOR THE EQUIPOTENTIALS AS

$$x^n + y^n + z^n = a^n \qquad \text{where } n \text{ is even}$$

IF $n=2$ WE HAVE THE SURFACE OF A SPHERE. AS n APPROACHES INFINITY THIS APPROACHES A CUBE. THIS IS BECAUSE THE DOMINATING NUMBER, OR THE LARGEST x, y, OR z, IS APPROXIMATELY a SO WE HAVE A CUBE. BUT I CAN'T GO ON; I AM SORRY. I AM JUST TOO DUMB AND I WISH I HADN'T BROUGHT UP THIS DAMN PROBLEM.

DIELECTRICS

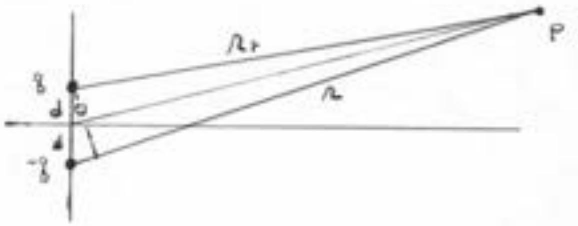
WE NOW COME TO ANOTHER KIND of MATERIAL. This TIME The charge DEPENDS ON The FIELD ACTING ON Them. So far we have discussed conductors And at The other END of The LINE IS The Perfect ~~DIELECTRIC~~ INSULATOR which does NOT SUPPORT Charge MOTION. Since There is NOTHING LIKE A Perfect INSULATOR we have AN intermediate CLASS of MATTER AND This is our dielectric MATERIAL. If The Applied field is NOT TOO Great, The AMOUNT of charge DISPLACEMENT is Proportional TO The field Applied.

The ATOM is MADE UP of ELECTRONS AND The NUCLEUS IN A SYMMETRICAL MANNER. When A field is Applied Across The ATOM The ELECTRON cloud is DISTORTED And The NUCLEUS is LIKEWISE offset from ITS original POINT. The LOCATION of AN ION of CERTAIN ENERGY qQ is GIVEN by The DISTRIBUTION $e^{-q\varphi/kT}$; for The POSITIVE IONS. For The NEGATIVE IONS we have $e^{+q\varphi/kT}$. The TOTAL Charge distribution SATISFIES POISSON'S EQUATION:

$$\frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left[e^{-q\varphi/kT} - e^{+q\varphi/kT}\right] = \nabla^2\varphi$$

ATOM
+ = NUCLEUS
\- = CENTER ELECTRON CLOUD

The ELECTRON TAKES A comprised POSITION when The field is Turned on because IT WANTS TO STAY by The NUCLEUS. The AMOUNT of SEPARATION is PROPORTIONAL TO FIELD. We will ONLY DEAL WITH This SPECIAL LINEAR dielectric behavior. The resulting Electric property which The ATOM Exhibits is dipolar. In order To understand This BETTER LET'S find The field DUE TO A SINGLE DIPOLE:



The field AT P is

$$\varphi_P = \frac{q}{r_+} - \frac{q}{r_-}$$

$$r_- = r$$

$$r_+ = r - d\cos\theta$$

$$\varphi = \frac{q}{r\left(1-\frac{d\cos\theta}{r}\right)} - \frac{q}{r} = q\frac{d\cos\theta}{r^2}$$

The QUANTITY qd is CALLED ~~be~~ The DIPOLE MOMENT of The ATOM AND DENOTED by P. IN VECTOR NOTATION

$$\vec{P} = q\vec{d}$$

So

$$\varphi = \frac{\vec{P}\cdot\vec{r}}{r^3}$$

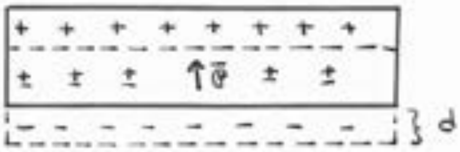
For a lump of dielectric material we have to add all these little dipoles. If we have n atoms per unit volume, then the total dipole moment per unit volume is

$$\vec{P} = nq\vec{d}$$

We will consider the case where

$$\vec{P} \propto \vec{E}$$

Now what field do we get from all these dipoles? It is too complicated to add up all the individual fields so we go to an easier technique where we determine the resulting charge density. So consider a block of dielectric material in an electric field. Inside the block we have a lot of plus and minus charges. With the field applied the plus and minuses do not cancel out. The little surface layer of negative charge does not lie within the material and this surface layer is



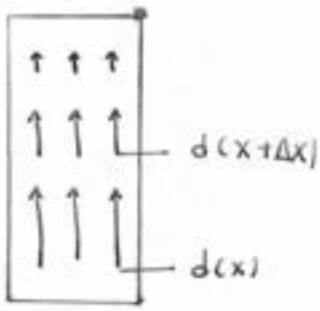
$$\sigma = \rho d = Nqd = nP = \text{amount dipole moment per unit volume}$$

For uniform polarization

$$\sigma = \vec{P}\cdot\vec{n}$$

What happens when $\vec{P}$ varies from place to place? Suppose we have some continuous variation. This happens when the electrons move through different distances and a charge density ρ is accumulated.

Let's find the charge in and out of a little volume $(\Delta x \cdot \text{Area})$



$$\underset{\text{charge in bottom}}{\rho\, d(x)\ \text{Area}} + \underset{\text{charge out top}}{\rho\, d(x+\Delta x)\ \text{Area}}$$

$$\frac{\partial(\rho d)}{\partial x}\,\Delta x\ \text{Area} = \frac{\partial P_x}{\partial x}\, d\text{vol}$$

Thus

$$\rho_{\substack{\text{due to}\\ \text{polarization}}} = -\frac{\partial P_x}{\partial x}$$

In three dimensions

$$\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$$

The total field satisfies a new Poisson's equation

$$\nabla^2\varphi = \frac{\rho_{pol} + \rho_{\text{all nice charges}}}{\varepsilon_0}$$

As an added thought the constant of ~~portion~~ proportionality is χ, the electric susceptibility.

$$\vec{P} = \chi\varepsilon_0\vec{E}$$

DIELECTRICS CONTINUED

LET'S GO BACK TO OUR DISCUSSION OF DIELECTRICS AND SUMMARIZE WHAT WE KNOW ABOUT DIELECTRICS AND ELECTROSTATICS AS A WHOLE. WE HAD $\bar{P}$ REPRESENTING THE DIPOLE MOMENT PER UNIT VOLUME OF THE MATERIAL AND PROCEEDED TO DERIVE THE RELATION:

$$-\bar{\nabla}\cdot\bar{P} = \rho_{POL}$$

where $\bar{P} = \epsilon_0 \chi \bar{E}$ $\quad \chi$ = ELECTRIC SUSCEPTIBILITY

FOR OUR PREVIOUS DISCUSSIONS ELECTROSTATICS CAN BE SUMMARIZED IN TWO EQUATIONS

$$\bar{E} = -\nabla\varphi \quad \text{AND} \quad \bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$$

where ρ = TOTAL CHARGE DISTRIBUTION ~~DUE~~ TO ALL SOURCES AND THEREFORE

$$\rho = \rho_{POL} + \rho_{ALL\ OTHER}$$

AND THEN

$$\epsilon_0 \bar{\nabla}\cdot\bar{E} = \rho_{POL} + \rho_{ALL\ OTHER}$$

NOW WE CAN COMBINE THIS EQUATION WITH THE FIRST ONE TO WRITE

$$\bar{\nabla}\cdot(\epsilon_0\bar{E} + \bar{P}) = \rho_{ALL\ OTHERS}$$

AND FOR HISTORICAL REASONS WE CALL $\epsilon_0\bar{E} + \bar{P}$ THE DISPLACEMENT VECTOR $\bar{D}$

$$\bar{D} = \epsilon_0\bar{E} + \bar{P}$$

AND WE CAN WRITE:

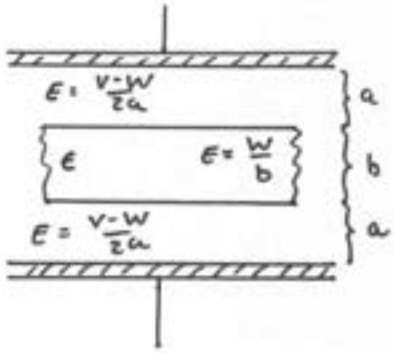
$$\bar{\nabla}\cdot\bar{D} = \rho_{ALL\ OTHER} \quad \text{AND} \quad \bar{E} = -\bar{\nabla}\varphi \quad \text{or} \quad \bar{\nabla}\times\bar{E} = 0$$

IF $\bar{P}$ IS GIVEN BY $\bar{P} = \epsilon_0\chi\bar{E}$, WE CAN SUBSTITUTE THIS INTO OUR EQUATION FOR $\bar{D}$ AND FIND

$$\bar{D} = \epsilon_0(1+\chi)\bar{E} = \epsilon\bar{E}$$

where $\epsilon = \epsilon_0(1+\chi)$ THE PERMITTIVITY OF THE DIELECTRIC

LET'S WORK A PROBLEM TO SHOW HOW TO USE THESE EQUATIONS. CONSIDER A SLAB OF DIELECTRIC BETWEEN TWO PLATES HELD AT A POTENTIAL:



IF WE LET W BE THE POTENTIAL DIFFERENCE ACROSS THE DIELECTRIC, THEN THE ELECTRIC FIELDS ARE GIVEN IN THE FIGURE: →.
WE MUST REQUIRE THE NORMAL COMPONENT OF $\bar{D}$ PRESERVED SO WE MUST HAVE

$$\frac{\epsilon}{\epsilon_0}\frac{W}{b} = \frac{V-W}{2a}$$

or

$$W = \frac{V}{1 + \frac{\epsilon}{\epsilon_0}\left(\frac{2a}{b}\right)}$$

Thus the electric field is given by

$$E = \frac{V-W}{2a} = \frac{V\left[1 - \frac{1}{1+\frac{\epsilon}{\epsilon_0}\left(\frac{2a}{b}\right)}\right]}{2a} = \frac{\frac{\epsilon}{\epsilon_0} V}{1+\frac{\epsilon}{\epsilon_0}\frac{2a}{b}} = \frac{\sigma}{\epsilon_0}$$

so
$$\sigma = \frac{\epsilon V}{1+\frac{\epsilon}{\epsilon_0}\frac{2a}{b}}$$

To calculate the capacitance between the plate

$$C = \frac{\epsilon \text{ Area}}{V} = \frac{\epsilon \text{ Area}}{b\left(1+\frac{\epsilon}{\epsilon_0}\frac{2a}{b}\right)}$$

Now I can discuss this same problem another way in which we think only in terms of $\bar{E}$. Since $\bar{P} = \epsilon_0 \chi \bar{E}$, we have $\bar{P} = \epsilon_0 \chi \frac{W}{b}$. But we also know $\sigma_{pol} = \bar{P} = \epsilon_0 \chi \frac{W}{b}$.

By Gauss' Theorm

$$E_{out} - E_{in} = \frac{\sigma_{pol}}{\epsilon_0}$$

$$\frac{V-W}{2a} - \frac{W}{b} = \chi \frac{W}{b}$$

or

$$\frac{V-W}{2a} = (1+\chi)\frac{W}{b} = \frac{\epsilon}{\epsilon_0}\frac{W}{b}$$

But I must point out we knew $\bar{P}$ to be a linear function of the electric field, i.e. $\bar{P} = \epsilon_0 \chi \bar{E}$. We could have a non-linear dielectric or some other arrangement where the problem becomes much more difficult to solve.

22 JAN 68

COMMENTS ON A NEW Theory

ref: FEYNMAN LECTURES ON ASTRONOMY P. 172

Four hours AGO I NOTICED something which has PUZZLED ME For SOME TWENTY YEARS. I Think I MAY be ON The WAY TO UNDERSTANDING SOMETHING ABOUT QUANTUM ELECTRODYNAMICS (Q.E.D) THAT WAS NEVER before KNOWN. IT IS NOT REALLY A fundamental LAW of PHYSICS THAT I THINK I have COME UP WITH but rather A NEW WAY TO UNDERSTAND A rather Difficult Theory. I HAVEN'T DONE MUCH THINKING SINCE MY brainstorm This AFTERNOON but SINCE YOU WANT ME TO DISCUSS The SUBJECT A LITTLE-I WILL. BUT I'LL COME BACK NEXT WEEK AND TELL YOU WHAT I DID WRONG because IT'S SO SIMPLE I CAN'T believe SOMEONE ELSE HASN'T been Through The ARGUMENT.

SOME PEOPLE SAY if YOU TAKE QUANTUM Theory And relativity And ADD TO IT SOME PROPOSITION, CALL IT X, which IS A STATEMENT of CAUSALITY, i.e., The future IS ONLY Affected by The PAST, The NET RESULT of This SUMMATION IS The Ability TO Prove A whole LOT of THINGS. CENTRAL TO The LIST of Provable Things IS The EXISTENCE of ANTIPARTICLES. IN fact The LAWS Predicting The precise BEHAVIOR of ANTIPARTICLES IS KNOWN or derivable And IS CALLED The TCP INVARIANCE. This TIME-CHARGE-PARITY RULE SAYS, QUITE SIMPLY, THAT AN POSITRON ACTS JUST LIKE AN ELECTRON'S MOTION IN A MOVIE running BACKWARD And viewed Through A Mirror.

IN Addition TO Proving The EXISTENCE of ANTIPARTICLES WE CAN ALSO Deduce The EXCLUSION PRINCIPLE which SAY NO TWO ELECTRONS CAN be IN The SAME STATE. Further You CAN Prove That This is A LOCAL Theory. I'LL EXPLAIN WHAT THIS MEANS IN A few MINUTES.

Now The Thing THAT Bothers ME IS where The EXCLUSION PRINCIPLE COMES from. The confusion comes from The fact That NO ONE IS EXPLICIT AS TO WHAT X REALLY MEANS. SOME PEOPLE SAY THAT X MEANS THAT There EXIST LOCAL OPERATORS such AS $\varphi_a(t,\bar{X})$ And $\varphi_b(t,\bar{X})$ which EXIST OUTSIDE of The LIGHT CONE. THAT IS TO SAY TWO EVENTS a And b OCCURRING SIMULTANEOUSLY TO AN observer CANNOT DISTURB each other And Therefore IT IS POSSIBLE TO MEASURE QUANTITIES IN SMALL REGIONS AROUND The EVENT TO Absolute Precision. IN other words OPERATORS MUST EXIST AT A POINT; They MUST be hermitian; And They MUST SATISFY The COMMUTAR REGATION:

$\boxtimes_a$ $\boxtimes_b$

$$\varphi_a(t,\bar{X}_1)\varphi_b(t,X_2) - \varphi_a(t,\bar{X}_2)\varphi_b(t,\bar{X}_1) = 0$$

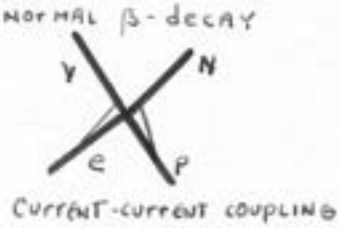
Now I DON'T KNOW WHAT This MEANS Therefore I CAN'T UNDERSTAND A BASIC ASSUMPTION LEADING TO A CONCLUSION which I KNOW IS TRUE, NAMELY, THAT ANTIPARTICLES DO EXIST. I have TO STATE The ASSUMPTIONS IN A form which I CAN UNDERSTAND Them SO I AT LEAST CONVINCE MYSELF AS TO The VALIDITY of The ARGUMENT.

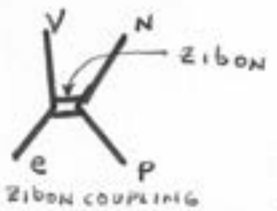
Now I would like to start with just quantum theory and relativity and then prove the exclusion principle must follow. But I don't think this can be done. The reason being the sum of the probabilities of all events happening must add up to one. So far that has never been done without the addition of hypothesis X. It would be truly significant to show that the sum is true while not satisfying causality. I think I have come up with a situation which does just that.

I have chosen a system with electron and no positrons; which is relativistically invariant and not local. Now I know this is not the way nature is because we do have pair production. But I want to understand the premise and to do that I must study my theory to see if I have any infinitely diverging energies or something screwball like that. So I have a world with just electrons and light; from there I must work out what the world would be like - how the atoms would be made and what their spectra would be. Perhaps I must modify electrodynamics down in the short ranges ($\sim 10^{-14}$ cm) where current theory runs afoul. Next week I'll tell you where I made my mistake.

Theory Looks Good

I can report that I don't have any problems yet in my Theory that I talk about last time. I have worked out what my corresponding energies levels for the "Hydrogen" atom would be and I am only off by an amount on the order of the Lamb shift but the difference is not a Lamb effect. I want to work out some perturbation problems to see if I have any waves being generated that are non-causal. I have found waves going backward in time but they are damped out and I don't have a problem with them. Further I have found out that my new objects (call them zibons) interact with photons which in turn can interact with normal things like positrons. [But zibons do violate causality. The zibon, so far, lives for all eternity and I have not worked out how they can be created suddenly.] One reaction which might create the zibon is in β decay where the incident electron and proton do not hit but a coupled by a mystic meson particle or something. Since we know nothing about this strange particle then it is a candidate for my theory.



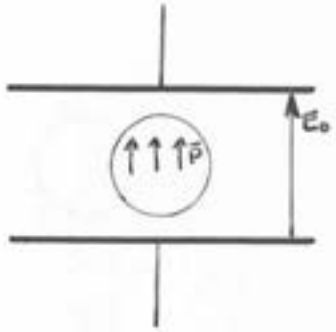


I'll work some more on it during the next week and let you know more about the theory next time

Back to Dielectrics

Returning to the more conventional E&M topic of dielectrics, I'd like to work a problem. Consider a ball in a dielectric with a uniform field across it. We would like to find $\vec{E}$, $\vec{D}$, and $\vec{P}$ inside the dielectric sphere. To do this imagine a polarization

$$\vec{P} = \epsilon_0 \chi \vec{E}_{inside}$$



We assume for simplicity the sphere is uniformly polarized so the charge distribution on the surface is given by the vertical component of the polarization,

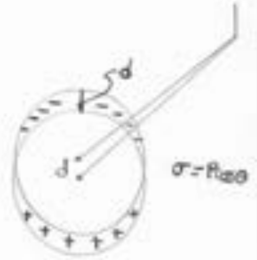
$$\sigma = P\cos\theta$$

What is the field due to such a charge distribution? We can imagine two spheres, one of positive charge density and the other of negative charge density, superimposed on one another to within a distance d between their centers. Remember for a uniformly distributed sphere, the potential inside and outside are given by

$$\varphi_{in} = \frac{4\pi\rho}{3\,4\pi\epsilon_0}\left(\frac{r^2}{2} - \frac{3a^2}{2}\right)$$

$$\varphi_{out} = \frac{-4\pi\rho a^3}{3\,4\pi\epsilon_0 r}$$

OUTSIDE OF THE DIPOLE IS A POTENTIAL GIVEN BY THE GRADIENT OF φ IN THE z DIRECTION



$$\varphi_{\substack{\text{DIPOLE}\\ \text{OUTSIDE}}} = d\frac{\partial\varphi}{\partial z} = \frac{d\rho}{3\epsilon_0}\frac{a^3 z}{r^3}$$

AND,

$$\varphi_{INSIDE} = \frac{d\rho z}{3\epsilon_0}$$

NOTE AT THE SURFACE THE TWO ARE EQUAL OR CONTINUOUS

SINCE $P = d\rho$ $\qquad \varphi_{OUT} = \frac{P}{3\epsilon_0}\frac{a^3}{r^3}z \qquad$ where $P = \frac{4\pi P a^3}{3}$

THEN THE ELECTRIC FIELDS ARE GIVEN BY

$$E_z = \frac{Pa^3}{3\epsilon_0}\left[\frac{1}{r^3} - \frac{3z^2/r^2}{r^3}\right]$$

$$E_{INSIDE_z} = \frac{P}{3\epsilon_0}$$

SO THE TOTAL ELECTRIC FIELDS ARE

$$E_{TOTAL\ OUT} = E_0 + \text{dipole field}$$

$$E_{TOTAL\ INSIDE} = E_0 - \frac{P}{3\epsilon_0}$$

Thus

$$\overline{P} = \epsilon_0\chi\overline{E}_{TOTAL\ INSIDE} = \epsilon_0\chi\left[\overline{E_0} - \frac{\overline{P}}{3\epsilon_0}\right]$$

or

$$\overline{P} = \frac{\epsilon_0\chi\overline{E_0}}{(1+\frac{1}{3}\chi)}$$

IN TERMS OF E_{TOTAL}

$$\overline{E}_{TOTAL\ IN} = \overline{E_0} - \frac{\chi\overline{E_0}}{3+\chi} = \frac{3\overline{E_0}}{3+\chi} = \frac{3\overline{E_0}}{K+2}$$

where $\chi = K-1$

ALSO

$$\overline{D} = \epsilon_0\overline{E}_0 + P = \epsilon_0\overline{E}_0 + \frac{\epsilon_0\chi\overline{E_0}}{1+\frac{1}{3}\chi} = \epsilon_0\overline{E_0}\left(\frac{3+4\chi}{3+\chi}\right)$$

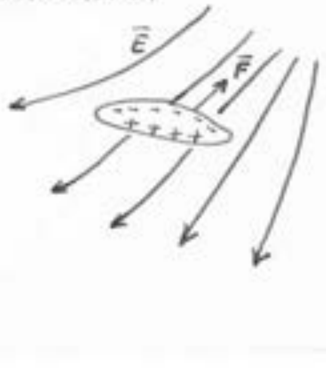
$$= \left(\frac{4K-1}{K+2}\right)\epsilon_0\overline{E_0}$$

ref. VOL II Chapter 10

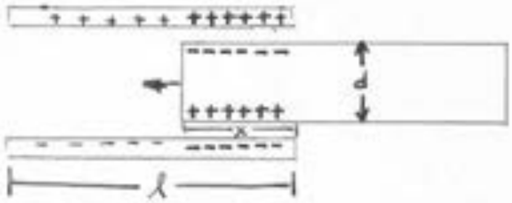
FORCES WITH DIELECTRICS

IT IS A FAIRLY GENERAL FACT THAT A COMBED RUBBED THROUGH YOUR HAIR WILL PICK UP SMALL PIECES OF PAPER. NOW PAPER IS AN INSULATOR AND DOES NOT GET A CHARGE. BUT THE PAPER WILL ACQUIRE A DIPOLE MOMENT DUE TO THE POLARIZATION INDUCED BY THE ELECTRIC FIELD OF THE COMB. SINCE THE ATTRACTING CHARGES ARE CLOSER THAN THE REPELLING ONES THERE IS A NET ATTRACTION. A DIELECTRIC OBJECT IN A NONUNIFORM FIELD FEELS A FORCE TOWARD REGIONS OF HIGHER FIELD STRENGTH. FOR SMALL OBJECTS THE FORCE TURNS OUT TO BE PROPORTIONAL TO THE GRADIENT OF THE SQUARE OF THE ELECTRIC FIELD

$$F \propto \nabla(E^2)$$



Now I'd like to show a way to work problems like this for non-uniform electric fields. We can use the conservation of energy to find the various forces of dielectrics. Consider a parallel plate capacitor with a slab of dielectric between it; there will be a force driving the dielectric further in. I will ignore end effects of the plates which implies $d << l$. Why will the slab be pulled in? The mechanism of attraction is quite complicated because it involves end effects and it's difficult to find the fields around these edges. Let's calculate the electrostatic energy of the system as a function of x, the inserted distance. Since we do have parallel plate condensers



$$Q_{metal} = C_{metal} V \qquad \text{where } C_{diel} = K\epsilon_0 \frac{\text{Area}}{\text{sep}}$$
$$Q_{diel.} = C_{diel} V$$

The total charge is

$$Q_{TOTAL} = VC$$

where

$$C = \frac{\epsilon_0 L(l-x)}{d} + \frac{\epsilon_0 x K L}{d} \qquad L = \text{width}$$

Therefore

$$U = \frac{1}{2}\frac{Q^2}{C} = U(x)$$

Now we know $F_x = -\frac{\partial U}{\partial x}$ so we must differentiate

$$F = -\frac{1}{2}Q^2\frac{\partial}{\partial x}\frac{1}{C} = +\frac{1}{2}\frac{Q^2}{C^2}\frac{\partial C}{\partial x} = \frac{1}{2}V^2\frac{\partial C}{\partial x}$$

Now

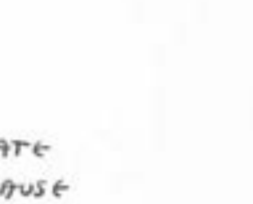
$$\frac{\partial C}{\partial x} = -\frac{\epsilon_0 L}{d} + \frac{\epsilon_0 K L}{d} = (K-1)\frac{\epsilon_0 L}{d}$$

So

$$F = \frac{1}{2}V^2(K-1)\frac{\epsilon_0 L}{d}$$

Note it is incorrect to write for U $\frac{1}{2}CV^2$ and differentiate because you get the wrong sign on the force. This happens because the battery source supplying the voltage is doing work twice as fast as the force due to electrostatic force.

An interesting problem is whether or not two ~~spherical~~ parallel condensers would attract eachother more when they are ~~not~~ hot than when cold. Suppose the two plates are connected by an inductance. What is the force on the condenser



$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2 d}{\epsilon_0 A}$$

where $C = \frac{\epsilon_0 A}{d}$

The force is then
$$F_1 = \frac{\partial U}{\partial a} = \frac{1}{2}\frac{Q^2}{\epsilon_0 A} = \frac{1}{2}\frac{Q^2}{aC}$$
Now the energy in any harmonic oscillation in one degree is $\frac{1}{2}kT$ so the mean square energy is just that
$$\frac{1}{2}\frac{Q^2}{C} = \frac{kT}{2}$$
Therefore the force is
$$F = \frac{kT}{2a}$$
So the force is proportional to the temperature and we would expect hotter objects to attract stronger.

If the oscillator is operating at frequency ω, then more correctly I should write
$$\frac{kT}{2} = \frac{1}{2}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right]$$
so that
$$F = \frac{1}{2a}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right]$$
When kT is much greater than $\hbar\omega$, the objects are so hot that
$F = \frac{\hbar\omega}{4a} + \frac{\hbar\omega}{2a}\frac{1}{(1+\frac{\hbar\omega}{kT}-1)} \doteq \frac{kT}{2a}$ is a good approximation.

But at zero temperature the quantum mechanical dipole fluctuations still produces a force of attraction
$$F \propto \frac{\hbar\omega}{4a}$$
which is the zero point motion

We can also estimate the force between to small objects as a function of the radial distance. The energy of this simple dipole is

$$U = -p\cdot\vec{E}$$
and the force given by
$$F = \nabla U = -p\cdot\vec{\nabla}E = +p\frac{P}{R^4}$$
$$F = \frac{\overline{P^2}}{R^7}$$
where $\overline{P^2}$ is the mean square moment of the system which is a constant characteristic of the various molecules

This is, in fact, the van der Vaal interaction force. In this discussion we have assumed $\frac{R}{c} <$ time of fluctuation of oscillator so the fields are not distorted.

INSIDE DIELECTRICS

I would now like to discuss what goes on inside dielectrics. First I want to consider the simplest case and that is a monatomic gas. For a gas the density of atoms is very slight so when an electric field is placed across the gas the variations in each little dipole is also very small. Thus the effects of one dipole on another are unimportant for our case. The dipoles result when the electric field distorts the atomic charge distribution as we have discussed previously. The induced dipole moment is called electronic polarization.

Because the dipoles do not interact the field at each atom is given by the average value of the applied electric field

$$\vec{E}_{AT\ ATOM} = \vec{E}_{AVE}$$

if we suppose the atoms have the property that the polarization is proportional to the applied field, i.e.,

$$\bar{p} = \alpha \epsilon_0 \bar{E}_{AT\ ATOM}$$

where p is the dipole moment per atom and α depends on the crap going on inside the atom. It is called the polarizability.

Thus we see

$$\bar{p} = \alpha \epsilon_0 \bar{E}_{AVE}$$

If there are N atoms per cubiccentimeter, the total polarization of the gas is

$$\bar{P} = Np = N\alpha\epsilon_0 E_{AT\ ATOM} = N\alpha\epsilon_0 \bar{E}_{AVE}$$

Recall from earlier lectures

$$\bar{P} = \chi\epsilon_0 \bar{E}_{AVE}$$

so we have

$$\chi = N\alpha$$

or

$$K-1 = N\alpha \rightarrow K = 1+N\alpha$$

where K is the dielectric constant for the gas.

How much is α? I'd like to make a crude calculation were I derive α. Actually the calculation is quite difficult since it lies in quantum mechanics and we don't know any of that subject yet. I'll remind you of the classical model of the atom in which we consider it as a simple one degree harmonic oscillator. The equation of motion is given by:

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = F_{EXT} = q_e E$$

ω_0 is the resonant frequency of the atom where the atom absorbs in the D.C. case the displacement is given by

$$x = \frac{q_e E}{m\omega_0^2}$$

Since we have defined the dipole moment of an atom as

$$p = q_e x$$

we have

$$p = \frac{q_e^2 E}{m\omega_0^2}$$

We have an estimate of α then

$$\alpha = \frac{q e^2}{m\omega_0^2 \epsilon_0}$$

Now in our world we have several systems of units. I'll use a new definition which you won't believe but I'll let the charge on an electron be q_e in coulomb such that I can define a pure number which I'll call e^2

$$e^2 = \frac{q_e^2}{4\pi\epsilon_0}$$

By definition I am in the MKS but can now easily convert to MKS by using this number e^2. Thus I can write for α

$$\alpha = \frac{4\pi e^2}{m\omega_0}$$

To see if the dielectric constant is about right consider a hydrogen atom with an ionization given by

$$U \sim \frac{1}{2}\frac{me^4}{\hbar^2}$$

Now remember I have chosen to represent the whole hydrogen spectra by a single frequency ω_0 so the approximation is very crude. I can write then

$$\hbar\omega_0 = \frac{1}{2}\frac{Me^4}{\hbar^2} = \text{Rydberg constant}$$

or

$$\omega_0 = \frac{1}{2}\frac{me^4}{\hbar^3}$$

Substituting into α

$$\alpha \sim 16\pi\left[\frac{\hbar^2}{me^2}\right]^3$$

But $\frac{\hbar^2}{me^2} = 0.528\,\text{Å}$ a Bohr radius so finally

$$\alpha = 16\pi\, a_0^3$$

which is, incidentally just about the volume of an atom. To find K we know at standard pressures

$$N = 2.69\times10^{19}\ \text{atom/cm}^3$$

Then

$$K = 1 + (2.69\times10^{+19})\times(0.528\times10^{-8})^3\,16\pi = 1.00020$$

The experiment value for K is 1.00026 so the theory is about right.

Polar Molecules

Another kind of gas has a dielectric constant which is different than the one calculated. The molecules have a permanent dipole moment like HCl or H_2O. When no field is on, the dipoles are randomly directed. When the field is on the dipoles tend to orientate themselves along the field. The energy of the dipole is

$$U = -\overline{p}_0\cdot\overline{E}$$

Along the field lines we have

$$U = -p_0 E \cos\theta$$

The minus says U is minimum when $\bar{p}_0$ is along $\bar{E}$. The probability of various orientations is given as proportional to $e^{-U/kT} = e^{+p_0 E \cos\theta / kT}$. The mean dipole moment is then

$$\bar{p} = \frac{\int_{-1}^{+1} p_0 \cos\theta \, e^{+p_0 E \cos\theta / kT} \, d\cos\theta^{(\Omega)}}{\int_{-1}^{1} e^{p_0 E \cos\theta / kT} \, d\cos\theta}$$

For normal temperatures and field the exponent is small so we approximate

$$e^{p_0 E \cos\theta / kT} = 1 + \frac{p_0 E \cos\theta}{kT}$$

so

$$\langle \bar{p} \rangle = \frac{\int_{-1}^{+1} p_0 \cos\theta \left(1 + \frac{p_0 E \cos\theta}{kT}\right) d\cos\theta}{1}$$

The denominator being about one

Integrating we get

$$\langle \bar{p} \rangle = \frac{p_0^2 E}{3kT}$$

So we know

$$\alpha = \frac{p_0^2}{3kT\epsilon_0}$$

or in another way

$$\chi = \frac{N p_0^2}{3kT\epsilon_0}$$

The bigger the moment the bigger the force to align it. At higher temperatures there is more misalignment so α must be bigger.

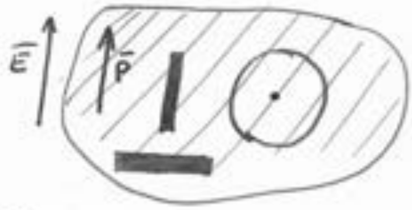
Higher density dielectrics, liquids and solids

I'll go on to talk about more dense dielectrics but I'd like to do it on a historical basis. We want to know what the polarizability of a crystal. The dipole moments per atom is still given by

$$\bar{P} = \alpha \epsilon_0 \bar{E}_{\text{at atom}}$$

but $\bar{E}_{\text{at atom}}$ no longer equals $\bar{E}_{\text{avg}}$ because each dipole now affects the other so we have a technical problem of how to find $\bar{E}_{\text{at atom}}$. To make this calculation Lorentz proposed a very eloquent and accurate technique. He proposed to consider the atom in question to be in a hole which can be take out of the rest of the dielectric which is considered a continuum. It turns out the problem depends on the shape of the hole.

If we have a long thin slot, the field inside must just cancel out the external field since $\nabla \times \bar{E} = 0$ so



$$\bar{E}_0 = \bar{E}$$

For a slot perpendicular to the field we can use Gauss' law to find

$$E_0 = E + \frac{P}{\epsilon_0} \quad \text{or} \quad \epsilon_0 \bar{E}_0 = \bar{D}_0 = \bar{D}$$

Now let's consider a spherical hole. The ball is uniformly polarized and if we consider the charge is frozen in we can write

$$E = E_{hole} + E_{plug} \quad \text{by superposition}$$

where E_{hole} is the field inside the hole and E_{plug} inside a sphere. We have work out the sphere before and it is just

$$E_{plug} = -\frac{P}{3\epsilon_0}$$

So

$$E_{hole} = E + \frac{P}{3\epsilon_0}$$

Therefore the dipole moment is

$$\bar{P} = N\alpha\epsilon_0 \left[E_{Ave} + P/3\epsilon_0 \right]$$

or

$$\bar{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3}} \epsilon_0 E$$

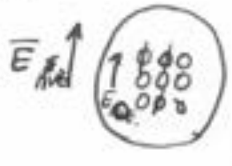
or

$$K - 1 = \frac{N\alpha}{1 - \frac{N\alpha}{3}} \quad : \text{CLAUSIUS-MOSOTTI EQUATION}$$

Note if $N\alpha$ is small, as for a gas, we get our old equation $K = 1 + N\alpha$. For a liquid $N\alpha$ is on the order of unity not 3. This means we can pack atoms only so close before repulsion gets too big.

The above calculation works for a solid crystal as well. It is just a special case where we sum the dipole effects to get the same answer. Consider a cubic crystal inside a big sphere with some E_0. The average field inside is

$$E_{Ave} = E_0 - P/3\epsilon_0$$



But E_0 is not the complete field inside. We must sum the dipole fields. It will ture out though this sum is zero

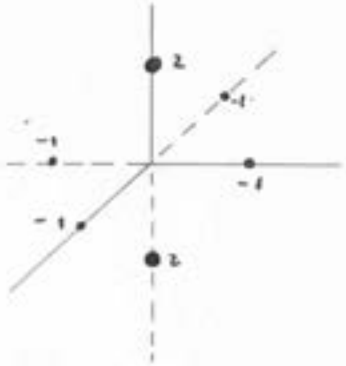
$$E_0' = E_0 + \sum \text{field from all dipoles} = E_0$$

Consider the z-component only then

$$E_z = \nabla \frac{Pz}{4\pi\epsilon_0 r^3} = P\left(\frac{r^2 - 3z^2}{r^5}\right)$$

where the function of r is an angular function given by $\frac{1}{r^3}(1 - 3\cos^2\theta)$.

We can see from this function when $\theta = 0$ the field goes as $2P/r^3$. And when $\theta = \pi/2$ the field goes a $-P/r^3$. The net field due to the superposition is zero.



Therefore, we conclude for a crystal

$$E_{ATOM} = E_{AVE} + P/3\epsilon_0$$

DIELECTRIC WITH VARYING POLARIZATION

Suppose we have a dielectric with a polarization which is no longer uniform. We have then at atom a some dipole moment P_a and at b we have P_b. If we want the field at a the problem is more complicated. We must consider the effect of the varying polarizations on eachother and can do so as follows:

$$E_{AT\,a} = \gamma_{aa} P_a + \gamma_{ab} P_b + E_{AVE} + P/3\epsilon_0$$

$$E_{AT\,b} = \gamma_{ba} P_a + \gamma_{bb} P_b + E_{AVE} + P/3\epsilon_0$$

The γ's represent dipole sums and for a sphere they are all zero. Thus we can find the individual dipole moments

$$P_a = \alpha_a \epsilon_0 E_{AT\,a} \,, \quad P_b = \alpha_b \epsilon_0 E_{AT\,b}$$

And the total ~~po~~ moment is

$$P = N_A P_A + N_b P_b$$

As a more general case the total dipole moment can be expressed in terms of the tensor γ_{ij} and the local average field about the atom. That is,

$$\bar{P} = \gamma_{ij} \overline{E}_{AVE.\,LOCALLY}$$

This equation together with

And

$$\bar{\nabla} \cdot \bar{E}_{AVE} = \bar{\nabla} \cdot \bar{P} + \rho_{EXT}$$

$$\nabla \times \bar{E}_{AVE} = 0$$

describes all of electrodynamics and is a elegantly simple statement. It is interesting to show that the dipole tensor does not depend on the shape of the dielectric in the problem.

I'll consider some real odd shape configuration shown here. The plates are charged producing some polarization inside as well as some average field. Consider $\bar{P}$ to be frozen in after the field is turned on. I'll try to find the field at atom a in the following manner. I hope I am not missing anything. For precision I'll do it in several steps.

The field at atom a does not depend on the microscopic polarization $\bar{P}$ averaged around a. Find E at a means we have really solved the smoothed out dipole polarization and subtracted the concentrated polarization. That is, we know E for a uniform polarization. To find E at atom a due to all the other oddly polarized atoms we do the following thing. Take out the polarization in a cell about 30 atomic diameters wide. Then the following fields are superimposed:

E_{ave} which is produced by the smooth polarization and external charges

$E'_{at\ atom}$ produced by smooth polarization with the cell empty (this is hard to do)

$E''_{at\ atom}$ due to ~~smooth polarization~~ all dipoles + ext. charges

We can write

$$E'' = E_{ave} + E(\text{due dipoles} - \text{due to smooth})$$

The difference is what we are after and it is the difference <u>locally</u> which depends on terms like $\gamma_{aa} \bar{P}_a + \gamma_{ba} \bar{P}_b$ etc. But this sum is rapidly converging to a definite value. This is true because the dipole field falls off as $1/R^3$. If I consider E_{ave} continuous over a cell of 30 atomic diameter, the dipole interactions will fall off rapidly with distance. That is, the sum converges so fast the local field doesn't know it's in an irregular shaped dielectric. And that's it.

ELECTRODYNAMICS IN RELATIVISTIC NOTATION

I THINK I'LL GO ON TO THE SUBJECT OF ELECTRODYNAMICS AND GET AWAY FROM ELECTROSTATICS WHICH HAS OCCUPIED OUR TIME FOR QUITE A WHILE. USUALLY WHEN COVERING ELECTROMAGNETIC THEORY THE NEXT SUBJECT WOULD BE MAGNETOSTATICS BUT I'M GOING TO SKIP THAT AND LEAVE IT FOR HOMEWORK.

