Name: **Due: Fri. Nov. 6**

Math 210: Homework 6 - Fall 2015

1. Let $a_1 = 2, a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers $n \geq 1$. PF

   We proceed by Induction.

   Assume n = 1

   So, $a_1 = 2 = 2^1$

   Assume that $a_{k+2} = 5a_{k+1} - 6a_k$ for all $k \geq 1$ and $a_k = 2^k$

   Let $k = n + 1$

   $a_{k+2} = 5a_{k+1} - 6a_k$

   which equals $(5)2^{k+2} - (6)2^{k+1}$

   $= (5)(2^2)(2^k) - (6)(2^k)(2)$

   $= 20(2^k) - 12(2^k)$

   $= 8(2^k)$

   $= 2^3(2^k)$

   $= 2^k + 3$

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2. Use induction to show that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$$

   PF

   Let $n = 1$, $\sum_{i=1}^{1} \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{3} = \frac{n}{2n+1}$.

   Therefore $n = 1$ is true.

   Suppose $\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$ for all $\mathbb{N}k \geq 1$

   Consider $\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)}$

$= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$= \frac{k}{(2k+1)} + \frac{1}{(2k+2-1)(2k+2+1)}$

$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{(k+1)}{(2k+3)}$

$= \frac{(k+1)}{(2(k+1)+1)}$

The result then follows by the Principle of Induction.

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3. (Problem 6.44) Consider the sequence $f_0, f_1, f_2, \dots$ where

$$f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5 \text{ and } f_5 = 8.$$

The terms in this sequence are called *Fibonacci Numbers*.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.

(b) Prove that $2 \mid f_n$ if and only if $3 \mid n$.

PF

a.) $F1 = F2 = 1$ and $Fn = Fn - 1 + Fn - 2$ for $n \leq 3$

b.) We proceed by Induction.

For n=1, $f3n - 2, f2n - 1, f3n = f1, f2, f3 = 1,\ 1,\ 2$

Notice f1 = f2 = 1, which is odd, while f3 = 2 which is even.

Therefore the it is true for n = 1.

Let $k \geq 1$, for some $k \in \mathbb{N}$, where $f3k - 2, f3k - 1, f3k = (2l + 1, 2m + 1, 2n)$, where l, m, n are $\in \mathbb{Z}$

Observe that for f_n we have $f3k + 1 = f3k + f3k - 1$

So $f3k + 1$ is the sum of $f3k$, an even number, and $f3k - 1$, an odd number.

Therefore $f3k + 1$ is odd

Similarly $f3k + 2$ is the sum of $f3k$, an even number, and $f3k + 1$, an odd number.

Therefore $f3k+2$ is odd

Lastly, $F3k+3 = F3k+2+F3k+1$

So $F3k+3$ is the sum of two odd integers and is therefore even.

The result then follows by the Principle of Induction.

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4. Use induction to show that the Fibonacci numbers satisfy the formula

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 0.$$

PF

For $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 0$. let $n = 0$

Then $f_1 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^2 - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^2$

So $f_1 = 1$

Let $f_n = f_{n-2} + f_{n-1}$ and $f_{n+1} = f_{n-1} + f_n$

Then $f_{k+1} = f_{k-1} + f_k = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{k+2} - (\frac{1-\sqrt{5}}{2})^{k+2}]$

So $f_k = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k+1}$

And $f_{k-1} = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^k - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^k$

Then $f_{k+1} = \frac{1}{\sqrt{5}}[((\frac{1+\sqrt{5}}{2})^{k+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{k+1}] + \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^k - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^k]$

$= \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{k+2} - (\frac{1-\sqrt{5}}{2})^{k+2}]$

the result follows the Principle of Mathematical Induction.

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5. Use induction to show that for all $n \in \mathbb{N}$, the Fibonacci numbers satisfy:

$$f_0 + f_1 + \ldots + f_n = f_{n+2} - 1$$

PF

Let $f_i = f_{n+2} - 1$

Then when $n = 1$, $f_3 - 1 = 2 - 1 = 1$

So $f_i = f_1 = 1$

Hence, $n = 1$ is true

Now let $k \in \mathbb{N}$ and suppose True for $n = k$

$$\sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1}$$

Where $f_i = f_{k+2} - 1$

So $f_i = (f_{k+2} - 1) + f_{k+1}$

$= f_{k+3} - 1$

This holds true for $n = k + 1$ so the result follows the Principle of Mathematical Induction.

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