CS/MATH 335: PROBABILITY, COMPUTING, AND GRAPH THEORY
HOMEWORK 1
DUE IN CLASS FRIDAY, SEPTEMBER 23, 2016

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Partner Problems

Roughly in order of difficulty

(1) □

(2) Mitzenmacher 2.13

a.) According to the problem we have $2n$ different coupons and n pairs of coupons. From this information we know that that $X = \sum_{i=1}^{n} x_i$. We also need to calculate the probability of getting a coupon from a new pair when we have $i-1$ pairs. Notice the the probability of choosing a specific coupon is $= \frac{1}{2n}$, hence we get $E[x_i] = \frac{1}{2n}(2n - 2(i-1) = \frac{n-i+1}{n}$. To find $E[X]$ we do the following:

$\Rightarrow E[X] = E[\sum_{i=1}^{n} x_i]$

$\Rightarrow \sum_{i=1}^{n} E[x_i]$

$\Rightarrow \sum_{i=1}^{n} \frac{n-i+1}{n}$

$\Rightarrow n \sum_{i=1}^{n} \frac{1}{n}$

b.)

(3) Mitzenmacher 2.7

a.) Using the information provided that X and Y are independent geometric random variables, $Pr(X = Y) = \sum_x (1-p)^{x-1} p (1-q)^{x-1} q$

$\Rightarrow \sum_x [(1-p)(1-q)]^{x-1} pq$

So, $Pr(X = Y) = \frac{pq}{p+q-pq}$

b.) $E[max(X, Y)] = E[X] + E[Y] - E[min(X, Y)]$. Notice $E[min(X, Y)] = \frac{1}{p+q-pq}$. Hence, $E[max(X, Y)] = \frac{1}{p} + \frac{1}{q} - \frac{1}{p+q-pq}$

c.) To find the $Pr(min(X, Y) = k)$ we can split the event into two events:

$\Rightarrow Pr(X = k, Y \geq k) + Pr(X > k, Y = k)$

$\Rightarrow Pr(X = k)Pr(Y \geq k) + Pr(X > k)Pr(Y = k)$

$\Rightarrow$ Notice $Pr(X > k) = Pr(X \geq k) - Pr(x = k) = (1 - p)^{k-1}(1 - p)$

$\Rightarrow Pr(min(X, Y) = k) = (1-p)^{k-1}p(1-q)^{k-1} + (1-p)^{k-1}(1-p)(1-q)^{k-1}q$

$\Rightarrow = [(1 - p)(1 - q)]^{k-1}(p + q - pq)$

d.)

$E[X|X \leq Y] = \sum_z x\frac{Pr(X=x)\cap x\leq Y}{Pr(X\leq Y)}$

Now we compute compute $Pr(X \leq Y)$:

$\Rightarrow \sum_z Pr(X = z)Pr(z \leq Y)$

$\Rightarrow \sum_z (1 - p)^{z-1}p(1 - q)^{z-1}$

$\Rightarrow p\sum_z [(1 - p - q + pq)]^{z-1}$

$\Rightarrow \frac{p}{p+q-pq}$

Hence we get:

$\Rightarrow \frac{p}{p+q-pq}\sum_x x(1 - p)^{x-1}p(1 - q)^{x-1}$

$\Rightarrow (p + q - pq)\sum_x x(1 - p - q + pq)^{x-1}$

$\Rightarrow E[X|X \leq Y] = \frac{1}{p+q-pq}$

(4) Mitzenmacher 3.7

Individual Problems

(5) $\boxed{\text{Mitzenmacher 3.1}}$

Since X is being chosen uniformly at random from $[1, n]$ we know that $P(X = x) = \frac{1}{n}$. Using the formula $V(X) = E[X^2] - (E[X])^2$ we get:

$\Rightarrow \sum_{x=1}^{n} x^2(\frac{1}{n})$ - $(\sum_{x=1}^{n} x(\frac{1}{n}))^2$

$\Rightarrow \frac{1}{n}(1 + 4 + 9 + ... + n^2) - (\frac{1}{n}(1 + 2 + 3 + ... + n))^2$

$\Rightarrow \frac{1}{n}(\frac{n(n+1)(2n+1)}{6}) - \frac{1}{n}(\frac{n(n+1)}{2})^2$

$\Rightarrow \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$

$\Rightarrow \frac{(n+1)(4n+2-3n-3)}{12}$

$\Rightarrow \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$