WE HAVE BEEN PIDDLING AROUND SO LONG IN ELECTROSTATICS THAT IT SEEMS HARD TO BELIEVE WE ONCE COVERED THE BASICS OF RELATIVITY. I'LL REMIND YOU OF FOUR-VECTORS AND THE LORENTZ TRANSFORMATION BETWEEN TWO COORDINATES SYSTEMS, FOR EXAMPLE FOR UNIFORM VELOCITY IN THE z-DIRECTION:

$$x = x' \qquad y = y' \qquad z = \frac{z' - vt}{\sqrt{1-v^2}} \qquad t = \frac{t' - vz'}{\sqrt{1-v^2}}$$

OR

$$x' = x \qquad y' = y \qquad z' = \frac{z + vt}{\sqrt{1-v^2}} \qquad t' = \frac{t + vz'}{\sqrt{1-v^2}}$$

ANOTHER EXAMPLE OF A FOUR-VECTOR IS THE MOMENTUM FOUR-VECTOR

$$p_x = \frac{m_0 \dot{x}}{\sqrt{1-v^2}} \qquad p_y = \frac{m_0 \dot{y}}{\sqrt{1-v^2}} \qquad p_z = \frac{m_0 \dot{z}}{\sqrt{1-v^2}} \qquad E = \frac{m_0}{\sqrt{1-v^2}}$$

AN INVARIANT QUANTITY UNDER SUCH A TRANSFORMATION IS

$$E^2 - p^2 = m^2$$

ANOTHER FOUR-VECTOR IS GIVEN BY THE ELECTRICAL POTENTIAL A_μ: A_x, A_y, A_z, ϕ AND ALSO THE FOUR-CURRENT j_x, j_y, j_z, ρ. REMEMBER ALSO THE PROPER FORCE AS DEFINED EARLIER TRANSFORMED LIKE A FOUR VECTOR.

SO FAR I HAVE WRITTEN A FEW EQUATIONS REPRESENTING MY KNOWLEDGE OF ELECTRICITY AS

$$\bar{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \vec{\nabla} \times \vec{A}$$

$$\bar{F} = q(\bar{E} + \vec{v} \times \bar{B})$$

THE FIRST TWO EQUATIONS IMPLY THAT

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} + \frac{1}{c}\frac{\partial \bar{B}}{\partial t} = 0$$

THE QUESTION WE WOULD LIKE TO ANSWER IS WHAT LAW DETERMINES HOW THE FIELDS ARE PRODUCED. TO START THE ANSWER WE'LL FIND THE POTENTIAL RATHER THAN THE FIELD BECAUSE IT IS EASIER TO CALCULATE. WE KNOW FOR STATIC FIELDS THE POTENTIAL MUST SATISFY

$$\nabla^2 \varphi = -\rho/\epsilon_0$$

Now let's go on to a more complicated situation. This time let the charge q, say, be moving along the z axis. We know if it is at the origin the position at some point a distance r out is given by $\varphi = \frac{q}{4\pi\epsilon_0 r}$. This is all we really know; that and the superposition principle. For a moving charge what is φ. The easiest way to find φ is to consider the charge at the origin and the point moving in a straight line past it. Then we just have to transform back using our laws. The four potential is given by

$$\varphi = \frac{\varphi' - vA_z'}{\sqrt{1-v^2}} \qquad A_z = \frac{A_t' - v\varphi'}{\sqrt{1-v^2}} \qquad A_x = A_x' \qquad A_y = A_y'$$

For the prime system the charge is at $x'=0$, $y'=0$, $z'=0$ for all t'. While in the unprimed system it is at $x=y=0$ and $z=-vt$. To show this is true consider the transformation

$$x = x' \qquad y = y' \qquad z = \frac{0 - vt'}{\sqrt{1-v^2}} \qquad t = \frac{t' - 0}{\sqrt{1-v^2}} \qquad \text{or } z = -vt$$

Now the primed potential is given by

$$\varphi' = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x')^2 + (y')^2 + (z')^2}} \qquad \text{for all } t'$$

In the stationary system $A' = 0$ so the unprimed potentials are given by

$$\varphi = \frac{\varphi'}{\sqrt{1-v^2}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1-v^2}}\frac{1}{\sqrt{(x')^2 + (y')^2 + (z')^2}}$$

$$A_z = -v\varphi$$

$$A_x = 0$$

$$A_y = 0$$

Now φ is no good to us in the above form so we must transform to the unprimed system with our transformation law

$$\varphi = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1-v^2}}\frac{1}{\sqrt{x^2 + y^2 + \frac{(z+vt)^2}{(1-v^2)}}}$$

or

$$\varphi = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(1-v^2)(x^2+y^2) + (z-vt)^2}}$$

The electric field is now computed

$$\bar{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

Since we only have a z component this equation reduces to

$$E_z = -\frac{\partial\varphi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t}$$

$$E_z = \frac{(z-vt)}{\left[(1-v^2)(x^2+y^2)+(z-vt)^2\right]^{3/2}} - \frac{v^2(z-vt)}{\left[(1-v^2)(x^2+y^2)+(z-vt)^2\right]^{3/2}}$$

$$= \frac{(z-vt)\ (1-v^2)}{\left[(1-v^2)(x^2+y^2)+(z-vt)^2\right]^{3/2}}$$

If we look at the field lines we can understand this result a little bit better. Consider first a point with $z=0$, the E only has x and y components, i.e.,

$$E_x \propto x \qquad E_y \propto y \quad \text{and} \quad E_z \propto z - vt$$

The electric field is radial from the charge. For $v=0$ the lines form a sphere. For $v \sim c$ the field lines are still radial through the charge but now are squashed by a factor of $\sqrt{1-v^2}$ in the direction of motion. The field lines become stronger on the sides while weaker ahead and behind of the direction of travel. Right angles to the motion $z-vt=0$ so the total field strength is $\sqrt{E_x^2+E_y^2} =$ $E = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{1-v^2}}\frac{1}{x^2+y^2}$ so the field is like a coulomb $1/r^2$ but the factor $1/\sqrt{1-v^2}$ which is always greater than one increases the strength. This answer is what we expect if we are naive but since were not we expect the answer to be more complicated.

We ought to talk a minute on the magnetic field lines. Recall $B = \nabla \times \bar{A}$ which means the B-field circulates around a current. Now the B lines are not considered real because they don't exist in the other system. That is the B lines can suddenly appear to a moving observer since $\bar{B} = \bar{v} \times \bar{E}/c^2$. We can also have the case where we have only B lines and no E lines as in the case of a permanent magnet. When the magnet is moved the B lines do the same hockus-pockus and E lines suddenly appear all over the place. If you have E and B lines in the beginning, there is no simple way to predict the results. So be careful here.

ELECTRODYNAMICS IN FOUR DIMENSIONAL NOTATION

I'd like to propose a guessing game by asking what you think a likely possibility would be for a general law of electricity. Suppose I give you the static case $\nabla^2\varphi = -\rho/\epsilon_0$. How would you make ~~to~~ this into an invariant form? A likely candidate would be to add a second time derivative so that we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -\rho/\epsilon_0$$

This is not right though because it is only partially correct; we must add three more equations to complete the four vector set, i.e.,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -\rho/\epsilon_0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_{x,y,z} = -j_{x,y,z}/\epsilon_0$$

This set of equations is now relativistically invariant and can be written more simply using the D'Alembertian operator $\Box^2$,

$$\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 = \nabla_\mu \nabla_\mu$$

In the new notation

$$\Box^2\phi = \rho/\epsilon_0 \quad , \quad \Box^2\bar{A} = \frac{j}{\epsilon_0}$$

or better yet

$$\Box^2 A_\mu = \frac{j_\mu}{\epsilon_0}$$

A_μ is not unique since it is possible to add a four vector χ, say to A_μ such that $\Box^2 A_\mu$ is still j_μ/ϵ_0 because $\Box^2\chi = 0$. The added four vector implies a wave solution through space. But we shall impose the physical condition that there are no free waves in space. That is, all waves have a source and we don't have any God given waves. Although there a some people looking for a wave that has existed through all times. Thus A_μ must have a source and j_μ is that source.

There exists then an arbitrariness in choosing φ and $\bar{A}$ but that is just the way the real world operates. We can ask only two questions about the world: what is the state of the world or what laws govern the phenomena and what is in the world, i.e. why are there laws saying waves must interact in the first place? There is no satisfactory formulation of the latter question as you might imagine. We must make certain assumptions about what the initial conditions were. There exists then an arbitrariness in what you assume and what you prove. So don't feel your way of proving something is the best way; one man's assumption is another man's conclusions.

Quotes Notes: "One man's assumption is another man's conclusion"

MORE ON THE LAWS OF ELECTRODYNAMICS

I'd LIKE TO START OUT BY WRITING DOWN THE THINGS WE KNOW FROM LAST TIME. THE D'ALEMBERTIAN OPERATOR IS GIVEN BY $\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ (NOTE THIS IS THE NEGATIVE OF THE DEFINITION IN THE ref.)
THE LAWS OF ELECTRODYNAMICS (ED) THEN TAKE THE FORM:

$$\left.\begin{array}{l} \Box^2 \varphi = -\rho/\epsilon_0 \,, \quad \Box^2 \bar{A} = -\dot{j}/\epsilon_0 \\ \text{TOGETHER WITH} \\ \bar{E} = -\nabla\varphi - \frac{\partial \bar{A}}{\partial t} \,, \quad \bar{B} = \bar{\nabla} \times \bar{A} \,, \quad \bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) \end{array}\right\} \quad \mathrm{I}$$

NOW I WANT TO SAY SOMETHING ABOUT ρ AND $\dot{j}$ AND HOW THEY ARE RELATED TO q. I WAS A LITTLE SLOPPY HERE BY WRITING THE FORCE DUE TO ONE DISCRETE PARTICLE WHEN IN FACT I MUST EXTEND THE LAW TO COVER A LARGE NUMBER OF PARTICLES AND IN THE LIMIT A CONTINUOUS CHARGE DISTRIBUTION. TO BE EXPLICIT THEN I MUST WRITE

$$\rho(x,t) = \sum_i q_i \, \delta^3(X - X_i(t))$$
$$\dot{j}(x,t) = \sum_i q_i \, \bar{v}_i \, \delta^3(X - X_i(t))$$

WHERE $\delta^3(X - X_i)$ IS THE THREE DIMENSIONAL DELTA FUNCTION WE MUST CONSIDER THE FORCE DENSITY OR FORCE PER UNIT VOLUME TO MAKE THE LAW RELATIVISTICALLY INVARIANT, I.E.,

$$\vec{f} = \rho \bar{E} + \dot{j} \times \bar{B}$$

THIS FORCE IS PART OF A FOUR VECTOR AND GIVES THE THREE SPATIAL COMPONENTS. IF WE ADD THE RATE OF ENERGY CHANGE PER UNIT VOLUME OR BETTER THE RATE OF DOING WORK PER SEC, WE HAVE A COMPLETE FOUR VECTOR, I.E.,

$$f_t = \bar{F}_i \cdot \bar{v}_i = \dot{j} \cdot \bar{E} \quad = \text{TIME COMPONENT OF 4-FORCE}$$

OUR SUPER DUPER NOTATION CAN NOW GIVE US THE FOLLOWING SET OF EQUATIONS REPRESENTING ALL OF ELECTRODYNAMICS

$$\left.\begin{array}{l} \Box^2 A_\mu = -j^\mu/\epsilon_0 \\ F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu \\ f_\mu = j_\nu F_{\mu\nu} \end{array}\right\} \quad \mathrm{II}$$

THE SECOND EQUATION IS EXPANDED OUT BACK ON PAGE 48 AND GIVES THE DIFFERENTIALS EQUATIONS FOR $\bar{E}$ AND $\bar{B}$. THE THIRD EQUATION IS, OF COURSE, THE EQUATIONS OF MOTION. SO WE TWO EQUIVALENT FORMULATIONS OF ELECTRODYNAMICS

WE CAN FORMULATE THESE EQUATIONS STILL ANOTHER WAY, THAT BEING, BY THE LEAST ACTION PRINCIPLE. (BACK ON PAGE 59 WE DISCUSSED THIS).

The force LAW CAN be derived from The ACTION EXPRESSION

$$F = q\int\left[A_t\,dt - \bar{A}\cdot\bar{v}\right]dt$$

or more compactly

$$S = q\int A_\mu \frac{dx_\mu}{ds}\,ds$$

This is for The INTERACTION of TWO PARTICLES for MANY WE HAVE

$$S = \Sigma_i\, q_i \int A_\mu \frac{dx^i_\mu}{dS}\,ds$$

or better yet

$$S = \int j_\mu(x,t)\,A_\mu(x,t)\,dt\,dx\,dy\,dz$$

The result of ~~the~~ VARY The ACTION WILL YIELD The EQUATIONS $F_{\mu\nu}$ AND f_μ for The MINIMUM CONDITION. A Third WAY of WRITING The LAWS of ELECTRODYNAMICS IS

$$\left.\begin{aligned} \Box^2 A_\mu &= -j^\mu/\epsilon_0 \\ \delta S &= \delta \int j_\mu(x,t)\,A_\mu(x,t)\,d\tau_\mu = 0 \end{aligned}\right\}\ \mathrm{III}$$

IN The CONTINUUM form There ARE NOT INFINITIES TO CAUSE difficulties There ARE INFINITIES IN The DISCRETE PARTICLE CASE when The TOTAL field ACTING ON A PARTICLE IS PARTLY DUE TO ITSELF. SINCE The field GOES AS $1/R^2$, There ENTERS IN AN INFINITE field STRENGTH. WE'LL DISCUSS This problem MORE LATER.

NOW I HAVE BEEN UTTERLY REPETITIOUS ON PURPOSE BUT, PERHAPS, MADE ALL THIS APPEAR MORE COMPLICATED THAT IT IS. BUT SIT AN CONTEMPLATE The JEWEL YOU HAVE before YOU. YOU CAN PICK ANY form (I, II, or III) YOU LIKE. If LIKE RELATIVITY OR NON-RELATIVISTIC form, PARTICLES or CONTINUOUS, YOU HAVE A choice. ONCE YOU PICK A ~~B~~ PARTICULAR form, Though, IT IS BEST TO STICK WITH IT SO AS NOT TO be confused. IN These EQUATIONS LIES AN EXPLANATION of ALL PHYSICAL PHENOMENA. IT IS indeed wonderful TO find SUCH A SIMPLE ~~set~~ SET of EXPRESSIONS. Of course, WE HAVE NOT CONSIDERED QUANTUM Theory YET BUT WE'LL COME TO THAT LATER.

WE ARE NOW AT A CRITICAL POSITION where NOW WE CAN deduce MANY THINGS. WE could GO INTO ELECTROSTATICS BUT WE HAVE covered THAT. WE COULD GO INTO AN EXPLANATION Of VARIOUS MAGNETIC EFFECTS. There ARE A MULTITUDE of PHENOMENA OPEN for DISCUSSION AND I DON'T KNOW which WAY TO GO. Therefore, I'LL LEAVE IT UP TO YOU AND YOU CAN decide where YOU WANT TO GO.

IT MAKE BE GOOD AS AN EXERCISE TO Show how MAXWELL'S EQUATIONS CAN be deduced from The PRINCIPLE of LEAST ACTION.

WE HAVE TO GUESS AT A FORM FOR THE ACTION AND A GOOD TRY MIGHT BE,

$$S = \frac{1}{2}\int (\partial_\nu A_\mu)(\partial_\mu A_\mu)\, d\tau_\mu$$

THIS IS SHORT HAND FOR A VERY LONG EXPRESSION

$$\int \left(\frac{\partial\varphi}{\partial t}\right)^2 - (\nabla\varphi)^2 - \left(\frac{\partial A_x}{\partial t}\right)^2 - \left(\frac{\partial A_x}{\partial x}\right)^2 - \left(\frac{\partial A_x}{\partial y}\right)^2 - \ldots \text{ETC.}$$

LET'S CHANGE THE ~~PATH~~ FIELD BY A SMALL AMOUNT $A_\mu = \underline{A_\mu} + \delta A_\mu$ WHERE $\underline{A_\mu}$ IS THE NATURAL FIELD RESULTING FROM ALL THE CHARGES IN THE UNIVERSE. THUS,

$$S = \frac{1}{2}\int \left[\partial_\nu(\underline{A_\mu} + \delta A_\mu)\right]\left[\partial_\mu(\underline{A_\mu} + \delta A_\mu)\right] d\tau_\mu$$

$$= \frac{1}{2}\int\Big\{\left[(\partial_\nu \underline{A_\mu})(\partial_\mu \underline{A_\mu})\right] + (\partial_\nu \delta A_\mu\, \partial_\mu \underline{A_\mu}) + (\partial_\nu \underline{A_\mu}\, \partial_\mu \delta A_\mu) + \partial_\nu \delta A_\mu \partial_\mu \delta A_\mu\Big\}\, d\tau_\mu$$

IGNORING THE LAST TERM WHICH IS SECOND ORDER WE HAVE

$$S = \frac{1}{2}\int (\partial_\nu \underline{A_\mu}\, \partial_\mu \underline{A_\mu})\, d\tau_\mu + \int (\partial_\nu \delta A_\mu\, \partial_\mu \underline{A_\mu})\, d\tau_\mu$$

or

$$S = \bar{S} + \int(\partial_\nu \delta A_\mu\, \partial_\mu A_\mu)\, d\tau_\mu$$

FOR THE FIRST ORDER TERM TO BE ZERO

$$\delta S = \int(\partial_\nu \delta A_\mu\, \partial_\mu A_\mu)\, d\tau_\mu = 0$$

WE MUST ISOLATE δA_μ BY INTEGRATING BY PARTS AS WE HAVE DONE BEFORE AND WE GET

$$\delta S = -\int \left[\partial_\nu(\partial_\mu A_\mu)\right]\delta A_\mu\, d\tau_\mu$$

FOR δS TO BE ZERO WE MUST HAVE

$$-\partial_\nu \partial_\mu A_\mu = 0$$

SO WE GUESSED PRETTY GOOD BUT WE FORGOT A TERM IN THE ACTION; IT IS

$$+\frac{1}{\epsilon_0}\int j_\mu A_\mu\, d\tau_\mu$$

SO VARYING THE ACTION WE GET

$$\delta S = \int\left(-\partial_\nu \partial_\mu A_\mu + \frac{1}{\epsilon_0} j_\mu\right)\delta A_\mu\, d\tau_\mu = 0$$

OR

$$\partial_\nu \partial_\mu A_\mu = -j_\mu/\epsilon_0 = \Box^2 A_\mu$$

I MUST POINT OUT THERE IS A ~~PRICE~~ HIDDEN GEM IN THIS FORMULATION WHICH HAS GONE UNNOTICED AND I MUST POINT IT OUT.

When we vary the action due to the interaction of mater, we wrote

$$\delta S = \delta \int j_\mu(1) A_\mu(1)\, dt_1 = \int j_\mu\, \delta A_\mu\, dt_\mu$$

So if I break the action up into two parts and multiply through by ϵ_0

$$\epsilon_0 \delta S = -\frac{\epsilon_0}{2}\int (\partial_\nu \partial_\mu A_\mu)\, dt_\mu + \int j_\mu \delta A_\mu\, dt_\mu = 0$$

we get the idea that there may be a more general form of the action which contains all of ED and mechanics and can be written as one superduper action of the world

$$S = \Sigma_i m_i \int dS_i + \Sigma_i g_i \int A_\mu(X_i, t) \cdot \frac{dX^i_\mu}{dt}\, dt + \frac{\epsilon_0}{2}\int (\partial_\nu A_\mu)\partial_\nu A_\mu\, dt_\mu$$

The middle term is just another way to write

$$\int j_\mu(1)\, A_\mu(1)\, d t_\mu$$

Now S is a minimum with respect to variations of $X_i(t)$ (the path of the particular particle) and yield the result

$$\frac{d}{ds}\left(m_i \frac{dX^i_\mu}{ds}\right) = g_i \frac{dX^i_\mu}{ds} F_{\mu\nu}(\text{at } X_i)$$

(Note: we worked this equation out on page 98)
This variation of the super action gives the dynamical motion of a particle in the fields you are assuming.

If S is a minimum with respect to $A_\mu(1)$, we get Maxwell's equations $\Box^2 A_\mu = -j_\mu/\epsilon_0$. Thus S contains all of electrodynamics and mechanics of classical physics.

Retarded waves

For these equations of ED to conform to reality we must add something else. That being, all waves have a source and that the cause preceeds effects. This is true because $\Box^2 X = 0$ but still has a solution so that $A_\mu \to A_\mu + X$ still gives $\Box^2(A_\mu + X) = -j_\mu/\epsilon_0$. Thus the new potential don't give the ~~new~~ same field in general, different motions, and different predictions. All this implies the idea of retarded wave solutions.

To prove this we must disguse the solutions to the equation

$$\Box^2 \psi(1) = -S(1)$$

The source S is a function of x and t so it wiggles and makes waves as some function of position and time. We want to analyze the solution when S exists only at the origin and shakes up and down. We will write the source as

$$S(R,t) = \delta^3(R)\,\sigma(t)$$

the strenth $\sigma(t)$ varies with time so we can think of σ as the same thing as the charge density ρ and the whole problem

BECOMES ANALOGOUS TO SOLVING

$$\nabla^2 \varphi = -\rho/\epsilon_0 = -\frac{1}{\epsilon_0} \delta^3(R) g(t)$$

which HAS AS ITS SOLUTION

$$\varphi = \frac{g(t)}{4\pi\epsilon_0 r}$$

The PROBLEM IS The SAME BUT WE ONLY FORGOT TO THROUGH IN The SECOND TIME DERIVATIVE. WHAT WE REALLY WANT TO SOLVE IS

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -\delta^3(R)\sigma(t)$$

WHEN WE SOLVE THIS WE HAVE SOLVED ALL WE NEED TO SOLVE.

There ARE MANY WAYS TO SOLVE THIS EQUATION. I'LL GUESS The SOLUTION IS GIVEN AS SPHERICALLY SYMMETRIC FROM THE SOURCE IN WHICH CASE

$$\psi(x,y,z,t) = f(r,t) \quad \text{where } r = \sqrt{x^2+y^2+z^2}$$

Then TO FIND The LAPLACIAN

$$\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 f}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial f}{\partial r}\frac{1}{r} - \frac{\partial f}{\partial r}\frac{x^2}{r^3}$$

ADDING The SIMILAR Y & Z COMPONENTS

$$\nabla^2 \psi = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$$

OR KNOWING The LAPLACIAN IN SPHERICAL COORDINATES IMMEDIATELY

$$\nabla^2 \psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) = \frac{1}{r}\frac{d^2}{dr^2}(rf) \quad \text{AS A more useful form}$$

Therefore we have

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = -\sigma(t)\,\delta^3(R)$$

For $r \neq 0$ everywhere except the origin

$$\frac{\partial^2}{\partial r^2}(rf) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rF) = 0$$

if $F = rf$

$$\frac{\partial^2}{\partial r^2} F = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} F$$

Now suppose we had The EQUATION

$$\frac{dF}{dr} = \frac{1}{c}\frac{dF}{dt}$$

AND WANTED The SECONDED DERIVATIVE

$$\frac{d^2 F}{dr^2} = \frac{1}{c}\frac{\partial}{\partial t}\frac{\partial F}{\partial r} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial F}{\partial t}\right) = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

After LARGE MENTAL EFFORT WE CAN SHOW THE MINUS SIGN ALSO IS A SOLUTION. WE WANT A SPECIAL SOLUTION where

$$F_1 = g(r+ct) \quad \text{OR}$$

$$F_2 = h(r-ct)$$

SINCE THE TWO SOLUTIONS CAN BE JOINED TO FORM ANOTHER SOLUTION WE HAVE

$$F = g(r+ct) + h(r-ct)$$

OUR SOLUTION THEN BECOMES

$$\psi = \frac{1}{r} g(r+ct) + \frac{1}{r} h(r-ct)$$

BUT SO FAR I HAVE'NT CONSIDERED THE RIGHT SIDE OF THE EQUATION YET AND THIS WITH DETERMINE WHAT HAPPENS AT THE ORIGIN.

THE CLOSER WE COME TO THE SOURCE OR ORGIN THE MORE SIGNIFICANT THE SPACE DERIVATIVE BECOMES AND THE LESS THE TIME DERIVATIVE BECOMES. THUS WE HAVE IN THE LIMIT AS $r \to 0$ $\nabla^2 \psi \propto 1/r$ WHICH HAS AS ITS SOLUTION

$$\psi = \frac{\sigma}{4\pi r}$$

ALL WE CAN DETERMINE ABOUT $\sigma(t)$ IS THAT AT THE ORIGIN

$$g(ct) + h(-ct) = \sigma(t)$$

AS A SPECIAL CASE, TRY $g=0$, THEN $h(-ct) = \sigma(t)$ OR $h(x) = \sigma(-x/c)$. THEN WE COULD WRITE $k(r-ct) = \sigma(t - r/c)$ WHICH YIELDS

$$\psi = \frac{\sigma(t - r/c)}{4\pi r}$$

THE OTHER POSSIBILITY IS $h=0$ AND THIS GIVES

$$\psi = \frac{\sigma(t + r/c)}{4\pi r}$$

THE ANSWER SAYS THAT AT A ~~TIME~~ DISTANCE FAR AWAY FROM THE SOURCE THE TIME IS NOT THE TIME OF THE MEASUREMENT BUT DEPENDS ON HOW FAR AWAY THE POINT IS. THE MOST GENERAL SOLUTION IS GIVEN BY

$$\psi(R,t) = \frac{\sigma(t - r/c)}{4\pi r} + \frac{k(t - r/c) - k(t + r/c)}{r}$$

IF WE CAN ONLY DETERMINE THE DIFFERENCE OF g AND h, THIS IS THE BEST WE CAN DO. IF $k = \sigma/4\pi$, WE GET THE OTHER SOLUTION, I.E. $\frac{\sigma(t+r/c)}{4\pi r}$ AND AT THE ORIGIN WE GET THE RIGHT ANSWER.

WHAT THIS SAYS IS THAT SOURCES PRODUCE EFFECTS LATER THAN THEIR CAUSE. THE k's ARE GENERALLY 0 UNLESS THERE ARE SOURCES WE DIDN'T CONSIDER. THE SOLUTION CAN BE WRITTEN AS

$$\psi(R_1,t) = \int \frac{S(R_2, t - r_{12}/c)}{4\pi r_{12}} \, dV_2$$

THIS IS THE ANSWER TO OUR EQUATION OF ED. SO ACTUALLY WE HAVE

$$A_\mu(R,t) = \int \frac{j_\mu(R_2, t - r_{12}/c)}{4\pi \epsilon_0 r_{12}} \, dV_2$$

THIS IS THE RETARDED WAVE SOLUTION WHICH SAYS EFFECTS CAN NOT PRECEED CAUSES. THERE IS NO LOGICAL REASON FOR THIS BUT WE ARE COMFORTABLE WITH THE THOUGHT AT TIME $= -\infty$ ALL FIELDS ARE 0 AND CONDITIONS WERE FIXED. IF GOD HAS GIVEN US WAVES WITH NO SOURCES THEN WE ARE IN A LITTLE TROUBLE.

ref: (1) Interaction with the Absorber as the Mechanism of Radiation, Rev. Mod. Phys. Vol 17 # 2 & 3 April–July 1945, Wheeler and Feynman

ADVANCED AND RETARDED WAVES

At the end of last lecture a good question came up which I gave a poor answer to; I'd like to go over it again in more detail. We had Maxwell's equation

$$\Box^2 A = -j/\epsilon_0$$

with the solution

$$A(R,t) = \int \frac{j(R_2, t - r_{12}/c)}{4\pi\epsilon_0 r_{12}}\, dV_2$$

We were forced to accept as a possible solution to the equation an advanced solution where $j = j(R_2, t + r_{12}/c)$ and a combination of half advanced and half retarded waves, this is called the symmetric solution.

If we consider the two equations with the advanced and retarded solutions, then

$$\Box^2 A_{RET} = -j/\epsilon_0$$
$$\Box^2 A_{Adv} = -j/\epsilon_0$$

and the difference is obviously

$$\Box^2 (A_{RET} - A_{Adv}) = 0$$

This equation describes a free wave, i.e. a wave which does not interact with matter. It started in the infinite past is going straight on through to the infinite future. It cannot be turned on or off. The most general to Maxwell's equation can now be written as

$$A = A_{ret.} + \text{free wave}$$

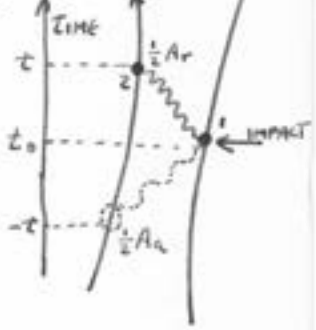
What I'd like to show is that there aren't any free waves at all and to do that suppose God's will was that

$$A = \int \left[\frac{j(R_2, t - r_{12}/c) + j(R_2, t + r_{12}/c)}{4\pi\epsilon_0 r_{12}}\right] dV_2$$

$$= \tfrac{1}{2} A_{ret} + \tfrac{1}{2} A_a$$

was the correct solution in nature. If some guy considered a retarded field only, he would fend his answer differs for he has
$A = A_{RET} + \frac{1}{2}(A_{Adv} - A_{retard})$

Consider a physical situation in which the world consists of two charges and one charge is hit, it jiggles, and its resulting current makes a wave which effects the other a little latter. If we were prejudiced as we observed the event in the symmetric world, the total vector potential would yield $\frac{1}{2}A_r + \frac{1}{2}A_a$. Now we would have to say that something, not from the source, struck particle 2 to kill one half the retarded field we expected. At the same time we must have a "free wave" wipe out the half advanced wave field $\frac{1}{2}A_a$. In other words we need to add to our retarded field, the free wave $\chi = \frac{1}{2}A_a - \frac{1}{2}A_r$



Advanced And Retarded Vector Potential

I'd like to go back over part of last week's lecture because I didn't say too much about it. I would like to contemplate the two solutions I developed for Maxwell's equations. We have a symmetric and asymmetric solution

$$A_{sym} = \tfrac{1}{2} A_{retarded} + \tfrac{1}{2} A_{advanced}$$

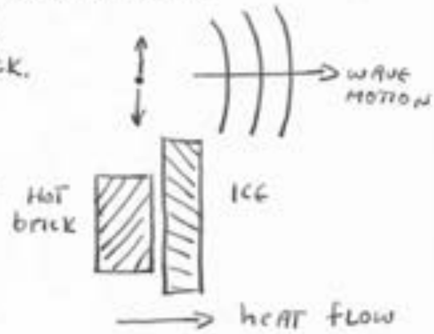
$$A_{asym} = A_{retarded} + \chi$$

Both vector potentials are admissable solutions to Maxwell's equation

$$\Box^2 A = - j/\epsilon_0$$

The non-symmetric solution we are familiar with but not the symmetric form because it admits an advanced wave solution which appears to contradict our belief that causes preceed effects.

Let's suppose we go back to good old fashion physics ideas that limit effects from occurring before causes. The question is why isn't the advanced solution admitted. Take a simple example of an electromagnetic oscillator radiating waves into space and nearby is a block of ice and a ~~br~~ hot brick. If the ice melts we know there has been a transfer of heat from the brick. The melting has cause the closed two body system to have an increase in entropy. Rising entropy implies the time arrow. We will discover the entropy increase corresponds to the same phenomena as waves getting farther away from their source in time. Could a world exist in which ice melts but at the same time waves come inward? The answer is yes but it is not obvious. The thing I want to get across is that the very same cause results in one way entropy and outward traveling waves.

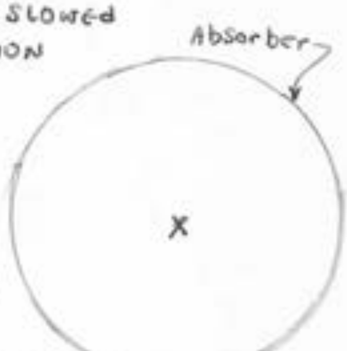


The law of increasing entropy is based on statistics and assumes the past was more organized than the present. Therefore, we talk about the quality of the history of the universe. We make the cosmological assumption that matter was tightly packed ~~something~~ sometime in the past is currently expanding. At one time in the past light had a mean free path because the universe was closed and light coexisted in thermal equilibrium with matter. At this time it was meaningless to say the current density was either $j = j(r_4, t - r_4/c)$, $j = j(r_4, t + r_{12}/c)$ or an combination of the two because in equilibrium time has no meaning. Light is locally in equilibrium with matter. After the expansion begins the vector potential is physically describable by a retarded wave solution because it doesn't make any sense to say waves could be sent into the past, i.e., before the equilibrium state.

In other words the irreversibility of heat conduction is just like the irreversibility of the emission of radiation. Both phenomena really are describable in terms of statistical mechanics connected with the asymmetry of the initial conditions with respect to time. A warm object in a cool surrounding will most probably look heat to that surrounding. Likewise, radiation will most probably be radiated outward, i.e., the oscillating source will lose energy to any absorber.

Now I am going to take a different cosmological model and get the same answer. This time I will assume matter flows in only one direction, outward from the big bang. Further I must assume that all light radiated is absorbed sooner or latter, i.e., nothing creeps in or out. Suppose the world is enclosed by absorbing walls. The theory I have says if I use Asymmetric, I get the same answer as with Aretarded. But it turns out Ared is more convenient to work with and easier to analyze.

The fact that the wall is absorbing means as particles are pushed into the walls, the motion is damped out and slowed down. We will start by assuming the symmetric solution



$$A = \tfrac{1}{2}A_A + \tfrac{1}{2}A_R$$

With this vector potential I can get the laws of force on a charge placed at the center of this enclosure,

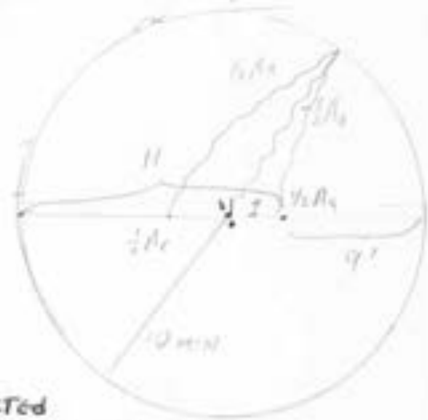
$$\bar{F} = q(\bar{E}_{SYM} + \bar{v} \times \bar{B}_{SYM})$$

Suppose we made up some other law from a "stupid theory"

$$\bar{F} = q\underbrace{(\bar{E}_{ret} + \bar{v} \times \bar{B}_{retard})}_{\text{Stupid Theory}} + q\underbrace{(\bar{E}_{\frac{1}{2}A_a - \frac{1}{2}A_r} + \bar{v} \times \bar{B}_{\frac{1}{2}A_a - \frac{1}{2}A_r})}_{\text{correction to stupid theory to get "right" answer}}$$

If the inside was all black and a light turned on and latter turned off no net motion will result. The fields in the correction produce effects but they must come from $-\infty$ and go on out to $+\infty$. In otherwords, the correction field activate the absorber but never gets through to the inside and produce unusual effects. The excitation of the absorber generates new currents and fields which add nicely to give zero, i.e., the correction field must not lose energy. Thus we are left with the result that the symmetric and antisymmetric solutions give the same physical laws. I'll work out an example of how all this works.

Let's see how this works. Consider a black wall absorber around a charge. The wall is 10 light minutes away and we'll start at 12:00. An experiment will be preformed one minute away from the origin. If the charge is hit at 12:00 at time $= t + 1$ min the field is ½R and at $t = -1$ the field must be ½A. Now the effect of the oscillations of the charge will radiate and strike the absorber at $t = +10$ minutes. The absorber is set into motion and all the constituent particles generate a field. This field is equal to half the retarded minus half the advanced field of the sources. The radiation field of the absorber arrives back at the source to cancel the advanced waves and add to the retarded waves to give the expected results, i.e., the ½ advanced wave of the absorber arrives back at the source at 12:00 not at 12:20 like the retarded wave. Actually we must be a little more careful in considering the cancellation of the advanced waves. ~~Remember~~ Remember the ½ advanced wave at the source occurs at ~~12~~ 11:59. The diagram shows the differences in time between the experiment and the ends of the diameter. Careful summing of the ½ advanced fields from the absorber will prove the cancellation. All advanced fields are cancelled by inferference. Their effects show up directly only in the force of radiative reaction.



SELF-FORCE AND RADIATION RESISTANCE

ref. Vol II Chap. 28

You probably know that an accelerating charge radiates by retarded waves which carry away energy. That is, you must do extra work on the charge as opposed to pushing along an uncharged particle which will not radiate. Where does the retarding force come from? If we consider the electron to be a ball of finite radius, then forces within the ball do not cancel when the electron is accelerating due to the time delay. Action and reaction are not exactly equal and the electron exerts a force on itself that tries to hold back the acceleration.

The self-force can be written as a series for the non-relativistic case.

$$F = \alpha \frac{e^2}{a c^2} \ddot{x} - \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \gamma \frac{e^2 a}{c^4}\ddddot{x} + \cdots$$

where α & γ are numbers about equal to 1. The coefficient $\alpha \frac{e^2}{a c^2}$ is like a different mass than inertial mass; it is called the electromagnetic mass. If the charge is a uniform sphere $\alpha = 2/3$. The second term is the rate at which work is done and therefore corresponds to radiation resistance.

If The electron radius, a, goes to zero, The third Term (And all higher order Terms) go To zero. The second Term is constant. The first Terms becomes infinite due to the infinite electromagnetic mass. The infinity Arises because we have let A "point" electron act on itself. If you say it cannot act on itself, you Throw Away A very nice baby with The bath water.

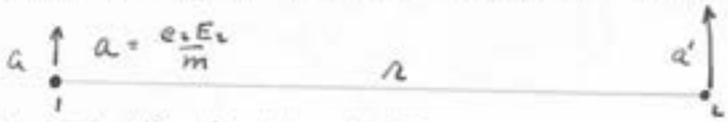
A way To get Around The difficulty is to use our symmetric solution for The vector potential. The formula for the self-~~act~~ Force due to The half Advanced And retarded waves,

$$F = \frac{\alpha e^2}{a c^2}\ddot{X} + \frac{2e^2}{3c^3}\dddot{X} + \gamma \frac{e^2 a}{c^4}\ddddot{X} + \cdots$$

If we modify The rule for self-force And say it is equivalent to ½ The difference of The retarded And Advanced field, All the even power terms go out,

$$F = -\frac{2}{3}\frac{e^2}{c^2}\dddot{X} + \text{higher Terms}$$

When I was A young Boy back at MIT, I Thought The idea An electron could not Act on itself was stupid. Furthermore I felt sure I could Prove it was stupid. Consider, I said, to accelerating charges A distance r Apart going At different accelerations a And a'.



$$a' = \frac{e_1 E_1}{m} = \frac{e_1^2 a}{r}$$

The field At 2 due To 1 is just

$$E_1 = \frac{e_1 a}{r}$$

Back At 1 The force due To

$$f = e_2 E_2 = \frac{e_2^2 a'}{r} = \frac{e_2^2 e_1 E_1}{m r} = \frac{e_1^2 e_2^2 a}{m r^2}$$

Well, I didn't get The right Answer And worse what I explained was reflected light. I forgot to consider the delay Across the distance r. So I went to Dr. Wheeler with The Great "Proof" And was crushed by my stupidity. But he said let's go from your answer. First, he said you must consider The total effect of many particles, i.e., N per unit volume. The number of particles in A spherical shell will be $4\pi N r^2 dr$. The total force is

$$F = \int_0^r \frac{a e_1^2 e_2^2}{m r^2} 4\pi r^2 dr N$$

But immediately we see as $N \to \infty$ the force becomes ~~w~~ likewise becomes infinite.

wheeler went right on just as if he had work the whole thing out just before I came in. He said we haven't taken into consideration the phase lag between the outgoing disturbance and the returning reaction. This turns out to be 90° and thus we can easily take this by throwing in $i = \sqrt{-1}$ into our integral. Now, he went on, the incoming advanced force acting on the source due to the motion of a typical interaction between two charges and propagates at the speed of light. The retarded wave traveling outward combines with the secondary fields from the absorber and the total disturbance travels at a different speed, c/n n = refractive index. Thus we must consider how far down we must go before we get out of phase and that is just

$$\frac{\lambda}{n-1}$$

since we're considering a medium of low density, the refractive index is known,

$$n-1 = 4\pi \frac{Ne_2^2}{m_2\omega^2}$$

so our answer now looks like

$$F = i\, a\, \frac{e_1^2 e_2^2}{m_2} \frac{4\pi N \lambda}{4\pi N e_2^2}\, m_2\omega^2 = i\omega a c e_1^2 \qquad \text{since } c = \lambda\omega$$

if we consider the primary acceleration is given according to

$$a = a_0 e^{-i\omega t}$$

we can replace $i\omega$ by $-\frac{\partial}{\partial t}$ so we now have

$$F = -\dot{a} c e_1^2$$

I have lost some factors of c along the way. But also I don't have $2/3$ either but that's because I forgot to average over $\sin^2\theta$ which is $2/3$ so I get

$$F = -\tfrac{2}{3}\,\dot{a}\,\frac{e_1^2}{c^3}$$

I was amazed; it was the greatest day of my life. I could hardly believe he hadn't worked it out before. He told me to go home and figure out just how much advanced wave I needed and to come back with the right answer. It turned out to be 1/2 just as expected.

