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Math 210: Homework 3 - Spring 2015

1. (Problem 3.10) Prove that if a and c are odd integers, then $ab + bc$ is even for every integer b.

   PF

   Suppose that a, c are odd integers

   Since a, c are odd, there exists $t, h \in \mathbb{Z}$ such that $a = 2t + 1$ and $b = 2h + 1$

   It follows that,

$$\begin{aligned} ab + bc &= b(a + c) \\ &= b(2t + 1 + 2h + 1) \\ &= b(2t + 2h + 2) \\ &= 2b(t + h + 1) \end{aligned} \tag{1}$$

   Therefore, ab + bc is even because b(t + h + 1) $\in \mathbb{Z}$ and 2 times any integer is an even number. //

2. (Problem 3.20) Let $x \in \mathbb{Z}$. Prove that $3x + 1$ is even if and only if $5x - 2$ is odd.

   PF

   For the forward direction

   Assume $3x$ + 1 is even

   Therefore, $3x$ + 1 = $2k$ for some $k \in \mathbb{Z}$

   So,

$$\begin{aligned} 5x - 2 &= 2x + (3x + 1) - 3 \\ &= 2x + (3x + 1) - 3 \\ &= 2x + 2k - 3 \\ &= 2x + 2k - 4 + 1 \\ &= 2(x + k - 2) + 1 \end{aligned} \tag{2}$$

$5x$ - 2

= $2x$ + ($3x$ + 1) - 3

= $2x$ + $2k$ - 3

= $2x$ + $2k$ - 4 + 1

= 2(x + k - 2) + 1

Therefore, $5x$ - 2 is odd because (x + k - 2) $\in \mathbb{Z}$ and 2 times any integer + 1 is an odd number.

For the other direction

Assume $5x$ - 2 is odd

Therefore, $5x$ - 2 = $2k$ + 1 for some $k \in \mathbb{Z}$

So,

$$\begin{aligned} 3x + 1 &= -2x + (5x - 2) + 3 \\ &= -2x + 2k + 1 + 3 \\ &= -2x + 2k + 4 \\ &= 2(-x + k + 2) \end{aligned} \tag{3}$$

Therefore $3x$ + 1 is even because (-x + k + 2) $\in \mathbb{Z}$ and 2 times any integer is an even number.

Thus $3x + 1$ is even if and only if $5x - 2$ is odd. //

3. (Problem 3.30) Let $x, y \in \mathbb{Z}$. Prove that $x - y$ is even if and only if x and y are of the same parity. PF

   Suppose that x and y are of the same parity.

   Case 1: x and y are even.

   Therefore, $x = 2k$ and $y = 2t$ for some $k, t \in \mathbb{Z}$

   So, $x - y$

   $= 2k$ - $2t$

   $= 2(k - t)$

   Since $(k - t)$ in $\in \mathbb{Z}$ $x - y$ is even

   Case 2: x and y are odd.

   Therefore, $x = 2k + 1$ and $y = 2t+ 1$ for some $k, t \in \mathbb{Z}$

   So, $x - y$

   $= 2k + 1$ - $2t + 1$

   = 2(k - t + 1)

   Since $(k - t + 1)$ in $\in \mathbb{Z}$ $x - y$ is odd. //

4. (Problem 4.8) In Result 4.4, it was proved for an integer x that if $2 \mid (x^2 - 1)$, then $4 \mid (x^2 - 1)$. Prove that if $2 \mid (x^2 - 1)$, then $8 \mid (x^2 - 1)$.

   PF

   Assume 2 — (x^2 - 1)

   So x^2 - 1 = $2y$ for some $y \in \mathbb{Z}$

   Thus, x^2 = 2y + 1 is an odd integer.

   By theorem 3.12 x is also an odd integer.

   So $x = 2z + 1$ for some $z \in \mathbb{Z}$

   Consider x^2 - 1 = $(4b + 1)^2$ - 1

   $= 16b^2 + 8b$

$= 8(2b + b)$

Since $(2b + b)$ is an integer, 8 — x^2 - 1

5. (Problem 4.16) Let $a, b \in \mathbb{Z}$. Prove that if $a^2 + 2b^2 \equiv 0 \pmod 3$, then either a and b are both congruent to 0 modulo 3 or neither is congruent to 0 modulo 3.

   PF

   Contrapositive

   Suppose $a^2 + 2b^2 \not\equiv 0 \pmod 3$

   Assume that ONE of a or b is $\equiv$ to 0 modulo 3

   Case 1: $a \equiv 0 \pmod 3$ and $b \not\equiv 0 \pmod 3$, then 3 — a and $a = 3j$ for some $j \in \mathbb{Z}$. And $b \not\equiv 0 \pmod 3$, so $3 \not| b$, then $3 \mid b^2 - 1$, then $b^2 - 1 = 3k$ for some $k \in \mathbb{Z}$.

   Then, $a^2 + 2b^2$

   $=(3j)^2 + 2(3k + 1)$

   $= 3(3j^2 + 2k) + 2.$

   let $t = 3j^2 + 2k$

   Since $a^2 + 2b^2 = 3t$+2, then $a^2 + 2b^2 \equiv 2 \pmod 3$ and $a^2 + 2b^2 \not\equiv 0 \pmod 3$

   Case 2: $b \equiv 0 \pmod 3$ and $a \not\equiv 0 \pmod 3$, then 3 — a and $a = 3j$ for some $j \in \mathbb{Z}$. And $b \not\equiv 0 \pmod 3$, so $3 \not| b$, then $3 \mid b^2 - 1$, then $b^2 - 1 = 3k$ for some $k \in \mathbb{Z}$.

   Thus, a^2-1 $= 3h$ for some $h \in \mathbb{Z}$ and $b = 3l$ for some $l \in \mathbb{Z}$

   Then, $a^2 + 2b^2$

   $= (3h + 1)$ +2(3l)^2

   $= 3(h + 6l)$ +1

   Let $z = 3(h + 6l)$

   Since $a^2 + 2b^2 = 3z$+1, then $a^2 + 2b^2 \equiv 1 \pmod 3$ and $a^2 + 2b^2 \not\equiv 0 \pmod 3$

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