CS/MATH 335: PROBABILITY, COMPUTING, AND GRAPH THEORY
HOMEWORK 8
DUE IN CLASS TUESDAY, NOVEMBER 15, 2016
TAYLOR HEILMAN INDIVIDUAL

Show all work to ensure full credit.

Partner Problems

Roughly in order of difficulty. CORRECT VERSION.

(1) Mitzenmacher 6.6.

(2) Suppose you are given a graph G and you are 3 coloring the vertices. Prove that there exists a coloring in which two thirds of the edges have distinct colors on their ends.

(3) Mitzenmacher 6.7. Hint: sample and modify.

(4) Mitzenmacher 6.9. Hint: part (a) is pretty easy. For part (b), follow your nose, looking at a random ranking (there are $n!$) and thinking about how many edges in the tournament disagree with a given ranking. In the middle of your analysis, you may find yourself looking at an event like $P(X_i \geq \frac{51}{100}\binom{n}{2})$. This is a job for Chernoff bounds! The "sufficiently large n" part comes at the very end, to show that a certain probability is strictly less than 1.

Individual Problems

(5) Streaming Exercises from Ullman's book:
- (a) Exercise 4.3.1
- (b) Exercise 4.5.4
- (c) Exercise 4.6.1
- (d) Exercise 4.6.2

a.) Given the fact that we have 8 billion bits, 1 billion members of the set S and 3 hash functions we see that $n = 8(10^9), m = 10^9, k = 3$. Thus the $pr(bit = 0) = e^{\frac{-3(10^9)}{8(10^9)}}$ which simplifies to $\frac{1}{e^{\frac{3}{8}}}$ and $pr(bit = 1) = (1 - e^{\frac{-3(10^9)}{8(10^9)}})$ which simplifies to $(1 - \frac{1}{e^{\frac{3}{8}}})$. Hence, the probability of getting a false positive is the probability of hashing to a 1 from every hash function, which is $= (1 - \frac{1}{e^{\frac{3}{8}}})^3 = .0305$.

Following the same thought process the probability of a false positive with 4 hash functions is simply $(1 - \frac{1}{e^{\frac{4}{8}}})^4 = .0239$

b.) Given the stream: 3 1 4 1 3 4 2 1 2

We compute the values of each variable. From these values we use the formula: $n(3v^2 - 3v + 1)$ where $n = 9$ and $v = X.val$ to find the estimate.

we get

$X_1 = X_1.element = 3, X_1.value = 2$, estimate = 63

$X_2 = X_2.element = 1, X_2.value = 3$, estimate = 171

$X_3 = X_3.element = 4, X_3.value = 2$, estimate = 63

$X_4 = X_4.element = 1, X_4.value = 2$, estimate = 63

$X_5 = X_5.element = 3, X_5.value = 1$, estimate = 9

$X_6 = X_6.element = 4, X_1.value = 1$, estimate = 9

$X_7 = X_7.element = 2, X_1.value = 2$, estimate = 63

$X_8 = X_8.element = 1, X_1.value = 1$, estimate = 9

$X_9 = X_9.element = 2, X_1.value = 1$, estimate = 9

the average of the estimate comes out to = 51, which is the exact value of the third moment. We can confirm this by calculating the third moment:

$2^3 + 3^3 + 2^3 + 2^3 = 51$

c.) Given the following bit stream: ..1011011000101110110010110 we divide the bitstream into the following buckets:

$\underline{..101}$ $\underline{101100010}$ $\underline{11101}$ $\underline{1001}$ 0 $\underline{1}$ $\underline{1}$ 0

To estimate the amount of ones in the last 5 bits we count the first two buckets with size 1, then we count half of the bucket at position $t - 4$ that has size=2. Thus for our estimation get 1+1+1= 3, which is the correct answer of 3.

To estimate the amount of ones in the last 15 bits we follow the same algorithm. We count the windows at positions $(t - 1)size = 1, (t - 2)size = 1, (t - 4)size = 2, and(t - 8)size = 4$. Lastly we count half the window at position $(t - 14)$ that has size=4. Thus we get 1+1+2+4+2= 10 which is also 1 off from the correct answer of 9.

d.) Given the stream 1001011011101 I found 4 different ways to partition the stream into buckets. Since there are 8 ones in the stream, we can have..

One bucket of size eight:

$\underline{1001011011101}$

Two buckets of size 4:

$\underline{1001011}$ 0 $\underline{11101}$

Two buckets of size 2 and one bucket of size 4:

$\underline{1001011}$ 0 $\underline{11}$ $\underline{101}$

Two buckets of size 1, one bucket of size 2, and one bucket of size 4:

$\underline{1001011}$ 0 $\underline{11}$ $\underline{1}$ 0 $\underline{1}$

BONUS Individual Problem:

(6) Prove that $R(3, 3, \ldots, 3) \leq 3 \cdot r!$, where there are r copies of 3 on the left. THEN, using that result, prove that if $n \geq 3 \cdot r!$ then no matter how the set $[n] = \{1, 2, \ldots, n\}$ is partitioned into r classes, there must be a solution of the equation $x + y = n$ where all three numbers are in the same class. This shows an application of Ramsey Theory to Number Theory.