Dirac claimed that total field should be the sum of 1/2 the advanced and retarded. According to his theory the force on the i^{th} particle is given by,

$$\overline{f_i} = \sum_{j \neq i} \overline{F}_{j\,sym}$$

where

$$F_{j\,sym} = q\left[\overline{E}_{(\frac{1}{2}A_a + \frac{1}{2}A_r)} + \overline{v} \times B_{(\frac{1}{2}A_a + \frac{1}{2}A_r)}\right]$$

This summation can be expressed as

$$f_i = \sum_{\text{All } i} F_{j\,\text{sym}} - \sum F_{i\,\text{sym}}$$

Or another way to express the answer

$$f_i = \sum_{j \neq i} \left(\tfrac{1}{2} F_{j\,\text{Ret}} + \tfrac{1}{2} F_{j\,\text{Adv}} \right)$$

This expression can be broken down into three parts

$$f = \sum_{j \neq i} F_{j\,(\text{ret})} + \left[\tfrac{1}{2} F_{i\,(\text{ret})} + \tfrac{1}{2} F_{i\,(\text{Adv})} \right] - \sum_{\text{all } j} \left(\tfrac{1}{2} F_{j\,\text{ret}} - \tfrac{1}{2} F_{j\,(\text{Adv})} \right)$$

The first term is just the old theory of retarded waves. The second term describes the self-force and gives rise to radiative damping. The third term vanishes for a complete absorber.

To show the third term is zero for a complete absorber we know outside the absorber,

$$\tfrac{1}{2} \sum_j F_{j\,(\text{Ret})} + F_{j\,(\text{Adv})} = 0$$

Since it vanishes everywhere and for all times we must have

$$\sum_j F_{j\,\text{Ret}} = 0 \qquad \sum_j F_{j\,\text{Adv}} = 0$$

outside independently. In other words if one did not vanish we allow for an outgoing wave while the other represents a converging wave. But complete destructive interference between two such waves is impossible. Thus, they must vanish independently. From this conclusion we deduce

$$\sum_j \left(\tfrac{1}{2} F_{j\,\text{Ret}} - \tfrac{1}{2} F_{j\,\text{Adv}} \right) = 0 \quad \text{Also outside}$$

This field, different from the sum, has no singularities within the absorber. Since it vanishes outside at all times, it must be zero inside. Therefore,

$$\sum_j \left(F_{j\,\text{ret}} - F_{j\,\text{Adv}} \right) = 0 \quad \text{Everywhere}$$

I spent a lot of time work on the symmetry field case. During that time I developed a lot of powerful tools to help solve the problem which I never did. When the Lamb shift was discovered I learned how quick I could solve the problem using these techniques. So the rules work just as well in the retarded case.

ASSOCIATED WITH THE IDEA OF ADVANCED FIELDS IS THE IDEA OF ADVANCED EFFECTS. THE DAMPING TERM DOES LEAD TO AN ADVANCED EFFECT CALLED PRE-ACCELERATION. SUPPOSE A PARTICLE OF MASS m IS BEHAVING ACCORDING TO THE CONDITION

$$m\ddot{x} = \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \text{ANY FORCES APPLIED}$$

SUPPOSE FOR A MOMENT THAT THE ADDITIONAL FORCE IS JUST

$$F = m\omega_0^2 x$$

THE USUAL DAMPED OSCILLATOR EQUATION IS

$$m\ddot{x} = -\gamma\dot{x} + m\omega_0^2 x$$

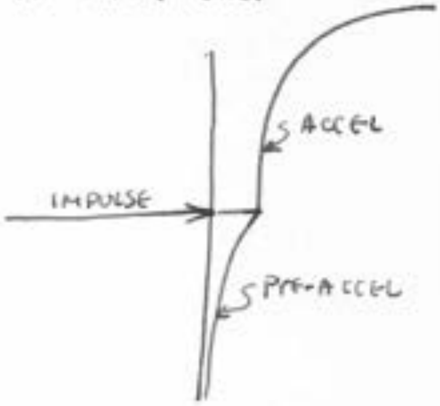
IF WE ASSUME $x \sim e^{i\omega_0 t}$ THEN WE CAN FIND

$$\gamma \sim \frac{2e^2}{3c^3}\frac{\omega_0^2}{m}$$

IF WE IGNORE THE $m\omega_0^2 x$ TERM FOR A MINUTE WE FIND

$$\ddot{x} = e^{\frac{3}{2}\frac{c^3 m}{e^2}t}$$

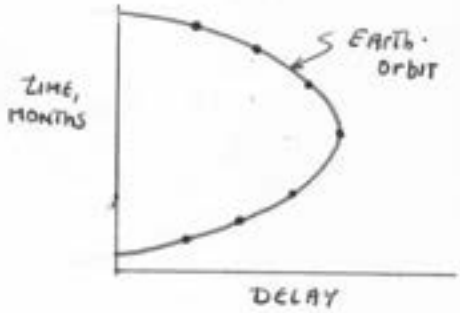
THIS IS A NON-DAMPED ACCELERATION. IT RECEIVES A PRE-ACCELERATION FOR $\tau \sim \frac{2}{3}\frac{e^2}{mc^2}$ SECONDS BEFORE THE IMPULSE AND JUST EXPLODES AFTER IT. DIRAC THOUGHT WE ~~BETTER~~ BETTER HAVE SOME DAMPING AND THE IDEA OF AN ANTICIPATORY RESPONSE 10^{-23} SECONDS BEFORE THE CAUSE IS AN UNACCEPTABLE IDEA. A WAY AROUND IT IS TO MAKE THE ACTION A MINIMUM FOR VARIATIONS IN ASYMPTOTICALLY SMALL AT EACH END.



NOISES FROM SPACE

SINCE WE ARE FREE TO TALK ABOUT ANYTHING, I LIKE TO REPORT TO YOU ON A NEW DISCOVERY. IT WAS MADE BY A GUY NAMED HEWISH IN RADIO ASTRONOMY. HE WAS STUDYING THE SCINTILLATIONS OF THE SOLAR WIND USING A RECEIVER OPERATING BETWEEN 80 AND 85 MEGA CYCLES WITH A BANDWIDTH OF 100 KILOCYCLES. HE NOTICED WHEN SET AT A FREQUENCY SAY 81.5 MEGACYCLES THAT HE WOULD GET LITTLE PIPS THAT KEPT REPEATING THEMSELVES. ON A MORE CAREFUL STUDY HE REALIZED THE PIPS WERE AS NARROW AS HIS OPERATING BANDWIDTH AND AFTER A LITTLE ELECTRONIC FOOLING AROUND FOUND THE PIPS WERE 80 KILOCYCLES WIDE. THE SIGNAL WAS SHIFTING DOWN WITH A RATE OF 5 MEGACYCLES PER SECOND. THIS MEANS IT PASSES ITS OWN WIDTH IN A LITTLE OVER 0.016 SEC OR ROUGHLY 1/60TH OF A SECOND. THERE IS NO INDICATION WHERE THE SIGNAL STARTS FROM ONLY THAT IT PASSES THROUGH THIS 80-85 MC RANGE.

THE REAL MYSTERY IS THAT THE PIP HAS BEEN REPEATING ITSELF EVERY 1.337279 ± .000002 SECONDS FOR THE PAST SIX MONTHS. NOW BECAUSE THE EARTH ROTATES AROUND ITS ORBIT PLANE THERE IS A SLIGHT SHIFT DUE TO THE TIME DELAYS ACROSS THE VARIOUS PARTS OF THE PATH. ONCE THIS ERROR IS CORRECTED FOR AND THE DOPPLER SHIFT DUE TO THE SOURCE RECESSION IS ACCOUNTED FOR WE CAN PLOT TIME IN MONTH VS. DELAY AND FIND THE POINTS FALL PRECISELY ON A PLOT OF THE EARTH ORBIT.



THE SOURCE FOLLOWS SIDEREAL TIME SO ACCURATELY THAT THIS IS NO PARALLAX IN THE MEASUREMENT. THE SOURCE IS AT LEAST 1000 ASTRONOMICAL UNITS AWAY SO IT IS DEFINITELY NOT WITH OUR SOLAR SYSTEM OR MAN MADE, I.E., A SATELLITE. ALSO THE STRENGTH OF THE SIGNAL VARIES LIKE MAD VERY IRREGULARLY. MAYBE IT IS SOME MODULATED SIGNAL FROM ANOTHER SOLAR SYSTEM!

THERE ARE FOUR QUESTIONS WE WOULD LIKE TO ASK:

(1). WHAT'S GOT SUCH AN ACCURATE PERIOD
(2). HOW CAN THE PULSE BE SO SHARP
(3). WHY DOES THE FREQUENCY SHIFT DOWN
(4). WHY DOES THE AMPLITUDE VARY SO MUCH

STARTING WITH THE LAST QUESTION AND GOING BACKWARD.

IF THERE ARE ANY VARIATIONS IN THE SOURCE OR MORE LOGICALLY IF THERE ARE VARIATIONS IN THE STUFF BETWEEN US AND THE SOURCE, MAYBE THIS WOULD ACCOUNT FOR THE FLUCTUATIONS. IN FACT THE ATMOSPHERE AND/OR THE SOLAR WIND ACTS LIKE A SCINTILLATOR TO JIGGLE THE ELECTRONS AT A SECOND OR TWO INTERVAL. WE THINK THERE ARE ENOUGH ELECTRONS BETWEEN US AND IT TO ACCOUNT FOR THIS PHENOMENA.

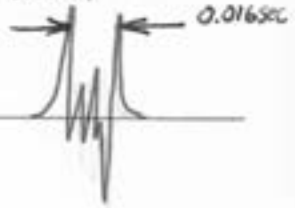
WHAT MAKES The frequency shift DOWN? ThAT IS A LITTLE EASIER, Perhaps, To ANSWER. If we hAVE A DILUTE ELECTRON GAS The INDEX of refraction IS GIVEN by

$$N - 1 = \frac{n e^2}{m \omega^2} \int \text{distance to object}$$

SINCE The index IS frequency dependent AND SUBSEQUENTLY The Group VELOCITY, The WAVES would NOT TRAVEL AT The speed of LIGHT. If There ARE A free ELECTRONS between US, This could EXPLANE The VARIATION. NOW WE CAN ARGUE ALL DAY how MANY ELECTRONS There MAY be but IN The direction of The SOURCE IT SEEMS OKAY TO ASSUME 0.2 electrons per cubic CENTIMETER WHICH GIVES AS A DISTANCE To The OBJECT of 65 PAR SECS or 200 LIGHT YEARS.

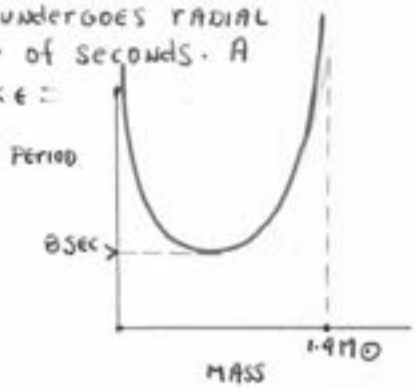
FROM The RATE of Drift of The SIGNAL IT IS POSSIBLE TO FIGURE OUT The TOTAL DELAY of The PIP. IT TURNS OUT TO TAKE About 8 Seconds for The SIGNAL TO SWEEP DOWN To 80-85 MC from INFINITE FREQUENCY.

HOW DO WE GET SUCH A SHARP PULSE IN 0.016 SECONDS. IT IS ALMOST CERTAIN The PULSE IS NO SO sharp but really MAde up of SHOCKS, NOISE, ELECTRIC AND MAGNETIC field DISTORTIONS TO GIVE A PULSE MORE LIKE:



Due TO The PULSE WIDTH of 0.016 Seconds WE WOULD EXPECT The SOURCE TO be ON The order of 5000 KM, i.e., The DISTANCE IT TAKES LIKE TO TRAVEL IN 0.016 SECONDS. This IS SMALLER ThAN The ordinary STAR AND PUTS IT IN The white dwarf RANGE.

NOW WHAT could hAVE SUCH AN ACCURATE PERIOD SO AS NOT TO VARY More ThAN 1 PART IN 10^7 OVER SIX MONTHS? Perhaps IT A CHARACTERISTIC ROTATION or VIBRATION of A White dwarf. IN white dwarfs The degenerate electron GAS supports The WEIGHT of The STAR And UNDERGOES RADIAL OSCILLATIONS. The LOWEST RADIAL MODE IS ON The Order of Seconds. A PLOT of PERIOD VERSUS MASS LOOKS SOMETHING LIKE:



The MINIMUM PERIOD TURNS OUT TO be About 8 SECONDS. The REASON The PERIOD RISES AGAIN IS ThAT The RESTORING force LOWERS AS The MASS INCREASES AND EVENTUAL GIVES IN. WE MUST GO TO A higher MODE THAN The SECOND TO PICK UP A 1.337279 PERIOD. Perhaps, Though SOME higher MODE IS BEING DRIVEN by SOME MOTION AND The lower MODES GET LOST. BUT There IS NO SIMPLE MODEL TO GIVE 1.337279 AS ITS NATURAL PERIOD. MAYbe IT IS SOME NON-RADIAL or QUADRIPOLE MODE WITH A high Q DUE TO STRONG GRAVITATIONAL WAVE GENERATION. BUT NOW The MUMBLING STARTS And The Theory GETS WORSE.

SO LETS TURN TO ANOTHER POSSIBLE SOURCE, THE NEUTRON STARS. NOW NO ONE HAS EVER OBSERVED A NEUTRON STAR. IT IS SUPPOSED THAT NEUTRONS EXIST AS A DEGENERATE. IT IS EASIER TO PACK ON OF THESE STARS BECAUSE IT TAKES LESS ENERGY. THE ENERGY IS GIVEN BY

$$E = \frac{P^2}{2M}$$

SINCE THE MASS OF A NEUTRON IS NEARLY 2000 TIMES THAT OF AN ELECTRON, THE SAME ENERGY CAN PACK THE MATERIAL TIGHTER. HOWEVER IT TURNS OUT THE FREQUENCY OF SUCH A STAR IS 10^{-3} - 10^{-2} SECONDS AND NOT LONG ENOUGH. TO GET THE RIGHT PERIOD THE STAR MUST BE MUCH LESS THAN THE SUN APPROXIMATELY $10^{-5}\ M_\odot$.

SUPPOSE WE CONSIDER A ~~NE~~ WHITE DWARF STAR ROUND OUR SUN AND WE ASSUME THE MASSES AND ANGULAR MOMENTUM ARE THE SAME

$$L_{SUN} = L_{W.D}$$

$$\frac{2}{5} m_\odot r_\odot^2 \omega_\odot = \frac{2}{5} M_{W.D}\, r_{W.D}^2\, \omega_{W.D}$$

ASSUME $r_\odot \sim 600{,}000$ KM $\quad r_{W.D} \sim 6000$ KM $\quad \omega_\odot = 27d \doteq 2.3\times10^5$ sec

THEN

$$\omega_{W.D} = 10^4\, \omega_\odot$$

RETURN TO ELECTRODYNAMICS

I'D LIKE TO RETURN TO ELECTRODYNAMICS AND SHOW YOU HOW MAXWELLS EQUATION CAN BE DERIVED FROM AN ACTION PRINCIPLE NOT IN TERMS OF ANY FIELDS. THE ACTION IS GIVEN BY

ACTION = SUM INDIVIDUAL ACTION + SUM INTERACTIONS

$$S = \sum_i \int m_i\, dS_i + \text{INTERACTION TERM}$$

TO KEEP RELATIVISTIC INVARIANCE I WILL WRITE THE COORDINATES OF AN EVENT, (t, x, y, z), IN TERM OF THE PARAMETER α SUCH THAT $t(\alpha)$, $x(\alpha)$, $y(\alpha)$, $z(\alpha)$. THE PATH IS THEN DESCRIBED BY A FOUR VECTOR $z_\mu(\alpha)$. THE PROPER TIME ALONG THE PATH IS GIVEN BY

$$ds^2 = dz_\mu\, dz_\mu$$

OR

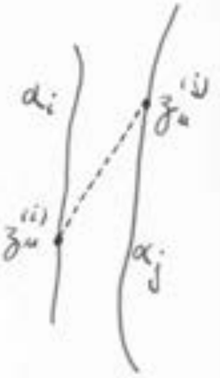
$$\left(\frac{ds}{d\alpha}\right)^2 = \left(\frac{dz_\mu}{d\alpha}\right)\left(\frac{dz_\mu}{d\alpha}\right)$$

WE CAN WRITE THE FIRST TERM AS

$$\sum_i \int m_i \sqrt{\frac{dz_\mu^{(i)}}{d\alpha}\,\frac{dz_\mu^{(i)}}{d\alpha}}\; d\alpha$$

THIS ACTION IS VARIED WHEN WE TAKE OUR MINIMUM.

NOW WE CONSIDER THE CASE OF TWO PARTICLES MOVING ON THEIR RESPECTIVE PATHS α_i AND α_j AND INTERACTING AT THE SAME TIME. THE INTERACTION TERM IS GIVEN BY



$$\sum_{\substack{i,j \\ \text{PAIRS} \\ \text{(NO SELF-ACTION)}}} e_i e_j \iint \delta\left\{\left[\left(z_\mu^j(\alpha^j) - z_\mu^i(\alpha^i)\right)\right]\left[z_\mu^i(\alpha^i) - z_\mu^j(\alpha^j)\right]\right\} \frac{dz_\nu^j}{d\alpha^j}\frac{dz_\nu^i}{d\alpha^i}\, d\alpha_i\, d\alpha_j$$

THE SQUARE DELTA FUNCTION IMPLIES THE PARTICLES NEVER INTERACT UNLESS THEY ARE A SPECIFIC DISTANCE AWAY GIVEN BY THE SQUARE OF THE PROPER DISTANCE $t^2 - x^2 - y^2 - z^2$. THUS THE TOTAL ACTION IS WRITTEN AS,

$$S = \sum_i \int m_i \sqrt{\frac{dz_\mu^{(i)}}{d\alpha}\frac{dz_\mu^{(i)}}{d\alpha}}\, d\alpha + \sum_{i,j} e_i e_j \iint \delta\left[(z_\mu^j - z_\mu^i)(z_\mu^i - z_\mu^j)\right] \frac{dz_\nu^i}{d\alpha^i}\frac{dz_\nu^j}{d\alpha^j}\, d\alpha_i\, d\alpha_j$$

JUST FOR FUN TO SEE IF WE CAN DEDUCE ANYTHING FROM THIS MESS, LET'S CONSIDER A CASE WHERE ONE PARTICLE IS AT REST AND THE OTHER IS VARIED. LET'S SAY THE ONE AT REST IS j AND DENOTE THE TWO TIMES AS $T = \alpha^j$, $t = \alpha^i$ WHERE WE KNOW $z_\mu^j = 0$ BUT $x_i(t), y_i(t), z_i(t)$ ARE PERMITTED TO VARY. THE ACTION BECOMES

$$S = m \int \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2} + ee' \iint \delta\left[(T-t)^2 - (x^2 - y^2 - z^2)^2\right] dt\, dT$$

THE INTERACTION INTEGRAL IS OF THE FORM

$$\int \delta\, f(x)\, dx = \sum_{\text{ROOTS}} \frac{1}{|f'(x_0)|} \quad \text{WHERE } f(x_0) = 0$$

THE WHOLE INTEGRAL REDUCES QUICKLY TO

$$\int \underline{ee'}\, \delta\left[(T-t)^2 + (x^2 - y^2 - z^2)\right] dt\, dT = \frac{ee'}{2(T-t)}$$

WE KNOW THE ONLY TIME THE INTEGRAL HAS A VALUE IS WHEN THE ARGUMENT IS ZERO OR WHEN

$$(T-t)^2 = (x^2 + y^2 + z^2) = r^2(t)$$

OR

$$(T - t) = \pm r(t) \quad \text{WHICH HAS 2 ROOTS}$$

WE CAN NOW PUT THE INTEGRATION OVER BIG T BACK IN AND GET

$$\int \frac{ee'}{r(t)}\, d\tau$$

WHICH IS JUST THE COULOMB POTENTIAL FOR TWO CHARGES AT REST. THE FIRST PART OF THE ACTION IS JUST THE KINETIC ENERGY.

Another case would be to vary the action for a change in one particle only, $i=0$ $\delta z^{(0)}_\mu(\alpha^0)$. This considers the change in 0 under the influences of all the other interactions. The action becomes

$$S_0 = m_0 \int \sqrt{\frac{dz^0_\mu}{d\alpha}\frac{dz^0_\mu}{d\alpha}}\, d\alpha + e_0 \int \frac{dz^0_\nu}{d\alpha^0}\, d\alpha^0 A_\nu(z^0_\mu(\alpha^0))$$

Where we have defined

$$A^0_\nu(z^0_\nu(\alpha)) = \sum_{j\neq 0} e_j \int \delta\left[z^j_\mu(\alpha^j) - z^0_\mu(\alpha^0)\right]^2 \frac{dz^j_\mu}{d\alpha^j}\, d\alpha^j$$

This is a wave function defined in space with half advanced and half retarded components. If you work out all the details you will get the set of equations

$$\frac{d}{dS}\left(m_0 \frac{dz_\mu}{dS}\right) = e_0 \frac{dz_\mu}{dS}\left(\frac{\partial A_\mu}{dx_\nu} - \frac{\partial A_\nu}{dx_\mu}\right)_{\text{at } x=z}$$

The derivatives enter in the expansion of the variation in a Taylor series.

What I have then is Maxwell's equations

$$\Box^2 A_\nu = e_j \int \delta^4(x - z(\alpha))\, \frac{dz_\nu}{d\alpha}\, d\alpha = j_\nu(\alpha)$$

$$= e_j \int \delta(t-t')\, \delta^3(\mathbf{x} - \mathbf{Z}(t')) \left\{ \begin{matrix} 1 \\ \dot{z} \\ \dot{y} \\ \dot{x} \end{matrix} \right\} dt'$$

My proof will be complete if I show that

$$\Box^2 \delta\left[(x_\mu - z_\mu)(x_\mu - z_\mu)\right] = \delta^4(x-z)$$

which I can do by consider the solution to the wave equation

$$\Box^2 \psi = S(t)\, \delta^3(x)$$

with the pulse at the origin. The solution is half advanced and half retarded

$$\psi = \int \frac{S(t-r) + S(t+r)}{2r}$$

with has as it Green's function

$$\psi = \frac{1}{2r}\left[\delta(t-r) + \delta(t+r)\right]$$

Dirac pointed out the right side is just the equivalent of

$$\delta(t^2 - r^2)$$

So we have proved the theorem.

For a good account of the development of Feynman's view of electrodynamics see Physics Today, August 1966 "The Development of the Space-Time View of Quantum Electrodynamics" by R. Feynman.

THE MAXWELL EQUATIONS

Ref: VOL II chapter 18, 21

I'LL START ONCE AGAIN BY WRITING DOWN ALL THE LAWS OF ELECTRODYNAMICS

$$\Box^2 \bar{A} = \dot{j}/\epsilon_0 \qquad \rightarrow \qquad \bar{A}(1,t) = \int \frac{\dot{j}(2, t - r_{12}/c)}{4\pi\epsilon_0 r_{12}} dV_2$$

$$\Box^2 \varphi = \rho/\epsilon_0 \qquad \rightarrow \qquad \varphi(1,t) = \int \frac{\rho(2, t - r_{12}/c)}{4\pi\epsilon_0 r_{12}} dV_2$$

$$\bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{f} = \rho\bar{E} + \dot{j} \times \bar{B}$$

$$\bar{B} = \nabla \times A$$

WITH THESE LAWS WE CAN GO IN MANY DIRECTIONS. FOR INSTANCE WE COULD STUDY PHENOMENA OCCURRING IN SHORT DISTANCES OR AS $c \to \infty$. THIS IMPLIES STUDYING QUASISTATIC FIELDS WHICH INVOLVES ELECTROSTATICS, MAGNETICS AND INDUCTION. WE COULD STUDY THE GENERAL CONSEQUENCES OF THESE LAWS WITHOUT GIVING SPECIFIC EXAMPLES. THIS MEANS WRITING MAXWELL'S EQUATIONS AND THE LAW OF ENERGY CONSERVATION. THE OTHER AREA OF INTEREST WOULD BE THE CASE OF LONG DISTANCES OR THE WAVE ZONES. HERE WE MUST CONCERN OURSELVES WITH DELAYS OF EFFECTS.

I THINK I'LL TALK ABOUT MAXWELL'S EQUATIONS BECAUSE WE HAVE DONE ELECTROSTATICS AND YOU WOULD GET BORED TO GO THROUGH MAGNETICS. BESIDES YOU ALL KNOW THE MATERIAL ANYWAY AND WE'RE JUST PLAYING GAMES. SO I MUST FINISH THE ABOVE SET OF EQUATIONS. IF I START WITH THE SOLUTIONS $\bar{A}(1,t)$, $\varphi(1,t)$ AND THE LAWS FOR $\bar{E}$, $\bar{B}$, AND $\bar{f}$, I CAN DEDUCE MAXWELL'S EQUATIONS. FIRST I MUST ADD THE ASSUMPTION THAT CHARGE IS CONSERVED. THIS CAN BE WRITTEN AS

$$\frac{\partial \rho}{\partial t} = -\bar{\nabla}\cdot\dot{j}$$

IF I HAVE A VOLUME OF CHARGE, THE TOTAL CHARGE IS GIVEN BY

$$Q = \int \rho \, dV$$

NOW THE RATE OF CHANGE OF THE CHARGE INSIDE, $\frac{dQ}{dt}$, IS EQUIVALENT TO THE FLOW OF CHARGE THRU THE SURFACE, I.E.,

$$\frac{dQ}{dt} = \frac{d}{dt}\int \rho \, dV = -\int \dot{j}\cdot\bar{n}\, dS$$

BUT BY THE PURELY MATHEMATICAL PROPOSITION GAUSS' LAW THE SURFACE CAN BE CHANGED TO A VOLUME INTEGRAL SO THAT

$$\int \frac{d\rho}{dt} dV = -\int \bar{\nabla}\cdot\dot{j}\, dV$$

IF WE ASSUME THE VOLUME IS FIXED.

Because we know what $\partial\rho/\partial t$ equals, let's calculate the time derivative of the scalar potential

$$\frac{\partial \phi}{\partial t} = \int \frac{\partial\rho/\partial t\,\big|_{t-\mathcal{R}_{12}/c}}{4\pi\epsilon_0 \mathcal{R}_{12}}\, dV_2 = -\int \frac{\nabla_2 \cdot \bar{j}(\mathbb{R}_2, t-\frac{\mathcal{R}_{12}}{c})}{4\pi\epsilon_0 \mathcal{R}_{12}}\, dV_2$$

I must be careful when I take the divergence of $\bar{j}$. The operator ∇_2 only works on $\mathbb{R}_2$ not on $\mathcal{R}_{12}$ which is a function of $\mathbb{R}_2$. I want $\partial\rho/\partial t$ at $t = \frac{\mathcal{R}_{12}}{c}$ which means I want

$$\left[\nabla_2 \cdot j(\mathbb{R}_2, t')\right]_{t' = t - \frac{\mathcal{R}_{12}}{c}}$$

Suppose I consider some idiot thing so that

$$\frac{\partial \varphi}{\partial t} = -\int \nabla_2 \cdot \left[\frac{j(\mathbb{R}_2, t-\frac{\mathcal{R}_{12}}{c})}{4\pi\epsilon_0 \mathcal{R}_{12}}\right] dV_2 - \int \nabla_{12} \cdot \frac{j(\mathbb{R}_2, t-\frac{\mathcal{R}_{12}}{c})}{\mathcal{R}_{12}}\, dV_2$$

The last operator only works on $\mathcal{R}_{12}$ and, therefore, compensates for the two terms I get from the first integral where I let ∇_2 work on the whole argument. If I change the first integral to a surface integral and let the integration go to infinity where there are no currents, the first term becomes zero. The second term can be expressed as

$$-\nabla_1 \cdot \int \frac{j(\mathbb{R}_2, t-\frac{\mathcal{R}_{12}}{c})}{4\pi\epsilon_0 \mathcal{R}_{12}}\, dV_2$$

so we have

$$\frac{\partial \varphi}{\partial t} = -\bar{\nabla}\cdot\bar{A} \qquad \left[\text{Actually } \bar{\nabla}\cdot\bar{A} = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right]$$

This is called a gauge transformation and relates φ and $\bar{A}$.

Now recall that $\bar{B} = \bar{\nabla}\times\bar{A}$ and we can express $\bar{B}$ in another way. Compute the curl of B, i.e.,

$$\bar{\nabla}\times\bar{B} = \bar{\nabla}\times(\bar{\nabla}\times\bar{A})$$

By mathematical identities

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = -\nabla^2\bar{A} + \bar{\nabla}(\bar{\nabla}\cdot\bar{A})$$

This comes from the relation

$$\bar{a}\times(\bar{b}\times\bar{c}) = \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b})$$

Now we notice from our starting laws

$$\frac{\partial^2\bar{A}}{\partial t^2} - \nabla^2\bar{A} = \bar{j}/\epsilon_0$$

So we have

$$\begin{aligned}\nabla\times B &= j/\epsilon_0 - \frac{\partial^2\bar{A}}{\partial t^2} - \nabla\left(\frac{\partial\varphi}{\partial t}\right)\\ &= j/\epsilon_0 + \partial/\partial t\left[-\frac{\partial\bar{A}}{\partial t} - \nabla\varphi\right]\end{aligned}$$

BUT $\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi$ SO WE HAVE

$$\bar{\nabla}\times\bar{B} - \frac{\partial \bar{E}}{\partial t} = \dot{J}/\epsilon_0$$

NOW LET'S CALCULATE $\bar{\nabla}\cdot\bar{E}$

$$\bar{\nabla}\cdot\bar{E} = \nabla\cdot\left(-\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi\right) = -\frac{\partial\, \nabla\cdot A}{\partial t} - \nabla^2\varphi$$

$$= \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2\varphi \qquad \text{SINCE } -\nabla\cdot A = \frac{\partial \varphi}{\partial t}$$

Therefore

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$$

THIS IS JUST ELECTROSTATICS.

SINCE $\nabla\cdot(\nabla\times A) \equiv 0$ WE FIND $\bar{\nabla}\cdot\bar{B} = 0$ AS THE THIRD MAXWELL EQUATION. THE LAST IS FOUND BY TAKING THE CURL OF $\bar{E}$,

$$\bar{\nabla}\times\bar{E} = -\bar{\nabla}\times(\bar{\nabla}\varphi) - \frac{\partial}{\partial t}\nabla\times A$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

SINCE $\nabla\times(\nabla\varphi) = 0$, THE X COMPONENT FOR EXAMPLE IS

$$\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}$$

THE COMPLETE SET OF MAXWELL'S EQUATIONS IS WRITTEN AS

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0 \qquad \bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t} \qquad c^2\bar{\nabla}\times\bar{B} = \dot{J}/\epsilon_0 + \frac{\partial\bar{E}}{\partial t}$$

THESE FOUR EQUATIONS PLUS THE CONSERVATION OF CHARGE RULE COMPLETELY DESCRIBE THE ELECTROMAGNETIC FIELD PHENOMENA. THE CONTRIBUTION MAXWELL MADE TO E&M WAS THE ADDITION OF THE TERM $\partial E/\partial t$ WHICH HE NEEDED TO AVOID AN INCONSISTENCE IN THE SET OF EQUATIONS. WITH THIS TERM HE WAS ABLE TO EXPLAIN A WHOLE CLASS OF PHENOMENA.

TO SHOW HOW THIS NEW TERM WORKS SUPPOSE WE ARE ADDING CHARGE TO A SPHERE OF JELLO AND IT IS LEAKING RADIALLY OUTWARD SO FROM

$$\bar{\nabla}\cdot\dot{j} = -\frac{\partial\rho}{\partial t}$$

WE HAVE

$$\frac{\partial Q(r)}{\partial t} = -4\pi r^2 j(r)$$

SINCE $\bar{B}$ CAN'T HAVE ANY PARTICULAR DIRECTION AROUND A CLOSED LOOP ON A SPHERE $\nabla\times B = 0$. BUT $E = Q(r)/4\pi\epsilon_0 r^2$ SO THAT

$$\frac{\partial Q}{\partial t} = 4\pi\epsilon_0 r^2 \frac{\partial E}{\partial t}$$

THEN

$$\frac{\partial\bar{E}}{\partial t} = -\dot{J}/\epsilon_0$$

. THE TWO SOURCES CANCEL AND $\bar{\nabla}\times\bar{B} = 0$.

FIELD ENERGY

Now I'd like to try to find out about the conservation of energy. I can get somewhere in this study if I suppose that energy in space has a certain density. We will let U equal the or represent the energy density in the field, i.e., the amount of energy per unit volume. Let the vector $\bar{S}$ represent the energy flux of the field (i.e., the flow of energy per unit time across a unit area perpendicular to the flow).

$$\bar{S} = \text{POYNTING VECTOR} = \frac{\text{flow of energy}}{cm^2\, sec}$$

Similar to the conservation of charge, we can write the "local" law of energy conservation in the field

$$-\frac{d}{dt}(\text{Energy inside vol}) = \underset{\text{surface}}{\text{flow of energy thru}} + \underset{\text{inside } V}{\text{work done on matter}}$$

Since power = rate of doing work = $\bar{j}\cdot\bar{E}$, the quantity $\bar{E}\cdot\bar{j}$ must equal the loss of energy per unit time and per unit volume. The above law can be written as

$$-\frac{d}{dt}\int U\,dV = \int \bar{S}\cdot\bar{N}\,d\text{Surf} + \int \bar{E}\cdot\bar{j}\,dV$$
$$= \int \nabla\cdot S\,dV + \int E\cdot j\,dV$$

The energy equation for the E-M fields becomes

$$-\frac{\partial U}{\partial t} = \bar{\nabla}\cdot\bar{S} + \bar{E}\cdot\bar{j}$$

OR

$$\bar{E}\cdot\bar{j} = -\frac{\partial U}{\partial t} - \bar{\nabla}\cdot\bar{S}$$

From this equation we can discover the form for U & S; so we will work this out.

Recall from Maxwell's equations

$$\bar{j} = \epsilon_0\,\bar{\nabla}\times\bar{B} - \epsilon_0\frac{\partial\bar{E}}{\partial t}$$

so we have

$$-\frac{\partial U}{\partial t} - \bar{\nabla}\cdot\bar{S} = \epsilon_0(\bar{\nabla}\times\bar{B})\cdot\bar{E} - \epsilon_0\,\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}$$

It is easy to rewrite the last term as $\partial/\partial t\,(\frac{1}{2}\epsilon_0\,\bar{E}\cdot\bar{E})$ so $\frac{1}{2}\epsilon_0\,\bar{E}\cdot\bar{E}$ is at least one part of U. Now it take a little mathematical gymnastics to get the first term into some form of a divergence. We know $(\bar{\nabla}\times\bar{B})\cdot\bar{E} = \bar{E}\cdot(\bar{\nabla}\times\bar{B})$ and also that $(\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}\cdot(\bar{b}\times\bar{c})$ so we have

$$\bar{\nabla}\cdot(\bar{B}\times\bar{E})$$

This looks like the answer but its wrong. $\bar{\nabla}$ operates on both $\bar{B}$ and $\bar{A}$ where initially it only worked on $\bar{B}$

If we define the operator ∇_B as the gradient only working on $\bar{B}$, we can rewrite the divergences as

$$\bar{\nabla}\cdot(\bar{B}\times\bar{E}) = \bar{\nabla}_B\cdot(\bar{B}\times\bar{E}) + \bar{\nabla}_E\cdot(\bar{B}\times\bar{E})$$

Going back we have $\bar{E}\cdot\bar{\nabla}_B\times\bar{B}$, since $\bar{a}\cdot\bar{b}\times\bar{c} = \bar{b}\cdot\bar{c}\times\bar{a}$ and the last term is equal to $\bar{B}\cdot\bar{E}\times\nabla_E$. To get the operator in front of its argument, $\bar{B}\cdot(\bar{E}\times\bar{\nabla}_E) = -\bar{B}\cdot(\bar{\nabla}_E\times\bar{E})$

Thus we have

$$\bar{\nabla}\cdot(\bar{B}\times\bar{E}) = \bar{E}\cdot(\bar{\nabla}\times\bar{B}) - \bar{B}\cdot(\bar{\nabla}\times\bar{E})$$

Returning to our starting equation

$$\bar{E}\cdot\bar{j} = \epsilon_0\,\bar{\nabla}\cdot(\bar{B}\times\bar{E}) + \epsilon_0\,\bar{B}\cdot(\nabla\times\bar{E}) - \frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon_0\,\bar{E}\cdot\bar{E}\right)$$

Remember from Maxwell's equations $\nabla\times E = -\partial B/\partial t$

$$\bar{E}\cdot\bar{j} = c^2\epsilon_0\,\nabla\cdot(B\times E) - \frac{\partial}{\partial t}\left(\frac{c^2}{2}\epsilon_0\,\bar{B}\cdot\bar{B} + \tfrac{1}{2}\epsilon_0\,\bar{E}\cdot\bar{E}\right)$$

Finally identifying terms

$$U = \frac{\epsilon_0}{2}\,\bar{E}\cdot\bar{E} + \frac{\epsilon_0 c^2}{2}\,\bar{B}\cdot\bar{B}$$

$$\bar{S} = \epsilon_0 c^2\,\bar{E}\times\bar{B}$$

(See the reference for some examples of how these formulas work out).

MAXWELL STRESS TENSOR

LAST TIME WE DEVELOPED FROM MAXWELL'S LAWS OF ELECTROMAGNETICS: $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$, $\nabla \times E = -\frac{\partial B}{\partial t}$, $\bar{\nabla}\cdot\bar{B} = 0$, $\bar{\nabla}\times\bar{B} = \frac{\partial \bar{E}}{\partial t} + j/\epsilon_0$ PLUS $\bar{F} = \rho\bar{E} + \bar{j}\times B$ TO EQUATIONS DESCRIBING THE ENERGY DENSITY IN SPACE AND THE RATE OF FLOW OF ENERGY AROUND IN SPACE. THE TWO EQUATIONS ARE:

$$u = \frac{\epsilon_0}{2}(\bar{E}\cdot\bar{E} + c^2\bar{B}\cdot\bar{B})$$

$$\bar{S} = \epsilon_0 c^2\, \bar{E}\times\bar{B}$$

BECAUSE WE KNOW MASS AND ENERGY ARE ONE IN THE SAME THING THE RATE OF CHANGE OF ENERGY OF A SYSTEM OF PARTICLE IS ANALOGOUS TO A RATE OF CHANGE OF MOMENTUM OF THE SYSTEM. CONSIDER FOR EXAMPLE A BOX OF PARTICLES EACH WITH MASS, m, VELOCITY v AND TOTAL NUMBER OF n, THE MASS FLOWING OUT OF THE BOX PER SEC WILL BE nmv WHERE m IS THE KINETIC MASS NOT THE REST MASS. THIS IS THE SAME AS THE MOMENTUM DENSITY mvn. WE CAN NOW MAKE A THEOREM THAT THE ENERGY FLOW, SAY IN THE X DIRECTION, PER SEC IS c^2 TIMES THE MOMENTUM DENSITY PER UNIT VOLUME

$$\frac{du}{dt} = Nv\,mc^2 = N\frac{m_0}{\sqrt{1-v^2/c^2}}\,v\,c^2$$

IT IS SOMEWHAT AWKWARD TO WORK WITH THE LAWS OF CONSERVATION OF ENERGY AND ANGULAR MOMENTUM SO WE INVENT A NEW THING. WE WANT TO BE ABLE TO DESCRIBE THE AMOUNT OF MOMENTUM IN THE X DIRECTION WHICH FLOWS IN THE X DIRECTION THRU A SURFACE PERPENDICULAR TO X. WE CAN CALL THIS QUANTITY T_{xx}. IF THE X COMPONENT OF MOMENTUM FLOWS IN THE Y OR Z DIRECTIONS WE LIKEWISE HAVE T_{xy} AND T_{xz}. IN OTHER WORDS, EACH TERM T_{ij} REPRESENTS THE FLOW OF THE i-TH COMPONENT OF MOMENTUM THROUGH A UNIT AREA PERPENDICULAR TO THE j DIRECTION.

THIS IS JUST LIKE A STRESS, A FORCE PER UNIT AREA. SINCE FORCE IS JUST THE TIME RATE RATE OF CHANGE OF THE MOMENTUM WE BEGIN TO SEE THE CONNECTION BETWEEN MECHANICS AND ~~AN~~ ELECTROMAGNETICS. SINCE STRESS IS SPECIFIED BY NINE NUMBERS IT IS A TENSOR OF THE SECOND RANK. THUS WE ASSUME THE TERM T_{ij} MUST REALLY BE A 4 DIMENSIONAL TENSOR $T_{\mu\nu}$ WHERE μ AND ν ARE t, x, y, z. BECAUSE WE ALSO KNOW THE TIME COMPONENT OF THE FORCE IS IDENTIFIED WITH ENERGY, WE CALL $T_{\mu\nu}$ THE STRESS-ENERGY TENSOR.

IN $T_{\mu\nu}$ THEN ν TELLS US WHAT FLOWS, NAMELY ENERGY (t), MOMENTUM IN THE X, Y, Z - DIRECTIONS. μ THEN IDENTIFIES DENSITY AND FLOW IN THE X, Y, Z - DIRECTION. SINCE μ AND ν CAN BE t, X, Y, OR Z SEPARATELY WE SHOULD IDENTIFY SOME TERMS.

$$T_{tx} = X \text{ flow of energy} = \epsilon_0 c^2 (E \times B)_x$$
$$T_{ty} = Y \text{ " " "} = \epsilon_0 c^2 (E \times B)_y$$
$$T_{tz} = Z \text{ " " "} = \epsilon_0 c^2 (E \times B)_z$$

$$T_{xt} = \text{DENSITY of X MOMENTUM}$$
$$T_{yt} = \text{" " Y "}$$
$$T_{zt} = \text{" " Z "}$$

$$T_{tt} = \text{ENERGY DENSITY} = \epsilon_0/2 \left(\bar{E}\cdot\bar{E} + c^2 \bar{B}\cdot\bar{B} \right)$$

$$T_{xx} = \text{X FLOW of X MOMENTUM}$$
$$T_{yy} = \text{Y " " Y "}$$
$$T_{zz} = \text{Z " " Z "}$$

$$T_{xy} = \text{Y flow of X MOMENTUM} \qquad \text{ETC.}$$
$$T_{xz} = \text{Z flow of X "}$$

IN MATRIX NOTATION

$$\begin{bmatrix} T_{xt} & T_{xx} & T_{xy} & T_{xz} \\ T_{yt} & T_{yx} & T_{yy} & T_{yz} \\ T_{zt} & T_{zx} & T_{zy} & T_{zz} \\ T_{tt} & T_{tx} & T_{ty} & T_{tz} \end{bmatrix} = T_{\mu\nu}$$

NOW THAT WE HAVE ESTABLISHED THE BASIC FORM OF THE MAXWELL STRESS TENSOR LET'S SEE IF WE CAN GO BACK AND UNDERSTAND IT. FOR INSTANCE, WHAT IS T_{xx}? IS IT THE STRESSES IN THE FIELD? IF SO LET'S SEE IF WE CAN FIND AN EXPRESSION FOR IT IN TERMS OF THE ELECTRIC AND MAGNETIC FIELDS. IN FACT, LET'S LOOK FOR THE GENERAL EXPRESSIONS FOR $T_{\mu\nu}$ IN TERMS OF $\bar{E}$ AND $\bar{B}$.

IN A UNIT VOLUME OF SPACE WE CAN GUESS AT SOME RELATION LIKE

$$\frac{d}{dt}\int (\text{MOMENTUM DENSITY})\, d\,\text{Vol} = \int (\text{SOME OUTFLOW})\, d\,\text{surface} + \sum \text{MOMENTUM GIVEN TO MATTER PER SEC.}$$

THE LAST TERM IS JUST THE FORCE ON THE MATTER INSIDE THE VOLUME AND CAN BE EXPRESSED AS $\int F\, d\,\text{VOL}$

Now the second term is some surface integral like the one we had when charge was flowing outward, namely

$$\int \dot{j} \cdot \bar{n} \; d\,\text{surf}$$

Since $\dot{j}$ is the current density, we might make an intelligent guess and substitute for it $\bar{\bar{T}}$, the three dimensional component of the 4-vector so that we have

$$\int \bar{T} \cdot \bar{n} \; d\,\text{surf}$$

But this expression is very confusing because we want the momentum flowing in the x direction only, say, to be dotted with the unit normal in the three directions. In other words ~~be~~ we must be more explicit and write something like

$$T_{x()} \cdot \bar{n} = T_{xx} n_x + T_{xy} n_y + T_{xz} n_z$$

What I mean by this crazy thing is the dot ~~goes~~ goes with the second index. The first index tells us what is flowing.

Let's go on and consider what we have for the x direction only. Since T_{xt} is the momentum density in the x-direction, we can write our little conservation law as,

$$\frac{d}{dt} \int T_{xt} \, d\,\text{Vol} = \int \bar{T}_{x()} \cdot \bar{n} \; d\,\text{Surf} + \int f_x \, d\,\text{Vol}$$

I would like to change the surface integral to a volume integral but again we have to be careful using Gauss' theorem,

$$\int \bar{T}_{x()} \cdot \bar{n} \; d\,\text{surf} = \int \bar{\nabla} \cdot \bar{T}_{x()} \, d\,\text{Vol}$$

where the divergence is

$$\bar{\nabla} \cdot \bar{T}_{x()} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}$$

Because I am considering a unit volume which is not changing in tim we can write the following differential equation,

$$\frac{\partial}{\partial t} T_{xt} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} + f_x$$

What I have to do now is get this equation into an expression with $\bar{E}$ and $\bar{B}$. To do that I start by substituting for f_x the term $\rho E_x + (\dot{j} \times \bar{B})_x$ since $\bar{f} = \rho \bar{E} + \dot{j} \times \bar{B}$. Since I know Maxwell's equations I can substitute for ρ and j, i.e.,

$$\rho = \epsilon_0 \bar{\nabla} \cdot \bar{E}$$
$$\dot{j} = \epsilon_0 \bar{\nabla} \times \bar{B} - \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

REARRANGING I HAVE

$$\frac{\partial T_{xt}}{\partial t} + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = \rho E_x + (j_y B_z - j_z B_y)$$

NOW I MUST EXPAND OUT THE LAST TERM,

$$\rho E_x + (j_y B_z - j_z B_y) = (\bar{\nabla}\cdot\bar{E})E_x + \left[(\nabla \times B)_y - \frac{\partial E_y}{\partial t}\right]B_z - \left[(\nabla\times B)_z - \frac{\partial E_z}{\partial t}\right]B_y$$

I HAVE TO START UNRAVELING THIS MESS AND I'LL START WITH THE TIME DERIVATIVES

$$\text{Rhside} = \frac{\partial E_x}{\partial x}E_x + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z}E_z - \frac{\partial}{\partial t}\left[E_y B_z - E_z B_y\right] + \left(E_y\frac{\partial B_z}{\partial t} - E_z\frac{\partial B_y}{\partial t}\right) + \bar{B}\text{ JUNK}$$

SINCE $\frac{\partial B}{\partial t} = -\bar{\nabla}\times\bar{E}$, WE HAVE $\frac{\partial B_z}{\partial t} = -(\bar{\nabla}\times\bar{E})_z$ AND $\frac{\partial B_y}{\partial t} = -(\bar{\nabla}\times\bar{E})_y$.

$$\text{Rhside} = \frac{\partial E_x}{\partial x}E_x + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z} - \frac{\partial}{\partial t}\left[E_y B_z - E_z B_y\right] + E_y\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) +$$
$$E_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)B_z + \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)B_y$$

IF I ADD THE TERM $B_x(\bar{\nabla}\cdot\bar{B}) = 0$ TO THIS EQUATION I FIND SOME NICE SYMMETRY FROM WHICH I CAN WRITE ALL THE E TERMS AS

$$\frac{\partial E_x}{\partial x}E_x + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z}E_x - E_y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + E_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

THE $\bar{B}$ TERM IS JUST LIKE THIS AND, THEREFORE, I WON'T WRITE IT OUT. BUT WE CAN WRITE THIS EXPRESSION IN A DIFFERENT FORM YET BY BREAKING IT UP INTO THE X, Y, Z, COMPONENTS,

$$\frac{\partial}{\partial x}\left(\tfrac{1}{2}E_x^2 - \tfrac{1}{2}E_y^2 - \tfrac{1}{2}E_z^2\right)$$
$$\frac{\partial}{\partial y}\left(E_y E_x\right)$$
$$\frac{\partial}{\partial z}\left(E_z E_x\right)$$

I'M OFF BY A NEGATIVE SIGN SOMEWHERE BUT I CAN FINALLY MAKE THE IDENTIFICATION:

$$T_{xx} = -\frac{\epsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right)$$
$$T_{xy} = -\epsilon_0 E_x E_y$$
$$T_{xz} = -\epsilon_0 E_{xz}$$

THIS IS NOT QUITE THE FINISHED ANSWER BECAUSE I DO HAVE THE B-TERMS WHICH I FORGOT EARLIER. IF I PUT THEM IN, I GET,

$$T_{xx} = -\frac{\epsilon_0}{2}\left[E_x^2 - E_y^2 - E_z^2 + c^2\left(B_x^2 - B_y^2 - B_z^2\right)\right]$$

$$T_{xy} = -\epsilon_0 E_x E_y + B_x B_y$$

$$T_{xz} = -\epsilon_0 E_x E_z + B_x B_z$$

The form of EQUATION I have just developed can be written in the following short form

$$T_{ij} = -\epsilon_0\left(\bar{E}_i E_j + B_i B_j\right) + \frac{\epsilon_0}{2}\delta_{ij}\left(\bar{E}\cdot\bar{E} + \bar{B}\cdot\bar{B}\right)$$

IN 4-VECTOR NOTATION I CAN WRITE

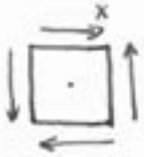
$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = f_{\mu\nu}$$

THIS TENSOR INVARIANT CAN be written IN INVARIANT form using $\bar{E}$ AND $\bar{B}$.

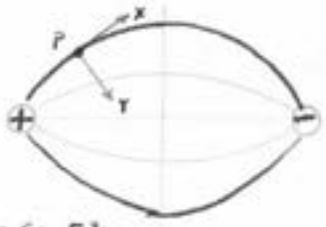
The MAXWELL STRESS TENSOR $T_{\mu\nu}$ IS A SYMMETRIC TENSOR, I.E.,

$$T_{\mu\nu} = T_{\nu\mu}$$

Briefly, I CAN show This is true. If I have AN INFINITESIMAL AREA with A force ACTING IN The + direction. This force CREATES A TORQUE About The SMALL MOMENT ARM AND The SQUARE STARTS TO TURN. BUT IT DOESN'T. There is AND EQUAL AND OPPOSITE force ACTING IN The Y DIRECTION TO PREVENT This MOTATION AND Thus PREVENT ANGULAR MOMENTUM from BEING INCREASED. TO PREVENT DISTORTION of The AREA WE NEED ALL four force TO MAINTAIN STATIC EQUILIBRIUM.



LET'S LOOK AT A SIMPLE SITUATION where Two CHARGES ARE CONNECTED BY The LINES of force:



SUPPOSE WE WANTED The force ON A PARTICLE AT POINT P, DUE TO The ROTATIONAL INVARIANCE of The TENSOR WE CAN POSITION our AXIS SO THAT $E_x, E_y = 0$ AND $E_z = E$

Then $T_{xx} = -\frac{\epsilon_0}{2}E^2 \quad T_{yy} = +\frac{\epsilon_0}{2}E^2 \quad T_{zz} = \frac{\epsilon_0}{2}E^2$

BY INTEGRATING The STRESSES over The AREA WE CAN find The forces. WE SEE, however, There IS A TENSION IN The direction PERPENDICULAR TO The field. IN The Y DIRECTION There IS A COMPRESSION EQUAL but OPPOSITE TO The TENSION. Thus field LINES repel.

If WE CONSIDER A MAGNETIC field ACTING WITH The ELECTRIC field, WE GET ALL of ELECTROMAGNETIC Theory.

I have been sloppy with my dimensional analysis and have the equations pretty messed up. of course my speed of light has been assumed to be unity for us and I want to go back and put them in correctly. To do that let me show you some easy mental ways to get things straight.

The energy is given in terms of

$$U = \epsilon_0 E^2$$

Since I know $E = \frac{F}{Q}$ I find $U = \epsilon_0 E^2 = \epsilon_0 \frac{F^2}{Q_2}$. But from coulomb's law $F = \frac{Q^2}{\epsilon_0 R^2}$ so I find

$$U \propto \frac{F}{R^2} = \frac{\text{force}}{\text{Area}} = \text{TENSION}$$

The rate of flow of energy per area per sec is given in units of

$$\frac{\text{ENERGY}}{\text{area sec}} = \frac{FR}{R^2 T} = \frac{F}{RT}$$

The Field Momentum of a moving charge ref. Vol II Chap 28

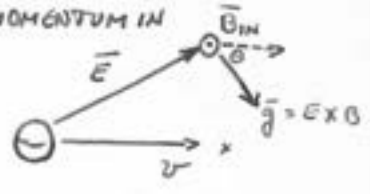
Suppose we consider a small charge q, radius a, moving along a straight line with velocity v. If the charge is all located on the surface and the velocity is zero, the energy in the field is just

$$T_{tt} = \frac{\epsilon_0}{2} E^2 = \frac{q^2}{32\pi^2 \epsilon_0 r^4}$$

To find the total energy over all space

$$U = \int T_{tt}\, 4\pi r^2 dr = \int_a^\infty \frac{q^2}{8\pi \epsilon_0 r^2} dr = \frac{1}{8} \frac{q^2}{4\pi\epsilon_0 a}$$

Now if the charge is moving we have associated with the charge a magnetic field $\bar{B} = \frac{\bar{v}}{c^2} \times \bar{E}$. This changes the energy density. To find how much a change we experience let's calculate the momentum in the field to first order, i.e.,



$$\bar{g} = \epsilon_0 \bar{E} \times B$$

Since $E = \frac{Q}{4\pi\epsilon_0 r^2}$ the magnitude of $\bar{B}$ is

$$|B| = \frac{Q}{4\pi\epsilon_0 r^2} \frac{V}{c^2} \sin\theta$$

We want the motion in the x direction so we need another $\sin\theta$

$$g_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \frac{V}{c^2} \sin^2\theta$$

The total momentum is

$$P = \int_0^\pi \int_a^\infty \frac{Q^2}{16\pi^2 \epsilon_0 r^4} \frac{V}{c^2} \sin^2\theta \; 2\pi r^2 \sin\theta \, d\theta \, dr = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0} \frac{V}{ac^2}$$

The electromagnetic mass of the system is given by

$$m_{el.} = \frac{2}{3} \frac{Q^2}{4\pi\epsilon_0 a c^2}$$

According to relativity we know mass and energy are one in the same thing.

$$U = m'c^2$$

where $m' = m'_{el}$. Since we already found U

$$m'_{elec} = \frac{U}{c^2} = \frac{1}{8} \frac{q^2}{\pi\epsilon_0 a c^2}$$

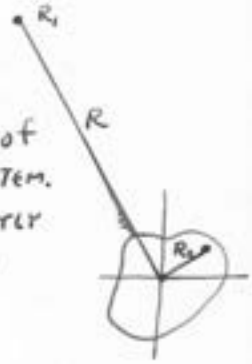
We find the two masses are not the same and we find our equations don't work. But we have forgotten something - the "something holding the charge together. We have assumed the repulsive forces in the charge are just balanced by some internal mechanism holding it all together. The nonelectrical forces are called Poincare stresses. When we add these stresses to the electromagnetic mass, relativity becomes consistent again.

THE POTENTIALS FROM A MOVING CHARGE

LET'S EXAMINE NOW THE FIELDS PRODUCED FROM MOVING CHARGES AND TO DO THAT WE MUST FIND THE VECTOR AND SCALAR POTENTIALS. IN PARTICULAR I WANT TO FIND THE POTENTIALS VERY FAR FROM THE SOURCE WHERE I WILL IGNORE TERMS IN $1/R^2$ AND ONLY KEEP THOSE THAT GO AS ONE OVER THE DISTANCE. THE VECTOR AND SCALAR POTENTIALS ARE GIVEN BY

$$\bar{A}(\bar{R}_1,t_1) = \int \frac{\vec{j}(\bar{R}_2, t - r_{12}/c)}{4\pi\epsilon_0 r_{12}}\, dV_2$$

$$\varphi(\bar{R}_1,t) = \int \frac{\rho(R_2, t - r_{12}/c)}{4\pi\epsilon_0 r_{12}}\, dV_2$$



SUPPOSE IN SOME LIMITED REGION OF SPACE THERE IS A SMALL BLOB OF CHARGE. IN THE CHARGE AGGREGATE WE ESTABLISH A COORDINATE SYSTEM. THE SOURCE INTEGRATION IS OVER THE RANGE OF R_2. NOW r_{12} IS NEARLY CONSTANT AND EQUAL TO R. HOWEVER, THERE ARE SMALL DELAYS WHICH WE HAVE IGNORED. THEN $\bar{A}$ COMES OUT VERY NEARLY

$$\bar{A} \sim \frac{1}{4\pi\epsilon_0 R}\left\{\int \vec{j}\, d\,\text{Vol}\right\}_{t-R/c}$$

THE INTEGRAL BEING THE TOTAL CURRENT.

IF I ASSUME NOW THE CHARGE IS AN ELECTRON MOVING ALONG WITH SOME VELOCITY v THEN

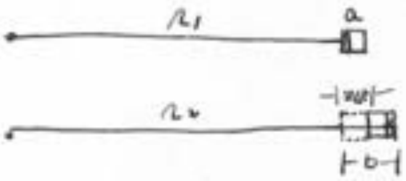
$$\int \vec{j}\, d\text{Vol} = \int \rho v\, d\text{Vol} = QV$$

SO IF THE ELECTRONS AREN'T GOING TOO FAST AND WE ARE FAR ENOUGH AWAY FROM THEM,

$$A \sim \frac{QV}{4\pi\epsilon_0 R} \quad \text{AND} \quad \phi \sim \frac{Q}{4\pi\epsilon_0 R}$$

LET'S BE MORE SPECIFIC AND SEE WHAT APPROXIMATION WE HAVE MADE AND WHAT THE CORRECT ANSWER SHOULD BE. TO GET A FEEL FOR WHAT WE MISSED CONSIDER A SINGLE CUBICAL CHARGE OR ELECTRON MOVING ALONG THE X AXIS. IN FINDING THE POTENTIALS WE MUST WORRY ABOUT $t - r_{12}/c$. SUPPOSE THE CUBE HAS A WIDTH "a" WHICH IS MUCH LESS THAN THE DISTANCE r_{12}. THE TOTAL CHARGE CONTAINED IN THIS CUBE IS $Q = \rho a^3$.

AT A PARTICULAR INSTANT WE DETERMINE THE FIELD DUE TO THIS MOVING CHARGE. BECAUSE THE CONTRIBUTION FROM THE BACK OF THE CHARGE IS ~~diffe~~ DELAYED SLIGHT FROM THAT DUE TO THE FRONT VOLUME ELEMENT WE MUST FIND THE DELAYED EFFECT.



THE TIME DELAY τ CORRESPONDS TO THE TIME DIFFERENCE THAT A SIGNAL EMITTED FROM THE BACK FAI TO THE TIME THE FROM ELEMENT EMITS. "b" IS THE LENGTH OF THE CUBE OF CHARGE INCREASED BY THE DISTANCE MOVED DURING τ, I.E.,

$$b = c\tau$$

But during t the object moves a distance

$$b - a = v t$$

The effects coming from the charge are not integrated over a but rather b so $Q \neq \rho a^3$. Instead we have $Q = \rho a^3 b$ or

$$Q \frac{b}{a}$$

It is easy to show $\frac{b}{a} = \frac{c}{c-v} = \frac{1}{1-v/c}$ so that

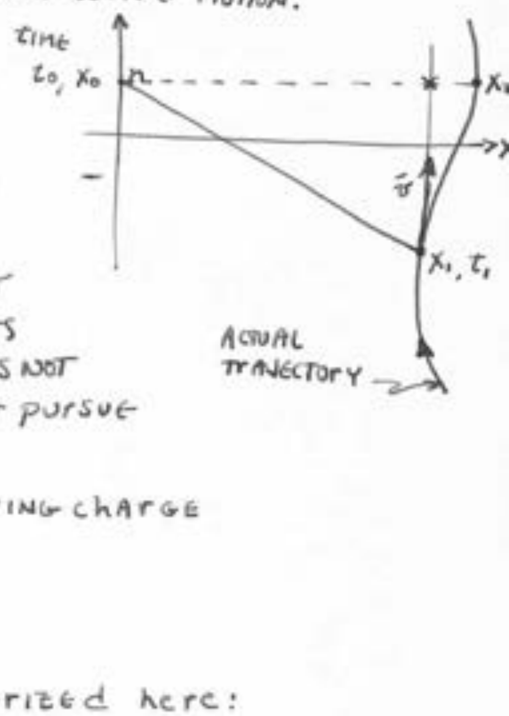
$$\bar{A} = \frac{Q}{4\pi\epsilon_0 R} \frac{\bar{v}}{1-v/c} = \frac{Q\bar{v}}{4\pi\epsilon_0 R} \frac{1}{\left(r - \frac{\bar{v}\cdot\bar{r}}{c}\right)_{ret}}$$

$$\phi = \frac{Q}{4\pi\epsilon_0 R} \frac{1}{[1 - v/c]_{ret.}}$$

If the velocity of the charge is not directed toward the observer, it is only the component in the observer's direction that counts. Also, the analysis does not depend on the shape of the object; this is true because the final answer does not depend on "a"!

The above potentials for a point charge are called the Lienard-Wiechert potentials.

A remark here: these potentials do not depend on anything else than the velocity and position of the particle in the past. Suppose we want to find the potentials do to some weird motion. If we are observer at some time t_0 and at a place x_0, the light we receive must have originated at some time in the past t_1. At that time, t_1, it appears the charge is moving in a straight line with velocity, $\bar{v}$. The potentials are the same if the charge follows it real path or if it moves in a straight line. If the particle has a straight line motion, we should be able to predict its position at time $= t_0$. Since the particle is not there but at x_0, we know the particle didn't pursue the straight line motion.



To find the fields $\bar{E}$ and $\bar{B}$ due to the moving charge we must determine

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$$\bar{E} = -\nabla\varphi - \frac{\partial \bar{A}}{\partial t}$$

The math is long and the results are summarized here:

The electric field is given by

$$\bar{E} = \frac{-q}{4\pi\epsilon_0}\left[\frac{e_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\bar{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\bar{e}_{r'} \right]$$

And the magnetic field is

$$\bar{B} = -\bar{e}_{r'} \times \bar{E}/c$$

The first term in the electric field is just the Coulomb law. $\bar{e}_{r'}$ is the unit vector in the direction of the observer the prime denotes how far it was when the effect was radiated. The Coulomb's law is not correct since we do not know the instantaneous position of the source. The retarded time r'/c tell's us how far in the past motions will effect the charge now.

The second term, $\frac{r'}{c}\frac{d}{dt}\left(\frac{\bar{e}_{r'}}{r'^2}\right)$ is like nature is trying to correct for the wrong Coulomb configuration. Nature guesses what the right field should be and corrects nearly 100%. The second derivative become important for long distance effects.

The first two terms both vary as $1/R^2$ for large distances. The third term is important for long distance since it does not fall off as $1/R^2$, this terms depends only on the acceleration of the unit vector to the observer. At long distances this term goes as $1/R$ and gives the law of radiation

If we consider the charge is undergoing very small motions, say up and down. The lateral displacement is $X(t)$. The angle that the unit vector $\bar{e}_{r'}$ is displaced is X/r. Since R is nearly constant, the x component of $\ddot{e}_{r'}$ is just

$$E = \frac{-q\ddot{X}}{4\pi\epsilon_0 r} = \frac{-q a_x}{4\pi\epsilon_0 r}\Big|_{\text{ret time}}$$

a_x is the component of the acceleration perpendicular to the line of sight. If the charges moves in and out towards us it does not wiggle and, therefore, has no acceleration.

Let's see if we can find the electric and magnetic fields far away from a moving charge where the vector potential is given by

$$\bar{A} = \frac{Q\bar{V}}{4\pi\epsilon_0 R}$$

As time varies so does R and $\bar{V}$ so when we find $\bar{E}$ and $\bar{B}$ we must be careful.

NOW $\bar{B} = \bar{\nabla} \times \bar{A}$ SO THE z COMPONENT IS GIVEN BY

$$B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}$$

THE FIRST TERM IS DETERMINED IN THE FOLLOWING WAY

$$\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}\left[\frac{Q v_x (t - r/c)}{r}\right] = \left[-\frac{Q v_x y}{r^3} - \frac{Q \dot{v}_x}{r}\frac{\partial r}{\partial y}\right]_{ret}$$

$$= \frac{Q \dot{v}_{ret}}{r}\frac{y}{r} = \frac{Q \dot{v}}{r}\bar{e}_y$$

HERE I HAVE IGNORED TERMS IN $1/R^2$ AND KEPT ONLY $1/r$ TERMS.

$$\frac{\partial A_y}{\partial x} = -\frac{Q \dot{v}_y}{4\pi\epsilon_0 r} e_x$$

THUS,
$$B_z = \frac{Q}{4\pi\epsilon_0 r}\left(\dot{v}_x \bar{e}_y - \dot{v}_y \bar{e}_x\right)$$

GENERALLY THEN

$$\bar{B} = -\frac{Q}{4\pi\epsilon_0 r}\left(\bar{a} \times \bar{e}\right)_{ret}$$

NOTICE AGAIN IT IS THE PERPENDICULAR COMPONENT THAT BECOMES IMPORTANT. ALSO $\bar{B}$ IS PROPORTIONAL TO THE ACCELERATIONS, ~~of~~

FOR THE ELECTRIC FIELD WE HAVE

$$\bar{E} = \bar{e} \times \bar{B} = -\frac{Q}{4\pi\epsilon_0 r}\bar{e} \times (\bar{e} \times \bar{a})$$

USING THE IDENTITY $a \times (b \times c) = \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b})$

$$\bar{E} = -\frac{Q}{4\pi\epsilon_0 r}\left[\bar{e}(\bar{a}\cdot\bar{e}) - \bar{a}\right]$$

WE CAN DEDUCE THIS RESULT SINCE

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi$$

THE FIRST TERM IS EASY AND IS JUST

$$\frac{\partial \bar{A}}{\partial t} = \frac{Q \dot{\bar{v}}}{4\pi\epsilon_0 r}$$

THE SECOND TERM IS A LITTLE TOUGHER BECAUSE

$$\nabla\varphi = \nabla\frac{Q}{4\pi\epsilon_0 r} \propto \frac{1}{r^2}$$

AND WE SPELL DEATH TO THE UNBELIEVER SINCE WE DO NOT KEEP $1/r^2$ TERMS. WE DID SOMETHING WRONG AND THAT WAS TO USE THE WRONG POTENTIAL. WE SHOULD HAVE USED

$$\varphi = \frac{Q}{4\pi\epsilon_0\left(r - \frac{\bar{r}\cdot\bar{v}}{c}\right)}$$

EXPANDING TO SECOND ORDER

$$\varphi = \frac{Q}{4\pi\epsilon_0 r} + \frac{Q \cdot v_r (t - r/c)}{4\pi\epsilon_0 r}$$

WITH SOME MATH. WE CAN SHOW

$$\nabla\varphi = \frac{Q}{4\pi\epsilon_0 r}\bar{e}(\bar{a}\cdot\bar{e})$$

DIPOLE RADIATOR

TODAY I WOULD LIKE TO DISCUSS SOME ELECTROMAGNETIC PHENOMENA INVOLVING LIGHT, INFRARED RADIATION, X-RAYS AND GENERALLY PHENOMENA OF LARGE NUMBER OF PHOTONS.

SUPPOSE WE CONSIDER A CHARGE MOVING UP AND DOWN IN AN OSCILLATORY FASHION. THE DISTANCE TRAVELED IN ONE OSCILLATION WILL BE CONSIDERED SMALL COMPARED TO THE DISTANCE LIGHT WOULD TRAVEL IN ONE PERIOD. WHAT WE ARE GOING TO STUDY IS A SIMPLE DIPOLE RADIATOR.

WHAT IS THE FIELD OUT AT POINT P? WELL, WE ALREADY KNOW WHAT IT IS. IT IS GIVEN BY THE EQUATION,

$$E = \frac{-q\,a_\perp\,(\text{Retarded})}{4\pi\epsilon_0 R c^2}$$

IF WE CONSIDER THE SPECIAL CASE OF HARMONIC MOTION, THE DISPLACEMENT OF THE CHARGE IS GIVEN BY

$$x = x_0 \cos\omega t'$$

THEN THE ACCELERATION IS,

$$a = -\omega^2 x_0 \cos\omega t'$$

WHERE $t' = t - R/c$ THE RETARDED TIME. IF WE PUT THIS INTO THE ABOVE EXPRESSION FOR $\bar{E}$ AND PROJECT THE FIELD NORMAL TO THE LINE OF SIGHT WE FIND

$$E = +q\,\frac{\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 R c^2}\cos\omega(t - R/c)$$

THE MAGNETIC FIELD IS GIVEN BY

$$\bar{B} = -\bar{e}_R \times \frac{\bar{E}}{c}$$

AND WOULD BE DIRECTED INTO THE PAGE AND HAVE A VALUE OF

$$B = -q\,\frac{\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 R c^3}\cos\omega(t - R/c)$$

THE INTENSITY OF THE RADIATION OR THE ENERGY RADIATED PER cm^2 PER SEC IS GIVEN BY THE POYNTING VECTOR

$$\text{FLUX} = \bar{S} = \epsilon_0 c^2 (\bar{E}\times\bar{B}) = \epsilon_0 c^2 E^2$$

SINCE $\bar{E}\times\bar{B} = \bar{E}\times(\bar{e}_R\times\bar{E}) = \bar{e}_R(\bar{E}\cdot\bar{E}) - \bar{E}(\bar{e}_R\cdot\bar{E}) = \bar{e}_R|E|^2$

SO THE ENERGY IS RADIATED RADIALLY AND GIVEN BY THE SQUARE OF E.

$$\text{Intensity, } I = \frac{q^2 \sin^2\theta\, \omega^4 x_0^2 \cos^2 \omega(t - R/c)}{(4\pi)^2 \epsilon_0 R^2}$$

But the intensity is really the average of the flux over a long period of time so we have to average the $\cos^2$ term which is just $1/2$. Thus

$$I = \frac{q^2 \sin^2\theta\, \omega^4 x_0^2\, 1/2}{(4\pi)^2 \epsilon_0 R^2}$$

The energy that the source can deliver decrease as we get farther away; it decreases inversely as the square of the distance.

If we were considering an antenna where a current $i = q\dot{x} = q\omega x_0$ was running up and down, we could substitute for x_0 and find the intensity as a function of the current.

To find the total radiation coming from the dipole, it is necessary to integrate the intensity over all directions. That is I must average $\sin^2\theta$ over all angles. This I know already is $2/3$ because there is 0 radiation along one axis and radiation along the other 2 so I get $2/3$ for all directions. The power of the dipole is then

$$P = \frac{2}{3}\, \frac{q^2 \omega^4 x_0^2\, 1/2}{4\pi\epsilon_0}$$

That is, multiply I by $2/3$ and $4\pi R^2$.

If I want to find the energy in the oscillator, I know a harmonic oscillator has an energy given by the sum of the kinetic and potential energy, i.e.,

$$\mathcal{E} = \frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2}$$

substituting for x, $x_0 \cos\omega t$ we get

$$\mathcal{E} = \frac{m x_0^2 \omega^2}{2} \sin^2 \omega t' + \frac{m\omega^2 x_0^2}{2} \cos^2 \omega t = \frac{m x_0^2 \omega^2}{2}$$

Thus the total power radiated per sec is

$$P = \frac{2}{3}\, \frac{q^2 \omega^2 \mathcal{E}}{4\pi\epsilon_0\, m}$$

Radiation Damping

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We would like to find out how much energy is lost in one oscillation. That is to say the charge will not oscillate forever but eventually stop. We have to find the Q of the oscillator where

$$Q = \frac{\text{total energy content of the oscillator}}{\text{energy loss per radian}}$$

or

$$\frac{1}{Q} = \frac{\text{energy radiated in time } (T/2\pi = 1/\omega)}{\text{energy content}}$$

There fore,

$$\frac{1}{Q} = \frac{\frac{\frac{2}{3} g^2 \omega^2 \mathcal{E} \times \frac{1}{\omega}}{4\pi\epsilon_0}}{\mathcal{E}} = \frac{2}{3}\frac{g^2\omega}{4\pi\epsilon_0 m}$$

If we substitute $e^2 = g^2/4\pi\epsilon_0$ we have

$$\frac{1}{Q} = \frac{2}{3}\frac{e^2\omega}{mc^2}$$

The quantity $\frac{e^2}{mc^2}$ is the classical radius of the electron and given as

$$r_0 = \frac{e^2}{mc^2} = 2.82 \times 10^{-13} \text{cm}$$

The quantity $1/Q$ is then,

$$\frac{1}{Q} = \frac{2}{3}\frac{(2.8 \times 10^{-13}\text{cm})}{\lambda/2\pi}$$

For light which has a wavelength of say 6280 Å we find

$$\frac{1}{Q} = 10^{-8}$$

And it takes about 10^8 radians or 10^7 oscillations for an atomic oscillator to lose $1/e$ of its energy, i.e., about a third of its energy. For X-rays with $\lambda \sim 10^{-13}$ cm we get a tremendous damping since $Q \sim 1$. Further the width of the spectral lines of an atom are given approximately by

$$\frac{\Delta\nu}{\nu} \sim \frac{1}{Q}$$

Also we can put in the mass of the neutron to find the Q's of nuclear vibrations.

So far we have only consider the special case of harmonic motion. More generally the power radiated from a charge moving in any old fashion is given by

$$P = \frac{2}{3}\frac{|\bar{a}(t-R/c)|^2}{4\pi\epsilon_0 c^3} = \frac{2g^2a'^2}{3\,4\pi\epsilon_0 c^3}$$

This formula, however, is not exactly right because the exact time when the energy is liberated cannot be precisely defined. We only determine the power over a full cycle not a part of one. The difficulty is seen better by examining another formula given by a man moving by with velocity v. He would say

$$P' = -\frac{2}{3}\frac{e^2}{c^3}\,\bar{v}\cdot\frac{d\bar{a}}{dt}$$

But

$$P' = -\frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}(\bar{a}\cdot\bar{v}) + \frac{2}{3}\frac{e^2}{c^3}(\bar{a}\cdot\bar{a})$$

Over a full cycle the d/dt term averages to zero and $P = P'$. The two results are the same.

Actually, the second expression is the correct one. It allows us to explain some paradoxes the other formula permits. To see how the formula works consider a charge moving in and out of a field. Upon entering the field the acceleration is abruptly changed from zero to some final value. And when it leaves the field it returns to its initial acceleration. It is because the two changes in acceleration occur that we get a net radiation. It is only when a charge is accelerating or de accelerating that we get any radiation.

But why must a charge radiate power in the the first place? If we have no charge on a ball it will not radiate. But when a ball is charged there is an electromagnetic interaction between the pieces of charges around the ball. When the ball is at rest all the Vol II chap 28

pieces of charge repell equally so the forces balance in pairs. When the charge is being accelerated the forces do not balance because of the retarded effects on each piece. In (b) the force on α due to ρ depends on the position of ρ at an earlier time. When all the interactions are considered the forces do not cancel as in (c). The charge exerts a force on itself to hold back the acceleration.

This force is given by the equation.

$$F = \frac{2}{3}\frac{Q^2}{4\pi\epsilon_0}\frac{a}{R} - \frac{2}{3}\frac{Q^2}{4\pi\epsilon_0}\dot{a} + \gamma R\ddot{a} + \text{higher terms}$$

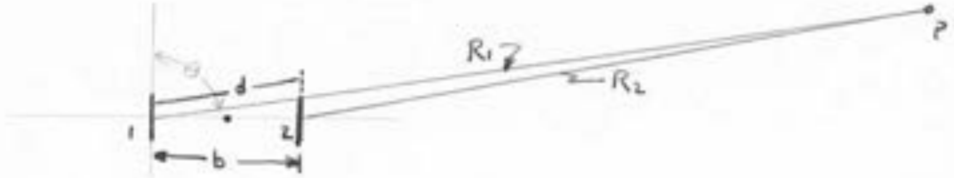
If we redefine the mass to include the electromagnetic mass and thus rid ourselves of the infinity if we let $R \to 0$ then when we do let $R \to 0$ we have

$$F = -\frac{2}{3}\frac{Q^2}{4\pi\epsilon_0}\dot{a}$$

Since the work done on the charge is just $\vec{V}\cdot\vec{F}$ we have the power radiated as $$P = \frac{2}{3}\frac{Q^2}{4\pi\epsilon_0}\vec{V}\cdot\dot{\vec{a}}$$

It is the $d\bar{a}/dt$ in the bootstrap force that is required to have energy conservation in radiating systems. We can't throw it away. The problem we created, though, is the infinite mass of a point charge. To get out of that difficulty we must mumble a lot.

I INTENDED TO START TO TELL YOU SOME INTERESTING FACTS ABOUT TWO DIPOLES OSCILLATING TOGETHER AND I'LL DO IT NOW



IF THE TWO CHARGES ARE SEPARATED BY A DISTANCE b, THE FIELD AT POINT P WILL BE THE SUM OF THE TWO DIPOLE FIELDS OVER THE DISTANCE R_1 AND R_2. BUT R_1 IS LONGER THAN R_2 BY $b\sin\theta$.

$$E = E_1 + E_2$$

$$E = -\frac{q\, a_\perp (t - R_1/c)}{4\pi\epsilon_0 R_1 c^2} - \frac{q\, a_\perp (t - R_2/c)}{4\pi\epsilon_0 R_2 c^2}$$

SINCE WE ASSUME THE OSCILLATORS ARE MOVING TOGETHER WE HAVE

$$E = +\frac{q\omega^2 x_0}{4\pi\epsilon_0 c^2}\sin\theta\left[\frac{\cos\omega(t - R_1/c)}{R_1} + \frac{\cos\omega(t - R_2/c)}{R_2}\right]$$

IF WE PUT IN $R_1 - b\sin\theta$ FOR R_2 OR FIRST JUST $R_1 - d$ WE HAVE

$$E = \frac{q\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2}\left[\frac{\cos\omega(t - R_1/c)}{R_1} + \frac{\cos\omega\left(t - \frac{R_1}{c} + \frac{d}{c}\right)}{R_1 - d}\right]$$

NOTICE IF I REWRITE THE COS TERMS AS

$$\cos\omega t \quad \text{AND} \quad \cos\omega(t+\Delta)$$

I CAN GET A CANCELLATION IN FIELDS TO FIRST ORDER POWERS OF $\frac{d}{R_1}$.

THAT IS, IF I REWRITE E AS

$$E = \frac{q\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t + \left(1 + \frac{d}{R_1}\right)\cos\omega(t+\Delta)\right]$$

I WILL GET NO RADIATION AT P IF $\omega\Delta = \frac{\omega d}{c} = \pi$ SINCE $\cos\omega(t+\Delta) = \cos\omega t\cos\Delta - \sin\omega t\sin\Delta$ AND $\Delta = \pi$.

IF THE OSCILLATORS ARE SITTING ON TOP OF EACHOTHER SO THAT $d = 0$, I GET TWICE THE FIELD OF ONE OSCILLATOR AT P. BUT SINCE THE ENERGY GOES AS E^2 WHERE $E = 2E_1$, I FIND I HAVE TWICE AS MUCH ENERGY THAN I WOULD GUESS.

WE MIGHT STUDY THE EFFECT OF CHANGING d TO SOME NICE VALUES LIKE λ, $\lambda/2$, $\lambda/4$, ETC, I.E. $\omega\Delta = \frac{d}{\lambda} = n\pi$. SINCE d IS MUCH LESS THAN R_1 LET'S FINISH THE ABOVE EXPANSION.

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t + \cos\omega(t+\Delta) + \frac{d}{R_1}\cos\omega(t+\Delta)\right]$$

If I now limit myself to radiation which decreases a 1/R, I will forget the last term and only consider the first two. If I expand the $\cos\omega(t+\Delta)$ out so I have $\cos\omega t\cos\omega\Delta - \sin\omega t\sin\omega\Delta$, I can now write

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t\,(1+\cos\omega\Delta) - \sin\omega t\,\sin\omega\Delta\right]$$

If the phase angle $\omega\Delta$ is π we have seen that $E = 0$. If we consider some other phase angles like $\omega\Delta = 0, 2\pi, 4\pi$, etc, we see

$$E = \underline{2}\,\frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\cos\omega t \qquad \omega\Delta = n\pi \quad \text{where } n = 0, 2, 4, \text{etc}$$

If $\omega\Delta = \pi/4,\ 9\pi/4$, etc.

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t\,(1.707) - .707\sin\omega t\right]$$

If $\omega\Delta = \pi/2,\ 5\pi/2$

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t - \sin\omega t\right] = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\cos\omega t\left[1 - \tan\omega t\right]$$

$$E = E_1\left[1 - \tan\omega t\right]$$

If $\omega\Delta = \frac{3\pi}{4}$

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t\,(.3) - .707\sin\omega t\right]$$

If $\omega\Delta = \frac{5\pi}{4}$

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[.3\cos\omega t + .707\sin\omega t\right]$$

If $\omega\Delta = \frac{3\pi}{2}$

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[\cos\omega t + \sin\omega t\right] = E_1(1 + \tan\omega t)$$

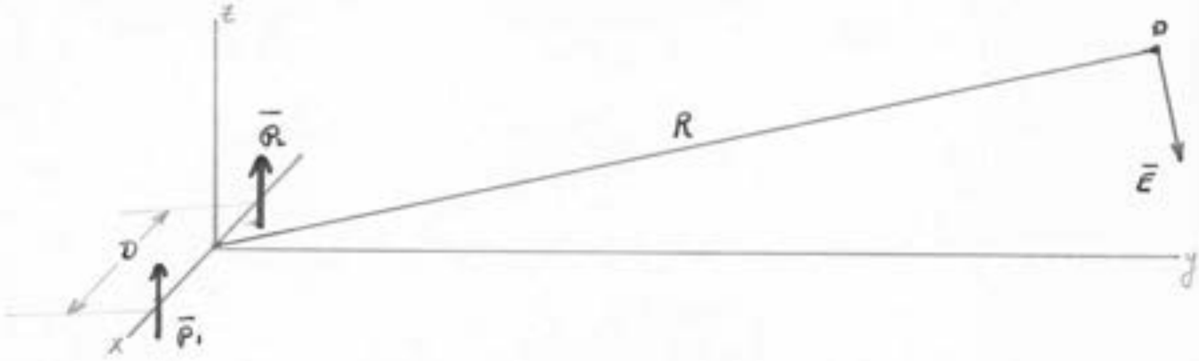
If $\omega\Delta = \frac{7\pi}{4}$

$$E = \frac{q\,\omega^2 x_0 \sin\theta}{4\pi\epsilon_0 c^2 R_1}\left[(1.3)\cos\omega t + (.707)\sin\omega t\right]$$

Ref. FEYNMAN VOL I
Chap 29

TWO DIPOLE OSCILLATOR

I WANT TO TALK ABOUT A MORE INTERESTING CASE of dipole oscillators were we have two dipoles working together. I will assume the distance separating the charges is much less than the distance from which I am observing the motion.



Recall if I have only one charge, the electric field out at point O is given by

$$\bar{E} = -\frac{1}{4\pi\epsilon_0}\frac{\ddot{\bar{P}}_\perp}{R}\ (\text{retarded})$$

where $\bar{p}$ is the dipole moment of the charge and given by $q\bar{x}$. For simplicity I'll again assume harmonic motion of the charge so that the dipole moment is given by

$$\bar{P} = \bar{P}_0\, e^{i\omega t}$$

where $e^{i\omega t}$ has its usual meaning of being $\cos\omega t + i\sin\omega t$. The only term of real significance is the cosine term but I'll leave it in exponential form. Thus the electric field from one oscillator is given by

$$\bar{E} = \frac{\omega^2}{4\pi\epsilon_0}\frac{\bar{P}_0}{R}\, e^{i\omega t}\, e^{-i\omega R/c}$$

Here I have calculated the field at the retarded time $t - R/c$ and, therefore, have a phase factor entering the formula, i.e., $\varphi = \omega R/c$ = phase angle. This phase angle can be expressed in terms of the wavelength, λ, in the following form

$$\varphi = \frac{2\pi R}{\lambda}$$

Now we come to the question of two oscillators, maybe in phase maybe not. The separation distance will be D and the distance to the observer is much larger than D so we consider the two oscillators at the same point. But let's be careful! The field at point O is the superposition of the fields from oscillator 1 and 2 and is therefore,

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

CONSIDER THE SIMPLE CASE WHERE THE OSCILLATORS ARE MOVING IN AND OUT OF THE PAPER. REMEMBER IT IS ONLY THE COMPONENT OF THE ACCELERATION PERPENDICULAR TO THE LINE OF SIGHT THAT IS CRITICAL. THUS WE HAVE MAXIMIZED THE FIELD IN THIS CASE. THE TWO DISTANCES R_1 AND R_2 ARE TO THEIR RESPECTIVE OSCILLATORS AND DIFFER BY A SMALL AMOUNT $D\cos\theta$ WHERE

$$R_2 = R_1 - D\cos\theta$$

WE CAN WRITE THE TOTAL FIELD AT O AS

$$E = \frac{\omega^2}{4\pi\epsilon_0 R}\left[P_1 e^{i\omega t} e^{-i\omega R_1/c} + P_2 e^{i\omega t} e^{-i\omega R_2/c}\right]$$

NOW EVEN THOUGH R_1 AND R_2 ARE MUCH GREATER THAN $D\cos\theta$, I CANNOT CONSIDER THE PHASE ANGLES CONSTANT. SMALL VARIATIONS IN D CAN MEAN BIG PHASE CHANGE EVEN THOUGH THE OBSERVER WOULD NEVER KNOW THE DIFFERENCE.

I AM REMINDED OF MY YOUTHFUL DAYS AT LOS ALAMOS WHEN I WAS WORKING ON THE BOMB. I BECAME QUITE GOOD ON VERY QUICKLY WORKING VERY COMPLICATED MATHEMATICAL PROBLEMS. I CHALLENGED ALL MY ASSOCIATES TO A DUAL - IF THEY COULD GIVE ME A PROBLEM IN FIFTEEN SECONDS, I COULD GIVE THEM AN ANSWER IN THREE MINUTES TO CORRECT VALUES LESS THAN A PERCENT ERROR. I GOT SOME REAL CRAZY ONES BUT MOST OF THE TIME CAME OUT ON TOP. BUT THERE WAS MY OLD COLLEGE NEMESIS, PAUL OWENS, WHO WAS ALWAYS SMARTER THAN ME. HE CAME OVER ONE DAY AND ASKED WHAT WAS GOING ON. AFTER BEING TOLD, HE SAID, "ALRIGHT FEYNMAN GIVE ME THE TANGENT OF 10^{10}" I KNEW I WAS STUMPED RIGHT AWAY - THERE WAS NO WAY OF QUICKLY FINDING OUT WHAT PHASE OF OSCILLATION I'D BE IN WAY OUT AT 10^{10}.

THAT ALSO BRINGS TO MIND THE TIME I WAS PLAYING WITH ONE OF THOSE RETRACTABLE TAPE MEASURES THAT YOU PUSH A BUTTON AND THE TAPE FLIES BACK INTO THE CASE. BUT BECAUSE OF THE INERTIA THE TAPE WOULD ALWAYS FLY BACK AND SLAP MY HAND. I WORKED AND WORKED ON HOLDING IT RIGHT TO AVOID THE PAIN BUT NOTHING WORKED. I WENT TO PAUL AND ASKED HOW HE HELD IT BECAUSE IT NEVER BOTHERED HIM. HE SHOWED ME AND FOR SEVERAL WEEKS I KEPT ON TEARING UP MY HAND. FINALLY, I ASKED HIM HOW HE DID IT SO IT DIDN'T HURT; HIS REPLY - WHO SAID IT DIDN'T HURT!

BACK TO THE PROBLEM. THE ELECTRIC FIELD CAN BE WRITTEN IN THE FOLLOWING FORM,

$$E = \frac{\omega^2}{4\pi\epsilon_0 R}\left[e^{i\omega t} e^{-i\omega R_L/c} f\right]$$

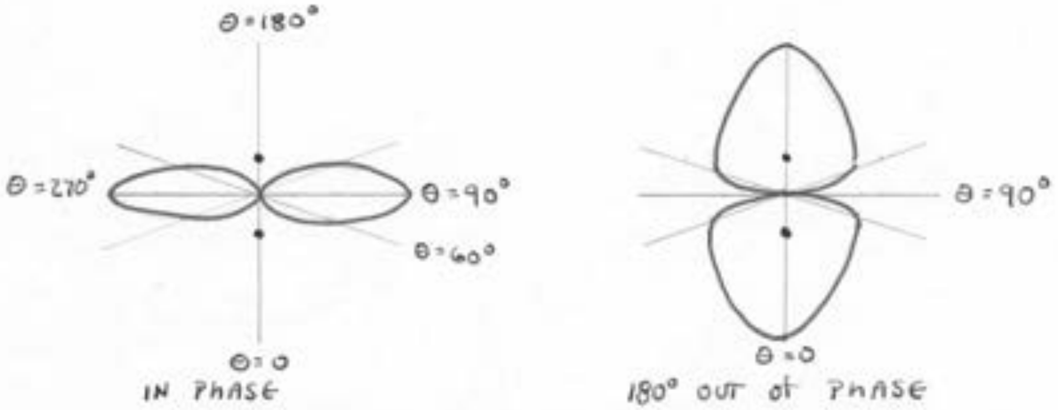
where I have used the function f to denote the quantity

$$f = \left(P_1 e^{-\frac{i\omega}{c} D\cos\theta} + P_2\right)$$

Since R may be defined to be the distance to the center of the dipole pair, we can say $R \approx R_L$ and the only concern ourselves with the very interesting term, f.

If I rewrite the phase angle as $\frac{2\pi D\cos\theta}{\lambda}$, I see when $D\cos\theta$ is of the order of λ we can get a substantial phase difference. In fact, if $D\cos\theta = \lambda/2$ and the two dipoles are of equal strength, then $f = 0$ since $e^{-i\pi} = -1$. If D, itself, is $\frac{\lambda}{2}$ and we vary the viewing angle θ, the field strength distribution varies is a complicated manner as we go through 2π radians. In the case just mentioned $\theta = 0$ or it could have been π. In either case the electric field is zero. If we turn $90°$ to $\theta = \pi/2$ or $3\pi/2$ we find that the phase lag is zero so that $f = 2P_0$; the dipole of one charge equals P_0. Therefore, the electric field is twice what we get from one dipole and the intensity or energy flowing past us is proportional to E-squared so we get 4 times the flux from one dipole. At an intermediate angle of $60°$, we find the field is $\sqrt{2}$ times as strong or the intensity is twice as great.

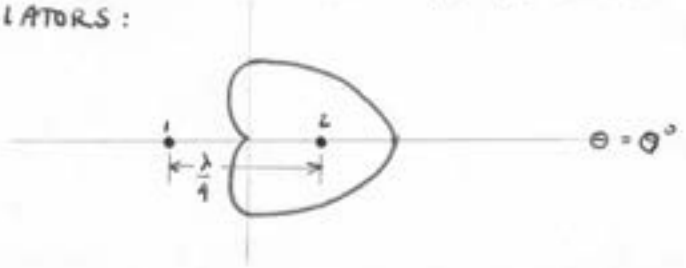
Had the oscillators been started $180°$ out of phase so that $P_1 = -P_2$ then the values for E and the intensity would have been reversed at $\theta = 0, \pi$ and $\pi/2, 3\pi/2$. That is, we now have zero intensity perpendicular to the line joining the dipoles. The following figure show the two cases just mentioned:



Radiation patterns from two equal dipoles separated by a distance $\lambda/2$

The previous remarks show a very interesting characteristic of electromagnetic waves - they can be directionally sent by proper phasing of closely spaced oscillators. But in the prior case we see in the figures if we are interested in broadcasting in only one direction, say along $\theta = 90°$ for the in phase oscillators we are wasting power in the other direction. There is another easy way to overcome this problem.

Suppose we have the following arrangement with two equal oscillators:



If dipole 2 is started at 90° out of phase with 1, i.e., it starts oscillating a quarter of a cycle after number one and the separation distance is $\lambda/4$, we now get a greater intensity along $\theta = 90°$. The reason is a signal arriving at a point far out on $\theta = 0°$ receives the signal from # 1 a short time later than it would get anything from # 2. But two is slowed down by a 90° lag so the fields add again in phase. Along the $\theta = \pi$ line the fields cancel because a signal leaving # 1 arrives at the observer; when the signal from # 2 comes in it is delayed through $\lambda/4$ but lags another $\lambda/4$ and, therefore, destructively added to # 1. Along $\theta = \pi/2$ and $3\pi/2$ will add to give an intensity twice that from one dipole.

Other dipole arrays can be devised to eliminate all the wasted power and radiate the beam into the desired direction. This is done by adding more dipoles and cleverly arranging their spacing and phasing.

The more general formula for a pair of dipole oscillators would involve separate dipole moments,

$$P_1 = |P_1| e^{i\varphi_1} \quad \text{and} \quad P_2 = |P_2| e^{i\varphi_2}$$

Then the function f becomes

$$f = |P_1| e^{i(\varphi_1 - \delta)} + |P_2| e^{i\varphi_2}$$

where δ is the phase lag between the two oscillators. The intensity is given by the square of f or

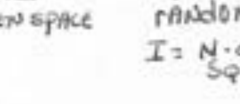
$$I \sim |f|^2 = |P_1|^2 + |P_2|^2 + 2|P_1||P_2| \cos(\varphi_1 - \delta - \varphi_2)$$

The TOTAL POWER from TWO such OSCILLATORS IS Therefore NOT The SUM of The TWO INDEPENDENTLY BUT CAN be FOUR TIMES AS LARGE AS ONE. Such MIGHT be The CASE of TWO RADIO STATIONS OPERATING CLOSE TO EACH OTHER AT The SAME frequency. If ONE TURNS ON AT The PROPER INSTANT IT COULD MAKE The COSINE TERM ZERO AND The TWO STATIONS destructively INTERFER. OR The could Add IN PhASE. BUT ON The AVERAGE The TOTAL POWER IS TWICE ONE AS expected. This IS A result of The random phase relationship which exists between The TWO. AS The POWER GOES IN AND OUT of The MAXIMUMS AND MINIMUMS, They AVERAGE TO 2.

The previous EXPLANATION of RADIATING DIPOLES helps EXPLAIN ANOTher very INTERESTING PhENOMENA. Because AN ATOM WILL radiate when The ELECTRONS ARE DRIVEN by SOME EXTERNAL force, IT IS POSSIBLE TO GET very LARGE INTENSITIES from The ATOMS if The bunch TOGEther. This IS The CASE for WATER Droplets IN The SKY. When They AGGREGATE IN A region where There ARE billions And billions of WATER MOLECULES MOVING TOGEther WE SEE CLOUDS. This IS why WE don't SEE WATER MOLECULES Above A Stream or LAKE, i.e., There Aren't ENOUGh TO make Their PRESENCE VISIBLE.

If The observer IS LOOKING up AT The SKY And The INCIDENT SUNLIGHT EXCITES The ATOMS IN The WATER MOLECULE TO RADIATE, IT IS ONLY The SUNLIGHT ThAT EXCITES The MOTION Perpendicular TO OUR LINE OF SIGHT THAT IS IMPORTANT. Thus ONLY A SMALL fraction of The INCIDENT IS POLARIZED IN The right DIRECTION TO DO MUCH Good. The LIGHT DRIVES The ATOMS AT A frequency WHICH MAKE blue LIGHT more Predominant. The TOTAL INTENSITY from ALL The ATOMS IS EQUAL TO The SUM of EACH ATOM And This IS proportional TO The number of ATOMS PER UNIT VOLUME. If The ATOMS vibrate TOGEther And There Are N of Them, The TOTAL ELECTRIC field IS N TIMES AS STRONG AS ONE of Them. The INTENSITY IS N-SQUARED AS much AS ONE ATOM could put out.

When The WATER MOLECULES condense TO form SMALL DROPLETS, They START TO work together. IT IS when The droplets reach A SIZE About EQUAL TO 5000 Å THAT The INTENSITY reaches A MAXIMUM. Since A WATER MOLECULE IS About 2 Å IN DIAMETER, WE CAN GET About 10^{9} MOLECULES IN This Droplet. The intensity Does become QUITE Large And The Cloud becomes visible. If The Droplets Are randomly SPACED The INTENSITY IS Greater Than if They Are evenly SPACED.



ATTEMPTS TO MODEL THE ELECTRON

IN THE PAST MANY DIFFERENT APPROACHES HAVE BEEN SOUGHT TO SATISFACTORILY MODELLING THE ELECTRON. WE HAVE SEEN THE DIFFICULTY ENCOUNTERED WITH THE INFINITE SELF ENERGY ON THE CLASSICAL SCALE. WHAT IS THE SIZE OF THE ELECTRON? WHAT HAPPENS WHEN INCIDENT LIGHT HAS A WAVELENGTH COMPARABLE TO AN ELECTRON DIAMETER? WILL THE HIGHER RESISTANCE TERMS IN THE SELF-FORCE EXPANSION DRASTICALLY EFFECT THE DAMPING?

RETURNING FOR THE MOMENT TO CLASSICAL CALCULATIONS, THE ELECTRON CAN BE ROUGHLY DETERMINED BY EQUATING THE KINETIC AND POTENTIAL ENERGIES, I.E.,

$$c^2 m = \frac{e^2}{R_0}$$

$$R_0 = \frac{e^2}{mc^2} \approx 2.8 \times 10^{-13}\ \text{cm}$$

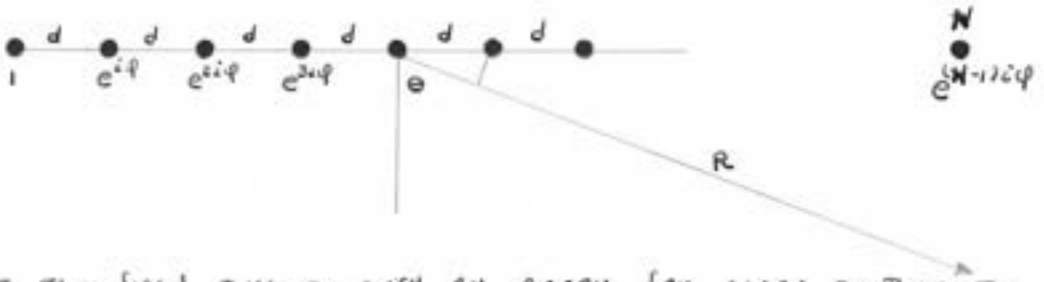
SO WHEN WE HAVE λ ON THE ORDER OF R_0 WE CAN EXPECT SOME FUNNY THINGS TO HAPPEN. UNFORTUNATELY WE HAVE NO EXPERIMENTS CONDUCTED AT THIS LEVEL; THIS CORRESPONDS TO PHOTONS HAVING AN ENERGY ON THE ORDER OF 200 MeV. THIS IS STRANGE BECAUSE THE ENERGY LEVEL IS NOT RIDICULOUSLY HIGH.

WE HAVE PROBED OUR VARIOUS THEORIES TO WAVELENGTH OR DISTANCES 5-10 TIME SMALLER THAN R_0. BUT IN THIS VERY SMALL REGION WE NEED QUANTUM THEORY TO EXPLAIN SOME OF THE EVENTS TAKING PLACE. UNFORTUNATELY THAT THEORY IS MODELED WITH A POINT CHARGE FOR THE ELECTRON. ALL WE CAN SAY IS EXPERIMENTS VERIFY THE THEORY TO ~~THAN~~ DISTANCES ON THE ORDER OF 10^{-14} CM. UNFORTUNATELY THE INERTIAL MASS TERM GIVEN ABOVE AS e^2/R_0c^2 DOES BLOW-UP AS $R_0 \to 0$. EVEN THE ACTUAL INERTIAL MASS FORMULA WHEN THE ELECTROMAGNETIC MASS IS INCLUDED DIVERGES SINCE $m \propto \ln R_0$. BECAUSE THE MASS GOES AS $\ln R_0$ A VARIATION OF 1/1000 ONLY CHANGES THE MASS BY 3 OR 4 %. SO WE HAVE REAL DIFFICULTY IN EXPERIMENTALLY VERIFYING THE VARIOUS MODELS.

jtn insert: What was wrong with the "ether" model?

NO ONE HAS MADE A SATISFACTORY MODEL OF THE ELECTRON. NATURE ALWAYS SEEMS TO WORK BY SIMPLER MODELS AND THIS CONFUSES US. A GOOD EXAMPLE OF THIS IS OUR MODEL OF THE ETHER TO EXPLAIN MAXWELL'S EQUATIONS. IT WAS POSTULATED THAT ELECTROMAGNETIC PROPAGATION WAS A DISTORTION OF A JELLO LIKE SUBSTANCE CALLED THE ETHER. THE DYNAMIC NATURE OF THE JELLO WOULD YIELD MAXWELL'S EQUATION - IT WAS THOUGHT. ALL EXPERIMENTS FAILED TO SHOW THE PRESENCE OF AN ETHER. SO WE GOT TIRED OF THE ETHER THEORY AND SCRAPPED IT BECAUSE IT WAS TOO COMPLICATED AND DIDN'T AGREE WITH REALITY. THE MODEL WAS MORE COMPLICATED THAN ~~M~~ ELECTROMAGNETIC THEORY SO IT WAS WORTHLESS.

I NOW WANT TO DISCUSS THE RADIATION PATTERN COMING FROM A NUMBER OF OSCILLATORS, ALL DRIVEN TO THE SAME AMPLITUDE, ALL SEPARATED BY THE SAME DISTANCE d AND EACH OSCILLATOR OUT OF PHASE WITH ITS NEIGHBOR BY $e^{i\varphi}$. THE ARRAY I HAVE PROPOSED IS THEN THE FOLLOWING:



IF WE WANT THE FIELD DUE TO SUCH AN ARRAY FAR AWAY SO THAT THE MEAN DISTANCE TO THE ARRAY, R, IS VERY SMALL COMPARED TO THE TOTAL LENGTH OF THE ARRAY $D = Nd$, THEN WE CAN WRITE THE SUPERPOSITION OF EACH FIELD AS,

$$E_T = \frac{a_r}{R} e^{i\omega(t - \frac{R}{c})} \left[1 + e^{i(\varphi - \frac{\omega d \sin\theta}{c})} + e^{2i(\varphi - \frac{\omega d \sin\theta}{c})} + \cdots \right.$$

WHERE WE HAVE AN EFFECTIVE PHASE DIFFERENCE ψ BETWEEN EACH OSCILLATOR DUE TO THE SMALL VARIATIONS FROM R TO EACH OSCILLATOR, I.E.,

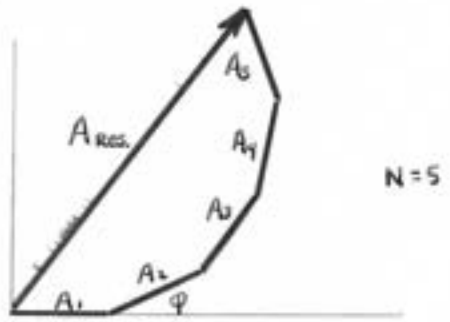
$$\psi = \varphi - \frac{\omega d \sin\theta}{c}$$

φ IS THE BUILT IN PHASE DELAY AND $\frac{\omega d \sin\theta}{c}$ IS THE VARYING PHASE ANGLE DUE TO THE SPACING AND OBSERVATION DIRECTION

WE HAVE THEN TO SUM UP THE FOLLOWING SERIES

$$S = 1 + e^{i\psi} + e^{2i\psi} + \cdots + e^{(N-1)\psi}$$

GEOMETRICALLY THIS IS LIKE SUMMING A SERIES OF SMALL AMPLITUDES EACH VARYING BY φ



WE COULD PROCEED GEOMETRICALLY TO FIND $A_{RESULTANT}$ OR S BUT LET'S DO IT MATHEMATICALLY.

BY SIMPLE REASONING WE KNOW WE HAVE A GEOMETRIC SERIES. IF WE MULTIPLY S BE $e^{i\varphi}$ WE HAVE

$$e^{i\psi} S = e^{i\psi} + e^{2i\psi} + \cdots + e^{i(N-1)\psi} + e^{i(N)\psi}$$

NOW SUBTRACT S FROM BOTH SIDES

$$(e^{i\psi} - 1) S = e^{iN\psi} - 1$$

OR

$$S = \frac{e^{iN\psi} - 1}{e^{i\psi} - 1}$$

THIS THEN IS THE SUM OR AMPLITUDE OF THIS SERIES. WHAT WE ARE REALLY AFTER IS THE MAGNITUDE OR ABSOLUTE VALUE OF THIS QUANTITY. THEREFORE WE MUST EVALUATE A TERM LIKE,

$$|e^{i\alpha} - 1|^2$$

THIS CAN BE DONE BY WRITING

$$(e^{i\alpha} - 1)(e^{-i\alpha} - 1) = 2 - e^{i\alpha} - e^{-i\alpha}$$

EXPANDING THE EXPONENTIALS WE GET

$$2(1 - \cos\theta) = 4 \sin^2\frac{\theta}{2}$$

IF WE GO BACK TO S WE NOW HAVE A FORMULA FOR THE INTENSITY, I.E., THE SQUARE OF THE ELECTRIC FIELD,

$$I \propto \frac{\sin^2 \frac{N\psi}{2}}{\sin^2 \frac{\psi}{2}}$$

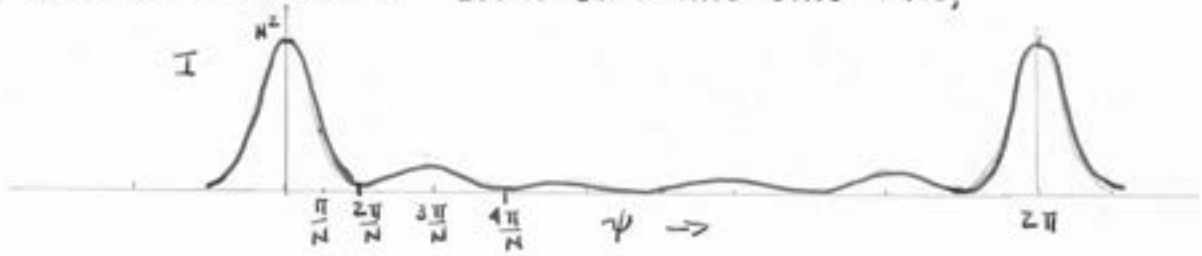
THIS EXPRESSION THEN RELATES THE INTENSITY OF A WHOLE SERIES OF OSCILLATORS TO ONE OSCILLATOR WORKING BY ITSELF. WE CAN CHECK THIS EQUATION TO SEE IF IT IS RIGHT. IF N=0, THE INTENSITY IS ZERO AS EXPECTED. IF N=1, WE GET THE INTENSITY FROM ONE RADIATOR. BUT N MAY BE A HUNDRED OR MORE SO WE HAVE TO STUDY THE INTENSITY FOR SEVERAL DIFFERENT CASES.

CONSIDER WHAT HAPPENS IF $\psi = 0$ OR WHEN $\varphi = \frac{\omega d \sin\theta}{c}$, PERHAPS, WE BETTER LETE ψ BE ABOUT ZERO BUT NOT EQUAL TO IT OR ELSE WE WILL GET 0/0. SO FOR VERY SMALL ψ WE MAY WRITE IF N IS LARGE

$$I \propto \frac{N^2\psi^2}{\psi^2} = N^2$$

THUS THE INTENSITY IS N SQUARED WHAT ONE WOULD BE IF THEY ALL ADDED IN PHASE. IF $\psi = \frac{2\pi}{N}$, THEN WE HAVE $I = 0$ AND THE INTENSITY FALLS TO A MINIMUM. BEYOND THIS POINT THE INTENSITY CURVE BUMPS UP AND DOWN AS A FUNCTION OF ψ BUT LATTER ON WHEN $\psi = 2\pi$ WE FIND ANOTHER MAXIMUM.

The plot of this function looks something like this,

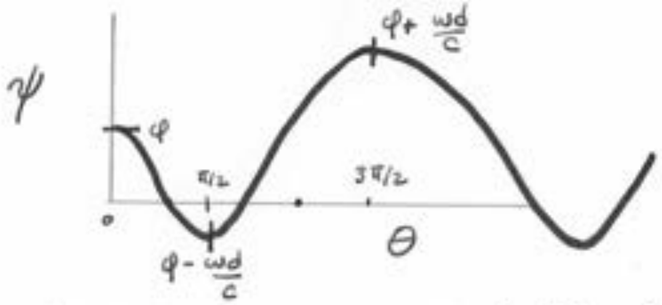


We can approximate some of the points on this curve if N is big then $\psi = \frac{2\pi}{N}$ is very small an to a good approximation around the maximum,

$$I \propto 4 \frac{\sin^2 \frac{N\psi}{2}}{\psi^2}$$

if we are at $\psi = \frac{\pi}{N}$ then $I = \frac{1}{(\frac{\pi}{2N})^2} \simeq .4 N^2$ so we are about four tenths of the way to the maximum when at $\psi = \pi/N$. The bigger N gets, the narrower the area under the maximum.

It is possible to arrange the array so it will be very powerful but have a limited number of spikes. That is, we can get rid of all the maxima of $2\pi, 4\pi, 6\pi$, etc by carefully selecting ψ. If we plot ψ as a function of θ the viewing angle, we get the following curve,



If d is many wavelengths, we get a lot of peaks but if d is less than a wavelength, then we can limit ourselves to only one main beam. We build our antenna so the main beam falls within $\pm \omega d/c$ of φ.

We can calculate the halfwidth intensity at some critical angle θ_0 in the following way

$$\varphi - \frac{\omega d}{c} \sin\theta_0 = 2\pi k$$

where k gives the spikes number

The width of the main beam $\Delta\theta$ is the width of the spike at π/N so

$$\varphi - \frac{\omega d}{c} \sin(\theta_0 + \Delta\varphi) = 2\pi k - \frac{\pi}{N}$$

If $\Delta\theta$ is small, then we can write

$$\varphi - \frac{\omega d}{c}\sin\theta_0 - \Delta\theta \frac{\omega d}{c}\cos\theta_0 = 2\pi k - \frac{\pi}{N}$$

BUT THIS IS NOTHING MORE THAN

$$\Delta\Theta \frac{\omega d}{c} \cos\Theta_0 = \frac{\pi}{N}$$

OR

$$\Delta\Theta = \frac{\pi c}{N d \omega \cos\Theta_0}$$

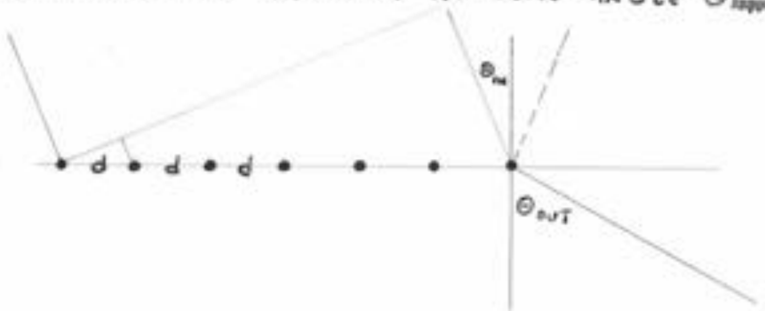
SINCE $Nd = D$ THE TOTAL SOURCE WIDTH AND $D\cos\Theta_0$ IS THE PROJECTED SOURCE LENGTH TO THE LINE OF SIGHT WE HAVE AN EXPRESSION FOR THE BEAM WIDTH AS A FUNCTION OF WAVE LENGTH AND ARRAY WIDTH,

$$\Delta\Theta = \frac{\lambda}{2D}$$

WE CAN NOW PLAY ALL SORTS OF GAMES AND TRY TO TAYLOR THE BEAM SHAPE TO OUR NEEDS. WE MIGHT WANT TO SMOOTH OUT THE SIDE BUMPS AND ACCENTUATE THE ENERGY IN THE MAIN LOBE. YOU MIGHT DESTROY THE NICE EVEN SPACING BETWEEN OSCILLATORS OR, PERHAPS, CHANGE THE PHASE SOME PARTICULAR AMOUNT PER INCH. FURTHER IT IS POSSIBLE TO GO TO A CONTINUUM OF CHARGE RUNNING ALONG A WIRE AND CALCULATE THE BEAM ~~BEAM~~ PATTERN BY WRITING SOME FACTOR DEPICTING THE PHASE AND STRENGTH OF EACH LITTLE ELEMENT OF CHARGE.

SCATTERING FROM ANTENNAS

WE SHOULD LOOK AT ANOTHER CASE OF RADIATORS WORKING IN PHASE BUT THIS TIME INSTEAD OF BEING DRIVEN BY A CURRENT OR LOCAL FORCE FIELD, A DISTANCE SOURCE CAN CREATE AN ELECTRIC FIELD WHICH WILL DRIVE THE ELECTRONS TO OSCILLATE. THIS PHENOMENA OCCURS WITH LIGHT AND IS CALLED SCATTERING. THE INDUCED MOTIONS WILL GENERATE NEW WAVES AND THIS IS CALLED A 'SCATTERED WAVE. LET'S CONSIDER AN INCIDENT LIGHT BEAM STRIKING THE ANTENNA AT SOME ANGLE Θ_{in}



EACH OSCILLATOR IS EXCITED A SHORT PHASE DELAY LATER DUE TO THE EXTRA DISTANCE REQUIRED FOR LIGHT TO TRAVEL. THAT PHASE IS GIVEN BY

$$\varphi = \frac{\omega d}{c} \sin\Theta_{in}$$

AS LIGHT STRIKES THE OSCILLATORS AND THEY RERADIATE IN SOME DIRECTION Θ_0, THE EFFECTIVE ~~WA~~ PHASE DIFFERENCE IS NOW

$$\psi = \frac{\omega d \sin\Theta_{out}}{c} - \frac{\omega d \sin\Theta_{in}}{c}$$

From the previous equation we see strong intensities occurring at even multiples of 2π. Let's consider the case when $\psi = 0$. Then we have the condition

$$\sin\theta_{out} = \sin\theta_{in}$$

which for small angles is just

$$\theta_{out} = \theta_{in}$$

or

$$\theta_{out} = \pi - \theta_{in}$$

This is interesting because when $\theta_{out} = \theta_{in}$ this implies the scattered radiation is propagated in the same direction as the radiated energy from the driven oscillators. In other words the light goes right on through the array and can, perhaps, interfer with the radiators.

When $\theta_{out} = \pi - \theta_{in}$ or at the supplement angle we get another scattered wave and this is the reflected beam of light. The angle of reflection is the same as the angle of incidence. Thus when the spacing between scatters or radiators is less than λ, we get a reflected and transmitted beam.

SCATTERED LIGHT VOL I CHAP 31

We have come then to understand reflected light from a random collecton of scatters. For example, if the scattersers are atom located between us and the source, they will be driven to radiate by the source field and create their own field. This field adds to the source's field to give us a total field

$$E = E_{source} + E_{scatterer}$$

Thus the original field is modified due to the presence of these radiators. Because the intensity of radiation goes as E squared we see a cross term appearing in the expression, i.e., $E_{source}\, E_{SCAT}$. This cross term represses a loss in intensity suffered by the source field as it acts to drive the radiators. The radiators are damped by their own resistance and thus what we see is not what was actual emitted, say from a star as its light excites intermediary material.

As a brief summary of what happens, because it is important to appreciate with the principle of superposition we can now explain virtually all interactions of electromagnetic waves and matter,

The sources makes a field, E_s

The field E_s acts on charges and their motion causes a scattering

The currents arising from the scattering generate a new field E_{SCAT}.

The total observed field is $E_{source} + E_{SCAT}$.

I haven't discussed the mechanism of how the scattered light generates the new field E_{SCAT}. I have only said such a field exists after light hits the scatterers. I'll worry about the mechanisms latter

BY CONSIDERING THE RESULTING FIELD AS A SUPERPOSITION OF THE SOURCE FIELD AND ALL OTHER FIELDS DUE TO CHARGES MOVING, I.E.,

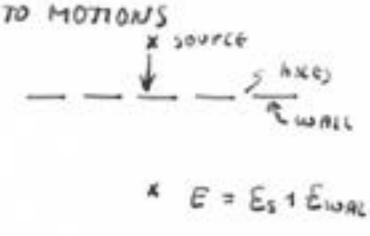
$$E = E_{source} + \sum_{\text{All other charges}} E_{\text{each charge}}$$

IT IS NOT NECESSARY TO MODIFY MAXWELL'S EQUATIONS. IN ORDER WORDS, BY SUBSTITUTING A $\vec{D}$ AND $\vec{H}$ INTO MAXWELL'S EQUATIONS YOU ARE COMPLICATING THE MATHEMATICAL REPRESENTATION OF A PHYSICALLY SIMPLE PROBLEM TO UNDERSTAND.

IF WE APPROACH PROBLEMS WITH THE ABOVE OUTLOOK THEY BECOME MUCH EASIER TO SOLVE. FOR INSTANCE, CONSIDER A BLACK PIECE OF PAPER BETWEEN A SOURCE AND THE OBSERVER. WE ALL KNOW THE OBSERVER DOESN'T SEE ANY LIGHT COMING THROUGH – BUT WHY. IF WE SAY THE FIELD AT THE OBSERVER IS THE FIELD DUE TO THE SOURCE, E_s, THEN SOMEHOW THE CHARGES AROUND US MUST EXACTLY CANCEL E_s OR ELSE WE WOULD SEE SOME LIGHT. THE ONLY THING AROUND IS THE BLACK PAPER; DOES IT GENERATE A FIELD, $-E_s$? HOW DOES IT KNOW TO DO THAT?

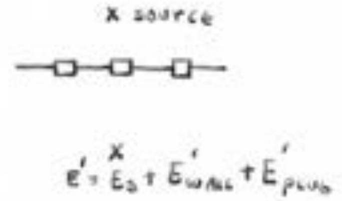
OF COURSE, THE CHARGES IN THE FACE OF THE PAPER WILL BE SET IN MOTION BY THE INCIDENT FIELD, E_s. THE FIELD DUE TO THE SCATTERERS AT THE TOP OF THE SHEET ARE BALANCED OUT BY THE FIELD FROM THE OTHER SIDE – THE PAPER DOES HAVE SOME THICKNESS. THERE IS NO RESIDUAL FIELD ~~CAUSE~~ BECAUSE THERE IS ENOUGH OPPORTUNITY TO FINALLY GET THINGS QUIETED DOWN. IN OTHER WORD E_s IS SUFFICIENTED ATTENUATED AS IT GOES THROUGH THE PAPER SO NO LIGHT COMES THROUGH. OF COURSE, IF THE SHEET IS THIN ENOUGH SOME LIGHT WILL COME THROUGH.

CONSIDER ANOTHER CASE WHERE WE WANT THE FIELD FROM A SOURCE PASSING A SCREEN WITH HOLES IN IT. WE EXPECT THE FIELD TO BE THE SUM OF THE SOURCE FIELD AND THE SCATTERED FIELD DUE TO MOTIONS IN THE LITTLE PIECES. THE FIELD IS FOUND USING OUR NEW TOOL.



LET'S PUT PLUGS IN THE HOLES AND MAKE THEM OF THE SAME MATERIAL AS THE WALLS. NOW WE KNOW THE FIELD E' MUST BE ZERO WITH THE PLUGS – WE JUST DETERMINED THIS. IF WE SUBTRACT E FROM E', WE GET,

$$E = (E_{wall} - E'_{wall}) - E'_{plug}$$



IF THE HOLES ARE NOT TOO SMALL (SAY MANY WAVELENGTHS), $E_{wall} \approx E'_{wall}$ EXCEPT FOR A SMALL EDGE EFFECT. SO NEGLECTING THESE EFFECTS, WE FIND, $E = -E'_{plug}$

THUS THE HOLES DIFFRACT LIGHT JUST LIKE A LINEAR ARRAY OF PLUGS AS WE HAVE DISCUSSED – ALMOST. REMEMBER WE ONLY HAVE AN APPROXIMATE ANSWER FOR LARGE HOLES.

Scattering of Light

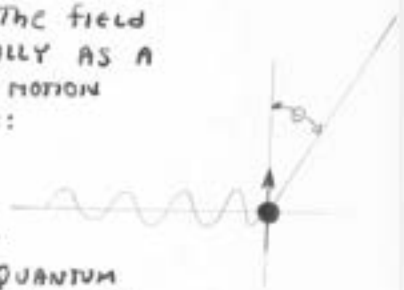
Last time we were discussing the scattering of light from a linear array of atom ~~ar~~ oscillators. From there we understood the phenomena of diffraction grating. We would nod like to extend our discussion of such first scattering approximations to the case of a diffused gas, i.e., where the index of refraction is about equal to one. In this case of matter and light interacting we can ignore the interaction of the scattered field back on the source field. In other words $E_{scattered}$ is much less than E_{source} so to a distant observer

$$\bar{E}_{observed} = \bar{E}_{source} + \bar{E}_{\substack{\text{total from}\\ \text{all scatterers}}} \approx \bar{E}_{source}$$

This approximation is what we called the first scattereting approximation.

Latter we will have to consider the case of material with a higher index of refraction say 1.2 or 1.3 and discuss some real things. In this case the scattered field is not ignorable and we must find out what effect it has on the total observed field. We now have to consider that the field at each atom is the sum of the source field and the field from all the neighboring chatges.

Let's return to the case of scattering from a dilute gas and study the simplest case of light striking an atom. The atom has a diameter which is much less than the incident wavelength. The atom will respond dynamically to the field and the resulting motion can be described mechanically as a simple harmonic motion. If we considered a damped motion due to some resistive force, the response is described as:



$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = qE$$

Now I must caution you that an atom does not behave as a classical harmonic oscillator. The atom obeys quantum mechanical predictions which is the better explanation of how nature really behaves. Since the problem is easy to solve using a harmonic oscillator model, physicists chose to use it. The success of classical physics depends on the use of the proper model for the problem in question.

If the incident beam has an electric field strength of $E = E_0 e^{i\omega t}$, then the atom will vibrate up and down and radiate in all directions. The atom's response to the field is given as

$$x = \frac{(q/m) E_0}{-\omega^2 + \omega_0^2 + i\gamma\omega}$$

Now to calculate the scattered field, E_{scat}, at some point R away from the atom we know the rule from previous lectures

$$E_{scat} = -\frac{q\, a_\perp (t - R/c) \sin\theta}{4\pi\epsilon_0 c^2 R}$$

Since x is given above we can find the acceleration and put it in the equation to get

$$E_{scat} = \frac{-q^2 E_0}{m\, 4\pi\epsilon_0 c^2}\left(\frac{-\omega^2}{-\omega^2 + i\omega\gamma + \omega_0^2}\right) \sin\theta \; e^{i\omega t}\, \frac{e^{-i\omega R/c}}{R}$$

If we want to calculate the intensity of the radiated field, we have to square E_{scat},

$$I = |E_{scat}|^2 = \frac{\sin^2\theta}{R^2}\left|\frac{-q^2\omega^2}{m\, 4\pi\epsilon_0 c^2(-\omega^2 + i\omega\gamma + \omega_0^2)}\right|^2 |E_0|^2$$

This formula gives the $1/R^2$ dependence, $\sin^2\theta$ gives the $\measuredangle$ dependence, $|E_0|^2$ compares it to a single oscillator and the middle term gives the phase dependence. In order to get rid of some of these extra terms we will borrow the cross-section idea from nuclear physics. If we imagine a beam pointed at some target with a cross section area σ, the total energy passing through this small area depends on the solid angle subtended, $R^2 d\Omega$.

The infinitesimal cross section of scattering is given by

$$d\sigma = \sigma\, d\Omega = \sin^2\theta \left|\frac{q^2}{4\pi\epsilon_0 m c^2}\left(\frac{-\omega^2}{-\omega^2 + i\omega\gamma + \omega_0^2}\right)\right|^2 d\Omega$$

The quantity inside the absolute value sign is very important and should have a name but it doesn't. It measures the degree of the scatterer's response to light.

If we consider light in air where the natural frequencies are higher than the incident light, we can disregard the ω^2 in the denominator and find the scattering proportional to ω^4. Thus light of higher frequencies by a factor of two, say, is 16 times more intensely scattered than the lower frequency. This is why blue light, which is about twice the frequency of red light, is scattered more and explains why the sky is blue.

For polarized light in a given direction

$$\sigma = (e_{in} \cdot e_{out})^2 r_0^2 \left|\frac{\omega^2}{\omega^2 - \omega_0^2 - i\gamma\omega}\right|^2$$

IF WE WANT THE TOTAL SCATTERING CROSS SECTION WE MUST INTEGRATE $d\sigma$ which IS EASY TO DO. If we recall $e^2 = \frac{q^2}{4\pi\epsilon_0}$ AND THE ~~sign~~ $\sin^2\theta$ INTEGRATES TO $2/3$, THEN

$$\sigma = \frac{8\pi R_0^2}{3} \, \omega^4 \left| \frac{1}{(-\omega^2 + i\omega\gamma + \omega_0^2)} \right|^2$$

where $R_0 = \frac{e^2}{mc^2} = 2.8 \times 10^{-13}\,\text{cm}$, THE CLASSICAL ELECTRON RADIUS

OR.

$$\sigma = \frac{8\pi R_0^2 \, \omega^4}{3\left[(\omega^2 - \omega_0^2) + \omega^2\gamma^2\right]}$$

TAKING THE CASE FOR VERY LOW NATURAL FREQUENCIES CORRESPONDS TO COMPLETELY UNBOUND ELECTRONS WITH NO RESISTANCE, I.E., THE X-RAYS, WE MAY WRITE

$$\sigma = \frac{8\pi R_0^2}{3} = 10^{-25}\ \text{cm}^2$$

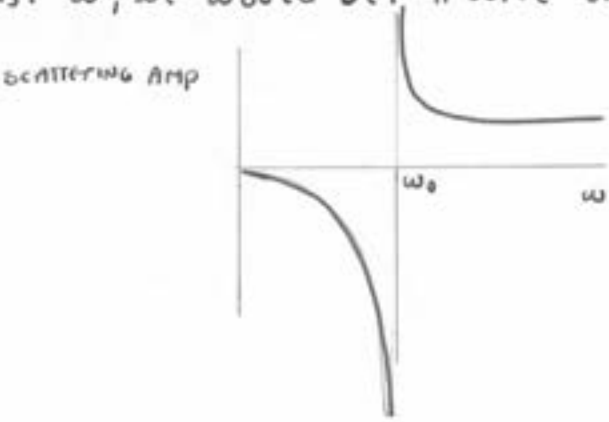
THIS IS KNOWN AS THE THOMPSON SCATTERING CROSS SECTION. Here we have have ASSUMED NEGLIGIBLE DAMPING.

ref VOL I chap 23

LET'S EXAMINE MORE CLOSELY THE CASE WHERE THE ~~RAD~~ RESISTANCE TERM IS AGAIN QUITE SMALL AND WE CAN WRITE

$$\sigma = \frac{8\pi R_0^2}{3} \frac{\omega^4}{(\omega_0^4 - \omega^2)}$$

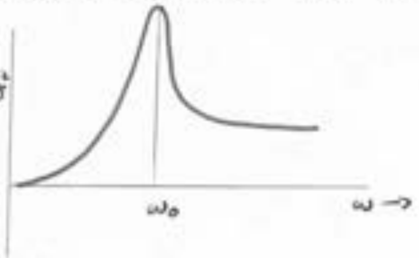
AS $\omega \to \omega_0$ THE SCATTERING BECOMES INFINITE. THIS IS CALLED APPROACHING RESONANCE. BUT NATURE WOULDN'T LET US GET BY WITH INFINITE SCATTERING SO WE BETTER PUT IN A LITTLE DAMPING. If we call $\frac{-\omega^2}{-\omega^2 + \omega_0^2 + i\omega\gamma}$ THE SCATTERING AMPLITUDE AND PLOT IT AGAINST ω, WE WOULD GET A CURVE LIKE THE FOLLOWING:



THIS IS A PLOT OF THE REAL PART OF THE AMPLITUDE ONLY. WHEN $\omega = \omega_0$ THE FUNCTION BECOMES IMAGINARY AND JUMPS TO THE OTHER SIDE OF THE AXIS AND EVENTUALLY SETTLES DOWN TO A CONSTANT VALUE.

I CAN PLOTT THE AMPLITUDE SQUARED AND GET A CURVE LIKE:

$$|AMP|^2 = \frac{\omega^4}{(\omega_0^2-\omega^2)^2 - \omega^2\gamma^2}$$

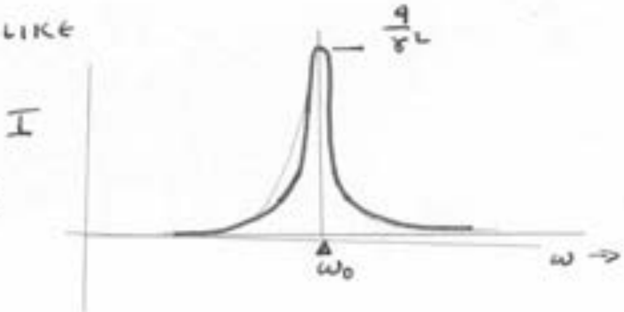


WHEN THE AMPLITUDE IS CLOSE TO RESONANCE WE MAY APPROXIMATE THE INTENSITY AS

$$I = \left| \frac{\omega_0^2}{(\omega_0-\omega)(\omega_0+\omega) - \frac{i\gamma}{2}} \right|^2 \approx \left| \frac{\omega_0^2}{2\omega_0(\omega_0 - \omega + \frac{i\gamma}{2})} \right|^2$$

THE CURVE NOW LOOKS LIKE

$$I = \frac{1}{(\omega-\omega_0)^2 + \frac{\gamma^2}{4}}$$



WHEN $\omega = \omega_0$ AND γ IS VERY SMALL, I IS VERY BIG. WHEN $I = \frac{1}{2} I_{MAX}$, THE WIDTH OF THE CURVE IS γ. SINCE $\gamma = \Delta\omega$, IT APPEARS THE FREQUENCY IS SMEARED OVER A SMALL RANGE OF VALUES. WHEN ω IS WITHIN $\pm \gamma/2$ OF ω_0 THE INTENSITY GETS BIG FAST. A MEASURE OF THE SPIKE IS THE Q OF THE CURVE. PREVIOUSLY WE SAW THAT $Q = \omega_0/\Delta\omega$; THEREFORE $Q = \omega_0/\gamma$. TO FIND OUT WHAT ~~ω~~ γ IS RECALL Q FOR AN ATOM IS ON THE ORDER OF 10^8. FOR $\lambda = 6000$ Å, $\omega_0 = 10^{15}$ CYCLE/SEC

$$\gamma = 10^7 \text{ CYCLE/SEC}$$

γ IS THEN THE WIDTH OF THE SPECTRAL LINES. SINCE THE ONLY REASON FOR LOSS OF ENERGY IS DUE TO RADIATION RESISTANCE, WE HAVE FOUND THE LINES FOR A FREELY RADIATING ATOM.

$$\Delta\lambda = \frac{\lambda}{Q} = \frac{4\pi r_0}{3} = 1.18 \times 10^{-14}\,m$$

THIS IS THE SMALLEST THE LINE WIDTH CAN BE. DUE TO INHOMOGENEITIES IN THE MATERIAL, THE RESPONSE TO FIELDS IS VARYING AND THE FREQUENCY BECOMES DISTRIBUTED AROUND ω_0

$$\sigma_{max} \propto \lambda^2$$

SINCE WE KNOW NOW AN OSCILLATOR WILL RADIATE LIGHT EVEN IF IT IS SITTING ALONE IN SPACE, WE CAN FIND OUT WHAT HAPPENS IF IT IS SUDDENLY STRUCK AND BEGINS TO OSCILLATE. IF γ IS SMALL, WE CAN WRITE

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0$$

TRYING THE SOLUTION $x = x_0 e^{i\alpha t}$ WE HAVE

$$-\alpha^2 + i\gamma\alpha + \omega_0^2 = 0$$

NOW IF $\alpha = \omega_0 + \epsilon$ WHERE ϵ IS SOME SMALL AMOUNT WE MAY WRITE

$$-\omega_0^2 - 2\omega_0\epsilon - \epsilon^2 - i\gamma\omega_0 - i\gamma\epsilon + \omega_0^2 = 0$$

IGNORING ϵ^2, $\gamma\epsilon$ TERMS WE FIND

$$\epsilon = i\gamma/2$$

THEN

$$\alpha = \omega_0 + i\gamma/2$$

THE RESPONSE IS THEN GIVEN AS

$$x = x_0 e^{i\omega_0 t} e^{-i\frac{\gamma}{2}t}$$

WHERE WE SHOULD HAVE TAKEN $\alpha = \omega_0 - \epsilon$ SINCE WE KNOW THE MOTION IS TO BE DAMPED OUT IN TIME - BUT SLOWLY. USING THE VALUE FOR γ FOUND ON THE PREVIOUS PAGE, WE SEE IT TAKES ABOUT 10^7 CYCLE FOR THE DISTURBANCE TO DIE TO $1/e$.

YOU MIGHT TRY ADDING THE NEXT RESISTANCE TERM TO THE EQUATION, I.E., $c\dddot{x}$ AND SEE WHAT YOU GET. NOW YOU WILL FIND 3 POLES. TWO ARE AT $\pm\omega_0$ BUT THE THIRD IS A VERY HIGH FREQUENCY IMAGINERY ROOT. IT CORRESPONDS TO THE TIME IT TAKES LIGHT TO CROSS THE ELECTRON. NOW WE GET INTO PROBLEM MODELING THE ELECTRON.

QUANTUM THEORY OF SCATTERING

IF I NOW CALL $\frac{e^2}{mc^2}\left(\frac{-\omega^2}{\omega_0^2-\omega^2+i\omega\gamma}\right)$ THE SCATTERING AMPLITUDE FOR A SINGLE OSCILLATOR, I CAN EXTEND MY RESULTS TO AN ATOM WHICH IS LIKE A LITTLE BOX MADE UP OF A LOT OF LITTLE OSCILLATORS WORKING AT DIFFERENT FREQUENCIES. IF THERE ARE 2 OSCILLATORS IN THE ATOM I MAY WRITE THE AMPLITUDE AS

$$\text{SCAT. AMP} = \frac{e^2}{m}(-\omega^2)\left\{\frac{f_1}{\omega_0^2-\omega_1^2+i\omega_1\gamma} + \frac{f_2}{\omega_0^2-\omega_2^2+i\omega_2\gamma}\right.$$

I HAVE INTRODUCED A FRACTION f_1 AND f_2 DEPICTING THE DIFFERENT STRENGTHS OF THE OSCILLATOR AS COMPARED TO A NORMAL ELECTRON. I CAN NOW CARRY MY SUMMATION TO A LARGE NUMBER OF ELECTRONS, I.E.,

$$\text{TOTAL SCAT. AMP} = \frac{e^2}{m}(-\omega^2)\sum_j \frac{f_j}{\omega_0^2-\omega_j^2+i\omega_j\gamma}$$

ω_j are the absorption frequencies at each level in the atom. The width of each line is characterized by γ. The oscillator sum rule is now useful; it says that for an atom with a number of electrons in it

$$\Sigma_j f_j = Z$$

Using this line of analysis, classical theory succeeded in predicting what a box full of harmonic oscillators would. It was when experiments uncovered a number of bumps in the intensity versus frequency curve that attention turned to modelling the atom on the harmonic oscillator. Heisenberg pursued the analogy to working out all the scattering amplitudes but his theory found it difficult to get the energy levels. He had to go through a complicated matrix algebra. Schroedinger, working from the bound wave theory, calculated the energy levels easily but for it hard to get the scattering amplitudes. Then it was realized that the two theories were the same.

FORWARD SCATTERING

So far we have only been concerned with scattering along some line making an angle θ with the incident light. We have have noticeably omitted one special direction, i.e., scattering in the forward direction. There is a phase correction which has to be made to our previous analysis to consider the variations in distance between the atom and the origin of the system. The following diagram may help.

The electric field at the atom is now a little out of phase with the driving field by an amount $e^{-i\omega z/c}$. Further when viewed at a very small angle θ, the scattered field will have a value

$$E_{scatter} = E_{scatter\ if\ z=0} \cdot \underset{(delay)}{e^{-i\omega z/c}} \cdot e^{+\frac{i\omega z}{c}\cos\theta}$$

In other words the new field appears to be the field due to an ~~cha~~ oscillator at the origin as before but now shifted in phase by an amount $kz(1-\cos\theta)$ where $k = \omega/c$. Therefore

$$E_{scatter} = \underset{normally}{E_{out}} \, e^{-ikz(1-\cos\theta)}$$

If the atom is displaced a little in the y direction, I can write this result in vector notation. I will use the notation that $\vec{K}$ denotes the propagation direction, then I can write

$$E_{scatter} = E_{out}\, e^{-i\bar{K}_{in}\cdot\bar{R}_{atom}}\, e^{+i\bar{K}_{out}\cdot\bar{R}_{atom}}$$
$$= E_{out}\, e^{i(\bar{K}_{out}-\bar{K}_{in})\cdot\bar{R}}$$

This result gives the diffraction from a crystal. For molecular scattering I may have a variety of different scattering centers each responding to the incident field with a unique amplitude a_j. We can measure the strength of the scattered wave as a function α where

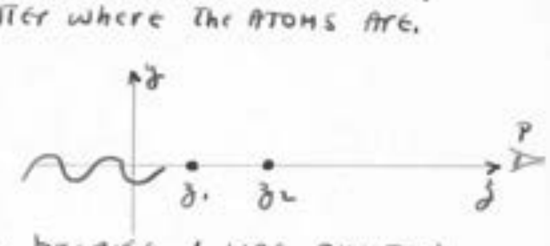
$$\alpha = \sum_j a_j e^{i(\bar{K}_{out}-\bar{K}_{in})\cdot\bar{r}_i}$$

Thus

$$\vec{E}_{scatter} = \alpha E_{incident}$$

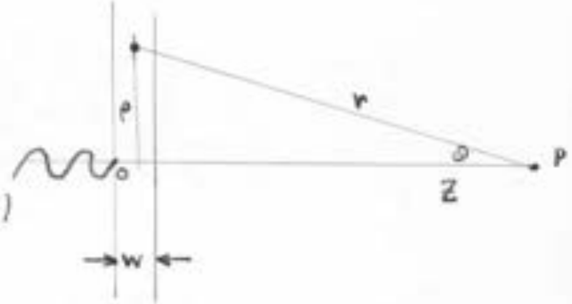
When $\lambda \approx r_i$, i.e., for x-rays, the scatter is intense and depend on where the atoms are. If $\lambda >> r$, then the phases average out to give a total field N times as strong.

If SCATTERING OCCURS IN THE FORWARD DIRECTION, THEN $K_{OUT} = K_{IN}$, AND EVERY ATOM CONTRIBUTES IN PHASE NO MATTER WHERE THE ATOMS ARE. IMAGINE TWO ATOMS A DISTANCE z_1 AND z_2 FROM AN ORIGIN. IF A FIELD COMES ALONG IT WILL DRIVE #1 BEFORE IT DRIVES #2. THE TIME LAG IS $(z_2 - z_1)/c$. SINCE THE FIELD ARRIVING AT THE OBSERVER IS THE SUM OF 1 AND 2 THEY ARE IN PHASE. THIS IS TRUE BECAUSE 1 WAS EXCITED EARLIER THAN 2 AND ITS SIGNAL TAKES $(z_2 - z_1)/c$ LONGER TO REACH P.



FIELD FROM A THIN SHEET OF GAS   ref. VOL I CHAPT 30.

CONSIDER A THIN SHEET OF GAS BEING DRIVEN BY AN INCIDENT FIELD, WHAT IS THE RESULTING FIELD TO AN OBSERVER. IF THE WIDTH OF THE SHEET IS W AND THERE ARE N ATOMS / CC, THEN THE FIELD SHOULD BE THE SUM OF ALL THE INDIVIDUAL FIELDS. IF EACH ATOM IS CONSIDERED TO BE DRIVEN AT THE SAME FREQUENCY, THE, WE CAN WRITE THE FIELD FROM ONE ATOM AS.



$$E_{ATOM} = \frac{q}{4\pi\epsilon_0 c^2} \frac{\omega^2 \left(\frac{q E_0}{m}\right)}{(-\omega^2 + \omega_0^2 + i\omega\gamma)} \frac{\cos\theta}{r} e^{i\omega(t - \frac{r}{c})}$$

WE MUST INTEGRATE OVER THE WHOLE SURFACE TO FIND THE TOTAL FIELD AT P.

$$\text{TOTAL FIELD AT P} = \frac{q^2}{4\pi\epsilon_0 m} \left(\frac{\omega^2}{-\omega^2 + \omega_0^2 + i\omega\gamma}\right) N W \int_{\rho=0}^{\rho=\infty} \frac{e^{i\omega(t - r/c)}}{r} 2\pi r \, dr$$

SINCE $r^2 = \rho^2 + z^2$ WE CAN WRITE $r\,dr = \rho\,d\rho$ AND REWRITE THE INTEGRAL,

$$I = \int_{r=z}^{r=\infty} e^{-i\omega r/c} \, dr$$

THIS INTEGRAL IS EASY INTEGRATE BUT A LITTLE DIFFICULT TO EVALUATE

$$I = -\frac{c}{i\omega}\left[e^{-i\omega\infty} - e^{-i\frac{\omega z}{c}}\right]$$

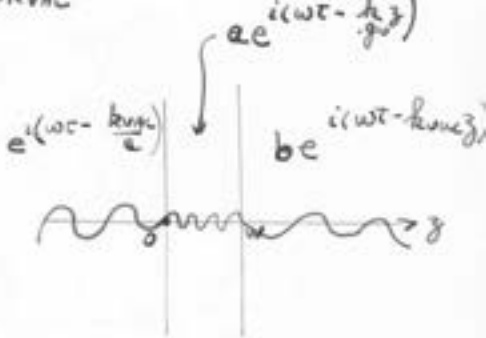
THE TERM $e^{-i\omega z/c}$ CAUSES NO PROBLEM BUT $e^{-i\infty}$ DOES. SINCE $e^{-i\infty}$ MEANS $\cos(-\infty)$, THE TERM BECOMES INDEFINITE. HOWEVER, PHYSICALLY SPEAKING SUCH A TERM HAS MEANING AND USUALLY CAN BE TAKEN TO BE ZERO. THE SURFACE CHARGE CANNOT BE INFINITE IN EXTENT. IF THE UPPER LIMIT IS TAKEN TO BE L AND THE LIMIT OF ∞ APPROACHED, THIS TERM DOES APPROACH ZERO.

The final result can be written now.

$$E_{\substack{\text{forward}\\ \text{scatter}}} = \frac{2\pi g^2 N E_0 W}{4\pi\epsilon_0 m}\left(\frac{\omega^2}{-\omega^2+\omega_0^2+i\gamma\omega}\right)\frac{c}{i\omega}\, e^{i\omega(t-z/c)}$$

How can we best interpret this result? Some people like to say that the light has a modified wavelength inside the sheet. Or the sheet has and index of of refraction, n. We can write

$$\lambda_{gas} = \frac{\lambda_{vac}}{n} \quad \text{or} \quad k_{gas} = n\, k_{vac}$$



Since the incident wave must match the wave in the media · at $z=0$, we know $a=1$. Further matching boundary conditions at $z=W$ we get

$$e^{i(\omega t - k_{gas} W)} = b\, e^{i(\omega t - k_{vac} W)}$$

$$b = e^{i(k_{vac} - k_{gas})W}$$

We can write the outgoing wave as

$$E_{\text{out wave}} = E_0\left[e^{-i(k_{vac} - k_{gas})W}\right] e^{i(\omega t - z/c)}$$

Since W is infinitesimal I can expand the first term to get

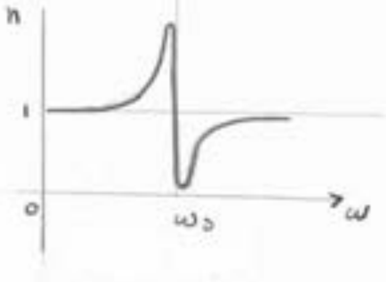
$$E_{\text{out wave}} = E_0\left[1 - i k (n-1) W\right] e^{i(\omega t - z/c)}$$

Thus we see the outgoing wave is the sum of two waves. The field $E_0 e^{i(\omega t - z/c)}$ is just the source field. The second field is the field from all the atoms in the sheet. The factor $[1 - ik(n-1)W]$ gives the intensity of the forward wave.

If we compare the result just obtained with the one previously given at the top of the page, it is possible to find n as a function of ω, i.e.,

$$n = 1 + \frac{N q_e^2}{2\epsilon_0 m\left(-\omega^2+\omega_0^2+i\omega\gamma\right)}$$

A plot of n as a function of ω is called the dispersion curve. It displays the same funny behavior around the resonance frequency, ω_0, that we talked about last time.



DISPERSION CURVE

TYPICALLY $n > 1$ AND THE CURVE HAS A POSITIVE SLOPE; THIS IS TERMED NORMAL DISPERSION. WHEN ω IS VERY NEAR ω_0, THE SLOPE IS NEGATIVE AND n DECREASES WITH ω. THIS IS CALLED ANOMALOUS DISPERSION. ACCOMPANYING ANOMALOUS DISPERSION IS APPRECIABLE ABSORPTION.

THE ABSORPTION COMES ABOUT BECAUSE n IS A COMPLEX NUMBER AND CAN BE WRITTEN AS THE SUM OF A REAL AND IMAGINARY PART, I.E.,

$$n = n_R - i\,n_{imag.}$$

$$n = 1 + \frac{q^2}{2m\epsilon_0}\frac{N\,(-\omega^2+\omega_0^2)}{\left[(-\omega^2+\omega_0^2)^2+(\gamma\omega)^2\right]} + \frac{i\,q^2 N\gamma\omega}{2m\epsilon_0\left[(-\omega^2+\omega_0^2)^2+(\gamma\omega)^2\right]}$$

SINCE THE WAVES IN THE MATERIAL VARY IN SPACE AS

$$e^{-ik_{gas}z} = e^{-in\frac{\omega}{c}z}$$

AND SUBSTITUTING FOR n, $n_R - i\,n_I$ THE WAVE BECOMES PROPORTIONAL TO

$$e^{-in_R\frac{\omega}{c}z}\,e^{-n_I\frac{\omega}{c}z}$$

WE SEE A REMARKABLE THING THAT THE OUT GOING WAVE IS NOT THE SAME SIZE AS THE INCIDENT WAVE BUT IS WEAKER. THE RATIO OF THE OUT GOING INTENSITY TO THE INGOING IS

$$\frac{I_{out}}{I_{in}} = e^{-2n_I\frac{\omega}{c}z}$$

IF I CALL $\alpha = 2n_I\frac{\omega}{c}$ THE LOGARITHMIC ADSORPTION COEFFICIENT, THE DECREASE IN I WITH PENETRATION z IS

$$\frac{dI}{dz} = -\alpha e^{-\alpha z}$$

THE INTENSITY OF THE OUTGOING WAVE IS THUS DECREASED AS MIGHT BE EXPECTED SINCE THE OSCILLATOR IS DAMPED BY A FRICTION FORCE. THIS IMPLIES A LOSS OF POWER WHICH IS EQUIVALENT TO THE WORK DONE ON THE OSCILLATORS IN SHEET.

I CAN FIND THE WORK DONE ON THE OSCILLATORS FROM THE EQUATION OF MOTION

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = q E_{in} e^{i\omega t}$$

THE WORK DONE BY THE FORCE IS JUST $q E_{in} e^{i\omega t}\dot{x}$ WHERE $\dot{x} = i\omega x$. THEREFORE,

$$\text{Work} = q E_{in}\,\frac{i\omega q}{-\omega^2+\omega_0^2+i\gamma\omega}\,e^{i\omega t}$$

I AM GOING TO NEED A LITTLE THEOREM IN A MINUTE THAT SAYS THE AVERAGE of TWO COMPLEX NUMBERS a AND b, SAY IS GIVEN AS

$$\overline{ab} = a^* b + b^* a$$

The POWER LOSS PER OSCILLATOR IS GIVEN AS THE REAL PART of $\frac{i\omega}{-\omega^2+\omega_0^2+i\gamma\omega}$ or $\frac{\omega^2\gamma}{(-\omega^2+\omega_0^2)^2+\gamma^2\omega^2}$ So The TOTAL POWER LOSS PER UNIT AREA IS

$$\text{Power}_{\text{LOSS}} = E_{\text{INCID}}^2 \frac{e^2}{m} \frac{\omega^2 \gamma \; NW}{\left[(-\omega^2+\omega_0^2)^2+\gamma^2\omega^2\right]}$$

NOW The POWER IN The OUTGOING BEAM IS DETERMINED from the formula,

$$E_{\text{forward}} = E_0 e^{i\omega(t-z/c)} \frac{e^2}{m}\left(\frac{\omega^2}{-\omega^2+\omega_0^2+i\gamma\omega}\right)\frac{c}{i\omega} 2\pi N\omega$$

The TOTAL field IS

$$E_{\text{TOTAL}} = E_{\text{IN}} + E_{\text{for}} = (1+f)\, E_0 e^{i\omega(t-z/c)}$$

where

$$f = \frac{e^2}{m}\left(\frac{\omega^2}{-\omega^2+\omega_0^2+i\gamma\omega}\right)\frac{c}{i\omega} 2\pi N\omega$$

The TOTAL POWER IS GIVEN AS

$$|E_{\text{TOT}}|^2 = |E_0|^2 (1+f)(1+f^*) = |E_0|^2 (1+f+f^*+ff^*)$$

SINCE $ff^* \propto W^2$ IT IS NEGLIGIBLE AND WE ARE LEFT WITH

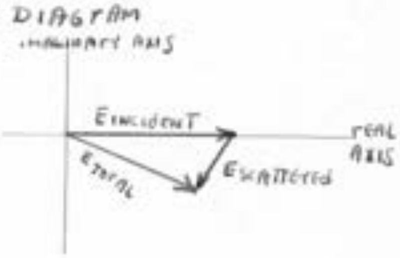
$$|E_{\text{TOT}}|^2 = |E_0|^2 + (f+f^*)\,|E_0|^2$$

where The first TERM IS The INCIDENT INTENSITY AND The SECOND CORRESPONDS TO LOSSES. I NEED The REAL PART of $f+f^*$ AND IT IS

$$\text{Real}(f+f^*) = \frac{4\pi\,\gamma\,\omega^2\, NW}{(-\omega^2+\omega_0^2)^2+(\gamma\omega)^2}$$

I HAVE MESSED UP A 4π SOMEWHERE BUT THAT HAS TO DO WITH DEFINING THE ABSOLUTE POWER IN TERMS of The POINTING VECTOR SO I WON'T WORRY ABOUT IT. The IMPORTANT THING IS THAT The CONSERVATION of ENERGY DOES HOLD AND WE CAN ACCOUNT for The POWER LOSSES.

The LOSSES CAN BE PICTURED BY DRAWING A COMPLEX DIAGRAM of The INCIDENT AND SCATTERED WAVES. AS RESONANCE IS APPROACHED The SCATTERED WAVE SUBTRACTS MORE AND MORE from The INCIDENT WAVE. While far from RESONANCE $E_{\text{SCATT.}}$ IS VERY SMALL AND $E_{\text{TOTAL}} \approx E_{\text{INC}}$.



When we wrote the index of refraction as a complex number, we saw the wave propagating as $e^{i\omega(t-\frac{n_R}{c}z)}\, e^{-n_I\frac{\omega}{c}z}$. In other words the phase was changing at a velocity given as

$$v_{ph} = \frac{c}{n_{real}}$$

If $n_{real} > 1$ as for most materials then v_{ph} is less than c. But for some cases say with X-rays or free electrons $n_{real} < 1$ and $v_{ph} > c$. This does not mean signals travel faster than c, but rather only beats in the wave train. The rate at which signals propagate is given by the group velocity defined as

$$v_{group} = \frac{d\omega}{dk} = \frac{1}{dk/d\omega}$$

If we write $\quad n = 1 - \frac{a}{\omega^2}$ and $k = n\frac{\omega}{c} = \frac{\omega}{c} - \frac{a}{\omega c}$

$$v_{gr} = \frac{c}{1 + \frac{a}{\omega^2}}$$

Now we have been incomplete in our logic here due to the problem of anomalous dispersion where the width of a wave group is wider than the frequency band about ω_0.

We can establish a condition which n has to satisfy so we never have trouble with signals going faster than c. We would like to find enough absorption so when we go through resonance, things won't get screwed up.

Let's imagine ~~imagi~~ a medium through which a burst of light travels and rattles the atoms. The forward scattered wave is given as $\quad E_{for} = f(\omega)\, E_0\, e^{i\omega t}$

where $f(\omega)$ is like the parameter α earlier discussed. Now the source field can be written as a Fourier sum

$$E_0(t) = \int \mathcal{E}_0(\omega)\, e^{i\omega t}\, \frac{d\omega}{2\pi}$$

where the initial condition is $\quad E_0(t) = 0$ for $t < 0$. The forward wave can be written as

$$E_{for.} = \int f(\omega)\, \mathcal{E}_0(\omega)\, e^{i\omega(t - z/c)}\, \frac{d\omega}{2\pi}$$

We'll first consider the special source field where $\mathcal{E}_0$ is a delta function, $E_0(t) = \delta(t)$. In this case the Fourier transform is a constant, $\mathcal{E}_0(\omega) = 1$. Thus the output for a sudden pulse can be written as

$$F(t) = \int_{-\infty}^{\infty} f(\omega)\, e^{i\omega t}\, \frac{d\omega}{2\pi} = 0 \quad \text{for } t < 0$$

where $\mathcal{E}_0(\omega) = 1$

Now the problem is reduced to asking about the properties of $f(\omega)$. If $F(t) = 0$ for $t < 0$, then surely

$$\int_{-\infty}^{0_-} F(t)\, e^{-i\nu t} e^{\epsilon t}\, dt = 0 \qquad \text{ALL } \nu$$

where ϵ is any real positive quantity. Thus we can write

$$\int \frac{d\omega}{2\pi} f(\omega) \int_{-\infty}^{0} e^{i\omega t} e^{-i\nu t} e^{\epsilon t}\, dt = 0$$

The integration can be done as before and we have

$$I = \int \frac{d\omega}{2\pi} \frac{f(\omega)}{(i\omega - i\nu + \epsilon)} = 0 \qquad \text{ALL } \nu, \text{ ANY } \epsilon > 0$$

rewriting
$$\frac{1}{i\omega - i\nu + \epsilon} = \frac{-i}{\omega - \nu - i\epsilon} = \frac{-i}{(\omega-\nu)^2 + \epsilon^2}\left[(\omega-\nu) + i\epsilon\right]$$

The integral becomes

$$I = \int \frac{d\omega}{2\pi} \frac{\text{Prin. Value}}{\omega - \nu} f(\omega) + \int \frac{d\omega}{2\pi}\, i\,\delta(\omega-\nu) f(\omega) = 0$$

or

$$f(\nu) = \frac{i}{\pi} \int d\omega \frac{\text{P.V}}{\omega - \nu} f(\omega)$$

The conclusion here is that the real and imaginary parts of $f(\omega)$ are not separably determinable. Once one is determined, the other is as well. If you know the index of refraction, then you know exactly the absorption and vice versa.

Rewriting this expression we find

$$\frac{n^2-1}{n^2+2} = \frac{1}{3}\chi = \frac{4}{3}\frac{\pi e^2 N}{m(-\omega^2+\omega_0^2+i\gamma\omega)}$$

This equation is called the Clausius Mosotti equation. Note since $\frac{n^2-1}{n^2+2}$ is proportional to the electron density, it follows the tighter the electrons are packed the bigger the change in this fraction. But this is not strickly true since distortions due to compression tend to alter the natural frequencies by disturbing the oscillators.

The above equation should give the same result for the index of refraction of a gas previously found. If n is close to 1 so we can approximate n^2+2 as 3 and n^2-1 as $2(n-1)$, then we find that

$$n-1 = \frac{\chi}{2}$$

or

$$n = 1 + \frac{2\pi e^2 N}{m(-\omega^2+\omega_0^2+i\gamma\omega)}$$

This answer does agree with our other answer.

Notice also that n is a complex number that can be written as

$$n = n_R - i n_I$$

The resulting electric field thus has its amplitude exponentially decayed as $e^{-\omega n_I \omega/c}$. That is to say the wave is absorbed as it goes into the material.

Waves in metals

Another interesting case is for waves inside of metals. In metals the electrons are not bound to any particular atom and are free to move about. To consider the effect of these free electrons on the resulting field we start by noting that there is no $\omega_0^2 x$ term in the equation of motion. This simplies implies no restoring force. This time, however, the field at the atoms is just the average field since the conduction electrons are free to move about and their fields average out.

For a metal then we can write

$$n^2 = 1 + \chi = 1 + \frac{4\pi e^2 N}{m(-\omega^2+i\gamma\omega)}$$

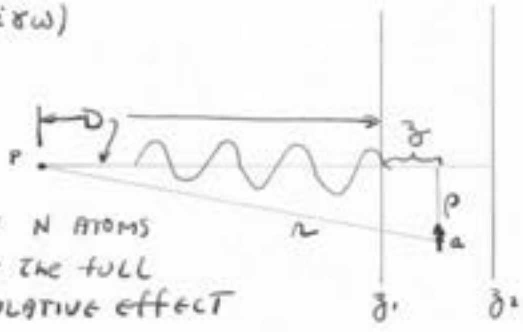
Reflected Light

I have been asked to work out the backward wave coming from a layer of material which has an electric field incident on it. If we consider the incident field varies sinusoidally in time and only has a disturbance in the z direction, then an atom in the material will have a response given by

$$x = \frac{q E_0 e^{i\omega t} e^{-ikz}}{m(-\omega^2 + \omega_0^2 + i\gamma\omega)}$$

The expression for the field at P due to an atom at "a" is given as

$$E_a = \frac{q}{4\pi\epsilon_0} \frac{\text{acceleration retard}}{r}$$



As we have seen before. If there are N atoms per cc ~~where~~ we must integrate over the full volume of the material to get the cumulative effect of all the atoms. That is,

$$E = \int_{z_1}^{z_2} \frac{-\omega^2 q^2 E_0 e^{i\omega t} e^{-ikz} e^{-ikr}}{4\pi\epsilon_0 m(-\omega^2 + \omega_0^2 + i\gamma\omega)\, r} \, 2\pi\rho\, d\rho\, N\, dz$$

since $r^2 = \rho^2 + (D+z)^2$ we can integrate to find

$$E = \frac{-\omega^2 q^2 E_0 e^{i\omega t} e^{-ikD}}{4\pi\epsilon_0 (-\omega^2 + \omega_0^2 + i\gamma\omega)} \cdot \frac{1}{ik}\left(e^{-2ikz_1} - e^{-2ikz_2}\right)$$

The first term in the parenthesis indicates reflection from the front surface of the layer while the second term gives the back layer reflection. The negative sign indicates they are 180° out of phase. Thus we can think of the backward scattered light ~~can~~ as being composed of two interfering reflections as if the intermediary material was not present.

We'll come back to this subject latter because the above analysis is not too satisfactory and it needs a better treatment.

Refractive index of Dense Material

I'd like to go on and discuss a problem which I have not attempted to explain yet. I want to get rid of the idea that the only field at the atom is due to external source. I now want to consider the effect of local fields created by neighboring atoms and discuss the resulting behavior of the atom under study. To do this I must begin talking about the polarization of matter.

Let's begin by considering the ~~atoms~~ electrons to be bound by a force proportional to its displacement and thus respond harmonically to a disturbing field. The equation of motion is given by

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2 x) = q\,E_{\text{at atom}}$$

where the disturbing force is proportional to the local field about the electron due to the original source and the neighboring oscillators. I want to solve for the electron's response and then find the resulting currents. With the currents I can then find the fields from Maxwell's equations. I'll then add the source field to find $E_{\text{at atom}}$ so I have a self consistent analysis. Recall Maxwell's equations where:

$$\bar{\nabla}\cdot\bar{E} = \bar{\rho}/\epsilon_0 \qquad \bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{B} = \frac{\partial E}{\partial t} + \dot{\jmath}/\epsilon_0 \qquad \bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

The question is what are ρ and $\dot{\jmath}$ in this case?

If the oscillator is isotropic all the resulting fields will be time varying with the same sinusoidal frequency and go as $e^{i\omega t}$. Thus

$$x = \frac{q\,E_{\text{at atom}}}{m(-\omega^2 + \omega_0^2 + i\gamma\omega)}$$

The resulting currents can now be computed. First the polarization density, $\bar{P}$, of the material is defined as

$$\bar{P} = N\bar{p}$$

where $\bar{p}$ = the induced dipole moment of the atom $= qx$ or

$$\bar{p} = \frac{q^2/m\ \bar{E}}{-\omega^2 + i\gamma\omega + \omega_0^2}$$

since $\bar{p}$ is proportional to $\bar{E}$, we can rewrite this expression as

$$\bar{p} = \epsilon_0\alpha\bar{E}$$

where α = atomic polarizability = $\dfrac{q^2/m\epsilon_0}{(-\omega^2 + i\gamma\omega + \omega_0^2)}$

Thus we have for the total polarization density

$$\bar{P} = \epsilon_0 N\alpha\bar{E}$$

Note at high frequencies α is small and there is not much response. At low frequencies there can be a strong response. α is also complex so the polarization may be shifted in phase from the incident $\bar{E}$.

IN PREVIOUS DISCUSSIONS WE FOUND THAT THE POLARIZATION GIVES RISE TO A CHARGE DENSITY GIVEN AS

$$\rho_{POL} = -\bar{\nabla}\cdot\bar{P}$$

NOW BECAUSE THE CHARGES ARE MOVING THERE IS A POLARIZATION CURRENT. THE CURRENT DENSITY IS GIVEN AS

$$\bar{j}_{POL} = Nq\bar{v} = Nq\dot{x}$$

SINCE $\bar{P} = qNX$ WE HAVE THAT $qN\dot{x} = \dot{P}$ OR THAT

$$\bar{j}_{POL} = \dot{\bar{P}}$$

IF WE CONSIDER OTHER CHARGES THAT THOSE INDUCED BY THE POLARIZATION, WE HAVE TO CONSIDER THE TOTAL CHARGE DENSITY TO BE THE SUM OF TWO CONTRIBUTIONS,

$$\rho = \rho_{POL} + \rho_{OTHER} = -\bar{\nabla}\cdot\bar{P} + \rho_{OTHER}$$

LIKEWISE FOR THE CURRENT WE HAVE

$$\bar{j} = \bar{j}_{POL} + \bar{j}_{OTHER} = \dot{\bar{P}} + \bar{j}_{OTHER}$$

MAXWELL'S EQUATIONS ARE MODIFIED THEN TO INCLUDE THESE TOTAL CHARGE DENSITY AND TOTAL CURRENT DENSITY,

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho_{POL} + \rho_{OTHER}}{\epsilon_0} = -\frac{\bar{\nabla}\cdot\bar{P}}{\epsilon_0} + \frac{\rho_{OTHER}}{\epsilon_0}$$

$$\bar{\nabla}\times\bar{B} = \frac{\partial\bar{E}}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial\bar{P}}{\partial t} + \bar{j}_{OTHER}$$

$$\bar{\nabla}\cdot\bar{B} = 0 \quad \text{AND} \quad \bar{\nabla}\times\bar{E} = -\partial\bar{B}/\partial t$$

AS A HISTORICALLY ASIDE IT IS COMMON TO REWRITE THESE EQUATIONS BY SUBSTITUTING FOR $\epsilon_0\bar{E} + \bar{P}$ THE VECTOR $\bar{D}$. AND THUS WRITING

$$\bar{\nabla}\cdot\bar{D} = \rho_{OTHER} \qquad \bar{\nabla}\times B = \frac{1}{\epsilon_0}\frac{\partial D}{\partial t} + j_{OTHER}$$

SO FAR WE HAVEN'T RELATED $\bar{E}_{AT\ ATOM}$ TO $\bar{E}$ WHERE $\bar{E}$ IS THE AVERAGE FIELD OVER THE WHOLE MATERIAL. IF WE CONSIDER THE CASE WHERE THE INCIDENT LIGHT HAS A WAVELENGTH MUCH GREATER THAN THE ATOMIC SPACING, SAY 5000 Å TO 2 Å, THEN THE LOCAL FIELD IS INCREASED OVER THE AVERAGE FIELD. THE AMOUNT OF INCREASE WE WORKED OUT BEFORE WHEN WE CONSIDERED AN ATOM IN A SPHERICAL HOLE IN A DIELECTRIC. THE LOCAL FIELD IS GIVEN AS

$$\bar{E}_{AT\ ATOM} = \bar{E} + \frac{\bar{P}}{3\epsilon_0}$$

SINCE $\bar{P} = \epsilon_0 N\alpha \bar{E}_{ATOM} = \epsilon_0 \chi \bar{E}_{ATOM}$ WHERE $\chi = \dfrac{q^2 N}{m\epsilon_0(-\omega^2 + \omega_0^2 + i\gamma\omega)}$ WE CAN WRITE

$$E_{ATOM} = E_{AVE} + \frac{\chi}{3}E_{ATOM}$$

FINALLY, THEN WE CAN WRITE E_{ATOM} IN TERMS OF E_{AVE},

$$\overline{E}_{ATOM} = \frac{1}{1-\chi/3}\,\overline{E} \quad \text{AND} \quad \overline{P} = \epsilon_0 \frac{\chi}{1-\chi/3}\,\overline{E}$$

MOTION OF WAVES IN A MATERIAL

LET'S BEGIN WITH THE SIMPLEST CASE OF A WAVE GOING IN THE z-DIRECTION. IF THE ELECTRIC FIELD IS POLARIZED IN THE x DIRECTION

$$E_x = E_0\, e^{i(\omega t - kz)}$$

THE OPERATION $\partial/\partial t$ GIVES A FACTOR OF $i\omega$ AND $\overline{\nabla}$ GIVES $-ik\bar{e}_z$

SOLVING MAXWELL'S EQUATIONS, WE HAVE FOR THE CASE OF NO OTHER CHARGES,

$$\bar{\nabla}\cdot\bar{E} = -\frac{\nabla\cdot P}{\epsilon_0}$$

WHICH GIVES $\quad -ik\tilde{E}_z = +\frac{ik}{3}P_z = ik\left(\frac{\chi}{1-\chi/3}\right)E_z$

THIS CONDITION IS SATISFIED ONLY IF $E_z = 0$. THIS IS NOTHING NEW. FROM $\nabla\cdot\bar{B} = 0$ WE CAN SHOW $B_z = 0$. FURTHER $E_Y = 0$, $B_x = 0$, $E_x = E_0$ AND $B_y = 0$. THAT IS FOR THE LAST TERM

$$(\nabla\times B)_x = \left(\frac{\partial E}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial P}{\partial t}\right)_x$$

$$ikB_y = i\omega\left(E + \frac{\chi}{1-\chi/3}E\right) = i\omega\left(\frac{1+2/3\chi}{1-\frac{1}{3}\chi}\right)E_x$$

AND FROM $\nabla\times E = -\partial B/\partial t$ WE GET

$$(\bar{\nabla}\times\bar{E})_y = -\frac{\partial B_Y}{\partial t} = -i\omega B_Y$$

OR

$$-ikE_x = -i\omega B_Y$$

TO SATISFY THESE CONDITIONS WE MUST REQUIRE THAT

$$k^2 = \omega^2\,\frac{(1+\frac{2}{3}\chi)}{(1-\frac{1}{3}\chi)}$$

IT IS A LITTLE MORE CONVENTIONAL TO DEFINE THE ABOVE CONDITION IN TERMS OF THE INDEX OF REFRACTION, n, WHERE

$$n = \frac{kc}{\omega}$$

THIS IMPLIES THAT THE FIELD GOES AS

$$E_x = E_0\, e^{i\omega(t - nz/c)}$$

IF $c=1$, WE CAN WRITE

$$n^2 = \frac{1+\frac{2}{3}\chi}{1-\frac{1}{3}\chi} = \frac{3+2\chi}{3-\chi}$$

NOTE WE CAN WRITE $B_Y = nE_x$ AND FOR A VACUUM $B = E$.

The DAMPING FACTOR γ IS IMPORTANT here BECAUSE IT TELLS US how FAST The ELECTRONS ARE SLOWED DOWN AS The SCATTER About. If WE SOLVE The EQUATION

$$m(\ddot{x} + \gamma\dot{x}) = qE$$

for The CASE of NO force WE find THAT

$$\dot{x} = v = v_0 e^{-\gamma t}$$

The velocity, v, is The AVERAGE drift velocity of The electrons. The velocity decays with A TIME CONSTANT of

$$\tau = \frac{1}{\gamma}$$

γ MEASURES The LIFETIME of The ELECTRON'S MOMENTUM. Further it is related to ELECTRICAL CONDUCTIVITY.

For A CONSTANT D.C. Field, $\dot{x}$ is CONSTANT AND GIVEN BY

$$\dot{x} = \frac{q}{m\gamma} E = \frac{qE}{m} \tau$$

where qE/m is The ACCELERATION ON The ELECTRON. The QUANTITY $q/m\gamma$ is called The MOBILITY, μ. The current density, j, is GIVEN AS

$$j = Nqv = \frac{Nq^2}{m\gamma} E = \sigma E$$

The conductivity, σ, is defined in The Above EQUATION.

Now WE CAN REWRITE The INDEX of refraction in ALL sorts of WAYS USING These NEW SYMBOLS. ONE SUCH EXPRESSION IS

$$n^2 = 1 + \frac{\sigma/\epsilon_0}{i\omega(1 + i\omega\tau)}$$

PLASMA frequency

If WE consider A DILUTE GAS IN which The ELECTRONS ARE NEARLY free, $\gamma \approx 0$ AND WE CAN write The INDEX of refraction AS

$$n^2 = 1 - \frac{4\pi N e^2}{m\omega^2}$$

For waves of high frequencies The index becomes real while AT low frequencies The index CAN be pure IMAGINARY implying tremendous Absorption below some critical frequency. If we write

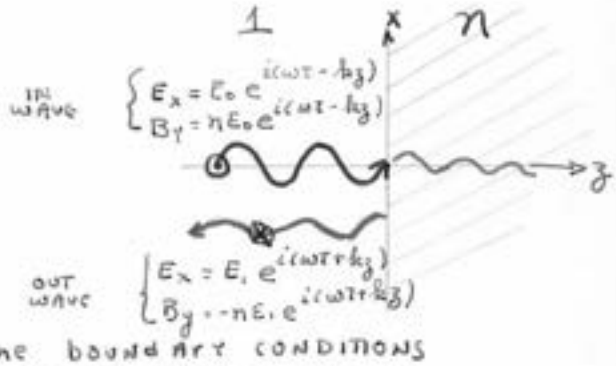
$$n^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

where $\omega_p^2 = 4\pi N e^2/m$ is called The PLASMA frequency, WE SEE when $\omega = \omega_p$ we are AT The critical frequency. For $\omega < \omega_p$ The index is IMAGINARY. When $\omega >> \omega_p$ The index is real And The MATERIAL is TRANSPARENT. This is why METALS ARE TRANSPARENT TO X-RAYS. IT is Also why The ionosphere reflects below A certain frequency And is TRANSPARENT Above IT.

NOW I WANT TO TALK ABOUT REFLECTIONS OF WAVES FROM ONE MEDIA TO ANOTHER. FIRST WE WILL CONSIDER THE CASE OF REFLECTIONS FROM A SURFACE WHICH SEPARATES TWO REGIONS OF WIDELY VARYING INDEX OF REFRACTION. BY THIS I MEAN n CHANGES RAPIDLY WITHIN A DISTANCE WHICH IS MUCH LESS THAN A WAVELENGTH, I.E., ABOUT 5000 Å.

LET'S FIRST CONSIDER THE CASE OF A WAVE MOVING THROUGH A VACUUM AND STRIKING A MATERIAL WITH INDEX n.

I WILL ASSUME THAT THE WAVE IS POLARIZED IN THE X-DIRECTION. NOW WE CAN FIND THE BACK-SCATTERED WAVE AND FIND ITS PHASE AND DETERMINE HOW IT INTERFERES WITH THE INCOMING WAVE, BUT IT IS A LITTLE MORE DIFFICULT TO FIND THE RESULTING AMPLITUDE AND INTENSITY OF THE REFLECTED LIGHT SO WE'LL DO IT.



THE PROBLEM WE MUST SOLVE INVOLVES THE BOUNDARY CONDITIONS AT THE SURFACE. TRADITIONALLY THE BOUNDARY CONDITIONS WERE EXPRESSED LIKE B_{normal} AND $E_{tangential}$ ARE CONTINUOUS OR SOME CRAP LIKE THAT. IT IS GIVEN TO YOU IN SOME FORM THAT CAN ONLY BE RETAINED FOR ABOUT 20 MINUTES. BUT IT IS MUCH EASIER TO WORK DIRECTLY FROM THE DIFFERENTIAL EQUATIONS OF MAXWELL AND APPLY A LITTLE PHYSICAL SENSE TO THE PROBLEM.

FIRST LET'S WRITE OUT MAXWELL'S EQUATIONS FOR THE MATERIAL

$$\bar{\nabla}\cdot\bar{B} = 0, \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{1}$$

$$\bar{\nabla}\cdot\bar{E} = -\frac{\bar{\nabla}\cdot\bar{P}}{\epsilon_0}, \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -\frac{1}{\epsilon_0}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) \tag{2}$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t},$$

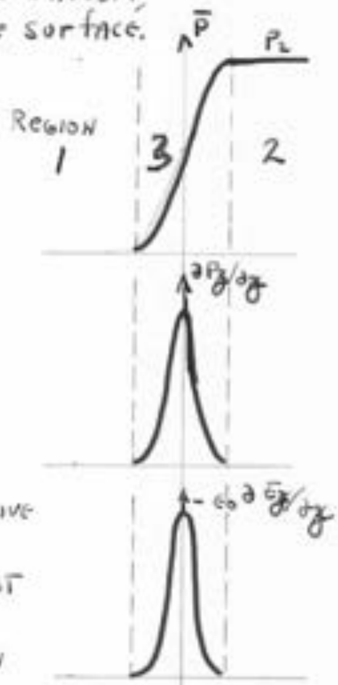
$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} \end{aligned} \tag{3}$$

$$\bar{\nabla}\times\bar{B} = \frac{\partial\bar{E}}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial P}{\partial t},$$

$$\begin{aligned} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \frac{\partial E_x}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial P_x}{\partial t} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= \frac{\partial E_y}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial P_y}{\partial t} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \frac{\partial E_z}{\partial t} + \frac{1}{\epsilon_0}\frac{\partial P_z}{\partial t} \end{aligned} \tag{4}$$

The whole question of boundary conditions and continuity of the wave across the surface reduces to a study of how the field varies with derivatives taken with respect to z and evaluate at the boundary.

Let's study equation (2) first and see what it tells us. Remember that we now have three distinct regions of interest: the vacuum, the material, and the infinitesimal region on either side of the surface. The important thing to remember is that while passing through region 3 no magic occurs. Even though the physical properties may change quite rapidly, the wave functions are still continuous across the region. Since region 1 is a vacuum there is no polarization vector. Therefore, we must build up a polarization during region 3. The change in $\bar{P}$ implies derivatives exist and these we shall study. Since nothing is happening in the x and y directions, the only terms of interest in (2) are



$$\frac{\partial E_z}{\partial z} = -\frac{1}{\epsilon_0}\frac{\partial P_z}{\partial z}$$

We know P_z is rapidly changing in region 3 so its derivative can be quite steep - even approaching a delta function. If the right side is big then so must the left, i.e., it too must be a delta function for the equality to hold. If the above equation is integrated with respect to z across region 3, we must have that

$$E_{x_2} - E_{x_1} = -\frac{1}{\epsilon_0}(P_{x_2} - P_{x_1})$$

We could rewrite this as,

$$\epsilon_0 E_{x_2} + P_{x_2} = \epsilon_0 E_{x_1} + P_{x_1}$$

or using the notation $\bar{D} = \epsilon_0 \bar{E} + \bar{P}$ we have the condition that $\bar{D}$ is continuous across the boundary. Note if P_1 was not zero, that this same analysis holds true so the condition is general

If we now look at equation (3), we get no information from the ~~first or~~ last equation since it contains no derivatives in z. In the first equation we have $\partial E_y/\partial z$ but $\partial B_x/\partial t$ and $\partial E_z/\partial y$ are both zero. Therefore, if $\partial E_y/\partial z$ was great, there isn't anything to balance the equation out. Therefore E_y cannot have any jump going into region 2 from region 1, i.e.,

$$E_{y2} - E_{y1} = 0$$

By the same argument

$$E_{x2} - E_{x1} = 0$$

GOING ON TO EQUATION (1), AND USING THE SAME ARGUMENT AS WE JUST DID, WE CONCLUDE THAT

$$B_{z2} = B_{z1}$$

FROM EQUATIONS (4) WE GET THE FINAL TWO CONDITIONS

$$B_{x2} = B_{x1}$$

$$B_{y2} = B_{y1}$$

WITH THE SIX RELATIONS WE JUST DEVELOPED, WE CAN GO ON TO SOLVING PHYSICAL PROBLEMS WHERE THE SOLUTIONS HAVE TO CROSS A SHARP BOUNDARY. NOW WE CAN GO BACK TO THE PROBLEM WE STARTED.

AT THE SURFACE WHERE $z = 0$ THE CONDITION ON THE ELECTRIC FIELD GIVES

$$E_0 + E_1 = E_2$$

AND THE CONDITION ON THE MAGNETIC FIELD GIVES

$$E_0 - E_1 = n E_2$$

THESE TWO EQUATIONS CAN BE SOLVED FOR E_1 AND E_2 AND GIVE

$$E_1 = \frac{1-n}{1+n} E_0 \qquad E_2 = \frac{2n}{1+n} E_0$$

E_1 IS THE REFLECTED AMPLITUDE AND E_2 IS THE TRANSMITTED AMPLITUDE. THE INTENSITIES ARE GIVEN AS THE SQUARE OF THESE AMPLITUDES OR

$$\frac{I_1}{I_0} = \frac{|1-n|^2}{|1+n|^2} \qquad \frac{I_2}{I_0} = \frac{|2n|^2}{|1+n|^2}$$

AS AN EXAMPLE OF REFLECTION WATER HAS A $n = 4/3$ SO

$$\frac{I_1}{I_0} = |\tfrac{1}{7}|^2 \sim 2\%$$

REFLECTION FROM METALS

TO ANSWER THE QUESTION WHY ARE METALS SHINY WE ONLY HAVE TO REMEMBER THAT THE INDEX OF REFRACTION IS PURE IMAGINARY. THE INDEX CAN BE WRITTEN AS $n = -i n_I$ SO THE INTENSITY OF THE REFLECTED WAVE

$$\frac{I_1}{I_0} = \frac{|1 + i n_I|^2}{|1 - i n_I|^2} = \frac{1 + n_I^2}{1 + n_I^2} = 1$$

THEREFORE FOR METALS WITH PURE IMAGINARY INDICES THERE IS TOTAL REFLECTION. WITH LARGE IMAGINARY INDEXES METALS CAN REFLECT VISIBLE LIGHT QUITE WELL. BUT A LARGE INDEX IMPLIES STRONG ABSORPTION. THUS ANY GOOD ABSORBER WILL NOT ABSORB MUCH BECAUSE IT IS A GOOD REFLECTOR.

ORIGIN OF THE REFLECTION COEFFICIENT

LAST TIME I DEVELOPED THE FORMULA FOR LIGHT OR WAVES REFLECTING FROM A SURFACE. WE FOUND THE AMPLITUDE OF THE REFLECTED WAVE WAS A FUNCTION OF THE INDEX OF REFRACTION OF THE MATERIAL CAUSING THE REFLECTION, VIZ., $A_R = A_i \left(\frac{n-1}{n+1}\right)$

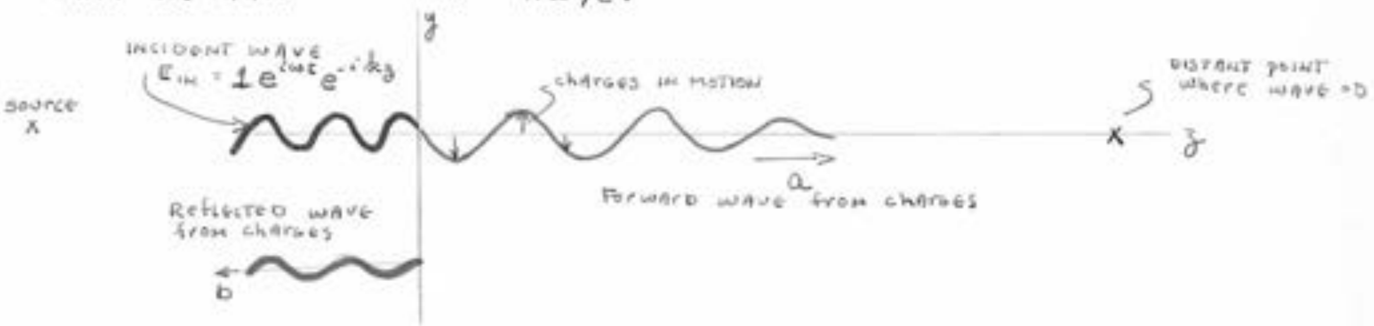
OUR QUESTION IS WHY THIS PARTICULAR EXPRESSION. WHAT I'D LIKE TO SHOW YOU IS THAT THIS SAME RESULT IS OBTAINED IF WE GO BACK TO MY EARLIER WORK. REMEMBER THAT I SAID THAT ALL WAVE PHENOMENA MUST BE CONSIDERED ORIGINATING FROM ALL THE SOURCES IN THE WORLD. NOT ONLY ~~MUST~~ WE INCLUDE ALL THE EXTERNAL SOURCES IN THE WORLD BUT ALSO THE WAVES PRODUCED BY THE INTERNAL CURRENTS FROM THE MOVING CHARGES IN THE MATTER.

WHAT I WANT TO SHOW IS THAT THE INCIDENT WAVE SETS THE CHARGES IN MOTION. THESE MOVING CHARGES GENERATE CURRENTS. THESE CURRENTS PRODUCE BOTH A FORWARD AND BACKWARD MOVING WAVE. THE FORWARD WAVE JUST HAPPENS TO BE OF THE RIGHT AMPLITUDE AND PHASE TO INTERFER WITH THE INCIDENT WAVE. IN PRODUCING THIS CANCELLING WAVE, THE CHARGES PRODUCE THE BACKWARD WAVE WHICH IS THE REFLECTED WAVE THAT WE MEASURE. FOR A RANDOM DISTRIBUTION OF OSCILLATORS, THE CURRENTS PRODUCE TWO FIELDS WHICH ARE IN THE RATIO OF $1-n/n+1$

LET'S ANALYZE THE SITUATION WHERE A POLARIZED WAVE STRIKES A SURFACE AT RIGHT ANGLES. WE ALREADY KNOW THE CHARGES IN THE MEDIA WILL PRODUCE CURRENTS WHICH VARY AS

$$j \propto e^{i\omega t} e^{-ik'z}$$

WHERE WE HAVE FOUND $k' = n\omega/c$.

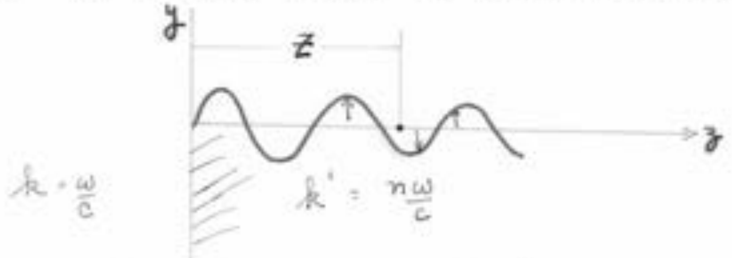


I HAVE DENOTED THE FORWARD AND REFLECTED WAVES AS a AND b RESPECTIVELY AND IT IS THE RATIO

$$\frac{b}{a} = \frac{1-n}{n+1}$$

THAT I WANT TO ESTABLISH. WHAT WILL HAPPEN, WE'LL FIND THAT $a = -1$ (RECALL WE CHOSE THE INCIDENT WAVE TO HAVE AMPLITUDE 1) THEREFORE THE REFLECTED AMPLITUDE $b = A_r = n-1/n+1$

LET'S LOOK AT WHAT IS GOING ON INSIDE THE MEDIA.



X

AT SOME GREAT DEPTH IN THE MATERIAL THE FIELD OF WAVE WILL BE ZERO. WHAT WE WANT TO DO IS TO PICK SOME REFERENCE POINT SAY Z AND ASK WHAT CONTRIBUTION WILL COME FROM THE MOVING CHARGES AROUND THAT POINT; INTEGRATING FROM $z = 0$ TO $z = X$ WHERE $X = \infty$. THE PHASE WILL VARY AS $e^{i\omega[t-(Z-z)]}$ ALONG THE z AXIS. THE FORWARD WAVE, Φ, CAN BE WRITTEN AS

$$\text{Forward wave} = \int_0^\infty e^{i\omega[t-(Z-z)]}\, e^{-ik'z}\, dz$$

THIS CAN BE REDUCED BY FACTORING OUT $e^{i\omega(t-Z)}$ AND WRITING

$$\text{Forward wave} = a\, e^{i\omega(t-Z)}$$

WHERE,

$$a = \int_0^\infty e^{i(\omega z - k'z)}\, dz$$

SINCE $k = \omega$ IN THE VACUUM,

$$a = \int_0^\infty e^{i(kz - k'z)}\, dz = \int_0^\infty e^{i(k-k')z}\, dz$$

PREFORMING THE INTEGRATION AS I HAVE DONE BEFORE, I FIND

$$a = \frac{-1}{i(k-k')}$$

I CAN NOW FIND THE REFLECTED AMPLITUDE SINCE I ONLY HAVE TO CHANGE THE SIGN IN THE PHASE, I.E.,

$$b = \int_0^\infty e^{i(-kz - k'z)}\, dz$$

INTEGRATING

$$b = \frac{-1}{i(-k-k')} = \frac{1}{i(k+k')}$$

THE RATIO CAN NOW BE FOUND

$$\frac{b}{a} = -\frac{(k-k')}{(k+k')}$$

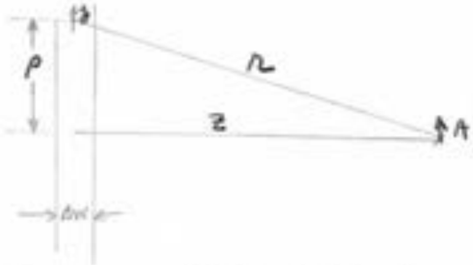
SUBSTITUTING $k' = nk$

$$\frac{b}{a} = \frac{n-1}{n+1}$$

WHICH IS THE ANSWER EXCEPT I MESSED UP THE SIGN SOMEWHERE.

The analysis I have used is valid when integrating over the full surface area because the ~~En~~ field produced by each little ~~current~~ charge is proportional to the resulting current. If we consider some thin slap of matter located a distance z from the observation point, the vector potential produced by each current is

$$A = \frac{j(t - r/c)}{4\pi\epsilon_0 r}$$



The vector potential is always in the same direction as j for the polarized case we are studying. Actually j is the current density and we want to integrate over an area, i.e.

$$A_{total} = \int \frac{j(t - r/c)}{4\pi\epsilon_0 r} 2\pi\rho\, d\rho$$

Since $r^2 = z^2 + \rho^2$ implies $r\,dr = \rho\, d\rho$, the integral simplifies to

$$A_{total} = \int \frac{j(t - r/c)}{2\epsilon_0}\, dr$$

Now remember the current is given as the time derivative of the polarization.

$$j = \dot{P}$$

if we put this in the integral and multiply & divide by c

$$A_{total} = \frac{c}{2\epsilon_0} \int_z^\infty \left.\frac{dP}{dt}\right|_{t-\frac{r}{c}} d\left(\frac{r}{c}\right) = \frac{c}{2\epsilon_0} \int_z^\infty \left.\frac{dP}{dt}\right|_{t-\frac{r}{c}} dt$$

Thus the integral can be evaluated

$$A_{total} = -\frac{c}{2\epsilon_0} P\left(t - \frac{z}{c}\right)$$

Since the field is given by the time derivative of the vector potential,

$$E = \frac{dA}{dt} = -\frac{c}{2\epsilon_0} j(t - z/c)$$

Because we should have integrated over a volume

$$E = -\frac{c}{2\epsilon_0} j\left(t - \frac{z}{c}\right) \Delta W$$

Since z is an arbitrary unit we see that the field at the point can be found even up to the slab. Thus our analysis does hold.

The waves arriving at the observation point will be reinforced because all the waves constructively interfer. Even though the currents are ~~a~~ generated out of phase the time delay in excitation means the waves will arrive at X at the same time. That is, a ~~be~~ point closer to X is out of phase with a point a little farther away but because it takes longer for the disturbance to reach the second, they will enforce. ~~enforce~~

If the wave was not damped as it went through the matter, each cycle will contribute equal amounts in and out of phase. Since the wave varies sinusoidally we have a series that looks like,

$$1 - 1 + 1 - 1 + 1 \cdots$$

Now if the wave gets weaker each cycle by a small amount γ, then the series becomes,

$$1 - \gamma + \gamma^2 - \gamma^3 + \gamma^4 - \gamma^5 + \cdots = \frac{1}{1+\gamma} = \frac{1}{2}$$

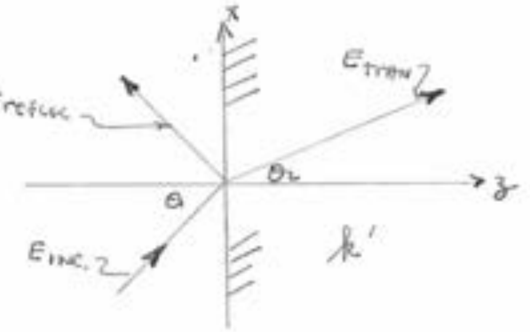
The sum of this series is 1/2 where we consider the material to be broken up into a number of zones. ~~We~~ Since the first zone makes a positive contribution, we think of the first half zone contributing the whole 1/2. But this is only a short hand for the integral,

$$\int_0^\infty e^{ikx}\,dx = \int_0^\infty \cos kx\,dx + i\int_0^\infty \sin kx\,dx$$

The $\cos kx$ integral gives zero. The second integral is integrated over the first half period $\frac{\pi}{k}$ to give i/k. This first half zone lasts longer than the others and is shifted in phase 90° so it takes a while to catch up with the other waves before it starts to interfer.

Reflected and transmitted waves from ~~OB~~ acute incidence ref Vol II Chapt 33

Now we'll take up the case of reflected waves from a surface when it hits at an angle. The Y axis will be out of the paper and we will assume the incident wave is polarized in the Y direction. The three fields are as follows



$$E_{inc} = E_0 e^{i(\omega t - k_z z - k_x x)}$$
$$E_{tran} = a e^{i(\omega t - k_z' z - k_x' x)}$$
$$E_{ref} = b e^{i(\omega t + k_z'' z - k_x'' x)}$$

First I want to point out that all the oscillations are at the same frequency ω, i.e., $\omega = \omega' = \omega''$. Now we can go on and write a few more relations which we can get easily,

$$k_x = k\sin\theta, \quad k_z = k\cos\theta, \quad k = \omega/c \text{ (vacuum)} \quad k_x^2 + k_z^2 = \omega^2$$
$$k_x' = k'\sin\theta_2 \quad k_z' = k\cos\theta_2 \quad k' = n\omega/c \quad (k_x')^2 + (k_z')^2 = \left(\frac{n\omega}{c}\right)^2$$

If the material is homogeneous in the x direction, all the waves must fit together at the boundary for all values of x. We must have

$$E_{inc} + E_R = E_{tran}$$

or

$$E_0 e^{i(\omega t - k_z z + k_x x)} + b e^{i(\omega t + k_z'' z - k_x'' x)} = a e^{i(\omega t - k_z' z - k_x' x)}$$

The easiest part is the coefficients of x which all must be the same,

$$k_x = k_x' = k_x''$$

Further we know the magnitudes of the various k's are given by

$$k_x^2 + k_z^2 = \omega^2$$
$$k_x''^2 + k_z''^2 = \omega^2$$
$$(k_x')^2 + (k_z')^2 = n^2\omega^2$$

The first two equations imply that $k_z''^2 = k_z^2$ or $k_z'' = \pm k_z$. The positive sign gives us another incident wave and therefore is not meaningful. Thus we find $k_z'' = -k_z$.

Finally for the reflected wave we write from the third equation above

$$(k_z')^2 = n^2\omega^2 - k_x^2 = n^2\omega^2 - \omega^2 \sin^2\theta.$$

or

$$k_z' = \omega\sqrt{n^2 - \sin^2\theta_1}$$

I might rewrite $k_x = k_x'$ into the more familiar form

$$k \sin\theta_1 = k' \sin\theta_2$$

or

$$\frac{\sin\theta_1}{\sin\theta_2} = n$$

This is Snell's law and the general form is,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$

An interesting case worth mentioning is when n is less than one and $\sin^2\theta_1$ is sufficiently small so that k_z' is imaginary. The critical angle at which k_z' becomes zero is

$$\sin\theta_1 = n$$

The coefficient of reflection in this case is given by

$$R = \frac{k_z - k_z'}{k_z + k_z}$$

Since k_z' is pure imaginary, the intensity of the reflected wave can be written as

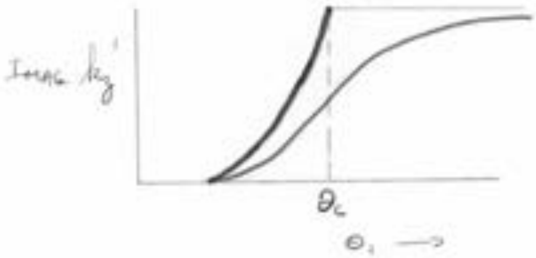
$$|R|^2 = \frac{|k_z - i\alpha|^2}{|k_z + i\alpha|^2} = 1$$

Therefore, we find no ~~reflec~~ transmission while the reflection is 100%. We might study this critical point a little closer to see what is going on.

If n is complex with a small imaginary part, we can write

$$k_z' = \sqrt{(1-\epsilon) - i t - \sin^2\theta_1} \simeq \sqrt{1 - \sin^2\theta_1} - \frac{i t}{2\sqrt{(1-\epsilon) - \sin^2\theta}}$$

If $\sin^2\theta_1$ eats the real part of k_y', then k_y' becomes more imaginary. If we plot the imaginary part of k_y' against θ_1 we get a curve something like,



If there isn't any absorption at all the curve will jump at θ_c, but normally it doesn't. If we are at $\theta_c = \alpha$ just to the left say $\theta = \alpha - \epsilon$ then the reflection coefficient can be written as

$$|R|^2 = \frac{|\cos\theta_1 - \sqrt{n^2 - \sin^2\theta_1}|^2}{|\cos\theta_1 + \sqrt{n^2 - \sin^2\theta_1}|^2}$$

$$|R|^2 = \frac{|\cos(\alpha-\epsilon) - \sqrt{\sin^2\alpha - \sin^2(\alpha-\epsilon)}|^2}{|\cos(\alpha-\epsilon) + \sqrt{\sin^2\alpha - \sin^2(\alpha-\epsilon)}|^2}$$

$$= \left| \frac{1-\sqrt{2\epsilon}\,n}{1+\sqrt{2\epsilon}\,n} \right| = 1 - 4\sqrt{2\epsilon}\,n$$

We see if $n \approx 1$ and $\epsilon = \frac{1}{100}$ radian, the intensity of the reflected wave is already down by almost 50%.

While I am playing with the reflection coefficient ~~ou~~ I will develop a historical equation, Remember

$$R = \frac{k_y - k_y'}{k_y + k_y'}.$$

If I now insert the cosine terms into this equation I get

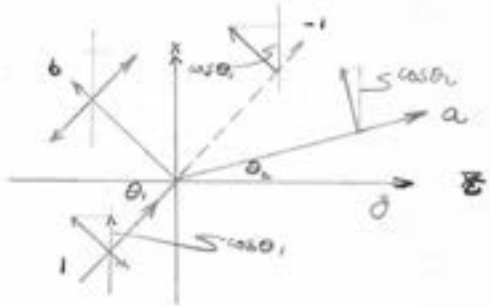
$$R = \frac{k\cos\theta_1 - k'\cos\theta_2}{k\cos\theta_1 + k'\cos\theta_2} = \frac{\cos\theta_1 - n\cos\theta_2}{\cos\theta_1 + n\cos\theta_2}$$

Since $n = \frac{\sin\theta_1}{\sin\theta_2}$ I can continue

$$R = \frac{\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2}{\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2} = \frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}$$

This is called Fresnel's relation or something. But I want to point out that the best form for understanding R is $\frac{k_y - k_y'}{k_y + k_y'}$. Here it is easy to see the forward and backward phase relationship that exists.

SUPPOSE NOW The INCIDENT LIGHT HITS The SURFACE AT AN ANGLE AND AT The SAME TIME POLARIZED NORMAL TO The DIRECTION of PROPAGATION. Both The reflected AND refracted WAVES will ALSO be POLARIZED NORMAL TO Their PROPAGATION DIRECTION.



TO ANALYZE This CASE we first realize ANY POLARIZATION horizontal TO The z AXIS will NOT be SEEN by AN observer AT z. More important The refracted WAVE MUST produce A WAVE which JUST CANCELS The INCIDENT WAVE: SINCE IT IS ONLY The TRANSVERSE COMPONENT of The POLARIZED refracted WAVE PROJECTED ONTO The INCIDENT WAVE THAT CAN CANCEL The WAVE, WE GET for a,

$$a = -1 = \frac{c\cos(\theta_1-\theta_2)}{k_z-k_z'}$$

c here is some coefficient gives The strength of The current making The wave. Now For The reflected wave

$$b = R = \frac{c\cos(\theta_1+\theta_2)}{k_z+k_z'}$$

This TIME WE WANT The SUM of The TWO ANGLES. Now GETTING rid of The TERM "c"

$$R = \frac{k_z-k_z'}{k_z+k_z'}\;\frac{\cos(\theta_1+\theta_2)}{\cos(\theta_1-\theta_2)}$$

If WE write This out IN TERMS of The k's WE GET

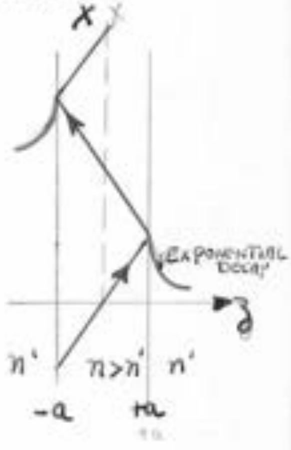
$$R = \frac{k_z-k_z'}{k_z+k_z'}\;\frac{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)}{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}$$

This can be simplified to The following form

$$R = \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}$$

CONFINED WAVES

Now I want to talk about waves that are confined by a sheet of material or some other thing to go along a certain paths You will recall last time we talked about a wave incident on a surface which has a lower index n than the material where the wave originates. There was some critical angle beyond which the wave is wholly reflected and ~~outside~~ inside the material with lower n the wave dies out exponentially. If we place two sheets close to eachother, we should be able to send the wave up the channel without any loses. So let's study the problem and see what we can understand about these confined waves.



We'll assume that the electric field is polarized in the Y direction so we can write

$$E = E_y = f(z)\, e^{-i(\omega t - kx)}$$

First we want to write down Maxwell's equations inside and outside the dielectric. Let's assume $n' = 1$, i.e., air is outside the conductor. Then

OUTSIDE : $\bar{\nabla}\cdot\bar{E} = 0 \quad \bar{\nabla}\cdot\bar{B} = 0 \quad \bar{\nabla}\times\bar{B} = \frac{\partial \bar{E}}{\partial t} \quad \bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$

INSIDE : $\bar{\nabla}\cdot\bar{E} = \frac{\bar{\nabla}\cdot\bar{P}}{\epsilon_0} \quad \bar{\nabla}\cdot\bar{B} = 0 \quad \bar{\nabla}\times\bar{B} = \frac{\partial \bar{E}}{\partial t} + \frac{\partial \bar{P}}{\partial t} \quad \bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$

We have to find out what $\bar{B}$ looks like and the best way to get $\bar{B}$ is from

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Writing this out completely

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \longrightarrow \frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z}$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \longrightarrow 0 = 0 \quad \text{As it should be. } \bar{B} \text{ is never } \parallel \text{ to } \bar{E}$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \longrightarrow \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}$$

From the first equation we get

$$\frac{\partial B_x}{\partial t} = \frac{\partial f(z)}{\partial z}\, e^{-i(\omega t - kx)}$$

Integrating we have

$$B_x = \frac{i}{\omega}\frac{\partial f(z)}{\partial z}\, e^{-i(\omega t - kx)}$$

From the third equation

$$\frac{\partial B_z}{\partial t} = -ik\, f(z)\, e^{-i(\omega t - kx)}$$

Upon integrating

$$B_z = +i\frac{k}{\omega}\, f(z)\, e^{-i(\omega t - kx)}$$

WE CAN WRITE for $\overline{B}$

$$\vec{B} = \left(\frac{i f'(z)}{\omega} \vec{e}_x + \frac{k}{\omega} f(z) \vec{e}_z \right) e^{-i(\omega t - k x)}$$

THIS $\vec{B}$ DOES INFACT SATISFY THE CONDITION $\nabla \cdot B = 0$ INSIDE AND OUT.

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{i^2 k}{\omega} f'(z) + \frac{k}{\omega} f'(z) = 0$$

ALSO WE NOT $\vec{\nabla} \cdot \vec{E} = 0$ OUTSIDE SINCE THIS REDUCES TO $\partial E_Y / \partial y = 0$

The MEANING-FUL EQUATIONS ARE $\nabla \times B = -\partial E/\partial t$ AND $\nabla \times B = -\partial E/\partial t + \partial P/\partial t$. If WE INTRODUCE A NEW VECTOR $\overline{D} = \epsilon_0 E + P = n^2 E$ where we have replaced The DIELECTRIC CONSTANT ϵ WITH n^2 SINCE $\epsilon = K\epsilon_0$ AND $n^2 = K$. Therefore we CAN write for INSIDE THE MATERIAL

$$\nabla \times B = -\frac{\partial (n^2 E)}{\partial t}$$

WE HAVE SIMPLIFIED THE PAIR OF EQUATIONS where OUTSIDE $n^2 = 1$. SO LET'S WRITE ALL THREE EQUATION OUT for $\nabla \times B = -\partial E/\partial t$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -\frac{\partial E_Y}{\partial t} = \text{■}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{\partial E_z}{\partial t} = 0$$

The SECOND EQUATION IS THE ONLY HELPFUL ONE AND differENTIATING OUT

$$\frac{i f''(z)}{\omega} - \frac{k}{\omega}(ik) f(z) = -i\omega n^2 f(z)$$

where I HAVE DIVIDED OUT The EXPONENTIAL AND DECIDED TO LEAVE n^2 IN. Thus I HAVE TWO SECOND ORDER DIFFERENTIAL EQUATIONS TO SOLVE

$$f''(z) = (k^2 - n^2\omega^2) f(z) \qquad \text{~~OUTSIDE~~ INSIDE}$$

$$f''(z) = (k^2 - \omega^2) f(z) \qquad \text{OUTSIDE}$$

SINCE $f''(z)$ IS PROPORTIONAL TO $f(z)$ ITSELF The ONLY FUNCTIONS WE CAN CONSIDER ARE SINES, COSINES, OR EXPONENTIALS.

OUTSIDE WE KNOW The WAVE DIES OUT EXPONENTIALLY SO WE COULD TRY The SOLUTIONS $f(z) \propto e^{\pm Kz}$ where $K^2 = k_x^2 - \omega^2$. For $z > a$ $f(z) = \alpha e^{-Kz}$ AND for $z < -a$ $f(z) = \beta e^{+Kz}$. INSIDE The DIELECTRIC The WAVE IS NOT DAMPED SO WE CAN TRY A SOLUTION which IS The LINEAR COMBINATION of $A e^{ik_z z}$ AND $B e^{-ik_z z}$. Here $k_z^2 = \omega^2 n^2 - k_x^2$; The free WAVE INSIDE HAS A REAL k_z. The TASK NOW IS TO MATCH BOUNDARY CONDITIONS AT THE TWO SURFACES.

AT $z = a$ $\quad \alpha e^{-\chi a} = A e^{ik_z a} + B e^{-ik_z a}$

AT $z = -a$ $\quad \beta e^{-\chi a} = A e^{-ik_z a} + B e^{+ik_z a}$

I HAVE TO USE THE FACT THAT B IS CONTINUOUS ACROSS THE FACES WHICH GIVES ME TWO MORE EQUATIONS;

$$-\chi \alpha e^{-\chi a} = ik_z A e^{ik_z a} - ik_z B e^{-ik_z a}$$

$$+\chi \beta e^{-\chi a} = ik_z A e^{-ik_z a} - ik_z B e^{ik_z a}$$

WE HAVE 4 EQUATIONS IN FOUR UNKNOWNS. THIS IS A DIFFICULT MATHEMATICAL MESS TO SOLVE BUT LET'S TRY.

FIRST WE GET RID OF α AND β

$$A e^{ik_z a} + B e^{-ik_z a} = -i\frac{k_z}{K} A e^{ik_z a} + i\frac{k_z}{K} B e^{-ik_z a}$$

or

$$A\left(1 + i\frac{k_z}{K}\right) e^{ik_z a} + B e^{-ik_z a}\left(1 - i\frac{k_z}{K}\right) = 0$$

ALSO

$$A\left(1 - i\frac{k_z}{K}\right) e^{-ik_z a} + B\left(1 + i\frac{k_z}{K}\right) e^{ik_z a} = 0$$

FROM THESE TWO EQUATIONS WE CAN FIND

$$e^{2ik_z a}\left(1 + i\frac{k_z}{K}\right)^2 = e^{-2ik_z a}\left(1 - i\frac{k_z}{K}\right)^2$$

$$e^{ik_z a}\left(1 + i\frac{k_z}{K}\right) = \pm\, e^{-ik_z a}\left(1 - i\frac{k}{K}\right)^2$$

THE TWO SIGNS INDICATE TWO SOLUTIONS EXIST WHICH CAN BE EITHER THE SAME OR HAVE DIFFERENT SIGN

THE TRANSCENDAL EQUATION WHICH GOVERNS THE MATCHING OF THESE WAVE SOLUTIONS IS

$$e^{2ika} = \frac{K - ik}{K + ik}$$

WE COULD WRITE THIS DIFFERENTLY AS

$$e^{ik_z a} + i\frac{k_z}{K} e^{ik_z a} = e^{-ik_z a}(1 - ik_z/K)$$

$$\frac{e^{ik_z a} - e^{-ik_z a}}{e^{ik_z a} + e^{+ik_z a}} = -i\frac{k_z}{K}$$

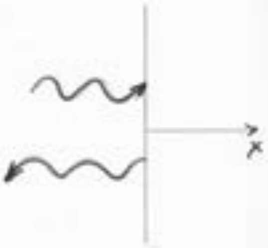
THE TERM ON THE LEFT IS THE HYPERBOLIC TANGENT OF THE ARGUMENT $ik_z a$

$$\tanh ik_z a = -i\frac{k_z}{K}$$

Cavity Resonators

I'd like to talk about something interesting involving waves in metal cans. The walls of the can contain charges which move all over as the field squish and squash around inside the can. The result of all this motion is that the can will ring, when this happens we call the device a cavity resonator.

The problem we must consider first is to examine what happens when a wave hits a metal wall. We did this before and we found the wave was reflected and the amplitude of the reflected wave varied from the incident wave by an amount



$$R = \frac{1-n}{1+n}$$

We note that if n became quite large and approached infinity or if it was highly imaginary, the reflected amplitude was -1, i.e., the wave was total reflected. Of course, this is never the case even for the most perfect metal and some loses occur in the walls. But disregarding this for the moment we can write the two formulas for the electric field,

$$E_{IN} = 1e^{i\omega t - ikx}$$

$$E_{OUT} = -1\,e^{i\omega t + ikx}$$

At the surface $x=0$ the sum of the two fields is zero, i.e.,

$$E = E_{IN} + E_{OUT} = i\,e^{i\omega t}\sin kx = 0 \quad \text{at } x=0$$

For a perfect metal the surface is the place of zero tangential electric field.

In the real world metals can have a very large index n and for those materials we can write the index as

$$n = -in'$$

The reflection coefficient is given as

$$R = \frac{1+in'}{1-in'}$$

This can be rewritten in terms of some phase shift θ where $\tan\theta = 1/n'$ as

$$R = -1\left(\frac{in'+1}{in'-1}\right) = -1\,\frac{e^{-i\theta}}{e^{+i\theta}} = -1e^{-2i\theta}$$

Thus the outgoing wave can be expressed in terms of θ as

$$E_{OUT} = -e^{-2i\theta}e^{i\omega t}e^{ikx}$$

where

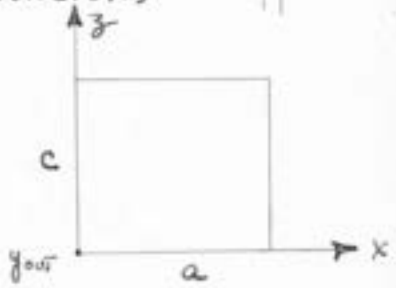
$$E_{IN} = e^{i\omega t}e^{-ikx}$$

If these two waves are now added, we find that

$$E = e^{-i\theta} e^{i\omega t} \sin(kx - \theta)$$

and E is no longer zero at the surface $x = 0$ but rather at the point $x = \theta/k$. When n' is large, θ is about equal to $1/n'$ so that $x = \frac{1}{n'k}$. Since $k = 2\pi/\lambda$ we have the zero tangential electric field plane at $x = \frac{\lambda}{2\pi n'}$. For waves with λ on the order of a few centimeters and n' in the thousands, the physical surface and the imaginary $E = 0$ surface are almost coincident but not quite. I will consider the ideal surface where $E = 0$ as the true surface of the metal and disregard the effect hereafter.

Now let's consider a rectangular can of the dimensions shown in the side drawing. The y-axis is out of the paper and the dimension associated with the can is "b". We'll begin by writing Maxwell's equation inside



$$\nabla \cdot E = 0 \quad \nabla \cdot B = 0 \quad \nabla \times E = -\partial B/\partial t \quad \nabla \times B = \partial E/\partial t$$

We'll assume all the fields vary sinusoidally so that $\partial/\partial t$ is the same as multiplying by $i\omega$. I prefer to work in terms of E so I want to eliminate B. To do this I notice that

$$\nabla \times (\nabla \times E) = \frac{\partial}{\partial t}(\nabla \times B)$$

But $\nabla \times B = -\partial E/\partial t$, therefore

$$\nabla \times (\nabla \times E) = -\frac{\partial^2 E}{\partial t^2}$$

Somewhere I worked out $\nabla \times (\nabla \times E)$ and showed that it equals $\nabla(\nabla \cdot E) - \nabla^2 E$ so we have two equations to solve

$$\nabla^2 E = \frac{\partial^2 E}{\partial t^2} \qquad \text{AND} \qquad \nabla \cdot E = 0$$

The first equation can be written as

$$\nabla^2 \bar{E} = -\omega^2 \bar{E}$$

Let's assume E is entirely in the z direction; then the second equation $\nabla \cdot E = 0$ tells us that $\partial E_z/\partial z = 0$ which implies E can only depend on x and y. Let's write

$$E = f(x, y)\, e^{i\omega t}$$

The first equation can be used to obtain the following equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = -\omega^2 f(x,y)$$

NOW THERE IS NO DIRECT WAY TO SOLVE THIS EQUATION UNLESS YOU KNOW THE ANSWER BEFORE HAND. YOU HAVE TO GUESS AT A SOLUTION. ONE THAT WE MIGHT TRY IS

$$f(x,y) = \sin k_x x \ \sin k_y y$$

IF THIS FUNCTION IS PUT INTO THE EQUATION WE FIND THAT A SOLUTION RESULTS IF

$$\omega^2 = k_x^2 + k_y^2$$

THIS IS NOT THE ONLY SOLUTION WE COULD HAVE TRIED. WE COULD HAVE USED $f(x,y) = \cos k_x x \sin k_y y$ or $f(x,y) = \cos k_x x \cos k_y y$.

BUT WE MUST APPLY OUR BOUNDARY CONDITION THAT REQUIRES E TANGENTIAL IS ZERO ON ALL SIDES OF THE CAN. BECAUSE WE CHOSE A SINE COSINE FUNCTION OF X AND Y WE KNOW RIGHT AWAY THE EQUATION IS SATISFIED ON THE TOP AND BOTTOM OF THE CAN. BUT WE ALSO REQUIRE THAT $f(x,y) = 0$ AT $x = 0$ AND $x = a$. AT $x = 0$ WE HAVE $\sin k_x x \big|_{x=0} = 0$. THUS WE SEE WHY ALL SINES ARE NEEDED. NOW AT $x = a$, $f(x,y) = 0$ WHICH WILL HAPPEN ONLY IF

$$\sin k_x a = 0 \ \rightarrow \ k_x a = \pi$$

AND LIKEWISE

$$\sin k_y b = 0 \ \rightarrow \ k_y b = \pi$$

WE FIND THAT k_x AND k_y CANNOT BE ARBITRARY BUT RATHER DEPEND ON THE LENGTH AND WIDTH OF THE GUIDE. FURTHER THE FREQUENCY OF THE PROPAGATING WAVE IS LIKEWISE RESTRICTED TO CERTAIN VALUES.

$$\omega^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$$

THIS IS THE ONLY FREQUENCY THAT IS PERMITTED TO OSCILLATE INDEFINITELY. THE CHARACTER OF THE WAVE THAT FITS INTO THE CAN IS SHOWN HERE. THE FIELD IS STRONGEST IN THE CENTER AND GRADUALLY FALLS TO ZERO ALONG THE WALL.

IN THE SIDE PICTURE THE MAGNETIC FIELD CIRCULATES ABOUT THE ELECTRIC FIELD; THE CIRCULATION IS GREATEST IN THE MIDDLE. AS E VARIES BETWEEN A MAXIMUM AND MINIMUM VALUE, THE B FIELD LIKEWISE VARIES. AS E COLLAPSES B IS GOING AROUND AND WHEN E DISAPPEAR ALL THE ENERGY IS IN THE B FIELD. BUT B CAN'T DISAPPEAR AND AS IT IS CHANGING AN ELECTRIC FIELD IS BEING BUILT UP. THE ENERGY THUS ALTERNATES BETWEEN THE ELECTRIC AND MAGNETIC FIELDS. THUS IT IS LIKE THE EXCHANGE BETWEEN KINETIC AND POTENTIAL ENERGY OF A HARMONIC OSCILLATOR.

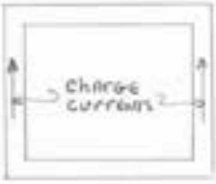
NOW THE INTERESTING QUESTION IS WHAT ARE THE CHARGES DOING IN THE CAN. HERE A REAL MIRACLE IS GOING ON BECAUSE THE CHARGES ALWAYS ARRANGE THEMSELVES SO NOTHING GETS OUT AND ONLY AN INTERNAL E AND B ARE PRODUCED.

WHEN ALL THE ENERGY IS IN THE MAGNETIC FIELD (I.E., A 1/4 CYCLE AFTER THE ELECTRIC FIELD IS A MAXIMUM) THE CURRENT IN THE WALL IS A MAXIMUM. POSITIVE AND NEGATIVE CHARGE IS MOVED TO AND FROM THE TOP AND BOTTOM OF THE CAN. WHEN THE E-FIELD IS MAXIMUM, ~~ALL~~ THE POSITIVE AND NEGATIVE CHARGES ARE HIGHEST IN THE MIDDLE OF THE CAN. AS THE ENERGY GOES INTO THE ~~E~~ MAGNETIC FIELD, AN INDUCTANCE IS BUILT UP WHICH KEEPS THE CURRENTS MOVING EVEN PAST THE MAXIMUM. THE INDUCTANCE ACTS LIKE AN INERTIA TO RESIST REVERSAL. THE B-FIELD WILL OSCILLATE BACK AND FORTH FOR MANY OSCILLATION BEFORE IT STARTS TO DIE OUT.

E_{MAX}

IT IS POSSIBLE TO KEEP THE FIELDS GOING INSIDE THE CAVITY IF A SMALL HOLE IS DRILLED IN THE BOTTOM AND A WIRE FEED INSERTED. IF THE CHARGE IS PUT IN AT THE WRIGHT FREQUENCY, ENERGY WILL BE SUPPLIED THAT IS LOST TO THE WALLS AND THE CAVITY WILL KEEP ON RESONATING

B_{MAX}



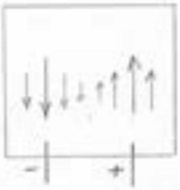
SO FAR I HAVE TALKED ABOUT THE LOWEST MODE OF THE RESONATOR. CERTAINLY I DID NOT CHOSE THE ONLY SOLUTION TO THE DIFFERENTIAL EQUATIONS. IN FACT THERE ARE A WHOLE SET OF SOLUTIONS WHICH CAN BE WRITTEN AS

$$k_x a = n\pi \qquad k_y b = m\pi$$

AND

$$\omega^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

FOR THE NEXT HIGHEST MODE $n=2$ $m=1$ TO GIVE $\omega^2 = \frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2}$
THIS MODE LOOKS LIKE THE SIDE DRAWING, THIS MODE CAN BE EXCITED BY TWO FEEDS PLACED A QUARTER OF THE WAY IN FROM EITHER SIDE. THE FEEDS HAVE OPPOSITE SIGNS.



NOW I HAVE ONLY TALKED ABOUT ONE CLASS OF RESONATORS NAMELY THE RECTANGULAR SHAPED ONES. BUT CAVITY RESONATORS CAN BE ALL SHAPES, EACH DIFFERENT SHAPES HAVING A RESONANCE CONDITION. HOWEVER, THERE NEED NOT BE A SOLUTION FOR EVERY FREQUENCY.

APPLICATION OF VARIATIONAL PRINCIPLE IN SOLVING RESONATOR PROBLEMS

IN PRINCIPLE EACH CAVITY WILL HAVE CERTAIN RESONANT MODES BUT OFTEN THEY ARE QUITE DIFFICULT TO WORK OUT. I'D LIKE TO SHOW YOU A TECHNIQUE FOR APPROXIMATING THE MODES USING THE VARIATIONAL PRINCIPLE.

LET'S TAKE AN EASY EXAMPLE for which I KNOW THE ANSWER TO SHOW YOU HOW THE IDEA WORKS. LET'S CONSIDER A CYLINDRICAL DRUM which HAS AN ELECTRIC FIELD PROPAGATING IN THE z DIRECTION. THE MATHEMATICS IS JUST THE SAME AS OUR LAST CASE UP TO THE EQUATION

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\omega^2 f$$

THIS EQUATION MUST BE SATISFIED AT THE SURFACE, I.E., $f = 0$ WHEN $x^2 + y^2 = A^2$

SUPPOSE I HAVE THE INTEGRAL, I

$$I = \frac{\int \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dx\, dy}{\int \varphi^2\, dx\, dy}$$

which I WANT TO MAKE A MINIMUM for SOME φ. THE USUAL PRACTICE IS TO VARY φ BY LETTING A SMALL QUANTITY $\Delta\varphi$ BE ADDED. THEN THE INTEGRAL CAN BE VARIED IN THE FOLLOWING CRYPTIC FORM

$$I = \frac{\text{NUM}}{\text{DENOM}} = \frac{N}{D}$$

$$\delta I = \frac{N + \Delta N}{D + \Delta D} - \frac{N}{D} = \frac{\Delta N}{D} - \frac{N \Delta D}{D^2}$$

THIS VARIATION IS SET EQUAL TO ZERO AND THE CONDITION which MUST BE SATISFIED IS

$$\delta N = \frac{N}{D}\delta D$$

NOW THE FIRST ORDER CHANGE IN N CAN BE COMPUTED

$$N = \int \left(\frac{\partial\varphi}{\partial x}\right)^2 dx\, dy \rightarrow \int \left(\frac{\partial\varphi + \partial(\delta\varphi)}{\partial x}\right)^2 dx\, dy$$

THIS BECOMES

$$\delta N = +2 \int \delta\varphi \left[-\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2}\right] dx\, dy$$

SIMILARLY

$$\delta D = 2\int \delta\varphi\, \varphi\, dx\, dy$$

FROM which WE FIND

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -\frac{N}{D}\varphi$$

AND WE FIND $\omega^2 = N/D$. THE TASK IS TO KEEP TRYING VARIOUS φ's UNTIL I IS MINIMUM.

LET'S TRY A φ which MIGHT WORK SAY

$$\varphi = \left(1 - \frac{r^2}{A^2}\right)$$

THIS IS A PARABOLA which FITS THE BOUNDARY CONDITION THAT $\varphi = 0$ AT $r = A$ AND AT THE CENTER $\varphi = 1$ IS A MAXIMUM. THE FUNCTION IS A PARABOLA AND, THEREFORE, SMOOTH AT THE CENTER.

SINCE $r^2 = x^2 + y^2$ WE HAVE

$$\varphi = 1 - \frac{x^2+y^2}{A^2}$$

AND

$$\frac{\partial \varphi}{\partial x} = -\frac{2x}{A^2}, \quad \frac{\partial \varphi}{\partial y} = -\frac{2y}{A^2}$$ which gives $\left(\frac{\partial \varphi}{\partial x}\right)^2 = \frac{4x^2}{A^4}$

AND $\left(\frac{\partial \varphi}{\partial y}\right)^2 = \frac{4y^2}{A^4}$. Now

$$I = \frac{N}{D} = \frac{\int_0^A \frac{4r^2}{A^4}\, r\,dr}{\int_0^A \left(1-\frac{r^2}{A^2}\right)^2 r\,dr} = \frac{1}{\frac{A^2}{2} - \frac{A^2}{2} + \frac{A^2}{6}}$$

Therefore

$$\omega^2 = \frac{N}{D} = \frac{6}{A^2}$$

OR

$$\omega = \frac{\sqrt{6}}{A} = \frac{2.449}{A}$$

I HAPPEN TO KNOW THE REAL SOLUTION IS THE FIRST ROOT OF THE BESSEL FUNCTION $J_0(x)$ AND IS

$$\omega = \frac{2.405}{A}$$

SO WE ARE PRETTY DAMN CLOSE ON THIS CRUDE TRY.

MORE ON THE VARIATIONAL PRINCIPLE
IN SOLVING FOR FREQUENCIES

LAST TIME WE HAD A VARIATIONAL PRINCIPLE FOR SOLVING A DIFFERENTIAL EQUATION OF THE FORM $\nabla^2\varphi - \omega^2\varphi = 0$. THE INTEGRAL WHICH WE WANTED TO MINIMIZE WAS

$$I = \frac{\int\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]}{\int \varphi^2 \, dx\,dy}$$

THE QUESTION ONE MIGHT ASK IS WHY THIS INTEGRAL FORM? HOW DO I KNOW THE INTEGRANDS ARE THE RIGHT FUNCTIONS? WELL, EXPERIENCE HAS A LOT TO DO WITH IT. NOT ALL DIFFERENTIAL EQUATION COME FROM A VARIATIONAL PRINCIPLE SO YOU MUST KNOW WHICH ONES DO. THE ABOVE INTEGRAL IS ACTUALLY THE SOLUTION FOR THE EQUATION

$$\nabla^2\varphi - \frac{\partial^2\varphi}{\partial t^2} = 0$$

IF $\varphi = e^{i\omega t}\psi$, THEN THE EQUATION JUST REDUCES TO $\nabla^2\psi - \omega^2\psi = 0$ WHICH IS WHAT WE HAD BEFORE.

WHEN WE WENT THROUGH THE CYLINDER PROBLEM LAST TIME WE TRIED A PARABOLIC SOLUTION $\varphi = 1 - r^2/a^2$ WHICH SATISFIED THE BOUNDARY CONDITION THAT $\varphi = 0$ AT $r = a$. THE FREQUENCY THAT WE FOUND WAS $\omega = 2.45/a$. NOW WE DIDN'T HAVE A PARAMETER TO VARY IN THE INTEGRAL SO A BETTER THING TO DO WOULD BE TO EXPAND $\omega^2 = I$ IN TERMS OF SOME PARAMETER α, FOR INSTANCE

$$\omega^2 = A + B\alpha + C\alpha^2 + \cdots$$

SOMEONE TRIED THE FUNCTION,

$$\varphi = 1 + \alpha - \frac{r^2}{a^2} - \alpha\frac{r^4}{a^4}$$

AND VARIED α AND FOUND A MINIMUM OCCURRING AT $\alpha = -.2809$. THIS GAVE A SOLUTION OF $\omega = 2.40502/a$. THE EXACT SOLUTION IS $2.4048/a$. SO YOU SEE WITH A VERY SMALL EXPANSION OF OUR INITIAL SOLUTION, WE ARE AWFULLY CLOSE TO THE REAL RESULT.

I WANT TO POINT OUT THAT YOU SHOULDN'T BE CARELESS WHEN TRYING A SOLUTION. THE VARYING PARAMETER MUST MAKE SENSE; A FUNCTION LIKE $\varphi = \alpha - \alpha r^2/a^2$ DOESN'T ACCOMPLISH A THING. THE CLASS OF TRIAL SOLUTIONS MUST INCLUDE ONLY THOSE SOLUTIONS WHICH FIT THE PHYSICAL CONDITIONS OF THE PROBLEM. IF THERE IS A WIRE OR SOMETHING ELSE INSIDE, THEN YOU MUST ADJUST THE TRIAL SOLUTION TO TAKE INTO CONSIDERATION THE RESULTANT VARIATIONS IN THE LOCAL FIELD AROUND THE CHARACTERISTIC SINGULARITY. FOR INSTANCE, IF A WIRE RAN DOWN A CAVITY CYLINDRICAL IN SHAPE WE MIGHT TRY

$$\varphi = 1 - \frac{r^2}{a^2} + \ln r$$

I NOW WANT TO TALK ABOUT A CAVITY WITH A PIECE OF DIRT IN IT. THIS CHUNK OF DIELECTRIC WILL HAVE SOME POLARIZATION CHARGE INDUCED ON IT AND THE FIELD AROUND IT WILL BE DISTORTED. THE VARIATION IN $\bar{E}$ WILL CAUSE A SHIFT IN FREQUENCY. WE WOULD LIKE TO KNOW HOW MUCH ω CHANGES IF THE BLOB IS SMALL COMPARED TO THE DIMENSIONS OF THE BOX.

WE HAVE TO WRITE DOWN MAXWELL'S EQUATIONS FOR THE CASE,

$$\nabla\cdot E = \frac{\rho}{\epsilon_0} = -\frac{\nabla\cdot P}{\epsilon_0} \Rightarrow \bar{\nabla}\cdot(K\bar{E}) = 0 \;,\; \nabla\cdot B = 0 \;,\; \nabla\times E = -\frac{\partial B}{\partial T}$$

AND

$$\nabla\times B = \frac{\partial E}{\partial t} + j = \frac{\partial}{\partial t}(KE)$$

NOTICE IF WE TAKE $\nabla\times(\nabla\times E)$ WE GET

$$\nabla\times(\nabla\times E) = \frac{\partial^2}{\partial t^2}(KE)$$

IF $\bar{E} = E_0 e^{i\omega t}$ THEN WE GET $\nabla\times(\nabla\times E) = -\omega^2 K\bar{E}$. NOW THE SOLUTION TO THIS EQUATION CAN BE FOUND BY VARYING THE INTEGRAL I

$$I \geq \omega^2 = \frac{\int(\bar{\nabla}\times\bar{E})\cdot(\nabla\times E)\,dVOL}{\int K\bar{E}\cdot\bar{E}\,dVOL}$$

IF THE DIELECTRIC IS SMALL THE SOLUTION WILL BE CLOSE TO THE ORIGINAL SOLUTION E_0, I.E., $\nabla\times(\nabla\times E_0) = -\omega^2 E_0$. THIS TELLS US THAT

$$\int(\nabla\times E_0)\cdot(\nabla\times E_0)\,dVOL = \omega_0^2\int E_0\cdot E_0\,dVOL.$$

NOW WITH A SMALL DISTURBANCE K IS ABOUT EQUAL TO 1 LIKE THE SURROUNDING VACUUM. LET'S ADD A SMALL AMOUNT λ TO K, I.E.,

$$K = 1+\lambda$$

THEN THE INTEGRAL EQUATION BECOMES

$$\omega^2 > \frac{\int(\nabla\times E_0)\cdot(\nabla\times E_0)\,dVOL}{\int(E\cdot E)\,dVOL + \int\lambda E\cdot E\,dVOL.}$$

AGAIN IF THE FIELD DOES NOT VARY

$$\omega^2 > \frac{\int(\nabla\times E_0)\cdot(\nabla\times E_0)\,dVOL}{\int E_0^2\,dVOL + \int\lambda E_0^2\,dVOL}$$

THE NUMERATOR IS JUST ω_0^2 SO LET'S REWRITE THIS AS

$$\omega^2 = \frac{\omega_0^2}{1+\frac{\int\lambda E_0^2\,dVOL}{\int E_0^2\,dVOL}} = \frac{\omega_0^2}{1+\bar{\lambda}}$$

LET'S DEFINE

$$\bar{\lambda} = \frac{\int\lambda E_0^2\,dVOL}{\int E_0^2\,dVOL}$$

$\bar{\lambda}$ IS AN AVERAGE VALUE WHICH IS WEIGHTED OVER ALL SPACE. ITS VALUE DEPENDS ON HOW STRONG THE E_0 IS WHERE YOU PUT THE DIELECTRIC. THAT IS, IF E_0 IS 0 AT THE POINT YOU PUT THE DIRT THE FREQUENCY CHANGES IS VERY SMALL. IF THE FIELD IS RAPIDLY CHANGING AT THE POINT YOU PUT THE DIRT ω^2 IS VERY DIFFERENT FROM ω_0^2. RECALL THAT E^2 IS THE ENERGY CONTENT OF THE FIELD. $\bar{\lambda}$ IS THEN,

$$\bar{\lambda} = \frac{\lambda\,(\text{VOLUME OF DIRT}) \cdot E_0^2\,(\text{AT DIRT})}{(\text{VOLUME OF BOX})\,(E_0^2 \text{ OF BOX})}$$

IF K IS LARGE SAY 5, THEN λ CAN BE 4 AND THE LOCAL VARIATIONS IN THE FIELD CAN BE QUITE SEVERE AND I HAVE NOT CONSIDERED THIS CASE.

I MIGHT ALSO POINT OUT THAT ω^2 MEASURES THE RATIO OF MAGNETIC FIELD ENERGY TO ELECTRIC FIELD ENERGY, I.E.,

$$\omega^2 = \frac{\int (\nabla\times E)\cdot(\nabla\times E)\,d\text{vol}}{\int K E\cdot E\,d\text{vol}} = \frac{\omega_0^2 \int B\cdot B\,d\text{vol}}{\int K E\cdot E\,d\text{vol}}$$

SINCE $\nabla\times E = -\frac{\partial B}{\partial t} = +\omega_0^2 B$ IF $\bar{B} = B e^{i\omega_0 t}$

ANOTHER VARIATIONAL PRINCIPLE WHICH YOU MIGHT HAVE TO WORK OUT IS A CHANGE IN THE POSITION OF THE WALL. FOR INSTANCE CONSIDER A SLIGHT DISTORTED SPHERE THAT IS NOW AN ELLIPSE. THE WAY I GO ABOUT ~~IN~~ SOLVING THIS EQUATION IS TO TAKE DIFFERENT COORDINATE SYSTEMS ALONG THE X, AND Y AXIS. ONE UNIT ALONG THE X AXIS MAY BE .4 UNITS ALONG THE Y AXIS. MY EQUATION FOR THE SURFACE IS STILL $x^2+y^2 = a^2$ BUT NOW I HAVE TO CORRECT FOR MY SCREWED UP COORDINATE SYSTEM. SINCE $\chi = \epsilon\, dx$ AND $Y = \eta\, dy$, THE INTEGRAL I BECOMES

$$I = \frac{\int \left[\frac{1}{\xi^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{\eta^2}\left(\frac{\partial\varphi}{\partial y}\right)^2 \right] \xi\eta\,dx\,dy}{\int \varphi^2\, \xi\eta\, dx\,dy}$$

THE BOUNDARY CONDITION IS THAT $\varphi = 0$ AT $x^2+y^2 = a^2$. NOW THE PERTURBATION IS IN THE DIFFERENCE FROM 1 OF $1/\xi^2$ AND $1/\eta^2$ SINCE THAT IS THE ONLY THING DIFFERENT FROM THE PREVIOUS INTEGRAL.

EIGENVALUES AND EIGEN FUNCTIONS

I'D LIKE TO TALK FOR A FEW MINUTES ON THE GENERALITY OF THE EQUATION WE JUST SOLVED, I.E., $\nabla\times(\nabla\times E) = -\omega^2 E$. NOW THERE ISN'T A SOLUTION FOR EVERY ω. WE FOUND THE LOWEST ω HAD A CERTAIN VALUE CORRESPONDING TO A PARTICULAR SPATIAL DISTRIBUTION OF E. BUT THERE IS A WHOLE SEQUENCE OF FREQUENCIES ($\omega_1^2, \omega_2^2, \omega_3^2$, ETC) WHICH ALSO CORRESPOND TO UNIQUE SPATIAL DISTRIBUTIONS.

The problem is just like any mechanically or electrically vibrating system. The shape of E for each frequency is called the mode shape. The functions themselves are called eigen or characteristic functions while the frequencies are call eigen values. The general equation is then

$$\nabla \times (\nabla \times E_N) = -\omega_N^2 E_N$$

If I bunch all the ∇ operators together and denote them by L, then I can rewrite this equation in general notation as

$$L\psi = -\lambda_N \psi$$

L is called a linear operator. It satisfies the equation or property that

$$L(\psi_1 + \psi_2) = L\psi_1 + L\psi_2$$

I should point out while each eigen value can be different, the boundary conditions are the same for all the eigenfunctions.

There are some relations which exist amoung the different mode shapes which I would like to develope. Suppose we had another mode, m, which satisfied the equation

$$\nabla \times (\nabla \times E_m) = -\omega_m^2 E_m$$

I will assume that no two frequencies are the same; that is, there is no degeneracy in the eigenvalues. Since the above equation cannot be used to determine the size of the eigen function, we will normalize it in the following way,

$$\int E_N \cdot E_N \, dVOL = 1$$

Notice now if we multiply E_m with $\nabla \times (\nabla \times E_N)$ and integrate,

$$\int E_n \cdot \nabla \times (\nabla \times E_m) \, dVOL = -\omega_m^2 \int (E_n \cdot E_m) \, dVOL$$

integrating the left side by parts

$$\int E_N \cdot (\nabla \times (\nabla \times E_m)) \, dVOL = \int (\nabla \times E_N) \cdot (\nabla \times E_m) \, dVOL + \int (E_N \times N) \cdot (\nabla \times E_m) \, dsurf$$

Now $E_N \times \bar{N}$ is just the tangential component of E but this is zero. Therefore,

$$\int (\nabla \times E_m) \cdot (\nabla \times E_n) \, dVOL = -\omega_n^2 \int E_m \cdot E_n \, dVOL$$

If we do the same thing to the other eigenequation, we find that

$$\int (\nabla \times E_m) \cdot (\nabla \times E_n) dVOL = -\omega_m^2 \int E_n \cdot E_m \, dVOL.$$

Therefore, we conclude that $\omega_m^2 = \omega_n^2$? Is something wrong? No. The two eigenvalues could different if $\int E_m \cdot E_n \, dVOL = 0$ or $E_m \cdot E_N = 0$. This condition implies the two modes are orthogonal to eachother.

DEGENERATE EIGENVALUES

As a side remark, suppose we had a case where two modes were the same, i.e. $\omega_N = \omega_M$. Then the two equations would read

$$\nabla \times (\nabla \times E_N) = -\omega_N^2 E_N$$
$$\nabla \times (\nabla \times E_M) = -\omega_N^2 E_M$$

To see if there is a solution lets try a combination of the two functions,

$$E = \alpha E_N + \beta E_M$$

Thus

$$\nabla \times [\nabla \times (\alpha E_N + \beta E_M)] = -\omega_N^2 (\alpha E_N + \beta E_M)$$

Suppose the functions E_N and E_M are normalized and $\int E_N \cdot E_M \, d\mathrm{vol} \neq 0$.
Let's determine if

$$\int E_N \cdot E \, d\mathrm{vol} = 0$$

$$\int (E_N \cdot \alpha E_N + E_1 \beta E_L) = \alpha + \beta \chi = 0$$

where $\chi = \int E_N \cdot E_m$. Now we know $\beta = -\frac{\alpha}{\chi}$ and can find

$$E = \alpha \left(E_N - \frac{E_m}{\chi} \right)$$

This is not the same as E_N or E_m.

jtn note: This ends Volume 2